CONVERGENCE ANALYSIS FOR THE NUMERICAL BOUNDARY CORRECTOR FOR ELLIPTIC EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS

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Abstract. We develop the convergence analysis for a numerical scheme proposed for approximating the solution of the elliptic problem

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i} a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega,$$

where the matrix $a(y) = (a_{ij}(y))$ is symmetric positive definite and periodic with period Y. The major goal is to develop a numerical scheme capturing the solution oscillations in the ϵ scale on a mesh size $h > \epsilon$ (or $h >> \epsilon$). The proposed method is based on asymptotic analysis and on numerical treatments for the boundary corrector terms, and the convergence analysis is based on asymptotic expansion estimates and finite elements analysis. We obtain discretization errors of $O(h^2 + \epsilon^{3/2} + \epsilon h)$ and $O(h + \epsilon)$ in the L^2 norm and the broken H^1 semi-norm, respectively.

Key words. Finite elements, homogenization, elliptic equations, multiscaling, boundary layer, mixed finite elements.

AMS subject classifications. 65N30, 35B27,

1. Introduction.

This paper develops the convergence analysis of the numerical scheme proposed in [43] to approximate u_ϵ, the solution of the problem:

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i}(a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon) = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $a(y) = (a_{ij}(y))$ is a positive symmetric definite matrix and $\epsilon \in (0, 1)$ is the periodicity parameter. We assume the $a_{ij} \in L^\infty_{\text{per}}(Y)$, i.e. $a_{ij} \in L^\infty(\mathbb{R}^2)$ and Y-periodic, $Y = (0, 1)^2$, and there exists a positive constant γ_a such that $a_{ij}(y)\xi_i\xi_j \geq \gamma_a\|\xi\|^2$ for all $\xi \in \mathbb{R}^2$ and $y \in Y$. We always use the Einstein summation convention, i.e. repeated indices indicate summation, except for the index k, which refers to variables or functions associated to edges of the polygonal domain Ω.

We note that when the mesh size $h > \epsilon$, standard finite element methods do not yield good numerical approximations; see [27]. Recently, new numerical methods have been proposed for solving the Problem (1.1) such as the multi-scale finite element methods [23, 26, 4, 13, 21], the residual-free bubble function methods [11, 5, 6, 38, 12], and the generalized FEM for homogenization problems [39]. There are also related methods for the case the homogenized equation is not known; see the heterogeneous multiscale method [18, 19, 2] and [22, 20] . The numerical method considered here, opposed to the methods in [5, 26, 38, 4, 11] is based strongly on the asymptotic expansion of u_ϵ. We also explore the periodicity of the matrix a to obtain a very efficient method for approximating u_ϵ.

One of the first mathematical tools used to handle this problem was homogenization theory [8, 9]. Based on this theory a first order expansion of u_ϵ plus a boundary corrector term is considered and then each term is numerically approximated in

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[42, 43]. These methods were designed to work with a mesh size $h > \epsilon$ (or $h >> \epsilon$), however they also work in the case $h < \epsilon$. The article [42] presents the numerical algorithm when the domain Ω is a rectangular region, while [43] generalizes the method to the case where the domain Ω is a convex polygon with rational boundary normals. This generalization is possible due to the Lagrange multiplier space introduced to approximate $\partial_\eta u_0$ on $\partial\Omega$.

The convergence analysis for the numerical method is performed in two parts. First we estimate the error between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$ in L^2 and H^1 norms, where ϕ_ϵ denotes the theoretical approximation for the boundary corrector term θ_ϵ. The theory developed for approximating θ_ϵ is similar to the one proposed in [3, 34]. We note that Propositions 6.1 and 6.4, which estimates the error between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon$ on the H^1 and L^2 norms, respectively, extend the results in [3, 34]. More specifically, Proposition 6.1 gives the same error estimate of Theorem 2.2 in [3], however here we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \le 1/2$ while in Theorem 2.2 in [3] it is assumed $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in H^1_{per}(\Omega)$. We also note that Propositions 6.1 and 6.4 generalize respectively, Propositions 2.1 and 2.3 from [34]. In Proposition 6.1 we assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \le 1/2$, and $\Omega \subset \mathbb{R}^{2,3}$, while in Proposition 2.1 from [34] it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$. In Proposition 6.4 we assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, χ^j and $\chi^{ij} \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \le 1/2$, and $\Omega \subset \mathbb{R}^{2,3}$, while in Proposition 2.3 from [34] it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^3(\Omega)$ and $\Omega \subset \mathbb{R}^2$. The importance of considering a theory that handles the case $a_{ij} \in L^\infty_{per}(Y)$ comes from applications to composite materials where the coefficients a_{ij} are often piecewise constant; see also Theorem 1.1 from [32] which gives conditions on the discontinuities of the functions a_{ij} so that χ^j and $\chi^{ij} \in W^{1,\infty}_{per}(Y)$. We also observe that Proposition 2.1 from [34] is used in the convergence analysis of the numerical methods presented in [23, 27, 38], and therefore the analysis presented here can be helpful for extending the convergence proofs of these numerical methods assuming less regularity on a or u_0. In the second part of the convergence analysis we use finite elements theory to estimate the error due to the discrete approximation. The main difficulty here lies in the fact that we use a discrete approximation of $\partial_\eta u_0$ as Dirichlet boundary condition for the boundary corrector problem. We observe that if u_0^h is a finite element approximation for u_0, then $\partial_\eta u_0^h$ does not necessarily belong to the trace of the finite element space used to obtain u_0^h, hence we introduce the Lagrange multiplier space to approximate $\partial_\eta u_0$ and we develop error estimates between $\partial_\eta u_0$ and its discrete approximation in $W^{1,1-1/p}$ spaces; see Lemma 4.3.

To simplify the exposition we perform the analysis in the case $\Omega = (0,1)^2$, although the same theory holds in the case $\Omega = \prod_{i=1}^2 (a_i, b_i)$, $a_i < b_i \in \mathbb{R}$. We note that Propositions 6.1 and 6.4 are proved in the case $\Omega \subset \mathbb{R}^d$ $d = 2, 3$, is a convex domain and $Y = (0,1)^d$. The analysis presented here can also be extended to the case where the domain Ω is a convex polygon with rational boundary normals; see [41].

We now introduce some norms and semi-norms. Let $B \subset \mathbb{R}^2$ be an open set and define

$$\|v\|_{m,\infty,B} = \max_{|\alpha| \le m} \{\operatorname{ess.}\sup_{x \in B} |\partial^\alpha v(x)|\},$$
$$|v|_{m,\infty,B} = \max_{|\alpha| = m} \{\operatorname{ess.}\sup_{x \in B} |\partial^\alpha v(x)|\},$$

and for $1 \le q < \infty$

$$\|v\|_{m,q,B} = \left(\int_B \sum_{|\alpha|\le m} |D^\alpha v|^q dx \right)^{1/q},$$

$$|v|_{m,q,B} = \left(\int_B \sum_{|\alpha|= m} |D^\alpha v|^q dx \right)^{1/q}.$$

We also define the non-conforming norms related to a partition $\mathcal{T}_h = K_1, K_2, ..., K_N$ of B by

$$\|v\|_{m,h} = \sqrt{\sum_{K_j \in \mathcal{T}_h} \|v\|^2{}_{H^m(K_j)}}.$$

Throughout this paper we do not make reference to the domain B, or to the coefficient q when $B = \Omega$, or $q = 2$, respectively. In what follows c denotes a generic constant independent of ϵ and mesh parameters.

This paper is organized as follows. Section 2 introduces the asymptotic expansion of u_ϵ considered in [42, 43], describes a theoretical approximation for the boundary corrector term, and presents the main theorems for estimating the errors due to the asymptotic expansion approximation. Section 3 describes the numerical algorithm, Section 4 treats the discretization errors due to the finite element approximation, and Section 5 presents the numerical experiments. The Appendix contains the proofs of the main results from Section 2.

2. Theoretical Approximation.

2.1. The Asymptotic Expansion.

Consider the following anzats

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots, \tag{2.1}$$

where the functions $u_j(x, y)$ are Y periodic in y. Using (2.1) in Equation (1.1) and matching the terms with the same order in ϵ, one may define functions u_j such that $u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon)$ approximates u_ϵ, for instance if $u_0 \in C^2(\Omega)$ and $\chi^j \in W^{1,\infty}(Y)$ we have

$$\|u_\epsilon(x) - u_0(x, x/\epsilon) - \epsilon u_1(x, x/\epsilon)\|_1 \le c\epsilon^{1/2}\|u_0\|_{2,\infty}$$

where the constant c depends on a, χ^j and Ω. These terms are defined below; for more details, including the proof of the above inequality see [9, 29].

Let $\chi^j \in H^1_{\text{per}}(Y)$, i.e. $\chi^j \in H^1_{loc}(\mathbb{R}^2)$ and Y-periodic, be the weak solution with zero average over Y of

$$\nabla_y \cdot a(y) \nabla_y \chi^j = \nabla_y \cdot a(y) \nabla_y y_j = \frac{\partial}{\partial y_i} a_{ij}(y), \tag{2.2}$$

and define the matrix

$$A_{ij} = \frac{1}{|Y|} \int_Y a_{lm}(y) \frac{\partial}{\partial y_l}(y_i - \chi^i) \frac{\partial}{\partial y_m}(y_j - \chi^j) dy. \tag{2.3}$$

It is easy to check that the matrix A is symmetric positive definite. Define $u_0 \in H_0^1(\Omega)$ as the weak solution of

$$-\nabla . A \nabla u_0 = f \quad \text{in} \quad \Omega, \qquad u_0 = 0 \quad \text{on} \quad \partial\Omega, \tag{2.4}$$

and let

$$u_1(x, \frac{x}{\epsilon}) = -\chi^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}(x). \tag{2.5}$$

Note that $u_0 + \epsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$ imposed for u_ϵ. In order to overcome this, the boundary corrector term $\theta_\epsilon \in H^1(\Omega)$ is introduced as the solution of

$$-\nabla \cdot a(x/\epsilon) \nabla \theta_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \theta_\epsilon = -u_1(x, \frac{x}{\epsilon}) \quad \text{on} \quad \partial\Omega, \tag{2.6}$$

hence $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon \in H_0^1(\Omega)$. Propositions 6.1 and 6.6 provide error estimates between u_ϵ and $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon$ in the norms $\| \cdot \|_1$ and $\| \cdot \|_0$, respectively.

We also define the term u_2, which is used in the proof of Proposition 6.4. Set

$$b_{ij} = -a_{ij} + a_{im} \frac{\partial \chi^j}{\partial y_m} + \frac{\partial}{\partial y_m}(a_{mi}\chi^j)$$

and observe that $\overline{b}_{ij} = A_{ij}$, where $\overline{b}_{ij} = \int_Y b_{ij} dy$. Define $\chi^{ij} \in H^1_{per}(Y)$ as the weak solution with zero average over Y of

$$\nabla_y \cdot a \nabla_y \chi^{ij} = b_{ij} - \overline{b}_{ij} \tag{2.7}$$

and let

$$u_2(x, \frac{x}{\epsilon}) = -\chi^{ij} \left(\frac{x}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_i \partial x_j}(x). \tag{2.8}$$

2.2. Boundary Corrector Approximation. The coefficients $a_{ij}(x/\epsilon)$ and the boundary values $-u_1(x, \frac{x}{\epsilon})$ in the Equation (2.6) are highly oscillatory, hence it is not a trivial problem to obtain a good discrete approximation for θ_ϵ . We propose an analytical approximation for θ_ϵ, denoted by ϕ_ϵ, which satisfies the oscillating boundary condition and is suitable for numerical approximation. The approximation for θ_ϵ proposed here is similar to the one used in [3, 34].

Note that u_0 vanishes on $\partial\Omega$, therefore $\nabla u_0|_{\partial\Omega} = \eta \partial_\eta u_0$, where η denotes the unity outward normal vector to $\partial\Omega$ and $\partial_\eta u_0$ denotes the unity outward derivative of u_0 on $\partial\Omega$. Hence in order to obtain the approximation ϕ_ϵ for θ_ϵ, we introduce the following decomposition $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$, where

$$-\nabla \cdot a(x/\epsilon) \nabla \tilde{\theta}_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \tilde{\theta}_\epsilon = (\chi^j(\frac{x}{\epsilon})\eta_j - \chi^*) \partial_\eta u_0 \quad \text{on} \quad \partial\Omega \tag{2.9}$$

and

$$-\nabla \cdot a(x/\epsilon) \nabla \bar{\theta}_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \bar{\theta}_\epsilon = \chi^* \partial_\eta u_0 \quad \text{on} \quad \partial\Omega, \tag{2.10}$$

where $\chi^*|_{\Gamma_k} = \chi^*_k$, $k \in \{e, w, n, s\}$ are properly chosen constants defined in Subsection 2.2.1, and $\Gamma_e = \{1\} \times [0,1]$, $\Gamma_w = \{0\} \times [0,1]$, $\Gamma_n = [0,1] \times \{1\}$, and $\Gamma_s = [0,1] \times \{0\}$. In Remark 2.1 we show that $\chi^* \partial_\eta u_0$ and $\chi^j(\frac{x}{\epsilon})\eta_j \partial_\eta u_0 \in H^{1/2}(\partial\Omega)$, therefore the Problems (2.9) and (2.10) are well posed. Later in this section we define the functions $\tilde{\phi}_\epsilon$ and $\bar{\phi}_\epsilon$, which are the approximations for $\tilde{\theta}_\epsilon$ and $\bar{\theta}_\epsilon$ respectively, and define $\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$.

Remark 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and assume $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. We have by Theorem A.2 [37] that $\partial_\eta u_0|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$ and $\|\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \le c\|u_0\|_2$, therefore*

$$\|\chi^* \partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \le c(\chi^*)\|u_0\|_2.$$

Note also that $u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$ and $\frac{\partial u_1}{\partial x_l} = -\left(\frac{\partial \chi^j}{\partial x_l}\right)\frac{\partial u_0}{\partial x_j} - \chi^j\left(\frac{\partial^2 u_0}{\partial x_l \partial x_j}\right)$. If we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$, for $p \ge 2$ and $q > 2$ or $p > 2$ and $q \ge 2$, by a direct application of Sobolev embedding Theorem (5.4 [1]) we obtain $u_1 \in H^1(\Omega)$. In addition, from regularity theory of elliptic equations we obtain $\chi^j \in L^\infty(Y) \cap H^1(Y)$ (see Theorem 13.1 [30] and 4.28 [15]), hence we also have $u_1|_{\Gamma_k} \in H^{1/2}_{00}(\Gamma_k)$.

2.2.1. Calculating the Constants χ^*_k. We define the constants χ^*_k such that the function $\tilde{\phi}_\epsilon$ decays exponentially to zero away from the boundary and satisfies the Dirichlet boundary condition $\tilde{\phi}_\epsilon(x) = -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0(x)$ for $x \in \partial\Omega$.

