Density of hyperbolicity and homoclinic bifurcations for attracting topologically hyperbolic sets.

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Abstract

Given a topologically hyperbolic attracting set of a smooth three dimensional Kupka-Smale diffeomorphism, it is proved under some hypothesis over the dissipation rate, that the set is either hyperbolic or the diffeomorphisms is C^1- approximated by another one exhibiting either a heterodimensional cycle or a homoclinic tangency.

1 Introduction and statements.

In the direction to describe the long range behavior of trajectories for "most" systems (residual, dense, etc.) within the space of all dynamical systems, once of the goal is to identify the dynamical mechanism underlying any generic behavior.

It was briefly thought in the sixties that this could be realized by the so-called hyperbolic ones. Under this assumption, it is proved that the limit set decomposes into a finite number of disjoint transitive sets such that the asymptotic behavior of any orbit is described by the dynamics in the trajectories in those finite transitive sets (see [S]).

Hyperbolicity was soon realized to be a less universal property than was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. Two mechanisms were identified that leads to generic (meaning generic perturbation of the initial system) nonhyperbolic behavior:

1. *heterodimensional cycle*, meaning the presences of two periodic points of different indices linked through the intersection of their stable and unstable manifolds (see [AS], [Sh], [D1]);

2. *homoclinic tangency*, meaning non-transversal intersection of the stable and unstable manifold of a periodic point (see [N1], [N2], [N3]).

The mentioned mechanisms are relevant due to the dynamical consequences involving their presences:

1. the first mechanism is related to the existence of non-hyperbolic robust transitive systems (see [D1], [BDPR]);

2. the second ones related to the existence of residual subsets of diffeomorphisms displaying infinitely many periodic attractors.

In the early 80's Palis conjectured (see [P] and [PT]) that those bifurcations are the maim obstruction to hyperbolicity:

Every C^r diffeomorphism of a compact manifold M can be C^r approximated by one which is hyperbolic or by one exhibiting a heterodimensional cycle or by one exhibiting a homoclinic tangency.

To be precise, a hyperbolic diffeomorphism means a diffeomorphism such that its limit set (the closure of the accumulation points of any orbit) is hyperbolic. A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \leq C\lambda^n \text{ and } |Df^{-n}_{/E^u(x)}| \leq C\lambda^n \;\; \forall x \in \Lambda, \;\; n \in \mathbb{N}.$$

Moreover, a diffeomorphism is called Axiom A, if the non-wandering set is hyperbolic and it is the closure of the periodic points.

We recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \; dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \; dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f. A point of intersection of these manifolds is called a homoclinic point.

Definition 1 Homoclinic tangency. *We say that f exhibits a homoclinic tangency if there is a periodic point p such that there is a point $x \in W^s(p) \cap W^u(p)$ with $T_xW^s(p) + T_xW^u(p) \neq T_xM$. Given an open set V, we say that the tangency holds in V if p and x belongs to V.*

The above conjecture was proved to be true for the case of surfaces and the C^1 topology (see [PS1]).

Theorem ([PS1]): *Let M^2 be a two dimensional compact manifold. Every $f \in Diff^1(M^2)$ can be C^1-approximated either by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism*

In dimensions higher than two, the theorem stated above is false, due to another kind of homoclinic bifurcation that breaks the hyperbolicity in a robust way: the so-called heterodimensional cycles (see [D1] and [D2]).

Definition 2 Heterodimensional cycle. *We say that f exhibits a heterodimensional cycle if there are two hyperbolic periodic points q and p of different stable index (the number, counted with multiplicity, of contractive eigenvalues), such that $W^u(q) \cap W^s(p) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$. Given an open set V, we say that the cycle holds in V if p, q and the points where the stable and unstable manifolds intersects belongs to V.*

It is remarkable to say that for a compact manifold with dimension larger and equal than three, there are C^1-open sets of diffeomorphisms containing a dense set of diffeomorphisms exhibiting a tangency and a dense set of diffeomorphisms exhibiting a heterodimensional cycle (see [D1] and [BD]). On the other hand, the conjecture formaulated by Palis, states that the systems exhibiting either a tangency or a heterodimensional cycle are dense in the complement of the hyperbolic ones.

A weak form of hyperbolicity introduced independently by Mañé, Liao and Pliss, as a first step in the attempt to prove that structurally stable systems satisfy a hyperbolic condition on the tangent map, is the so called dominated splitting:

Definition 3 *An f-invariant set Λ is said to have a dominated splitting, if the tangent bundle over Λ is decomposed in two invariant subbundles $T_\Lambda M = E \oplus F$, and such that there exist $C > 0$ and $0 < \lambda < 1$ with the following property:*

$$|Df^n_{|E(x)}||Df^{-n}_{|F(f^n(x))}| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

If the bundle $T_\Lambda M$ is decomposed in more than two directions, i.e.: if $T_\Lambda M = \oplus_{i=1}^k E_i$ then it is said that the decomposition is a dominated splitting if for any $1 \leq j \leq k-1$ follows that

$$|Df^n_{|\oplus_{i=1}^j E_i(x)}||Df^{-n}_{|\oplus_{i=j+1}^k E_i(f^n(x))}| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

Related to the notion of dominated splitting, there is a well known result proved in [HPS] that states that for any point $x \in \Lambda$ there are manifolds $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$ (not dynamically defined) tangents to the subbundles E and F respectively, which are usually called *local tangent manifolds*. It is natural to ask which is the relation of this tangent submanifolds with the local stable and unstable manifolds.
To precise, let us first recall the definition of local stable and unstable manifold of size ϵ (where ϵ is a positive constant):

$$W^s_\epsilon(x) = \{y \in M : \ dist(f^n(y), f^n(x)) \to 0 \, n \to \infty, dist(f^n(y), f^n(x)) < \epsilon\},$$

$$W^u_\epsilon(x) = \{y \in M : \ dist(f^n(y), f^n(x)) \to 0 \, n \to -\infty, dist(f^n(y), f^n(x)) < \epsilon\}.$$

To be concise, $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are called the local stable and unstable manifold respectively.

Observe that if Λ is hyperbolic, then follows that the tangent manifolds to E and F are contained in the local stable and unstable manifold respectively. However, the converse is false: it may happen that the tangent manifolds are dynamically defined and Λ is not hyperbolic. Taking into account this observation, we introduce the next definition:

Definition 4 Topologically hyperbolic sets: *Given a compact invariant set exhibiting a dominated splitting $E \oplus F$, it is said that the set Λ is a topologically hyperbolic set if it is maximal invariant (i.e.: $\Lambda = \cap_{\{n \in \mathbb{Z}\}} f^n(U)$ for some neighborhood U) and the local tangent manifold to E is contained in the local stable manifold and the local tangent manifold to F is contained in the local unstable manifold. It this case, it is said that E is topologically contractive and F is topologically expansive.*

In other words, it is said that a compact invariant set Λ is topologically hyperbolic if it is maximal invariant and for each point, the local stable and unstable manifolds are two complementary submanifolds of size independent of the point. Roughly speaking we may say that a set Λ is topologically hyperbolic if its dynamic is conjugated to a hyperbolic dynamic. In fact, topologically hyperbolic sets share the same dynamical properties of the hyperbolicity. In particular, transitive topologically hyperbolic sets are homoclinic classes (see subsection 2 for details).

With this definition in mind we could reformulates the Palis's conjecture in the following terms: *Every C^r diffeomorphism of a compact manifold M can be C^r approximated by one which is topologically hyperbolic (its limit set is decomposed in a finite number of topologically hyperbolic invariant set) or by one exhibiting a heterodimensional cycle or by one exhibiting a homoclinic tangency.*

In the paper [Pu] it is proved that this weak version of the Palis's conjecture holds for *attracting homoclinic class of a smooth diffeomorphisms acting on a three dimensional compact manifold.* To be precise, first we have to introduce more definitions.

Definition 5 Homoclinic class. *Given a periodic point p, we define the homoclinic class associated to p as the closure of the set $\{W^s(p) \cap W^u(p)\}$.*

Definition 6 Attracting homoclinic class. *Given a homoclinic class we say that H_p is an attracting homoclinic class if there exists an open set U such that $H_p \subset U$ and $H_p = \cap_{n>0} f^n(U)$*

Different kind of examples of three dimensional attracting homoclinic classes can be found: the solenoid attractor, the Plykin attractor (both hyperbolic), the Henon attractor (that it can be approximated by a map exhibiting a tangency; see [BeCa], [V] and [U]), or partially hyperbolic attractors (which can be approximated by a map exhibiting a heterodimensional cycle; see [M], [BD] and [BV] for these kind of examples).

In [Pu] the following theorem was proved:

Theorem A ([Pu]): *Let $f \in Diff^2(M^3)$. Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class such that all the periodic points in H_p are hyperbolic. Then it follows that either*

1. *H_p is hyperbolic or*
2. *there exists g C^1-arbitrarily close to f exhibiting a homoclinic tangency in U or,*
3. *there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U or,*
4. *H_p is a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive, E_1 is contractive, E_2 is a one dimensional subbundle and E_3 is topologically expansive*

To get a complete answer to the Palis's conjecture for the case of attracting homoclinic class, we have to deal with the last alternative of theorem A. The answer in this case is given under an extra hypothesis related to the rate of dissipation.

Definition 7 Normally dissipative invariant sets. *Let H be a topologically hyperbolic set exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$. If there exists $d < 1$ such that for any $x \in H$ holds that*

$$|Df_{|E_1(x)}| \frac{|Df_{|E_3(x)}|}{|Df_{|E_2(x)}|} < d.$$

then we say that H is normally dissipative.

In few words it is said that a compact invariant set Λ is normally dissipative if the rate of contraction along the direction the direction E_1 is smaller than the rate of domination between the direction E_2 and E_3. The normal dissipative condition it is used to prove that the strong stable foliation is C^1.

Maim theorem: *Let $f \in Diff^2(M^3)$. Let $H = \cap_{n>0} f^n(U)$ be an attracting transitive topologically hyperbolic set exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive and E_3 is topologically expansive. Let us also assume that H is normally dissipative and all the periodic points in H are hyperbolic. Then it follows that either*

1. *H is hyperbolic or*
2. *there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U.*

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2 Preliminaries to maim theorem.

First we introduce a series of results about topologically hyperbolic homoclinic classes. Some of these results are proved in [Pu] and are related to the dynamics and structure of topologically hyperbolic sets and the continuation of them for perturbed systems. In subsection 2.1 we states some results about invariant manifolds of topologically hyperbolic set and the continuation of those manifolds for perturbation of the initial system. In the subsection 2.2 we state a series of results related to the strong stable holonomy induced by the strong stable foliation. In subsection 2.3 we state some results related to the dynamics and structure of topologically hyperbolic sets and the continuation of them for perturbed systems. In subsection 2.4 we get state results for non-hyperbolic topologically hyperbolic sets. In the last subsection, we state a series of results that allows to to distinguish the "two dimensional case" from the genuinely higher dimensional dynamics.

2.1 Invariant manifolds of a topological hyperbolic set.

Recall that we are assuming that H_p is a topological hyperbolic set exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_1 is contractive. Under this hypothesis, first it is conclude that we can assume that there exists a positive constant $\lambda_s < 1$ such that

$$|Df_{|E_1}| < \lambda_s.$$

Moreover, for each points $x \in H_p$ it can be defined local embedded manifolds tangent to the subbundles E_1, E_2, E_3. More precisely, there exist continuous functions

$$\phi^{cs} : H_p \to Emb^1(D_1, M), \quad \phi^{ss} : H_p \to Emb^1(I_1, M),$$

$$\phi^{c} : H_p \to Emb^1(I_1, M), \quad \phi^{u} : H_p \to Emb^1(I_1, M)$$

where $I_1 = (-1, 1)$, $I_\epsilon = (-\epsilon, \epsilon)$; $D_1 = \{z \in R^2 : ||z|| < 1\}$; $D_\epsilon = \{z \in R^2 : ||z|| < \epsilon\}$ such that for any $x \in H_p$ it is defined

$$W^{cs}_\epsilon(x) = \phi^{cs}(x)D_\epsilon; \;\; W^{ss}_\epsilon(x) = \phi^{ss}(x)I_\epsilon; \;\; W^{c}_\epsilon(x) = \phi^{c}(x)I_\epsilon; \;\; W^{u}_\epsilon(x) = \phi^{u}(x)I_\epsilon$$

and verifying

1. $T_xW^{cs}_\epsilon(x) = E(x)$, $T_xW^{ss}_\epsilon(x) = E_1(x)$, $T_xW^{c}_\epsilon(x) = E_2(x)$, $T_xW^{u}_\epsilon(x) = F(x)$
2. $W^{cs}_\epsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0, dist(f^n(x), f^n(y)) < \epsilon\}$,
3. $W^{ss}_\epsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) < \lambda_s^n, dist(f^n(x), f^n(y)) < \epsilon\}$
4. $W^{c}_\epsilon(x) \subset W^{cs}_\epsilon(x) = W^{s}_\epsilon(x)$
5. $W^{u}_\epsilon(x) = \{y \in M : dist(f^{-n}(x), f^{-n}(y)) \to 0, dist(f^{-n}(x), f^{-n}(y)) < \epsilon\}$.

The next proposition states that for topologically hyperbolic set it is possible to get a hyperbolic metric (not necessarily coherent with a riemannian structure).

Proposition 2.1 *Given a topologically hyperbolic set, follows that there exists an adapted metric dist compatible with the topology, and there exist constants $\epsilon > 0$ and $0 < \lambda_0 < 1$ such that*

1. *if $y \in W^{cs}_\epsilon(x)$ then*
$$dist(f^n(x), f^n(y)) < \lambda_0^n dist(x,y).$$
2. *if $y \in W^u_\epsilon(x)$ then*
$$dist(f^{-n}(x), f^{-n}(y)) < \lambda_0^n dist(x,y).$$

Proposition 2.2 *Given a transitive topologically hyperbolic set Λ, follows that for any periodic point p in Λ holds that*
$$\Lambda = H_p$$
where H_p is the homoclinic class associated to p. Moreover, the dynamic on Λ is conjugated to a subshift of finite symbols.

Proof: The proof is similar to the the proof of the same proposition formulated for maximal invariant hyperbolic sets. ∎

Corollary 2.1 *Given a topologically hyperbolic set Λ, follows that there is a finite number of periodic points $p_1, ..., p_n$ such that*
$$\Lambda = \cup_i H_{p_i}.$$
Moreover, the homoclinic classes are disjoints.

Now, we state a result about the continuation of a dominated splitting and the associated tangent manifolds.

Lemma 2.1.1 *Let $f \in Diff^r(M)$ ($r \leq 1$) and Λ be a compact maximal invariant topologically hyperbolic set of f exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive.*

There exists an open neighborhood $\mathcal{U}$ of f in $Diff^r(M)$ and an open neighborhood U of Λ such that for each $g \in \mathcal{U}$ and any subbundle E_i there exists a continuous function, $T_g : \Lambda_g \to T_{\Lambda_g}M$ and $\phi^i_g : \Lambda_g \times Diff(M) \to Emb^1(D, M)$ such that for any $g \in \mathcal{U}$ and $x \in \Lambda_g$ it is defined the dominated splitting $E_1(g) \oplus E_2(g) \oplus E_3(g)$ and the manifold tangent to $E_i(g)$ is given by $W^{E_i}_\epsilon(x, g) = \phi^i_g(x)D_\epsilon$ and verifying

1. $T_x W^{E_i(g)}_\epsilon(x, g) = E_i(g, x)$,
2. *if $g(W^{E_i(g)}_\epsilon(x, g)) \subset B_\epsilon(g(x))$ then $g(W^{E_i(g)}_\epsilon(x, g)) \subset W^{E_i(g)}_\epsilon(g(x), g)$,*
3. *if $g^{-1}(W^{E_i(g)}_\epsilon(x)) \subset B_\epsilon(g^{-1}(x))$ then $g^{-1}(W^{E_i(g)}_\epsilon(x, g)) \subset W^{E_i(g)}_\epsilon(g^{-1}(x), g)$.*
4. *the maps $g \in \mathcal{U} \to T_g$ and $g \in \mathcal{U} \to Emb^1(D, M)$ are continuous.*

Remark 2.1 *If one of the subbundles of the dominated splitting is hyperbolic, then it remains hyperbolic after a C^r-perturbation of the system.*

We take a small neighborhood V of H_p and for g C^k- close to f we take the set

$$\Lambda_g = \Lambda_g(V) = Closure(\cap_{\{n\in\mathbb{Z}\}} g^n(V)).$$

From lemma 2.1.1 and previous remark, follows that for any g close to f there is a dominated splitting

$$E_1^s(g) \oplus E_2(g) \oplus E_3(g),$$

such the subbundle $E_1^s(g)$ is contractive in $\Lambda_g(V)$.
In the sequel, we denote with $W_\epsilon^{cs}(x,g)$ the tangent manifold to $E_1(g)\oplus E_2(g)$, with $W_\epsilon^{ss}(x,g)$ the tangent manifold to $E_1^s(g)$, with $W_\epsilon^c(x,g)$ the tangent manifold to $E_2(g)$, with $W_\epsilon^u(x,g)$ the tangent manifold to $E_3(g)$. Observe that the tangent manifolds $W_\epsilon^c(x)$ and $W_\epsilon^u(x)$ are not necessarily contained in the stable and unstable manifold respectively. However, form results stated in [PS4] follows that $W_\epsilon^u(x)$ is unique and dynamically defined. With $W_\epsilon^{cu}(x,g)$ we note the manifold

$$W_\epsilon^{cu}(x,g) = \cup_{\{z\in W_\epsilon^c(x,g)\}} W_\epsilon^u(z,g).$$

In some cases, given positive numbers $\epsilon_u < \epsilon$ we take

$$W_{\epsilon,\epsilon^u}^{cu}(x,g) = \cup_{z\in W_{\epsilon^u}^u(x,g)} W_\epsilon^c(z,g).$$

Now, we study how the dynamic of a perturbed map behave related to the distance introduce in proposition 2.1. Observe that the adapted metric not necessary is coming from a riemannian metric so even the distance along the center manifold are contracted exponentially this does not imply that the derivative is either contractive or expansive along the respective subbundles. In particular, we cannot expect that a perturbation of the initial map contracts distances along the center manifold. However, some contraction along the center stable manifold is kept when the points are not close enough one to each other. This is the statement of the next lemma.

Lemma 2.1.2 *Let dist, ϵ and λ the distances and the constants introduced in proposition 2.1. Then, for any $\gamma < r$ there exist a neighborhood $\mathcal{U}$ of f and λ_1 with $\lambda < \lambda_1 < 1$ such that for any $g \in \mathcal{U}$ holds:*

1. *if $y \in W_\epsilon^{cs}(x,g)$ follows that:*
   (a) *if $dist(x,y) > \gamma$ then $dist(g(x),g(y)) < \lambda_1 dist(x,y)$,*
   (b) *if $dist(x,y) < \gamma$ then $dist(g(x),g(y)) < \gamma$;*
2. *if $y \in W_\epsilon^{cu}(x,g)$ follows that:*
   (a) *if $dist(x,y) > \gamma$ then $dist(g^{-1}(x),g^{-1}(y)) < \lambda_1 dist(x,y)$,*
   (b) *if $dist(x,y) < \gamma$ then $dist(g^{-1}(x),g^{-1}(y)) < \gamma$.*

Moreover, the distance dist remains contractive along $E_1^s(g)$.

2.2 Strong stable foliation and strong stable holonomy map.

Remark 2.2 *Observe that since H_p is an attractor follows that the subbundle $x \to E_1(x)$ is defined in a unique way in a whole neighborhood of H_p; i.e.: there exists a neighborhood U of H_p where it is defined an invariant continuous subbundle contracted by Df.*

Lemma 2.2.1 *Let f be a C^2-diffeomorphisms and let us suppose that there exists a positive constant $d < 1$ such that for any $x \in H_p$ follows that*

$$|Df_{|E_1(x)}|\frac{|Df_{|E_3(x)}|}{|Df_{|E_2(x)}|} < d.$$

Then it follows $E_1(x)$ is a C^1-subbundle.
If g is C^2-close to f then $E_1(.,g)$ is a C^1-subbundle that it is C^1-close to $E_1(.,f)$.

See [HPS] for the proof.

Corollary 2.2 *The local strong stable foliation is a C^1-foliation.*

Definition 8 Strong stable holonomy.
Let $x \in H_p$ and let us take a neighborhood $B(x)$ of x contained in U. Let us consider the map

$$\Pi^{ss} : B(x) \to W^{cu}_\epsilon(x)$$

defined as

$$\Pi^{ss}(z) = W^{ss}_\epsilon(z) \cap W^{cu}_\epsilon(x).$$

This map is called the strong stable holonomy.
In some cases, we note

$$\Pi^{ss}_{f,x}$$

to specify that the projection is done over $W^{cu}_\epsilon(x)$ and it is associated to the map f.
If there is a pair of points x, y such that $y \in W^{ss}_\epsilon(x)$ we also consider

$$\Pi^{ss} : W^{cu}_\epsilon(y) \to W^{cu}_\epsilon(x).$$

In some cases, we note

$$\Pi^{ss}_{f,y,x}$$

to specify that the projection is done from $W^{cu}_\epsilon(y)$ to $W^{cu}_\epsilon(x)$ and it is associated to the map f.

