Non-Matching Mortar Discretization Analysis for the Coupling Stokes-Darcy Equations

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Abstract

We consider the coupling across an interface of fluid and porous media flows with Beavers-Joseph-Saffman transmission conditions. Under adequate choices of Lagrange multipliers on the interface we analyze inf-sup conditions and optimal a priori error estimates associated to the continuous and to the discrete formulations of the Stokes-Darcy system. We allow the meshes of the two regions to be nonmatching across the interface. Using mortar finite element analysis and appropriate scaled norms we show that the constants that appear on the a priori error bounds do not depend on the viscosity, permeability and ratio of mesh parameters. Numerical experiments are presented.

Keywords: inf-sup condition, error estimates, mortar finite elements, multiphisics, porous media flow, incompressible fluid flow, Lagrange multipliers, saddle point problems, nonmatching grids, discontinuous coefficients

1 Introduction

We analyze the coupling across an interface of fluid and porous media flows. Such problem appears in several applications like well-reservoir coupling in petroleum engineering, transport of substances across groundwater and surface water, and (bio)fluid-organ interactions. More precisely, we consider the following situation: an incompressible fluid in a region Ω_f can flow both ways across an interface Γ into a saturated porous medium domain Ω_p. The model studied here consists of *Stokes equations* in the fluid region Ω_f and *Darcy law* for the filtration velocity in the porous medium region Ω_p. The transmission conditions we consider on the interface Γ are the Beavers-Joseph-Saffman conditions

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[BJ67, JM00, Saf71] which are widely accepted on the scientific community. In this paper we study inf-sup conditions and a priori error estimates associated to the continuous and discrete formulations of this Stokes-Darcy system. There are previous works addressing such issues [LSY03, RY05, DQ04, BH06] as well as related problems such as Stokes-Laplacian systems [DMQ02, QVZ02, DIS04], Stokes-Navier Stokes [GRW04, QV99], and preconditioned iterative methods [DQ03, DIS04, DQ04, GS06], among others [MTW02, AL02].

This paper is organized as follows: In Section 2 we discuss norms and semi-norms of dual spaces on subsets. The differential systems are introduced in Section 3, where velocity and normal flux are considered as the boundary data for the Stokes part $\Gamma_f = \partial\Omega_f \backslash \Gamma$ and the Darcy part $\Gamma_p = \partial\Omega_p \backslash \Gamma$, respectively; for other formulations and boundary data see [DMQ02, DQ03]. The transmission conditions are then introduced on the interface Γ, known as Beavers-Joseph-Saffman conditions. In Section 4 we analyze the weak formulation of the continuous model and we discuss the choice $H^{1/2}(\Gamma)$ as a space for Lagrange multipliers in order to couple the two regions. In [LSY03], Layton, Schieweck, and Yotov have developed existence and uniqueness of the weak solution of this problem. They were able to show the inf-sup condition on a smaller space $H_{00}^{1/2}(\Gamma)$. In this paper, we use tools developed in Section 2 and in [LSY03] to develop a complete analysis for the inf-sup condition with Lagrange multipliers on the $H^{1/2}(\Gamma)$ space. We note that in the physical point of view the space $H^{1/2}(\Gamma)$ is the correct choice since the Lagrange multipliers are related to the Darcy pressure on the interface Γ and their values at the end points of Γ where touch $\partial\Omega$ are not prescribed; we note however the case where the pressure is imposed as a boundary condition for the Darcy side Γ_p, the space $H_{00}^{1/2}(\Gamma)$ would be the correct choice; see [DMQ02]. In Section 5 we derive the discrete inf-sup condition and a priori error estimates. We consider second order Taylor Hood elements space for the free flow region Ω_f and lower order Raviart-Thomas for the Darcy region Ω_p. In [LSY03], Layton, Schieweck and Yotov have developed a priori error estimates for the matching case, while in [RY05] Riviere and Yotov, and also in [BH06] Burman and Hansbo, have considered the non-matching case using discontinuous Garlekin finite element discretizations. In this paper we pay a special attention to the constants appearing in the a priori error estimates. We develop an analysis based on mortar finite elements techniques [BMP94, Woh00] and scaled norms in order to obtain constants independent of the permeability and the viscosity. In addition, these constants do not depend also on the ratio of mesh parameters, and the estimates depend only on the local mesh. We note also that a constructive proof based on Fortin's criteria [BF91] for the discrete inf-sup condition is provided, where a constructive proof for the inf-sup condition for the Stokes system is provided in the Appendix. In Section 7 we test numerically the algorithms and in Section 8 we make some conclusions.

2 Preliminaries and notations

Let Ω be a Lipschitz continuous domain and let $\Gamma \subset \partial\Omega$ and $\Gamma^c := \partial\Omega \setminus \Gamma$ be of non-vanishing $(n-1)$-dimensional measure and with nonempty interior with respect to $\partial\Omega$.

To avoid the proliferation of constants, we will use the notation $A \preceq B$ to represent the inequality $A \leq (constant) \cdot B$.

Lemma 1. *Given $\mu \in H^{1/2}(\Gamma)$, define $E_\Gamma^{1/2}\mu := \gamma_0\phi$ where γ_0 is the trace on $\partial\Omega$ and ϕ is the weak solution of*

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \\ \phi = \mu & \text{on } \Gamma \\ \partial_{\boldsymbol{\eta}}\phi = 0 & \text{on } \Gamma^c. \end{cases}$$

Then $E_\Gamma^{1/2}\mu \in H^{1/2}(\partial\Omega)$ and $\|E_\Gamma^{1/2}\mu\|_{H^{1/2}(\partial\Omega)} \preceq \|\mu\|_{H^{1/2}(\Gamma)}$.

For $\mu \in H^{1/2}(\Gamma)$ let $E_{00,\Gamma}^{1/2}\mu$ denote the extension by zero on Γ^c. Remember that $E_{00,\Gamma}^{1/2}\mu \in H^{1/2}(\partial\Omega)$ if and only if $\mu \in H_{00}^{1/2}(\Gamma)$. We have the following result:

Lemma 2. *For all $\mu \in H^{1/2}(\partial\Omega)$ there exist $\mu_\Gamma \in H^{1/2}(\Gamma)$ and $\mu_{\Gamma^c} \in H_{00}^{1/2}(\Gamma^c)$ such that $\mu = E_\Gamma^{1/2}\mu_\Gamma + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$. This decomposition is unique.*

Proof. Let $\mu \in H^{1/2}(\partial\Omega)$. Take $\mu_\Gamma = \mu|_\Gamma$ and $\mu_{\Gamma^c} = \phi|_{\Gamma^c}$ where $\phi = \mu - E_\Gamma^{1/2}\mu_\Gamma$. Observe that $\mu_\Gamma \in H^{1/2}(\Gamma)$ and

$$\|E_\Gamma^{1/2}\mu_\Gamma\|_{H^{1/2}(\partial\Omega)} \preceq \|\mu_\Gamma\|_{H^{1/2}(\Gamma)} \leq \|\mu\|_{H^{1/2}(\partial\Omega)}$$

so $\phi \in H^{1/2}(\partial\Omega)$. Observe also that $E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c} = \phi$ because μ and $E_\Gamma^{1/2}\mu_\Gamma$ coincide on Γ. For the uniqueness, if $0 = E_\Gamma^{1/2}\mu_\Gamma + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$ then $E_\Gamma^{1/2}\mu_\Gamma$ is the trace of the weak solution of the problem: $-\Delta\phi = 0$ in Ω, $\phi = 0$ on Γ, $\partial_{\boldsymbol{\eta}}\phi = 0$ on Γ^c, then $\mu_\Gamma = 0$. □

We have two dual spaces associated with Γ, $H_{00}^{-1/2}(\Gamma)$ the dual of $H_{00}^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ the dual space of $H^{1/2}(\Gamma)$. The first one is larger than the second.

Definition 1. *If $f \in H^{-1/2}(\partial\Omega)$, then $f|_{\Gamma^c} = 0$ means by definition that:*

$$\langle f, E_{00,\Gamma^c}^{1/2}\phi\rangle_{\partial\Omega} = 0 \quad \text{for all } \phi \in H_{00}^{1/2}(\Gamma^c).$$

A useful result related with this definition is the following:

Lemma 3. *If $f \in H^{-1/2}(\partial\Omega)$, there are $f_\Gamma \in H^{-1/2}(\Gamma)$ and $f_{\Gamma^c} \in H_{00}^{-1/2}(\Gamma^c)$ such that, for all $\mu \in H^{1/2}(\partial\Omega)$, $\mu = E_\Gamma^{1/2}\mu_\Gamma + E_{00,\Gamma^c}^{1/2}\mu_{\Gamma^c}$ as in Lemma 2, we have:*

$$\langle f, \mu\rangle_{\partial\Omega} = \langle f_\Gamma, \mu_\Gamma\rangle_\Gamma + \langle f_{\Gamma^c}, \mu_{\Gamma^c}\rangle_{\Gamma^c}. \tag{1}$$

Proof. For $\mu_\Gamma \in H^{1/2}(\Gamma)$ and $\mu_{\Gamma^c} \in H_{00}^{1/2}(\Gamma^c)$ define:

$$\langle f_\Gamma, \mu_\Gamma \rangle_\Gamma := \langle f, E_\Gamma^{1/2} \mu_\Gamma \rangle_{\partial\Omega} \qquad \langle f_{\Gamma^c}, \mu_{\Gamma^c} \rangle_{\Gamma^c} := \langle f, E_{00,\Gamma^c}^{1/2} \mu_{\Gamma^c} \rangle_{\partial\Omega}.$$

We obtain

$$\langle f_\Gamma, \mu_\Gamma \rangle_\Gamma \leq \|f\|_{H^{-1/2}(\partial\Omega)} \|E_\Gamma^{1/2} \mu_\Gamma\|_{H^{1/2}(\partial\Omega)} \preceq \|f\|_{H^{-1/2}(\partial\Omega)} \|\mu_\Gamma\|_{H^{1/2}(\Gamma)},$$

and so $f_\Gamma \in H^{-1/2}(\Gamma)$, analogously $f_{\Gamma^c} \in H_{00}^{-1/2}(\Gamma^c)$. Moreover:

$$\langle f_\Gamma, \mu_\Gamma \rangle_\Gamma + \langle f_{\Gamma^c}, \mu_{\Gamma^c} \rangle_{\Gamma^c} = \langle f, E_\Gamma^{1/2} \mu_\Gamma + E_{00,\Gamma^c}^{1/2} \mu_{\Gamma^c} \rangle_{\partial\Omega} = \langle f, \mu \rangle_{\partial\Omega}.$$

□

Remark 1. *In particular, if $f \in H^{-1/2}(\partial\Omega)$ and $f|_{\Gamma^c} = 0$, see Definition 1, we have from (1) that:*

$$\langle f, \mu \rangle_{\partial\Omega} = \langle f_\Gamma, \mu_\Gamma \rangle_\Gamma.$$

Then functionals in $H^{-1/2}(\partial\Omega)$ which are zero restricted to $\partial\Omega \setminus \Gamma$ can be identified with functionals in $H^{-1/2}(\Gamma)$.

Remark 2. *Given $f_\Gamma \in H^{-1/2}(\Gamma)$ we can define $f \in H^{-1/2}(\partial\Omega)$ by $\langle f, \mu \rangle_{\partial\Omega} := \langle f_\Gamma, \mu_\Gamma \rangle_\Gamma$, where $\mu = E_\Gamma^{1/2} \mu_\Gamma + E_{00,\Gamma^c}^{1/2} \mu_{\Gamma^c}$ as in Lemma 2. We have a similar result for $f_\Gamma \in H_{00}^{-1/2}(\Gamma^c)$.*

Define the space $\boldsymbol{H}(\mathrm{div}, \Omega)$ by

$$\boldsymbol{H}(\mathrm{div}, \Omega) := \left\{ \boldsymbol{v}_p \in \boldsymbol{L}^2(\Omega) \;\middle|\; \nabla \cdot \boldsymbol{v}_p \in L^2(\Omega) \right\},$$

with the norm $\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div}, \Omega)} := \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|^2_{L^2(\Omega)}$. For next result see [WTW00].

Lemma 4. *For each $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\eta} = \langle \boldsymbol{u} \cdot \boldsymbol{\eta}, 1 \rangle_{\partial\Omega} = 0$ we have that*

$$\sup_{\substack{\phi \in H^{1/2}(\partial\Omega) \\ \phi \neq \text{ constant}}} \frac{\langle \boldsymbol{u} \cdot \boldsymbol{\eta}, \phi \rangle_{\partial\Omega}}{|\phi|_{H^{1/2}(\partial\Omega)}} \preceq \|\boldsymbol{u} \cdot \boldsymbol{\eta}\|_{H^{-1/2}(\partial\Omega)} \leq \sup_{\substack{\phi \in H^{1/2}(\partial\Omega) \\ \phi \neq \text{ constant}}} \frac{\langle \boldsymbol{u} \cdot \boldsymbol{\eta}, \phi \rangle_{\partial\Omega}}{|\phi|_{H^{1/2}(\partial\Omega)}}.$$

with a constant which depends only on Ω.

Using an argument similar to the one given in [WTW00] we have:

Lemma 5. *For each $\mu \in H^{-1/2}(\Gamma)$ with $\int_\Gamma \mu = \langle \mu, 1 \rangle_\Gamma = 0$*

$$\sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq \text{ constant}}} \frac{\langle \mu, \phi \rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}} \preceq \|\mu\|_{H^{-1/2}(\Gamma)} \leq \sup_{\substack{\phi \in H^{1/2}(\Gamma) \\ \phi \neq \text{ constant}}} \frac{\langle \mu, \phi \rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}$$

with a constant which depends only on Ω and Γ.

Proof. Observe that if α is a constant $\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \alpha\rangle_\Gamma = \alpha\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, 1\rangle_\Gamma = 0$ and for $\phi \in H^{1/2}(\Gamma)$ non constant we have

$$\frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_\Gamma}{\|\phi\|_{H^{1/2}(\Gamma)}} \leq \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}},$$

then

$$\|\boldsymbol{u}\cdot\boldsymbol{\eta}\|_{H^{-1/2}(\Gamma)} = \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq\text{constant}}} \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_\Gamma}{\|\phi\|_{H^{1/2}(\Gamma)}} \leq \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq\text{constant}}} \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}$$

which gives the right inequality.

Using Poincaré Inequality, there exists a positive constant which depends only on Ω and Γ, such that

$$\|\psi\|^2_{H^{1/2}(\Gamma)} \preceq |\psi|^2_{H^{1/2}(\Gamma)}$$

holds for all $\psi \in H^{1/2}(\Gamma)$, with $\int_\Gamma \psi = 0$. For $\phi \in H^{1/2}(\Gamma)$ non constant we have that

$$\psi := \phi - \int_\Gamma \phi \neq 0$$

and

$$\frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \psi\rangle_\Gamma}{\|\psi\|_{H^{1/2}(\Gamma)}} = \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle_\Gamma}{\|\psi\|_{H^{1/2}(\partial\Omega)}} \succeq \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle}{|\psi|_{H^{1/2}(\Gamma)}} = \frac{\langle \boldsymbol{u}\cdot\boldsymbol{\eta}, \phi\rangle}{|\phi|_{H^{1/2}(\Gamma)}}.$$

□

This gives an equivalent norm in the subspace of $H^{-1/2}(\Gamma)$ of zero average functionals.