Associated to each side of Ω define the functions v_k, $k \in \{e, w, n, s\}$ as

1. Let $G_e = \{(-\infty, 0] \times [0,1]\}$ and v_e the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_e = 0 \text{ in } G_e,\\
&v_e(0, y_2) = \chi^1(1/\epsilon, y_2) \quad \text{for} \quad 0 < y_2 < 1,\\
&v_e(y_1, \cdot) \quad [0,1]\text{-periodic for} \quad -\infty < y_1 < 0,\\
&\text{and} \quad \partial_{y_i} v_e \exp(-\gamma y_1) \in L^2(G_e) \quad i = 1, 2.
\end{aligned}$$

2. Let $G_w = \{[0, \infty) \times [0,1]\}$ and v_w the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_w = 0 \quad \text{in} \quad G_w,\\
&v_w(0, y_2) = -\chi^1(0, y_2) \quad \text{for} \quad 0 < y_2 < 1,\\
&v_w(y_1, \cdot) \quad [0,1]\text{-periodic for} \quad 0 < y_1 < \infty,\\
&\text{and} \quad \partial_{y_i} v_w \exp(\gamma y_1) \in L^2(G_w) \quad i = 1, 2.
\end{aligned}$$

3. Let $G_n = \{[0,1] \times (-\infty, 0]\}$ and v_n the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_n = 0 \quad \text{in} \quad G_n,\\
&v_n(y_1, 0) = \chi^2(y_1, 1/\epsilon) \quad \text{for} \quad 0 < y_1 < 1,\\
&v_n(\cdot, y_2) \quad [0,1]\text{-periodic for} \quad -\infty < y_2 < 0,\\
&\text{and} \quad \partial_{y_i} v_n \exp(-\gamma y_2) \in L^2(G_n) \quad i = 1, 2.
\end{aligned}$$

4. Let $G_s = \{[0,1] \times [0, \infty)\}$ and v_s the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_s = 0 \quad \text{in} \quad G_s,\\
&v_s(y_1, 0) = -\chi^2(y_1, 0) \quad \text{for} \quad 0 < y_1 < 1,\\
&v_s(\cdot, y_2) \quad [0,1]\text{-periodic for} \quad 0 < y_2 < \infty,\\
&\text{and} \quad \partial_{y_i} v_n \exp(\gamma y_2) \in L^2(G_s) \quad i = 1, 2.
\end{aligned}$$

The above problems have been studied by several authors, see [36, 33, 29, 34]. Theorem 10.1 in Section 10.4 from [33] guarantees the existence of a unique solution for each of the above equations. In addition, by Theorem 3 [36] there exists constants χ_k^*, such that

$$|v_k(y) - \chi_k^*| \leq c\exp(\gamma y \cdot \eta_k) \text{ as } \; y \cdot \eta_k \to -\infty,$$

where η_k denotes the unity outward normal on Γ_k.

2.2.2. Approximating $\tilde{\theta}_\epsilon$. We note by Remark 2.1 that $(u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$. Thus, we can split $\tilde{\theta}_\epsilon = \sum_{k\in\{e,w,n,s\}} \tilde{\theta}_\epsilon^k$ where

$$L_\epsilon \tilde{\theta}_\epsilon^k = 0 \text{ in } \Omega, \quad \text{and} \quad \tilde{\theta}_\epsilon^k = \begin{cases} -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0 & \text{on } \Gamma_k \\ 0 & \text{on } \partial\Omega \setminus \Gamma_k. \end{cases} \tag{2.11}$$

We approximate $\tilde{\theta}_\epsilon^k$ by $\tilde{\phi}_\epsilon^k$ given as

$$\begin{aligned} \tilde{\phi}_\epsilon^e(x_1, x_2) &= \varphi_e(x_1) \left(v_e(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_e^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2), \\ \tilde{\phi}_\epsilon^w(x_1, x_2) &= -\varphi_w(x_1) \left(v_w(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_w^* \right) \frac{\partial u_0}{\partial x_1}(x_1, x_2), \\ \tilde{\phi}_\epsilon^n(x_1, x_2) &= \varphi_n(x_2) \left(v_n(\frac{x_1}{\epsilon}, \frac{x_2 - 1}{\epsilon}) - \chi_n^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2), \\ \tilde{\phi}_\epsilon^s(x_1, x_2) &= -\varphi_s(x_2) \left(v_s(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_s^* \right) \frac{\partial u_0}{\partial x_2}(x_1, x_2), \end{aligned} \tag{2.12}$$

where φ_k are nonnegative smooth functions satisfying

$$\varphi_e(s) = \varphi_n(s) = \begin{cases} 1 & \text{if } \; s \in [2/3, 1] \\ 0 & \text{if } \; s \in [0, 1/3], \end{cases} \qquad \varphi_w(s) = \varphi_s(s) = \begin{cases} 0 & \text{if } \; s \in [2/3, 1] \\ 1 & \text{if } \; s \in [0, 1/3]. \end{cases}$$

Hence

$$\tilde{\phi}_\epsilon = \sum_{k\in\{e,w,n,s\}} \tilde{\phi}_\epsilon^k \tag{2.13}$$

approximates $\tilde{\theta}_\epsilon$, and $\tilde{\phi}_\epsilon = \tilde{\theta}_\epsilon$ on the boundary of Ω.

2.2.3. Approximating $\bar{\theta}_\epsilon$. The boundary condition imposed on Equation (2.10) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi} \in H^1(\Omega)$ the weak solution of

$$-\nabla \cdot A \nabla \bar{\phi} = 0 \text{ in } \Omega, \qquad \bar{\phi} = \chi^* \partial_\eta u_0 \text{ on } \partial\Omega. \tag{2.14}$$

By Propositions 6.3 and 6.5, we have that $\bar{\phi}$ is a good approximation for $\bar{\theta}_\epsilon$ only on the L^2 norm, since $\|\bar{\phi} - \bar{\theta}_\epsilon\|_0$ is $O(\epsilon)$ and $\|\bar{\phi} - \bar{\theta}_\epsilon\|_1$ is $O(1)$. We note, however, that the asymptotic expansion considered here to approximate u_ϵ is given by $u_0 + \epsilon u_1 + \epsilon\bar{\theta}_\epsilon + \epsilon\tilde{\theta}_\epsilon$, and by a triangular inequality we obtain $\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\bar{\phi} - \epsilon\tilde{\theta}_\epsilon\|_1 \leq c\epsilon + \|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\theta_\epsilon\|_1$. Hence, when estimating the error on the H^1 norm between u_ϵ and its theoretical approximation, the contribution due to the approximation of $\bar{\theta}_\epsilon$ by $\bar{\phi}$ is $O(\epsilon)$.

2.2.4. Approximating u_ϵ. We finally define the theoretical approximation for u_ϵ as $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon$, where

$$\phi_\epsilon = \tilde{\phi}_\epsilon + \bar{\phi}. \tag{2.15}$$

Note that $\phi_\epsilon|_{\partial\Omega} = \theta_\epsilon|_{\partial\Omega}$, therefore $u_0 + \epsilon u_1 + \epsilon\phi_\epsilon = 0$ on $\partial\Omega$.

2.2.5. Error estimates. The following theorems provide error estimates between u_ϵ and $u_0 - \epsilon u_1 - \epsilon\phi_\epsilon$ on the H^1 and L^2 norms. Theorem 2.1 estimates the error on the H^1 norm, while Theorems 2.2 and 2.3 estimate the error on the L^2 norm. Theorem 2.2 assumes more regularity on u_0 and less regularity on a that is assumed in Theorem 2.3.

Theorem 2.1. *Let u_ϵ be the solution of the Problem (1.1), u_0, u_1 and ϕ_ϵ defined by Equations (2.4), (2.5) and (2.15), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, $\chi^j \in W^{1,q}_{per}(Y)$, v_e and $\nabla(v_e - \chi^*_e)exp(-\gamma y_1) \in L^s(G_e)$, for $1/s + 3/p \leq 1$, $s \geq 2$ and $1/p+1/q \leq 1/2$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_1 \leq c\epsilon\|u_0\|_{2,p}.$$

Proof. See Subsection 6.1 □

Theorem 2.2. *Let u_ϵ be the solution of Problem (1.1), u_0, u_1, ϕ_ϵ, $\bar{\phi}$ and χ^{ij} defined by Equations (2.4), (2.5), (2.15), (2.14) and (2.7), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, and $\bar{\phi} \in W^{2,p}(\Omega)$ and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p > 2$ and $1/p+1/q \leq 1/2$. Assume also $\chi^j \in W^{1,\infty}(Y)$, v_e and $\nabla(v_e - \chi^*_e)exp(-\gamma y_1) \in L^\infty(G_e)$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_0 \leq c\epsilon^{3/2}\|u_0\|_{3,p}.$$

Proof. See Subsection 6.2 □

Theorem 2.3. *Let u_ϵ be the solution of Problem (1.1), u_0, u_1 and ϕ_ϵ be defined by Equations (2.4), (2.5) and (2.15), respectively. Assume $a_{ij} \in C^{1,\beta}_{per}(Y)$, $\beta > 0$, $u_0 \in H^3(\Omega)$. Then there exists a constant c independent of ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\phi_\epsilon(\cdot)\|_0 \leq c\epsilon^{3/2}\|u_0\|_3.$$

Proof. See Subsection 6.3 □

REMARK 2.2. *Due to the Proposition 6.2, which under the hypothesis of Theorems 2.2 and 2.3 gives that $\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_0$ is $O(\epsilon^{1/2})$, we obtain a factor $\epsilon^{3/2}$ in these theorems, rather than ϵ^2 as in Propositions 6.4 and 6.6.*

3. Finite Element Approximation. We now describe how to approximate the terms u_0, u_1, $\tilde{\phi}_\epsilon$ and $\bar{\phi}$ numerically.

- Approximate the solution of Problem (2.2) with a second order accurate conforming finite element on a partition $\mathcal{T}_{\hat{h}}(Y)$. Denote these solutions by $\chi^j_{\hat{h}}$.
- Define $A^{\hat{h}}_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i_{\hat{h}})\frac{\partial}{\partial y_m}(y_j - \chi^j_{\hat{h}})dy.$

- Let $V^h(\Omega)$ be a conforming second order accurate finite element space on a mesh $\mathcal{T}_h(\Omega)$ and let $V_0^h(\Omega) = V^h(\Omega) \cap H_0^1(\Omega)$. Define $u_0^{h,\hat{h}} \in V_0^h(\Omega)$ as the solution of

$$\int_\Omega A^{\hat{h}} \nabla u_0^{h,\hat{h}} \cdot \nabla v^h dx = \int_\Omega f v^h dx, \qquad \forall v^h \in V_0^h(\Omega).$$

- Since $\partial_\eta u_0$ appears as boundary condition imposed in Equation (2.14), it is important to obtain a good discrete approximation for it. In oder to approximate $\partial_\eta u_0$ we define $Y^h = V^h(\Omega)|_{\partial\Omega}$, $Y_k^h = Y^h|_{\Gamma_k}$ and $Y_{0,k}^h = \{\lambda^h \in Y_k^h; \lambda^h = 0 \text{ at } \partial\Gamma_k\}$. Let $\lambda_k^{h,\hat{h}} \in Y_{0,k}^h$ be the solution of

$$\int_{\Gamma_k} \lambda_k^{h,\hat{h}} \phi^h d\sigma = \int_\Omega A_{ij}^{\hat{h}} \partial_i u_0^{h,\hat{h}} \partial_j \phi^h dx - \int_\Omega f \phi^h dx, \tag{3.1}$$

$\forall\ \phi^h \in V^h(\Omega)$, such that $\phi^h|_{\partial\Omega\setminus\Gamma_k} = 0$. Later in Proposition 4.3 we show that $\lambda_k^{h,\hat{h}}$ is a good approximation for $A\nabla u_0 \cdot \eta_k$ on Γ_k, hence we approximate $\partial_\eta u_0$ by $\mu^{h,\hat{h}}$ where

$$\mu^{h,\hat{h}}|_{\Gamma_k} = \lambda_k^{h,\hat{h}} / A_{l_k l_k}^{\hat{h}},\ l_k = \begin{cases} 1 & \text{if} \quad k = e, w \\ 2 & \text{if} \quad k = n, s. \end{cases}$$

- We observe that we use $\mu^{h,\hat{h}}$ as the approximation for $\partial_\eta u_0$ in Equation (3.5), hence in order to guarantees that the final numerical approximation for u_ϵ satisfies the zero Dirichlet boundary condition we define the approximation for ∇u_0 as

$$\Psi^{h,\hat{h}} = \nabla u_0^{h,\hat{h}} + \sum_{k\in\{e,w,n,s\}} E_k^h(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)\eta^k. \tag{3.2}$$

Here $E_k^h(\cdot)$ denotes a non-conforming discrete extension of $\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k$ by zero on Ω. More specifically, $E_k^h(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)(z) = 0$, if z is a vertex of $\mathcal{T}_h(\overline{\Omega}) \setminus \Gamma_k$, $E_k^h(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)(z) = \mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k(z)$, if z is a vertex of Γ_k, and $E_k^h(\mu^{h,\hat{h}} - \nabla u_0^{h,\hat{h}} \cdot \eta^k)|_{K_i} \in V^h(\Omega)|_{K_i}$, $\forall\ K_i \in \mathcal{T}_h(\Omega)$.
- Define

$$u_1^{h,\hat{h}}(x, x/\epsilon) = -\Psi_j^{h,\hat{h}}(x) \chi_{\hat{h}}^j(x/\epsilon). \tag{3.3}$$

Note that this leads to a nonconforming approximation for u_1 in the partition $\mathcal{T}_h(\Omega)$.
- Let τ be a positive integer and $G_e^\tau = \{y \in \mathbb{R}^2;\ -\tau \leq y_1 \leq 0 \text{ and } 0 \leq y_2 \leq 1\}$. Define $\tilde{v}_e \in H^1(G_e^\tau)$ as the weak solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y) \nabla_y \tilde{v}_e = 0 \ \text{ in } \ G_e^\tau, \\
&\tilde{v}_e(y) = \chi_{\hat{h}}^1(1/\epsilon, y_2) \ \text{ on } \ \{y \in G_e^\tau, y_1 = 0\}, \\
&\partial_\eta \tilde{v}_e = 0 \ \text{ on } \{y \in G_e^\tau;\ y_1 = -\tau\}, \\
&\text{and } v_e(y_1, 0) = v_k(y_1, 1) \ \text{ for } \ -\tau \leq y_1 \leq 0.
\end{aligned}$$

Let $v_e^{\hat{h},\tau}$ be a numerical approximation of $\tilde{v}_e$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G_e^\tau)$, and define

$$\chi_e^{*,\hat{h},\tau} = \int_0^1 v_e^{\hat{h},\tau}(-\tau, y_2)dy_2.$$

The other cases $k \in \{w, n, s\}$ are treated similarly.

- Observe that the term $v_e(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ appears in Equation (2.12). The approximation $v_e^{\hat{h},\tau}$ is defined in G_e^τ, hence we have defined $v_e^{\hat{h},\tau}(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ only when $x_1 \geq 1 - \epsilon\tau$. Since the functions $v_e - \chi_e^*$ decays exponentially to zero in the $-\eta_e$ direction, its is natural to define the following approximation

$$\tilde{\phi}_\epsilon^{e,h,\hat{h},\tau}(x_1, x_2) = \begin{cases} \left(v_e^{\hat{h},\tau}\left(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon}\right) - \chi_e^{*,\hat{h},\tau}\right)\Psi^{h,\hat{h}} & \text{if } x_1 > 1 - \epsilon\tau \\ 0 & \text{otherwise.} \end{cases}$$

- Let

$$\tilde{\phi}_\epsilon^{h,\hat{h},\tau} = \sum_{k\in\{e,w,n,s\}} \tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}, \tag{3.4}$$

where the others terms $\tilde{\phi}_\epsilon^{k,h,\hat{h},\tau}$ are defined in a similar way.

