CONFORMAL DEFORMATIONS TO SCALAR-FLAT METRICS WITH CONSTANT MEAN CURVATURE ON THE BOUNDARY

FERNANDO C. MARQUES

ABSTRACT. Let (M^n, g) be a compact manifold with nonempty boundary and finite Sobolev quotient $Q(M^n, \partial M)$. We prove that there exists a conformal deformation which is scalar-flat and has constant boundary mean curvature, if $n = 4$ or 5 and the boundary is not umbilic. In particular we prove such existence for any smooth and bounded open set of the Euclidean space, finishing the remaining cases of a theorem of J. F. Escobar.

CONTENTS

1. INTRODUCTION

The classical Riemann mapping theorem is one of the most famous results in Mathematics. It states that any simply connected, proper domain of the plane is conformally diffeomorphic to a disk. This theorem is certainly false in higher dimensions, since the only bounded open subsets of $\mathbb{R}^n$, $n \geq 3$, that are conformally diffeomorphic to Euclidean balls are the Euclidean balls themselves. Nevertheless, we could ask how close to a ball one can get by means of a conformal deformation of the metric. In that spirit, J. F. Escobar proved in [7] (the proof in dimension 6 appeared in [10]) the following beautiful result:

Theorem 1.1 (Escobar). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain of the Euclidean space, $n \neq 4, 5$. Then there exists a smooth function $u > 0$ such*

---

that $\tilde{g} = u^{\frac{4}{n-2}}g$ is scalar-flat and has constant mean curvature on the boundary $\partial\Omega$.

In fact this is a special case of the following problem:

Yamabe Problem

Let (M^n, g) be an n-dimensional, compact, Riemannian manifold with boundary ∂M, $n \geq 3$. Is there a conformally related metric $\tilde{g}$ with zero scalar curvature and boundary ∂M of constant mean curvature?

That general question was addressed for the first time by Escobar in [7], where he gave an affirmative answer for a large class of manifolds (see Theorem 2 in that paper). The present author studied a higher dimensional case in [17], where the Weyl tensor on the boundary plays a natural role. We should also note that the classical Yamabe problem, for manifolds without boundary, was completely solved after the works of Yamabe([20]), Trudinger([19]), Aubin([2]) and Schoen([18]).

The problem we are interested in is equivalent, in analytical terms, to finding a smooth positive solution to the nonlinear boundary-value problem:

$$(1.1) \qquad \begin{cases} \Delta_g u - \frac{n-2}{4(n-1)}R_g u = 0 & \text{in} \quad M, \\ \frac{\partial u}{\partial \eta} + \frac{n-2}{2}h_g u = \frac{n-2}{2}cu^{\frac{n}{n-2}} & \text{on} \quad \partial M, \end{cases}$$

where Δ_g denotes the Laplace-Beltrami operator of the metric g, R_g is the scalar curvature, h_g is the mean curvature of the boundary with respect to g, η is the outward unit normal vector to ∂M and c is a constant. In fact, given a solution u to problem (1.1), the metric $\tilde{g} = u^{\frac{4}{n-2}}g$ is scalar-flat and its boundary mean curvature is equal to c. Note that the equations (1.1) are specially interesting due to the boundary nonlinearity.

The solutions to equations (1.1) can also be seen as critical points of the functional

$$Q(\phi) = \frac{\int_M(|\nabla\phi|^2 + \frac{n-2}{4(n-1)}R_g\phi^2)dv_g + \frac{n-2}{2}\int_{\partial M}h_g\phi^2 d\sigma_g}{(\int_{\partial M}|\phi|^{2\frac{(n-1)}{(n-2)}}d\sigma_g)^{\frac{(n-2)}{(n-1)}}}.$$

Here dv_g and $d\sigma_g$ denote the volume forms of M and ∂M, respectively.

In [7], Escobar introduced the Sobolev quotient

$$Q(M, \partial M) = \inf\{Q(\phi) : \phi \in C^1(\overline{M}), \phi \not\equiv 0 \text{ on } \partial M\},$$

proving that it is conformally invariant and always satisfies

$$Q(M, \partial M) \leq Q(B^n, \partial B^n),$$

where B^n denotes the unit ball in $\mathbb{R}^n$ endowed with the Euclidean metric.

Under the hypothesis that $Q(M, \partial M)$ is finite (which is the case if $R_g \geq 0$), he also showed that the strict inequality

$$Q(M, \partial M) < Q(B^n, \partial B^n)$$

assures the existence of a minimizing solution to problem (1.1).

The main result of the paper is:

Theorem 1.2. *Let (M^n, g) be a compact manifold with nonempty boundary, $n = 4$ or 5. Assume the boundary ∂M is not umbilic. Then*

$$Q(M^n, \partial M) < Q(B^n, \partial B).$$

If $Q(M^n, \partial M)$ is finite, then there exists a smooth function $u > 0$ such that $\tilde{g} = u^{\frac{4}{n-2}} g$ is scalar-flat and has constant boundary mean curvature.

In the case of a smooth bounded domain $\Omega \subset \mathbb{R}^n$, we know the boundary is not umbilic unless the domain is a ball (in which case the existence is trivial). We also have that $Q(\Omega, \partial\Omega)$ is finite because the Euclidean metric is scalar-flat.

Therefore we finish the remaining cases of Theorem 1.1 as a corollary:

Corollary 1.3. *Theorem 1.1 is also true in dimensions 4 and 5.*

The proof of Theorem 1.2 is based on explicitly constructing a function ϕ, with support in a small halfball around a nonumbilic point $P \in \partial M$, such that

(1.2) $$Q(\phi) < Q(B^n, \partial B^n).$$

The usual strategy in this kind of problem (which goes back to Aubin, [2]) consists in defining the function ϕ, in the small halfball, as one of the standard entire solutions to the corresponding Euclidean equations. In our context those are

$$U_\varepsilon(x, t) = \left(\frac{\varepsilon}{(\varepsilon + t)^2 + |x|^2} \right)^{\frac{n-2}{2}}.$$

The next step would be to expand the quotient of ϕ in powers of ε and, by exploiting the local geometry around P, show that inequality (1.2) holds if ε is small. In order to simplify the asymptotic analysis we use conformal Fermi coordinates (introduced in [17]), which play the same role the conformal normal coordinates (see [16]) did in the classical Yamabe problem.

The difficulty arises because, when $3 \leq n \leq 5$, the first correction term in the expansion has the wrong sign. When $n = 3$, Escobar proved the strict inequality by applying the Positive Mass Theorem (a global construction originally due to Schoen, [18]). This argument does not work when $4 \leq n \leq 5$ because the metric is not sufficiently flat around the nonumbilic point $P \in \partial M$.

Our point of view in this article is that the test functions U_ε are not optimal in dimensions 4 and 5, but the problem should still be local. This kind of phenomenon does not appear in the solution of the Yamabe problem for manifolds without boundary.