Remark 2.3 *Observe that there is a neighborhood $\mathcal{U}$ of f, and positive constants $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ such that given a pair of points x, y verifying $W^{ss}_{\epsilon_1}(x) \cap W^{cu}_{\epsilon_2}(y) \neq \emptyset$ then for any $g \in \mathcal{U}$, $z_1 \in h_g^{-1}(x)$ and $z_2 \in h_g^{-1}(y)$ then $W^{ss}_{\epsilon_3}(z_1) \cap W^{cu}_{\epsilon_4}(z_2) \neq \emptyset$. Without loss of generality and to avoid notation we assume that $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4$. Moreover, in what folow, any neigborhood of f satisfies the present remark.*

The strong stable holonomy, in general, is a continuous map. However, under the assumption of normally dissipativeness it is possible to conclude that the strong stable holonomy is smooth.:

Corollary 2.3 *Let $\Pi^{ss}: W^{cu}_\epsilon(y) \to W^{cu}_\epsilon(x)$ be the strong stable holonomy induced by the subbundle E_1. It follows that it is a C^1-map. In particular, there exists a constant C_0 such that*

$$C_0^{-1}d(z_1, z_2) < d(\Pi^{ss}(z_1), \Pi^{ss}(z_2)) < C_0 d(z_1, z_2),$$

where in this case dist is the distance induced by the riemannian metric restricted to the local center unstable manifold.

2.3 Continuation of topologically hyperbolic maximal invariant sets.

Recall that given a neighborhood V of H_p, we define for any diffeomorphisms g nearby f the following set:

$$\Lambda_g(V) = Closure(\cap_{\{n\in\mathbb{Z}\}} g(V)).$$

Theorem E1: *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class. There exists a neighborhood $\mathcal{U}$ of f and V of H_p such that for any $g \in \mathcal{U}$ follows that there is a continuous map*

$$h_g : \Lambda_g(V) \to H_p$$

such that

$$h_g \circ g = f \circ h_g.$$

Moreover, the map $g \to h_g$ is continuous with g and h_g is close to the Identity map.

Assuming that Λ has a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles, then it is possible to show that the map h_g is onto and it is possible to get better description of the continuation of the homoclinic class for perturbation of the initial system.

Now, given a periodic point q, we take $\lambda_2(q)$ and $\lambda_3(q)$ the eigenvalues of $D_q f^{n_q}$ (n_q being the period of q) associated to the subbundles $E_2(q)$ and $E_3(q)$ respectively. Given λ_2 and λ_3 such that $0 < \lambda_2 < 1 < \lambda_3$, we take the set of periodic point

$$Per_{\lambda_2\lambda_3}(f) = \{q \in Per(f) : |\lambda_2(q)| < \lambda_2, \ |\lambda_3(q)| > \lambda_3\}.$$

Lemma 2.3.1 *There exist positive constants $\lambda_2^0 < 1 < \lambda_3^0$ such that for any λ_2 and λ_3 such that $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$ follows that the periodic points of f with center eigenvalue smaller than λ_2 and unstable eigenvalue larger than λ_3, are dense in H_p.*

Moreover, given a transitive invariant set Λ contained in H_p follows that there exists a neighborhood V of Λ such that

$$Per_{\lambda_2\lambda_3}(f/V) = \{q \in Per(f) : \mathcal{O}(q) \subset V, \ |\lambda_2(q)| < \lambda_2, \ |\lambda_3(q)| > \lambda_3\}$$

verifies that $\Lambda \subset Closure(Per_{\lambda_2\lambda_3}(f/V))$.

Theorem E2: *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class. If H_p has a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles, then there exists λ_2^0 and λ_3^0 with $0 < \lambda_2^0 < 1 < \lambda_3^0$ such that for any λ_2, λ_3 with $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$, there exist a neighborhood $\mathcal{U}$ of f, λ_2^1, λ_3^1 with $\lambda_2 < \lambda_2^1 < 1 < \lambda_3^1 < \lambda_3$ and a neighborhood V of H_p such that*

$$H_p = Closure(Per_{\lambda_2\lambda_3}(f/V)) \text{ and,}$$

$$h_g : Per_{\lambda_2^1\lambda_3^1}(g/V) \to Per_{\lambda_2\lambda_3}(f/V)$$

is a homeomorphisms. In particular, it follows that h_g is onto.

The previous result is proved using that the periodic points with eigenvalue exponentially far from one are dense in H_p (see lemma 2.3.1) and later it is shown that those points has a well defined analytic continuation for any g in an uniform neighborhood of f.

In what follows, we consider neighborhood $\mathcal{U}^{1,2}$ of f given by C^2-maps that they are C^1-close to f. In the next proposition, it is characterized the pre image by h_g of a point that does not belong to the stable manifold of a periodic point. In this case, it is proved that the pre image by h_g is contained in a center stable disc and the local center unstable manifold is contained in the unstable manifold. Observe that this result is only valid for at least C^2-maps.

Proposition 2.3 *Let $f \in Diff^2(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles. Let h_g be the semicongugacy introduced in theorem E2. Then there exists a neighborhood $\mathcal{U}^{1,2}$ of f such that for any $g \in \mathcal{U}^{1,2}$ it follows that given $z' \in h_g^{-1}(z)$ either*

1. *$h_g^{-1}(z) \cap W_\epsilon^{cu}(z', g)$ is a single point or*
2. *z belongs to the stable manifold of a periodic points q_z and $W_\epsilon^{cu}(z', g) \cap h_g^{-1}(z)$ is a compact arc verifying that its $\omega-$limit is a periodic arc such that one of its extremal points is a continuation of q_z.*

Corollary 2.4 *Let q be a hyperbolic periodic point of f in H_p. There exists a neighborhood $\mathcal{U}^{1,2} = \mathcal{U}^{1,2}(q, f)$ such that for any $g \in \mathcal{U}^{1,2}$ follows that if $z \in W^u(q)$ and z does not belong to the stable manifold of some periodic point, then $h_g^{-1}(z)$ is a single point.*

A similar result to the one obtained in propositions 2.3 and corollary 2.4 can be stated for points that belong to the stable manifold of a periodic point.

Proposition 2.4 *Let q be a hyperbolic periodic point of f in H_p. There exists a neighborhood $\mathcal{U}^{1,2} = \mathcal{U}^{1,2}(q, f)$ such that for any $g \in \mathcal{U}^{1,2}$ follows that if $z \in W^s(q)$ then $h_g^{-1}(z)$ is contained in the stable manifold of $h_g^{-1}(q)$.*

Corollary 2.5 *If $h_g^{-1}(x)$ is a single point then $W_\epsilon^{cs}(x, g) \subset W_\epsilon^s(x, g)$ and for any $\delta > 0$ there exists $n = n(\delta)$ such that $\ell(g^n(W_\delta^u(x, g))) > \epsilon$ and $\ell(g^k(W_\delta^u(x, g))) < \epsilon$ for $0 \leq k < n$.*

Lemma 2.3.2 *Let $x \in H_p$. Let us assume that for any $g \in \mathcal{U}$ follows that $h_g^{-1}(x)$ is a single point. Then, the map*

$$g \in \mathcal{U} \to h_g^{-1}(x)$$

is a continuous function.

Proof:

Let us assume that it is false. Therefore, there exist a a sequences $\{g_n\}$ converging to some g_0 in $\mathcal{U}$ such that $\{h_{g_n}^{-1}(x)\}$ does not converge to $h_{g_0}(x)$. Let z be an accumulation point of $\{h_{g_n}^{-1}(x)\}$. By theorem E_2 follows that $x = h_{g_n}(h_{g_n}^{-1}(x)) \to h_{g_0}(z)$. Therefore, $x = h_{g_0}(z)$ which implies that $z = h_{g_0}(x)$. A contradiction.

■

Notation 2.1 *Let $z \in H_p$ such that for any g close to f, $h_g^{-1}(z)$ is a single point. In this case we note*

$$z_g := h_g^{-1}(z).$$

Lemma 2.3.3 *Let x, y in H_p such that $y \in W_\epsilon^{ss}(x)$ and such that for any g close to f, $h_g^{-1}(x)$ and $h_g^{-1}(x)$ are single points. There exists a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ follows that*

$$y_g \in W_\epsilon^{cs}(x_g).$$

Proof: If it not the case, follows that $W_\delta^u(y_g, g) \setminus \{y_g\}$ for some δ small (such that δ is arbitrarialy small provided that $|g - f|_1$ is sufficuiently small) intersects $W_\epsilon^{cs}(x_g)$. Let us note this point with z. From corollary 2.5 follows that $dist(g^n(z), g^n(x_g))$ goes to zero, and there exists a positive integer n such that $dist(g^n(y_g), g^n(z)) > \epsilon$. Therefore, there exists n such that $dist(g^n(y_g), g^n(x_g)) > \epsilon\,(*)$. On the other hand, $dist(g^n(y_g), f^n(y))$ and $dist(g^n(x_g), f^n(x))$ are small, provided that $|g - f|_1$ is sufficuiently small. Since $y \in W_\epsilon^{ss}(x)$ follows that $dist(f^n(x), f^n(y))$ is small for large n and so $dist(g^n(y_g), g^n(x_g))$ is also small for n large; which is a contradiction with $(*)$.

■

Let $x \in H_p$ such that for any g close to f, $h_g^{-1}(x)$ is a single point. Then, we can take a neighborhood $B(x)$ of x such that for each g we take

$$W_{\epsilon^c, \epsilon^u}^{cu}(x_g)$$

and we can define the strong stable holonomy from

$$\Pi_g^{ss} : B(x) \to W_{\epsilon^c, \epsilon^u}^{cu}(x_g).$$

Lemma 2.3.4 *Let x, y in H_p such that $y \in W_\epsilon^{ss}(x)$ and such that for any g close to f, $h_g^{-1}(x)$ and $h_g^{-1}(x)$ are single points. There exists $\mathcal{U}$ and ϵ^u and $B(x)$ such that for any $g \in \mathcal{U}$ follows that*

$$W_{\epsilon^c, \epsilon^u}^{cu}(x_g) \setminus W_\epsilon^u(x_g, g)$$

$$W_{\epsilon^c, \epsilon^u}^{cu}(x) \setminus \Pi_g^{ss}(W_\epsilon^u(y_g, g))$$

has two connected components.

Proof: It is immediat from the transversality of the local manifolds.

■

2.4 Weak hyperbolic periodic points and heteroclinic cycles

The next lemma states that under the assumption of dominated splitting over a homoclinic class for a C^2 diffeomorphisms in a three dimensional manifold, holds that if the subbundle E_2 is not hyperbolic then there are periodic points contained in H_p, homoclinically related to p, such that the eigenvalue associated to the center subbundle is close to one.

Lemma 2.4.1 *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive, and E_2 is one dimensional. Then it follows that if E_2 is not contractive, then for any δ there exists a periodic point q with period n_q and homoclinically related to p such that $(1-\delta)^{n_q} < |Df^{n_q}_{|E_2(q)}| < 1$ (in this case we say that q has $\delta-$weak contraction along the center direction). Moreover, the periodic points with weak contraction are dense in H_p.*

Lemma 2.4.2 *Let q_δ be a periodic point that has $\delta-$weak contraction along the center direction. Then there exists a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $q \in Per_{\lambda_c \lambda_u}(f/V_1)$ follows that the q_g is homoclinically related with q_δ.*

For any periodic points in the homoclinic class follows that they exhibits a transverse intersection of its stable and unstable manifold. If this intersection holds along the strong stable and unstable manifolds we say that there is a strong homoclinic connection:

Definition 9 Strong homoclinic connection. *Given a periodic point q, we say that it has a strong homoclinic connection if the strong stable and strong unstable manifolds of q has an intersection.*

Now, let assume that there is a periodic point with weak contraction (expansion) along the center direction and also exhibiting a strong homoclinic connection. In this case, after a C^1 perturbation, it is possible to show that it is created a heterodimensional cycle.

Proposition 2.5 *Given $\delta_0 > 0$, there exists δ such that if there is a periodic point with $\delta-$weak contraction (expansion) along the central direction and exhibiting and strong homoclinic connection, then there is g $C^1 - \delta_0-$close to f exhibiting a heterodimensional cycle.*

Now we reformulate a lemma proved in [H] and already to stated in previous subsection, that allows to connect the strong stable and unstable manifolds when they are orbits that accumulates on both manifolds.

Lemma ([H]): *(C^1- connecting lemma) Let $f \in Diff^r(M^n)$ and let p be a periodic point such that there are points x in the strong unstable manifold and y in the strong unstable manifold, a sequence of points x_n accumulating in x and points $f^{k_n}(x_n)$ in the forward orbit of the sequences x_n accumulating on y. Then, there is a diffeomorphisms g C^1-close to f such that p remains periodic for g, x is in the strong unstable manifold, y is in the strong unstable manifold and y is in the forward orbit of x.*

2.5 Actual "two dimensional" situation.

Even though our ambient manifold is three dimensional, it may happen that the homoclinic class that we are considering are contained in a two dimensional submanifolds and therefore it could turn out that the attractors are actually two dimensional. In fact, to get examples of this kind a situation, let us consider an attractor for a surface diffeomorphism f (for instance a Plykin attractor or a Henon attractor), and then, let us embed this surface inside a in three dimensional manifold in a such a way that the three dimensional diffeomorphism coincides with f on the surface and such that this surface is invariant and normally hyperbolic for the new dynamics. First we start recalling the definition of normally hyperbolic submanifold.

Definition 10 *We say that an invariant submanifold S is normally hyperbolic if there is a splitting $T_S M = E^s \oplus F \oplus E^u$ such that*

1. *E^s is contractive;*
2. *there is $\lambda < 1$ such that $|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}| < \lambda$ for any $x \in S$;*
3. *E^u is expansive;*
4. *there is $\lambda < 1$ such that $|Df_{|F(x)}||Df^{-1}_{|E^u(f(x))}| < \lambda$ for any $x \in S$;*
5. *$T_x S = F(x)$ for any $x \in S$.*

If it holds that $f \in Diff^r(M)$ and

$$|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}|^r < \lambda < 1 \quad |Df_{|F(x)}|^r|Df^{-1}_{|E^u(f(x))}| < \lambda < 1$$

it is said that S is $r-$normally hyperbolic and follows that S is C^r (see [HPS]).

Theorem 2.1 *([BC]) Let $f \in Diff^r(M)$ $(r \geq 1)$ be a diffeomorphism on a compact manifold M. Let Λ be a compact maximal invariant set exhibiting a dominated splitting $T_\Lambda = E^s \oplus F \oplus E^u$ where E^s is contractive and E^u is expansive. Let also assume that for every $x \in \Lambda$ holds that $W^{ss}_\epsilon(x) \cap \Lambda = \{x\}$ (where $W^{ss}_\epsilon(x)$ is the local strong stable manifold tangent to E^s) and $W^{uu}_\epsilon(x) \cap \Lambda = \{x\}$ (where $W^{uu}_\epsilon(x)$ is the local strong unstable manifold tangent to E^u) . Then, there exist two C^1-submanifold normally hyperbolic S and $\hat{S}$ such that,*

1. *$T_x S = F(x)$,*
2. *$S \subset \hat{S}$,*
3. *$\Lambda \subset S$, $f(S) \subset \hat{S}$ and $f^{-1}(S) \subset \hat{S}$.*

Applying the previous theorem to the homoclinic class H_p follows the next corollary:

Corollary 2.6 *Let H_p be a topological hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive, E_3 is topologically expansive, E_2, E_3 are one dimensional subbundle, and $\mathcal{T} = \emptyset$. Then there is a C^1-submanifold S containing H_p and such that $f_{|S}$ is a C^1-surface map exhibiting a dominated splitting.*

Even f is C^2, the submanifold obtained in theorem 2.1 it could be only C^1. In fact, if there is a periodic point q in H_p with stable eigenvalues λ_1 and λ_2 such that $0 < \lambda_1 < \lambda_2$ but $\lambda_2^2 < \lambda_1$ follows that S cannot be C^2.

On the other hand, for topologically hyperbolic sets of C^1-surfaces maps exhibiting some extra properties it is possible to obtain a well description of the limit set.

To be more precise, we have to introduce some definitions for two dimensional diffeomorphisms.

Let S be a surface and $f \in Diff^1(S)$. Let us assume that f has an invariant set Λ exhibiting a two dimensional dominated splitting $E \oplus F$. Recall that for each subbundle and for every point $x \in \Lambda$ we have associated the tangent manifolds $W_\epsilon^E(x)$ and $W_\epsilon^F(x)$.

Definition 11 *We say that $W_\epsilon^F(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W_\epsilon^F(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^{-i}(J)) \leq \delta$ for $0 \leq i \leq n$ then*

1. $\frac{|Df^{-n}_{/\tilde{F}(y)}|}{|Df^{-n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J))),$

2. $|Df^{-n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J))) \quad \tilde{F}(y) = T_y W_\epsilon^F(x).$

We say that $W_\epsilon^E(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W_\epsilon^E(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^i(J)) \leq \delta$ for $0 \leq i \leq n$ then

1. $\frac{|Df^{n}_{/\tilde{F}(y)}|}{|Df^{n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J))),$

2. $|Df^{n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J))) \quad \tilde{F}(y) = T_y W_\epsilon^E(x).$

With this definition in mind, it is possible to get the following result which is a generalization of the theorem B of [PS1] for C^1-maps on surfaces:

Theorem 2.2 *Let $f \in Diff^1(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set topologically hyperbolic and exhibiting a dominated splitting $E \oplus F$ such that any periodic point is a hyperbolic saddle periodic point. Moreover, assume that $W_\epsilon^E(x)$ and $W_\epsilon^F(x)$ has bounded distortion. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ...\mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

Proposition 2.6 *Let $f \in Diff^2(M)$ and let H_p be a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive, E_3 is topologically expansive. Let us assume that there exists a two dimensional C^1-normally submanifold S such $H_p \subset S$. Then, the tangent manifolds $W_\epsilon^{cu}(x) \cap S$ and $W_\epsilon^{cs}(x) \cap S$ have bounded distortion property.*

3 Proof of maim theorem.

From proposition 2.2 follows that we can assume in what follows that the set Λ is a homoclinic class and we denote H with H_p to indicate that we are dealing with a homoclinic class associated to p.

First we define the set

$$\mathcal{T} = \{x \in H_p : [W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset\}.$$

Then we consider the following options:

1. $\mathcal{T} = \emptyset$,
2. $\mathcal{T} \neq \emptyset$ and $Interior(\mathcal{T}) \neq \emptyset$,
3. $\mathcal{T} \neq \emptyset$ and $Interior(\mathcal{T}) = \emptyset$,

where the topology is the restricted topology to H_p.

Case 1. $\mathcal{T} = \emptyset$.

The theorem follows applying theorem 2.1, theorem 2.2 and proposition 2.6 stated here. For details see theorem C in [Pu].

Case 2. $\mathcal{T} \neq \emptyset$ **and** $Interior(\mathcal{T}) \neq \emptyset$.

In the case that the interior of $\mathcal{T}$ is not empty, from the fact that the periodic points with weak contraction are dense (see lemma 2.4.1), follows immediately that there exists a periodic point q with weak contraction along the center direction such that $[W^{ss}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$. Then, applying the C^1-connecting lemma, and proposition 2.5 (for details, see theorem D of [Pu]). In other words, we have proved the following proposition:

Proposition 3.1 *Let H_p be an attracting topological hyperbolic homoclinic class. If the interior of $\mathcal{T}$ is not empty then the thesis of the maim theorem follows.*

So it remains to consider the case that the interior of $\mathcal{T}$ is empty.

Case 3. $\mathcal{T} \neq \emptyset$ **and** $Interior(\mathcal{T}) = \emptyset$.

In the present case, we consider either if there exists a periodic point q such that $[W^{ss}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$ or for all periodic point q follows that $[W^{ss}_\epsilon(q) \setminus \{q\}] \cap H_p = \emptyset$. In the first case, we apply the following proposition to f:

Proposition 3.2 *Let $g \in Diff^r(M^3)$ and $\delta > 0$ such that*

1. *g has two hyperbolic periodic points q_1 and q_2 verifying*
   (a) *q_1 and q_2 are homoclinically connected,*
   (b) *$W^u(q_2) \cap W^{ss}_\epsilon(q_1) \neq \emptyset$;*
2. *there exists a periodic points q_δ with $\frac{\delta}{2}-$weak contraction along the center direction and homoclinically related with q_1.*

Then, there is $\hat{g}$ arbitrarily C^k-close to g and a periodic point $\hat{q}_\delta$ with $\delta-$weak contraction along the center direction and exhibiting a strong homoclinic connection.

The proof of this proposition can be found in [Pu].