Definition 2. *For $\mu \in H^{-1/2}(\Gamma)$, μ with zero average ($\langle\mu, 1\rangle_\Gamma = 0$), define:*

$$|\mu|_{H^{-1/2}(\Gamma)} := \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \phi\neq\ \textit{constant}}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}} = \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \int_\Gamma \phi=0, \phi\neq 0,}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}.$$

We have the following result:

Lemma 6. *For $\phi \in H^{1/2}(\Gamma)$ with $\int_\Gamma \phi = 0$ we have:*

$$|\phi|_{H^{1/2}(\Gamma)} = \sup_{\substack{\mu\in H^{-1/2}(\Gamma)\\ \langle\mu,1\rangle=0}} \frac{\langle \mu, \phi\rangle_\Gamma}{|\mu|_{H^{-1/2}(\Gamma)}}.$$

Proof. Consider $(H^{1/2}(\Gamma) \cap L^2_0(\Gamma))^*$, the dual space of $H^{1/2}(\Gamma) \cap L^2_0(\Gamma)$, and observe that a functional $\mu_0 \in (H^{1/2}(\Gamma) \cap L^2_0(\Gamma))^*$ can be extended to one in $H^{1/2}(\Gamma)^*$, say μ, by the following formula: $\langle\mu, \lambda\rangle := \langle\mu_0, \lambda_0\rangle$ where $\lambda \in H^{1/2}(\Gamma)$ and $\lambda_0 := \lambda - \int_\Gamma \lambda$. □

3 P.D.E model

In general, $\Omega_f,\ \Omega_p \subset \Re^n$, $\Omega = \mathrm{int}(\overline{\Omega}_f \cup \overline{\Omega}_p)$, Ω_f and Ω_p are Lipschitz, so it is possible to define outward unit normal vectors, denoted by $\boldsymbol{\eta}_j$, $j = f, p$. The tangent vectors of Γ are denoted by $\boldsymbol{\tau}_1$ ($n = 2$), or $\boldsymbol{\tau}_l$, $l = 1, 2$ ($n = 3$).

Define $\Gamma_j := \partial\Omega_j \setminus \Gamma$, $j = f, p$. Fluid velocities are denoted by $\boldsymbol{u}_j : \Omega_j \to \Re^n$, $j = f, p$. Pressures are $p_j : \Omega_j \to \Re$, $j = f, p$.

As it was mentioned previously, the model for the fluid region is Stokes equations. It basically consists of conservation of mass and conservation of momentum, we have:

$$\left\{ \begin{array}{rll} -\nabla\cdot T(\boldsymbol{u}_f, p_f) & = \boldsymbol{f}_f & \text{in } \Omega_f \\ \nabla\cdot\boldsymbol{u}_f & = g_f & \text{in } \Omega_f \\ \boldsymbol{u}_f & = \boldsymbol{h}_f & \text{on } \Gamma_f. \end{array} \right. \tag{2}$$

Here $T(\boldsymbol{v}, p) := -pI + 2\nu\boldsymbol{D}\boldsymbol{v}$ where ν is the viscosity and $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + \nabla^T\boldsymbol{v})$ is the linearized strain tensor.

For the porous domain Ω_p it is used Darcy's law, i.e., $(\boldsymbol{u}_p, p_p)$ satisfies on Ω_p:

$$\left\{ \begin{array}{rlll} \boldsymbol{u}_p & = -\frac{\kappa}{\nu}\nabla p_p & \text{in } \Omega_p & \text{Darcy's law} \\ \nabla\cdot\boldsymbol{u}_p & = g_p & \text{in } \Omega_p & \\ \boldsymbol{u}_p\cdot\boldsymbol{\eta}_p & = h_p & \text{on } \Gamma_p. & \end{array} \right. \tag{3}$$

In general κ is a symmetric and uniformly positive definite tensor. κ represents the rock permeability and ν the fluid viscosity. For simplicity on the analysis we assume that κ is a real positive constant.

We also impose the compatibility condition

$$\int_{\Omega_f} g_f + \int_{\Omega_p} g_p - \int_{\Gamma_f} \boldsymbol{h}_f\cdot\boldsymbol{\eta}_f - \int_{\Gamma_p} h_p = 0. \tag{4}$$

The systems presented above must be coupled across the interface Γ. The following conditions are imposed (see [LSY03, DQ03, DMQ02, DQ04] and references therein):

Conservation of mass across Γ*:* It is expressed by:

$$\boldsymbol{u}_f\cdot\boldsymbol{\eta}_f + \boldsymbol{u}_p\cdot\boldsymbol{\eta}_p = 0 \text{ on } \Gamma. \tag{5}$$

This means that the fluid that is leaving a region enters in the other one.

Balance of normal forces across Γ*:* From Cauchy formula we see that

$$\Sigma(\boldsymbol{u}_f, p_f) := T(\boldsymbol{u}_f, p_f)\boldsymbol{\eta}_f$$

is the force on $\partial\Omega_f$ acting on the fluid volume inside Ω_f, i.e., Σ is the Cauchy stress (or traction) vector. The force on Γ from Ω_f is then $-\Sigma(\boldsymbol{u}_f, p_f)$.
The (only) force acting on the interface from Ω_p is the one given by p_p in the direction of $\boldsymbol{\eta}_p$ and must be equal to the component of Σ in this direction. We get

$$p_f - 2\nu\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f = p_p \quad \text{on } \Gamma. \tag{6}$$

The other components of Σ, i.e., $\Sigma \cdot \boldsymbol{\tau}_j$, $j = 1, d-1$, are more delicate and treated below.

Beavers-Joseph-Saffman condition: This condition is a kind of empirical law that gives an expression for the tangential components of Σ. It is expressed by:

$$\boldsymbol{u}_f \cdot \boldsymbol{\tau}_j = -\frac{\sqrt{\kappa}}{\alpha_f} 2\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\tau}_j \quad \text{on } \Gamma,\ j = 1, d-1. \tag{7}$$

In the general case, κ is a symmetric and uniformly positive definite tensor, and κ in (7) is replaced by $\tau_j \cdot \kappa \cdot \tau_j$.

A related condition is

$$(\boldsymbol{u}_f - \boldsymbol{u}_p)\cdot\boldsymbol{\tau}_j = -\frac{\sqrt{\kappa}}{\alpha_f} 2\boldsymbol{\eta}_1^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\tau}_j \qquad \text{on } \Gamma,\ j = 1, d-1,$$

which is known as the Beavers-Joseph condition. But it turns out in practice that the component of $\boldsymbol{u}_p$ in $\boldsymbol{\tau}_j$ is small compared with that of $\boldsymbol{u}_f$. When more general cases are considered, suitable interface conditions have to be imposed. An analytical way to find the right interface conditions is via homogenization (see [Hor96]).

4 Weak formulation and analysis

According to Appendix A, it is enough to consider the case $g_f = 0$ and $\boldsymbol{h}_f = \boldsymbol{0}$ in (2) and $h_p = 0$ in (3). Define:

$$\boldsymbol{X}_f := \left\{ \boldsymbol{v}_f \in H^1(\Omega_f)^n \ : \ \boldsymbol{v}_f\big|_{\Gamma_f} = \boldsymbol{0} \text{ and } \int_{\partial\Omega_f} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f = 0 \right\}, \qquad M_f := L_0^2(\Omega_f).$$

Velocities in $\boldsymbol{X}_f$ have zero mean flux across $\partial\Omega_f$, where $\boldsymbol{X}_f \subset H_0^1(\Omega_f, \Gamma_f)^n$ means that each component of $\boldsymbol{v}_f$ belong to $H^1(\Omega_f)$ and vanish on Γ_f.

For Ω_p we introduce the following spaces:

$$\boldsymbol{X}_p := \left\{ \boldsymbol{v}_p \in \boldsymbol{H}(\mathrm{div}, \Omega_p) \ : \ \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p\big|_{\Gamma_p} = 0 \text{ and } \int_{\partial\Omega_p} \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p = 0 \right\}, \quad M_p := L_0^2(\Omega_p).$$

The restriction of $\boldsymbol{v}_p \cdot \boldsymbol{\eta}_p$ to Γ_p is taken in the sense of Definition 1. Using Lemma 5 we have an equivalent norm for $\|\cdot\|_{H^{-1/2}(\partial\Omega_p)}$ involving only the semi-norm of $H^{1/2}(\partial\Omega_p)$. Note that $\boldsymbol{X}_p \subset \boldsymbol{H}_0(\mathrm{div}, \Omega_p, \Gamma_p)$, where $\boldsymbol{H}_0(\mathrm{div}, \Omega_p, \Gamma_p)$ is defined as the subspaces of $\boldsymbol{H}(\mathrm{div}, \Omega_p)$ of functions with vanishing normal component on Γ_p; see Definition 1.

Define $\boldsymbol{X} := \boldsymbol{X}_f \times \boldsymbol{X}_p$ with the usual norm, i.e., given $\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{X}$, then

$$\|\boldsymbol{v}\|_X^2 := |\boldsymbol{v}_f|^2_{H^1(\Omega_f)^n} + \|\boldsymbol{v}_p\|^2_{\boldsymbol{H}(\mathrm{div},\,\Omega_p)}.$$

In order to derive a weak formulation we first proceed formally and then we introduce the adequate rigorous framework.

We start with the Stokes equation (2). For all $\boldsymbol{v}_f \in X_f$ we have:

$$(-\nu \Delta \boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} + (\nabla p_f, \boldsymbol{v}_f)_{\Omega_f} = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f}. \tag{8}$$

From the Green formula we get:

$$\begin{aligned} -(\Delta \boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} &= (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{\Omega_f} - (\nabla \boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f)_\Gamma \\ &= (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{\Omega_f} - \langle \boldsymbol{\eta}_f^T \nabla \boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f \rangle_\Gamma - \sum_{l=1}^{n-1} \langle \boldsymbol{\tau}_l^T \nabla \boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f \cdot \boldsymbol{\tau}_l \rangle_\Gamma, \end{aligned}$$

and using that $\nabla \cdot \boldsymbol{u}_f = 0$ implies $\int_{\Omega_f} \nabla \boldsymbol{u}_f^T : \nabla \boldsymbol{v}_f = \int_{\partial\Omega_f} \nabla \boldsymbol{u}_f^T \boldsymbol{\eta}_f \cdot \boldsymbol{v}_f$, we get

$$\begin{aligned} -(\Delta \boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} &= 2(\boldsymbol{D}\boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{\Omega_f} - 2\langle \boldsymbol{\eta}_f^T \boldsymbol{D}\boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f \rangle_\Gamma \\ &\quad -2 \sum_{j=1}^{n-1} \langle \boldsymbol{\tau}_j^T \boldsymbol{D}\boldsymbol{u}_f \boldsymbol{\eta}_f, \boldsymbol{v}_f \cdot \boldsymbol{\tau}_j \rangle_\Gamma. \end{aligned}$$

For the second integral in (8) we have:

$$(\nabla p_f, \boldsymbol{v}_f)_{\Omega_f} = \langle p_f, \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f \rangle_\Gamma - (p_f, \nabla \cdot \boldsymbol{v}_f)_{\Omega_f}. \tag{9}$$

By replacing (9) in (8), using conditions (6) and (7), and defining, for all $\boldsymbol{u}_f, \boldsymbol{v}_f \in \boldsymbol{X}_f$ and $q_f \in M_f$:

$$a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) := 2\nu(\boldsymbol{D}\boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{\Omega_f} + \sum_{j=1}^{n-1} \frac{\nu \alpha_f}{\sqrt{\kappa}} \langle \boldsymbol{u}_f \cdot \boldsymbol{\tau}_j, \boldsymbol{v}_f \cdot \boldsymbol{\tau}_j \rangle_\Gamma \tag{10}$$

$$b_f(\boldsymbol{v}_f, q_f) := -(q_f, \nabla \cdot \boldsymbol{v}_f)_{\Omega_f} \tag{11}$$

we obtain *for all* $\boldsymbol{v}_f \in \boldsymbol{X}_f$ *and* $q_f \in M_f$

$$\left\{ \begin{array}{ll} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + b_f(\boldsymbol{v}_f, p_f) + \langle p_p, \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f \rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} \\ b_f(\boldsymbol{u}_f, q_f) & = 0. \end{array} \right.$$

Analogously, defining

$$a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) := (\frac{\nu}{\kappa}\boldsymbol{u}_p, \boldsymbol{v}_p)_{\Omega_p} \quad \text{for all } \boldsymbol{u}_p, \boldsymbol{v}_p \in \boldsymbol{X}_p, \tag{12}$$

$$b_p(\boldsymbol{v}_p, p_p) := -(p_p, \nabla\cdot \boldsymbol{v}_p)_{\Omega_p} \quad \text{for all } \boldsymbol{u}_p \in \boldsymbol{X}_p \text{ and } p_p \in M_p,$$

we have *for all* $\boldsymbol{v}_p \in \boldsymbol{X}_p$ *and* $q_p \in M_p$

$$\begin{cases} a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(\boldsymbol{v}_p, p_p) + \langle p_p, \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p\rangle_\Gamma & = 0 \\ b_p(\boldsymbol{u}_p, q_p) & = -(g_p, q_p)_{\Omega_p}. \end{cases}$$

Define $a : X \times X \to \Re$ and $b : X \times M \to \Re$ by:

$$a(\boldsymbol{u}, \boldsymbol{v}) := a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_p(\boldsymbol{u}_p, \boldsymbol{v}_p), \qquad b(\boldsymbol{v}, p) := b_f(\boldsymbol{v}_f, p_f) + b_p(\boldsymbol{v}_p, p_p), \tag{13}$$

then we obtain, using (5):

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \langle \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f + \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p, p_p\rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} \\ b(\boldsymbol{u}, q) & = -(g_p, q_p)_{\Omega_p} \\ \langle \boldsymbol{u}_f\cdot \boldsymbol{\eta}_f + \boldsymbol{u}_p\cdot \boldsymbol{\eta}_p, q_p\rangle_\Gamma & = 0. \end{cases}$$

To uncouple the two subproblems we use the Darcy pressure on the interface Γ. We point out that Stokes side can also be chosen for the coupling (not analyzed here) at the cost of losing the local conservation property. Introduce the Lagrange multiplier λ :

$$\lambda = p_p = p_f - 2\nu\boldsymbol{\eta}_f^T \boldsymbol{D}(\boldsymbol{u}_f)\boldsymbol{\eta}_f = p_f - 2\nu\boldsymbol{\eta}_f^T \nabla\boldsymbol{u}\boldsymbol{\eta}_f. \tag{14}$$

Then we get :

$$\begin{cases} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + b_f(\boldsymbol{v}_f, p_f) + \langle \boldsymbol{v}_f\cdot \boldsymbol{\eta}_f, \lambda\rangle_\Gamma & = (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} & \forall \boldsymbol{v}_f \in & \boldsymbol{X}_f \\ a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(\boldsymbol{v}_p, p_p) + \langle \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p, \lambda\rangle_\Gamma & = 0 & \forall \boldsymbol{v}_p \in & \boldsymbol{X}_p \\ b_f(\boldsymbol{u}_f, q_f) & = 0 & \forall q_f \in & M_f \\ b_p(\boldsymbol{u}_p, q_p) & = -(g_p, q_p)_{\Omega_p} & \forall q_p \in & M_p \\ \langle \boldsymbol{u}_f\cdot \boldsymbol{\eta}_f + \boldsymbol{u}_p\cdot \boldsymbol{\eta}_p, \mu\rangle_\Gamma & = 0 & \forall \mu \in & \Lambda \end{cases} \tag{15}$$

where the space Λ is defined next.