In order to prove the strict inequality (1.2), we introduce

$$(1.3) \qquad \psi_\varepsilon(x,t) = \varepsilon^{\frac{n-2}{2}} h_{ij} x^i x^j t \left(\frac{1}{(\varepsilon+t)^2 + |x|^2} \right)^{\frac{n}{2}},$$

where h_{ij} are the coefficients of the second fundamental form at P. Our test function ϕ is defined as

$$\phi = U_\varepsilon + \psi_\varepsilon,$$

around $P \in \partial M$.

When $n = 5$, this solves the problem because the first correction term in the expansion of $Q(\phi)$ is strictly negative, but if $n = 4$ one can check that the term actually vanishes. In order to deal with that we need to work with a small perturbation $\phi = U_\varepsilon + \psi_\varepsilon + \delta T$.

The motivation for the definition (1.3) came from blowup analysis ideas, by determining good enough approximations of solutions concentrating at a boundary point. We think it should be possible to apply this sort of reasoning to other similar problems.

Other works concerning conformal deformation on manifolds with boundary include [1], [4], [5], [8], [9], [11], [12], [13], [14] and [15].

2. Preliminaries and Statements

Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary, $n \geq 3$. In what follows, let R_g denote the scalar curvature of the metric g, and let h_g denote the mean curvature of ∂M.

We are interested in finding a conformal deformation $\tilde{g} = u^{\frac{4}{n-2}} g$ which is scalar-flat and has constant mean curvature on the boundary. This geometric problem is equivalent to finding a constant $c \in \mathbb{R}$ and a positive solution u to

$$(2.1) \qquad \begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u = 0 & \text{in} \quad M, \\ \frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u = \frac{n-2}{2} c u^{\frac{n}{n-2}} & \text{on} \quad \partial M. \end{cases}$$

The operator $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is called the *conformal Laplacian* of g.

In order to obtain the variational formulation of problem (2.1) we need to introduce the Sobolev quotient of M.

Given $\phi \in H^1(M)$, define

$$E(\phi) = \int_M (|\nabla_g \phi|^2 + \frac{n-2}{4(n-1)} R_g \phi^2) dv_g + \frac{n-2}{2} \int_{\partial M} h_g \phi^2 d\sigma,$$

and also

$$Q(\phi) = \frac{E(\phi)}{\left(\int_{\partial M} |\phi|^{2\frac{(n-1)}{(n-2)}} d\sigma\right)^{\frac{(n-2)}{(n-1)}}}.$$

One can now check that solutions to equations (2.1) are the positive functions which are critical points of Q.

The Sobolev quotient $Q(M, \partial M)$ is defined as

$$Q(M, \partial M) = \inf\{Q(\phi) : \phi \in C^1(\overline{M}), \phi \not\equiv 0 \text{ on } \partial M\}.$$

It is not difficult to see ([7]) that the Sobolev quotient is a conformal invariant.

It follows from the work in [7] that whenever

$$-\infty < Q(M^n, \partial M) < Q(B^n, \partial B),$$

there is a minimizer of Q. This minimizer is automatically a solution to the problem (2.1).

The number $Q(B^n, \partial B)$ also appears as the best constant in the Sobolev-trace inequality:

$$\left(\int_{\partial \mathbb{R}^n_+} |\phi|^{2\frac{(n-1)}{(n-2)}} dx\right)^{\frac{n-2}{n-1}} \leq \frac{1}{Q(B^n, \partial B)} \int_{\mathbb{R}^n_+} |\nabla \phi|^2 dx dt,$$

for every $\phi \in H^1(\mathbb{R}^n_+)$. Here $\mathbb{R}^n_+ = \{(x,t) \in \mathbb{R}^n : t \geq 0\}$.

It was proven by Escobar [6], and independently by Beckner [3], that the equality is achieved by

$$U_\varepsilon(x,t) = \left(\frac{\varepsilon}{(\varepsilon+t)^2 + |x|^2}\right)^{\frac{n-2}{2}},$$

which is a solution to the boundary-value problem:

(2.2) $$\begin{cases} \Delta U_\varepsilon = 0 & \text{in } \mathbb{R}^n_+, \\ \frac{\partial}{\partial t} U_\varepsilon = -(n-2) U_\varepsilon^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

One can check

$$\int_{\mathbb{R}^n_+} |\nabla U_\varepsilon|^2 dx dt = (n-2) \int_{\partial\mathbb{R}^n_+} U_\varepsilon^{2\frac{(n-1)}{n-2}} dx,$$

and also

(2.3) $$Q(B^n, \partial B^n) = (n-2)\left(\int_{\partial\mathbb{R}^n_+} U_\varepsilon^{2\frac{(n-1)}{n-2}} dx\right)^{\frac{1}{n-1}}.$$

In the next section we will prove our main result:

Theorem 2.1. *Let (M^n, g) be a compact manifold with nonempty boundary, $n = 4$ or 5. Assume the boundary ∂M is not umbilic. Then*

$$Q(M^n, \partial M) < Q(B^n, \partial B).$$

If $Q(M^n, \partial M)$ is finite, then there exists a smooth function $u > 0$ such that $\tilde{g} = u^{\frac{4}{n-2}} g$ is scalar-flat and has constant boundary mean curvature.

The proof we will give in the next section works for any $n \geq 4$, but we only state the theorem in dimensions 4 and 5 because those were the only cases not covered by Escobar, when the boundary is not umbilic.

If the manifold is a smooth bounded domain $\Omega \subset \mathbb{R}^n$, we know the boundary is not umbilic unless the domain is a ball (in which case the existence is trivial). We also know the functional Q is bounded from below, since the euclidean metric is scalar-flat.

Therefore we get the following Riemann mapping-type theorem as a corollary:

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain of the Euclidean space, $n = 4$ or 5. Then there exists a smooth function $u > 0$ such that $\tilde{g} = u^{\frac{4}{n-2}} g$ is scalar-flat and has constant boundary mean curvature.*

3. Estimating the Sobolev Quotient

In this section we will prove the Theorem 2.1 by constructing a function ϕ with

$$Q(\phi) < Q(B^n, \partial B).$$

Let $P \in \partial M$ be a nonumbilic point, and choose geodesic normal coordinates $(x_1, \ldots, x_{n-1})$ on the boundary, centered at P. We say that $(x_1, \ldots, x_{n-1}, t)$ are the *Fermi coordinates* of the point $\exp_x(t\eta(x)) \in M$, where $\eta(x)$ denotes the inward unit vector normal to the boundary at $x \in \partial M$ and $t \geq 0$ is small. It is easy to see that in these coordinates $g_{tt} \equiv 1$ and $g_{ti} \equiv 0$. The expansion of the coefficients g_{ij} up to second order is given by (see Lemma 2.2 in [17]):

(3.1) $g_{ij} = \delta_{ij} - 2h_{ij}t$

$$-\frac{1}{3}\overline{R}_{ikjl}x_k x_l - 2h_{ij,k}tx_k + (-R_{titj} + h_{is}h_{sj})t^2 + O(|(x,t)|^3).$$

All the coefficients are computed at P, where h_{ij} denotes the second fundamental form with respect to the inward unit normal, and R_{titj}, $\overline{R}_{ikjl}$ denote components of the full Riemannian curvature tensors of M and ∂M, respectively.