- Let $\bar{\phi}^{h,\hat{h},\tau}$ be a second order accurate finite element approximation on a mesh of size h for the following equation (for the well posedness of the equation bellow see Remark 3.1)

$$-\nabla \cdot A^{\hat{h}}\nabla\psi = 0 \text{ in } \Omega, \quad \text{and} \quad \psi = \chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \text{ on } \partial\Omega. \tag{3.5}$$

- Approximate θ_ϵ by $\phi_\epsilon^{h,\hat{h},\tau} = \tilde{\phi}_\epsilon^{h,\hat{h},\tau} + \bar{\phi}^{h,\hat{h},\tau}$ and finally define the numerical solution for Equation (1.1) as

$$u_\epsilon^{h,\hat{h},\tau} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon\phi_\epsilon^{h,\hat{h},\tau}. \tag{3.6}$$

REMARK 3.1. *By construction $\mu^{h,\hat{h}}$ vanishes at the corners of Ω, therefore $\chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \in H^{1/2}(\partial\Omega)$. This implies that Equation (3.5) is well posed. In addition $\chi^{*,\hat{h},\tau}\mu^{h,\hat{h}} \in V^h|_{\partial\Omega}$, hence we can look for a numerical solution of Equation (3.5) in $V^h(\Omega)$.*

4. Finite Element Approximation Error Analysis. For the discrete error analysis we assume $\hat{h} = 0$ and $\tau = \infty$, i.e. $v_k^{\hat{h},\tau} = v_k$, $\chi_{\hat{h}}^j = \chi^j$ and $A^{\hat{h}} = A$, and for this reason we will note make reference to the indices τ and $\hat{h}$ when we make reference to the the numerical approximation for u_0, ∇u_0, $\bar{\phi}$, $\tilde{\phi}_\epsilon$ and u_ϵ, i.e. $u_\epsilon^h = u_\epsilon^{h,\hat{h},\tau}$ and similar for the other terms; an error analysis including the error due to the numerical approximation of the functions v_k and χ^j, and the matrix A is currently work under progress. We also assume that linear or bilinear finite elements are used to approximate u_0. Theorems 4.1 and 4.2 give the main results of this section. Theorem 4.1 provides error estimates for the broken H^1 semi-norm and the L^2 norm between the exact solution u_ϵ and its numerical approximation u_ϵ^h. Theorem 4.2 assumes more regularity from u_0 resulting in a better error estimate on the L^2 norm.

Theorem 4.1. *Let u_ϵ be the solution of the Problem (1.1), u_0, χ^j and u_ϵ^h be defined by Equations (2.2), (2.4) and (3.6), respectively, and the functions v_k and the* *constants χ_k^* be defined as in Subsection 2.2.1. Assume $a_{ij} \in L^{\infty}_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, $\chi^j \in W^{1,q}_{per}(Y)$, v_e and $\nabla(v_e - \chi_e^*)exp(-\gamma y_1) \in L^s(G_e)$, for $1/p + 1/q \le 1/2$ and $1/s + 3/p \le 1$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ and h such that*

$$|u_\epsilon - u_\epsilon^h|_{1,h} \le c(h+\epsilon)\|u_0\|_{2,p}$$

and

$$\|u_\epsilon - u_\epsilon^h\|_0 \le c(h^2 + \epsilon + \epsilon h)\|u_0\|_{2,p}.$$

Proof. By the triangular inequality we have

$$|u_\epsilon - u_\epsilon^h|_{1,h} \le |u_\epsilon - u_0 - u_1 - \phi_\epsilon|_1 + |u_0 - u_0^h|_{1,h} + \epsilon|u_1 - u_1^h|_{1,h}$$
$$+\epsilon|\bar{\phi} - \bar{\phi}^h|_{1,h} + \epsilon|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h},$$

and the theorem follows from Theorem 2.1, the approximation error (4.1), and Propositions 4.2, 4.3 and 4.4. ☐

Theorem 4.2. *Let u_ϵ be the solution of the Problem (1.1), χ^j, u_0, χ^{ij}, $\bar{\phi}$ and u_ϵ^h be defined by Equations (2.2), (2.4), (2.7), (2.14) and (3.6), respectively, and the functions v_k and the constants χ_k^* be defined as in Subsection 2.2.1. Assume $a_{ij} \in L^{\infty}_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, $\bar{\phi} \in W^{2,p}(\Omega)$ and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p > 2$ and $1/p + 1/q \le 1/2$. Also assume $\chi^j \in W^{1,\infty}(Y)$, and v_e and $\nabla(v_e - \chi_e^*)exp(-\gamma y_1) \in L^{\infty}(G_e)$. We also assume similar hypothesis for the other functions v_k. Then there exists a constant c independent of ϵ and h such that*

$$\|u_\epsilon - u_\epsilon^h\|_0 \le c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)\|u_0\|_{3,p}.$$

Furthermore, if $a_{ij} \in C^{1,\beta}_{per}(Y)$ and $u_0 \in H^3(\Omega)$, then

$$\|u_\epsilon - u_\epsilon^h\|_0 \le c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h)\|u_0\|_3.$$

Proof. The same proof of Theorem 4.1 holds here, except that (4.1) is replaced by (4.2) and Theorem 2.1 is replaced by Theorems 2.3 and 2.2. ☐

We now prove the propositions used in the proofs of Theorems 4.1 and 4.2.

For the approximation error of the term u_0 we use standard finite element analysis to obtain

$$\|u_0 - u_0^h\|_{1,p} \le ch\|u_0\|_{2,p}, \text{ for } 2 \le p \le \infty, \tag{4.1}$$

$$\|u_0 - u_0^h\|_{0,p} \le ch^2\|u_0\|_{2,p}, \text{ for } 2 \le p < \infty \tag{4.2}$$

and

$$\|u_0 - u_0^h\|_{0,\infty} \le ch^2 \ln(h)\|u_0\|_{2,\infty}; \tag{4.3}$$

see Corollary 7.1.2, Theorem 4.4.20 and inequality (7.5.4) from [10]. Let $\mathcal{I}^h$ be the usual local point-wise $\mathcal{P}_1$ or $\mathcal{Q}_1$ interpolate and $K \in \mathcal{T}_h(\Omega)$, then

$$|u_0 - u_0^h|_{2,p,K} \le |u_0 - \mathcal{I}^h u_0|_{2,p,K} + |\mathcal{I}^h u_0 - u_0^h|_{2,p,K}.$$

Using an interpolation error estimate, see Theorem 4.4.20 [10], we obtain

$$|u_0 - \mathcal{I}^h u_0|_{s,p,h} \le ch^{m-s}|u_0|_{m,p,h}, \text{ for } 0 \le s \le m, \tag{4.4}$$

and from an inverse inequality, see Lemma 4.5.3 [10], we have

$$|\mathcal{I}^h u_0 - u_0^h|_{2,p,K} \le ch^{-1}\|\mathcal{I}^h u_0 - u_0^h\|_{1,p,K}. \tag{4.5}$$

Finally from (4.4), (4.5) and (4.1) we obtain

$$\|u_0 - u_0^h\|_{2,p,h} \le c\|u_0\|_{2,p}. \tag{4.6}$$

In order to estimate the L^2 and the broken H^1 semi-norm of $u_1 - u_1^h$, (see Proposition 4.2) we note that $u_1 - u_1^h = (\partial_{x_j} u_0 - \Psi_j^h)\chi^j$ hence by a Cauchy inequality and the Sobolev embedding Theorem we obtain $\|u_1 - u_1^h\|_0 \le c\|\partial_{x_j} u_0 - \Psi_j^h\|_{0,p}\|\chi^j\|_{0,q}$ for $1/p + 1/q \le 1/2$. Therefore we have to estimate the error between Ψ^h and ∇u_0 on the L^p and on the broken $W^{1,p}$ semi-norm, (see Proposition 4.1) this is done by first estimating the error between $A\nabla u_0 \cdot \eta$ and λ^h in the trace space of $W^{1,p}(\Omega)$ over Γ_k in different norms; see Lemma 4.3. Lemmas 4.1 and 4.2 are auxiliary results used for obtaining Lemma 4.3.

Consider the following spaces:

Case $2 < p < \infty$**:** Since $W^{1-1/p,p}(\Gamma_k) \hookrightarrow C^0(\Gamma_k)$, we define the spaces $W_{00}^{1-1/p,p}(\Gamma_k) = \{\varphi \in W^{1-1/p,p}(\Gamma_k);\ \varphi = 0 \text{ on } \partial\Gamma_k\}$ equipped with the norm $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{W^{1-1/p,p}(\Gamma_k)}$.

Case $p = 2$**:** We set $W_{00}^{1-1/p,p}(\Gamma_k) = H_{00}^{1/2}(\Gamma_k)$ and $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{H_{00}^{1/2}(\Gamma_k)}$; see [31] for the definition of $H_{00}^{1/2}(\Gamma_k)$.

Case $1 < p < 2$**:** We define $W_{00}^{1-1/p,p}(\Gamma_k) = W^{1-1/p,p}(\Gamma_k)$ equipped with the norm $\|\cdot\|_{W_{00}^{1-1/p,p}(\Gamma_k)} = \|\cdot\|_{W^{1-1/p,p}(\Gamma_k)}$.

These spaces have the following important feature. Denote by $\tilde{\varphi}$ the extension by zero to $\partial\Omega \setminus \Gamma_k$ of a given function $\varphi \in W_{00}^{1-1/p,p}(\Gamma_k)$. Then by the Trace Theorem and the Lift Theorem 1.5.2.3 from [24] there exists a function $\psi_\varphi \in W^{1,p}(\Omega)$ such that $\psi_\varphi|_{\partial\Omega} = \tilde{\varphi}$ and

$$c_1\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le \|\psi_\varphi\|_{1,p} \le c\|\tilde{\varphi}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le c_2\|\varphi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}. \tag{4.7}$$

We also introduce the dual space of $W_{00}^{1-1/p,p}(\Gamma_k)$, denoted by $W^{-1+1/p,p'}(\Gamma_k)$, where $1/p + 1/p' = 1$.

The following inverse inequality is required in the proof of Lemma 4.3.

Lemma 4.1. *Let* $1 < p < \infty$ *and* $v^h \in Y_{0,k}^h$. *Then*

$$\|v^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le ch^{-1}\|v^h\|_{W^{-1+1/p',p}(\Gamma_k)}. \tag{4.8}$$

Proof. Consider the following inverse inequality (see Theorem 4.5.11 [10])

$$\|v^h\|_{s,q,\partial\Omega} \le ch^{-s}\|v^h\|_{0,q,\partial\Omega}, \quad \forall\, v^h \in Y^h,\ 1 \le q \le \infty \text{ and } 0 \le s \le 1. \tag{4.9}$$

Given $v^h \in Y_{0,k}^h$ let $\tilde{v}^h \in Y^h$ be the extension of v^h to $\partial\Omega \setminus \Gamma_k$ by zero. Using (4.7) and (4.9) we obtain

$$\begin{aligned} \|v^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} &\le c\|\tilde{v}^h\|_{W^{1-1/p,p}(\partial\Omega)} \\ &\le ch^{-1+1/p}\|\tilde{v}^h\|_{L^p(\partial\Omega)} = ch^{-1+1/p}\|v^h\|_{L^p(\Gamma_k)}. \end{aligned} \tag{4.10}$$

Let $\mathcal{P}_{0,k}$ denote the L^2 projector to $Y^h_{0,k}$ and assume that $v^h \in Y^h_{0,k}$. Then

$$\|v^h\|_{L^p(\Gamma_k)} = \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\langle v^h, \phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} = \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\langle v^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}.$$

By Theorem 1 in [17] we have

$$\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)} \le c\|\phi\|_{L^{p'}(\Gamma_k)} \quad 1 \le p' \le \infty. \tag{4.11}$$

Hence

$$\begin{aligned} \|v^h\|_{L^p(\Gamma_k)} &\le c \sup_{\phi \in L^{p'}(\Gamma_k)} \frac{\|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)} \|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}(\Gamma_k)}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\ &\le ch^{-1+\frac{1}{p'}} \|v^h\|_{W^{-1+\frac{1}{p'},p}(\Gamma_k)}, \end{aligned} \tag{4.12}$$

where on the last inequality we have used (4.10) for bounding $\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-1/p',p'}(\Gamma_k)}$. Combining inequalities (4.10) and (4.12) we obtain (4.8). ☐

The following lemma provide stability and error estimates concerning $\mathcal{P}_{0,k}$, the L^2 projector to $Y^h_{0,k}$. These results are required in the proof of Lemma 4.3.

Lemma 4.2. *Let $2 \le p < \infty$ and $\mathcal{P}_{0,k} : W^{-1+\frac{1}{p},p'}(\Gamma_k) \to Y^h_{0,k}$ be the L^2 projector to $Y^h_{0,k}$. Then we have*

$$\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le c\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\, \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{4.13}$$

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \le ch^{1-\frac{1}{p}} \|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \quad \forall\, \phi \in W_{00}^{1-\frac{1}{p},p}(\Gamma_k), \tag{4.14}$$

$$\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \le ch^{1-\frac{1}{p}} \|\phi\|_{L^{p'}(\Gamma_k)} \quad \forall\, \phi \in L^{p'}(\Gamma_k) \tag{4.15}$$

and

$$\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \le c\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} \quad \forall\, \phi \in W^{-1+\frac{1}{p},p'}(\Gamma_k). \tag{4.16}$$

Proof of (4.13):

Case $p > 2$: Observe that $\mathcal{P}_{0,k} : L^p(\Gamma_k) \to Y^h_{0,k}$ is stable in L^p and $W^{1,p}$, i.e. $\|\mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} \le c\|\phi\|_{L^p(\Gamma_k)}$ $\forall\, \phi \in L^p(\Gamma_k)$, and $\|\mathcal{P}_{0,k}\phi\|_{W^{1,p}(\Gamma_k)} \le c\|\phi\|_{W^{1,p}(\Gamma_k)}$ $\forall\, \phi \in W^{1,p}(\Gamma_k)$, respectively; see Theorems 1 and 2 in [17]. Since $W^{1-\frac{1}{p},p}(\Gamma_k) = [L^p(\Gamma_k), W^{1,p}(\Gamma_k)]_{1-1/p,p}$; see Theorem 12.2.3 in [10], we obtain the stability of $\mathcal{P}_{0,k}$ in $W^{1-\frac{1}{p},p}(\Gamma_k)$ by the real interpolation method, see Proposition 12.1.5 in [10], and the inequality (4.13) follows.

Case $p = 2$: By definition $H^{1/2}_{00}(\Gamma_k) = [L^2(\Gamma_k), H^1_0(\Gamma_k)]_{1/2}$ and the proof is analogue to the case $p > 2$.

Proof of (4.14):

Case $p > 2$: Let $\mathcal{I}^h : L^p(\Gamma_k) \to V^h(\Omega)|_{\Gamma_k}$ denote the standard $\mathcal{P}_1$ or $\mathcal{Q}_1$ interpolation operator. Then we have

$$\begin{aligned} \|\phi - \mathcal{P}_{0,k}\phi\|_{L^p(\Gamma_k)} &\leq \|\phi - \mathcal{I}^h\phi\|_{L^p(\Gamma_k)} + \|\mathcal{P}_{0,k}(\phi - \mathcal{I}^h\phi)\|_{L^p(\Gamma_k)} \\ &\leq c\|\phi - \mathcal{I}^h\phi\|_{L^p(\Gamma_k)}, \text{ by (4.11)} \\ &\leq ch^{1-\frac{1}{p}}\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}, \text{ by (4.4).} \end{aligned} \tag{4.17}$$

Case $p = 2$: Follows similarly to the case $p > 2$ by replacing $\mathcal{I}^h$ by the Clement interpolation operator defined by (2.13) in [40] and use the real interpolation method to obtain (4.17).