We have to choose a suitable function space Λ for λ. Observe that on the porous exterior boundary Γ_p we are using zero flux as boundary condition, i.e, $\boldsymbol{v}_p\cdot \boldsymbol{\eta}_p = 0$ on Γ_p. Recalling Definition 1, this means that

$$\langle \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p, E^{1/2}_{00,\Gamma_p}\phi\rangle_{\partial\Omega_p} = 0 \quad \text{for all } \phi \in H^{1/2}_{00}(\Gamma_p),$$

where $E^{1/2}_{00,\Gamma_p}$ denotes the extension by zero on $\Gamma_p^c = \Gamma$. Then, according to Lemma 3 and Remark 1 we can think of $\boldsymbol{v}_p\cdot \boldsymbol{\eta}_p$ as a distribution in $H^{-1/2}(\Gamma)$, more precisely, we can define $\boldsymbol{v}_p\cdot \boldsymbol{\eta}_p|_\Gamma \in H^{-1/2}(\Gamma)$ as

$$\langle \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p|_\Gamma, \phi\rangle_\Gamma := \langle \boldsymbol{v}_p\cdot \boldsymbol{\eta}_p, E^{1/2}_\Gamma \phi\rangle_{\partial\Omega}, \quad \phi \in H^{1/2}(\Gamma), \tag{16}$$

where $E_\Gamma^{1/2}$ is the extension operator defined in Lemma 1. In what follows we denote $\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p|_\Gamma$ simply by $\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p$. This is the main mathematical motivation for choosing Λ as $H^{1/2}(\Gamma)$ rather than $H_{00}^{1/2}(\Gamma)$. On the fluid exterior boundary Γ_f we are using Dirichlet boundary condition, i.e., $\boldsymbol{v}_f = \boldsymbol{0}$ on Γ_f. Then $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f|_\Gamma \in H_{00}^{1/2}(\Gamma)$ relatively to Ω_f. Then $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f|_\Gamma \in H_{00}^{1/2}(\Gamma)$ relatively to Ω_p. Here we use the fact that $H_{00}^{1/2}(\Gamma)$ which is the trace of $H_0^1(\Omega_f,\Gamma_f)$ is equivalent to the trace of $H_0^1(\Omega_p,\Gamma_p)$ if the shape and measure of Ω_f are of the same order of those of Ω_p. See [Gri85, Neč67]. Since $H_{00}^{1/2}(\Gamma)\subset H^{-1/2}(\Gamma)$ we conclude that $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f|_\Gamma \in H^{-1/2}(\Gamma)$. In what follows we denote $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f|_\Gamma$ simply by $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f$.

From the previous discussions we conclude that $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f + \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p \in H^{-1/2}(\Gamma)$ and so we choose for λ the subspace $\Lambda := H^{1/2}(\Gamma)$.

Other reason for choosing $H^{1/2}(\Gamma)$ instead of $H_{00}^{1/2}(\Gamma)$ is that the Lagrange multiplier in this case is pressure, see equation (14), and there is no physical reason for the pressure p_p to vanish on the interface relative boundary points (with respect to $\partial\Omega_j$) when flux boundary conditions are imposed. Choosing $H_{00}^{1/2}(\Gamma)$ as the spaces for pressure or related quantities makes more sense when pressure is given as boundary data as in [DMQ02].

Define $b_\Gamma : \boldsymbol{X}\times\Lambda \to \Re$ by:

$$b_\Gamma(\boldsymbol{v},\mu) := \langle \boldsymbol{v}_f\cdot\boldsymbol{\eta}_f,\mu\rangle_\Gamma + \langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p,\mu\rangle_\Gamma, \quad \boldsymbol{v}=(\boldsymbol{v}_f,\boldsymbol{v}_p)\in\boldsymbol{X}, \quad \mu\in\Lambda, \tag{17}$$

with the first duality pairing as in (16). From Lemma 3 we have the following

Lemma 7. $b_\Gamma : \boldsymbol{X}\times\Lambda\to\Re$ *defined in (17) and (16) is continuous.*

If λ is a constant function, we already have $b_\Gamma(\boldsymbol{v},\lambda)=0$ for $\boldsymbol{v}\in\boldsymbol{X}$ since $\boldsymbol{v}=(\boldsymbol{v}_f,\boldsymbol{v}_p)\in\boldsymbol{X}$ implies, by definition of $\boldsymbol{X}$, that $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f$ and $\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p$ have zero average on Γ. Then it is convenient to modify Λ to

$$\Lambda := H^{1/2}(\Gamma)\cap L_0^2(\Gamma) \quad \text{with norm } |\cdot|_\Lambda := |\cdot|_{H^{1/2}(\Gamma)}.$$

We finally arrive to the weak formulation of the problem: *find* $(\boldsymbol{u},p,\lambda)\in\boldsymbol{X}\times M\times\Lambda$ *satisfying, for all* $(\boldsymbol{v},q,\mu)\in\boldsymbol{X}\times M\times\Lambda$:

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + b_\Gamma(\boldsymbol{v},\lambda) & = \ell(\boldsymbol{v}) \\ b(\boldsymbol{u},q) & = g(q) \\ b_\Gamma(\boldsymbol{u},\mu) & = 0, \end{cases} \tag{18}$$

where

$$\ell(\boldsymbol{v}) := (\boldsymbol{f}_f,\boldsymbol{v}_f)_{\Omega_f} \text{ for all } \boldsymbol{v}\in\boldsymbol{X} \text{ and } g(q) := -(g_p,q_p)_{\Omega_p} \text{ for all } q\in M. \tag{19}$$

Remark 3. *We note however that the space we choose for* Λ *is richer than* $H_{00}^{1/2}(\Gamma)\cap L_0^2(\Gamma)$ *and therefore closer will be* $\boldsymbol{v}_f\cdot\boldsymbol{\eta}_f$ *and* $\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p$ *near the end points of the interface* Γ.

Define $\boldsymbol{V} = (\boldsymbol{V}_f, \boldsymbol{V}_p) := \{\boldsymbol{v} \in \boldsymbol{X} \,:\, b_\Gamma(\boldsymbol{v}, \mu) = 0 \;\; \forall\, \mu \in \Lambda\}$. The space $\boldsymbol{V}$ is closed because the linear map $B_\Gamma : \boldsymbol{X} \to \Lambda'$ defined by $B_\Gamma(\boldsymbol{v})\mu := b_\Gamma(\boldsymbol{v}, \mu)$ is continuous and $V = \text{Ker } B_\Gamma$. Then we can formulate problem (18) as:

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = \ell(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \\ b(\boldsymbol{u}, q) & = g(q) \quad \forall\, q \in M. \end{cases} \tag{20}$$

Now, define

$$\boldsymbol{Z} = (\boldsymbol{Z}_f, \boldsymbol{Z}_p) := \{\boldsymbol{v} \in \boldsymbol{X} \,:\, b(\boldsymbol{v}, q) = 0 \;\; \forall\, q \in M\}. \tag{21}$$

Then we can also formulate problem (18) as:

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{Z} \\ b_\Gamma(\boldsymbol{u}, \mu) & = 0 \qquad \forall\, \mu \in \Lambda. \end{cases} \tag{22}$$

Remark 4. *The Korn inequality implies that the bilinear form a_f defined in (10) is $\boldsymbol{X}_f$-*elliptic *(See [Bra01, Neč67]). The bilinear for a_p defined in (12) is* $\boldsymbol{H}(\text{div}^0, \Omega_p)$-elliptic*, here $\boldsymbol{H}(\text{div}^0, \Omega_p)$ consists of functions in $\boldsymbol{H}(\text{div}, \Omega_\text{p})$ with vanishing divergence, i.e., the kernel of bilinear form b_p. Then the bilinear form a defined in (13) is $\boldsymbol{X}_f \times \boldsymbol{H}(\text{div}^0, \Omega_p)$*-elliptic.

Remember that we are working with pressure of zero mean in each subdomain (and with velocities of zero mean flux on each subdomain), then from the usual *inf-sup* condition for Stokes we can easily derive the *inf-sup* condition needed in formulation (20).

Lemma 8. *There is a constant $\rho > 0$ such that*

$$\inf_{\substack{p \in M \\ p \neq 0}} \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V} \\ \boldsymbol{v} \neq 0}} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{\boldsymbol{X}} \|p\|_M} \geq \rho > 0.$$

Lemma 8, Remark 4 and the fact that $\big(\text{Ker } b \cap \boldsymbol{V}\big) \subset \big(\boldsymbol{X}_f \times \boldsymbol{H}(\text{div}^0, \Omega_p)\big)$ guarantees stability of the weak formulation (20). See [BF91, GR86].

Recall that $\boldsymbol{Z} \subset \boldsymbol{X}_f \times \boldsymbol{H}(\text{div}^0, \Omega_p)$. To see that the weak formulation (22) is stable, the next lemma shows that the inf-sup condition between spaces Z and Λ holds (see [BF91, GR86]). The proof presented here follows the same ideas as [LSY03], Lemma 3.4. The main difference is that we are working with zero mean functions and the function spaces involved.

Lemma 9. *There is a constant $\gamma > 0$ such that*

$$\inf_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \sup_{\substack{\boldsymbol{v} \in \boldsymbol{Z} \\ \boldsymbol{v} \neq 0}} \frac{b_\Gamma(\boldsymbol{v}, \lambda)}{\|\boldsymbol{v}\|_{\boldsymbol{X}} |\lambda|_\Lambda} \geq \gamma > 0.$$

Proof. Fix $\lambda \in \Lambda$, then $\lambda \in H^{1/2}(\Gamma)$ and $\int_\Gamma \lambda = 0$, in particular if $\lambda \neq 0$ then it is non constant. From Lemma 6 we have that there exists $\mu_\Gamma \in H^{-1/2}(\Gamma)$ such that $\langle \mu_\Gamma, 1\rangle_\Gamma = 0$ and:

$$\frac{\langle \mu_\Gamma, \lambda\rangle_\Gamma}{|\mu_\Gamma|_{H^{-1/2}(\Gamma)}} \geq \frac{1}{2}|\lambda|_{H^{1/2}(\Gamma)} = \frac{1}{2}|\lambda|_\Lambda. \tag{23}$$

From Remark 2 we introduce $\mu \in H^{-1/2}(\partial\Omega_p)$ given by:

$$\langle \mu, \phi\rangle_{\partial\Omega_p} := \langle \mu_\Gamma, \phi|_\Gamma\rangle_\Gamma, \tag{24}$$

with

$$|\mu|_{H^{-1/2}(\partial\Omega_p)} \leq C_1 |\mu_\Gamma|_{H^{-1/2}(\Gamma)} \tag{25}$$

and zero mean on $\partial\Omega_p$, $\langle \mu, 1\rangle_{\partial\Omega_p} = \langle \mu_\Gamma, 1\rangle_\Gamma = 0$.
By using the normal trace theorem, and a continuous Stokes problem (μ has zero mean on $\partial\Omega_p$) we can find $\boldsymbol{v}_p \in \boldsymbol{H}(\mathrm{div}, \Omega_p)$ with $\nabla\cdot\boldsymbol{v}_p = 0$ in Ω_p and such that:

$$\begin{aligned} \|\boldsymbol{v}_p\|_{\boldsymbol{H}(\mathrm{div},\,\Omega_p)} &\leq C|\mu|_{H^{-1/2}(\partial\Omega_p)} && (26)\\ \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p &= \mu. && (27) \end{aligned}$$

Observe that $\boldsymbol{v}_p \in \boldsymbol{X}_p$. In fact if $\phi \in H^{1/2}_{00}(\Gamma_p)$ then:

$$\langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, \phi\rangle_{\partial\Omega_p} = \langle \mu, \phi\rangle_{\partial\Omega_p} = \langle \mu_\Gamma, \phi|_\Gamma\rangle_\Gamma = \langle \mu_\Gamma, 0\rangle_\Gamma = 0$$

and $\langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, 1\rangle_{\partial\Omega_p} = \langle \mu_\Gamma, 1\rangle_\Gamma = 0$.

Choosing $\boldsymbol{v}_f = 0$, we have $\boldsymbol{v} := (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \boldsymbol{Z}$ and:

$$\begin{aligned} \frac{b_\Gamma(\boldsymbol{v}, \lambda)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} &= \frac{0 + \langle \boldsymbol{v}_p\cdot\boldsymbol{\eta}_p, E_\Gamma^{1/2}\lambda\rangle_{\partial\Omega_p}}{\|\boldsymbol{v}_p\|_{\boldsymbol{H}(\mathrm{div},\,\Omega_p)}} && \text{by (16)}\\ &\geq \frac{1}{C}\frac{\langle \mu, E_\Gamma^{1/2}\lambda\rangle_{\partial\Omega_p}}{|\mu|_{H^{-1/2}(\partial\Omega_p)}} && \text{by (26) and (27)}\\ &= \frac{1}{CC_1}\frac{\langle \mu_\Gamma, \lambda\rangle_\Gamma}{|\mu|_{H^{-1/2}(\Gamma)}} && \text{by (24) and (25)}\\ &\geq \frac{1}{CC_1}\frac{1}{2}|\lambda|_{H^{1/2}(\Gamma)} && \text{by (23).} \end{aligned}$$

□

For $(q, \mu) \in M \times \Lambda$ define $|(p, \mu)|^2_{M\times\Lambda} := \|p\|^2_M + |\mu|^2_\Lambda$. With Lemma 8 and Lemma 9 we can show:

Proposition 1. *There is a constant $\beta > 0$ such that:*

$$\inf_{\substack{(p,\lambda)\in M\times\Lambda\\(p,\lambda)\neq(0,0)}} \sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}\\ \boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda)}{\|\boldsymbol{v}\|_X |(p, \lambda)|_{M\times\Lambda}} \geq \beta > 0. \tag{28}$$

Proof. Given $(p,\lambda)\in M\times\Lambda$, if $p\neq 0$, from Lemma 8 there exists $\hat{\boldsymbol{v}}\in\boldsymbol{V}$ such that

$$\frac{b(\hat{\boldsymbol{v}},p)}{\|\hat{\boldsymbol{v}}\|_X}\geq\rho\|p\|_M>0,$$

where ρ independent of p. If $\lambda\neq 0$, from Lemma 9 there exists $\boldsymbol{z}\in\boldsymbol{Z}$ such that

$$\frac{b_\Gamma(\boldsymbol{z},\lambda)}{\|\boldsymbol{z}\|_X}\geq\gamma|\lambda|_\Lambda>0,$$

where γ independent of λ.