Since the Sobolev quotient $Q(M^n, \partial M)$ is a conformal invariant, we can assume (see Section 3 in [17]) there exist conformal Fermi coordinates, i.e., Fermi coordinates $(x_1, \ldots, x_{n-1}, t)$ centered at P such that

$$(3.2) \qquad dv_g = (1 + O(|(x,t)|^N))dxdt,$$

where N is arbitrarily large.

In general, from the expansion (3.1):

$$\det g = 1 - 2Ht$$
$$-\frac{1}{3}\overline{R}_{kl}x_kx_l - 2H_{,k}tx_k + (-R_{tt} - |h|^2 + 2H^2)t^2 + O(r^3).$$

Here H denotes the mean curvature, $|h|^2 = \sum_{i,j}(h_{ij})^2$, and R_{tt}, $\overline{R}_{kl}$ denote components of the Ricci tensors of M and ∂M, respectively.

Hence, in conformal Fermi coordinates, we have $H = H_{,k} = \overline{R}_{kl} = 0$, and $R_{tt} = -\sum_{i,j}(h_{ij})^2$ at P.

In what follows $B_\rho^+ = \{(x,t) \in \mathbb{R}_+^n : |x|^2 + t^2 \leq \rho^2\}$, $S_\rho^+ = \{(x,t) \in \mathbb{R}_+^n : |x|^2 + t^2 = \rho^2\}$, and $B_\rho^{n-1} = \{(x,0) \in \mathbb{R}^n : |x|^2 \leq \rho^2\}$.

Fix $A \in \mathbb{R}$, $\rho_0 > 0$ and $\varepsilon > 0$.

If

$$\psi_\varepsilon(x,t) \quad = \quad A\varepsilon^{\frac{n-2}{2}}h_{jk}x^jx^kt\left(\frac{1}{(\varepsilon+t)^2 + |x|^2}\right)^{\frac{n}{2}},$$

define

$$u_\varepsilon = U_\varepsilon + \psi_\varepsilon$$

for $(x,t) \in \mathbb{R}_+^n$, where U_ε is as in the previous section.

Note that

$$u_\varepsilon(x,t) = (1 + O(r))\,U_\varepsilon(x,t).$$

Hence, if ρ_0 is sufficiently small and A is fixed:

$$\frac{1}{2}U_\varepsilon \leq u_\varepsilon \leq 2U_\varepsilon \text{ in } B_{2\rho_0}^+.$$

Let $\eta(\rho)$ be a smooth cut-off function satisfying $\eta(\rho) = 1$ for $\rho \leq \rho_0$, $\eta(\rho) = 0$ for $\rho \geq 2\rho_0$, $0 \leq \eta \leq 1$ and $|\eta'(\rho)| \leq C\rho_0^{-1}$ if $\rho_0 \leq \rho \leq 2\rho_0$.

Define

$$\phi = \eta u_\varepsilon.$$

Let us first estimate $E_g(\phi)$ on $A_{\rho_0}^+ = B_{2\rho_0}^+ \setminus B_{\rho_0}^+$.

In what follows we will denote by C different positive constants.

First observe

$$|\nabla\phi|_g^2 \leq C|\nabla\phi|^2 \leq C(|\nabla\eta|^2u_\varepsilon^2 + \eta^2|\nabla u_\varepsilon|^2).$$

Now

$$
\begin{aligned}
\int_{A_{\rho_0}^+} |\nabla \eta|^2 u_\varepsilon^2 \, dx dt \ & \leq \ C \rho_0^{-2} \int_{A_{\rho_0}^+} U_\varepsilon^2 \, dx dt \\
& \leq \ C \rho_0^{-2} \varepsilon^{n-2} \int_{A_{\rho_0}^+} \left(\frac{1}{t^2 + |x|^2} \right)^{n-2} dx dt \\
& = \ C \rho_0^{-2} \varepsilon^{n-2} \int_{\rho_0}^{2\rho_0} \rho^{3-n} d\rho \\
& \leq \ C \varepsilon^{n-2} \rho_0^{2-n}.
\end{aligned}
$$

We also have

$$
\int_{A_{\rho_0}^+} \eta^2 |\nabla u_\varepsilon|^2 \, dx dt \leq C \varepsilon^{n-2} \int_{\rho_0}^{2\rho_0} \rho^{1-n} d\rho \leq C \varepsilon^{n-2} \rho_0^{2-n}.
$$

The other terms are estimated as follows:

$$
\int_{A_{\rho_0}^+} R_g \phi^2 \, dx dt \leq C \int_{A_{\rho_0}^+} U_\varepsilon^2 \, dx dt \leq C \varepsilon^{n-2} \rho_0^{4-n},
$$

and

$$
\int_{A_{\rho_0}^+ \cap \partial M} h_g \phi^2 \, dx \leq C \int_{A_{\rho_0}^+ \cap \partial M} U_\varepsilon^2 \, dx \leq C \varepsilon^{n-2} \rho_0^{3-n}.
$$

We conclude that

$$
E_{M \setminus B_{\rho_0}^+}(\phi) \leq C \varepsilon^{n-2} \rho_0^{2-n}.
$$

Now let us turn our attention to the quantity $E_{B_{\rho_0}^+}(\phi)$. Since N is as large as we want in equality (3.2), we can just assume $dv_g = dx dt$ in the expansions to come.

Using that $g_{tt} = 1$ and $g_{ti} = 0$ in Fermi coordinates, we obtain

$$
\int_{B_{\rho_0}^+} |\nabla \phi|_g^2 \, dx dt = \int_{B_{\rho_0}^+} |\nabla u_\varepsilon|^2 \, dx dt + \int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(u_\varepsilon)_i (u_\varepsilon)_j \, dx dt,
$$

where the indices on functions denote derivatives.

Now

$$
\int_{B_{\rho_0}^+} |\nabla u_\varepsilon|^2 \, dx dt = \int_{B_{\rho_0}^+} (|\nabla U_\varepsilon|^2 + |\nabla \psi_\varepsilon|^2) \, dx dt.
$$

We are using that

$$
\int_{B_{\rho_0}^+} \langle \nabla U_\varepsilon, \nabla \psi_\varepsilon \rangle \, dx dt = 0,
$$

since

$$
\int_{S_r^{n-2}} h_{ij} x^i x^j = \frac{H}{n-1} \int_{S_r^{n-2}} r^2 = 0.
$$

Note that from the properties (2.2) and the identity (2.3), we can get

$$(3.3) \qquad \int_{B_{\rho_0}^+} |\nabla U_\varepsilon|^2 dx dt \leq Q(B^n, \partial B^n) \left(\int_{B_{\rho_0}^{n-1}} U_\varepsilon^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{n-2}{n-1}}.$$

Here we are using that $\frac{\partial U_\varepsilon}{\partial \eta} < 0$ on $\partial B_{\rho_0}^+ \cap \mathbb{R}_+^n$, where η denotes the outward unit normal vector.