Proof of (4.15):

$$\begin{aligned} \|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} &= \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\ &= \sup_{v\in W_{00}^{1-\frac{1}{p},p}} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\ &= \sup_{v\in W_{00}^{1-\frac{1}{p},p}} \frac{\langle \phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\ &\leq \sup_{v\in W_{00}^{1-\frac{1}{p},p}} \frac{\|\phi\|_{L^{p'}(\Gamma_k)}\|v - \mathcal{P}_{0,k}v\|_{L^p(\Gamma_k)}}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\ &\leq ch^{1-\frac{1}{p}}\|\phi\|_{L^{p'}(\Gamma_k)}, \end{aligned} \tag{4.18}$$

where we have used (4.14) to obtain the last inequality.

Proof of (4.16):

$$\begin{aligned} \|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} &= \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \mathcal{P}_{0,k}\phi, v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\ &\leq c \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \mathcal{P}_{0,k}\phi, \mathcal{P}_{0,k}v\rangle}{\|\mathcal{P}_{0,k}v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}}, \quad \text{by} \quad (4.13) \\ &\leq c \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \phi, \mathcal{P}_{0,k}v\rangle}{\|\mathcal{P}_{0,k}v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \leq c\|\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}. \end{aligned}$$

□

The following lemma estimate the error between $A\nabla u_0 \cdot \eta$ and its numerical approximation λ^h. This lemma is used in the proof of Proposition 4.1.

Lemma 4.3. *Let λ^h be defined by Equation (3.1) and $\lambda = \partial_{\eta_A} u_0 = A_{ij}\partial_j u_0 \eta_i$, where η_i is the ith component of the normal vector to Γ_k. Assume that $u_0 \in W^{2,p}(\Omega)$. Then we have*

$$\|\lambda - \lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \leq c\|u_0\|_{2,p} \quad \textit{for} \quad 2 \leq p < \infty, \tag{4.19}$$

$$\|\lambda-\lambda^h\|_{L^p(\Gamma_k)} \le ch^{1-\frac{1}{p}}\|u_0\|_{2,p} \quad \textit{for} \quad 2\le p\le\infty \tag{4.20}$$

and

$$\|\lambda-\lambda^h\|_{W^{-1+1/p,p'}(\Gamma_k)} \le ch\|u_0\|_{2,p} \quad \textit{for} \quad 2\le p<\infty. \tag{4.21}$$

Proof of (4.19): From Remark 2.1 if $p=2$, or from the Sobolev embedding theorem if $p>2$, we have

$$\|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le c\|u_0\|_{2,p}. \tag{4.22}$$

In order to prove inequality (4.19) observe that

$$\|\lambda-\lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \le \|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)} + \|\lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)},$$

and

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} = \sup_{\phi\in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle\lambda^h,\phi\rangle}{\|\phi\|_{W^{-1+1/p,p'}(\Gamma_k)}}.$$

Since $\lambda^h\in Y_{0,k}^h$ then $\langle\lambda^h,\phi\rangle=\langle\lambda^h,\mathcal{P}_{0,k}\phi\rangle$, and using (4.16) we obtain

$$\|\lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \le c \sup_{\phi\in W^{-1+\frac{1}{p},p'}(\Gamma_k)} \frac{\langle\lambda^h,\mathcal{P}_{0,k}\phi\rangle}{\|\mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}. \tag{4.23}$$

Now we recall the discrete extension operator, $E_h: Y^h\to V^h(\Omega)$ defined by (5.5) in [40], which satisfies

$$\|E_h g\|_{1,p'} \le c\|g\|_{W^{1-1/p',p'}(\partial\Omega)}$$

for $g\in Y^h$. Hence if $g^h\in Y_{0,k}^h$ and $\tilde{g}^h$ denotes the extension of g^h by zero to $\partial\Omega\setminus\Gamma_k$ it follows

$$\|E_h\tilde{g}^h\|_{1,p'} \le c\|g^h\|_{W_{00}^{1-1/p',p'}(\Gamma_k)}. \tag{4.24}$$

Let $\tilde{\mathcal{P}}_{0,k}\phi$ denote the discrete extension of $\mathcal{P}_{0,k}\phi$ to $\partial\Omega\setminus\Gamma_k$ by zero. From the definition of λ^h and inequalities (4.24) and (4.1), we obtain

$$\begin{aligned}\langle\lambda^h,\mathcal{P}_{0,k}\phi\rangle &= \langle\lambda,\mathcal{P}_{0,k}\phi\rangle + a(u_0^h-u_0, E_h\tilde{\mathcal{P}}_{0,k}\phi)\\ &\le \|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)}\|\mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)} + ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-1/p',p'}(\Gamma_k)}\\ &\le c\left(\|\lambda\|_{W_{00}^{1-1/p,p}(\Gamma_k)} + c\|u_0\|_{2,p}\right)\|\mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)}.\end{aligned} \tag{4.25}$$

Here we used the inverse estimate (4.8) applied to $\mathcal{P}_{0,k}\phi$ to obtain (4.25) . Inequality (4.19) follows from (4.25), (4.23) and (4.22).

Proof of (4.21): We observe that

$$\begin{aligned}\|\lambda-\lambda^h\|_{W^{-1+1/p,p'}(\Gamma_k)} &= \sup_{\phi\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle\lambda-\lambda^h,\phi\rangle}{\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}}\\ &\le c \sup_{\phi\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle\lambda-\lambda^h,\phi-\mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} + c \sup_{\phi\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle\lambda-\lambda^h,\mathcal{P}_{0,k}\phi\rangle}{\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}}.\end{aligned} \tag{4.26}$$

In order to estimate the second term on the right hand side of (4.26) we use the definition of λ and λ^h, and the inequality (4.24) to obtain

$$
\begin{aligned}
\langle \lambda - \lambda^h, \mathcal{P}_{0,k}\phi\rangle &= a(u_0^h - u_0, E_h\tilde{\mathcal{P}}_{0,k}\phi) \\
&\le ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-1/p',p'}(\Gamma_k)} \\
&\le ch\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-1/p,p}(\Gamma_k)} \quad \text{since } p > p'. \qquad (4.27)
\end{aligned}
$$

For estimating the first term on the right hand side of (4.26) we note that

$$
\begin{aligned}
\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)} &= \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\langle \phi - \mathcal{P}_{0,k}\phi, v - \mathcal{P}_{0,k}v\rangle}{\|v\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}} \\
&\le \sup_{v\in W_{00}^{1-\frac{1}{p},p}(\Gamma_k)} \frac{\|\phi - \mathcal{P}_{0,k}\phi\|_{L^2(\Gamma_k)}\|v - \mathcal{P}_{0,k}v\|_{L^2(\Gamma_k)}}{\|v\|_{W_{00}^{1-1/p,p}(\Gamma_k)}} \\
&\le ch\|\phi\|_{W_{00}^{1-1/p,p}(\Gamma_k)}. \qquad (4.28)
\end{aligned}
$$

In the last inequality we used (4.14) and the fact that $W_{00}^{1-1/p,p}(\Gamma_k) \hookrightarrow H_{00}^{1/2}(\Gamma_k)$ for $p > 2$. Hence,

$$
\begin{aligned}
\langle \lambda - \lambda^h, \phi - \mathcal{P}_{0,k}\phi\rangle &\le \|\lambda - \lambda^h\|_{W_{00}^{1-1/p,p}(\Gamma_k)}\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+1/p,p'}(\Gamma_k)} \\
&\le ch\|u_0\|_{2,p}\|\phi\|_{W_{00}^{1-1/p,p}(\Gamma_k)}, \text{ by (4.19) and (4.28)}, \qquad (4.29)
\end{aligned}
$$

and the inequality (4.21) follows from (4.26), (4.27) and (4.29).

Proof of (4.20):

Case $2 \le p < \infty$: We have

$$
\|\lambda - \lambda^h\|_{L^p(\Gamma_k)} \le \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \phi - \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} + \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}}. \qquad (4.30)
$$

The first term on the right hand side of (4.30) is bounded as follows:

$$
\begin{aligned}
\sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \phi - \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\le \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\|\lambda - \lambda^h\|_{W_{00}^{1-\frac{1}{p},p}(\Gamma_k)}\|\phi - \mathcal{P}_{0,k}\phi\|_{W^{-1+\frac{1}{p},p'}(\Gamma_k)}}{\|\phi\|_{L^{p'}(\Gamma_k)}} \\
&\le ch^{1-\frac{1}{p}}\|u_0\|_{2,p}. \qquad (4.31)
\end{aligned}
$$

Here we have used (4.15) and (4.19) to arrive in (4.31). In order to estimate the second term on the right hand side of (4.30) we use the definition of λ and λ^h to obtain

$$
\begin{aligned}
\sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\langle \lambda - \lambda^h, \mathcal{P}_{0,k}\phi\rangle}{\|\phi\|_{L^{p'}(\Gamma_k)}} &\le \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\int_Y a_{ij}\partial_i(u_0 - u_0^h)\partial_j(E_h\tilde{\mathcal{P}}_{0,k}\phi)dy}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\
&\le ch \sup_{\phi\in L^{p'}(\Gamma_k)} \frac{\|u_0\|_{2,p}\|\mathcal{P}_{0,k}\phi\|_{W_{00}^{1-\frac{1}{p'},p'}}}{\|\mathcal{P}_{0,k}\phi\|_{L^{p'}(\Gamma_k)}} \\
&\le ch^{1-\frac{1}{p}}\|u_0\|_{2,p}, \text{ by (4.10)}.
\end{aligned}
$$

Case $p = \infty$: Let $z \in \Gamma_k$, then

$$|\lambda(z) - \lambda^h(z)| \le |\lambda(z) - \mathcal{P}_{0,k}\lambda(z)| + |\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)|. \tag{4.32}$$

For the first term of (4.32), by Theorem 3.1 in [44] there exists a positive constant c such that

$$|\lambda(z) - \mathcal{P}_{0,k}\lambda(z)| \le c\|\lambda - v^h\|_{0,\infty,\Gamma_k} + c\,\exp(-ch)\|\lambda - v^h\|_{0,1,\Gamma_k},\ \forall\, v^h \in Y_{0,k}. \tag{4.33}$$

The use of $\mathcal{Q}_1$ elements to approximate u_0 implies $A\nabla u_0^h \cdot \eta_k|_{\Gamma_k} \in Y_{0,k}$, therefore we can take $v^h = A\nabla u_0^h \cdot \eta_k$ in (4.33) and use (4.1) to obtain

$$\|\lambda - \mathcal{P}_{0,k}\lambda\|_{0,\infty} \le ch\|u_0\|_{2,\infty}. \tag{4.34}$$

When $\mathcal{P}_1$ elements are used $A\nabla u_0^h \cdot \eta_k$ is piecewise constant, hence $A\nabla u_0^h \cdot \eta_k|_{\Gamma_k} \notin Y_{0,k}$. We then consider a rectangular mesh $\tilde{\mathcal{T}}^h(\Omega)$ such that the approximation $\tilde{u}_0^h$ using bilinear elements on $\tilde{\mathcal{T}}^h(\Omega)$ for u_0 satisfies $A\nabla \tilde{u}_0^h \cdot \eta_k|_{\Gamma_k} \in Y_{0,k}$. Hence we take $v^h = A\nabla \tilde{u}_0^h \cdot \eta_k$ in (4.33) and use (4.1) to obtain (4.34).

To estimate the second term on the right hand side of (4.32) we follow ideas from [44]. Let $E_z \subset \Gamma_k$ denote an edge of an element $K_z \in \mathcal{T}^h(\Omega)$ such that $z \in E_z$, and define δ_z as the polynomial of degree 1 on E_z such that

$$\int_{E_z} \delta_z(s)v(s)ds = v(z),\ \text{for any } v \text{ polynomial of degree 1.}$$

Regard δ_z as extended by zero to $\Gamma_k \setminus E_z$ and denote by $\tilde{\delta}_z^h \in V^h(\Omega)$ the extension by zero of $\mathcal{P}_{0,k}\delta_z$ to Ω. Then we have

$$\begin{aligned} \lambda^h(z) - \mathcal{P}_{0,k}\lambda(z) &= \int_{\Gamma_k} \mathcal{P}_{0,k}(\lambda^h - \lambda)\delta_z ds = \int_{\Gamma_k} (\lambda^h - \lambda)\mathcal{P}_{0,k}\delta_z ds \\ &= \int_\Omega A_{ij}\partial_i(u_0 - u_0^h)\partial_j(\tilde{\delta}_z^h)dx \end{aligned} \tag{4.35}$$

where we have used the definition of λ^h to obtain (4.35). From (4.1) and (4.35) follows

$$|\lambda^h(z) - \mathcal{P}_{0,k}\lambda(z)| \le ch\|u_0\|_{2,\infty}\|\tilde{\delta}_z^h\|_{1,1}.$$

Using an inverse estimate followed by a Poincare inequality we have

$$\|\tilde{\delta}_z^h\|_{1,1} \le ch^{-1}\|\tilde{\delta}_z^h\|_{0,1} \le c\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k}.$$

Finally, we use the fact that $\|\mathcal{P}_{0,k}\delta_z\|_{0,1,\Gamma_k} \le c$, see Lemma 3.5 in [44], and (4.20) follows. $\square$

Proposition 4.1 estimates the error between ∇u_0 and its proposed numerical approximation Ψ^h. This Proposition is required in the proof of Proposition 4.2.

Proposition 4.1. *Let u_0 and Ψ^h be defined by Equations (2.4) and (3.2), respectively. Assume $u_0 \in W^{2,p}(\Omega)$ and that linear or bilinear finite elements are used to approximate u_0. Then for $2 \le p \le \infty$ we have*

$$\|(\nabla u_0 - \Psi^h)\cdot \nu\|_{0,p} \le ch\|u_0\|_{2,p},\quad \forall\, \nu \in \mathbb{R}^2 \text{ with } |\nu| = 1 \tag{4.36}$$

and

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{1,p,h} \le c\|u_0\|_{2,p}, \quad \forall\, \nu\in\mathbb{R}^2 \textit{ with } |\nu| = 1. \tag{4.37}$$

Proof of (4.36): From the triangular inequality we have

$$\|(\nabla u_0 - \Psi^h)\cdot\nu\|_{0,p} \le \|(\nabla u_0 - \nabla u_0^h)\cdot\nu\|_{0,p} + \|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p}. \tag{4.38}$$

Use (4.1) to estimate the first term on the right hand side of (4.38). For the second term, by the definition of Ψ^h, we have

$$\|(\nabla u_0^h - \Psi^h)\cdot\nu\|_{0,p} \le c \sum_{k\in\{e,w,n,s\}} \|E_k^h(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,p}.$$

Consider $k = e$ and that bilinear elements are used to approximate u_0; the other cases, $k\in\{w,n,s\}$ or when $\mathcal{P}_1$ elements are used, follow in a similar way. From definition, the function $E_e^h\left(\mu^h - \frac{\partial u_0^h}{\partial x_1}\right)$ is linear in the x_1 direction and equal to zero in $x_1 \le 1-h$, hence

$$\|E_e^h(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,p} \le h^{1/p}\left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,p,\Gamma_e}, \quad \text{if } \ 2\le p<\infty$$

or

$$\|E_e^h(\mu^h - \nabla u_0^h\cdot\eta^k)\|_{0,\infty} \le \left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,\infty,\Gamma_e}, \quad \text{if } \ p=\infty.$$

Case $2\le p<\infty$: The triangular inequality gives

$$\left\|\partial_{x_1}u_0^h - \mu^h\right\|_{0,p,\Gamma_e} \le \left\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\right\|_{0,p,\Gamma_e} + \left\|\partial_{x_1}u_0 - \mu^h\right\|_{0,p,\Gamma_e}. \tag{4.39}$$

In order to estimate the first term on the right hand side of (4.39), let $K\in\mathcal{T}_h(\Omega)$ containing an edge $E\subset\Gamma_k$. Applying a Trace Theorem we have

$$\begin{aligned}&\left\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\right\|_{0,p,E} \le \\ &\qquad c\left(h^{-1}\left\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\right\|_{0,p,K}^p + h^{p-1}\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\|_{1,p,K}^p\right)^{1/p}.\end{aligned} \tag{4.40}$$

From (4.1), (4.6) and (4.40) we obtain

$$\left\|\partial_{x_1}u_0^h - \partial_{x_1}u_0\right\|_{0,p,\Gamma_e} \le ch^{1-1/p}\|u_0\|_{2,p}. \tag{4.41}$$

For second term on the right hand side of (4.39), we apply the definition of λ and λ^h to obtain $\left\|\partial_{x_1}u_0 - \mu^h\right\|_{0,p,\Gamma_e} = A_{11}\left\|\lambda - \lambda^h\right\|_{0,p,\Gamma_e}$, and therefore from (4.20) we have

$$\left\|\partial_{x_1}u_0 - \mu^h\right\|_{0,p,\Gamma_e} \le ch^{1-1/p}\|u_0\|_{2,p}. \tag{4.42}$$

From (4.39), (4.41) and (4.42) we obtain

$$\|E_e(\mu^h - \nabla u_0^h\cdot\eta^e)\|_{0,p} \le ch\|u_0\|_{2,p},$$

and hence estimate (4.36) holds for $p<\infty$.