Observe that, if $p\neq 0$

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}\\ \boldsymbol{v}\neq 0}}\frac{b(\boldsymbol{v},p)+b_\Gamma(\boldsymbol{v},\lambda)}{\|\boldsymbol{v}\|_X}\geq\frac{b(\hat{\boldsymbol{v}},p)+b_\Gamma(\hat{\boldsymbol{v}},\lambda)}{\|\hat{\boldsymbol{v}}\|_X}=\frac{b(\hat{\boldsymbol{v}},p)+0}{\|\hat{\boldsymbol{v}}\|_X}\geq\rho\|p\|_M,$$

analogously, if $\lambda\neq 0$,

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}\\ \boldsymbol{v}\neq 0}}\frac{b(\boldsymbol{v},p)+b_\Gamma(\boldsymbol{v},\lambda)}{\|\boldsymbol{v}\|_X}\geq\frac{0+b_\Gamma(\boldsymbol{z},\lambda)}{\|\boldsymbol{z}\|_X}\geq\gamma|\lambda|_\Lambda,$$

then

$$\begin{aligned}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}\\ \boldsymbol{v}\neq 0}}\frac{b(\boldsymbol{v},p)+b_\Gamma(\boldsymbol{v},\lambda)}{\|\boldsymbol{v}\|_X}\quad&\geq\tfrac{\min\{\rho,\gamma\}}{2}\Big(\|p\|_M+|\lambda|_\Lambda\Big)\\ &\geq\tfrac{\min\{\rho,\gamma\}}{2}|(p,\lambda)|_{M\times\Lambda}.\end{aligned}$$

$\square$

Proposition 1 permit us to formulate problem (18) as

$$\left\{\begin{array}{lll}a(\boldsymbol{u},\boldsymbol{v})+c(\boldsymbol{v},(p,\lambda)) & =\ell(\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\boldsymbol{X}\\ c(\boldsymbol{u},(q,\mu)) & =\hat{g}(q,\mu) & \forall\,(q,\mu)\in M\times\Lambda,\end{array}\right.\tag{29}$$

where $c(\boldsymbol{v},(p,\mu)):=b(\boldsymbol{v},q)+b_\Gamma(\boldsymbol{v},\mu)$. Then (28) in Lemma 1 can be written as, there exists $\beta=\frac{\min\{\rho,\gamma\}}{2}>0$ such that:

$$\inf_{\substack{(p,\mu)\in M\times\Lambda\\ (p,\mu)\neq 0}}\ \sup_{\substack{\boldsymbol{v}\in\boldsymbol{X}\\ \boldsymbol{v}\neq 0}}\frac{c(\boldsymbol{v},(p,\mu))}{\|\boldsymbol{v}\|_X|\eta|_{M\times L}}\geq\beta>0.\tag{30}$$

This *inf*-*sup* condition, together with the fact that a is $\boldsymbol{X}_f\times\boldsymbol{H}(\mathrm{div}^0,\Omega_p)$-elliptic and a and c are bounded, (according to the abstract saddle point theory) guarantees the well-posedness of the problem (29) or (18). See [BF91, GR86].

5 Finite element approximation

In Section 3 it was presented the problem for the coupling fluid with porous media flow in its continuous form, while in Section 4 it was presented its variational formulation and wellposedness. Now a two dimensional non-matching grid finite element approximation is discussed. A discrete space for Lagrange multipliers is introduced and we establish the discrete inf-sup conditions related to the weak formulation (20) and (18). The extension of the results to the three dimensional case is straightforward.

5.1 Discretization

From now on we assume that Ω has polygonal boundary. Let $\mathcal{T}_j^{h_j}$ be a triangulation of Ω_j, $j = f, p$. We do not assume that they match at the polyhedral interface Γ. We choose the following spaces for the fluid region:

$$\boldsymbol{X}_f^{h_f} := \Big\{ \boldsymbol{u} \in \boldsymbol{X}_f \ : \ \boldsymbol{u}_K = \hat{\boldsymbol{u}}_K \circ F_K^{-1} \text{ on } K \text{ and } \hat{\boldsymbol{u}}_K \in P_2(\hat{K})^2 \Big\} \cap C^0(\overline{\Omega}_f)^2, \tag{31}$$

where $\boldsymbol{u}_K := \boldsymbol{u}|_K$ and

$$M_f^{h_f} := \Big\{ p \in M_f \ : \ p_K = \hat{p}_K \circ F_K^{-1} \text{ on } K \text{ and } \hat{p}_K \in P_1(\hat{K}) \Big\} \cap C^0(\overline{\Omega}_f), \tag{32}$$

that is, we use the $P2\backslash P1$ triangular Taylor-Hood finite elements of order two (see [BS91, BF91, GR86]).

We have the following:

Lemma 10 (Approximation of Taylor-Hood elements). *Suppose that $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Then, there exists a bounded linear operator $\boldsymbol{I}_{h_f}^{TH} : \boldsymbol{X}_f \to \boldsymbol{X}_f^{h_f}$ such that*

$$b_f\big(\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f, p_f^{h_f}\big) = 0 \quad \textit{for all } p^{h_f} \in M^{h_f}$$

and $\|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\|_{\boldsymbol{X}_f} \preceq \|\boldsymbol{v}_f\|_{\boldsymbol{X}_f}$, with constant independent of h_f. In addition we have,

$$\|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \preceq h_f^s |\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad s = 1, 2. \tag{33}$$

$$|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq h_f |\boldsymbol{v}_f|_{H^2(\Omega_f)^2} \tag{34}$$

$$\int_\Gamma \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f = \int_\Gamma \boldsymbol{v}_f \cdot \boldsymbol{\eta}_f$$

$$|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2}. \tag{35}$$

A constructive and apparently new proof using Fortin interpolation is given in Appendix B, or refer to [BS91, BF91, GR86].

A direct consequence of Fortin's criterion and the previous lemma is that, if $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$, then $(\boldsymbol{X}_f^{h_f}, M_f^{h_f})$ satisfy the inf-sup condition.

For the porous region we are going to use the lowest order Raviart-Thomas finite elements based on triangles. In general the Raviart-Thomas elements in a cell are defined by (see [Bra01, BF91, GR86]):

$$RT_k(K) := (P_k(K))^n + P_k(K)\boldsymbol{x},$$

and if $\boldsymbol{u} \in RT_k(K)$ then $\nabla\cdot\boldsymbol{u} \in P_k(K)$ and $\boldsymbol{u}\cdot\boldsymbol{\eta}|_{e_i} \in P_k(e_i)$, for all edge e_i. Then we choose:

$$\boldsymbol{X}_p^{h_p} := \{\boldsymbol{u} \in \boldsymbol{X}_p \,:\, \boldsymbol{u}|_K = \boldsymbol{u}_K \in RT_0(K)\}\,, \tag{36}$$

and

$$M_p^{h_p} := \left\{p \in M_p \,:\, p|_K = p_K = \hat{p}_K \circ F_K^{-1} \text{ on } K \text{ and } \hat{p}_K \in P_0(\hat{K})\right\}. \tag{37}$$

Observe that in the previous definitions the boundary conditions are included, i.e., for $\boldsymbol{v}_f^{h_f} \in \boldsymbol{X}_f^{h_f}$ we have $\boldsymbol{v}_f^{h_f} = \boldsymbol{0}$ on Γ_f and for $\boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p}$, $\boldsymbol{v}_p^{h_p}\cdot\boldsymbol{\eta} = 0$ holds on Γ_p. Moreover, if $\boldsymbol{v}_f^{h_f} \in \boldsymbol{X}_f^{h_f}$ we have $\int_\Gamma \boldsymbol{v}_f^{h_f}\cdot\boldsymbol{\eta}_f = 0$. The same holds for $\boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p}$.

Velocities of lowest order Raviar-Thomas finite elements, $RT_0(K)$, $K \in \mathcal{T}_p^{h_p}$, are of the form:

$$\boldsymbol{v}_p(x_1, x_2) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We have the following (see [Bra01, BF91, Gal04]):

Lemma 11 (Approximation of Raviar-Thomas elements). *For $K \in \mathcal{T}_p^{h_p}$ and $r > 2$, r fixed, define $\boldsymbol{I}_{h_p,K}^{RT} : \boldsymbol{H}(\mathrm{div}, K) \cap L^r(K)^2 \to RT_0(K)$ by*

$$\boldsymbol{I}_{h_p,K}^{RT}\boldsymbol{v}_p \cdot \boldsymbol{\eta}|_e = \frac{1}{|e|}\int_e \boldsymbol{v}_p \cdot \boldsymbol{\eta}$$

and define $\boldsymbol{I}_{h_p}^{RT} : \boldsymbol{H}(\mathrm{div}, \Omega_p) \cap L^r(\Omega_p)^2 \to RT_0$ locally by: $\boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p|_K = \boldsymbol{I}_{h_p,K}^{RT}\boldsymbol{v}_p$. Then

$$\int_{\Omega_p} \nabla\cdot\left(\boldsymbol{v}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\right) q_p^{h_p} = 0 \quad \textit{for all } q_p^{h_p} \in M_p^{h_p}$$

and $\|\boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\|_{\boldsymbol{H}(div,\,\Omega_p)} \preceq \|\boldsymbol{v}_p\|_{\boldsymbol{H}(div,\,\Omega_p)\,\cap\,L^r(\Omega_p)^2}$.

Moreover, if $\boldsymbol{v}_p \in H^1(\Omega_p)^2$ then

$$\|\boldsymbol{v}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\|_{L^2(\Omega_p)^2} \preceq h_p|\boldsymbol{v}_p|_{H^1(\Omega_p)^2}, \tag{38}$$

and

$$\|\nabla\cdot(\boldsymbol{v}_p - \boldsymbol{I}^{RT}_{h_p}\boldsymbol{v}_p)\|_{L^2(\Omega_p)} \preceq h_p|\nabla\cdot\boldsymbol{v}_p|_{H^1(\Omega_p)}.$$

By using the Fortin's idea we can establish the inf-sup condition for the spaces $(X_p^{h_p}, M_p^{h_p})$ defined in (36) and (37), respectively.

5.2 Discrete inf-sup condition

Let $\Gamma \cap \mathcal{T}_p^{h_p}$ be the trace on Γ of the porous side triangulation. We use piecewise constant Lagrange multipliers space:

$$\Lambda^{h_p} := \left\{\lambda \;:\; \lambda|_{e_p} = \lambda_{e_p} \;\text{ is constant for all edge } e_p \in \Gamma\cap\mathcal{T}_p^{h_p} \text{ and such that } \int_\Gamma \lambda = 0.\right\}.$$

We note that this choice leads to nonconforming finite elements associated to Λ since piecewise constant functions do not belongs to $H^{1/2}(\Gamma)$.
Define $h = (h_f, h_p)$, $\boldsymbol{X}^h := \boldsymbol{X}_f^{h_f} \times \boldsymbol{X}_p^{h_p}$ and

$$\boldsymbol{V}^h = (\boldsymbol{V}_f^{h_f}, \boldsymbol{V}_p^{h_p}) := \left\{\boldsymbol{v}^h \in \boldsymbol{X}^h \;:\; ([\![\boldsymbol{v}^h]\!]\cdot\boldsymbol{\eta}_p, \mu^{h_p})_\Gamma = 0 \;\; \forall\, \mu^{h_p} \in \Lambda^{h_p}\right\},$$

where $[\![\boldsymbol{v}^h]\!] := \boldsymbol{v}_f^{h_f} - \boldsymbol{v}_p^{h_p}$ for all $\boldsymbol{v}^h \in \boldsymbol{X}^h$. Also define

$$\boldsymbol{Z}^h(g) = (\boldsymbol{Z}_f^{h_f}, \boldsymbol{Z}_p^{h_p})(g_p) := \left\{\boldsymbol{v}^h \in \boldsymbol{X}^h \;:\; b(\boldsymbol{v}^h, q^h) = g(q^h) \;\; \forall\, q^h \in M^h\right\}, \quad (39)$$

with g defined in (19). For $\mu \in L^2_0(\Gamma)$ define $Q^{h_p}\mu$ as the L^2-projection on Λ^{h_p}, i.e., on the spaces of piecewise constant function on Γ.

Then using local arguments we have that $\|Q^{h_p}\mu - \mu\|_{L^2(\Gamma)} \preceq h_p^{1/2}|\mu|_{H^{1/2}(\Gamma)}$ and then

$$\begin{aligned}
|Q^{h_p}\mu - \mu|_{H^{-1/2}(\Gamma)} &= \sup_{\substack{\phi\in H^{1/2}(\Gamma)\\ \int_\Gamma \phi\neq 0}} \frac{\langle Q^{h_p}\mu - \mu, \phi\rangle_\Gamma}{|\phi|_{H^{1/2}(\Gamma)}}\\
&\leq \sup_{\phi\in H^{1/2}(\Gamma)} \frac{\|Q^{h_p}\mu - \mu\|_{L^2(\Gamma)}\|Q^{h_p}\phi - \phi\|_{L^2(\Gamma)}}{|\phi|_{H^{1/2}(\Gamma)}}\\
&\preceq h_p|\mu|_{H^{1/2}(\Gamma)},
\end{aligned}$$

i.e.,

$$|Q^{h_p}\mu - \mu|_{H^{-1/2}(\Gamma)} \preceq h_p|\mu|_{H^{1/2}(\Gamma)}. \quad (40)$$

For $z_p^{h_p} \in \Lambda^{h_p} = \boldsymbol{X}_p^{h_p}\cdot\boldsymbol{\eta}_p|_\Gamma$, (i.e., $z_p^{h_p}$ piecewise constant on Γ relatively to $\mathcal{T}_p^{h_p}$ and with zero mean on Γ) define $\boldsymbol{E}_p^{h_p} z_p^h \in \boldsymbol{X}_p^{h_p}$ as the discrete velocity solution of the problem

$$\begin{cases} a_p(\boldsymbol{E}_p^{h_p} z_p^{h_p}, \boldsymbol{v}_p^{h_p}) + b_p(\boldsymbol{v}_p^{h_p}, \hat{p}_p^{h_p}) & = 0; \quad \forall \boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p} \\ & \quad \text{such that } \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p = 0 \text{ on } \Gamma_p \\ b_p(\boldsymbol{E}_p^{h_p} z_p^{h_p}, q_p^{h_p}) & = 0 \quad \forall q_p^{h_p} \in M_p^{h_p} \\ \boldsymbol{E}_p^{h} z_p^{h_p} \cdot \boldsymbol{\eta}_p = \mu \text{ on } \Gamma. & \end{cases}$$

Then (see [Mat89])

$$\|\boldsymbol{E}_p^{h_p} z_p^{h_p}\|_{\boldsymbol{X}_p} \preceq \|z_p^{h_p}\|_{H^{-1/2}(\Gamma)} \preceq |z_p^{h_p}|_{H^{-1/2}(\Gamma)}. \tag{41}$$

We have the following:

Proposition 2. *Suppose that $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. For r fixed, $r > 2$, there exists a linear continuous operator*

$$\boldsymbol{\Pi}^h : \boldsymbol{V} \cap [\boldsymbol{X}_f \times L^r(\Omega_p)^2] \to \boldsymbol{V}^h$$

such that

$$b(\boldsymbol{\Pi}^h \boldsymbol{v} - \boldsymbol{v}, q^h) = 0 \text{ for all } q^h \in M^h$$

and

$$\|\boldsymbol{\Pi}^h \boldsymbol{v}\|_X \preceq \|\boldsymbol{v}\|_X. \tag{42}$$

Moreover assuming smooth enough solutions we have

$$\|\boldsymbol{v} - \boldsymbol{\Pi}^h \boldsymbol{v}\|_{\boldsymbol{X}} \preceq h_f |\boldsymbol{v}_f|_{H^2(\Omega_f)^2} + h_p \left(|\boldsymbol{v}_p|_{H^1(\Omega_p)^2} + |\nabla \cdot \boldsymbol{v}_p|_{H^1(\Omega_p)} \right). \tag{43}$$

Proof. Write $\boldsymbol{\Pi}^h(\boldsymbol{v}) = (\boldsymbol{\Pi}_f^{h_f} \boldsymbol{v}, \boldsymbol{\Pi}_p^{h_p} \boldsymbol{v})$ where $\boldsymbol{\Pi}_f^{h_f} \boldsymbol{v} := \boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f$ and

$$\Pi_p^{h_p} \boldsymbol{v} := \boldsymbol{I}_{h_p}^{RT} \boldsymbol{v}_p + \boldsymbol{E}_p^{h_p} \left(Q^{h_p} (\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p \right).$$

Then, for $\lambda^{h_p} \in \Lambda^{h_p}$ we have

$$\begin{aligned} ([\![\boldsymbol{\Pi}^h \boldsymbol{v}]\!] \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma &= (\boldsymbol{\Pi}_p^{h_p} \boldsymbol{v} \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma - (\boldsymbol{\Pi}_f^{h_f} \boldsymbol{v} \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma \\ &= (Q^{h_p}(\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p), \lambda^{h_p})_\Gamma - (\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma \\ &= 0 \qquad \text{by definition of } Q^{h_p}, \end{aligned}$$

we obtain $\boldsymbol{\Pi}^h \boldsymbol{v} \in \boldsymbol{V}^h$.