Therefore, in what follows, we assume that for all periodic point q follows that $[W^{ss}_\epsilon(q)\setminus\{q\}]\cap H_p = \emptyset$. However, recall that there exists a pair of points in the homoclinic class x, y such that $y \in W^{ss}_\epsilon(x)$. On one hand, observe that x is accumulated by a sequence $\{q_n\}$ of periodic points and so it follows that there is a sequences of points $\{q^*_n\}$ such that $q^*_n \in W^{ss}_\epsilon(q_n)$ and $q^*_n \to y$. Moreover, we can assume that the periodic points q_n have weak contraction along the center direction. On the other hand, the unstable manifold of p accumulates on y and therefore, the unstable manifold of the points q_n also accumulates on y. Observe that even if for any q_n holds that $[W^{ss}_\epsilon(q_n)\setminus\{q_n\}]\cap H_p = \emptyset$, since $y \in H_p$ and $q^*_n \to y$, it is natural to try to perform some kind of connecting lemma argument's with the goal to connect the unstable manifold of one of the points q_n with the local strong stable manifold of the same point. If this type of perturbation can be done, again, a heterodimensional cycle is created.

However, to use the connecting lemma, it is necessary to assume some restrictions over the orbits of the periodic points $\{q_n\}$. For instances, if the periodic points $\{q_n\}$ do not accumulate on y then it can be applied the connecting lemma. On the other hands, if it occurs that the periodic points $\{q_n\}$ do accumulate on y, then connecting lemma argument's can not be performed. In fact, if the pair of points x and y belongs to a minimal invariant set contained in H_p, then the situation mentioned above holds. Therefore, it is necessary to develop other techniques to deal with these type of situation. The rest of the paper is devoted to overcome these difficulties. Under this hypothesis, we show the following: *there is a C^1 suitable perturbations of f, exhibiting a pair of periodic points q_1 and q_2 homoclinically related and such that $W^u(q_2)\cap W^{ss}_\epsilon(q_1) \neq \emptyset$.* More precisely, by perturbation we are in the hypothesis of proposition 3.2. So the goal is to show that if the interior of $\mathcal{T}$ is empty, then for any $\delta > 0$ we can get by perturbation a diffeomorphisms g C^1-arbitrarily close to f verifying the hypothesis of proposition 3.2.

To get the pair of periodic points in the hypothesis of the proposition 3.2, we consider the pair of points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$. Recall that the are sequences $\{q_n\}$ and $\{p_n\}$ of periodic points in $Per_{\lambda_c\lambda_u}(f_{/V})$, the first accumulating on x and the second on y. Moreover, the local unstable manifold and the local strong stable manifold of the points q_n and p_n accumulate in the local unstable manifold and strong stable manifold of x and y respectively. Using that the periodic points $\{q_n\}$ and $\{p_n\}$ have well defined continuation for any diffeomorphisms g nearby f, named $\{q_n(g)\}$ and $\{p_n(g)\}$ respectively. The goal is to show that for some g nearby f, holds that $y_g \notin W^{ss}_\epsilon(h_g^{-1}(x))$ and the periodic points $\{q_n(g)\}$ and $\{p_n(g)\}$ accumulate on $h_g^{-1}(x)$ and $h_g^{-1}(y)$ respectively. Using this and that the local unstable and strong stable manifolds move continuously with the perturbation, it is proved that for some perturbation holds that the strong stable manifold of some of the points $\{p_n(g)\}$ intersects the local unstable manifold of some of the points $\{q_n(g)\}$. To get the periodic point with weak contraction along the center direction as in proposition 3.2 we proceed as follow: Let us take $\delta > 0$ and let q_δ be a periodic point with $\frac{\delta}{3}-$weak contraction along the center direction. We consider an arbitrarily small open neighborhood $\mathcal{U} = \mathcal{U}(\delta) \subset Diff^1(M^3)$ of f such that for any $g \in \mathcal{U}$ follows that q has analytic continuation and q_δ remains $\frac{\delta}{2}-$weak contractive. Then, recall from lemma 2.4.2, that for any periodic points in $Per_{\lambda_c\lambda_u}(f)$ and any $g \in \mathcal{U}$ follows that $h_g^{-1}(q)$ is homoclinically related with q_δ. Since $h_g^{-1}(q)$ is a single point we note it with $q(g)$.

To perform these arguments, we have to consider two alternatives. To introduce the mentioned alternative first let us consider the pair of points x, y such that $y \in W^{ss}_\epsilon(x)$. Let us consider the strong stable holonomy introduced in subsection 2.2,

$$\Pi^{ss} : W^{cu}_\epsilon(y) \to W^{cu}_\epsilon(x).$$

Observe that if $y \in W^{ss}_\epsilon(x)$ then $\Pi^{ss}(W^u(y)) \cap W^u(x) \neq \emptyset$.

The mentioned alternative splits in different parts related to the kind of intersection of $\Pi^{ss}(W^u_\epsilon(x))$ with $W^u_\epsilon(y)$.

Definition 12 Joint integrability: *We say that strong stable foliation and the strong unstable foliation are jointly integrable if there exist $0 < \epsilon_1 < \epsilon_2$ and $0 < \epsilon_0$ such that for any x and y in the homoclinic class with $y \in W^{ss}_{\epsilon_0}(x)$ holds that*

$$\forall\, z \in W^u_{\epsilon_1}(x) \text{ then } W^{ss}_\epsilon(z) \cap W^u_{\epsilon_2}(y) \neq \emptyset$$

in other words, for all $x, y \in \Lambda$ such that $y \in W^{ss}_\epsilon(x)$ follows that

$$\Pi^{ss}(W^u_{\epsilon_1}(y)) \subset W^u_{\epsilon_2}(x).$$

Without loss of generality, we can assume that $\epsilon = \epsilon_0 = \epsilon_1 = \epsilon_2$ and

$$\Pi^{ss}(W^u_\epsilon(y)) = W^u_\epsilon(x).$$

We consider independently *the case that the strong foliation are jointly integrable and the case that this does not happen.* More precisely:

Alternative:

1. *There exists $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$, there exist $0 < \epsilon_1 < \epsilon_2$ such that there is $z \in W^u_{\epsilon_1}(y)$ verifying that $W^{ss}_{\epsilon_1}(z) \cap W^{cu}_{\epsilon_2}(x) \neq \emptyset$ and $W^{ss}_{\epsilon_1}(z) \cap W^u_{\epsilon_2}(x) = \emptyset$.*

2. *The strong foliations are jointly integrable.*

In the case that the strong foliation are not jointly integrable, we can conclude there are a pair of periodic points p_x, p_y such that there is a pair of points x and y in the unstable manifold of p_x and p_y respectively such that they share the same strong stable leaf. Later, performing a suitable perturbation it is concluded the existence of a new diffeomorphisms verifying the hypothesis of proposition 3.2. In the case that the strong foliation are jointly integrable, it is necessary to perform another perturbation different that the one done in the previous case. The goal of the next subsection are devoted to consider both situations.

3.1 The strong foliations are not jointly integrable.

To study this situation we have to analyze different cases related to the type of intersection of the unstable manifolds:

Definition 13 *Given a pair of points $x, y \in H_p$, we say that $\Pi^{ss}(W^u_\epsilon(x))$* **intersects** *$W^u_\epsilon(y)$ if $y \in W^{ss}_\epsilon(x)$.*
We say that $\Pi^{ss}(W^u_\epsilon(x))$ **intersects transversally** *$W^u_\epsilon(y)$ if $\Pi^{ss}(W^u_\epsilon(x))$ intersects both components of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$.*
We say that $W^u_\epsilon(y)$ locally s-intersects transversally $W^u_\epsilon(x)$ if for any $\delta > 0$ follows that $\Pi^{ss}(W^u_\epsilon(x))$ intersects both components of $W^{cu}_\delta(y) \setminus W^u_\epsilon(y)$.
To avoid notation we say that $W^u_\epsilon(y)$ s-intersects (s-intersects transversally, locally s-intersects transversally) $W^u_\epsilon(x)$ and we note with $W^u_\epsilon(x) \cap_s W^u_\epsilon(y)$ the set $\Pi^{ss}_\epsilon(W^u_\epsilon(x)) \cap W^u_\epsilon(y)$.

Now we consider the following alternative:

1. **There are not transversal intersections:** for any $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$ follows that $W^u_\epsilon(x)$ does not s-intersect transversally $W^u_\epsilon(y)$.
2. **There are transversal intersections:** there is a pair $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$ and $W^u_\epsilon(x)$ s-intersect transversally $W^u_\epsilon(y)$.

3.1.1 Transversal intersections.

In the case that the strong foliation are not jointly integrable and there are transversal intersections, we get the following result (the proof is given in subsection 4.1):

Lemma 3.1.1 *Let H_p be a topologically hyperbolic attracting homoclinic class such that $\mathcal{T} \neq \emptyset$, the strong foliations are not jointly integrable and there are transversal intersections. Then, there are a pair of points x, y in the homoclinic class and a pair of periodic points p_x, p_y also in the homoclinic class such that:*

1. *$y \in [W^{ss}_\epsilon(x) \setminus \{x\}]$,*
2. *$x \in W^u(p_x)$ and $y \in W^u(p_y)$.*

Moreover, one of the following option holds:

1. *there exists a connected compact arc contained in $W^u_\epsilon(x) \cap \Pi^{ss}(W^u_\epsilon(y))$ or*
2. *$W^u_\epsilon(y)$ locally s-intersects transversally $W^u_\epsilon(x)$.*

In the first case of the options stated in lemma 3.1.1 we apply the next lemma.

Lemma 3.1.2 *If there exists a compact arc contained in $W^u_\epsilon(x) \cap \Pi^{ss}(W^u_\epsilon(y))$ follows that we get that there are two points x', y' in the unstable manifold of p_x and p_y respectively and there is a periodic point q such that*

1. *$y' \in [W^{ss}_\epsilon(x') \setminus \{x\}]$*
2. *$y', x', \in W^s(q)$.*

In this situation we apply the following lemma to finish the maim theorem:

Lemma 3.1.3 *Let $x \in W^u(p_x)$ and $y \in W^u(p_y)$ such that $y \in W^{ss}_\epsilon(x)$. Let us suppose that there exists a periodic point q such that $x, y \in W^s(q)$. Then, the hypothesis of proposition 3.2 holds.*

In the case $W^u_\epsilon(y)$ locally s-intersects transversally $W^u_\epsilon(x)$ we do not know if the hypothesis of lemma 3.1.1 imply that there are two periodic points in the hypothesis of proposition 3.2 or in the hypothesis of lemma 3.1.3. However, using that the intersection is transversal, it is possible to get a diffeomorphisms g C^1-close to f, that verifies the hypothesis of proposition 3.2.

To do that we consider two different cases related to $\omega(x)$ (where $\omega(x)$ is the closure of the accumulation points of the forward orbit of x):

1) If $x \notin \omega(x)$ then it is performed a perturbation such that for some g close to f holds that the "continuation" of the points x and y do not belong to the same strong stable leaf. Then considering an isotopy between the initial map and the perturbation, follows that for some map of the isotopy holds that there are two periodic points as in the thesis of proposition 3.2 (see lemma 3.1.4 in what follows).

2) If $x \in \omega(x)$ we prove that there exists a diffeomorphisms C^1-close to f such that the points x and y such that they belongs to the unstable manifold of p_x and p_y respectively, they share the same strong stable leaf and they are in the stable manifold of p_x. In few words, a C^1-connecting lemma preserving the strong stable foliations is proved. After that, by lemma 3.1.3 follows that a heterodimensional cycle can be created by perturbations.

Case 1. $x \notin \omega(x)$.

In the present case we use the following proposition and lemma.

Proposition 3.3 *Let p_x and p_y verifying that their local unstable manifold intersects transversally at x, y. Let us assume that $x \notin \omega(x)$. Then there exists g close to f such that $x_g \notin W^{ss}_\epsilon(y_g, g)$.*

Lemma 3.1.4 *Let us take x, y as in the thesis of lemma 3.1.1. Let us suppose that x and y do not belong to the stable manifold of some periodic point. Let us also assume that there exist $g_1 \in \mathcal{U}^{2,1}$, such that $x_{g_1} \notin W^{ss}_\epsilon(y_{g_1}, g_1)$. Then, there exists $\hat{g} \in \mathcal{U}^{2,1}$ such that it verifies the hypothesis of proposition 3.2.*

This finish the proof of maim theorem in case that $x \notin \omega(x)$. In fact, if the thesis of lema 3.1.3 holds then the proof is complete. If the hypothesis of lemma 3.1.3 does not hold, we apply the proposition 3.3 and then it follows that the hypothesis of lemma 3.1.4 holds and so the proof is finished.

Case 2. $x \in \omega(x)$.

In this situation we perform a C^1-connecting lemma perturbation that preserves the strong stable foliation. At this point it is used that H_p is normally dissipative.

Proposition 3.4 *Let H_p normally dissipative topologically hyperbolic homoclinic class such that the interior of $\mathcal{T}$ is empty. Let q be a periodic point such that there exists $x \in W^u_\epsilon(q)$ satisfying:*

1. *there exists $y \in W^{ss}_\epsilon(x) \cap W^u_\epsilon(p_y)$ for some periodic point p_y,*
2. *$\omega(x) \cap W^s_\epsilon(q) \neq \emptyset$*

Then there exists g C^1-close to f such that

1. *q and p_y are periodic points for g,*
2. *$x \in W^u_\epsilon(q, g)$, $y \in W^u_\epsilon(p_y, g)$,*
3. *$y \in W^{ss}_\epsilon(x, g)$,*
4. *$x \in W^s(q, g)$.*

Observe that the points x, y that satisfy the thesis of proposition 3.4 also verify the hypothesis of lemma 3.1.3. Therefore, to conclude the proof of maim theorem in this case, first it is applied the proposition 3.4 and then the lemma 3.1.3.

3.1.2 Non transversal intersections.

Using that they are not transversal intersections, it is proved that there is a pair of points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$, and that they belong to the unstable manifold of some periodic points p_x and p_y. Then, it is performed a perturbation which consists essentially in moving the unstable manifold of p_x and keeping the local unstable manifold of p_y unperturbed. This perturbation is performed with the goal in mind that the points x_g, y_g verify that $y_g \notin W^{ss}_\epsilon(x_g)$. The movement of the unstable arc of x related to the unstable arc of y, allows to show that the local strong stable manifold of the continuation of some periodic point q close to x intersects the local unstable manifold of y. Since y belongs to the unstable manifold of a periodic point, the hypothesis of proposition 3.2 holds.

Proposition 3.5 *Let H_p be a topologically hyperbolic attracting homoclinic class such that $\mathcal{T} \neq \emptyset$, the strong foliations are not jointly integrable, and there are not transversal intersections. Then, there is a pair of points x, y in H_p and a pair of periodic points p_x, p_y in H_p such that:*

1. *$y \in [W^{ss}_\epsilon(x) \setminus \{x\}]$,*
2. *$x \in W^u(p_x)$ and $y \in W^u(p_y)$.*

Observe that at this point we could apply the same strategy that was considered for the case of transversal intersection. However, we develop another strategy, suited for the case of non-transversal intersection, that do not use the hypothesis of normal dissipativeness.

More precisely, we prove:

Proposition 3.6 *Let $f \in Diff^r(M^3)$. Let H_p be a non-hyperbolic topologically hyperbolic attracting homoclinic class with nontransversal intersections and verifying the thesis of proposition 3.5. Then, for any $\delta > 0$ there exists a diffeomorphisms g arbitrarily C^1-close to f verifying the hypothesis of proposition 3.2.*

3.2 The strong foliations are jointly integrable.

Now we have to address the case that the strong foliations are jointly integrable. It is not clear if under the hypothesis of joint integrability it is possible to get two points x, y as in the proposition 3.1.1; i.e.: a pair o points x, y such that belong to the same strong stable leaf and contained in the unstable manifold of some periodic points.

However, using strongly that the strong foliation are jointly integrable, it is possible to perform a C^1-perturbation to get two periodic points as in the proposition 3.2.

For that, it is necessary the following theorem that state if the interior of $\mathcal{T}$ is empty, then there is a subset Λ such that E_3 is uniformly expansive on Λ and containing a pair of points x, y with $y \in W^{ss}_\epsilon(x)$.

Theorem 3.1 *Let H_p be a topologically hyperbolic attracting homoclinic class such that $\mathcal{T} \neq \emptyset$ and the interior of $\mathcal{T}$ is empty. Then, there is a compact transitive invariant subset Λ such that*

1. *there is a pair of points $x, y \in \Lambda$ such that $y \in W^{ss}_\epsilon(x)$,*
2. *E_3 is uniformly expansive in Λ.*

Observe that in theorem 3.1 is not assumed that the strong foliations are jointly integrable (the proof is given in section 7).

Then we apply the following proposition (the proof is given in section 6):

Proposition 3.7 *Let H_p be a normally dissipative topologically hyperbolic attracting homoclinic class such that the strong foliations are jointly integrable. Let us suppose that there exists a compact invariant set $\Lambda \subset H_p$ that verifies the thesis of theorem 3.1. Then, for any $\delta > 0$ there exists a diffeomorphisms g arbitrarily C^1-close to f verifying the hypothesis of proposition 3.2.*

As a consequences of the previous proposition, again we conclude the maim theorem when the strong foliation are jointly integrable.

4 Non jointly integrable case with transversal intersection.

In the present section we give the proof of lemma 3.1.1, 3.1.3 and 3.1.4. Later is given the proof of proposition 3.3 and 3.4. The proof of the last one is more intricated and it includes a series of technical lemmas.

4.1 Proof of lemma 3.1.1.

Let us consider the strong stable holonomy map $\Pi^{ss}_{f,x}$ defined from a neighborhood of x to the center-unstable manifold of x. Let us a take a periodic point p_x close to x and a periodic point p_y close to y. So, the local unstable manifold of p_x and p_y are closed to the local unstable manifold of x and y respectively and so $\Pi^{ss}(W^u_\epsilon(p_x))$ and $\Pi^{ss}(W^u_\epsilon(p_y))$ are closed to $W^u_\epsilon(x)$ and $\Pi^{ss}(W^u_\epsilon(y))$ respectively. Since $W^u_\epsilon(x)$ s-intersect transversally $W^u_\epsilon(y)$, follows that, $\Pi^{ss}(W^u_\epsilon(p_x))$ and $\Pi^{ss}(W^u_\epsilon(p_y))$ intersects transversally.

4.2 Proof of lemma 3.1.3:

Observe that there is a hyperbolic set $H = H(p_x, x, q)$ that contains p_x, x, and q. Moreover, it can also be assumed that there is a connected compact arc l_y contained in the unstable manifold of p_y that contains p_y and y such that $H \cap l_y = \emptyset$. So, it follows that there is a periodic point $\hat{q}$ arbitrarily close to x, contained in H, and with orbit uniformly disjoint from l_y: there exists $\delta > 0$ such that for any $\gamma > 0$ there is a periodic point $\hat{q}$ such that $d(\hat{q}, x) < \gamma$ and $d(f^i(\hat{q}), l_y) > \delta$ for any integer i. So, it is possible to perturb the intersection between l_y and $W^s(q)$ in a such a way that for the perturbation follows that $l_y \cap W^{ss}_\epsilon(\hat{q}) \neq \emptyset$ and l_y remains contained in the unstable manifold of p_y. Therefore, the proof of the proposition 3.2 is finished in this case.

■

4.3 Proof of lemma 3.1.4:

Since x and y do not belong to the stable manifold of some periodic point, by lemma 2.4 follows that for any smooth g C^1-close to f follows that $h_g^{-1}(x)$ and $h_g^{-1}(y)$ are single points. Therefore, for any $g \in \mathcal{U}$ we denote them with $x_g = h_g^{-1}(x)$ and $y_g = h_g^{-1}(y)$. From lemma 2.3.3 follows that x_g and y_g belongs to the same local center stable manifold.

First observe that if $\{q_n\}$ is a sequences of periodic points such that $q_n \in Per_{\lambda^c\lambda^u}(f/V)$ and x is an accumulation point of $\{q_n\}$, then x_g is an accumulation point of $\{q_n(g))\}$ where $q_n(g) := h_g^{-1}(q_n)$..

Let us consider a homotopy $\mathcal{F} = \{g_\eta\}_{0\leq\eta\leq 1}$ such that $g_\eta \in \mathcal{U}$ for any η, $g_0 = f$ and g_1 is the diffeomorphism in the hypothesis of the present lemma. We keep denoting g_0 with f.

For each $g \in \mathcal{F}$ let us take $W^{cu}_\epsilon(x_g, g)$. Using that $W^{cu}_\epsilon(x_g, g)$ is continuous with g and that $W^u_\epsilon(x_g, g)$ is contained inside a compact arc of the unstable manifold of a periodic point, for each g we can assume that

1. $x_g = x$,
2. $W^u_\epsilon(x_g, g) = W^u_\epsilon(x, f)$,

3. $W^c_\epsilon(x,g) = W^c_\epsilon(x,f)$ and

4. $W^{cu}_\epsilon(x_g,g) = W^{cu}_\epsilon(x,f)$.