Case $2=\infty$: We have

$$\left\|\partial_{x_1} u_0^h-\mu^h\right\|_{0,\infty,\Gamma_e} \leq \|\partial_{x_1} u_0^h-\partial_{x_1} u_0\|_{0,\infty,\Gamma_e}+\|\partial_{x_1} u_0-\mu^h\|_{0,\infty,\Gamma_e},$$

and applying (4.20) and (4.1) we have

$$\left\|\partial_{x_1} u_0-\mu^h\right\|_{0,\infty,\Gamma_e} \leq ch\|u_0\|_{2,\infty},$$

and hence estimate (4.36) follows for $p=\infty$.

Proof of (4.37): We have

$$\begin{aligned} \|(\nabla u_0^h-\Psi^h)\cdot\nu\|_{0,p} &\leq c\|(\nabla u_0-\Psi^h)\cdot\nu\|_{0,p}+\|(\nabla u_0-\nabla u_0^h)\cdot\nu\|_{0,p} \\ &\leq ch\|u_0\|_{2,p}, \quad \text{by} \quad (4.1) \text{ and } (4.36) \end{aligned} \tag{4.43}$$

and from an inverse inequality, see Lemma 4.5.3 from [10], follows that

$$\|(\nabla u_0^h-\Psi^h)\cdot\nu\|_{1,p,h} \leq c\|u_0\|_{2,p}.$$

Since

$$\|(\nabla u_0-\Psi^h)\cdot\nu\|_{1,p,h} \leq c\left(\|(\nabla u_0^h-\nabla u_0)\cdot\nu\|_{1,p,h}+\|(\nabla u_0^h-\Psi^h)\cdot\nu\|_{1,p,h}\right),$$

we obtain (4.37) from (4.6). $\square$

The following proposition estimates the error between u_1 and u_1^h. These estimates are required in the proof of Theorems 4.1 and 4.2.

Proposition 4.2. *Let u_1 and u_1^h be defined by (2.5) and (3.3), respectively. Assume that $u_0\in W^{2,p}(\Omega)$ and $\chi^i\in W^{1,q}_{per}(Y)$, for $1/p+1/q\leq 1/2$. Then there exists a constant c independent of ϵ and h such that*

$$|u_1-u_1^h|_{1,h} \leq c\|u_0\|_{2,p}\|\chi\|_{1,q,Y}\left(\frac{h^2}{\epsilon^2}+1\right)^{1/2} \tag{4.44}$$

and

$$\|u_1-u_1^h\|_0 \leq ch\|u_0\|_{2,p}\|\chi\|_{1,q,Y}, \tag{4.45}$$

where $\|\chi\|_{1,q,Y}=\sum_i\|\chi^i\|_{1,q,Y}$.

Proof of (4.44): We have

$$|u_1-u_1^h|_{1,h}^2 \leq 2\sum_{K_j\in\mathcal{T}_h(\Omega)}\int_{K_j}\sum_{j\in 1,2}((\partial_{x_i}u_0-\Psi_i^h)\partial_{x_j}\chi^i(\cdot/\epsilon))^2+(\chi^i(\cdot/\epsilon)\cdot\partial_{x_j}(\partial_{x_i}u_0-\Psi_i^h))^2dx. \tag{4.46}$$

For the first term on the right hand side of (4.46) we have

$$\begin{aligned} \sum_{K_j\in\mathcal{T}_h(\Omega)}\int_{K_j}\sum_{j\in 1,2}((\partial_{x_i}u_0-\Psi_i^h)\partial_{x_j}\chi^i(\cdot/\epsilon))^2dx &\leq |\partial_{x_i}u_0-\Psi_i^h|_{0,p}^2\|\partial_{x_j}\chi^i(\cdot/\epsilon)\|_{0,q}^2 \\ &\leq \epsilon^{-2}|\partial_{x_i}u_0-\Psi_i^h|_{0,p}^2\|\chi\|_{1,q,Y}^2 \leq c\epsilon^{-2}h^2\|u_0\|_{2,p}^2\|\chi\|_{1,q,Y}^2, \end{aligned} \tag{4.47}$$

where we have used (4.36) to obtain (4.47).

The second term on the right hand side of (4.46) is bounded by a Cauchy inequality, $\|\chi^i \partial_j(\partial_i u_0 - \Psi_i^h)\|_0^2 \le \|\chi\|_{0,q}^2 |\partial_i u_0 - \Psi_i^h|_{1,p,h}^2$.

Proof of (4.45): It follows from a direct application of Cauchy inequality and the approximation error estimate (4.1). $\square$

The following proposition estimates the error between $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$. This Proposition is required in the proof of Theorems 4.1 and 4.2.

Proposition 4.3. *Let $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$ be defined by (2.13) and (3.4), respectively. Assume that $u_0 \in W^{2,p}(\Omega)$ and $v_k \in W^{1,q}(G_k)$, for $1/p + 1/q \le 1/2$. Then*

$$|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} \le c \left(\frac{h^2}{\epsilon^2} + 1\right)^{1/2} \max_k \|v_k\|_{1,q,G_k} \|u_0\|_{2,p} \tag{4.48}$$

and

$$\|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h\|_0 \le ch \max_k \|v_k - \chi_k^*\|_{0,q,G_k} \|u_0\|_{2,p}. \tag{4.49}$$

Proof. From definition of $\tilde{\phi}_\epsilon$ and $\tilde{\phi}_\epsilon^h$ we have

$$|\tilde{\phi}_\epsilon - \tilde{\phi}_\epsilon^h|_{1,h} \le \sum_{k \in \{e,w,n,s\}} |\tilde{\phi}_\epsilon^k - \tilde{\phi}_\epsilon^{k,h}|_{1,h},$$

and the proposition follows from arguments similar to the ones given in the proof of Proposition 4.2. $\square$

Finally, we prove the last proposition used in the proof of Theorems 4.1 and 4.2. Proposition 4.4 estimates the error between $\bar{\phi}$ and $\bar{\phi}^h$.

Proposition 4.4. *Let $\bar{\phi}$ be defined by Equation (2.14), $\bar{\phi}^h$ be the finite element approximation to the Equation (3.5), and assume that $u_0 \in H^2(\Omega)$. Then we have*

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \le c\|u_0\|_2 \tag{4.50}$$

and

$$\|\bar{\phi} - \bar{\phi}^h\|_0 \le ch\|u_0\|_2. \tag{4.51}$$

Proof of (4.50): We note that $\chi^* \mu^h \in H^{1/2}(\partial\Omega)$, see Remark 3.1, hence we define $\psi \in H^1(\Omega)$ as the solution of

$$\nabla \cdot A \nabla \psi = 0 \text{ in } \Omega \quad \psi = \chi^* \mu^h \text{ on } \partial\Omega. \tag{4.52}$$

From regularity theory and (4.19) we have

$$\|\psi\|_1 \le \sum_k c\|\chi^* \mu^h\|_{H_{00}^{1/2}(\Gamma_k)} \le c\|u_0\|_2, \tag{4.53}$$

and from triangular inequality

$$\|\bar{\phi} - \bar{\phi}^h\|_1 \le \|\bar{\phi} - \psi\|_1 + \|\bar{\phi}^h - \psi\|_1.$$

Since $\chi^* \mu^h \in V^h(\Omega)$, the problem of finding $\bar{\phi}$ reduces to a conforming finite element problem, hence standard finite element analysis and (4.53) gives

$$|\bar{\phi}^h - \psi|_1 \le c\|u_0\|_2.$$

Finally, from regularity theory and Lemma 4.3 we obtain

$$\begin{aligned}|\bar{\phi}-\psi|_1 &\leq \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} \\ &\leq \sum_k \|\chi^*\mu^h - \chi^*\partial_\eta u_0\|_{H^{1/2}_{00}(\Gamma_k)} \leq c\|u_0\|_2.\end{aligned}$$

Proof of (4.51): From the triangular inequality

$$\|\bar{\phi}-\bar{\phi}^h\|_0 \leq c\|\bar{\phi}-\psi\|_0 + \|\bar{\phi}^h-\psi\|_0,$$

and from standard finite element analysis and (4.53) we obtain

$$\|\bar{\phi}^h-\psi\|_0 \leq ch\|\psi\|_1 \leq ch\|u_0\|_2.$$

Theorem 6.1 from [37] states

$$\begin{aligned}\|\bar{\phi}-\psi\|_0 &\leq c\Big(\sum_k \|\chi^*\partial_\eta u_0 - \chi^*\mu^h\|^2_{H^{-1/2}(\Gamma_k)}\Big)^{1/2} \\ &\leq ch\|u_0\|_2 \quad \text{by (4.21).}\end{aligned}$$

□

5. Numerical Results. As in [26] we consider the case

$$a(x) = \left(\frac{2+1.8\sin(2\pi x_1/\epsilon)}{2+1.8\cos(2\pi x_2/\epsilon)} + \frac{2+\sin(2\pi x_2/\epsilon)}{2+1.8\sin(2\pi x_1/\epsilon)}\right) I_{2\times 2}, \quad \text{and} \quad f(x) = -1.$$

We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method on a fine mesh with size h_f, which we call u^*_ϵ. Table 5.1 provide absolute errors estimates for $u^*_\epsilon - u^{h,\hat{h},\tau}_\epsilon$. We have used $\tau = 2$, $\hat{h} = 1/128$, $h_f = 1/2048$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi^j_{\hat{h}}$ and $v^{\hat{h},p}_k$.

TABLE 5.1
$u^*_\epsilon - u^{h,\hat{h},\tau}_\epsilon$ *error*

$\epsilon \downarrow \quad h \rightarrow$	1/8	1/16	1/32	1/64
		$\|\cdot\|_0$ error		
1/16	2.7085e-04	7.7993e-05		
1/32	2.6300e-04	6.6246e-05	1.7773e-05	
1/64	2.5388e-04	5.8069e-05	1.6020e-05	1.2137e-05
		$\lvert\cdot\rvert_{1,h}$ error		
1/16	0.0097	0.0067		
1/32	0.0086	0.0051	0.0036	
1/64	0.0086	0.0044	0.0025	0.0018

From Table 5.1, we see that for $\epsilon << h$ we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and H^1 semi norm , respectively. We observe that when we fix h and decrease ϵ the errors almost do not change. This is evidence that in this case the dominant error term is $O(h)$. Also looking at the diagonal values in this table we see clearly that the numerical error agrees with the theoretical rates from Theorems 4.1 and 4.2.

Table 5.2

$\epsilon = 1/64$, $h = 1/32$, $h_f = 1/1024$

	$\|\cdot\|_0$	$\vert\cdot\vert_{1,h}$
$u_\epsilon^* - u_0^{h,\hat{h}}$	0.0287	0.0215
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$	0.0213	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon\bar{\phi}^{h,\hat{h},\tau}$	5.0450e-05	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},\tau} + \tilde{\phi}_\epsilon^{h,\hat{h},\tau})$	5.1865e-05	0.0025

Table 5.2 shows the improvement obtained in the final approximation when the term $\phi_\epsilon^{h,\hat{h},\tau}$ is taken into account. It can be appreciated from this table that a better improvement on the $\|\cdot\|_0$ norm rather than on $|\cdot|_{1,h}$ semi norm is clearly seen. The improvement on the L^2 norm is an evidence that we were able to obtain, through the proper calculation of χ^*, the asymptotic L^2 behavior of the boundary corrector θ_ϵ in the interior of the domain Ω. We also note that the term $\tilde{\phi}_\epsilon$ primarily forces the final approximation $u_\epsilon^{h,\hat{h},\tau}$ to satisfy the zero Dirichlet boundary condition, and since it has support only in a thin boundary layer of $\partial\Omega$, then no much error improvement is obtained on the $|\cdot|_{1,h}$ semi norm.

We also consider the following example:

$$a(y) = \begin{cases} 2 & \text{if } 2/5 < y_1 < 3/5 \text{ or } 2/5 < y_2 < 3/5 \\ 1 & \text{otherwise.} \end{cases} \qquad \text{and } f = -1$$

Table 5.3
$u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$ *error*

$\|\cdot\|_0$ error, $h_f = 1/2000$

$\epsilon\downarrow$ $h\rightarrow$	1/10	1/20	1/40
1/20	4.8318e-04	1.3043e-04	
1/40	4.7578e-04	1.1954e-04	3.0805e-05
1/64	2.5388e-04	5.9446e-05	1.4414e-05

Table 5.4
$u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$ *error*

$|\cdot|_{1,h}$ error, $h_f = 1/2000$

$\epsilon\downarrow$ $h\rightarrow$	1/10	1/20	1/40
1/20	0.0180	0.0092	
1/40	0.0179	0.0090	0.0046
1/64	0.0086	0.0045	0.0026

We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method in a fine mesh of size h_f, which we call u_ϵ^*. Tables 5.3 and 5.4 provide absolute errors estimates for $u_\epsilon^* - u_\epsilon^{h,\hat{h},\tau}$, on the $\|\cdot\|_0$ norm and $|\cdot|_{1,h}$ semi norm for different values of h and ϵ. We have used $\tau = 2$, $\hat{h} = 1/128$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_e^{\hat{h},\tau}$.

Although the convergence analysis presented here are not intended for the quasi periodic case $a_{ij}(x, x/\epsilon)$ the numerical approximation presented here can be generalized for this case. This would be done by approximating matrix $a(x, x/\epsilon)$ by $\sum_j a^j(x/\epsilon) I_{K_j}(x)$, where I_{K_J} is the characteristic function for $K_j \in \mathcal{T}_k(\Omega)$, and then solving a cell problem in each sub-domain K_j.