Now we show (43). The proof of boundedness of $\boldsymbol{\Pi}^h$, (42), is similar. We observe that

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{\Pi}^h \boldsymbol{v}\|_{\boldsymbol{X}} &\leq \|\boldsymbol{v}_f - \boldsymbol{\Pi}_f^{h_f} \boldsymbol{v}_f\|_{\boldsymbol{X}_f} + \|\boldsymbol{v}_p - \boldsymbol{\Pi}_p^{h_p} \boldsymbol{v}_p\|_{\boldsymbol{X}_p} \\ &\leq \|\boldsymbol{v}_f - \boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f\|_{\boldsymbol{X}_f} + \|\boldsymbol{v}_p - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{v}_p\|_{\boldsymbol{X}_p} \\ &\quad + \|\boldsymbol{E}_p^{h_p} \left(Q^{h_p} (\boldsymbol{I}_{h_f}^{TH} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT} \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p \right) \|_{\boldsymbol{X}_p}. \end{aligned}$$

The bound (43) follows from the boundedness and approximations properties of $\boldsymbol{I}_{h_f}^{TH}$ and $\boldsymbol{I}_{h_p}^{RT}$, the boundedness of $\boldsymbol{E}_p^h$ in (41), the estimate

$$|Q^{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p) - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} \leq |Q^{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p) - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} + |\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p - \boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} + \|\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p\|_{H^{-1/2}(\Gamma)}, \quad (44)$$

and following three bounds:

1. Approximation property (40), boundedness of $\boldsymbol{I}_{h_f}^{TH}$ in (35) and the trace theorem give

$$\begin{aligned} |Q^{h_p}(\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p) - \boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\preceq h_p|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\ &\preceq h_p|\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\ &= h_p|\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p|_{H^{1/2}(\Gamma)} \\ &\leq h_p|\boldsymbol{v}_p|_{H^{1/2}(\Gamma)^2} \\ &\preceq h_p|\boldsymbol{v}_p|_{H^1(\Omega_p)^2}. \end{aligned}$$

2. The trace theorem and approximation properties of $\boldsymbol{I}_{h_f}^{TH}$ (Lemma 10) give

$$\begin{aligned} |\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p - \boldsymbol{v}_f\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &\preceq \|\boldsymbol{I}_{h_f}^{TH}\boldsymbol{v}_f - \boldsymbol{v}_f\|_{\boldsymbol{X}_f} \\ &\preceq h_f|\boldsymbol{v}_f|_{H^2(\Omega_f)^2}. \end{aligned}$$

3. The normal trace theorem and the approximation property of $\boldsymbol{I}_{h_p}^{RT}$ in (38) imply

$$\begin{aligned} |\boldsymbol{v}_f\cdot\boldsymbol{\eta}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} &= |\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\cdot\boldsymbol{\eta}_p|_{H^{-1/2}(\Gamma)} \\ &\preceq \|\boldsymbol{v}_p - \boldsymbol{I}_{h_p}^{RT}\boldsymbol{v}_p\|_{\boldsymbol{X}_p} \\ &\preceq h_p\left(|\boldsymbol{v}_p|_{H^1(\Omega_p)^2} + |\nabla\cdot\boldsymbol{v}_p|_{H^1(\Omega_p)}\right). \end{aligned}$$

□

By using the Fortin's Lemma we can establish the discrete *inf-sup* condition.

Lemma 12. *Suppose that $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Then $(\boldsymbol{V}^h, M^h)$ satisfies the discrete* inf-sup *condition, i.e., there is a constant $\beta > 0$ independent of h, such that:*

$$\inf_{\substack{q^h\in M^h\\ q^h\neq 0}} \sup_{\substack{\boldsymbol{v}^h\in\boldsymbol{V}^{h_p}\\ \boldsymbol{v}\neq 0}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_X\|q^h\|_M} \geq \beta > 0.$$

For $\lambda^{h_p} \in \Lambda^{h_p}$, define $\widetilde{\boldsymbol{u}}_p^{h_p} = \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}) \in \boldsymbol{X}_p^{h_p}$ as velocity solution of the discrete problem

$$\left\{ \begin{array}{lll} a_p(\widetilde{\boldsymbol{u}}_p^{h_p}, \boldsymbol{v}_p^{h_p}) + b_p(\boldsymbol{v}_p^{h_p}, p_p^{h_p}) & = -(\boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma & \boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p} \\ b_p(\widetilde{\boldsymbol{u}}_p^{h_p}, q_p^{h_p}) & = 0 & q_p^{h_p} \in M_p^{h_p} \end{array} \right. \tag{45}$$

and

$$|\lambda^{h_p}|^2_{\Lambda^{h_p}} := a_p(\widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}), \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})). \tag{46}$$

In order to see that $|\cdot|_{\Lambda^{h_p}}$ is a norm on Λ^{h_p}, observe that if λ^{h_p} is such that $|\lambda^{h_p}|_{\Lambda^{h_p}} = 0$, then $\widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^h)$ vanishes and if we take $\boldsymbol{v}_p^{h_p}$ in (45) such that (remember that $M_p^{h_p} \subset L_0^2(\Omega_p)$)

$$\left\{ \begin{array}{lll} \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p & = \lambda^{h_p}; & \\ b_p(\boldsymbol{v}_p^{h_p}, q_p^{h_p}) & = 0 & q_p^{h_p} \in M_p^{h_p} \end{array} \right.$$

and hence we see that $\|\lambda^{h_p}\|_{L^2(\Gamma)} = 0$, that is $\lambda^{h_p} = 0$. Then $|\cdot|_{\Lambda^{h_p}}$ is positive.

We have:

Lemma 13. *The spaces $(\boldsymbol{Z}^h, \Lambda^{h_p})$ satisfy the discrete* inf-sup *condition, i.e., there is a constant $\beta > 0$ such that:*

$$\inf_{\substack{\lambda^{h_p} \in \Lambda^{h_p} \\ \lambda^{h_p} \neq 0}} \sup_{\substack{\boldsymbol{v}^h \in \boldsymbol{Z}^h \\ \boldsymbol{v} \neq 0}} \frac{([\![\boldsymbol{v}^h]\!] \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma}{\|\boldsymbol{v}^h\|_{\boldsymbol{X}} |\lambda^{h_p}|_{\Lambda^{h_p}}} \geq \beta > 0.$$

Proof. Take $\lambda^{h_p} \in \Lambda^{h_p}$ and $\widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})$ the velocity solution of (45). Observe that we are working with piecewise linear velocities on the porous region and then with piecewise constant divergence velocities, then from the second equation in (45) we have $\nabla \cdot \widetilde{\boldsymbol{u}}_p^{h_p} = 0$. Take $\boldsymbol{u}^h = (\boldsymbol{0}, \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}))$, then from (45)

$$\frac{([\![\boldsymbol{u}^h]\!] \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma}{\|\boldsymbol{u}^h\|_{\boldsymbol{X}} |\lambda^{h_p}|_{\Lambda^{h_p}}} = \frac{a_p(\widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}), \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}))}{\|\widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{L^2(\Omega_p)} |\lambda^{h_p}|_{\Lambda^{h_p}}} = \kappa^{-\frac{1}{2}} > 0$$

$\square$

For $(p^h, \lambda^{h_p}) \in M^h \times \Lambda^{h_p}$ define $|(p^h, \lambda^{h_p})|^2_{M^h \times \Lambda^{h_p}} := \|p^h\|^2_M + |\lambda^{h_p}|^2_{\Lambda^{h_p}}$. Then using the same arguments of Proposition 1 we have

Proposition 3. *Under assumptions of Lemmas 12 and 13 we have*

$$\inf_{\substack{(p^h, \lambda^{h_p}) \in M^h \times \Lambda^{h_p} \\ (p^h, \lambda^h) \neq (0,0)}} \sup_{\substack{\boldsymbol{v}^h \in \boldsymbol{X}^h \\ \boldsymbol{v}^h \neq 0}} \frac{b(\boldsymbol{v}^h, p^h) + ([\![\boldsymbol{v}^h]\!], \lambda^{h_p})_\Gamma}{\|\boldsymbol{v}\|_X \|(p^h, \lambda^{h_p})\|_{M^h \times \Lambda^{h_p}}} \geq \beta > 0. \tag{47}$$

6 Error analysis

From now on we assume $\alpha_f = 0$ in (7). See Remark 6 for the case $\alpha_f \neq 0$. .
In this section the constants involved in the notation $\preceq$ are all independents not only of the mesh size but also independent of the parameters ν and κ. Introduce the following weighted norms

$$|\boldsymbol{v}_f|^2_{a_f} := a_f(\boldsymbol{v}_f, \boldsymbol{v}_f) = \nu \|\boldsymbol{v}_f\|^2_{\boldsymbol{X}_f} \tag{48}$$

$$\|\boldsymbol{v}_p\|^2_{a_p} := a_p(\boldsymbol{v}_p, \boldsymbol{v}_p) = \left(\frac{\nu}{\kappa}\right) \|\boldsymbol{v}_p\|^2_{L^2(\Omega_p)^2} \tag{49}$$

$$\|\boldsymbol{v}\|^2_a := a(\boldsymbol{v}, \boldsymbol{v}) = |\boldsymbol{v}_f|^2_{a_f} + \|\boldsymbol{v}_p\|^2_{a_p} \tag{50}$$

$$|\boldsymbol{v}_f|^2_{H^2_{a_f}(\Omega_f)^2} := \nu \|\boldsymbol{v}_f\|^2_{H^2(\Omega_f)^2}$$

$$|\boldsymbol{v}_p|^2_{H^2_{a_p}(\Omega_p)^2} = \frac{\nu}{\kappa} |\boldsymbol{v}_p|^2_{H^2(\Omega_p)^2}$$

$$\|q_f\|^2_{L^2_{a_f^{-1}}} := \frac{1}{\nu} \|q_f\|^2_{L^2(\Omega_f)}$$

$$\|q_p\|^2_{L^2_{a_p^{-1}}} := \frac{\kappa}{\nu} \|q_p\|^2_{L^2(\Omega_p)}$$

$$\|q_f\|^2_{L^2_{a^{-1}}} := \|q_f\|^2_{L^2_{a_f^{-1}}} + \|q_p\|^2_{L^2_{a_p^{-1}}}$$

$$\|q_f\|^2_{H^1_{a_f}(\Omega_f)} := \nu \|q_f\|^2_{H^1(\Omega_f)}$$

$$\|q_p\|^2_{H^1_{a_p}(\Omega_p)} := \frac{\kappa}{\nu} \|q_p\|^2_{H^1(\Omega_p)}.$$

We can write (34) in Lemma 10 as

$$|\boldsymbol{v}_f - \boldsymbol{I}^{TH}_{h_f} \boldsymbol{v}_f|_{a_f} \preceq h_f |\boldsymbol{v}_f|_{H^2_{a_f}(\Omega)^2} \tag{51}$$

and Lemma 11 as

$$\|\boldsymbol{v}_p - \boldsymbol{I}^{RT}_{h_p} \boldsymbol{v}_p\|_{a_p} \preceq h_p |\boldsymbol{v}_p|_{H^1_{a_p}(\Omega)^2}. \tag{52}$$

From Proposition 3 we get that $\boldsymbol{Z}^h(g) \cap \boldsymbol{V}^h$ is not empty, where $\boldsymbol{Z}^h(g)$ is defined in (39). Choose an $\boldsymbol{u}^h_0 \in \boldsymbol{Z}^h(g) \cap \boldsymbol{V}^h$ and solve: find $\boldsymbol{z}^h \in \boldsymbol{Z}^h(0) \cap \boldsymbol{V}^h$ such that

$$a(\boldsymbol{z}^h, \boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) - a(\boldsymbol{u}^h_0, \boldsymbol{v}^h) \quad \boldsymbol{v}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}^h(0). \tag{53}$$

This is possible because a is *elliptic* in $\boldsymbol{V}^h \cap \boldsymbol{Z}^h(0)$. Then $\boldsymbol{u}^h := \boldsymbol{z}^h + \boldsymbol{u}^h_0$ is the only solution of

$$\left\{ \begin{array}{lll} a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) + ([\![\boldsymbol{v}^h]\!] \cdot \boldsymbol{\eta}_p, \lambda^{h_p}) & = \ell(\boldsymbol{v}^h) & \forall \boldsymbol{v}^h \in \boldsymbol{X}^h \\ b(\boldsymbol{u}^h, q^h) & = g(q^h) & \forall q^h \in M^h \\ ([\![\boldsymbol{u}^h]\!] \cdot \boldsymbol{\eta}_p, \mu^{h_p}) & = 0 & \forall \mu^{h_p} \in \Lambda^{h_p}. \end{array} \right.$$

For any $\boldsymbol{w}^h \in \boldsymbol{Z}^h(g) \cap \boldsymbol{V}^h$ we have that $\boldsymbol{v}^h := \boldsymbol{u}^h - \boldsymbol{w}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}^h(0)$ and

$$\begin{aligned} a(\boldsymbol{v}^h, \boldsymbol{v}^h) &= a(\boldsymbol{u}^h, \boldsymbol{v}^h) - a(\boldsymbol{w}^h, \boldsymbol{v}^h) \\ &= \ell(\boldsymbol{v}^h) - a(\boldsymbol{w}^h, \boldsymbol{v}^h). \end{aligned}$$

Let $(\boldsymbol{u}, p, \lambda)$ be the solution of the continuous problem (15). We have

$$\begin{aligned} \ell(\boldsymbol{v}^h) &= a(\boldsymbol{u}, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p) + b_\Gamma(\boldsymbol{v}^h, \lambda) \\ &= a(\boldsymbol{u}, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p - q^h) + b_\Gamma(\boldsymbol{v}^h, \lambda), \end{aligned}$$

and then

$$a(\boldsymbol{v}^h, \boldsymbol{v}^h) = a(\boldsymbol{u} - \boldsymbol{w}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p - q^h) + b_\Gamma(\boldsymbol{v}^h, \lambda).$$

Note that, since we are using lowest order Raviar-Thomas elements, the porous side components of $\boldsymbol{Z}^h(0)$ defined in (39) are divergence free, i.e., $\boldsymbol{Z}_p^h(0) \subset \boldsymbol{Z}_p$ where $\boldsymbol{Z}_p$ is defined in (21). Then we have $b_p(\boldsymbol{v}_p^{h_p}, p_p - q_p^{h_p}) = 0$ and

$$\begin{aligned} \|\boldsymbol{u}^h - \boldsymbol{w}^h\|_a = \|\boldsymbol{v}^h\|_a &\le \|\boldsymbol{u} - \boldsymbol{w}^h\|_a + \|b\| \|p_f - q_f^h\|_{L^2_{a_f^{-1}}} \\ &\quad + \sup_{\boldsymbol{v}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}^h(0)} \frac{b_\Gamma(\boldsymbol{v}^h, \lambda)}{\|\boldsymbol{v}^h\|_a}. \end{aligned}$$

Then, for $\boldsymbol{w}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}^h(g)$ and for all $p^h \in M^h$ we have

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}^h\|_a &\le \|\boldsymbol{u} - \boldsymbol{w}^h\|_a + \|\boldsymbol{u}^h - \boldsymbol{w}^h\|_a \\ &\le 2\|\boldsymbol{u} - \boldsymbol{w}^h\|_a + \|b_f\| \|p_f - q_f^h\|_{L^2_{a_f^{-1}}} + \sup_{\boldsymbol{v}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}^h(0)} \frac{b_\Gamma(\boldsymbol{v}^h, \lambda)}{\|\boldsymbol{v}^h\|_a}, \end{aligned}$$

hence

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}^h\|_a \le \; & 2 \inf_{\boldsymbol{w}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}(g)} \|\boldsymbol{u} - \boldsymbol{w}^h\|_a + \|b_f\| \inf_{q_f^{h_f} \in M_f^{h_f}} \|p_f - q_f^h\|_{L^2_{a_f^{-1}}} \\ & + \sup_{\boldsymbol{v}^h \in \boldsymbol{V}^h \setminus \{0\}} \frac{b_\Gamma(\boldsymbol{v}^h, \lambda)}{\|\boldsymbol{v}^h\|_a}. \end{aligned} \tag{54}$$

To analyze the last term in this inequality we show that

$$|b_\Gamma(\boldsymbol{v}^h, \lambda)| \preceq h_p \nu^{-\frac{1}{2}} \|p_p\|_{H^{1/2}(\Gamma)} \|\boldsymbol{v}^h\|_a, \tag{55}$$

and this gives a bound for the consistency error.