Given positive numbers $\epsilon^u < \epsilon$ we take $W^{cu}_{\epsilon,\epsilon^u}(x)$. Now, for each $g \in \mathcal{F}$ we take $\Pi^{ss}_g : B \to W^{cu}_g(x)$ where B is a neighborhood that contains x and y_g for any $g \in \mathcal{F}$. Let us take L^+_f and L^-_f the connected components of $W^{cu}_{\epsilon,\epsilon^u}(x) \setminus \Pi^{ss}_f(W^u_\epsilon(y,f))$. Since $y_{g_1} \notin W^{ss}_\epsilon(x,g_1)$ then $\Pi^{ss}_{g_1}(y_{g_1}) \neq x$. Since $y_{g_1} \in W^{cs}_\epsilon(x)$ and $W^u_\epsilon(x)$ intersect transversally $\Pi^{ss}_f(W^u_\epsilon(y,f))$ follows that $\Pi^{ss}_g(y_{g_1}) \notin \Pi^{ss}_f(W^u_\epsilon(y,f))$. Therefore, follows that $\Pi^{ss}_{g_1}(y_{g_1}) \in L^+_f \cup L^-_f$. We can suppose that $\Pi^{ss}_{g_1}(y_{g_1}) \in L^+_f$. Moreover, we can also assume that for any $g \in \mathcal{F}$, which is not f, follows that $\Pi^{ss}_g(y_g) \in L^+_f$.

Taking ϵ^u sufficiently small, also follows that $\Pi^{ss}_{g_1}(W^u_\epsilon(y_{g_1},g_1)$ is also contained in L^+_f. Moreover, from lemma 2.3.4 we can also take ϵ^u small such that for any $g \in \mathcal{F}$ holds that $W^{cu}_{\epsilon,\epsilon^u}(x) \setminus \Pi^{ss}_g(W^u_\epsilon(y_g,g))$ also has two connected components. We denote with $L^+_{g_1}$ the connected component that does not contains x. Observe that therefore, $x \in L^-_{g_1}$.

Since $W^u_\epsilon(y,f)$ and $W^u_\epsilon(x)$ s-intersect transversally, there is a sequences of periodic points $\{q_n\}$ such that $q_n \in Per_{\lambda^c\lambda^u}(f/V)$, $q_n \to x$ and $\Pi^{ss}_f(q_n)$ belong to L^+_f. In fact, since $W^u_\epsilon(y,f)$ and $W^u_\epsilon(x)$ s-intersect transversally, there is a sequences of points $\{x_n\}$ in the local unstable manifold of x that accumulate on x and $x_n \in L^+_f$. Then for each x_n we take a periodic point q_n in $Per_{\lambda^c\lambda^u}(f/V)$ close to x_n and such that its projections by the strong stable holonomy are contained in L^+_f.

Since for g_1 also holds that $\Pi^{ss}_{g_1}(q_n(g_1)) \to x$ then follows that for n large enough holds that $\Pi^{ss}_{g_1}(q_n(g_1)) \in L^-_{g_1}$. Therefore, there is a sequences $\{q_n\}$ such that

$$\Pi^{ss}_f(q_n) \in L^+_f, \text{ and } \Pi^{ss}_{g_1}(q_n(g_1)) \in L^-_{g_1}.$$

Using that everything moves continuously, follows that there exists $\hat{g} \in \mathcal{F}$ such that for some n large

$$\Pi^{ss}_{\hat{g}}(q_n(\hat{g})) \in \Pi^{ss}_{\hat{g}}(W^u_\epsilon(y_{\hat{g}},\hat{g})).$$

This is equivalent to say that

$$W^{ss}_\epsilon(q_n(\hat{g})) \cap W^u_\epsilon(y_{\hat{g}},\hat{g}) \neq \emptyset.$$

Therefore, the thesis of the lemma follows.

■

4.4 Proof of proposition 3.3.

Since $x \notin \omega(x)$ we can get a neighborhood B of $f^{-1}(x)$ such that for any $z \in W^{cs}_\epsilon(x)$ follows that

$$f^k(z) \notin B \;\; \forall\, k > 0.$$

Using that, we construct a perturbation of f with support in B and such that it moves the unstable manifold of p_x in a small neighborhood of $f^{-1}(x)$ with the property that

$$g(f^{-1}(x)) \neq \Pi^{ss}_f(y), \text{ and } g(f^{-1}(x)) \in L^+_f \tag{1}$$

where L_f^+ is the connected component of $W_\epsilon^{cu}(x) \setminus \Pi_f^{ss}(W_\epsilon^u(y))$ such that its interior intersects $\Pi_f^{ss}(H_p \cap B(y))$. Since the support of the perturbation is localized in B it follows that $W_\epsilon^{cs}(x, f) = W_\epsilon^{cs}(g(f^{-1}(x), g)$ and $\mathcal{O}^+(g(f^{-1}(x))) = \{g^n(g(f^{-1}(x)))\}_{\{n\in\mathbb{N}\}}$ does not intersect the support of the perturbation and therefore follows that

$$x_g = g(f^{-1}(x)) \text{ and } y_g = y. \tag{2}$$

Moreover, it holds that

$$W_\epsilon^{ss}(g(f^{-1}(x)), g) = W_\epsilon^{ss}(g(f^{-1}(x)), f). \tag{3}$$

From 1, 2 and 3 follows that

$$\Pi_g^{ss}(x_g) \neq y_g.$$

4.5 Proof of proposition 3.4.

Roughly speaking, the perturbations that is done in the present proposition is a kind of C^1-closing lemma that preserves the strong direction; i.e.: it is performed a perturbation that sends the point z_1 to a point z_2 in such a way that the image of the local strong stable manifold of z_1 by the perturbation goes to the local strong stable of z_2.

A naive idea of the proof of the present lemma, it is to consider an endomorphism, "$\Pi^{ss} \circ f$", induced by the projection of the dynamic of f by the strong stable manifold. Since the strong stable foliation is C^1 then follows that the induced endomorphism is also C^1. Then, it is performed a C^1-connecting lemma for non-singular endomorphism (see [LW2] for closing lemmas for endomorphism). If it holds that any perturbation of the induced endomorphism can be performed as a projection by the strong stable foliation of a perturbation of f, we would conclude the thesis of the proposition 3.4.

To perform the connecting lemma preserving the strong stable leaves, it is followed and adapted the arguments used in the connecting lemma. However, instead to use the Euclidean cubes to localize the perturbation, it is used rectangles coherent with the splitting. These rectangles also verify that are uniformly large along the strong stable direction (see lemma 4.5.5) and arranged in tiles as is done in the connecting lemma.

Moreover, it is also proved a version of the closing lemma that keeps invariant the local strong stable leaves:

Lemma 4.5.1 C^1-closing lemma preserving the strong stable leaves. *Let $f \in Diff^1(M^3)$ and let H_p be an attracting topologically hyperbolic homoclinic class which is normally dissipative and such that the interior of $\mathcal{T}$ is empty. There exist ϵ_0 and a neighborhood U of H_p such that for any $\gamma > 0$ and $x \in H_p$ verifying $x \in \omega(x)$ there exists a positive integer $m = m(x, \gamma)$, a C^1-diffeomorphisms g, $\gamma - C^1-$close to f such that $f^m(x)$ is a periodic orbit of $g$with orbit in U and verifying that*

$$W_{\epsilon_0}^{ss}(f^m(x), g) = W_{\epsilon_0}^{ss}(f^m(x), f).$$

Before to give the proof, we explain the strategy of the C^1-closing lemma and latter we show how to adapt it to get the proof of lemma 4.5.1. The rest of this subsection is organized in the following way: In subsection 4.5.1 we explain the strategy of the C^1 closing lemma. In subsection 4.5.2 we build rectangles coherent with the splitting. In subsection 4.5.3 we give the proof of the C^1-closing lemma preserving the strong stable leaves. In 4.5.4, we explain the strategy of the connecting lemma. and in 4.5.5 we prove proposition 3.4.

4.5.1 Strategy of the proof of the C^1-closing lemma.

First we state two lemmas due to Pugh, used by him to get the closing lemma. The first one states that given two points close enough it is possible to get a diffeomorphisms C^1 close to the identity that sends one point to the other and such that it is the identity in the complement of a small neighborhood of the points (see local perturbations below). The second lemma states that it is possible to spread the perturbation along the orbit, in a way to obtain a safety zone to close the orbit. Since the C^1-closing lemma deals with local perturbation, it can be state for linear dynamics (see shortcoming procedure in what follows). Later we indicate a selection of points to apply the lemmas 4.5.2 and 4.5.3 to conclude the C^1 closing lemma.

1- Local perturbation.

We consider Euclidean rectangles $[-l, l]^d + \{x\}$ of radius l centered at x and we note it with $C(l)$ without expliciting the point where it is centered.

Lemma 4.5.2 *Let us consider an Euclidean rectangle $C(l)$. For all $\beta > 0$ there exists $r > 0$ such that for any $x_1, x_2 \in C(l)$ such that $d(x_1, x_2) < rd(x_1, \partial C(l))$ follows that there exists $h \in C^\infty(\mathbb{R}^d)$ such that*

1. $|h - Id|_1 < \beta$,
2. $h(x_1) = x_2$,
3. $h_{|C(l)^c} = Id$.

2- Shortcoming procedure.

Lemma 4.5.3 *Let $\mathbb{R}^d$ be endowed with some Euclidean metric $d(.,.)$ and let M_i be any sequence of linear isomorphisms of $\mathbb{R}^d$. Given $\eta > 0$ and $r > 0$ there exists an integer $N \geq 1$ and some basis of $\mathbb{R}^d$ such that the following holds; given any pair of points x, y in the cube $C(l)$ of radius l relative to this basis, there exists points z_i in the cube $C((1+\eta)l)$ of radius $(1+\eta)l$ such that $z_0 = z, z_N = y$ and*

$$d(M_i(z_{i-1}), M_i(z_i)) < rd(M_i(z_{i-1}), \partial M_i(C((1+\eta)l)))$$

for all $i = 1, ..., N$.

3- Selection of points to close orbits.

Let x be a non periodic point such that $x \in \omega(x)$. Let $\beta > 0$, and let $r = r(\beta) > 0$ be the positive constant given by lemma 4.5.2. We take $\eta = \frac{1}{3}$ and the linear maps $M_i = D_x f^i$. Then we take the integer

N given lemma 4.5.3. Now we take a neighborhood B of x such that the sets $\{f^i(B)\}_{\{0\leq i\leq N\}}$ are pairwise disjoint and the linear part $D_x f^i$ of f in the neighborhood B is close to f for any $0 \leq i \leq N$. Now, we take two consecutive iterates $f^m(x)$ and $f^n(x)$ (with $n > m$) sufficiently close such that $n - m > N$, $f^n(x), f^m(x) \in B$ and

$$d(f^n(x), f^m(x)) \leq d(f^j(x), f^k(x)) \ \forall \ m < j < k < n. \tag{4}$$

We take the minimal Euclidean cube $C(l) \subset B$ of radius l, where $l = d(f^n(x), f^m(x))$, that contains $f^n(x)$ and $f^m(x)$. Moreover, we take the points $f^n(x)$ and $f^m(x)$ close enough such that

$$C((1+\eta)l) \subset B.$$

Observe that from 4 follows that for any $0 < k < n - m$ holds

$$f^k(f^m(x)) \notin C((1+\eta)l),$$

and therefore follows that

$$f^k(f^{m+N}(x)) \notin \cup_{i\leq 0}^{N} f^i(C((1+\eta)l)) \ \forall \ 0 < k < n-m-N. \tag{5}$$

Now we select the sequences of points $\{z_i\}$ such that $z_0 = f^n(x), z_N = f^m(x)$ given by lemma 4.5.3. Then, the perturbation introduced on lemma 4.5.2 is performed along the orbit $\{f^{m+i}\}_{\{0\leq i\leq N\}}$. More precisely, for any $0 \leq i \leq N$ we take h_i such that h_i restricted to the complement of $f^i(C((1+\eta)l))$ is the identity and satisfying $h_i(f^i(z_{i-1}) = f^i(z_i)$. Taking $g = h_i \circ f$ in each $f^i(C((1+\eta)l))$ and observing that the sets $\{f^i(C((1+\eta)l))\}_{\{0\leq i\leq N\}}$ are pairwise disjoint it is obtained that $g^N(f^n(x)) = f^{m+N}(x)$. Since (5) holds then $g^{m+N+i}(x) = f^{m+N+i}(x)$ for $0 < i < n-m-N$ and therefore

$$g^{n-m}(f^n(x)) = f^n(x)$$

i.e.: $f^n(x)$ becomes a periodic point for g.

To adapt this criterium to select points, first we have to introduce rectangles coherent with the dominated splitting with the property that they have a uniform size (independent of the points involved in the perturbation) along the strong stable direction. At this point, we use strongly that the distance between two points is comparable to the distances of their images by the strong stable holonomy.

4.5.2 Rectangles coherent with the splitting.

Now we construct some kind of rectangle in terms of the splitting $E_1 \oplus E_2 \oplus E_3$. For that, we start defining the notion of rectangle coherent with the splitting.

Definition 14 *We say that a set R is a rectangle coherent with the splitting if*

$$R = int(h([-1,1]^3))$$

where $h : [-1,1]^3 \to M$ is an homeomorphism such that there exists points $x_{-1}, x_1, y_{-1}, y_1, z_{-1}, z_1$ in H_p verifying that

$$h(\{-1\} \times [-1,1]^2) \subset W^{su}_{\epsilon_0}(x_{-1}), \ \ h(\{1\} \times [-1,1]^2) \subset W^{su}_{\epsilon_0}(x_1),$$

$$h([-1,1]^2 \times \{-1\}) \subset W^{cs}_\epsilon(y_{-1}),\ \ h([-1,1]^2 \times \{1\}) \subset W^{cs}_\epsilon(x),$$

$$h([-1,1] \times \{-1\} \times [-1,1]) \subset W^{cu}_\epsilon(z_{-1}),\ \ h([-1,1] \times \{-1\} \times [-1,1]) \subset W^{cu}_\epsilon(x)$$

We call the set $h([-1,1]^2 \times \{-1\}) \cup h([-1,1]^2 \times \{1\})$ the unstable boundary of R.

Given $\epsilon^s, \epsilon^c, \epsilon^u$ and a point x we define the rectangle $R(\epsilon^s, \epsilon^c, \epsilon^u)(x)$ as the rectangle of center x and size $\epsilon^s, \epsilon^c, \epsilon^u$ in each axis. More precisely, $x = h(0,0,0)$ and

$$R(\epsilon^s, \epsilon^c, \epsilon^u)(x) = h((-\epsilon^s, \epsilon^s) \times (-\epsilon^c, \epsilon^c) \times (-\epsilon^u, \epsilon^u)).$$

Notation 4.1 *Given a rectangle R we use the following notation:*

$$W^{ss}_R(x) := W^{ss}_\epsilon(x) \cap R,\ \ W^u_R(x) := W^u_\epsilon(x) \cap R,\ \ W^{cu}_R(x) := W^{cu}_\epsilon(x) \cap R.$$

Lemma 4.5.4 *Assuming that the splitting is normally dissipative, it follows that it is possible to get a rectangle coherent with the splitting $R = int(h([-1,1]^3))$ where $h : [-1,1]^3 \to M$ is a C^1-map such that for any $(x_0, y_0, z_0) \in [-1,1]^3$ follows that*

$$h([-1,1] \times \{y_0\} \times \{z_0\}) \subset W^{ss}_\epsilon(x_0, y_0, z_0).$$

Lemma 4.5.5 *Let H_p be a topologically hyperbolic attracting homoclinic class such that the interior of $\mathcal{T}$ is empty. There is ϵ_0 such that for any $z \in H_p$, $\epsilon < \epsilon_0$ and $\delta > 0$ small there exists a pair of rectangles $\hat{R}, R$ such that*

1. *$z \in \hat{R} \subset R$,*
2. *$W^{ss}_\epsilon(z) \subset \hat{R}$,*
3. *for any $\hat{z} \in W^{ss}_\epsilon(z)$ follows that $W^{cu}_\delta(\hat{z}) \subset R \cap W^{cu}_{e_0}(\hat{z}) \subset W^{cu}_{2\delta}(\hat{z})$,*
4. *$R = int(h([-1,1]^3))$ for some homeomorphism $h : [-1,1]^3 \to M$ and $\hat{R} = h([a,b] \times [-1,1]^2)$ for a pair of points a, b such that $-1 < a < b < 1$;*
5. *$[R \setminus \hat{R}] \cap H_p = \emptyset$,*
6. *the rectangles $\hat{R}$ and R verify the thesis of lemma 4.5.4.*

The proof is given at the end of the subsection. Observe that in the hypothesis is only assume that the interior of $\mathcal{T}$ is empty. This lemma is also used in the proof of theorem 3.1.

Corollary 4.1 *Let x be a non periodic point. There exist rectangles $R_3 \subset R_2 \subset R_1$ centered at x and coherent with the splitting, such that*

1. *the rectangles $R_3 \subset R_2 \subset R_1$verify the thesis of lemma 4.5.4,*

2. if $z \in R_1 \cap H_p$ then $z \in R_3$,

3. if $z \in R_3$ and $f^n(z) \in R_3$ then

$$f^n(W^{ss}_{R_1}(z)) \subset R_2.$$

Lemma 4.5.6 *Let R be rectangle coherent with the splitting as in lemma 4.5.4. Then there exists $\gamma = \gamma(R_1)$ such that for any $x_1 \in R \cap H_p$ follows that $W^{ss}_{\gamma}(z) \subset R$ and if $x_2 \in H_p \cap R_1 \cap W^{cu}_{\epsilon}(x_1)$ then*

$$C^{-1}d(x_1, x_2) < d(z_1, z_2) < Cd(x_1, x_2),$$

where $z_1 \in W^{ss}_R(x_1)$ and $z_2 \in W^{cu}_{\epsilon}(z_1) \cap W^{ss}_R(x_2)$.

Proof: It follows from lemma 2.3.

■

4.5.3 Proof of lemma 4.5.1: C^1-closing lemma preserving the strong stable leaves.

To adapt the proof of the C^1-closing lemma to prove lemma 4.5.1, we need first an equivalent to lemma 4.5.2. Later we need a criterium to select points where to apply the perturbations and the shortcoming procedure.

Let $R = int(h([-1,1]^3))$ be a rectangle as defined in 14. Given a set of rectangles $R_3 \subset R_2 \subset R_1$ as in corollary 4.1, positive constants $\epsilon^s < \epsilon, \epsilon^c < \epsilon, \epsilon^u < \epsilon$ we take

$$R_j(\epsilon^s, \epsilon^c, \epsilon^u) = R(\epsilon^s, \epsilon^c, \epsilon^u)(x) \cap R_j \;\; j = 1, 2, 3. \tag{6}$$

1-Local perturbations preserving the strong stable leaves.

First we formulate a simple lemma that states that it is possible to perform small perturbations that sends local strong stable leaves to local strong stable leaves.

Lemma 4.5.7 *For any $\beta > 0$ there exists $r > 0$ such that for any Euclidean rectangle R coherent with the splitting, given any pair $x_1, x_2 \in R$ and connected arcs $l_{x_1} \subset W^{ss}_{\epsilon}(x_1) \cap R$, $l_{x_2} \subset W^{ss}_{\epsilon}(x_2) \cap R$ such that*

$$d(x_1, x_2) < r\, d(l_{x_j}, \partial R) \;\; j = 1, 2$$

follows that there exists $g \in C^{\infty}(\mathbb{R}^3)$ such that

1. $|g - Id|_1 < \beta$,

2. $g(l_{x_1}) = l_{x_2}$,

3. $g_{|R^c} = Id$

Proof:

Using the dominated splitting, we can assume that $x_2 \in W^{cu}_{\epsilon}(x_1)$. In fact, taking $\hat{x}_2 = W^{cu}_{\epsilon}(x_1) \cap W^{ss}_{\epsilon}(x_1)$ it follows that $dist(x_1, \hat{x}_2) \leq d(x_1, x_2)$. Therefore we can reformulate the lemma replacing x_2

by $\hat{x}_2$. Let $x_1(.)$ and $x_2(.)$ be a parameterization of the local strong stable manifold of x_1 and x_2; i.e.: $x_i \in C^2([-\epsilon,\epsilon], M^3)$ such that $x_i(t) \in W^{ss}_\epsilon(x_i)$. Moreover, we can assume that $x_i(0) = x_i$ and

$$x_1(t) \in W^{cu}_\epsilon(x_2(t)) \;\; \forall t \in [-\epsilon,\epsilon].$$

Observe that from lemma 4.5.6 there is a constant C such that

$$C^{-1}d(x_1,x_2) < d(x_1(t),x_2(t)) < Cd(x_1,x_2). \tag{7}$$

Claim 1 *Given $\beta > 0$ there exists $r > 0$ such that for any t if*

$$d(x_1(t),x_2(t)) < rd(x_1(t), \partial W^{cu}_R(x_2(t))) \;\; \forall t \in [-\epsilon,\epsilon]$$

then there is g_t defined in $\mathbb{R}^2$ such that

1. $|g_t - Id|_1 < C_0 d(x_1(t),x_2(t)) < \beta$,
2. $g_t(x_1(t)) = x_2(t)$,
3. $g_{|W^{cu}_R(x_2(t))^c} = Id$.

Proof of the claim: The proof is similar to the proof of lemma 4.5.2.