6. Appendix.

6.1. Proof of Theorem 2.1. By the triangular inequality we have

$$
\begin{aligned}
|u_\epsilon - u_0 - u_1 - \phi_\epsilon|_{1,h} \leq & |u_\epsilon - u_0 - u_1 - \theta_\epsilon|_1 \\
& + \epsilon|\bar{\theta}_\epsilon - \bar{\phi}|_1 + \epsilon|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon|_1,
\end{aligned}
$$

and the theorem follows from Propositions 6.1, 6.2 and 6.3. □

We now prove the propositions used in the proof of Theorem 2.1. The following proposition gives the same error estimate of Theorem 2.2 in [3], however here we assume $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(\Omega)$ for $1/p + 1/q \leq 1/2$ while in Theorem 2.2 in [3] it is assumed $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in H^1_{per}(\Omega)$. It also generalizes Proposition 2.1 from [34] where it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$. We note here that Theorem 1.1 from [32] gives conditions concerning the discontinuities of the functions a_{ij} such that $\chi^j \in W^{1,\infty}_{per}(Y)$. Finally, we observe that in the case $a_{ij} \in C^{1,\beta}_{per}(Y)$ a error estimate similar to Proposition 6.1 can be obtained in the case a zero Neumann boundary condition is used to define u_ϵ; see [35].

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.1) and u_0, u_1, and θ_ϵ be defined by Equations (2.4), (2.5) and (2.6), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{2,p}(\Omega)$, and $\chi^j \in W^{1,q}_{per}(Y)$ for $1/p + 1/q \leq 1/2$. Then there exists a constant c independent of u_0 and ϵ, such that*

$$
\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_1 \leq c\epsilon\|u_0\|_{2,p}.
$$

Proof. Define

$$
v_0(x, y) = a(y)\nabla_x u_0(x) + a(y)\nabla_y u_1(x, y) = a(y)(\nabla_y y_j - \nabla_y \chi^j(y))\frac{\partial u_0}{\partial x_j}(x). \tag{6.1}
$$

From the definition of χ^j we have

$$
\int_Y \left(a(y)(e_j - \nabla_y \chi^j(y)) - Ae_j\right) \nabla_y \phi(y) dy = 0, \quad \forall\, \phi \in H^1_{per}(Y).
$$

Since the vector $a(y)(e_j - \nabla_y \chi^j(y)) - Ae_j$ is Y periodic and has zero average entries over Y, by Lemma 6.1 there exists $\phi_j(y) \in H^1_{per}(Y)$ with zero average over Y such that

$$
a(y)(\nabla_y y_j - \nabla_y \chi^j(y)) - Ae_j = -curl_y \phi_j(y). \tag{6.2}
$$

Let

$$
\phi = \phi_j(y)\frac{\partial u_0}{\partial x_j}(x) \tag{6.3}
$$

and define

$$v_1(x,y) = -curl_x \phi(x,y) = \begin{pmatrix} -\phi_j(y)\frac{\partial^2 u_0}{\partial x_2 \partial x_j}(x) \\ \phi_j(y)\frac{\partial^2 u_0}{\partial x_1 \partial x_j}(x) \end{pmatrix}.$$

In the case $d=2$ we have $|curl_y \phi_j|_{0,q} = |\phi_j|_{1,q}$. Since $\chi^j \in W^{1,q}_{per}(Y)$ and ϕ_j has zero average over Y, we apply a Poincare inequality to obtain

$$\|\phi_j\|_{1,q,Y} \le c|curl_y \phi_j|_{0,q,Y} \le c(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y}).$$

In the case $d=3$ by the Remark 3.11 in [25] we also obtain that $\phi_j \in W^{1,q}_{per}(Y)^3$ if $\chi^j \in W^{1,q}_{per}(Y)$. From hypothesis $u_0 \in W^{2,p}(\Omega)$ for $1/p+1/q \le 1/2$, hence $v_1(x,x/\epsilon) \in L^2(\Omega)$ and $\|v_1\|_0 \le c(\|\chi^1\|_{1,q,Y} + \|\chi^2\|_{1,q,Y})\|u_0\|_{2,p}$. Moreover, by Lemma 6.1,

$$\nabla_x \cdot v_1(x,y) = 0, \tag{6.4}$$

and simple calculations give

$$\begin{aligned} \nabla_y \cdot v_1(x,y) &= \nabla_y \cdot curl_x \left(\phi_j(y)\partial_{x_j} u_0(x)\right) \\ &= -\nabla_x \cdot curl_y \left(\phi_j(y)\partial_{x_j} u_0(x)\right) \\ &= -\nabla_x \cdot v_0(x,y) - f. \end{aligned} \tag{6.5}$$

Let

$$z_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon)$$

and

$$\eta_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon).$$

Then

$$\begin{aligned} &a(x/\epsilon)\nabla z_\epsilon(x) - \eta_\epsilon(x) \\ &\quad = a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x,x/\epsilon) \\ &\qquad - a(x/\epsilon)\nabla_y u_1(x,x/\epsilon) - a(x/\epsilon)\nabla u_\epsilon(x) + v_0(x,x/\epsilon) + \epsilon v_1(x,x/\epsilon) \\ &\quad = \epsilon(v_1(x,x/\epsilon) - a(x/\epsilon)\nabla_x u_1(x,x/\epsilon)), \end{aligned}$$

and so

$$\|a(\cdot/\epsilon)\nabla z_\epsilon - \eta_\epsilon\|_0 \le \epsilon\|v_1(\cdot,\cdot/\epsilon) - a(\cdot/\epsilon)\nabla_x u_1(\cdot,\cdot/\epsilon)\|_0. \tag{6.6}$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H^1_0(\Omega)$ be the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall \psi \in H^1_0(\Omega), \tag{6.7}$$

hence

$$\begin{aligned} \int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon)dx &= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla(z_\epsilon - \epsilon\theta_\epsilon)dx \\ &= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx - \epsilon\int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla\theta_\epsilon dx \\ &= \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx. \end{aligned} \tag{6.8}$$

Now observe that

$$\int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a(\cdot/\epsilon)\nabla w_\epsilon \cdot (\nabla z_\epsilon - \eta_\epsilon)dx + \int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx. \tag{6.9}$$

In order to estimate the second term on the right hand side of (6.9) we apply the definition of η_ϵ to obtain

$$\begin{aligned}\int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon)) \cdot \nabla w_\epsilon(x)dx \\ &= \int_\Omega f w_\epsilon dx - \int_\Omega (v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon)) \cdot \nabla w_\epsilon(x)dx. \end{aligned}\tag{6.10}$$

We note that

$$\begin{aligned}\int_\Omega v_1(x, x/\epsilon) \cdot \nabla w_\epsilon(x)dx &= \int_\Omega \nabla \cdot v_1(x, x/\epsilon) w_\epsilon(x)dx \\ &= \int_\Omega (\nabla_x + 1/\epsilon\nabla_y) \cdot v_1(x, y)|_{(y=x/\epsilon)} w_\epsilon(x)dx \\ &= -\frac{1}{\epsilon}\int_\Omega (\nabla_x \cdot v_0 + f)w_\epsilon dx,\end{aligned}\tag{6.11}$$

where we have used (6.4) and (6.5) to obtain (6.11). Using the definition of v_0 we have

$$\int_\Omega v_0(x, x/\epsilon) \cdot \nabla w_\epsilon(x)dx = \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon))\frac{\partial u_0}{\partial x_j}(x) \cdot \nabla w_\epsilon(x)dx,$$

and by the chain rule we obtain

$$\begin{aligned}\int_\Omega v_0(x, x/\epsilon) \cdot \nabla w_\epsilon dx &= \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla \left(\frac{\partial u_0}{\partial x_j} w_\epsilon(x)\right) dx \\ &- \int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \left(w_\epsilon \nabla \frac{\partial u_0}{\partial x_j}(x)\right) dx.\end{aligned}\tag{6.12}$$

In this paragraph we evaluate the first term on the right hand side of (6.12). Let $(\frac{\epsilon}{3}Y_i)_{i=1,\dots,i_m}$ be a finite set of translated cells of $\frac{\epsilon}{3}Y$, recovering $\overline{\Omega}$, and consider a partition of unity ρ_i, such that $\text{supp}\rho_i \subset \frac{2\epsilon}{3}Y_i$, where $\frac{2\epsilon}{3}Y_i$ denotes the cell $\frac{2\epsilon}{3}Y$ centered in $\frac{\epsilon}{3}Y_i$. We note that

$$\text{supp}(\rho_i w_\epsilon) \subset \frac{2\epsilon}{3}Y_i \cap \overline{\Omega} \subset \epsilon Y_i \tag{6.13}$$

hence

$$\begin{aligned}&\int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla(\frac{\partial u_0}{\partial x_j} w_\epsilon(x))dx = \\ &\sum_{i=1:i_m} \int_{\epsilon Y_i} a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \nabla(\rho_i \frac{\partial u_0}{\partial x_j} w_\epsilon(x))dx = 0.\end{aligned}\tag{6.14}$$

Here to obtain (6.14) we first note that u_0 has a stable extension to $W^{2,p}(\mathbb{R}^2)$, which we also denote u_0 applying (6.13) we obtain that the function $\rho_i \partial_{x_j} u_0 w_\epsilon$ is defined uniquely as zero outside of Ω and since $1/p + 1/q \leq 1/2$ we obtain $\rho_i \partial_{x_j} u_0 w_\epsilon \in W^{1,q'}(\mathbb{R}^2)$ for $1/q' = 1 - 1/q$. We then observe that $\chi^j \in W^{1,q}_{per}(Y)$, $H^1_{per}(Y) \hookrightarrow W^{1,q'}_{per}(Y)$ and (2.2) implies

$$\int_Y a_{ij}(y) \partial_{y_l} (\chi^j - y_j) \partial_{y_m} \psi = 0, \quad \forall\, \psi \in W^{1,q'}_{per}(Y).$$

Finally, since $\rho_i \partial_{x_j} u_0 w_\epsilon$ has a compact support contained in the interior of ϵY_i, see (6.13), then $\rho_i \partial_{x_j} u_0 w_\epsilon \in W^{1,q'}_{per}(\epsilon Y_i)$ and (6.14) follows.

For the second term on the right hand side of equation (6.12), we use the definition of v_0 and it follows that

$$-\int_\Omega a(x/\epsilon)(e_j - \nabla_y \chi^j(x/\epsilon)) \cdot \left(w_\epsilon \nabla \frac{\partial u_0}{\partial x_j}(x) \right) dx = -\int_\Omega \nabla_x \cdot v_0(x, x/\epsilon) w_\epsilon(x) dx.$$

Hence

$$\int_\Omega v_0(x, x/\epsilon) \cdot \nabla w_\epsilon(x) dx = -\int_\Omega \nabla_x \cdot v_0(x, x/\epsilon) w_\epsilon(x) dx. \tag{6.15}$$

From Equations (6.10), (6.11) and (6.15) we obtain

$$\int_\Omega \eta_\epsilon \cdot \nabla w_\epsilon dx = 0,$$

and from (6.9)

$$\int_\Omega a(\cdot/\epsilon) \nabla w_\epsilon \cdot \nabla z_\epsilon dx = \int_\Omega a(\cdot/\epsilon)(\nabla z_\epsilon - \eta_\epsilon) \cdot \nabla w_\epsilon) dx. \tag{6.16}$$

From Equations (6.8) and (6.16) we have

$$\begin{aligned} \left| \int_\Omega g(z_\epsilon - \epsilon\theta_\epsilon) dx \right| &\leq c \| a(\cdot/\epsilon) \nabla z_\epsilon - \eta_\epsilon \|_0 \| w_\epsilon \|_1 \\ &\leq \epsilon \| v_1(\cdot, \cdot/\epsilon) - a(\cdot/\epsilon) \nabla_x u_1(\cdot, \cdot/\epsilon) \|_0 \| g \|_{-1} \quad \text{by (6.6)}. \end{aligned}$$

Dividing by $\|g\|_{-1}$ and taking the supremum for $g \neq 0$ we get

$$\begin{aligned} \| z_\epsilon(x) - \epsilon\theta_\epsilon \|_1 &\leq c\epsilon \| v_1(\cdot, \cdot/\epsilon) - a(\cdot/\epsilon) \nabla_x u_1(\cdot, \cdot/\epsilon) \|_0 \\ &\leq c\epsilon (\| \chi^1 \|_{1,q,Y} + \| \chi^2 \|_{1,q,Y}) \| u_0 \|_{2,p}. \end{aligned}$$

□

The following remark is used in the proof of Proposition 6.5.

REMARK 6.1. *Let $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ and define $u_\epsilon \in H^1(\Omega)$ as the weak solution of the following problem*

$$L_\epsilon u_\epsilon = f \quad in \ \ \Omega, \quad u_\epsilon = g \quad on \ \ \partial\Omega.$$

It is easy to see that Proposition 6.1 extends immediately to this case if u_0, defined as the solution of

$$-\nabla . A \nabla u_0 = f \quad in \ \ \Omega, \qquad u_0 = g \quad on \ \ \partial\Omega,$$

belongs to $W^{2,p}(\Omega)$.

The following corollary follows from Proposition 6.1 and is used in the proof of Proposition 6.5.

Corollary 6.1. *Let* $\Omega \subset \mathbb{R}^d$, $d = 2,3$, *be a convex domain,* u_ϵ *and* u_0 *be defined by Equations (1.1) and (2.4), respectively. Assume* $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{m,p}(\Omega)$ *and* $\chi^j \in W^{1,q}_{per}(Y)$ *for* $(m-1)p > 2$ *and* $1/p + 1/q \leq 1/2$. *Then there exists a constant c independent of* u_0 *and* ϵ *such that*

$$\|u_\epsilon - u_0\|_0 \leq c\epsilon \|u_0\|_{m,p}.$$

Proof. The hypothesis $u_0 \in W^{m,p}(\Omega)$, $(m-1)p > d$ implies $\partial_{x_i} u_0 \in C(\Omega)$, and $\chi^j \in C(Y)$ see Remark 2.1, therefore $\|u_1\|_0 \leq c\|u_0\|_{m,p}$. From the maximum principle $\|\theta_\epsilon\|_{0,\infty} \leq \|\partial_{x_i} u_0\|_{0,\infty,\partial\Omega} \|\chi^i\|_{0,\infty,\partial\Omega}$, and hence the corollary follows from Proposition 6.1. □

The following proposition estimates the H^1 norm of $\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon$, and is used in the proof of Theorem 6.1.

Proposition 6.2. *Let* u_0, $\tilde{\theta}_\epsilon$ *and* $\tilde{\phi}_\epsilon$ *be defined by Equations (2.4), (2.9) and (2.13), respectively, and the functions* v_k *be defined as in Subsection 2.2.1. Assume* $u_0 \in W^{2,p}(\Omega)$, *and* v_e *and* $\nabla(v_e - \chi^*_e) exp(-\gamma y_1) \in L^s(G_e)$ *for* $s \geq 2$ *and* $1/s + 3/p \leq 1$. *We also assume similar hypothesis for the other functions* v_k. *Then there exists positive constants* $0 < \delta(p,s) \leq 1/2$, *and* $c(\delta,\gamma)$ *independent of* ϵ *such that*

$$\begin{aligned}\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_1 \leq c(\delta,\gamma)\epsilon^\delta \|a\|_{0,\infty} \|u_0\|_{2,p} \max_k \big(&\|\nabla(v_k - \chi^*_k) exp(-\gamma y \cdot \eta^k)\|_{0,s,G_k} \\ &+ \|v_k - \chi^*_k\|_{1,s,G_k}\big).\end{aligned}$$

In addition, when $p, s \to \infty$ *then* $\delta \to 1/2$ *with* $c(\delta,\gamma)$ *bounded independent of* δ.