On the other hand

$$u_\varepsilon(x, 0) = U_\varepsilon(x, 0).$$

Hence

$$\int_{B_{\rho_0}^+} |\nabla U_\varepsilon|^2 dx dt \leq Q(B^n, \partial B^n) \left(\int_M \phi^{\frac{2(n-1)}{n-2}} dv_g \right)^{\frac{n-2}{n-1}}.$$

Now integration by parts gives:

$$
\begin{aligned}
(3.4) \quad \int_{B_{\rho_0}^+} |\nabla \psi_\varepsilon|^2 dx dt &= -\int_{B_{\rho_0}^+} \psi_\varepsilon \Delta \psi_\varepsilon dx dt \\
&\quad + \int_{S_{\rho_0}^+} \psi_\varepsilon \frac{\partial \psi_\varepsilon}{\partial r} d\sigma_{\rho_0} - \int_{B_{\rho_0}^{n-1}} \psi_\varepsilon \frac{\partial \psi_\varepsilon}{\partial t} dx \\
&= -\int_{B_{\rho_0}^+} \psi_\varepsilon \Delta \psi_\varepsilon dx dt + O(\varepsilon^{n-2} \rho_0^{4-n}).
\end{aligned}
$$

We are using the definition of ψ_ε to estimate the integral over the upper hemisphere $S_{\rho_0}^+$ and to observe that the integral over $B_{\rho_0}^{n-1}$ vanishes.

Let us now compute $\Delta \psi_\varepsilon$.

If $Z = (\varepsilon + t)^2 + |x|^2$, then $\Delta Z^{-\frac{n}{2}} = 2n Z^{-\frac{n+2}{2}}$ and

$$
\begin{aligned}
\Delta \psi_\varepsilon &= A \varepsilon^{\frac{n-2}{2}} \{ \Delta(h_{jk} x^j x^k t) Z^{-\frac{n}{2}} - n \langle \nabla(h_{jk} x^j x^k t), Z^{-\frac{n}{2}-1} \nabla Z \rangle \\
&\qquad\qquad + h_{jk} x^j x^k t \Delta(Z^{-\frac{n}{2}}) \} \\
&= -2n A \varepsilon^{\frac{n-2}{2}} h_{jk} x^j x^k Z^{-\frac{n+2}{2}} (2t + \varepsilon).
\end{aligned}
$$

We will use that

$$\int_{S_r^{n-2}} q = \frac{r^2}{d(d+n-3)} \int_{S_r^{n-2}} \Delta q,$$

if q is a homogeneous polynomial of degree d.

Therefore

$$\int_{S_r^{n-2}} h_{ij} h_{kl} x^i x^j x^k x^l = \frac{2(h_{ij})^2}{(n-1)(n+1)} \int_{S_r^{n-2}} r^4,$$

and then, from identity (3.4),

$$
\begin{aligned}
\int_{B_{\rho_0}^+} |\nabla \psi_\varepsilon|^2 dx dt &= 2nA^2 \varepsilon^{n-2} \int_{B_{\rho_0}^+} \frac{h_{ij} h_{kl} x^i x^j x^k x^l t(2t+\varepsilon)}{((\varepsilon+t)^2 + |x|^2)^{n+1}} dx dt \\
&\quad + O(\varepsilon^{n-2} \rho_0^{4-n}) \\
&= \frac{4nA^2 \varepsilon^{n-2} \sum_{i,j} (h_{ij})^2}{(n-1)(n+1)} \int_{B_{\rho_0}^+} \frac{|x|^4 t(2t+\varepsilon)}{((\varepsilon+t)^2 + |x|^2)^{n+1}} dx dt \\
&\quad + O(\varepsilon^{n-2} \rho_0^{4-n}).
\end{aligned}
$$

Now

$$
\int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(u_\varepsilon)_i (u_\varepsilon)_j dx dt =
$$

$$
= \int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij}) \{ (U_\varepsilon)_i (U_\varepsilon)_j \quad + \quad 2(U_\varepsilon)_i (\psi_\varepsilon)_j \} dx dt + E_1,
$$

where

$$
E_1 = \begin{cases} O(\varepsilon^2 \rho_0) & \text{if} \quad n = 4, \\ O(\varepsilon^3 \log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n = 5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}
$$

To see this note that

$$
\begin{aligned}
E_1 &= \int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(\psi_\varepsilon)_i (\psi_\varepsilon)_j dx dt \\
&= \varepsilon^{n-2} O\left(\int_{B_{\rho_0}^+} \frac{|x|^5}{((\varepsilon+t)^2 + |x|^2)^n} dx dt \right)
\end{aligned}
$$

and perform the change of variables (Lemma 3.4 in [7]):

$$
(3.5) \quad \int_{B_{\rho_0}^+} \frac{t^k x^\alpha dx dt}{((\varepsilon+t)^2 + |x|^2)^{n-l}} = \varepsilon^{k+m+2l-n} \int_{B_{\frac{\rho_0}{\varepsilon}}^+} \frac{s^k y^\alpha dy ds}{((1+s)^2 + |y|^2)^{n-l}},
$$

where $\alpha = (i_1, \ldots, i_m)$ and $x^\alpha = x_{i_1} \cdots x_{i_m}$.

From the expansion (3.1), we can compute

$$
(3.6) \quad g^{ij}(x,t) = \delta_{ij} + 2h_{ij} t
$$
$$
+ \frac{1}{3} \overline{R}_{ikjl} x_k x_l + 2h_{ij,k} t x_k + (R_{titj} + 3h_{ik} h_{kj}) t^2 + O(|(x,t)|^3).
$$

We also have

$$
(U_\varepsilon)_i(x,t) = (2-n) \varepsilon^{\frac{n-2}{2}} x_i \left(\frac{1}{(\varepsilon+t)^2 + |x|^2} \right)^{\frac{n}{2}}.
$$

Therefore

$$\int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(U_\varepsilon)_i (U_\varepsilon)_j dx dt =$$

$$= \int_{B_{\rho_0}^+} ((R_{titj} + 3h_{ik}h_{kj})t^2 + O(|(x,t)|^3))(U_\varepsilon)_i (U_\varepsilon)_j dx dt$$

$$= (n-2)^2 \varepsilon^{n-2} \int_{B_{\rho_0}^+} \frac{(R_{titj} + 3h_{ik}h_{kj})t^2 x^i x^j}{((\varepsilon+t)^2 + |x|^2)^n} dx dt + E_2$$

$$= \frac{(n-2)^2}{n-1} \varepsilon^{n-2} (R_{tt} + 3\sum_{i,j}(h_{ij})^2) \int_{B_{\rho_0}^+} \frac{t^2|x|^2}{((\varepsilon+t)^2 + |x|^2)^n} dx dt + E_2$$

$$= \frac{2(n-2)^2}{n-1} \varepsilon^{n-2} \sum_{i,j}(h_{ij})^2 \int_{B_{\rho_0}^+} \frac{t^2|x|^2}{((\varepsilon+t)^2 + |x|^2)^n} dx dt + E_2,$$

where

$$E_2 = \begin{cases} O(\varepsilon^2 \rho_0) & \text{if} \quad n = 4, \\ O(\varepsilon^3 \log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n = 5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}$$