■

Since it follows that $d(x_1,x_2) < r\, d(l_{x_j}, \partial R)$, then from 7 holds that

$$d(x_1(t),x_2(t)) < rd(x_1(t), \partial W^{cu}_R(x_2(t)))$$

and so for each t we can apply claim 1. Moreover, we can take the maps g_t in such a way that they moves continuously with t.

Since $\partial_t(x_i(t)) = E_1(x_i(t))$ and

$$SL(E_1(x_1(t)), E_1(x_2(t))) < C_0 d(x_1(0),x_2(0))$$

follows that the map

$$G(t,x) = g_t(x)$$

can be taken C^1-close to the identity.

Now we restrict the map $G(.,.)$ to

$$\hat{R} = \cup_{\{x_1(t)\in l_{x_1}\}} W^{cu}_R(x_1(t)).$$

Observe that $\hat{R} \subset R$. Let x_i^{+-} be the extremal points of l_{x_i}. Since

$$d(x_1^{+-}, x_2^{+-}) < rd(x_i^{+-}, \partial R),$$

$|G - Id|_1 < C_0 d(x_1(0),x_2(0))$ and $d(x_1(0),x_2(0)) < r\, d(l_{x_j}, \partial R)$ then follows that $G_{|\hat{R}}$ can be extended to R in such a way that

$$|G - Id|_1 < \beta, \;\text{ and } G_{|R^c} = Id.$$

■

Selection of points to close strong stable leaves.

Let x be a non periodic point such that $x \in \omega(x)$. Let $\beta > 0$, and let $r = r(\beta) > 0$ be the positive constant given by lemma 4.5.2. We take $\eta = \frac{1}{3}$ and the linear maps $M_i = D_x f^i$. Then we take the integer N given lemma 4.5.3. Now we take a neighborhood $B = B(W^{ss}_\epsilon(x))$ of the local strong stable manifold of x such that the sets $\{f^i(B)\}_{\{0\leq i\leq N\}}$ are pairwise disjoint. Observe that from the fact that f contracts along the local strong stable leaves follows that B can be taken close enough to $W^{ss}_\epsilon(x)$ such that the linear part $D_x f^i$ of f is close to f in the neighborhood B for any $0 \leq i \leq N$. Let also take $R_3 \subset R_2 \subset R_1$ rectangles coherent with the splitting around x given by corollary 4.1. We take the rectangles $R_j \cap B$ $(j = 1, 2, 3.)$ and we keep noting them with R_j.

Now, we take two consecutive iterates $f^m(x)$ and $f^n(x)$ (with $n > m$) sufficiently close such that $n - m > N$, $f^n(x), f^m(x) \in R_3$ and

$$d(W^{ss}_{R_1}(f^m(x)), W^{ss}_{R_1}(f^n(x))) \leq d(W^{ss}_{R_1}(f^j(x)), W^{ss}_{R_1}(f^k(x))) \ \forall \ m < j < k < n. \tag{8}$$

Let $y_0 \in W^{ss}_{R_1}(f^n(x))$ and $y_1 \in W^{ss}_{R_1}(f^m(x))$ such that $d(y_0, y_1) = d(W^{ss}_{R_1}(f^m(x)), W^{ss}_{R_1}(f^n(x)))$. We take the minimal rectangles $R_j(\epsilon, l, l)$ $(j = 1, 2, 3.)$ of radius l, where $l = d(y_0, y_1)$, that contains y_0 and y_1 (recall (6) for the definition of these rectangles). We note them with $R_j(l)$. We also take the minimal Euclidean cube $C(l) \subset R_3(l)$ of radius l, where $l = d(y_0, y_1)$, that contains y_0 and y_1. We also take the points $f^n(x)$ and $f^m(x)$ close enough such that $R_1((1 + \eta)l) \subset B$. Observe that the sets $\{f^i(R_1((1 + \eta)l))\}_{\{0\leq i\leq N\}}$ are pairwise disjoint.

Let

$$0 < \gamma < \frac{1}{2} \min_{z\in R_3} \{\ell(W^{ss}_{R_1}(z) \setminus W^{ss}_{R_2}(z)).$$

Claim 2 *For any* $0 < k < n - m$ *follows that*

$$W^{ss}_\gamma(f^k(f^m(x))) \cap R_3((1 + \eta)l) = \emptyset.$$

Proof of the claim: In fact, if the intersection is not empty from the election of γ and the rectangles R_j follows that

$$f^k(f^m(x)) \in R_1((1 + \eta)l)$$

and therefore if $\hat{y} = W^{ss}_{R_1}(f^k(f^m(x))) \cap W^{cu}_\epsilon(y_0)$ then

$$\min\{dist(\hat{y}, y_0), dist(\hat{y}, y_1)\} < dist(y_1, y_0)$$

which is a contradiction with (8).

■

From the previous claim follows that

$$W^{ss}_\gamma(f^{N+m}(x))) \cap R_3((1 + \eta)l) = \emptyset \tag{9}$$

$$f^k(W^{ss}_\gamma(f^{N+m}(x)))) \cap \cup^N_{i\leq 0} f^i(R_3((1 + \eta)l)) = \emptyset \ \forall \ 0 < k < n - m - N. \tag{10}$$

Now we select the sequences of points $\{z_i\}$ given by lemma 4.5.3 such that $z_0 = y_0, z_N = y_1$. Let us take now

$$l_i = f^i(W^{ss}_{R_3}(z_i)) \ \text{ and } \ l_{i-1} = f^i(W^{ss}_{R_3}(z_{i-1})).$$

Lemma 4.5.8 *For any points z_i follows that*

$$d(f^i(z_i), f^i(z_{i-1})) < r \min\{d(l_i, \partial f^i(R_1)), d(l_{i-1}, \partial f^i(R_1))\}.$$

Before to prove the lemma, let us show that how to finish the proof of lemma 4.5.1. In fact, the lemma 4.5.8 allows to perform the perturbation introduced on lemma 4.5.7 along the orbit $\{f^i(z_i)\}_{\{0\leq i\leq N\}}$. More precisely, for any $0 \leq i \leq N$we take h_i such that h_i restricted to the complement of $f^i(R_1((1+\eta)l))$ is the identity and satisfying

$$h_i(f^i(W^{ss}_{R_3}(z_{i-1})) = f^i(W^{ss}_{R_3}(z_{i-1})).$$

Taking $g = h_i \circ f$ in each $f^i(R_1((1+\eta)l))$ and observing that the sets $\{f^i(R_1((1+\eta)l))\}_{\{0\leq i\leq N\}}$ are pairwise disjoint it is obtained that

$$g^N(W^{ss}_{R_3}(f^n(x))) = f^N(W^{ss}_{R_3}(f^m(x))).$$

Since (9) holds then $g^{N+i}(W^{ss}_{R_3}(f^m(x))) = f^{N+i}(W^{ss}_{R_3}(f^m(x)))$ for $i < n-m-N$ and therefore

$$g^{n-m}(W^{ss}_{R_3}(f^n(x))) \subset W^{ss}_{R_3}(f^n(x)),$$

i.e., $W^{ss}_{R_3}(f^n(x))$ becomes a periodic strong stable leaf.

Proof of lemma 4.5.8: First we estimate the distance of the image of the strong stable leaves inside R_3 with the distance to the boundary of the images of R_1.

Claim 3 *There exists a constant C_0 such that for any point z_i follows that*

$$C_0^{-1}\gamma||Df^i_{|E_1(z_i)}|| < d(f^i(W^{ss}_{R_3}(z_i)), \partial f^i(R_1)) < C_0\gamma||Df^i_{|E_1(z_i)}||.$$

Proof of the claim: Observe first that from the domination follows that

$$d(f^i(W^{ss}_{R_3}(z_i)), \partial f^i(R_1)) = \ell(f^i(W^{ss}_{R_1}(z_i) \setminus W^{ss}_{R_3}(z_i))).$$

Observe now that for each connected component of $W^{ss}_{R_1}(z_i)\setminus W^{ss}_{R_3}(z_i)$ there is a point y_i in this connected component such that

$$\ell(f^i(W^{ss}_{R_1}(z_i) \setminus W^{ss}_{R_3}(z_i))) = \ell(W^{ss}_{R_1}(z_i) \setminus W^{ss}_{R_3}(z_i))||Df^i_{|E_1(y_i)}|| = \gamma||Df^i_{|E_1(y_i)}||.$$

Since $W^{ss}_{R_1}(z_i)$ is C^2 and there is K_0 such that

$$\Sigma_{k>0}\ell(f^k(W^{ss}_{R_1}(z_i))) < K_0,$$

follows that there is C_0 such that

$$C_0^{-1} < \frac{||Df^i_{|E_1(y_i)}||}{||Df^i_{|E_1(z_i)}||} < C_0.$$

■

Claim 4 *There exists a constant C_0 such that for any point $y_i \in W^{ss}_{R_1}(z_i)$ follows that*

$$C_0^{-1}(1+\eta)l||Df^i_{|E_1(z_i)}|| < d(f^i(z_i), \partial f^i(C((1+\eta)l))) < C_0(1+\eta)l||Df^i_{|E_1(z_i)}||.$$

Proof of the claim: The proof is similar to the proof of claim 3.

■

Now we can finish the proof of lemma 4.5.8. In fact, since

$$d(f^i(z_i), f^i(z_{i-1}) < rd(f^i(z_i), \partial f^i(C((1+\eta)l))),$$

from claim 3, claim 4 and that $l << \gamma$, it follows the lemma.

■

4.5.4 Strategy of the proof of the C^1−connecting lemma.

In the context of the connecting lemma, it is given two orbit segments $f^i(y)$, $0 \leq i \leq m$ and $f^{-i}(z)$, $0 \leq i \leq n$ such that $f^m(y)$ is close to $f^{-n}(z)$ and the goal is to find a perturbation for which y and z are in the same orbit (observe that this is enough to conclude the connecting lemma). The strategy introduced by Hayashi consists in to make perturbation at several places in order to shorten the two segments every times one of them comes closes to itself or to the other.

To perform that, it is taken a small Euclidean cube containing $f^m(y)$ and $f^{-n}(z)$, and later it is tiled this cube into Euclidean sub-cubes such that the sizes of each sub cube is comparable up to a factor of the size of its neighborhood.

Arrangement of Euclidean cubes.

For all $i \geq 0$ we take $\alpha_i = 1 + \Sigma_{j=0}^{i} 2^{-j}$ and the interval $[-2^i\alpha_i, 2^i\alpha_i - 1]$. For each $i \geq$ and $j \in [-2^i\alpha_i, 2^i\alpha_i - 1]$ we take the Euclidean rectangle (called tiles)

$$R^{i,k}(\epsilon) = [\epsilon\frac{k}{2^i}, \epsilon\frac{k+1}{2^i}]^3.$$

Then we take

$$R(\epsilon) = \cup_{i,k} R^{i,k}(\epsilon).$$

Then it is considered the following criterium of connecting intermediate orbits:

Criterium of connecting orbits: *Given two points of the orbits $f^i(y)$, $0 \leq i \leq m$ and $f^{-i}(y)$, $0 \leq i \leq n$ are selected if either:*

1. *they belong to the same tile and the distance between then is smaller to any intermediate point,*
2. *they belong to different neighbors tiles but the distance between then is much smaller than the size of the neighbors tiles.*

After that, the selected orbits are connected, using the shortcoming procedure introduced in the proof of the C^1 closing lemma, following that y and x are in the same orbit for the perturbed map.

In the direction to obtain the proof of proposition 3.4 we need to apply the previous criterium for rectangles coherent with the splitting.

4.5.5 Proof of proposition 3.4.

Arrangement of rectangles coherent with the splitting.

To adapt the strategy of the C^1-connecting lemma preserving the strong stable leaves we have to adapt the construction and arrangement of the tiles satisfying the following properties:

1. the tiles are built using the dominated splitting,
2. the tiles has uniform size along the strong stable direction,
3. the tiles are arranged in such a way that if two points in different tiles are closed then they are far from the strong stable boundary of the tiles.

Finishing the proof of proposition 3.4.

To do that, we take $z \in W^s_\epsilon(q) \setminus \{q\}$ such that z is accumulated by the positive orbit of x (recall that there exists $y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$). Now we take a rectangles $R_3 \subset R_2 \subset R_1$ as in corollary 4.1 and we take $h : [-1,1]^3 \to M$ as in lemma 4.5.4 such that $R_1 = h([-1,1]^3)$. We select those rectangles in such a way that if $f^k(x) \in R_1$ then follows that

$$[f^k(x), f^k(y)]^{ss} \subset R_3,$$

where $[f^k(x), f^k(y)]^{ss}$ is the connected arc contained in $W^{ss}_\epsilon(f^k(x))$ such that its extremal points are given by $f^k(x)$ and $f^k(y)$. This is possible taking R_1 small in the vertical direction in such a way that if $f^k(x) \in R_1$ then

$$\ell(f^k(W^{ss}_\epsilon(x)))) < \gamma < \frac{1}{2}\min_{z\in R_3}\{\ell(W^{ss}_{R_1}(z) \setminus W^{ss}_{R_2}(z)).$$

Therefore, since if $f^k(x) \in R_1$ then follows that $f^k(x) \in R_3$, it holds that $\ell(f^k(W^{ss}_\epsilon(x)))) \subset R_1$ and so, $f^k(y) \in R_1$ and since $f^k(y) \in H_p$ holds that $f^k(y) \in R_3$, which implies that $[f^k(x), f^k(y)]^{ss} \subset R_3$. Then, given $\beta > 0$ we select $r = r(\beta) > 0$ given by lemma 4.5.7 and we perform the following arrangement of the rectangles: For all $i \geq 0$ we take $\alpha_i = 1 + \Sigma^i_{j=0} 2^{-j}$ and the interval $[-2^i\alpha_i, 2^i\alpha_i - 1]$. For each $i \geq$ and $j \in [-2^i\alpha_i, 2^i\alpha_i - 1]$ we take the rectangle

$$R^{i,k}_j(\epsilon^c, \epsilon^u) = h([-l_0, l_0] \times [-\epsilon^c\frac{k}{2^i}, \epsilon^c\frac{k+1}{2^i}] \times [-\epsilon^u\frac{k}{2^i}, \epsilon^u\frac{k+1}{2^i}]) \cap R_j.$$

Then we take

$$R_j(\epsilon^c, \epsilon^u) = \cup_{i,k} R^{i,k}_j(\epsilon^c, \epsilon^u).$$

Observe that

$$[R_1(\epsilon^c, \epsilon^u) \setminus R_3(\epsilon^c, \epsilon^u)] \cap H_p = \emptyset.$$

Then, it is applied the connecting lemma criterium to connect orbit using the present arrangement of rectangles and the perturbation introduced in lemma 4.5.7. This conclude the proof of proposition 3.4.

4.5.6 Proof of lemma 4.5.5.

To prove the lemma 4.5.5 first we use the following lemma. In this lemma it is used explicitly that the interior of $\mathcal{T}$ is empty.

Lemma 4.5.9 *Let H_p be a topologically hyperbolic attracting homoclinic class such that the interior of $\mathcal{T}$ is empty. Then for every ϵ' there is a connected arc $l_x^+ \subset W_{\epsilon'}^{ss,+}(x)$ such that $l_x^+ \cap H_p = \emptyset$ and there is a connected arc $l_x^- \subset W_{\epsilon'}^{ss,-}(x)$ such that $l_x^- \cap H_p = \emptyset$, where $W_{\epsilon'}^{ss,+}(x)$ and $W_{\epsilon'}^{ss,-}(x)$ are the connected components of $W_{\epsilon'}^{ss}(x) \setminus \{x\}$.*

Proof of lemma 4.5.9:
Let us assume that the thesis of the lemma is false; i.e: there is $x \in H_p$ such that for instance $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ for some small ϵ_x. We consider two cases:

1. For some x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (for some small ϵ_x), there is $y \in W_{\epsilon}^{ss,+}(x)$ such that $\Pi^{ss}(W_\epsilon^u(y))$ does not coincide with $W_\epsilon^u(x)$ or
2. For any x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (some small ϵ_x) holds that for any $y \in W_{\epsilon}^{ss,+}(x)$ follows that $\Pi^{ss}_{f,x,y}(W_\epsilon^u(y)) = W_\epsilon^u(x)$; i.e.: $W_\epsilon^u(y) \subset W_\epsilon^{su}(x) = \cup_{\{z \in W_\epsilon^{ss,+}(x)\}} W_\epsilon^u(z)$

In other words, we are considering if for any x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (some small ϵ_x) given the point x then the strong foliation associated to x are either jointly integrable or it is not the case. Let us take

$$W_\epsilon^{us}(x) = \cup_{\{z \in W_\epsilon^{ss,+}(x)\}} W_\epsilon^u(z)$$

From the fact that we are assuming that $W_\epsilon^{ss,+}(x)$ is contained in H_p and from the fact that H_p is an attractor, follows that $W_\epsilon^{us}(x) \subset H_p$.

Given a point z_0 in $W_\epsilon^{us}(x)$ we consider the set $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ and observe that there is z_0 such that $W_\epsilon^{ss}(z_0)$ intersect transversally $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$, in the sense that $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ intersects both components of $W_\epsilon^s(z_0) \setminus W_\epsilon^{ss}(z_0)$. To check this assertion, it is enough to take z_0 such that that $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ intersects only one components of $W_\epsilon^s(z_0) \setminus W_\epsilon^{ss}(z_0)$ and $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ it is not contained in $W_\epsilon^{ss}(z_0)$. This point z_0 exists because otherwise it follows that the strong foliations are jointly integrable. Then, it holds immediately that we can choose another point $z_0' \in W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ such that $W_\epsilon^{ss}(z_0') \cap W_\epsilon^{us}(x)$ intersect both components of $W_\epsilon^s(z_0') \setminus W_\epsilon^{ss}(z_0')$.

Now, let z be any point close to z_0 contained in $H_p \cap W_\epsilon^s(z_0)$, so it follows that $W_\epsilon^{ss}(z)$ intersect transversally $W_\epsilon^s(z) \cap W_\epsilon^{us}(x)$ (this follows from the fact that $W_\epsilon^{ss}(z)$ is C^1-close to $W_\epsilon^{ss}(z_0)$). Now we have two options: If for some z close to z_0 holds that $z \notin W_\epsilon^{us}(x)$ or for any z close to z_0 holds that $z \in W_\epsilon^{us}(x)$. In the former, taking an small neighborhood of z follows that for any z' in this small neighborhood of z holds that $[W_\epsilon^{ss}(z') \setminus \{z'\}] \cap H_p \neq \emptyset$, and this implies that the interior of $\mathcal{T}$ is not empty, which is absurd. In the latter, from the fact that any point z close to z_0, follows that we can take a periodic point q such that $q \in W_\epsilon^{us}(x)$. We take $W_\epsilon^s(q)$ and $W_\epsilon^s(q) \cap W_\epsilon^{us}(x)$ and we can assume that $W_\epsilon^{ss}(q)$ intersect transversally the set $W_\epsilon^s(q) \cap W_\epsilon^{us}(x)$ (otherwise, using that the periodic points are dense it can be argued as before). Then we take $f^{n_q k}(W_\epsilon^s(q) \cap W_\epsilon^{su}(x))$ where n_q is the period of q and k is large positive integer. Observe that $f^{n_q k}(W_\epsilon^s(q) \cap W_\epsilon^{us}(x)) \subset W_\epsilon^s(q)$. Since

$W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ intersect transversally $W^{ss}_\epsilon(q)$ follows that $W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ is not invariant by $f^{n_q k}$, then there is $z \in f^{n_q k}(W^s_\epsilon(q) \cap W^{us}_\epsilon(x)) \setminus [W^s_\epsilon(q) \cap W^{su}_\epsilon(x)]$ close to q. Taking an small neighborhood of z follows that for any z' in this small neighborhood holds that $[W^{ss}_\epsilon(z') \setminus \{z'\}] \cap H_p \neq \emptyset$, and this implies that the interior of $\mathcal{T}$ is not empty, which is absurd. See figure 20.

In the second situation let us consider

$$W^{us}(x) = Closure(\cup_{\{n>0\}} \cup_{\{y \in W^{ss}_\epsilon(x)\}} f^n(W^u_\epsilon(f^{-n}(y))))$$

Observe that for any $y \in W^{us}(x)$ follows that there is ϵ_y such that $W^{us}_{\epsilon_y}(y) = W^{su}_{\epsilon_y}(y) \subset W^{us}(x)$. Let us take also

$$\Lambda_0 = Closure(\cup_{\{y \in W^{us}(x)\}} \alpha(y)).$$

Observe that Λ_0 is a topologically hyperbolic compact invariant set such that for any $z \in \Lambda_0$ follows that $W^u(z) \subset \Lambda_0$. To prove that, is enough to prove that for any $z \in \alpha(y)$ for some $y \in W^{us}(x)$ follows that $W^u_\epsilon(z) \subset \Lambda_0$ and observe that this immediate from the definition of $W^{us}(y)$.