To obtain (55) remember that $\lambda := p_p$ and $\lambda \in \Lambda$. Then

$$\begin{aligned} b_\Gamma(\boldsymbol{v}^h, p_p) &= \langle p_p, \boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f \rangle_\Gamma + \langle p_p, \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p \rangle_\Gamma \\ &= \langle p_p, \boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f \rangle_\Gamma + \langle Q^{h_p} p_p, \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p \rangle_\Gamma \qquad \boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p \text{ is constant in } e \\ &= \langle p_p - Q^{h_p} p_p, \boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f \rangle_\Gamma, \end{aligned}$$

hence

$$\begin{aligned}|b_\Gamma(\boldsymbol{v}^h, p_p)| &= |\langle p_p - Q^{h_p} p_p, \boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f \rangle_\Gamma| \\ &\leq |\boldsymbol{v}_f^{h_f} \cdot \boldsymbol{\eta}_f|_{H^{1/2}(\Gamma)} |p_p - Q^{h_p} p_p|_{H^{-1/2}(\Gamma)},\end{aligned}$$

which together with (40) give (55).

Then, from (54), (55) , (51), (52) and since $\|b_f\| \leq 1$, we have

Proposition 4. *Suppose that $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$. Then we have the following estimate*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^h\|_a \quad &\preceq h_f \|\boldsymbol{u}_f\|_{H^2_{a_f}(\Omega_f)^2} + h_f \|p_f\|_{H^1_{a_f}(\Omega_f)} \\ &+ h_p |\boldsymbol{u}_p|_{H^1_{a_p}(\Omega_p)^2} + h_p \nu^{-\frac{1}{2}} \|p_p\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Remark 5. *We note that from (14) we can estimate*

$$\begin{aligned}\|p_p\|_{H^{1/2}(\Gamma)} \quad &\leq \|p_f\|_{H^{1/2}(\Gamma)} + 2\nu |\boldsymbol{u}_f|_{H^{1/2}(\Gamma)^2} \\ &\preceq \|p_f\|_{H^{1/2}(\Gamma)} + \nu^{\frac{1}{2}} |\boldsymbol{u}_f|_{H^{1/2}_{a_f}(\Omega)^2},\end{aligned}$$

then the previous proposition implies

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^h\|_a \quad \preceq \quad & h_f \|\boldsymbol{u}_f\|_{H^2_{a_f}(\Omega_f)^2} + h_f \|p_f\|_{H^1_{a_f}(\Omega_f)} \\ & + h_p |\boldsymbol{u}_p|_{H^1_{a_p}(\Omega_p)^2} + h_p \nu^{-\frac{1}{2}} \|p_f\|_{H^{1/2}(\Gamma)} + h_p |\boldsymbol{u}_f|_{H^{1/2}_{a_f}(\Omega)^2}.\end{aligned}$$

Remark 6. *If $\alpha_f \neq 0$ we can use the norm*

$$a|\boldsymbol{v}_f|^2_{a^{\alpha_f}} := \nu |\boldsymbol{v}_f|^2_{H^1(\Omega_f)} + \sum_{j=1}^{n-1} \frac{\nu \alpha_f}{\sqrt{\kappa}} \|\boldsymbol{v}_f \cdot \boldsymbol{\eta}_f\|^2_{L^2(\Gamma)}$$

instead of $|\cdot|_{a_f}$ defined in (48). Then the first term in (54) can be bounded by

$$\inf_{\boldsymbol{w}^h \in \boldsymbol{V}^h \cap \boldsymbol{Z}(g)} \|\boldsymbol{u} - \boldsymbol{w}^h\|_a \preceq h_f |\boldsymbol{u}_f|_{H^2_{a_f}(\Omega_f)^2} + h_f^{\frac{3}{2}} |\boldsymbol{u}_f|_{H^1_\alpha(\Omega_f)^2}$$

where $|\boldsymbol{u}_f|_{H^1_\alpha(\Omega_f)^2} := \frac{\nu \alpha_f}{\sqrt{\kappa}} |\boldsymbol{u}_f|_{H^1(\Omega_f)^2}$.

To obtain an expression for the pressure error observe that for all $\boldsymbol{v}^h$ we have

$$\begin{aligned}b(\boldsymbol{v}^h, p^h - q^h) + ([\![\boldsymbol{v}^h]\!], \lambda^h - \nu^h) \quad &= \quad a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p - q^h) \\ &\quad + b_\Gamma(\boldsymbol{v}^h, \lambda) - ([\![\boldsymbol{v}^h]\!], \lambda^h).\end{aligned}$$

In particular this holds true for all $\boldsymbol{v}^h = (\boldsymbol{v}_f^{h_f}, 0)$ with $\boldsymbol{v}_f^{h_f} = 0$ on $\partial\Omega_f = \Gamma \cup \Gamma_f$, i.e.,

$$b_f(\boldsymbol{v}_f^{h_f}, p_f^{h_f} - q_f^{h_f}) = a_f(\boldsymbol{u}_f - \boldsymbol{u}_f^{h_f}, \boldsymbol{v}_f^{h_f}) + b(\boldsymbol{v}_f^{h_f}, p_f - q_f^{h_f}),$$

then, using the usual *inf-sup* condition for the fluid problem, we have that

$$\begin{aligned} \|p_f^{h_f} - q_f^{h_f}\|_{L^2_{a_f^{-1}}} &\leq \frac{1}{\beta} \sup_{\boldsymbol{v}_f^{h_f} \in \boldsymbol{X}_f^{h_f} \cap H_0^1(\Omega)} \frac{a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p - q^h)}{\|\boldsymbol{v}^h\|_{\boldsymbol{X}}} \\ &\leq \frac{1}{\beta}\|\boldsymbol{u_f} - \boldsymbol{u}_f^{h_f}\|_{a_f} + \frac{1}{\beta}\|p_f - q_f^{h_f}\|_{M_f} \\ &\leq \frac{1}{\beta}\|\boldsymbol{u} - \boldsymbol{u}^h\|_a + \frac{1}{\beta}\|p_f - q_f^{h_f}\|_{L^2_{a_f^{-1}}}. \end{aligned}$$

From the triangle inequality and the bound for the velocity error we have

$$\begin{aligned} \|p_f - p_f^{h_f}\|_{L^2_{a_f^{-1}}} &\leq \|p_f^{h_f} - q_f^{h_f}\|_{L^2_{a_f^{-1}}} + \|p_f - q_f^{h_f}\|_{L^2_{a_f^{-1}}} \\ &\leq \frac{1}{\beta}\|\boldsymbol{u} - \boldsymbol{u}^h\|_a + \frac{2}{\beta}\|p_f - q_f^{h_f}\|_{L^2_{a_f^{-1}}} \\ &\preceq h_f|\boldsymbol{u}_f|_{H^2_{a_f}(\Omega_f)^2} + h_f|p_f|_{H^1_{a_f}(\Omega_f)} \qquad (56) \\ &\quad + h_p|\boldsymbol{u}_p|_{H^1_{a_p}(\Omega_p)} + h_p\nu^{-\frac{1}{2}}|p_p|_{H^{1/2}(\Gamma)}. \qquad (57) \end{aligned}$$

Analogously we obtain

$$\begin{aligned} \|p_p - p_p^{h_p}\|_{M_p} &\preceq h_f|\boldsymbol{u}_f|_{H^2_{a_f}(\Omega_f)^2} + h_f|p_f|_{H^1_{a_f}(\Omega_f)} \\ &\quad + h_p|\boldsymbol{u}_p|_{H^1_{a_p}(\Omega_p)^2} + h_p\nu^{-\frac{1}{2}}|p_p|_{H^{1/2}(\Gamma)} + h_p\|p_p\|_{H^1_{a_p}(\Omega_p)} \\ &\preceq h_f|\boldsymbol{u}_f|_{H^2_{a_f}(\Omega_f)^2} + h_f|p_f|_{H^1_{a_f}(\Omega_f)} \\ &\quad + h_p|\boldsymbol{u}_p|_{H^1_{a_p}(\Omega_p)^2} + h_p\frac{1}{\kappa^{\frac{1}{2}}}\|p_p\|_{H^1_{a_p}(\Omega_p)}. \qquad (58) \end{aligned}$$

Now we analyze the approximation properties of Λ^{h_p} (see [ACY00]). Note that when $p_p \in L^2(\Gamma)$ we can define

$$\|p_p\|_{\Lambda^{h_p}} := \|Q^{h_p}p_p\|_{\Lambda^{h_p}} \qquad (59)$$

where $Q^{h_p}p_p$ is the L^2-projection of p_p on Λ^{h_p}.

Proposition 5. *If p_p and $\boldsymbol{u}_p$ are regular enough, then*

$$\|p - \lambda^{h_p}\|_{\Lambda^{h_p}} \preceq h_p\left(|p_p|_{H^2(\Omega_p)^2} + |\boldsymbol{u}_p|_{H^1(\Omega_p)^2}\right). \qquad (60)$$

Proof. Let $\hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})$ and $\hat{p}_p^{h_p}(\lambda^{h_p})$ solve

$$\begin{cases} a_p(\hat{\boldsymbol{u}}_p^{h_p}, \boldsymbol{v}_p^{h_p}) + b_p(\boldsymbol{v}_p^{h_p}, \hat{p}_p^{h_p}) & = -(\boldsymbol{v}_p^{h_p} \cdot \boldsymbol{\eta}_p, \lambda^{h_p})_\Gamma; \quad \boldsymbol{v}_p^{h_p} \in \boldsymbol{X}_p^{h_p} \\ b_p(\hat{\boldsymbol{u}}_p^{h_p}, q_p^{h_p}) & = -(g_p, q_p^{h_p})_{\Omega_p} \qquad q_p^{h_p} \in M_p^{h_p}. \end{cases} \tag{61}$$

By using (45) and (61) with $\boldsymbol{v}^{h_p} = \widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})$ in both equations we get

$$\begin{aligned} a_p\left(\widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}), \widetilde{\boldsymbol{u}}_p^{h_p}(p_p) \right. & - \left. \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\right) = \\ & = \left(\left[\widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\right] \cdot \boldsymbol{\eta}_p, p_p - \lambda^{h_p}\right)_\Gamma \\ & = a_p\left(\hat{\boldsymbol{u}}_p^{h_p}(p_p) - \hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}), \widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\right) \\ & \leq \|\hat{\boldsymbol{u}}_p^{h_p}(p_p) - \hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p} \|\widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p}, \end{aligned}$$

then $\|\widetilde{\boldsymbol{u}}_p^{h_p}(p_p) - \widetilde{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p} \leq \|\hat{\boldsymbol{u}}_p^{h_p}(p_p) - \hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p}$, and we estimate

$$\|\hat{\boldsymbol{u}}_p^{h_p}(p_p) - \hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p} \leq \|\hat{\boldsymbol{u}}_p^{h_p}(p_p) - \boldsymbol{u}_p\|_{a_p} + \|\boldsymbol{u}_p - \hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})\|_{a_p}. \tag{62}$$

In order to obtain (60) we use (62) and observe that $\hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p})$ is our discrete approximation of the porous side velocity, i.e., $\hat{\boldsymbol{u}}_p^{h_p}(\lambda^{h_p}) = \boldsymbol{u}_p^{h_p}$. Then the second term in last equation is estimated using (52). To estimate the other term use standard mixed method for problem (61). (See [GR86, ACY00]). □

7 Numerical Results

In this section we present a numerical test in order to verify the estimates proved above. We consider $\Omega_f = (1, 2) \times (0, 1)$ and $\Omega_p = (0, 1) \times (0, 1)$. We use $\alpha_1 = 0$.

The velocity solution for Stokes is given by $\boldsymbol{u}_f(x, y) = (y(1 - y), -x + 1 + 2(x - 1)y)$ with pressure $p_f(x, y) = -2x - \frac{\nu}{\kappa}y + 5/2 + \frac{5\nu}{12\kappa}$. Note that $\boldsymbol{u}_f$ is not divergence free.
The velocity solution for Darcy is $\boldsymbol{u}_p(x, y) = (1-2x+x^2+y-y^2, -1+x+2y-2xy)$ with pressure $p_p(x, y) = \frac{\nu}{\kappa}((1 - x)y(1 - y) - x + x^2 - \frac{x^3}{3} + \frac{3}{4} - y) + \frac{1}{2}$.

Note that the normal component of $\boldsymbol{u}_p$ has parabolic profile on the interface $\Gamma = 1 \times (0, 1)$ while its tangential component is zero. Note also that $\boldsymbol{D}\boldsymbol{u}_f = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$ on Γ, which it compatible with the choices $\alpha_1 = 0$ and continuity of pressures on Γ.

A similar example is presented in [BH06], where the term $\nabla \cdot \boldsymbol{D}\boldsymbol{u}_f$ is replaced by $\Delta \boldsymbol{u}_f$ in Stokes equation.

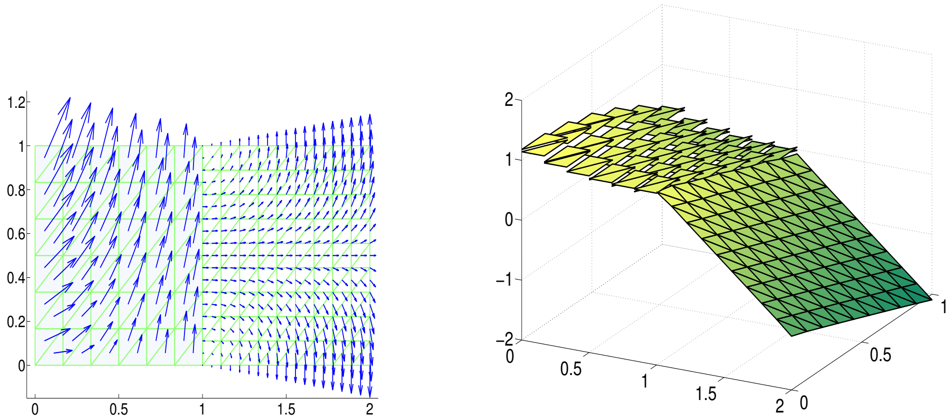

Figure 1: Computed velocities and pressures. On the porous side (left) we have plot the value of the velocity at the centroid of each triangle.