Proof. By definition

$$\|\tilde{\theta}_\epsilon - \tilde{\phi}_\epsilon\|_1 \leq \sum_{k \in \{e,w,n,s\}} \|\tilde{\theta}^k_\epsilon - \tilde{\phi}^k_\epsilon\|_1.$$

Consider the case $k = e$, the other cases are treated in a similar way. We denote $v^\epsilon_e(x) = v_e(\frac{x_1 - 1}{\epsilon}, \frac{x_2}{\epsilon})$ and $a^\epsilon(x) = a(x/\epsilon)$, and let $g \in H^1_0(\Omega)$. Then applying the definition of $\tilde{\phi}^e_\epsilon$ we obtain

$$\begin{aligned}\int_\Omega a^\epsilon \nabla(\tilde{\theta}^e_\epsilon - \tilde{\phi}^e_\epsilon)\nabla g dx &= \int_\Omega -a^\epsilon \nabla\left((v^\epsilon_e - \chi^*_e)\varphi_e \frac{\partial u_0}{\partial x_1}\right)\nabla g dx \\ &= -\int_\Omega \left(\varphi_e \frac{\partial u_0}{\partial x_1} a^\epsilon \nabla(v^\epsilon_e - \chi^*_e)\right)\nabla g dx - \int_\Omega \left((v^\epsilon_e - \chi^*_e) a^\epsilon \nabla\left(\varphi_e \frac{\partial u_0}{\partial x_1}\right)\right)\nabla g dx. \end{aligned} \tag{6.17}$$

We note that due to the Sobolev embedding Theorem 5.4 from [1], the integrals above are well defined. For the first term on the right hand side of Equation (6.17) we have

$$\begin{aligned}&\int_\Omega \left(\varphi_e \frac{\partial u_0}{\partial x_1} a^\epsilon \nabla(v^\epsilon_e - \chi^*_e)\right)\nabla g dx = \\ &\int_\Omega a^\epsilon \nabla(v^\epsilon_e - \chi^*_e)\nabla\left(\varphi_e \frac{\partial u_0}{\partial x_1} g\right) dx - \int_\Omega a^\epsilon \nabla(v^\epsilon_e - \chi^*_e) \cdot g \nabla\left(\varphi_e \frac{\partial u_0}{\partial x_1}\right) dx.\end{aligned} \tag{6.18}$$

We now estimate the first term of the right hand side of (6.18). Let $I_i = \{(i-1)\epsilon/6 - \epsilon/6 < x_2 < i\epsilon/6 + \epsilon/6, \}$, $i_m = 1 + \sup_{i \in \mathbb{N}}(i3/\epsilon < 1)$, and consider a partition of unity ρ_i of Ω, subject to $(0,1) \times I_i$. Let I_i^ϵ be the interval centered in I_i with $|I_i^\epsilon| = \epsilon$. Since $\text{supp}(\rho_i g) \subset [0,1] \times I_i^\epsilon$ we have

$$\int_\Omega a^\epsilon \nabla(v_e^\epsilon - \chi_e^*) \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1} g \right) dx = \sum_{i=0:i_m} \int_0^1 \int_{I_i^\epsilon} a^\epsilon \nabla(v_e^\epsilon - \chi_e^*) \nabla \left(\rho_i \varphi_e \frac{\partial u_0}{\partial x_1} g \right) dx_2 dx_1 = 0, \tag{6.19}$$

where to arrive in (6.19) we have used the definition of v_e and arguments similar to the ones used to obtain (6.14).

For the second term on the right hand side of Equation (6.18), we apply a Cauchy inequality to obtain

$$\left| \int_\Omega a^\epsilon \nabla(v_e^\epsilon - \chi_e^*) \cdot \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1} \right) g dx \right| \leq \|a\|_\infty |\varphi_e \nabla u_0|_{1,p} \left\| \nabla v_e^\epsilon \exp(-\gamma \frac{x_1 - 1}{\epsilon}) \right\|_{0,s} \left(\frac{\epsilon}{\gamma} \right)^{1/l} \left\| (\gamma/\epsilon)^{1/l} \exp(\gamma \frac{x_1 - 1}{\epsilon}) g \right\|_{0,l}, \tag{6.20}$$

where $1/l = 1 - 1/p - 1/s$. Taking $y_1 = (x_1 - 1)/\epsilon$ and $y_2 = x_2/\epsilon$, and exploring the $[0,1]$-periodicity of $v_e(y_1, \cdot)$ we have

$$\begin{aligned} \left\| \nabla(v_e^\epsilon - \chi_e^*) \exp(-\gamma \frac{x_1 - 1}{\epsilon}) \right\|_{0,s}^s &\leq (\frac{1}{\epsilon} + 1) \int_{-1/\epsilon}^0 \int_0^1 |\nabla_y v_e \exp(-\gamma y_1)|^s \epsilon^{2-s} dy_2 dy_1 \\ &\leq c\epsilon^{(1-s)} \|\nabla_y v_e \exp(-\gamma y_1)\|_{0,s,G_e}^s. \end{aligned} \tag{6.21}$$

Let $g_n \in C_0^\infty(\Omega)$, $g_n \to g$ in H^1 and $I_n = (0,1) \cap |g_n| > 0$, then integrating by parts in x_1

$$\left\| (\gamma/\epsilon)^{1/l} \exp(\gamma \frac{x_1 - 1}{\epsilon}) g_n \right\|_{0,l} = \left(\int_0^1 \int_{I_n} \frac{\gamma}{\epsilon} \exp(l\gamma \frac{x_1 - 1}{\epsilon}) |g_n|^l dx_1 dx_2 \right)^{1/l}$$

$$= \left(-\int_0^1 \int_{I_n} \frac{1}{l} \exp\left(l\gamma \frac{x_1 - 1}{\epsilon} \right) \frac{\partial |g_n|^l}{\partial x_1} dx_1 dx_2 \right)^{1/l} \tag{6.22}$$

$$\leq c \left(\left\| \exp\left(l\gamma \frac{x_1 - 1}{\epsilon} \right) \right\|_{0,r'} \|g_n\|_{0,s'(l-1)}^{l-1} \left\| \frac{\partial g_n}{\partial x_1} \right\|_0 \right)^{1/l} \tag{6.23}$$

$$\leq c(\Omega)(s'(l-1))^{(l-1)/l} \left(\frac{\epsilon}{r' l \gamma} \right)^{1/(r'l)} |g_n|_1^2. \tag{6.24}$$

To obtain (6.23) we have used a Cauchy inequality with $1/r' + 1/s' = 1/2$. In order to obtain (6.24), we note that the last inequality in the proof of Lemma 5.10 in [1] states

$$\begin{aligned} \|g_n\|_{0,s'(l-1)} &\leq 2^{(t-1)/t} \left(\frac{2t - t}{2 - t} \right) \|g_n\|_{1,t}, \quad \text{for} \;\; 2t/(2-t) = s'(l-1),\; 1 \leq t < 2 \\ &\leq 2^{(t-1)/t} \left(\frac{2t - t}{2 - t} \right) \text{vol}(\Omega)^{(1/t - 1/2)} \|g_n\|_1, \;\text{by Theorem 2.8 in [1]} \\ &\leq c(\Omega) \left(\frac{2t - t}{2 - t} \right) |g_n|_1, \;\text{by a Poincare inequality.} \end{aligned}$$

Hence (6.24) follows from (6.23). Taking the limit $n \to \infty$ we obtain inequality (6.24) for g.

Since $1/s + 3/p < 1$, there exists $r' > 2$ such that $1/lr' + 1/l + 1/s - 1 > 0$, and hence from (6.18), (6.19), (6.20), (6.21), and (6.24) it follows

$$\int_\Omega \varphi_e \frac{\partial u_0}{\partial x_1} a^\epsilon \nabla (v_e^\epsilon - \chi_e^*) \nabla g dx \leq c(\Omega, \gamma)(s'(l-1))^{(l-1)/l} \epsilon^{\delta'} \|a\|_\infty |\varphi_e \nabla u_0|_{1,p}$$
$$\|\nabla (v_e - \chi_e^*) \exp(-\gamma y_1)\|_{0,s,G_e} |g|_1, \tag{6.25}$$

where $\delta' = 1/lr' + 1/l + 1/s - 1$.

For estimating the second term on the right hand side of (6.17), we apply a Cauchy inequality with $1/r + 1/p = 1/2$ to obtain

$$\left| \int_\Omega (v_e^\epsilon - \chi_e^*) a^\epsilon \nabla \left(\varphi_e \frac{\partial u_0}{\partial x_1} \right) \cdot \nabla g dx \right| \leq \|a\|_{0,\infty} \left| \varphi_e \frac{\partial u_0}{\partial x_1} \right|_{1,p} \left(\epsilon \int_{G_e} (v_e - \chi_e^*)^r dy \right)^{1/r} |g|_1$$
$$\leq c(r) \epsilon^{1/r} \|a\|_{0,\infty} \left| \varphi_e \frac{\partial u_0}{\partial x_1} \right|_{1,p} \|v_e^\epsilon - \chi_e^*\|_{1,G_e} |g|_1, \tag{6.26}$$

where we have used the Sobolev embedding Theorem 5.4 in [1] to obtain the last inequality.

Taking $g = \tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e$ and using the ellipticity of a

$$\begin{aligned} |\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e|^2_{H_0^1(\Omega)} &\leq \gamma_a^{-1} \int_\Omega (a^\epsilon \nabla (\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e)) \cdot \nabla (\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e) dx \\ &\leq \frac{c(r)}{\gamma_a} \epsilon^\delta \|a\|_{0,\infty} |\varphi_e \nabla u_0|_{1,p} \left(\|\nabla (v_e - \chi_e^*) \exp(-\gamma y_1)\|_{0,s,G_e} \right. \\ &\quad \left. + \|\nabla (v_e - \chi_e^*)\|_{1,G_e} \right) |\tilde{\theta}_\epsilon^e - \tilde{\phi}_\epsilon^e|_{H_0^1(\Omega)}, \end{aligned}$$

where $\delta = \min\{\delta', 1/r\}$.

Observe that $s, p \to \infty$ implies $l \to 1$. Choosing $s' = 1/(l-1)$ in Inequality (6.24) we have that $(s'(l-1))^{(l-1)/l} \left(\epsilon/(r' l \gamma) \right)^{1/(r'l)} \to \epsilon^{1/2}/(2\gamma)$. In inequality (6.26) $p \to \infty$ implies $1/r \to 1/2$ and $c(r)\epsilon^{1/r} \to c\epsilon^{1/2}$. ∎

Finally, we prove the last proposition used in the proof of Theorem 6.1. Proposition 6.3 estimates the H^1 norm of $\bar{\phi} - \bar{\theta}_\epsilon$,

Proposition 6.3. *Let Ω be a convex polygon, and the functions u_0, $\bar{\theta}_\epsilon$ and $\bar{\phi}$ be defined by Equations (2.4), (2.10) and (2.14), respectively. Assume that $u_0 \in H^2(\Omega)$, then there exists a positive constant c independent of ϵ and u_0 such that*

$$\|\bar{\phi} - \bar{\theta}_\epsilon\|_1 \leq c \frac{\|a\|_{0,\infty,Y}}{\gamma_a} \|u_0\|_2.$$

Proof. Consider the notation $a^\epsilon(x) = a(x/\epsilon)$, the same will be used for a_{ij}. Since $(\bar{\phi} - \bar{\theta}_\epsilon) = 0$ on $\partial\Omega$ we have

$$\begin{aligned} \int_\Omega a_{ij}^\epsilon \frac{\partial (\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_i} \frac{\partial (\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j} dx &= \int_\Omega a_{ij}^\epsilon \frac{\partial \bar{\phi}}{\partial x_i} \frac{\partial (\bar{\phi} - \bar{\theta}_\epsilon)}{\partial x_j} dx \\ &\leq \|a\|_{0,\infty,Y} \left(\int_\Omega |\nabla \bar{\phi}|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla (\bar{\phi} - \bar{\theta}_\epsilon)|^2 dx \right)^{1/2}, \end{aligned}$$

and from the ellipticity of a we obtain

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq \frac{\|a\|_{0,\infty,Y}}{\gamma_a} |\bar{\phi}|_1.$$

The regularity theory gives that $|\bar{\phi}|_1 \leq c\|\chi^* \partial_\eta u_0\|_{H^{1/2}(\partial\Omega)}$, and since Ω is a convex polygon by Remark 2.1

$$|\bar{\phi} - \bar{\theta}_\epsilon|_1 \leq c\|u_0\|_2.$$

The proposition follows from a Poincare inequality. ▯

6.2. Proof of Theorem 2.2. Use a triangular inequality similar to the one used in the proof of Theorem 2.1 and Propositions 6.4, 6.2 and 6.5. □

We now prove the propositions used in the proof of Theorem 2.2. The following proposition generalizes Proposition 2.3 from [34], where it is assumed $a_{ij} \in C^{1,\beta}_{per}(Y)$, $u_0 \in H^3(\Omega)$ and $\Omega \subset \mathbb{R}^2$. We note here that Theorem 1.1 from [32] gives conditions concerning the discontinuities of the functions a_{ij} such that χ^j and $\chi^{ij} \in W^{1,\infty}_{per}(Y)$.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.1), and χ^j, u_0, u_1, θ_ϵ and χ^{ij} be defined by Equations (2.2), (2.4), (2.5), (2.6) and (2.7), respectively. Assume $a_{ij} \in L^\infty_{per}(Y)$, $u_0 \in W^{3,p}(\Omega)$, χ^j and $\chi^{ij} \in W^{1,q}_{per}(Y)$, for $p, q > d$ and $1/p + 1/q \leq 1/2$. Then there exists a constant c independent of u_0 and ϵ such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_0 \leq C\epsilon^2 \|u_0\|_{3,p} (\max_j \|\chi^j\|_{0,q} + \max_{kj} \|\chi^{kj}\|_{1,q}).$$

Proof.
Define the field v_1 by

$$(v_1(x, y))_k = -a_{ki}(y)\chi^j \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + a_{kl}(y) \frac{\partial \chi^{ij}}{\partial y_l} \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x), \tag{6.27}$$

hence

$$a(y)\nabla_x u_1(x, y) + a(y)\nabla_y u_2(x, y) = v_1(x, y). \tag{6.28}$$

Let $q(y) = \phi(y)$, ϕ defined by Equation (6.3) and let $\psi_{ij} \in W^{1,q}_{\mathrm{per}}(Y)$ such that

$$curl_y \psi_{1j} = \tilde{\psi}_{1j} = \begin{pmatrix} -a_{11}\chi^j + a_{1l}\partial_l \chi^{1,j} - c^1_{1j} \\ -a_{21}\chi^j + a_{2l}\partial_l \chi^{1,j} - \phi^{(3)}_j - c^2_{1j} \\ -a_{31}\chi^j + a_{3l}\partial_l \chi^{1,j} + \phi^{(2)}_j - c^3_{1j} \end{pmatrix},$$

$$curl_y \psi_{2j} = \tilde{\psi}_{2j} = \begin{pmatrix} -a_{12}\chi^j + a_{1l}\partial_l \chi^{2,j} + \phi^{(3)}_j - c^1_{2j} \\ -a_{22}\chi^j + a_{2l}\partial_l \chi^{2,j} - c^2_{2j} \\ -a_{32}\chi^j + a_{3l}\partial_l \chi^{2,j} - \phi^{(1)}_j - c^3_{2j} \end{pmatrix}$$

and

$$curl_y \psi_{1j} = \tilde{\psi}_{3j} = \begin{pmatrix} -a_{13}\chi^j + a_{1l}\partial_l \chi^{3,j} - \phi^{(2)}_j - c^1_{3j} \\ -a_{23}\chi^j + a_{2l}\partial_l \chi^{3,j} + \phi^{(1)}_j - c^2_{3j} \\ -a_{33}\chi^j + a_{3l}\partial_l \chi^{3,j} - c^3_{3j} \end{pmatrix},$$

where the constants c_{ij}^l are chosen such that each entry of the vectors $\tilde{\psi}_{ij}$ has integral zero over Y, e.g. $c_{1j}^1 = \int_Y -a_{11}\chi^j + a_{1l}\partial_l \chi^{1,j} dy$. It is easy to check that $\nabla_y \cdot \tilde{\psi}_{kj} = 0$, what guarantees by Lemma 6.1 the existence of such functions ψ_{kj}, and by Remark 3.11 in [25] we have

$$\|\psi_{kj}\|_{1,q} \leq c(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{6.29}$$