We are using symmetry arguments to observe that

$$\int_{B_{\rho_0}^+} (2h_{ij}t + \frac{1}{3}\overline{R}_{ikjl}x_k x_l + 2h_{ij,k}t x_k)(U_\varepsilon)_i (U_\varepsilon)_j dx dt = 0.$$

On the other hand, by integrating by parts and observing that $\psi_\varepsilon = 0$ when $t = 0$:

$$2\int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(U_\varepsilon)_i (\psi_\varepsilon)_j dx dt =$$

$$= 2 \int_{B_{\rho_0}^+} (2h_{ij}t + O(|(x,t)|^2))(U_\varepsilon)_i (\psi_\varepsilon)_j dx dt$$

$$= -4 \int_{B_{\rho_0}^+} h_{ij}t(U_\varepsilon)_{ij}\psi_\varepsilon dx dt + 4\rho_0^{-1} \int_{S_{\rho_0}^+} h_{ij}t(U_\varepsilon)_i \psi_\varepsilon x^j d\sigma_{\rho_0} + E_3,$$

where

$$E_3 = \begin{cases} O(\varepsilon^2 \rho_0) & \text{if} \quad n = 4, \\ O(\varepsilon^3 \log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n = 5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}$$

We can compute

$$(U_\varepsilon)_{ij}(x,t) = (2-n)\varepsilon^{\frac{n-2}{2}}\delta_{ij}\left((\varepsilon+t)^2 + |x|^2\right)^{-\frac{n}{2}}$$

$$+ n(n-2)\varepsilon^{\frac{n-2}{2}} x_i x_j \left((\varepsilon+t)^2 + |x|^2\right)^{-\frac{n+2}{2}}.$$

Therefore

$$2 \int_{B_{\rho_0}^+} (g^{ij} - \delta^{ij})(U_\varepsilon)_i(\psi_\varepsilon)_j dxdt =$$

$$= -4n(n-2)A\varepsilon^{n-2} \int_{B_{\rho_0}^+} \frac{h_{ij}h_{kl}x^ix^jx^kx^lt^2}{((\varepsilon+t)^2+|x|^2)^{n+1}} dxdt + E_3'$$

$$= -\frac{8n(n-2)}{(n-1)(n+1)}A\varepsilon^{n-2}\sum_{i,j}(h_{ij})^2 \int_{B_{\rho_0}^+} \frac{|x|^4t^2}{((\varepsilon+t)^2+|x|^2)^{n+1}} dxdt + E_3',$$

where $E_3' = E_3 + O(\varepsilon^{n-2}\rho_0^{4-n})$.

Now for the scalar curvature term:

$$\frac{n-2}{4(n-1)} \int_{B_{\rho_0}^+} R_g u_\varepsilon^2 dxdt = \frac{n-2}{4(n-1)} \int_{B_{\rho_0}^+} \left(R_g(0) + O(|(x,t)|)\right)u_\varepsilon^2 dxdt$$

$$= -\frac{n-2}{4(n-1)}\sum_{i,j}(h_{ij})^2 \int_{B_{\rho_0}^+} u_\varepsilon^2 dxdt + E_4,$$

where

$$E_4 = \begin{cases} O(\varepsilon^2\rho_0) & \text{if} \quad n=4, \\ O(\varepsilon^3\log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n=5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}$$

We are using that $R_g(P) = -(h_{ij})^2$ in conformal Fermi coordinates.

Since $u_\varepsilon^2 = U_\varepsilon^2 + 2U_\varepsilon\psi_\varepsilon + \psi_\varepsilon^2$, we get

$$\frac{n-2}{4(n-1)} \int_{B_{\rho_0}^+} R_g u_\varepsilon^2 dxdt = -\frac{n-2}{4(n-1)}(h_{ij})^2 \int_{B_{\rho_0}^+} U_\varepsilon^2 dxdt + E_5$$

$$= -\frac{n-2}{4(n-1)}(h_{ij})^2\varepsilon^{n-2} \int_{B_{\rho_0}^+} \frac{1}{((\varepsilon+t)^2+|x|^2)^{n-2}} dxdt + E_5,$$

where

$$E_5 = E_4 - \frac{n-2}{4(n-1)}\sum_{i,j}(h_{ij})^2 \int_{B_{\rho_0}^+} \psi_\varepsilon^2 dxdt.$$

After using the definition of ψ_ε and performing the change of variables (3.5) on the integral above, we get:

$$E_5 = \begin{cases} O(\varepsilon^2\rho_0) & \text{if} \quad n=4, \\ O(\varepsilon^3\log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n=5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}$$

The integral involving the mean curvature can be estimated as follows:

$$\int_{B_{\rho_0}^{n-1}} h_g u_\varepsilon^2 dx = \int_{B_{\rho_0}^{n-1}} O(|x|^2)u_\varepsilon^2 dx = \begin{cases} O(\varepsilon^2\rho_0) & \text{if} \quad n=4, \\ O(\varepsilon^3\log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n=5, \\ O(\varepsilon^3) & \text{if} \quad n \geq 6. \end{cases}$$

We are using that in conformal Fermi coordinates $H = H_k = 0$ at P.
Therefore,

$$(3.7) \quad E(\phi) \le Q(B^n, \partial B^n)\left(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} \, d\sigma_g\right)^{\frac{n-2}{n-1}}$$

$$+ \frac{4nA^2\varepsilon^2 \sum_{ij}(h_{ij})^2}{(n-1)(n+1)}\left\{2\int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} \, dyds\right.$$

$$+ \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^4 s}{((1+s)^2 + |y|^2)^{n+1}} \, dyds\Big\}$$

$$- \frac{8n(n-2)}{(n-1)(n+1)} A\varepsilon^2 \sum_{ij}(h_{ij})^2 \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} \, dyds$$

$$+ \frac{2(n-2)^2}{n-1} \varepsilon^2 \sum_{ij}(h_{ij})^2 \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{s^2|y|^2}{((1+s)^2 + |y|^2)^n} \, dyds$$

$$- \frac{n-2}{4(n-1)} \varepsilon^2 \sum_{ij}(h_{ij})^2 \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{1}{((1+s)^2 + |y|^2)^{n-2}} \, dyds + E',$$

where

$$E' = \begin{cases} O(\varepsilon^2 \rho_0^{-2}) & \text{if} \quad n = 4, \\ O(\varepsilon^3 \log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n = 5, \\ O(\varepsilon^3) & \text{if} \quad n \ge 6. \end{cases}$$

Let us divide the rest of the proof in two cases.