From the fact that $W^u(z) \subset \Lambda_0$ for any $z \in \Lambda_0$, follows that Λ_0 has local product structure. So $\Lambda_0 = \cap_{n \in \mathbb{Z}} f^n(V)$ for some V. Since it holds that whole unstable manifold of each points is contained in Λ_0 and the unstable manifolds are dense, follows that $H_p = \Lambda_0$.

On the other hand, observe that if $z \in \Lambda_0$, then $W^{us}_\epsilon(z) \subset \Lambda_0$ and $W^{us}_\epsilon(z) = W^{su}_\epsilon(z)$. It follows from the fact that $W^{us}_{\epsilon'}(f^{-n}(y)) \subset f^{-n}(W^{us}(x))$ and from the fact that for any $z \in W^{us}(x)$ there is ϵ_z such that $W^{us}_{\epsilon_z}(x) \subset W^{us}(x)$ and $W^{us}_{\epsilon_z}(x) = W^{su}_{\epsilon_z}(x)$. Therefore, $\mathcal{T} = H_p$ and so the interior is not empty, which is an absurd.

■

Let us continue now with the proof of lemma 4.5.5. Let us start taking a point $x \in H_p$. By the previous lemma, for any ϵ' there exist arcs l^+_x and l^-_x contained in opposite connected components of $W^{ss}_{\epsilon'}(x) \setminus \{x\}$ and such that $l^+_x \cap H_p = \emptyset$ and $l^-_x \cap H_p = \emptyset$. So, there exists $\gamma_x > 0$ such that $W^c_{\gamma_x}(l^+_x) \cap H_p = \emptyset$ and $W^c_{\gamma_x}(l^-_x) \cap H_p = \emptyset$, where $W^c_{\gamma_x}(l^{+-}_x) = \cup_{\{z \in l^{+-}_x\}} W^c_{\gamma_x}(z)$. Let z^-_x and z^+_x (they depends on the point x) in opposite connected components of $W^c_\epsilon(x)$ such that $W^{ss}_\epsilon(z^-_x) \cap W^c_{\gamma_x}(l^+_x) \neq \emptyset$, $W^{ss}_\epsilon(z^+_x) \cap W^c_{\gamma_x}(l^+_x) \neq \emptyset$ and $W^{ss}_\epsilon(z^-_x) \cap W^c_{\gamma_x}(l^-_x) \neq \emptyset$, $W^{ss}_\epsilon(z^+_x) \cap W^c_{\gamma_x}(l^-_x) \neq \emptyset$. Let us consider the region B_x in $W^s_\epsilon(x)$ bounded by $W^{ss}_\epsilon(z^-_x)$, $W^c_{\gamma_x}(l^+_x)$, $W^{ss}_\epsilon(z^+_x)$ and $W^c_{\gamma_x}(l^-_x)$. Now we take the rectangle given by:

$$R_x = \cup_{\{z \in B_x\}} W^u_\epsilon(z).$$

Observe that this rectangle verifies the property required in the thesis of lemma 4.5.5.

■

5 Non jointly integrable case with non transversal intersections.

5.1 Proof of proposition 3.5.

Observe that given a point in the homoclinic class, the local strong stable manifold of it, splits the local stable manifold in two disjoint sets; i.e.: $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$ has two disjoint connected components. Using this, we introduce the following definition:

Definition 15 Stable boundary point: *We say that a point x is a stable boundary point, if $H_p \cap W^{ss}_\epsilon(x)$ accumulates on x from only one connected component of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$.*

Now we prove that under the assumption of non integrability of the strong foliation and that there are not transversal intersection then either there are stable boundary points or the thesis of proposition 3.5 holds.

Lemma 5.1.1 *If there are not transversal intersection and the strong foliations are not jointly integrable, then for any compact set Λ such that $\mathcal{T}_\Lambda \neq \emptyset$ follows that either there exist stable boundary points in Λ or the thesis of proposition 3.5 holds.*

Proof:

Let us assume that there are not stable boundary points in Λ. Then, there are two situations to consider: for every $x \in \Lambda$ follows that either $[W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)] \cap H_p = \emptyset$ or the homoclinic class intersect both components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$.

Suppose that there is a point $x \in \Lambda$ such that $[W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)] \cap H_p = \emptyset$. Recall that there are not isolated point in the homoclinic class. In particular, there are periodic points nearby x. Therefore, if $[W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)] \cap H_p = \emptyset$ follows that the local unstable manifold of those periodic points intersects the local strong manifold of the point x. So, there are periodic points such that their local unstable manifold $s-$intersect each other and then the thesis of proposition 3.5 holds.

To finish the proof we show that the second option cannot hold. In fact, it is shown that if for every $x \in H_p$, the homoclinic class intersect both components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$ and the strong foliation are not jointly integrable then we can find a pair of points such that their local unstable manifold s-intersect transversally. Which is a contradiction with the hypothesis of the lemma.

To prove that, let us consider a pair of point x and y such that they belong to the same strong stable manifold. Moreover, since that we are assuming that the strong foliations are not jointly integrable, we can suppose that $\Pi^{ss}(W^u_\epsilon(x))$ does not coincide with $W^u_\epsilon(y)$. Then $\Pi^{ss}(W^u_\epsilon(x))$ is contained in the closure of one of the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ and there is a point x' in $\Pi^{ss}(W^u_\epsilon(x))$ which is properly contained in $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$. This is equivalent to say that, there is a point $y' \in W^u_\epsilon(y)$ such that is properly contained in $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$; i.e. $dist(y', \Pi^{ss}(W^u_\epsilon(x))) > r_0 > 0$. Since y is not a boundary points, we can take a point z close to y contained in $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$ such that $\Pi^{ss}(z)$ is contained in the same connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that contains x'. Moreover, follows that $\Pi^{ss}(z)$ is contained in the connected component of $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$ that does not contain y'. Therefore, since there are not transversal intersections, follows that $\Pi^{ss}(W^u_\epsilon(z))$ is contained in the closure of the connected component of $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$ that does not contain y' and so

$dist(y', \Pi^{ss}(W^u_\epsilon(z))) > r_0 > 0$. However, if z is close enough to y follows that $\Pi^{ss}(W^u_\epsilon(z))$ is close to $W^u_\epsilon(y)$ and in particular $\Pi^{ss}(W^u_\epsilon(z))$ is arbitrarily close to y' which is a contradiction.

■

Proof of proposition 3.5:

We start the proof with the following lemma:

Lemma 5.1.2 *Let x be a boundary point. Then, either it belongs to the unstable manifold of some periodic points or there exist a periodic points p_x such that $W^{ss}_\epsilon(x) \cap W^u(p_x) \neq \emptyset$.*

Proof: Let us suppose that there is a boundary point x which is not contained in the unstable manifold of any periodic point. Let us take the sequence $\{f^{-n}(x)\}_{n>0}$ and take n_1, n_2, n_3 arbitrarily large such that the points $f^{-n_1}(x), f^{-n_2}(x)$, and $f^{-n_3}(x)$ verify that $dist(f^{n_i}(x), f^{n_j}(x)) < \frac{\epsilon}{4}$. Observe that $f^{-n_i}(x) \notin W^u_\epsilon(f^{-n_j}(x))$ for $i \neq j$ $i, j = 1, 2, 3$. If it is not the case, follows that $f^{-n_i}(x)$ is contained in the local unstable manifold of a periodic point.

Claim 5 *The local unstable manifold of at least two of the three points $f^{-n_1}(x)$, $f^{-n_2}(x)$, $f^{-n_3}(x)$ s-intersects each other.*

Proof of the claim: Assume now that the local unstable manifold of the three points do not s-intersect each other. In this case, follows that there is one of the three points, for instance $f^{-n_2}(x)$ such that the unstable manifold of $f^{-n_1}(x)$ and $f^{-n_3}(x)$ intersects the stable manifold of $f^{-n_2}(x)$ on opposite connected components of $W^s_\epsilon(f^{-n_2}(x)) \setminus W^{ss}_\epsilon(f^{-n_2}(x))$ of it.
Now, taking

$$z_{n_2} = W^u_\epsilon(f^{-n_1}(x)) \cap W^s_\epsilon(f^{-n_2}(x)) \text{ and } z'_{n_2} = W^u_\epsilon(f^{-n_3}(x)) \cap W^s_\epsilon(f^{-n_2}(x))$$

follows that they belong to H_p and they are in different components of $W^s_\epsilon(f^{-n_2}(x)) \setminus W^{ss}_\epsilon(f^{-n_2}(x))$. Then, using that n_1, n_2, n_3 are arbitrarily large follows that $f^{n_2}(z_{n_2}) \to x$ and $f^{n_2}(z'_{n_2}) \to x$ as $n_2 \to +\infty$ accumulating on x from different components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$, which is a contradiction since we are assuming that x is a boundary point.

■

Let us suppose without loss of generality that the local unstable manifold of $f^{-n_1}(x)$ s-intersects the local unstable manifold of $f^{-n_2}(x)$, with n_1 and n_2 arbitrarily large.

Claim 6 *Without loss of generality we can assume that*

$$W^{ss}_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(f^{-n_2}(x)) \neq \emptyset.$$

Proof of the claim: Let us suppose that this is not the case. Therefore the unstable manifold of $f^{-n_2}(x)$ intersect the stable manifold of $f^{-n_1}(x)$ on one component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$. We claim that the point $f^{-n_1}(x)$ is only accumulated by points of the homoclinic class only in the same component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$ where the unstable manifold of $f^{-n_2}(x)$ intersects $W^s_\epsilon(f^{-n_1}(x))$. In fact, if this is not the case, using that n_1 and n_2 are arbitrary large follows that x is not a boundary point; i.e.: if there are points $z \in H_p$ close to $f^{-n_1}(x)$ in the opposite component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$

that contains $z_{n_1} = W^u_\epsilon(f^{-n_2}(x)) \cap W^s_\epsilon(f^{-n_1}(x))$ follows that x is accumulated by $f^{n_1}(z)$ and by $f^{n_1}(z_{n_1})$ from different connected components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$, which is an absurd since we are assuming that x is a boundary point. The same can be conclude for $f^{-n_2}(x)$.
Now we take

$$\Pi^{ss} : B_\epsilon(f^{-n_1}(x)) \to W^{cu}_\epsilon(f^{-n_1}(x)).$$

Let us also take L^+ and L^- the closure of the connected components of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus W^u_\epsilon(f^{-n_1}(x))$. Let us assume that $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ is contained in the closure of L^+. Let us take the point $\Pi^{ss}(f^{-n_3}(x))$ and observe that it can not be contained in the interior of the region bounded by $W^u_\epsilon(f^{-n_1}(x))$ and $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ (let us denote this region with $\hat{L}$), because otherwise follows that $W^u_\epsilon(f^{-n_1}(x))$ and $W^u_\epsilon(f^{-n_2}(x))$ intersects both connected components $W^s_\epsilon(f^{-n_3}(x))) \setminus W^{ss}_\epsilon(f^{-n_3}(x))$, which is an absurd because x is a boundary point (see proof of claim 5). Therefore, it follows that either $\Pi^{ss}(f^{-n_3}(x)) \in \hat{L}^c$ or one of the following options holds:

1. $\Pi^{ss}(f^{-n_3}(x)) \in W^u_\epsilon(f^{-n_1}(x))$,
2. $\Pi^{ss}(f^{-n_3}(x)) \in \Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$.

In the first option, changing $f^{-n_1}(x)$ and $f^{-n_2}(x)$ by $f^{-n_1}(x)$ and $f^{-n_3}(x)$ then the claim holds.
In the second option, changing $f^{-n_1}(x)$ and $f^{-n_2}(x)$ by $f^{-n_2}(x)$ and $f^{-n_3}(x)$ then the claim holds.
Now, let us consider the case that $\Pi^{ss}(f^{-n_3}(x)) \in \hat{L}^c$. Therefore, it follows the next options:

1. $\Pi^{ss}(f^{-n_3}(x)) \in L^-$,
2. $\Pi^{ss}(f^{-n_3}(x)) \in L^+ \setminus \hat{L}$.

In the first option follows that $\Pi^{ss}(W^u_\epsilon(f^{-n_3}(x)))$ is contained in the closure of L^- and since $f^{-n_1}(x)$ is accumulated by points in L^+ and since $f^{-n_1}(x)$ is boundary point (follows from the fact that x is a boundary point) follows that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(f^{-n_3}(x)))$. Therefore, changing $f^{-n_1}(x)$ and $f^{-n_2}(x)$ by $f^{-n_1}(x)$ and $f^{-n_3}(x)$ respectively, then the claim holds. In the last option, follows that $\Pi^{ss}(W^u_\epsilon(f^{-n_3}(x)))$ is contained in the closure of $L^+ \setminus \hat{L}$ and since $f^{-n_2}(x)$ is accumulated by points in $\hat{L}$ follows that $f^{-n_2}(x) \in \Pi^{ss}(W^u_\epsilon(f^{-n_3}(x)))$. Therefore, changing $f^{-n_1}(x)$ and $f^{-n_2}(x)$ by $f^{-n_2}(x)$ and $f^{-n_3}(x)$ respectively, then the claim holds.

■

Now, we consider two situations:

1. $W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x)) = W^u_\epsilon(f^{-n_1}(x))$,
2. $W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x))$ is properly contained in $W^u_\epsilon(f^{-n_1}(x))$.

Observe that the first situation can hold even if we are assuming that the strong foliation are not jointly integrable. In the first case, we can assume without loss of generality, that $n_2 < n_1$ and $n_1 - n_2$ is arbitrarily large. We take an arc l containing $f^{-n_1}(x)$ and such that $l \subset W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x))$. Then we take $f^k(l)$ where $k = n_1 - n_2$ and observe that $f^k(l) \subset W^u(f^{-n_2}(x))$ and $\Pi^{ss}(f^k(l))$ contains l (where Π^{ss} projects over $W^{cu}_\epsilon(f^{-n_1}(x))$). So, there is a periodic point q such that $W^u_\epsilon(q)$ contains

$\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and $\Pi^{ss}(W^u_\epsilon(f^{-n_1}(x)))$ (where the projection is done over the center unstable manifold of q). Therefore, the thesis of the lemma holds.
Now we study the second situation. Let us take $\Pi^{ss} : B(f^{-n_1}(x)) \to W^{cu}_\epsilon(f^{-n_1}(x))$. Let us take points $z_1 \in W^u_\epsilon(f^{-n_1}(x))$ such that $z_1 \notin \Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and $z_2 \in W^u_\epsilon(f^{-n_2}(x))$ such that $\Pi^{ss}(z_2) \notin W^u_\epsilon(f^{-n_1}(x))$.
Let us take the connected component of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus W^u_\epsilon(f^{-n_1}(x))$ and we note with $L^+_{f^{-n_1}(x)}$ the one that contains $\Pi^{ss}(z_2)$. Let us take the connected components of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus \Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and we note with $L^-_{f^{-n_2}(x)}$ the one that contains z_1; with $L^+_{f^{-n_2}(x)}$ we note the other component. Related to this components, observe that $L^+_{f^{-n_2}(x)} \subset L^+_{f^{-n_1}(x)}$ and $L^-_{f^{-n_1}(x)} \subset L^-_{f^{-n_2}(x)}$.
Let q_2 be a periodic point close to z_2; observe that since there is nontransversal intersections then $\Pi^{ss}(W^u_\epsilon(q_2))$ is either contained in the closure of $L^+_{f^{-n_2}(x)}$ or in the closure of $L^-_{f^{-n_2}(x)}$. Let us first consider that $\Pi^{ss}(W^u_\epsilon(q_2))$ is contained in the closure of $L^+_{f^{-n_2}(x)}$. In this case either follows that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_2))$ and so $W^{ss}_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(q_2)$ and then concluding the thesis of the lemma, or $f^{-n_1}(x)$ is accumulated by points from $L^+_{f^{-n_1}(x)}$. Let us consider now that $\Pi^{ss}(W^u_\epsilon(q_2))$ is contained in the closure of $L^-_{f^{-n_2}(x)}$. Then it implies that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_2))$ otherwise, if it is not the case, since $\Pi^{ss}(W^u_\epsilon(q_2))$ is close to $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ follows that $\Pi^{ss}(W^u_\epsilon(q_1))$ intersect transversally $W^u_\epsilon(f^{-n_1}(x))$ which is an absurd. Therefore, either the lemma is concluded or $f^{-n_1}(x)$ is accumulated by points from $L^+_{f^{-n_1}(x)}$. Let us suppose so that $f^{-n_1}(x)$ is accumulated by points from $L^+_{f^{-n_1}(x)}$. We take now a periodic point q_1 close to z_1; observe that $\Pi^{ss}(W^u_\epsilon(q_1))$ is contained in the closure of $L^-_{f^{-n_1}(x)}$: in fact, if it is not the case, since $\Pi^{ss}(W^u_\epsilon(q_1))$ is close to $W^u_\epsilon(f^{-n_1}(x))$ follows that $\Pi^{ss}(W^u_\epsilon(q_1))$ intersect transversally $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ which is an absurd. Now, if it happens that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_1))$ again the lemma is finished; if not follows that $f^{-n_1}(x)$ is accumulated by points from $L^-_{f^{-n_1}(x)}$. Therefore, either the lemma is concluded or $f^{-n_1}(x)$ is accumulated by points from $L^-_{f^{-n_1}(x)}$. Because also holds that $f^{-n_1}(x)$ is accumulated by points from $L^+_{f^{-n_1}(x)}$ and x is a boundary point it can not hold that $f^{-n_1}(x)$ is accumulated by points from $L^-_{f^{-n_1}(x)}$. This finish the proof of lemma 5.1.2.
Now, we can prove the following lemma:

Lemma 5.1.3 *Let us assume that there are not transversal intersection and there are stable boundary points. Then, if $x, y \in H_p$ are such that $y \in W^{ss}_\epsilon(x)$ follows that x and y are boundary points.*

Proof: Let us assume that the lemma is false. Let us take x, y such that $y \in W^{ss}_\epsilon(x)$, we can assume that for instances y is not a boundary point. Moreover, if the unstable foliations are not jointly integrable, we can also assume that $\Pi^{ss}_{f,y}(W^u_\epsilon(x))$ intersect the interior of one of the connected components of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$.
Let us take a periodic point q close to y such that $\Pi^{ss}(q)$ is in the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that its closure contains $\Pi^{ss}(W^u_\epsilon(x))$. Then, since $\Pi^{ss}(W^u_\epsilon(q))$ is close to $W^u_\epsilon(y)$, $\Pi^{ss}(W^u_\epsilon(q))$ is in the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that its closure contains $\Pi^{ss}(W^u_\epsilon(x))$. Since $\Pi^{ss}(W^u_\epsilon(x))$ and $W^u_\epsilon(y)$ do not coincide, follows that $\Pi^{ss}(W^u_\epsilon(q))$ intersects $W^u_\epsilon(x)$ transversally. Which is a contradiction because we are assuming that they are not transversal intersection.

■

Now, we proceed to show that the two previous lemma imply proposition 3.5.

Let $x, y \in \Lambda$ such that $y \in W^{ss}_{\epsilon}(x)$. By lemma 5.1.3 follows that they are boundary points. By lemma 5.1.2 either they belong to the unstable manifold of a periodic point or there exist two periodic points p_x and p_y such that $W^{ss}_{\epsilon}(x) \cap W^u(p_x) \neq \emptyset$ and $W^{ss}_{\epsilon}(y) \cap W^u(p_y) \neq \emptyset$. Taking $x' = W^{ss}_{\epsilon}(x) \cap W^u(p_x) \neq \emptyset$ and $y' = W^{ss}_{\epsilon}(y) \cap W^u(p_y) \neq \emptyset$, the proposition follows.

■

5.2 Proof of proposition 3.6.

Let $x, y, p_x, p_y \in H_p$ as in lemma 3.5. Now, we consider two cases: either x and y belongs to the stable manifold of some periodic point or it is not the case.

In the first case, we apply lemma 3.1.3. The rest of the subsection, deals with the second case. Since x and y do not belong to the stable manifold of some periodic point then $h_g^{-1}(x)$ and $h_g^{-1}(y)$ are single points and so from now on, for each g let us denote $x_g = h_g^{-1}(x)$ and $y_g = h_g^{-1}(y)$.

Given positive numbers $\epsilon_u < \epsilon$ we take $W^{cu}_{\epsilon^c,\epsilon^u}(x)$ and let us consider $\Pi^{ss} : B(x) \to W^{cu}_{\epsilon^c,\epsilon^u}(x)$. Let us take both connected components of $W^{cu}_{\epsilon^c,\epsilon^u}(x) \setminus W^u_{\epsilon}(x)$. Since there are not transversal points, follows that $\Pi^{ss}_f(W^u_{\epsilon}(y))$ is contained in one of the connected components of $W^{cu}_{\epsilon^c,\epsilon^u}(x) \setminus W^u_{\epsilon}(x)$. Let us denote these components as $L^+_f(x)$ and $L^-_f(x)$ where $L^+_f(x)$ is the connected component such that its closure contains $\Pi^{ss}_f(W^u_{\epsilon}(y))$. Moreover, we can assume that for any δ there is not an arc contained in $\Pi^{ss}_f(W^u_{\delta}(y)) \cap W^u_{\delta}(x)$. In fact, if it is not the case, we have that there is a point in $\Pi^{ss}_f(W^u_{\delta}(y)) \cap W^u_{\delta}(x)$ that belongs to the stable manifold of some periodic point and therefore we can apply the lemma 3.1.3. For each g, we take the two connected components of $W^{cu}_{\epsilon^c,\epsilon^u}(x) \setminus W^u_{\epsilon}(x_g)$ and we note it $L^+_g(x_g)$ and $L^-_g(x_g)$ respectively with the property that $L^+_g(x_g)$ and $L^-_g(x_g)$ move continuously with g.