Define

$$p(x,y) = \psi_{kj}(y)\frac{\partial^2 u_0}{\partial x_k \partial x_j}(x) \tag{6.30}$$

and let

$$v_2(x,y) = -curl_x p(x,y),$$

and a simple calculation gives

$$\nabla_y \cdot v_2 = -\nabla_x \cdot v_1, \quad \nabla_x \cdot v_2 = 0 \tag{6.31}$$

and

$$\begin{aligned} \|v_2(\cdot,\cdot/\epsilon)\|_0 &\leq c\|u_0\|_{3,p} \max_{kj} \|\psi_{kj}\|_{1,q,Y} \\ &\leq c\|u_0\|_{3,p}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}) \quad \text{by} \quad (6.29). \end{aligned} \tag{6.32}$$

Define

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon) - \epsilon^2 u_2(x, x/\epsilon)$$

and

$$\xi_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon) - \epsilon^2 v_2(x, x/\epsilon),$$

where v_0 is defined by (6.1). Then

$$\begin{aligned} a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon(x) &= a(x/\epsilon)\nabla u_\epsilon(x) - a(x/\epsilon)\nabla u_0(x) - \epsilon a(x/\epsilon)\nabla u_1(x, x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla u_2(x, x/\epsilon) \\ &\quad -a(x/\epsilon)\nabla u_\epsilon(x) + v_0(x, x/\epsilon) + \epsilon v_1(x, x/\epsilon) + \epsilon^2 v_2(x, x/\epsilon) \\ &= -a(x/\epsilon)\nabla_x u_0(x) - \epsilon a(x/\epsilon)\nabla_x u_1(x, x/\epsilon) - a(x/\epsilon)\nabla_y u_1(x, x/\epsilon) \\ &\quad -\epsilon^2 a(x/\epsilon)\nabla_x u_2(x, x/\epsilon) - \epsilon a(x/\epsilon)\nabla_y u_2(x, x/\epsilon) \\ &\quad +v_0(x, x/\epsilon) + \epsilon v_1(x, x/\epsilon) + \epsilon^2 v_2(x, x/\epsilon) \\ &= \epsilon^2(v_2(x, x/\epsilon) - a(x/\epsilon)\nabla_x u_2(x, x/\epsilon)), \quad \text{by (6.1), and (6.28).} \end{aligned}$$

From the definition of u_2 and (6.32) we obtain

$$\|a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon\|_0 \leq c\epsilon^2\|u_0\|_{3,p} \max_{kj}(\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}). \tag{6.33}$$

Define $\varphi_\epsilon \in H^1(\Omega)$ as the weak solution of

$$-\nabla \cdot a(x/\epsilon)\nabla\varphi_\epsilon = 0 \quad \text{in } \Omega, \quad \text{and} \quad \varphi_\epsilon(x) = u_2(x, x/\epsilon) \quad \text{on} \quad \partial\Omega. \tag{6.34}$$

We observe that the Sobolev embedding theorem and the hypothesis $p, q > d$, implies the function u_2 is continuous. Therefore, we use the maximum principle to obtain

$$\begin{aligned}\|\varphi_\epsilon\|_0 &\leq c\|\varphi_\epsilon\|_{0,\infty} \\ &\leq c\max_{ij}\|\chi^{ij}\|_{0,\infty,Y}\|\partial_{x_i x_j}u_0\|_{0,\infty} \\ &\leq c\max_{ij}\|\chi^{ij}\|_{1,q,Y}\|u_0\|_{3,p}. \qquad (6.35)\end{aligned}$$

Given $g \in L^2(\Omega)$, let $w_\epsilon \in H^1(\Omega)$ denotes the solution of

$$\int_\Omega a(x/\epsilon)\nabla w_\epsilon(x)\nabla\psi(x)dx = \int_\Omega g(x)\psi(x)dx, \quad \forall \psi \in H_0^1(\Omega). \qquad (6.36)$$

Since $\psi_\epsilon + \epsilon\theta_\epsilon + \epsilon^2\varphi_\epsilon \in H_0^1(\Omega)$ we obtain

$$\begin{aligned}\int_\Omega g(\psi_\epsilon + \epsilon\theta_\epsilon + \epsilon^2\varphi_\epsilon)dx &= \int_\Omega a(x/\epsilon)(\nabla\psi_\epsilon + \epsilon\nabla\theta_\epsilon + \epsilon^2\nabla\varphi_\epsilon)\nabla w_\epsilon(x)dx \\ &= \int_\Omega a(x/\epsilon)\nabla\psi_\epsilon\nabla w_\epsilon(x)dx, \qquad (6.37)\end{aligned}$$

where we have used the definition of θ_ϵ and φ_ϵ to obtain (6.37). We observe that

$$\int_\Omega a^\epsilon\nabla\psi_\epsilon\nabla w_\epsilon dx = \int_\Omega (a^\epsilon\nabla\psi_\epsilon - \xi_\epsilon)\cdot\nabla w_\epsilon dx + \int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon dx, \qquad (6.38)$$

and we estimate the second term on the right hand side of (6.38) as follows

$$\begin{aligned}\int_\Omega \xi_\epsilon\cdot\nabla w_\epsilon dx &= \int_\Omega (a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x,x/\epsilon) - \epsilon v_1(x,x/\epsilon) \\ &\quad -\epsilon^2 v_2(x,x/\epsilon))\cdot\nabla w_\epsilon(x)dx \\ &= \int_\Omega f w_\epsilon(x) + \nabla_x\cdot v_0(x,x/\epsilon)w_\epsilon(x) \\ &\quad -\epsilon v_1(x,x/\epsilon)\cdot\nabla w_\epsilon(x) + \epsilon\nabla_x v_1(x,x/\epsilon)w_\epsilon(x)dx, \qquad (6.39)\end{aligned}$$

here we used the definition of u_ϵ, (6.15), integration by parts and (6.31) to obtain (6.39). Using (6.27) we have

$$\begin{aligned}\int_\Omega v_1(x,x/\epsilon)\cdot\nabla w_\epsilon(x) = \int_\Omega \bigg(&-a_{ki}^\epsilon\chi_\epsilon^j\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x) \\ &+ a_{kl}^\epsilon\frac{\partial\chi_\epsilon^{ij}}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}(x)\bigg)\frac{\partial w_\epsilon}{\partial x_k}(x)dx. \qquad (6.40)\end{aligned}$$

Consider the partition of unit ρ_i defined in the proof of Proposition 6.1, then

$$\int_\Omega a_{kl}^\epsilon\frac{\partial\chi_\epsilon^{ij}}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}(x)dx =$$

$$= \sum_1^{i_m}\int_{\epsilon Y_i} a_{kl}^\epsilon\frac{\partial\chi_\epsilon^{ij}}{\partial y_l}\rho_i\frac{\partial^2 u_0}{\partial x_j\partial x_i}\frac{\partial w_\epsilon}{\partial x_k}dx$$

$$= \sum_1^{i_m} \int_{\epsilon Y_i} a_{kl}^{\epsilon} \frac{\partial \chi_\epsilon^{ij}}{\partial y_l} \frac{\partial}{\partial x_k} \left(\rho_i \frac{\partial^2 u_0}{\partial x_j \partial x_i} w_\epsilon(x) \right) - a_{kl}^{\epsilon} \frac{\partial \chi_\epsilon^{ij}}{\partial y_l} w_\epsilon(x) \frac{\partial}{\partial x_k} \left(\rho_i \frac{\partial^2 u_0}{\partial x_j \partial x_i} \right) dx$$

$$= \sum_1^{i_m} \int_{\epsilon Y_i} \epsilon^{-1} \left(a_{ij}^{\epsilon} - a_{ik}^{\epsilon} \frac{\partial \chi_\epsilon^{j}}{\partial_{y_k}} + A_{ij} \right) \rho_i \frac{\partial^2 u_0}{\partial x_j \partial x_i} w_\epsilon$$

$$+ a_{ki}^{\epsilon} \chi_\epsilon^j \left(\frac{\partial}{\partial_{x_k}} \left(\rho_i \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) \right) w_\epsilon(x) + \rho_i \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) \frac{\partial w_\epsilon}{\partial_{x_k}}(x) \right) dx$$

$$- \int_\Omega a_{kl} \frac{\partial \chi_\epsilon^{ij}}{\partial y_l} w_\epsilon(x) \frac{\partial}{\partial x_k} \left(\frac{\partial^2 u_0}{\partial x_j \partial x_i} \right) dx \tag{6.41}$$

$$= \int_\Omega \epsilon^{-1} \left(\nabla_x v_0 \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) - f \right) w_\epsilon(x) dx$$

$$- \int_\Omega a_{ki}^{\epsilon} \chi_\epsilon^j \frac{\partial^2 u_0}{\partial x_j \partial x_i} \frac{\partial w_\epsilon}{\partial_{x_k}}(x) dx - \int_\Omega \nabla_x \cdot v_1 dx. \tag{6.42}$$

Here we used the definition of χ^{ij} to arrive in (6.41), and from (6.39), (6.40) and (6.42) we obtain

$$\int_\Omega \xi_\epsilon \cdot \nabla w_\epsilon(x) dx = 0,$$

and hence from (6.33) and (6.38)

$$\left| \int_\Omega g(\psi_\epsilon + \epsilon \theta_\epsilon + \epsilon^2 \varphi_\epsilon) dx \right| \leq \|a^\epsilon \nabla \psi_\epsilon - \xi_\epsilon)\|_0 \|w_\epsilon\|_1$$
$$\leq c\epsilon^2 \|u_0\|_{3,p} (\|\chi^j\|_{0,q,Y} + \|\chi^{kj}\|_{1,q,Y}) \|g\|_{-1}.$$

Dividing by g and taking the supremum over g, we have

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon \theta_\epsilon - \epsilon^2 u_2 - \epsilon^2 \varphi_\epsilon\| \leq c\epsilon^2 \|u_0\|_{3,p} \max_{kj} (\|\chi^j\|_{0,q} + \|\chi^{kj}\|_{1,q}).$$

Observe that $u_2(x, x/\epsilon)$ and $\varphi_\epsilon(x)$ are bounded in $L^2(\Omega)$ by $\|u_0\|_{3,p} \max_{kj} \|\chi^{kj}\|_{1,q}$, independent of ϵ, see (6.35). Hence

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon \theta_\epsilon\| \leq c\epsilon^2 \|u_0\|_{3,p} (\max_j \|\chi^j\|_{0,q} + \max_{kj} \|\chi^{kj}\|_{1,q}).$$

□

The following proposition estimates the L^2 norm of $\bar{\phi} - \bar{\theta}_\epsilon$, and it is used in the proof of Theorem 2.2

Proposition 6.5. *Let u_0, χ^j, $\bar{\theta}_\epsilon$ and $\bar{\phi}$ be defined by (2.4), (2.2), (2.10) and (2.14), respectively. Assume that $u_0 \in W^{3,p}(\Omega)$, $\bar{\phi} \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{per}(Y)$, for $1/p + 1/q \leq 1/2$. Then we have*

$$\|\bar{\theta}_\epsilon - \bar{\phi}\|_0 \leq c\epsilon \|u_0\|_{3,p}.$$

Proof. Observe that $\bar{\phi} \in W^{2,p}(\Omega)$ and $p \geq 2$, hence from Corollary 6.1 and Remark 6.1 we obtain

$$\|\bar{\theta}_\epsilon - \bar{\phi}\|_0 \leq c\epsilon \|\bar{\phi}\|_{2,p}.$$

Since

$$\bar{\phi}|_{\partial\Omega} = \sum_k \varphi_k \chi_k^* \nabla u_0 \cdot \eta_k|_{\partial\Omega},$$

by regularity theory, see Theorems 4.3.1.4 and 4.3.2.4 [24], $\|\bar{\phi}\|_{2,p} \leq c(\chi^*)\|u_0\|_{3,p}$, and the proposition follows. □

6.3. Proof of Theorem 2.3. Use a triangular inequality similar to the one used in the Proof of Theorem 2.1 and Propositions 6.6, 6.2 and 6.5. Observe that if $a_{ij} \in C^{1,\beta}_{per}(Y), \beta > 0$, by regularity theory $\chi^j \in C^{1,\beta}_{per}$, $v_e \in C^{1,\beta}$ and $\nabla(v_e - \chi_e^*)\exp(-\gamma y_1) \in L^\infty(G_e)$; see Theorem 15.1 in [30] and Remark 6.4 in [34]. By the Sobolev embedding theorem $u_0 \in W^{2,\infty}(\Omega)$, hence Proposition 6.2 holds for $\delta = 1/2$. □

The following proposition is used in the proof of Theorem 2.3. Proposition 6.6 generalizes Proposition 2.3 from [34] to the case $\Omega \subset \mathbb{R}^3$.

Proposition 6.6. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a convex domain, u_ϵ be the solution of Problem (1.1), and u_0, u_1, and θ_ϵ be defined by Equations (2.4), (2.5) and (2.6), respectively. Assume $a_{ij} \in C^{1,\beta}(Y), \beta > 0$ and $u_0 \in H^3(\Omega)$. Then there exists a constant c independent of u_0 and ϵ, such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_0 \leq C\epsilon^2\|u_0\|_3.$$

Proof. Since $a_{ij} \in C^{1,\beta}(Y)$ by regularity theory $\chi^i \in C^{2,\beta}(Y)$, $\chi^{ij} \in C^1(Y)$ and by Theorem 3 in [7] we obtain

$$\|\varphi_\epsilon\|_0 \leq c\|u_2(\cdot, \cdot/\epsilon)\|_{0,\partial\Omega} \leq c\|u_0\|_3\|\chi^i j\|_{0,\infty},$$

where the function φ_ϵ is defined by (6.34) and we have used the trace theorem in the last inequality. The rest of the proof of follows exactly as the proof of Proposition 6.4. □

6.4. Auxiliary Result . The following lemma is used in the proof of Propositions 6.1 and 6.4.

Lemma 6.1. *A function $\mathbf{v} \in L^2_{per}(Y)^2$, $(\mathbf{v} \in L^2_{per}(Y)^3)$ satisfies*

$$\nabla \cdot \mathbf{v} = \mathbf{0}, \tag{6.43}$$

and $\int_Y v_i dy = 0$ iff there exists a function $\phi \in H^1_{per}(Y)$ $(\phi \in H^1_{per}(Y)^3)$ such that:

$$\mathbf{v} = curl\phi. \tag{6.44}$$

Proof. Similar to the proof of Theorem 3.4 from [25] using discrete Fourier transforms rather than continuous Fourier transforms, see [41]. □

7. Conclusions. We perform the convergence analysis for the proposed numerical method for approximating the solution of Equation (1.1). The error estimates obtained in the numerical experiments agree with the theoretical errors estimates from Theorems 4.1 and 4.2. The method presented here is strongly based on the periodicity of the coefficients a_{ij}, and for this reason it has relative low computational cost with optimal error convergence rate.

We generalize results found in the literature for estimating the error between u_ϵ and its first order asymptotic expansion $u_0 + \epsilon u_1$ approximation plus the boundary corrector term θ_ϵ. Such generalization permit us to develop sharp finite element error estimates with very weak assumptions on the regularity of $a(y)$, including composite materials applications.

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