Case $n = 5$. We will apply the change of variables $y = (1+s)z$ in order to compare the different integrals in expansion (3.7).

First

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n_+} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} \, dyds \\ &= \int_0^\infty \frac{s^2}{(1+s)^{n-1}} \, ds \int_{\mathbb{R}^{n-1}} \frac{|z|^4}{(1+|z|^2)^{n+1}} \, dz \\ &= \frac{2\sigma_{n-2}}{(n-2)(n-3)(n-4)} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}} \, dr. \end{aligned}$$

For the second integral:

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^n_+} \frac{|y|^4 s}{((1+s)^2 + |y|^2)^{n+1}} dy \, ds \\
&= \int_0^\infty \frac{s}{(1+s)^{n-1}} ds \int_{\mathbb{R}^{n-1}} \frac{|z|^4}{(1+|z|^2)^{n+1}} dz \\
&= \frac{\sigma_{n-2}}{(n-2)(n-3)} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}} dr.
\end{aligned}
$$

Now the fourth integral:

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^n_+} \frac{|y|^2 s^2}{((1+s)^2 + |y|^2)^n} dy \, ds \\
&= \int_0^\infty \frac{s^2}{(1+s)^{n-1}} ds \int_{\mathbb{R}^{n-1}} \frac{|z|^2}{(1+|z|^2)^n} dz \\
&= \frac{2\sigma_{n-2}}{(n-2)(n-3)(n-4)} \int_0^\infty \frac{r^n}{(1+r^2)^n} dr.
\end{aligned}
$$

And the last integral:

$$
\begin{aligned}
I_5 &= \int_{\mathbb{R}^n_+} \frac{1}{((1+s)^2 + |y|^2)^{n-2}} dy \, ds \\
&= \int_0^\infty \frac{ds}{(1+s)^{n-3}} \int_{\mathbb{R}^{n-1}} \frac{dz}{(1+|z|^2)^{n-2}} \\
&= \frac{\sigma_{n-2}}{n-4} \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(\phi) \leq {}& Q(B^n, \partial B^n) \Big(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g \Big)^{\frac{n-2}{n-1}} \\
&+ A^2 \varepsilon^2 \sum_{i,j} (h_{ij})^2 \Big\{ \frac{8n}{(n-1)(n+1)} I_1 + \frac{4n}{(n-1)(n+1)} I_2 \Big\} \\
&+ A\varepsilon^2 \sum_{i,j} (h_{ij})^2 \Big\{ -\frac{8n(n-2)}{(n-1)(n+1)} I_1 \Big\} \\
&+ \varepsilon^2 \sum_{i,j} (h_{ij})^2 \Big\{ \frac{2(n-2)^2}{n-1} I_4 - \frac{n-2}{4(n-1)} I_5 \Big\} + E,
\end{aligned}
$$

where

$$
E = \begin{cases}
O(\varepsilon^2 \rho_0^{-2}) & \text{if} \quad n = 4, \\
O(\varepsilon^3 \log(\frac{\rho_0}{\varepsilon})) & \text{if} \quad n = 5, \\
O(\varepsilon^3) & \text{if} \quad n \geq 6.
\end{cases}
$$

We are using that the differences between the integrals in expansion (3.7) and the corresponding integrals over the entire $\mathbb{R}^n_+$ do not change the order of the error terms.

Now integration by parts yields

$$(3.8) \qquad \int_0^\infty \frac{r^n}{(1+r^2)^n} dr = \frac{2n}{n+1} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}} dr.$$

For the same reason

$$\int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr = \frac{2(n-2)}{n-1} \int_0^\infty \frac{r^n}{(1+r^2)^{n-1}} dr,$$

and also

$$(3.9) \qquad \int_0^\infty \frac{r^n}{(1+r^2)^{n-1}} dr = \frac{2(n-1)}{n+1} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr,$$

so

$$\int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr = \frac{4(n-2)}{n+1} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr.$$

Now identity (3.9) and the equality

$$\int_0^\infty \frac{r^n}{(1+r^2)^{n-1}} dr = \int_0^\infty \frac{r^n}{(1+r^2)^n} dr + \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr$$

immediately imply

$$\int_0^\infty \frac{r^{n+2}}{(1+r^2)^n} dr = \frac{n+1}{n-3} \int_0^\infty \frac{r^n}{(1+r^2)^n} dr.$$

Therefore

$$(3.10) \qquad \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr = \frac{8n(n-2)}{(n-3)(n+1)} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}} dr.$$

Hence

$$I_2 = \frac{(n-4)}{2} I_1,$$

$$I_4 = \frac{2n}{(n+1)} I_1,$$

and

$$I_5 = \frac{4n(n-2)^2}{(n+1)} I_1.$$

Thus

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$+A^2\varepsilon^2 \sum_{i,j}(h_{ij})^2 I_1\{\frac{8n}{(n-1)(n+1)} + \frac{2n(n-4)}{(n-1)(n+1)}\}$$

$$-A\varepsilon^2 \sum_{i,j}(h_{ij})^2 I_1\{\frac{8n(n-2)}{(n-1)(n+1)}\}$$

$$+\varepsilon^2 \sum_{i,j}(h_{ij})^2 I_1\{\frac{4n(n-2)^2}{(n-1)(n+1)} - \frac{n(n-2)^3}{(n-1)(n+1)}\} + E.$$

Since $n = 5$, we get

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$+\frac{5\varepsilon^2 \sum_{i,j}(h_{ij})^2 I_1}{24}\{10A^2 - 24A + 9\} + C\varepsilon^3 \log(\frac{\rho_0}{\varepsilon}).$$

Choosing $A = 1$, and observing that $\sum_{i,j}(h_{ij})^2(P) > 0$ since $P \in \partial M$ is not umbilic, we obtain

$$E(\phi) < Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

for small ε.

This finishes the proof when $n = 5$.

Case $n = 4$.

We will again apply the change of variables $y = (1 + s)z$ to the integrals in expansion (3.7).