Now, we take the two connected components of $W^{cu}_{\epsilon^c,\epsilon^u}(x) \setminus \Pi^{ss}_g(W^u_{\epsilon}(y_g))$ and we note them with $L^+_g(\Pi^{ss}(y_g))$ and $L^-_g(\Pi^{ss}(y_g))$, taking in account that we note with $L^+_f(\Pi^{ss}(y))$, the connected component contained in $L^+_f(x)$.

Proposition 5.1 *Let us take x, y as in lemma 3.5. Let us suppose that they do not belong to the stable manifold of some periodic point. For any $\eta_0 > 0$ there exists a one parameter family $\mathcal{F} = \{g_\eta\}_{\eta \in [0,1]}$ such that for any $g \in \mathcal{F}$ follows that*

1. $|g - f| < \eta_0$,
2. $y \in W^u(p_y, g)$ *and* $g(f^{-1}(x)) \in W^u(p_x, g)$,
3. $g^{-n}(y) = f^{-n}(y)$ *and* $g^{-n}(f^{-1}(x)) = f^{-n}(f^{-1}(x))$ *for any* $n \geq 0$;

and for any $\eta > 0$ holds that:

1. $g_\eta(f^{-1}(x)) \neq \Pi^{ss}_{g_\eta}(y)$ *and*
2. $g_\eta(f^{-1}(x)) \in L^+_{g_\eta}(\Pi^{ss}_{g_1}(y))$.

Remark 5.1 *Observe that given $g \in \mathcal{F}$, x_g and y_g follows that they are not necessarily equal to $g(f^{-1}(x))$ and y respectively. In particular, the fact that $g(f^{-1}(x)) \neq \Pi_g^{ss}(y)$ does not imply that $x_g \neq \Pi_g^{ss}(y_g)$.*

Before to give the proof of the Proposition, let us show how it implies the proposition 3.6.

Proof of proposition 3.6: proposition 5.1 implies proposition 3.6.

Observe that x_g and y_g belongs to the same local center stable manifold. For each $g \in \mathcal{F}$ let us take $W_\epsilon^{cu}(x_g, g)$. Using that $W_\epsilon^{cu}(x_g, g)$ is continuous with g we can assume that $W_\epsilon^{cu}(x_g, g) = W_\epsilon^{cu}(x, f)$. Moreover, we also can assume that $W_\epsilon^{cs}(x, g) \cap W_\epsilon^{cu}(x, f) = W_\epsilon^{cs}(x, f) \cap W_\epsilon^{cu}(x, f)$. Observe that for any $g \in \mathcal{F}$ follows that $g(f^{-1}(x)) \in W_\epsilon^u(x_g, g)$ and $y \in W^u(y_g, g)$, therefore, $L_g^{+,-}(g(f^{-1}(x))) = L_g^{+,-}(x_g)$ and $L_g^{+,-}(\Pi_g^{ss}(y)) = L_g^{+,-}(\Pi_g^{ss}(y_g))$. In particular, if $g_1(f^{-1}(x)) \in L_{g_1}^+(\Pi_{g_1}^{ss}(y))$ follows that

$$W_\epsilon^u(x_{g_1}, g_1) \cap L_{g_1}^+(\Pi_g^{ss}(y_{g_1})) \neq \emptyset.$$

Then, follows that there is an arc in $W_\epsilon^u(x_{g_1}, g_1)$ contained in $L_{g_1}^+(\Pi_{g_1}^{ss}(y))$. Let us consider the maximal connected components of $W_\epsilon^u(x_{g_1}, g_1)$ that intersects $L_{g_1}^+(\Pi_{g_1}^{ss}(y))$ and let us denote it with $\omega(x_{g_1}, g_1)$.

We claim that there exists $z_0 \in \omega(x_{g_1}, g_1)$ such that $h_{g_1}(z_0)$ belongs to the stable manifold of some periodic point q_0. In fact, let us consider the map

$$h_{g_1} : \omega(x_{g_1}, g_1) \to W_\epsilon^u(x).$$

Taking into account that $h_{g_1}(x)$ is continuous then either $h_{g_1 | \omega(x_{g_1}, g_1)}$ is constant or $h_{g_1}(\omega(x_{g_1}, g_1))$ is a connected interval. In the first case, by lemma 2.4 follows that $h_{g_1}(\omega(x_{g_1}, g_1))$ belongs to the stable manifold of some periodic point. In the second case, since the intersection of any unstable arc with the stable manifolds of any periodic point in H_p is dense, the assertion also follows.

Now, for each η we take the set $h_{g_\eta}^{-1}(h_{g_1}(z_0))$. It holds that either the set is a single point or it is an interval I_η contained in the stable manifold of a periodic interval J_η, such that the extremal points of J_η are periodic points (see proposition 2.3). The proof of the proposition in the second case is more elaborated and contains the proof in the first case. So, we focus only in the second case. Observe that for any $\eta > 0$, from the fact that $g_\eta(f^{-1}(x)) \in L_{g_\eta}^+(\Pi_{g_\eta}^{ss}(y))$ follows that $I_\eta \cap L_{g_\eta}^+(\Pi_{g_\eta}^{ss}(y))$ is not empty. Also observe that $x_g \notin I_\eta$ because x does not belong to the stable manifold of some periodic point (see propositionconjugacionbis3). Let us take z_η^+ and z_η^- the extremal points of I_η such that z_η^+ is the extremal point of I_η closest to x_{g_η}. Let us take q_η^+ and q_η^- the extremal points of J_η such that $z_\eta^+ \in W^s(q_\eta^+)$ and $z_\eta^- \in W^s(q_\eta^-)$. Observe that q_η^+ and q_η^- are continuations of q_0 and there exists n_0 such that for any η follows that

$$g_\eta^{n_0}(z_\eta^+) \in W_\epsilon^s(q_\eta^+), \text{ and } g_\eta^{n_0}(z_\eta^-) \in W_\epsilon^s(q_\eta^-).$$

Now we consider two following obvious alternative:

1. There exists η_0 such that either $z_{\eta_0}^+ \in L_g^+(\Pi_{g_{\eta_0}}^{ss}(y))$ or $z_{\eta_0}^- \in L_g^+(\Pi_{g_{\eta_0}}^{ss}(y))$.

2. For all $\eta > 0$ follows that $z_\eta^+ \in L_{g_\eta}^-(\Pi_{g_\eta}^{ss}(y))$ and $z_\eta^- \in L_{g_\eta}^-(\Pi_{g_\eta}^{ss}(y))$.

To conclude the proof of the proposition, we show that the second case cannot occur and that in the first case there is a map g close to f exhibiting two points in the condition of lemma 3.1.3.

First case:

Let us suppose that there exists η_0 such that $z^+_{\eta_0} \in L^+_g(\Pi^{ss}_{g_{\eta_0}}(y))$. Since $W^u_\epsilon(x) \subset L^-_f(\Pi^{ss}_f(y))$ it holds that

$$z^+_0 \in L^-_f(\Pi^{ss}_f(y))$$

therefore follows that there exists η_1 such that

$$z^+_{\eta_1} \in \Pi^{ss}_{g_{\eta_1}}(W^u_\epsilon(y, g_{\eta_1})) \cap W^u_\epsilon(x_{g_{\eta_1}}, g_{\eta_1}).$$

Therefore, there are two points in the condition of lemma 3.1.3 and from that we conclude the proposition.

If it holds that $z^-_{\eta_0} \in L^+_g(\Pi^{ss}_{g_{\eta_0}}(y))$, we can repeat the argument and again we conclude the proposition.

Second case:

First observe that from the continuity of h_g follows that $x_{g_\eta} \to x$ as $\eta \to 0$. Moreover, it also holds that

$$z^+_\eta \to x \quad \eta \to 0.$$

In fact, if it is not the case, let us take the interval $[z^-_0, z^+_0]$ contained in $W^u_\epsilon(x)$defined as $[z^-_0, z^+_0] = \lim_{\eta\to 0}[z^-_\eta, z^+_\eta]$. Since z^+_η is the closest point to x_{g_η} it follows that z^+_0 is the closest point to x and so if $z^+_0 \neq x$ then $[z^-_0, z^+_0]$ is contained in the interior of $L^-_f(\Pi^{ss}_f(y))$, and therefore by continuity follows that for η close to 0 holds that $[z^-_\eta, z^+_\eta]$ is contained in the interior of $L^-_g(\Pi^{ss}_{g_\eta}(y))$ which is a contradiction because $I_\eta \subset [z^-_\eta, z^+_\eta]$ and $I_\eta \cap L^+_{g_\eta}(\Pi^{ss}_{g_\eta}(y))$ is not empty for any $\eta > 0$.

Therefore, $z^+_\eta \to x$ as $\eta \to 0$. Then,

$$d(g^{n_0}(z^+_\eta), g^{n_0}(x_\eta)) \to 0.$$

Since $g^{n_0}_\eta(z^+_\eta) \in W^s_\epsilon(q^+_\eta)$ and $q^+_\eta \to q_0$ it follows that x belongs to the stable manifold of q_0, which is a contradiction.

■

Proof of proposition 5.1:

To construct the one parameter family, we perturb the unstable manifold of p_x in a small neighborhood of $f^{-1}(x)$ with the property that for any $\eta > 0$ holds that

$$g_\eta(f^{-1}(x)) \neq \Pi^{ss}_f(y), \text{ and } g_\eta(f^{-1}(x)) \in L^+(\Pi^{ss}_f(W^u_\epsilon(y)))$$

where $L^+(\Pi^{ss}_f(W^u_\epsilon(y)))$ is the connected component of $W^{cu}_\epsilon(x) \setminus \Pi^{ss}_f(W^u_\epsilon(y))$ such that its interior intersects $\Pi^{ss}_f(H_p \cap B(y))$. Later it is showed that this property implies that for any $\eta > 0$ follows that

$$g_\eta(f^{-1}(x)) \neq \Pi^{ss}_{g_\eta}(y) \text{ and } g_\eta(f^{-1}(x)) \in L^+_{g_\eta}(\Pi^{ss}_{g_\eta}(y)).$$

We consider two cases:

1. $x \notin \omega(x)$,
2. $x \in \omega(x)$.

Case 1: $x \notin \omega(x)$.

Since $x \notin \omega(x)$ we can get a neighborhood B of $f^{-1}(x)$ such that for any $z \in W^{cs}_\epsilon(x)$ follows that

$$f^k(z) \notin B \ \ \forall \ k > 0.$$

Then it holds that

$$x_g = g(f^{-1}(x)) \text{ and } \Pi^{ss}_g(y_g) = \Pi^{ss}_g(g(f^{-1}(y_g)))$$

following immediately the thesis of the proposition. Therefore, it remains the case that $x \in \omega(x)$.

Case 2: $x \in \omega(x)$.

First we need a series of results that allows to localize and control the recurrences.

Lemma 5.2.1 *Let q be a periodic point. Let also assume that $[W^{ss}_\epsilon(q)\setminus\{q\}]\cap H_p = \emptyset$. Then, there exists $z_0, z_1 \in W^c_\epsilon(q)$ and $\epsilon' > 0$ such that if $z \in W^{cs}_{\epsilon'}(q)$ then there is $k > 0$ verifying that*

$$f^{-k}(z) \in W^{ss}_\epsilon([f^{m_q}(z_0), z_0)] \cup W^{ss}_\epsilon([f^{m_q}(z_1), z_1)])$$

where $W^{ss}_\epsilon([f^{m_q}(z_i), z_i)] = \cup_{\{z' \in [f^{m_q}(z_i), z_i]\}} W^{ss}_\epsilon(z')$ and $[f^{m_q}(z_i), z_i]$ is the connected arc of $W^c_\epsilon(q)$ that its extremal points are given by z_i and $f^{m_q}(z_i)$ $(i = 0, 1)$.

Moreover, given $N > 0$ there exists $d = d(N)$ such that if $z \in W^{cs}_\epsilon(q)$, $d(q, z) < d$ and $f^{-k}(z) \in W^{ss}_\epsilon([f^{m_q}(z_0), z_0)] \cup W^{ss}_\epsilon([f^{m_q}(z_1), z_1)])$ then $k > N$.

Coordinates in a neighborhood of x.

We take a small neighborhood R_0 of x and a map

$$H : R(\eta^0_s, \eta^0_c, \eta^0_u)(x) = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| < \eta^0_s, |\bar{y}| < \eta^0_c, |\bar{z}| < \eta^0_u, \} \to R_0$$

such that

1. $H(x) = (0, 0, 0)$;
2. $H(W^u_\epsilon(x) \cap R(\eta^0_s, \eta^0_c, \eta^0_u)(x)) = \{\bar{x} = 0, \bar{y} = 0\}$;
3. $H(W^{cu}_\epsilon(x) \cap R(\eta^0_s, \eta^0_c, \eta^0_u)(x)) = \{\bar{x} = 0\}$;
4. $H(\Pi^{ss}(W^u_\epsilon(y) \cap R(\eta^0_s, \eta^0_c, \eta^0_u)(x)))) \subset L^+ = \{\bar{y} \geq 0\}$;
5. $H(\Pi^{ss}(W^u_\epsilon(y) \cap \{\bar{z} > 0\})) \subset L^+ = \{\bar{y} > 0\}$;

We take $\eta_s < \eta^0_s$, $\eta_c < \eta^0_c$ and $\eta_u < \eta^0_u$ and a rectangle $R(\eta_s, \eta_c, \eta_u)(x) \subset R(\eta^0_s, \eta^0_c, \eta^0_u)(x)$ in $\mathbb{R}^3$ given by

$$R = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| < \eta_s, |\bar{y}| < \eta_c, |\bar{z}| < \eta_u, \}.$$

Now we apply lemma 5.2.1 to the point p_x such that x is contained in the local unstable manifold of p_x and we obtain the following corollary:

Corollary 5.1 *There exist $\eta_s^0, \eta_c^0, \eta_u^0$ such that if $z \in R(\eta_s, \eta_c, \eta_u)(x)$ then there exists $k > 0$ such that $W^u_\epsilon(f^{-k}(z)) \cap [W^{ss}_\epsilon([f^{m_q}(z_0), z_0)] \cup W^{ss}_\epsilon([f^{m_q}(z_1), z_1)])] \neq \emptyset$. Moreover, given $N > 0$ there exists $\eta_c < \eta_c^0$ such that if $z \in R(\eta_s^0, \eta_c, \eta_u^0)(x)$ and $f^k(z) \in R(\eta_s^0, \eta_c, \eta_u^0)(x)$ then $k > N$.*

Perturbation of f.

We take η_s, η_c, η_u small. For each η with $-\eta_c < \eta < \eta_c$ we take the map

$$T_\eta(\bar{z}, \bar{y}, \bar{z}) = (\bar{x}, \bar{y} + G^\eta(\bar{x}, \bar{y}, \bar{z}), \bar{z})$$

where

$$G^\eta(\bar{z}, \bar{y}, \bar{z}) = G_1^\eta(\bar{x}) G_2^\eta(\bar{y}) G_3^\eta(\bar{z})$$

and $G_1^\eta, G_2^\eta, G_3^\eta$ are C^r functions satisfying

1. $G^\eta_{1|[-\eta_s,\eta_s]^c} = 0$, $G^\eta_{2|[-\eta_c,\eta_c]^c} = 0$, $G^\eta_{3|[-\eta_u,\eta_u]^c} = 0$;
2. $G^\eta_{1|[-\eta_s,\eta_s]} \geq 0$, $G^\eta_{2|[-\eta_c,\eta_c]} \geq 0$, $G^\eta_{3|[-\eta_u,\eta_u]} \geq 0$;
3. $G_1^\eta(0) = \eta$, $G_2^\eta(0) = \eta$, $G_3^\eta(0) = \eta$;
4. $|G_1^{\eta\prime}| < \frac{\eta}{\eta_s}$, $|G_2^{\eta\prime}| < \frac{\eta}{\eta_c}$, $|G_1^{\eta\prime}| < \frac{\eta}{\eta_u}$.

Observe that

$$T_\eta(0,0,0) = (0, \eta^3, 0) \quad |\partial_x T_\eta| \leq \frac{\eta^3}{\eta_s}, \quad |\partial_y T_\eta| \leq \frac{\eta^3}{\eta_c}, \quad |\partial_z T_\eta| \leq \frac{\eta^3}{\eta_u}.$$

Therefore, taking η small enough follows that T is C^1-close to the identity map in R. Now we take g_η equal to f in the complement of $R = H(R(\eta_s, \eta_c, \eta_u)(x))$ and in R we take

$$g_\eta = H^{-1} \circ T_\eta \circ H \circ f.$$

Remark 5.2 *It follows that for any $g_\eta \in \mathcal{F}$ follows that*

$$dist(g_\eta(f^{-1}(x)), x) = \eta^3.$$

In particular this implies that

$$g_\eta(f^{-1}(x)) \in L_f^+(\Pi_f^{ss}(W^u_\epsilon(y))).$$

Lemma 5.2.2 *For any $g \in \mathcal{F}$ follows that if $z \in W^u_\epsilon(y)$ then*

$$dist(\Pi_g^{ss}(z), \Pi_f^{ss}(z)) < 2\lambda^N \frac{\eta^3}{\eta_s}.$$

Lemma 5.2.2 and remark 5.2 imply proposition 5.1:

Observe that it is enough to show that $dist(g(f^{-1}(x)), \Pi_g^{ss}(y)) > 0$. In fact,

$$\begin{aligned} dist(g(f^{-1}(x)), \Pi_g^{ss}(y)) &> dist(g(f^{-1}(x)), \Pi_f^{ss}(y)) - dist(\Pi_f^{ss}(y), \Pi_g^{ss}(y)) \\ &= dist(g(f^{-1}(x)), x) - dist(\Pi_f^{ss}(y), \Pi_g^{ss}(y)) \\ &> \eta^3 - 2\lambda^N \frac{\eta^3}{\eta_s} = \eta^3[1 - 2\frac{\lambda^N}{\eta_s}]. \end{aligned}$$

Therefore, recalling that $N = N(\eta_c) \to +\infty$ as $\eta_c \to 0$ (recall corollary 5.1), choosing η_c small enough such that

$$2\frac{\lambda^N}{\eta_s} < 1$$

we conclude the proposition.

■

Proof of lemma 5.2.2:

First we prove that for any $z \in W_\epsilon^u(y)$ follows that

$$Sl(E_1(z,g), E_1(z,f)) < 2\lambda^N \frac{\eta^3}{\eta_s}. \tag{11}$$

To do that, we use the next immediate claims:

Claim 7 *Let $g \in \mathcal{F}$ and let $Dg : T_zM \to T_{g(z)}M$. Let $w \in T_g(z)M$ be close to $E_1(g(z), f)$. Then*

$$SL(Dg^{-1}(w), E_1(z,f)) < \frac{\eta^3}{\eta_s} + \lambda SL(w, E_1(g(z), f)).$$

Claim 8 *Let $g \in \mathcal{F}$ and let $Df^n : T_zM \to T_{f^n(z)}M$. Let $w \in T_{f^n(z)}M$ be close to $E_1(f^n(z), f)$. Then*

$$SL(Df^{-n}(w), E_1(z,f)) < \lambda^n SL(w, E_1(f^n(z), f)).$$

Now we proceed to prove (11). Let $z \in W_\epsilon^u(y)$. Let us take the sequences n_i such that $z_i = g^{n_i}(z) \in R$. Let $k_i = n_i - n_{i-1}$ with $n_0 = 0$. Observe that $f^j(z_i) = g^j(z_i)$ for $0 \le j \le k_i - 1$. Observe that

$$E_1(z_i, g) = Dg^{-(k_i-1)}(E_1(g^{k_i-1}(z), g)) = Df^{-(k_i-1)}(E_1(f^{k_i-1}(z), g)).$$

Using claims 7 and 8, follows that

$$\begin{aligned} SL(E_1(z_i,g), E_1(z_i,f)) &\le \lambda^{k_i}[\frac{\eta^3}{\eta_s} + \lambda SL(E_1(z_{i+1},g)E_1(z_{i+1},f))] \\ &\le \lambda^N[\frac{\eta^3}{\eta_s} + \lambda SL(E_1(z_{i+1},g)E_1(z_{i+1},f))] \end{aligned}$$

Therefore,

$$SL(E_1(z,g), E_1(z,f)) \leq \lambda^{k_i} \frac{\eta^3}{\eta_s}[1 + \Sigma_{j>0}\lambda^{Nj}].$$

So, for N large (11) holds. To conclude the proof of the lemma, it is proved that (11) holds for any $z' \in W^{ss}_\epsilon(z)$ such that $z \in W^u_\epsilon(y)$.