In Figure 1 we show the computed solution of the coupled problem. On the porous side we have plot the velocity in the center of the each triangle. In Figure 2 we zoom part of the interface and plot the y component of the velocities.

In Figure 3 we show the behavior of the error (in the scaled norms defined in (48), (49) and (50)) with respect to the discretization parameters. Here we also show $\|p-\lambda^{h_p}\|_{\Lambda^{h_p}}$, i.e., the Lagrange multiplier approximation error in the discrete norm defined in (59).

We observe according to Figure 3 that the error in the norm $\|\cdot\|_a^2$ defined in (50) which is the sum of the fluid velocity and porous velocity errors in the scaled norms is of linear order. This agree with Proposition 4. Analogously, the pressure error is of linear order. This also agree with the result about the pressure error, see (57) and (58). We finally observe that the Lagrange multiplier error in the discrete norm defined in (59) is also of linear order.

8 Conclusion

We study the coupling across an interface of fluid and porous media flows, consisting of *Stokes equations* in the fluid region Ω_f and *Darcy law* for the filtration velocity in the porous medium region Ω_p. After discussing the adequate choice of a Lagrange multiplier space we present a complete analysis for the inf-sup and approximation results associated to the continuous and discrete formulations of this Stokes-Darcy system. The analysis considered here permit us to choose the space $H^{1/2}(\Gamma)$ as the Lagrange multipliers space. We have chosen second order Taylor Hood elements and lower order Raviart-Thomas elements as dis-

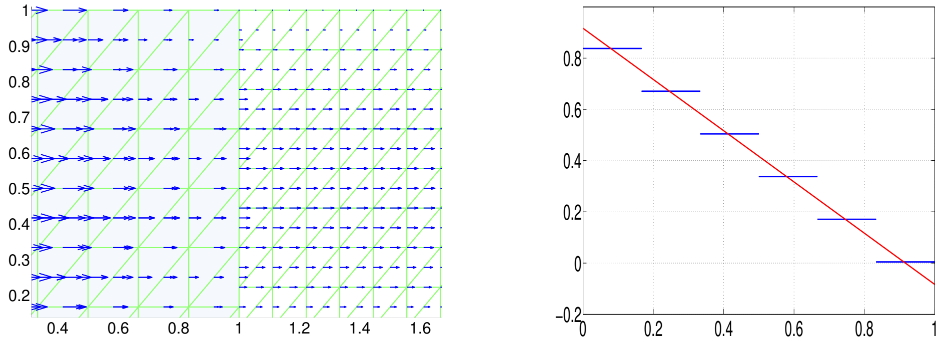

Figure 2: Zoom on the interface (left) and computed Lagrange multiplier compared with porous pressure on the interface (right). On the porous side (left) we have plotted the two values of the y component of the velocities at the midpoint of each edge. Recall that Raviart Thomas elements allows discontinuous tangential velocities on interior edges.

crete spaces for the free and porous medium subdomain, respectively. Optimal a priori discrete error estimates do not depend on meshes ratio were deduced using mortar analysis. Moreover, the constants that appear on those a priori errors do not depend on the parameters such as viscosity and permeabilities. The numerical experiment show the good agreements with our theoretical results.

A Two Remarks on the weak formulation

A.1 Nonhomogeneuous Boundary Conditions

The nonhomogeneous boundary condition can be reduced to the homogeneous case when $\boldsymbol{h}_f \in H^{1/2}(\Gamma_f)^2$. In fact, since $\boldsymbol{h}_f \in H^{1/2}(\Gamma_f)^2$ we can consider $\boldsymbol{\mu} := E_{\Gamma}^{1/2}\boldsymbol{h}_f \in H^{1/2}(\partial\Omega_f)^2$ and define:

$$\hat{\boldsymbol{h}}_f := \boldsymbol{\mu} + cE_{00,\Gamma}^{1/2}\boldsymbol{\theta}, \text{ in } \partial\Omega_f, \tag{63}$$

here $\boldsymbol{\theta} \in H_{00}^{1/2}(\Gamma)^2$ with $\int_\Gamma \boldsymbol{\theta}\cdot\boldsymbol{\eta}_f = 1$. Note that $\hat{\boldsymbol{h}}_f|_{\Gamma_f} = \boldsymbol{h}_f$. The constant c is chosen such that:

$$\int_{\Omega_f} g_f = \int_{\partial\Omega_f} \hat{\boldsymbol{h}}_f\cdot\boldsymbol{\eta}_f.$$

It is possible to construct $\boldsymbol{\omega}_f \in H^1(\Omega_f)^2$ such that[1]:

$$\boldsymbol{\omega}_f|_{\partial\Omega_f} = \hat{\boldsymbol{h}}_f \quad \text{and} \quad \nabla\cdot\boldsymbol{\omega}_f = g_f \quad \text{in} \quad \Omega_f.$$

[1]Take $\boldsymbol{w}$ solution of $\left\{\begin{array}{rll} -\Delta\boldsymbol{w} + \nabla p & = 0 & \text{in } \Omega_f \\ \nabla\cdot\boldsymbol{w} & = g_f & \text{in } \Omega_f \\ \boldsymbol{w} & = \hat{\boldsymbol{h}}_f & \text{on } \partial\Omega_f \end{array}\right.$, note that $\int_{\Omega_f} g_f = \int_{\partial\Omega_f} \hat{\boldsymbol{h}}_f\cdot\boldsymbol{\eta}_f$.

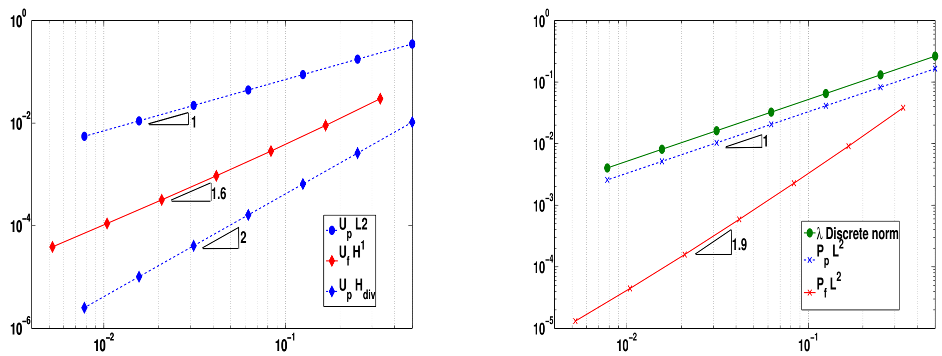

Figure 3: Velocities errors (right) and pressures errors (left).

Now put $\boldsymbol{u}_f = \boldsymbol{\omega}_f + \boldsymbol{\zeta}_f$ where $\boldsymbol{u}_f$ satisfies the nonhomogeneous system (2). So we are looking for $\boldsymbol{\zeta}_f$ that satisfy:

$$\left\{ \begin{array}{rll} -\mu\Delta\boldsymbol{\zeta}_f + \nabla p & = \boldsymbol{f}_f + \mu\Delta\boldsymbol{\omega}_f & \text{in } \Omega_f \\ \nabla\cdot\boldsymbol{\zeta}_f & = 0 & \text{in } \Omega_f \\ \boldsymbol{\zeta}_f & = 0 & \text{on } \Gamma_f \end{array} \right. \tag{64}$$

Analogously, on the porous region the nonhomogeneous case can be reduced to the homogeneous one. In this case we need $h_p \in H^{-1/2}(\Gamma_p)$. We can define $\hat{h}_p \in H^{-1/2}(\partial\Omega_p)$ by:

$$\langle \hat{h}_p, \mu\rangle_{\partial\Omega_p} := \langle \hat{\boldsymbol{h}}_f \cdot \eta_p, \mu|_\Gamma\rangle_\Gamma + \langle h_p, \mu|_{\Gamma_p}\rangle_{\Gamma_p}, \quad \forall \mu \in H^{1/2}(\partial\Omega_p), \tag{65}$$

and then use the (normal) trace theorem to get $\boldsymbol{\omega}_p \in \boldsymbol{H}(\text{div}, \Omega_p)$ such that:

$$\boldsymbol{\omega}_p \cdot \eta = \hat{h}_p.$$

Put $\boldsymbol{u}_p = \boldsymbol{\omega}_p + \boldsymbol{\zeta}_p$. Then we look for $\boldsymbol{\zeta}_p$ such that:

$$\left\{ \begin{array}{rll} \boldsymbol{\zeta}_p & = -\kappa\nabla p - \boldsymbol{\omega}_p & \text{in } \Omega_p \\ \nabla\cdot\boldsymbol{\zeta}_p & = g_p - \nabla\cdot\boldsymbol{\omega}_p & \text{in } \Omega_p \\ \boldsymbol{\zeta}_p \cdot \boldsymbol{\eta}_p & = 0 & \Gamma_p, \end{array} \right.$$

where the last equation follows from the definition of $\hat{h}_p$, (65). The compatibility condition is now:

$$\int_{\Omega_p} (g_p - \nabla\cdot\boldsymbol{\omega}_p) = 0,$$

which is equivalent to (4). If we look at the weak formulation, with $\boldsymbol{\omega}_j$ constructed as before define $\boldsymbol{\omega} := (\boldsymbol{\omega}_f, \boldsymbol{\omega}_p)$ we have to: *find* $(\boldsymbol{\zeta}, p, \lambda) \in \boldsymbol{X} \times M \times \Lambda$ *satisfying, for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M \times \Lambda$:

$$\left\{ \begin{array}{ll} a(\boldsymbol{\zeta}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_\Gamma(\boldsymbol{v}, \lambda) & = \ell(\boldsymbol{v}) - a(\boldsymbol{\omega}, \boldsymbol{v}) \\ b(\boldsymbol{\zeta}, q) & = g(q) + b(\boldsymbol{\omega}, q) \\ b_\Gamma(\boldsymbol{\zeta}, \mu) & = 0, \end{array} \right.$$

which is the same problem (18) with a different right hand side.

A.2 Equivalent weak formulation

Instead of the spaces $\boldsymbol{X}$ and M used to obtain the weak formulation (18) we can use:

$$\boldsymbol{Y}_f := \left\{ \boldsymbol{v}_f \in \boldsymbol{H}^1(\Omega_f) \;:\; \boldsymbol{v}_f\Big|_{\Gamma_f} = \boldsymbol{0} \right\}, \qquad N_f := L^2(\Omega_f).$$

and

$$\boldsymbol{Y}_p := \left\{ \boldsymbol{v}_p \in \boldsymbol{H}(\mathrm{div}, \Omega) \;:\; \boldsymbol{v}_p \cdot \boldsymbol{\eta}_p\Big|_{\Gamma_p} = \boldsymbol{0} \right\}, \qquad N_p := L^2(\Omega_p).$$

with $\boldsymbol{Y} := \boldsymbol{Y}_f \times \boldsymbol{Y}_p$ and

$$N := \left\{ (q_f, q_p) \in N_f \times N_p \;:\; \int_{\Omega_f} q_f + \int_{\Omega_p} q_p = 0 \right\},$$

and obtain an equivalent weak formulation. This is the weak formulation presented in [LSY03]. The advantage of the weak formulation (18) is that permit us to work with seminorms instead of norms and that the space Λ for the Lagrange multipliers is bigger and is not restricted to have zero value at the end points of the interface. The equivalence between the two formulations follows from the solvability condition and the divergence theorem.

B Approximation properties of Taylor-Hood finite elements

The domain of reference is Ω_f. In order to simplify the notation in some cases we omit the subscript that refers to the domain. In particular, all the operators defined in this section acts on velocities defined on Ω_f.

Let $\boldsymbol{\mathcal{Q}} : \boldsymbol{X}_f \to \boldsymbol{X}_f^h$ be Clement interpolation (see [Bra01, Cle75, SZ90]). It is know that $\boldsymbol{\mathcal{Q}}$ is bounded, i.e.,

$$|\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq |\boldsymbol{v}|_{H^1(\Omega_f)^2}, \tag{66}$$

and we have

$$\begin{aligned}
\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} &\preceq h^s |\boldsymbol{v}_f|_{H^s(\Omega_f)^2}, \quad s = 1, 2. &(67)\\
|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} &\preceq h |\boldsymbol{v}_f|_{H^2(\Omega_f)^2}, &(68)\\
|\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} &\preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2}, &(69)\\
\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L(\Gamma)^2} &\preceq h^{\frac{1}{2}} |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2}. &(70)
\end{aligned}$$

This interpolation is basically a Clement interpolation on Γ, i.e., values zero at the interface relative boundary points and a Clement interpolation at the interior nodes.

Given $K \in \mathcal{T}^h$ and e edge of K, let $\boldsymbol{\eta}_e^{(K)} = (\eta_e^1, \eta_e^2)$ denotes the normal to e exterior to K, $\boldsymbol{\tau}_e^{(K)} = (\tau_e^1, \tau_e^2)$ the tangential vector to e (with ∂K anticlockwise oriented), and x_e the midpoint of the edge e. Each interior edge belongs to two triangles K_1 and K_2. Let $\boldsymbol{\eta}_e$ denote one of the directions $\boldsymbol{\eta}_e^{(K_1)}$ or $\boldsymbol{\eta}_e^{(K_2)}$. For boundary edges $\boldsymbol{\eta}_e$ denotes $\boldsymbol{\eta}_e^{(K)}$. Analogously, for interior edges let $\boldsymbol{\tau}_e$ denote one of the directions $\boldsymbol{\tau}_e^{(K_1)}$ or $\boldsymbol{\tau}_e^{(K_2)}$, and for boundary edges $\boldsymbol{\eta}_e = \boldsymbol{\eta}_e^{(K)}$.