For the first integral :

$$
\begin{aligned}
I_{1,\frac{\rho_0}{\varepsilon}} &= \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} dy ds \\
&= \int_{B^+_{\frac{\rho_0}{\varepsilon}} \cap \{s \le \frac{\rho_0}{2\varepsilon}\}} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} dy ds + O(1) \\
&= \int_{\mathbb{R}^n_+ \cap \{s \le \frac{\rho_0}{2\varepsilon}\}} \frac{|y|^4 s^2}{((1+s)^2 + |y|^2)^{n+1}} dy ds + O(1) \\
&= \int_0^{\frac{\rho_0}{2\varepsilon}} \frac{s^2}{(1+s)^{n-1}} ds \int_{\mathbb{R}^{n-1}} \frac{|z|^4}{(1+|z|^2)^{n+1}} dz + O(1) \\
&= \left(\sigma_{n-2} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}} dr \right) \log(\frac{\rho_0}{\varepsilon}) + O(1).
\end{aligned}
$$

The second integral :

$$
\begin{aligned}
I_{2,\frac{\rho_0}{\varepsilon}} &= \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^4 s}{((1+s)^2 + |y'|^2)^{n+1}} dy ds \\
&= O(1).
\end{aligned}
$$

Now the fourth integral :

$$
\begin{aligned}
I_{4,\frac{\rho_0}{\varepsilon}} &= \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{|y|^2 s^2}{((1+s)^2 + |y|^2)^n} dy ds \\
&= \int_{B^+_{\frac{\rho_0}{\varepsilon}} \cap \{s \le \frac{\rho_0}{2\varepsilon}\}} \frac{|y|^2 s^2}{((1+s)^2 + |y|^2)^n} dy ds + O(1) \\
&= \int_{\mathbb{R}^n_+ \cap \{s \le \frac{\rho_0}{2\varepsilon}\}} \frac{|y|^2 s^2}{((1+s)^2 + |y|^2)^n} dy ds + O(1) \\
&= \int_0^{\frac{\rho_0}{2\varepsilon}} \frac{s^2}{(1+s)^{n-1}} ds \int_{\mathbb{R}^{n-1}} \frac{|z|^2}{(1+|z|^2)^n} dz + O(1) \\
&= \left(\sigma_{n-2} \int_0^\infty \frac{r^n}{(1+r^2)^n} dr \right) \log(\frac{\rho_0}{\varepsilon}) + O(1).
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
I_{5,\frac{\rho_0}{\varepsilon}} &= \int_{B^+_{\frac{\rho_0}{\varepsilon}}} \frac{1}{((1+s)^2+|y|^2)^{n-2}}dyds \\
&= \int_{B^+_{\frac{\rho_0}{\varepsilon}}\cap\{s\le\frac{\rho_0}{2\varepsilon}\}} \frac{1}{((1+s)^2+|y|^2)^{n-2}}dyds + O(1) \\
&= \int_{\mathbb{R}^n_+\cap\{s\le\frac{\rho_0}{2\varepsilon}\}} \frac{1}{((1+s)^2+|y|^2)^{n-2}}dyds + O(1) \\
&= \int_0^{\frac{\rho_0}{2\varepsilon}} \frac{ds}{(1+s)^{n-3}} \int_{\mathbb{R}^{n-1}} \frac{dz}{(1+|z|^2)^{n-2}} + O(1) \\
&= \left(\sigma_{n-2} \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}}dr \right) \log(\frac{\rho_0}{\varepsilon}) + O(1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(\phi) \le Q(B^n,\partial B^n)(&\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}} \\
&+ A^2\varepsilon^2(h_{ij})^2\{\frac{8n}{(n+1)(n-1)}I_{1,\frac{\rho_0}{\varepsilon}}\} \\
&- A\varepsilon^2(h_{ij})^2\{\frac{8n(n-2)}{(n-1)(n+1)}I_{1,\frac{\rho_0}{\varepsilon}}\} \\
&+ \varepsilon^2(h_{ij})^2\{\frac{2(n-2)^2}{n-1}I_{4,\frac{\rho_0}{\varepsilon}} - \frac{n-2}{4(n-1)}I_{5,\frac{\rho_0}{\varepsilon}}\} + O(\varepsilon^2\rho_0^{-2}).
\end{aligned}
$$

Define

$$
J = \sigma_{n-2} \int_0^\infty \frac{r^{n+2}}{(1+r^2)^{n+1}}dr.
$$

Therefore, using identities (3.8) and (3.10):

$$
\begin{aligned}
E(\phi) \le Q(B^n,\partial B^n)(&\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}} \\
&+ (h_{ij})^2 J\varepsilon^2 \log(\frac{\rho_0}{\varepsilon})\{\frac{8nA^2}{(n+1)(n-1)} - \frac{8n(n-2)A}{(n-1)(n+1)} \\
&+ \frac{4n(n-2)^2}{(n-1)(n+1)} - \frac{8n(n-2)^2}{4(n-3)(n-1)(n+1)}\} + O(\varepsilon^2\rho_0^{-2}).
\end{aligned}
$$

Since $n = 4$, we get

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$+\frac{32}{15}(h_{ij})^2 J\varepsilon^2 \log(\frac{\rho_0}{\varepsilon})\{A^2 - 2A + 1\} + O(\varepsilon^2 \rho_0^{-2}).$$

The optimal choice is $A = 1$, by which we lose the logarithmic singularity:

$$(3.11) \qquad E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}} + O(\varepsilon^2 \rho_0^{-2}).$$

In order to solve the problem we will need to consider a small perturbation of ψ_ε :

$$\psi_{\varepsilon,\delta} = \psi_\varepsilon + \delta T,$$

where

$$(3.12) \qquad T = \varepsilon^{\frac{n-2}{2}} h_{ij} x^i x^j \left(\frac{1}{(\varepsilon + t)^2 + |x|^2} \right)^{\frac{n-1}{2}}.$$

Define $u_\varepsilon = U_\varepsilon + \psi_{\varepsilon,\delta}$ and $\phi = \eta u_\varepsilon$ as before.
Following the previous computations we get:

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$+2\delta \int_{B_{\rho_0}^+} \langle \nabla \psi_\varepsilon, T \rangle dxdt$$

$$+4\delta \int_{B_{\rho_0}^+} h_{ij} t (U_\varepsilon)_i (T)_j dxdt$$

$$+\delta^2 \int_{B_{\rho_0}^+} |\nabla T|^2 dxdt + O(\varepsilon^2 \rho_0^{-2}).$$

We are using that

$$\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g \geq \int_{B_{\rho_0}^{n-1}} u_\varepsilon^{\frac{2(n-1)}{(n-2)}} d\sigma_g,$$

and

$$\int_{B_{\rho_0}^{n-1}} u_\varepsilon^{\frac{2(n-1)}{(n-2)}} dx = \int_{B_{\rho_0}^{n-1}} (U_\varepsilon + \delta T)^{\frac{2(n-1)}{(n-2)}} dx$$

$$= \int_{B_{\rho_0}^{n-1}} \left(U_\varepsilon^{\frac{2(n-1)}{(n-2)}} + \frac{2(n-1)}{n-2} \delta T U_\varepsilon^{\frac{n}{n-2}} + \delta^2 O(T^2 U_\varepsilon^{\frac{2}{n-2}}) \right) dx$$

$$= \int_{B_{\rho_0}^{n-1}} U_\varepsilon^{\frac{2(n-1)}{(n-2)}} dx + O(\delta^2 \varepsilon^2).$$