Let $\phi_i^{(K)}$, $i = 1, 2, 3$, be the Taylor-Hood basis functions based on the mid-points of the edges of K. Let $\boldsymbol{\psi}_i^{(K)} := \phi_i^{(K)} \boldsymbol{\eta}_{e_i}$, $i = 1, 2, 3$, and $\boldsymbol{\vartheta}_i^{(K)} := \phi_i^{(K)} \boldsymbol{\tau}_{e_i}$, $i = 1, 2, 3$. Observe that:

$$\textstyle\int_K \boldsymbol{\psi}_i^{(K)} \cdot \boldsymbol{\eta}_{e_i} \neq 0, \qquad \boldsymbol{\psi}_i^{(K)} \cdot \boldsymbol{\tau}_{e_i} = 0 \quad i = 1, 2, 3.$$

$$\boldsymbol{\vartheta}_i^{(K)}(x_{e_i}) \cdot \boldsymbol{\tau}_{e_i} \neq 0, \quad \boldsymbol{\vartheta}_i^{(K)} \cdot \boldsymbol{\eta}_{e_i} = 0 \quad i = 1, 2, 3.$$

Now consider the following subspaces of $\boldsymbol{X}_f^h$:

$$\boldsymbol{W}_{\boldsymbol{\eta}}^h := \left\{ \boldsymbol{v}_f^h \in \boldsymbol{X}_f^h : v|_K \in \operatorname{Span}\{\boldsymbol{\psi}_1^{(K)}, \boldsymbol{\psi}_2^{(K)}, \boldsymbol{\psi}_3^{(K)}\} \right\} \cap \boldsymbol{X}_f^h$$

and

$$\boldsymbol{W}_{\boldsymbol{\tau}}^h := \{ \boldsymbol{v}_f^h \in \boldsymbol{X}_f^h : v|_K \in \operatorname{Span}\{\boldsymbol{\vartheta}_1^{(K)}, \boldsymbol{\vartheta}_2^{(K)}, \boldsymbol{\vartheta}_3^{(K)}\} \} \cap \boldsymbol{X}_f^h.$$

Note that if $\boldsymbol{v}_f^h \in \boldsymbol{W}_{\boldsymbol{\eta}}^h$ then $\boldsymbol{v}_f^h \cdot \boldsymbol{\eta}|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $\boldsymbol{v}_f^h \cdot \boldsymbol{\tau}|_{\partial\Omega_f} = 0$. Also note that if $\boldsymbol{v}_f^h \in \boldsymbol{W}_{\boldsymbol{\tau}}^h$ then $\boldsymbol{v}_f^h \cdot \boldsymbol{\tau}|_\Gamma \in H_{00}^{1/2}(\Gamma)$ and $\boldsymbol{v}_f^h \cdot \boldsymbol{\eta}|_{\partial\Omega_f} = 0$.

Let $\boldsymbol{\Pi}_\eta : \boldsymbol{X}_f \to \boldsymbol{W}_{\boldsymbol{\eta}}^h$ be (locally) defined by :

$$\boldsymbol{\Pi}_\eta \boldsymbol{v}_f \in \operatorname{Span}\{\boldsymbol{\psi}_1^{(K)}, \boldsymbol{\psi}_2^{(K)}, \boldsymbol{\psi}_3^{(K)}\}, \quad \text{s.t.} \int_{e_i} \boldsymbol{\Pi}_\eta \boldsymbol{v}_f \boldsymbol{\eta} = \frac{1}{|e_i|} \int_{e_i} \boldsymbol{v}_f \boldsymbol{\eta}_{e_i}, \;\; i = 1, 2, 3,$$

for all $K \in \mathcal{T}^h$. In other words, $\boldsymbol{\Pi}_\eta \boldsymbol{v}_f = \alpha_1 \boldsymbol{\psi}_1 + \alpha_2 \boldsymbol{\psi}_2 + \alpha_3 \boldsymbol{\psi}_3$, where

$$\alpha_i := \frac{\int_{e_i} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_{e_i}}{|e_i| \int_{e_i} \boldsymbol{\psi}_i \cdot \boldsymbol{\eta}_{e_i}} = \frac{\int_{e_i} \boldsymbol{v}_f \cdot \boldsymbol{\eta}_{e_i}}{|e_i| \int_{e_i} \phi_i^{(K)}}.$$

From a trace theorem and a scaling argument we have that:

$$|\alpha_i|^2 \preceq \frac{1}{h^2} \|\boldsymbol{v}_f\|_{L^2(K)^2}^2 + |\boldsymbol{v}_f|_{H^1(K)^2}^2.$$

Then

$$|\boldsymbol{\Pi}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq \max_{1 \le i \le 3} |\alpha_i|^2 \preceq \frac{1}{h^2} \|\boldsymbol{v}_f\|_{L^2(\Omega_f)^2}^2 + |\boldsymbol{v}_f|_{H^1(\Omega_f)^2}^2$$

and

$$\|\boldsymbol{\Pi}_\eta \boldsymbol{v}_f\|^2_{L^2(\Omega_f)^2} \preceq h^2 \max_{1\le i\le 3} |\alpha_i|^2 \preceq \|\boldsymbol{v}_f\|^2_{L^2(\Omega_f)^2} + h^2|\boldsymbol{v}_f|^2_{H^1(\Omega_f)^2}. \tag{71}$$

Observe that

$$\int_K \nabla\cdot\boldsymbol{\Pi}_\eta \boldsymbol{v}_f = \int_{\partial K} \boldsymbol{\Pi}_\eta(\boldsymbol{v}_f)\cdot\boldsymbol{\eta} = \int_{\partial K} \boldsymbol{v}_f\cdot\boldsymbol{\eta} = \int_K \nabla\cdot\boldsymbol{v}_f.$$

We also have

$$\|\boldsymbol{\Pi}_\eta \boldsymbol{v}_f\|_{L^2(\Gamma)} \preceq \|\boldsymbol{v}_f\|_{L^2(\Gamma)}. \tag{72}$$

Define $\boldsymbol{\Upsilon}_\eta : \boldsymbol{X}_f \to \boldsymbol{X}_f^h$ by:

$$\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f := \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f + \boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f), \tag{73}$$

then we have the following results:

Lemma 14. *The operator $\boldsymbol{\Upsilon}_\eta$ defined in (73) is bounded*

$$|\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq \|\boldsymbol{v}_f\|_{H^1(\Omega_f)^2}, \tag{74}$$

moreover,

$$\|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \preceq h^s \|\boldsymbol{v}_f\|_{H^s(\Omega_f)^2} \quad s = 1, 2. \tag{75}$$

and

$$|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)} \preceq h|\boldsymbol{v}_f|_{H^2(\Omega_f)^2}. \tag{76}$$

We also have

$$|\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \preceq \|\boldsymbol{v}_f\|_{H^{1/2}(\Gamma)^2}, \tag{77}$$

and

$$\int_e \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\cdot\boldsymbol{\eta}_e = \int_e \boldsymbol{v}_f\cdot\boldsymbol{\eta}_e \quad \textit{for all edge } e. \tag{78}$$

Proof. From (71) we have, for $s = 1, 2$,

$$\begin{aligned}
\sum_{K\in\mathcal{T}^h} \|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f \quad &- \quad \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|^2_{L^2(K)^2} \preceq \sum_{K\in\mathcal{T}^h} \left(\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|^2_{L^2(K)^2} + h^2|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|^2_{H^1(K)^2}\right) \\
&\preceq \quad h^{2s}|\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2} + h^{2(s-1)+2}|\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2} \quad \text{by (67) and (66).} \\
&\preceq \quad h^{2s}|\boldsymbol{v}_f|^2_{H^s(\Omega_f)^2}.
\end{aligned} \tag{79}$$

Then, using an inverse estimate (see [Bra01]) and (79) we get

$$|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{H^1(K)^2} \preceq \frac{1}{h}\|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|_{L^2(K)^2} \preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2},$$

and hence

$$\begin{aligned}
|\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} \quad &\le |\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{H^1(\Omega_f)^2} \quad \text{by definition of } \boldsymbol{\Upsilon}_\eta \\
&\preceq |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} \preceq \|\boldsymbol{v}_f\|_{H^1(\Omega_f)^2}.
\end{aligned}$$

To show (75) we have that

$$
\begin{aligned}
\|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} &= \|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}} \boldsymbol{v}_f - \boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \quad \text{by definition of } \boldsymbol{\Upsilon}_\eta. \\
&\leq \|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \\
&\preceq h^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2} + h^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad \text{by (67) and (79)} \\
&\preceq h^s|\boldsymbol{v}_f|_{H^s(\Omega_f)^2}, \quad s = 1, 2.
\end{aligned}
$$

Analogously we get (76). To proof (77) observe that

$$
\begin{aligned}
|\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} &\leq |\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} + |\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{H^{1/2}(\Gamma)^2} \\
&\preceq \|\boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{H^{1/2}(\Gamma)^2} + h^{-\frac{1}{2}}|\boldsymbol{\Pi}_\eta(\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f)|_{L^2(\Gamma)^2} \\
&\preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} + h^{-\frac{1}{2}}\|\boldsymbol{v}_f - \boldsymbol{\mathcal{Q}}\boldsymbol{v}_f\|_{L^2(\Gamma)^2} \quad \text{by (69) and (72)} \\
&\preceq |\boldsymbol{v}_f|_{H^{1/2}(\Gamma)^2} \quad \text{by (70).}
\end{aligned}
$$

The last assertion, (78), is straightforward. □

Given $q_f \in M_f^h$, define (locally) $\widehat{\boldsymbol{\Pi}}_\tau q_f \in \boldsymbol{W}_{\boldsymbol{\tau}}^h$ by

$$\widehat{\boldsymbol{\Pi}}_\tau q_f|_K \in \operatorname{Span}\{\boldsymbol{\vartheta}_1^{(K)}, \boldsymbol{\vartheta}_2^{(K)}, \boldsymbol{\vartheta}_3^{(K)}\}$$

with

$$\widehat{\boldsymbol{\Pi}}_\tau q_f(x_{e_i}) \cdot \boldsymbol{\eta} = 0 \text{ and } \widehat{\boldsymbol{\Pi}}_\tau q_f(x_{e_i}) \cdot \boldsymbol{\tau} = \nabla q_f(x_{e_i}) \cdot \boldsymbol{\tau}$$

for all interior edge e_i and $\widehat{\boldsymbol{\Pi}}_\tau q|_e = 0$ for all edges on Γ_f. Note that $\widehat{\boldsymbol{\Pi}}_\tau q_f$ is zero at the vertices of all elements of $\mathcal{T}^h$ and observe that $\widehat{\boldsymbol{\Pi}}_\tau q_f \in H^1(\Omega_f)^2$ because the above equation are consistent in neighbor triangles which gives $\widehat{\boldsymbol{\Pi}}_\tau q_f$ continuous (see [Bra01], Chapter II, theorem 5.2).

Lemma 15. *Suppose that $\mathcal{T}_f^{h_f}$ is non-degenerate and has no triangle with two edges on $\partial\Omega_f$ and consider the operator $\widehat{\boldsymbol{\Pi}}_\tau$ defined in (B). Then*

$$\|\widehat{\boldsymbol{\Pi}}_\tau q_f^h\|_{\boldsymbol{L}^2(\Omega_f)^2} \preceq |q_f^h|_{H^1(\Omega_f)} \quad \textit{for all } q_f^h \in M_f^h$$

and there exists a positive constant such that:

$$\int_{\Omega_f} \widehat{\boldsymbol{\Pi}}_\tau q_f^h \cdot \nabla q_f^h \succeq |q_f^h|^2_{H^1(\Omega_f)} \succeq \|q_f^h\|^2_{L^2(\Omega_f)} \quad \textit{for all } q_f^h \in M_f^h.$$

From Lemma 15 and the boundedness of $\widehat{\boldsymbol{\Pi}}_\tau$, the spaces $\boldsymbol{W}_{\boldsymbol{\tau}}^h$ (with the $L^2(\Omega_f)$-norm) and M_f^h (with the $H^1(\Omega_f)$-norm) satisfy the inf-sup condition independent of h with respect to the bilinear form defined (in (11)) by:

$$b_f(\boldsymbol{v}_f, q_f) := -(q_f, \nabla\cdot \boldsymbol{v}_f)_{\Omega_f} \quad \text{for all } \boldsymbol{v}_f \in \boldsymbol{X}_f \text{ and } q_f \in M_f,$$

and observe that if $\boldsymbol{v}_f \in \boldsymbol{W}_{\boldsymbol{\tau}}^h$ then $\boldsymbol{v}_f \cdot \boldsymbol{\eta} = 0$ on $\partial\Omega_f$ and then $b_f(\boldsymbol{v}_f, q_f) = \int_{\Omega_f} \boldsymbol{v}_f \cdot \nabla q_f$ by the Green formula.

Then, according to the Brezzi's splitting theorem, see [Bra01, GR86], we can always obtain a stable solution $\boldsymbol{w} \in \boldsymbol{W}^h_{\boldsymbol{\mathcal{T}}}$ of:

$$\left\{ \begin{array}{lll} (\boldsymbol{w}, \boldsymbol{v}_f)_{\Omega_f} + b_f(\boldsymbol{w}, p_f) & = (\boldsymbol{z}, \boldsymbol{v}_f)_{\Omega_f} & \forall\, \boldsymbol{v}_f \in \boldsymbol{W}^h_{\boldsymbol{\mathcal{T}}} \\ b_f(\boldsymbol{w}, q_f) & = b_f(\boldsymbol{z}, q_f)_{\Omega_f} & \forall\, q_f \in M^h_f \end{array} \right. \tag{80}$$

where $\boldsymbol{z} \in L^2(\Omega_f)^2$.

Given $\boldsymbol{z}$, denote by $\boldsymbol{\Upsilon}_\tau \boldsymbol{z}$ the solution of (80), then

$$\|\boldsymbol{\Upsilon}_\tau \boldsymbol{z}\|_{L^2(\Omega_f)^2} \preceq \|\boldsymbol{z}\|_{L^2(\Omega_f)^2} \tag{81}$$

and $b_f(\boldsymbol{\Upsilon}_\tau \boldsymbol{z}, q^h) = b_f(\boldsymbol{z}, q^h)$ for $q^h_f \in M^h_f$.

In order to proof Lemma 10 define

$$\boldsymbol{I}^{TH}_{h_f} \boldsymbol{v}_f := \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f + \boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f).$$

Observe that:

$$\begin{array}{rcll} |\boldsymbol{I}^{TH}_{h_f} \boldsymbol{v}_f|_{H^1(\Omega_f)} & \leq & |\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f|_{H^1(\Omega_f)^2} + |\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)|_{H^1(\Omega_f)^2} & \\ & \preceq & |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + \dfrac{1}{h}\|\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} & \text{by (74) and inverse estimate.} \\ & \preceq & |\boldsymbol{v}_f|_{H^1(\Omega_f)^2} + \dfrac{1}{h}\|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \quad \text{by (81)} & \\ & \preceq & |\boldsymbol{v}_f|_{H^1(\Omega_f)} + |\boldsymbol{v}_f|_{H^1(\Omega_f)} \quad \text{by (75).} & \end{array}$$

Then the operator $\boldsymbol{I}^{TH}_{h_f}$ is bounded (with constant independent of h). In addition for $p^h_f \in M^h_f$ we get:

$$\begin{array}{rcl} b_f\big(\boldsymbol{I}^{TH}_{h_f} \boldsymbol{v}_f, p^h_f\big) & = & b_f\big(\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p^h_f\big) + b_f\big(\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f), p^h_f\big) \\ & = & b_f\big(\boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p^h_f\big) + b_f\big(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f, p^h_f\big) \quad \text{by definition of } \boldsymbol{\Upsilon}_\tau. \\ & = & b_f\big(\boldsymbol{v}_f, p^h_f\big). \end{array}$$

To obtain (33) observe that from definition of $\boldsymbol{I}^{TH}_{h_f}$ we have:

$$\begin{array}{rcl} \|\boldsymbol{v}_f - \boldsymbol{I}^{TH}_{h_f} \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} & \leq & \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{\Upsilon}_\tau(\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f)\|_{L^2(\Omega_f)^2} \\ & \preceq & \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} + \|\boldsymbol{v}_f - \boldsymbol{\Upsilon}_\eta \boldsymbol{v}_f\|_{L^2(\Omega_f)^2} \quad \text{by (81)} \\ & \preceq & h^s |\boldsymbol{v}_f|_{H^s(\Omega_f)^2} \quad s = 1, 2. \quad \text{by (75)} \end{array}$$

The proof of (34) is similar. Inequality (35) is obtained from (77).

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