Now

$$2\delta \int_{B_{\rho_0}^+} \langle \nabla \psi_\varepsilon, T \rangle dx dt + 4\delta \int_{B_{\rho_0}^+} h_{ij} t(U_\varepsilon)_i (T)_j dx dt =$$

$$= -2\delta \int_{B_{\rho_0}^+} T \Delta \psi_\varepsilon dx dt - 4\delta \int_{B_{\rho_0}^+} h_{ij} t(U_\varepsilon)_{ij} T dx dt$$

$$+ \delta \int_{S_{\rho_0}^+} \{2T \frac{\partial \psi_\varepsilon}{\partial r} + 4\rho_0^{-1} h_{ij} t(U_\varepsilon)_i T x^j\} d\sigma_{\rho_0}$$

$$- 2\delta \int_{B_{\rho_0}^{n-1}} T \frac{\partial \psi_\varepsilon}{\partial t} dx$$

$$= -2\delta \int_{B_{\rho_0}^+} T(\Delta \psi_\varepsilon + 2h_{ij} t(U_\varepsilon)_{ij}) dx dt$$

$$- 2A\delta \varepsilon^{\frac{n-2}{2}} \int_{B_{\rho_0}^{n-1}} T h_{jk} x^j x^k Z^{-\frac{n}{2}} dx + \delta O(\varepsilon^2).$$

We are integrating by parts, estimating the integral over $S_{\rho_0}^+$, and using that

$$\frac{\partial \psi_\varepsilon}{\partial t} = A\varepsilon^{\frac{n-2}{2}} h_{jk} x^j x^k Z^{-\frac{n}{2}},$$

when $t = 0$. Recall $Z = (\varepsilon + t)^2 + |x|^2$.

Note that

$$\Delta \psi_\varepsilon + 2h_{ij} t(U_\varepsilon)_{ij} = -2nA\varepsilon^{\frac{n-2}{2}} h_{jk} x^j x^k Z^{-\frac{n+2}{2}} (2t + \varepsilon)$$
$$+ 2n(n-2)\varepsilon^{\frac{n-2}{2}} h_{ij} x_i x_j t Z^{-\frac{n+2}{2}}.$$

Hence, if $n = 4$ and $A = 1$, after a cancellation we get

$$\Delta \psi_\varepsilon + 2h_{ij} t(U_\varepsilon)_{ij} = \varepsilon^{\frac{n}{2}} O(|x|^2 Z^{-\frac{n+2}{2}}).$$

Therefore, in this case,

$$E(\phi) \le Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$- 2\delta \varepsilon^{\frac{n-2}{2}} \int_{B_{\rho_0}^{n-1}} T h_{jk} x^j x^k Z^{-\frac{n}{2}} dx$$

$$+ \delta^2 \int_{B_{\rho_0}^+} |\nabla T|^2 dx dt + O(\varepsilon^2 \rho_0^{-2}).$$

From the definition (3.12):

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$-\frac{4\delta\varepsilon^{n-2}}{(n-1)(n+1)}(h_{ij})^2 \int_{B_{\rho_0}^{n-1}} \frac{|x|^4}{(\varepsilon^2 + |x|^2)^{n-\frac{1}{2}}} dx$$

$$+\delta^2 \int_{B_{\rho_0}^+} |\nabla T|^2 dx dt + O(\varepsilon^2 \rho_0^{-2}).$$

Hence

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$-\frac{4\delta\varepsilon^2}{(n-1)(n+1)}(h_{ij})^2 \int_{B_{\frac{\rho_0}{\varepsilon}}^{n-1}} \frac{|y|^4}{(1 + |y|^2)^{n-\frac{1}{2}}} dy$$

$$+O(\delta^2\varepsilon^2 \log(\frac{\rho_0}{\varepsilon})) + O(\varepsilon^2 \rho_0^{-2}).$$

Therefore, for some $c > 0$,

$$E(\phi) \leq Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}}$$

$$-c(h_{ij})^2 \delta\varepsilon^2 \log(\frac{\rho_0}{\varepsilon}) + O(\delta^2\varepsilon^2 \log(\frac{\rho_0}{\varepsilon})) + O(\varepsilon^2 \rho_0^{-2}).$$

Since $\sum_{i,j}(h_{ij})^2 > 0$ at the nonumbilic point $P \in \partial M$, we find

$$E(\phi) < Q(B^n, \partial B^n)(\int_{\partial M} \phi^{\frac{2(n-1)}{(n-2)}} d\sigma_g)^{\frac{n-2}{n-1}},$$

if $\delta > 0$ is sufficiently small

This finishes the proof of the Theorem.

Fernando Coda Marques, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil

Current address :   Stanford University, Stanford, CA

E-mail address :   coda@impa.br

References

[1] M. Ould Ahmedou. A Riemann mapping type theorem in higher dimensions. I. The conformally flat case with umbilic boundary. In *Nonlinear equations: methods, models and applications (Bergamo 2001)*, volume 54 of *Progr. Nonlinear Differential Equations Appl.*, pages 1–18. Birkhäuser, 2003.

[2] T. Aubin. Équations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.*, 55:269–296, 1976.

[3] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Annals of Mathematics*, 138(1):213–242, 1993.

[4] S. Brendle. A generalization of the Yamabe flow for manifolds with boundary. *Asian J. of Math.*, 6(4):625–644, 2002.

[5] Z. Djadli, A. Malchiodi, and M. Ould Ahmedou. Prescribing scalar and boundary mean curvature on the three-dimensional half sphere. *J. Geom. Anal.*, 13(2):255–289, 2003.

[6] J. F. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana Math. J.*, 37:687–698, 1988.

[7] J. F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Annals of Mathematics*, 136:1–50, 1992.

[8] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Diff. Geom.*, 35:21–84, 1992.

[9] J. F. Escobar. Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. *Indiana Univ. Math. J.*, 45(4):917–943, 1996.

[10] J. F. Escobar. Conformal metrics with prescribed mean curvature on the boundary. *Calc. Var.*, 4:559–592, 1996.

[11] J. F. Escobar and G. Garcia. Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary. *J. Funct. Anal.*, 211(1):71–152, 2004.

[12] V. Felli and M. Ould Ahmedou. Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries. *Math. Z.*, 244(1):175–210, 2003.

[13] V. Felli and M. Ould Ahmedou. Some geometric equations with critical nonlinearity at the boundary. *Pacif Journal of Math.*, 218(1):75–100, 2005.

[14] Z. C. Han and Y. Y. Li. The Yamabe problem on manifolds with boundary: existence and compactness results. *Duke Math. Journal*, 99(3):489–542, 1999.

[15] Z. C. Han and Y. Y. Li. The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature. *Comm. Anal. Geom.*, 8(4):809–869, 2000.

[16] J. Lee and T. Parker. The Yamabe problem. *Bull. Amer. Math. Soc.*, 17:37–91, 1987.

[17] F. C. Marques. Existence results for the Yamabe problem on manifolds with boundary. *Indiana Univ. Math. Journal*, 54(6):1599–1620, 2005.

[18] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geometry*, 20:479–495, 1984.

[19] N. Trudinger. Remarks concerning the conformal deformation of a Riemannian structure on compact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 22(3):165–274, 1968.

[20] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.