■

6 Joint integrable case. Proof of proposition 3.7.

Recall that in the present situation we can not guarantee that there is a pair of periodic points such that their unstable manifolds s-intersect. So, in general, the perturbations performed in the previous sections can not be carried out in the context of joint integrability.
Before to start, we need more information about the points x and y in particular when either x or y are not boundary points. More precisely, using that the strong foliations are jointly integrable, we can prove the following lemma.

Lemma 6.0.3 *Let H_p be a topologically hyperbolic homoclinic class. Let also assume that the strong foliations are jointly integrable. Then, for any $z \in H_p$ one of the next options holds:*

1. *for any positive integer n_0 and a positive constant r, there exist positive integers n_1, n_2, n_3 such that*
   (a) *$n_i > n_0$ for $i = 1, 2, 3$,*
   (b) *$dist(f^{-n_i}(z), f^{-n_j}(z)) < r$ for $i, j = 1, 2, 3$,*
   (c) *the local unstable manifold of $f^{-n_1}(z)$ and $f^{-n_3}(z)$ intersects different connected components of $W^s_\epsilon(f^{-n_2}(z)) \setminus W^{ss}_\epsilon(f^{-n_2}(z))$;*
2. *$z \in W^u(q)$ for some periodic point q;*
3. *there exits a pair of periodic points q_1 and q_2 such that the local strong stable manifold of each point intersect the unstable manifold of the other point.*

Proof: The proof is similar to some part of the proof of proposition 3.1.1. If the first item does not hold, then follows that there exist n_1 and n_2 arbitrarily large such that either $f^{-n_2}(z) \in W^u_\epsilon(f^{-n_1}(z))$ or $[W^{ss}_\epsilon(f^{-n_2}(z)) \setminus \{f^{-n_2}(z)\}] \cap W^u_\epsilon(f^{-n_1}(z)) \neq \emptyset$. In the first case, follows that z belong to the unstable manifold of some periodic point. In the second case, from the joint integrability follows that $W^u_\epsilon(f^{-n_2}(z)) \subset W^{su}_\epsilon(f^{-n_1}(z))$ and arguing as item (ii) of claim 6 2.1 of lemma 5.1.2, the third situation follows.

■

Let us take the points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$. Observe that if either x or y verify the third item them it follows that f satisfies the proposition 3.2. If both points satisfy the second item, since the strong foliations are jointly integrable, follows that we are in the hypothesis of lemma 3.1.2 and so then we can apply lemma 3.1.3 to finish the proof of proposition 3.7.

Therefore, it what follows we assume that at least one of the points x, y, verifies the first item of lemma 6.0.3. Then, one of the following options holds:

1. **Case A:** the points x and y satisfy the first item of lemma 6.0.3.
2. **Case B:** either x or y satisfy the first item of lemma 6.0.3 and the other point satisfies the second item of lemma 6.0.3.

Moreover, we can assume that the points x, y are contained in the set Λ given by theorem 3.1.

Given a point $z \in \Lambda$ that verifies the first item of lemma 6.0.3 we can show that the point z is accumulated by periodic points in a neighborhood of Λ converging on z from both connected components of $W^s_\epsilon(z) \setminus W^{ss}_\epsilon(z)$. As the consequences of lemma 6.0.3 we get:

Lemma 6.0.4 *Let $z \in \Lambda$ such that verifies the first item of lemma 6.0.3. Then, for any small open neighborhood V of Λ follows that there are periodic points in $\cap_{n\in\mathbb{Z}} f^n(V)$ such that the local unstable manifold of these periodic points intersects different connected components of $W^s_\epsilon(z) \setminus W^{ss}_\epsilon(z)$.*

Proof:

Let us take a periodic point q_{21} with orbit in V and close to $f^{-n_1}(z)$ such that the local unstable manifold of q_{21} intersect the same connected components of $W^s_\epsilon(f^{-n_2}(z)) \setminus W^{ss}_\epsilon(f^{-n_2}(z))$ where the local unstable manifold of $f^{-n_1}(z)$ intersects $W^s_\epsilon(f^{-n_2}(z)) \setminus W^{ss}_\epsilon(f^{-n_2}(z))$. Let us take a periodic point q_{23} with orbit in V and close to $f^{-n_3}(y)$ such that the local unstable manifold of q_{23} intersect the same connected components of $W^s_\epsilon(f^{-n_2}(z)) \setminus W^{ss}_\epsilon(f^{-n_2}(z))$ where the local unstable manifold of $f^{-n_3}(z)$ intersects $W^s_\epsilon(f^{-n_2}(z)) \setminus W^{ss}_\epsilon(f^{-n_2}(z))$. So, observe that there are arcs γ_{21} and γ_{32} of the local unstable manifold of q_{21} and q_{32} such that $f^i(\gamma_{21}) \subset V$, $f^i(\gamma_{32}) \subset V$ for $0 \le i \le n_2$, and intersecting different connected components of $W^s_\epsilon(z) \setminus W^{ss}_\epsilon(z)$. Using a dense orbit in Λ_f (recall that Λ_f is transitive), follows that there are discs D_{21} and D_{32} of the local stable manifold of q_{21} and q_{32} such that $f^{-i}(D_{21}) \subset V$ for $0 \le i \le k_2$ and k_2 large, $f^{-i}(D_{32}) \subset V$ for $0 \le i \le k_3$ and k_3 large, such that $f^{-k_2}(D_{21})$ and $f^{-k_3}(D_{32})$ intersect the local unstable manifold of z.

Moreover, we can suppose that $f^{n_2}(\gamma_{21})$ intersects $f^{-k_2}(D_{21})$ and $f^{n_2}(\gamma_{32})$ intersects $f^{-k_3}(D_{32})$. Then, there homoclinic points z_{21} and z_{32} of q_{21} and q_{32} respectively, with orbits in V and such that their local unstable manifolds intersects the local stable manifold of y in different connected components of $W^s_\epsilon(z) \setminus W^{ss}_\epsilon(z)$. Then, we can get a pair of periodic point, each one arbitrarily closed to each homoclinic point. This conclude the proof.

■

As a consequences of previous lemma and using lemma 2.3.1 and that the subbundle E_3 is expansive restricted to Λ follows that there is $\lambda_c < 1$ and a neighborhood V_1 of Λ such that $\Lambda \subset Closure(Per_{\lambda_c}(f/V_1))$ and therefore for any $z \in \Lambda$ there are periodic points in $Per_{\lambda_c}(f/V_1)$ accumulating on z such that the local unstable manifold of these periodic points intersects different connected components of $W^s_\epsilon(z) \setminus W^{ss}_\epsilon(z)$. Now we take the two connected components of $W^{cu}_\epsilon(z) \setminus W^u_\epsilon(z)$ and we denote it with $L^l(z)$ and $L^r(z)$.

Lemma 6.0.5 *Let $z \in \Lambda$ such that verifies the first item of lemma 6.0.3. Then, follows that*

1. *there is a sequences $\{q^l_n\} \in Per_{\lambda_c}(f/V_1)$ accumulating on z and such that $\Pi^{ss}_{f,z}(q^l_n) \in L^l(z)$;*
2. *there is a sequences $\{q^r_n\} \in Per_{\lambda_c}(f/V_1)$ accumulating on z and such that $\Pi^{ss}_{f,z}(q^r_n) \in L^r(z)$.*

Recall, that for any $g \in \mathcal{U}$ the map h_g^{-1} is well defined over the periodic points in $Per_{\lambda_c}(f/V_1)$.

Now, fixed a sequences verifying the lemma 6.0.4, for each $g \in \mathcal{U}$ we define

$$z^l_g = \lim_n q^l_n(g) \quad \text{and} \quad z^r_g = \lim_n q^r_n(g).$$

Remark 6.1 *Observe that if $h_g^{-1}(z)$ is a single point then $z_g^l = z_g^r$.*

Remark 6.2 *Observe that z_g^l and z_g^r move continuously with g and they belong to $H_p(g)$ for any g close to f.*

For each $g \in \mathcal{U}$, given z and the points z_g^r and z_g^l, we define $L^r(z_g^r)$ and $L^l(z_g^l)$ as the continuation of $L^r(z)$ and $L^l(z)$ respectively.

Lemma 6.0.6 *Let $z \in \Lambda_f$ such that verifies the first item of lemma 6.0.3 and let $\{q_n^l\} \in Per_{\lambda_c}(f/V_1)$ be the sequence such that $\Pi_f^{ss}(q_n^l) \in L^l(z)$. Then either*

1. *for any $g \in \mathcal{U}$ and any q_n follows that $\Pi_g^{ss}(q_n(g)) \in L^l(z_g^l)$ or*
2. *there is a $g \in \mathcal{U}$ arbitrarily close to f and q_n such that $\Pi_g^{ss}(q_n(g)) \in W_\epsilon^u(z_g^l)$, i.e.: $W_\epsilon^{ss}(q_n(g), g) \cap W_\epsilon^u(z_g^l, g) \neq \emptyset$.*

The same statement follows for z_g^r.

Proof: It follows immediately from the continuity with g of x_g^l, $q_n(g)$, the local strong stable manifolds and local strong unstable manifolds.

■

Lemma 6.0.7 *If the second item of lemma 6.0.6 holds, follows that there is a map g C^1-close to f and a periodic point q_n such that $q_n(g)$ exhibits a strong homoclinic connection. Therefore, the proposition 3.7 is finished in this case.*

Proof: If there is a $g \in \mathcal{U}$ and q_n such that $W_\epsilon^{ss}(q_n(g), g) \cap W_\epsilon^u(z_g^l, g) \neq \emptyset$, using that z_g^l and $q_n(g)$ belongs to $H_p(g)$ with $g_n(g)$ homoclinically related with $p(g)$, follows that the strong stable manifold and the unstable manifold of $q_n(g)$ accumulates one into the other. Using the connecting lemma, the results follows. ■

Therefore, in what follows, we assume that the first item of lemma 6.0.6 holds. In the next sections we study the case A and we show how the proof is adapted to deal with the case B.

6.1 Case A:

We can apply lemma 6.0.5 for both points x and y. Moreover, recall that we are also assuming that the first item of lemma 6.0.6 holds. Therefore, in what follows we assume that

1. there is a sequences $\{q_n^l\} \in Per_{\lambda_c}(f/V_1)$ accumulating on x such that for any $g \in \mathcal{U}$ follows that $q_n^l(g)$ accumulates on x_g^l and $\Pi_g^{ss}(q_n^l(g)) \in L^l(x_g^l)$,
2. there is a sequences $\{q_n^r\} \in Per_{\lambda_c}(f/V_1)$ accumulating on x such that for any $g \in \mathcal{U}$ follows that $q_n^r(g)$ accumulates on x_g^r and $\Pi_g^{ss}(q_n^r(g)) \in L^r(x_g^r)$,

3. there is a sequences $\{p_n^l\} \in Per_{\lambda_c}(f/V_1)$ accumulating on y such that for any $g \in \mathcal{U}$ follows that $p_n^l(g)$ accumulates on y_g^l and $\Pi_g^{ss}(p_n^l(g)) \in L^l(y_g^l)$,

4. there is a sequences $\{p_n^r\} \in Per_{\lambda_c}(f/V_1)$ accumulating on y such that for any $g \in \mathcal{U}$ follows that $p_n^r(g)$ accumulates on y_g^r and $\Pi_g^{ss}(p_n^r(g)) \in L^r(y_g^r)$.

In what follows, we have to study what happen to the right or left continuation of x and y. We do the study just for the right continuation; the same follows for the left one. To avoid notation, we denote with x_g and y_g the right continuation of x and y respectively.

To finish the proof of proposition 3.7, we consider the following options:

1. there exists $g \in \mathcal{U}$ arbitrarily close to f such that the points x_g and y_g verifies that
$$W_\epsilon^u(y_g, g) \cap W_\epsilon^{ss}(x_g, g) = \emptyset;$$

2. or for all $g \in \mathcal{U}$ follows that
$$W_\epsilon^u(y_g, g) \cap W_\epsilon^{ss}(x_g, g) \neq \emptyset.$$

In other words, as in the case of transversal intersection, we also consider here the following two situations:

1) If for some g close to f holds that the "continuation" (right or left) of the points x and y do not belong to the same strong stable leaf, then taking an isotopy between the initial map and the perturbation, follows that for some map of the isotopy holds that there are two periodic points as in the thesis of proposition 3.2.

2) If it occurs that for any g close to f holds that the "continuation" (right or left) of the points x and y have the property that they belong to the same strong stable leaf, then it is performed a perturbation such that the local unstable manifold of the "continuation" of the points x and y are not jointly integrable, and this allows to find two periodic points as in the hypothesis of proposition 3.2.

Remark 6.3 *We do not know if the second option can occur for an open set of diffeomorphisms; therefore a strategy is developed assuming that the second option can occur.*

Now we start analyzing the first case.

1. There exists $g \in \mathcal{U}$ such that $W_\epsilon^u(y_g, g) \cap W_\epsilon^{ss}(x_g, g) = \emptyset$.

To avoid notation, we denote $W_\epsilon^{u(ss)}(y_g, g)$ with $W_\epsilon^{u(ss)}(y_g)$, except it is necessary to clarify. The same for x_g.

Lemma 6.1.1 *Let us assume that there exists $g \in \mathcal{U}$ such that $W_\epsilon^u(y_g) \cap W_\epsilon^{ss}(x_g) = \emptyset$. Then there exists $\hat{g} \in \mathcal{U}$ such that the thesis of proposition 3.7 holds for $\hat{g}$.*

Proof: Let us consider a homotopy $\mathcal{F} = \{g_\eta\}_{0\leq\eta\leq 1}$ such that $g_\eta \in \mathcal{U}$ for any η, $g_0 = f$ and g_1 is a diffeomorphism that verifies the hypothesis of the present lemma. We take the two connected components of $W_\epsilon^{cu}(y) \setminus W_\epsilon^u(y)$ and we note with $L^+(y)$ the right connected component (recall that we are dealing with the right continuation of x and y, so $L^+(y)$ is equal to L_y^r). The same is done for x.

For each $g \in \mathcal{F}$ let us also take $W^{cu}_\epsilon(y_g)$ and the two connected components of $W^{cu}_\epsilon(y_g) \setminus W^u_\epsilon(y_g)$. We note with $L^+(y_g)$ the right connected component and with $L^-(y_g)$ the other component. Using that $W^{cu}_\epsilon(y_g)$ and $W^u_\epsilon(y_g)$ are continuous with g, it follows that $L^+(y_g)$ and $L^-(y_g)$ move continuously with g (observe that since $y_g = y^r_g$ then $L^-(y_g)$ do not necessarily coincides with $L^l(y^l_g)$, in fact, this is the case if $y^l_g \neq y^r_g$). The same is done with x_g.

Now, for each $g \in \mathcal{F}$ we take $\Pi^{ss}_g : B \to W^{cu}_g(y_g)$ where B is a neighborhood that contains x_g and y_g for any $g \in \mathcal{F}$.

Since $W^u_\epsilon(y_{g_1}) \cap W^{ss}_\epsilon(x_{g_1}) = \emptyset$ then $\Pi^{ss}_{g_1}(x_{g_1})$ is contained in one of the connected components of $W^{cu}_\epsilon(y_{g_1}) \setminus W^u_\epsilon(y_{g_1})$. We can suppose that

$$\Pi^{ss}_{g_1}(x_{g_1}) \in L^+(y_{g_1}).$$

Taking a reduced center unstable manifold, we can also assume that for any $z \in W^u_\epsilon(x_g)$ it follows that $\Pi^{ss}_{g_1}(z) \in L^+(y_{g_1})$. By lemma 6.0.5, 6.0.6, and the fact that x_g and y_g are the right continuation of x and y respectively, follows that there exist a pair of periodic points q_x and q_y of f such that q_x is arbitrarily close to x, q_y is arbitrarily close to y and such that

1. $q_x \in Per_{\lambda_c}(f/V_1)$ and $q_y \in Per_{\lambda_c}(f/V_1)$,
2. $\Pi^{ss}_{f,x}(q_x) \in L^+(x)$ and $\Pi^{ss}_{f,y}(q_y) \in L^+(y)$
3. for any $g \in \mathcal{U}$ holds that $q_x(g)$ and $q_y(q)$ are close to x_g and y_g respectively,
4. for any $g \in \mathcal{U}$ holds that $\Pi^{ss}_{g,x_g}(q_x(g)) \in L^+(x_g)$ and $\Pi^{ss}_{g,y_g}(q_y(g)) \in L^+(y_g)$,

Moreover, we can suppose that for any $g \in \mathcal{F}$ follows that $q_x(g)$ and $q_y(g)$ belong to B.

Now, for each g we take

$$W^{cu}_\epsilon(y_g) \setminus \Pi^{ss}_g(W^u_\epsilon(q_y(g)))$$

and we note the both connected components with $L^+(q_y(g))$ and $L^-(q_y(g)))$. Again we can choose the connected components $L^\pm(q_y(g))$ in such a way that they move continuously with g. Moreover, we choose $L^+(q_y)$ as the connected component that verifies that it is contained in $L^+(y)$.

Using that $\Pi^{ss}_{g_1}(x_{g_1}) \in L^+(y_{g_1})$ we can choose the periodic q_y close enough to y in such a way that it verifies:

$$\Pi^{ss}_{g_1}(W^u_\epsilon(x_{g_1})) \subset L^+(q_y(g_1)). \tag{12}$$

Since $\Pi^{ss}_{g,x_g}(q_x(g)) \in L^+(x_g)$, from (12) follows that

$$\Pi^{ss}_{g_1}(q_x(g_1)) \subset L^+(q_y(g_1)). \tag{13}$$

The periodic point q_x can be also choose close enough to x in such a way that it verifies:

$$dist(\Pi^{ss}_f(q_x), \Pi^{ss}_f(x)) < dist(\Pi^{ss}_f(q_y), y). \tag{14}$$

From (14) follows that

$$\Pi_f^{ss}(q_x) \in L^-(q_y). \tag{15}$$

Using that the maps $g \to \Pi_g^{ss}(q_x(g))$ and $g \to \Pi_g^{ss}(q_y(g))$ move continuously with g from (13) and (15) follows that there is $\hat{g} \in \mathcal{F}$ such that

$$\Pi_{\hat{g}}^{ss}(q_x(\hat{g})) \in \Pi_{\hat{g}}^{ss}(W_\epsilon^u(q_y(\hat{g})))$$

therefore,

$$W_\epsilon^{ss}(q_x(\hat{g}))) \cap W_\epsilon^u(q_y(\hat{g})) \neq \emptyset$$

and so the lemma follows.

In the case that

$$\Pi_{g_1}^{ss}(x_{g_1}) \in L^-(y_{g_1})$$

since $x_g \in W_\epsilon^{cs}(y_g)$ for any g, it follows that

$$\Pi_{g_1}^{ss}(y_{g_1}) \in L^+(x_{g_1})$$

and so we can repeat the same analysis replacing x by y.

■

2. For every $g \in \mathcal{U}$ follows that $W_\epsilon^u(y_g) \cap W_\epsilon^{ss}(x_g) \neq \emptyset$.

Given the pair x, y, observe that there is a periodic point p_0 close to them such that $W_\epsilon^u(x) \cap W_\epsilon^s(p_0) \neq \emptyset$ and $W_\epsilon^u(y) \cap W_\epsilon^s(p_0) \neq \emptyset$. Without loss of generality, we can assume that the point p_0 is fixed. We can take a disc D contained in $W_\epsilon^s(p_0)$ such that $W_\epsilon^u(x) \cap D \neq \emptyset$ and $W_\epsilon^u(y) \cap D \neq \emptyset$. We take the points

$$x^- \in W_\epsilon^u(x) \cap D, \text{ and } \ y^- \in W_\epsilon^u(y) \cap D.$$

Observe that it could occur that $x^- = x$ and $y^- = y$. We can also suppose that for any g close to f, the point p_0 remains fixed and the disc D remains contained in $W_\epsilon^s(p_0)$. Now, for each $g \in \mathcal{U}$ we considerer the points

$$x_g^- = W_\epsilon^u(x_g, g) \cap D \text{ and } y_g^- = W_\epsilon^u(y_g, g) \cap D.$$

If it holds that there is $g \in \mathcal{U}$ such that

$$y_g^- \notin W_\epsilon^{ss}(x_g^-)$$

then we use proposition 6.1 below that proves proposition 3.7.
If it holds that for every $g \in \mathcal{U}$ holds that

$$y_g^- \in W_\epsilon^{ss}(x_g^-)$$

then there is performed a C^1-suitable perturbation (see proposition 6.3) to show that the strong foliation associated to these points are not jointly integrable and then we show that this implies the proposition 3.7. We consider both cases separately.

Remark 6.4 *We do not know if the second option can occur for an open set of diffeomorphisms; therefore a strategy is developed assuming that the second option can occur.*

2.1. There is $g \in \mathcal{U}$ such that $y_g^- \notin W_\epsilon^{ss}(x_g^-)$.

Proposition 6.1 *If there exists $g \in \mathcal{U}$ such that $y_g^- \notin W_\epsilon^{ss}(x_g^-)$, then follows that there exists $\hat{g} \in \mathcal{U}$ such that the proposition 3.7 holds for $\hat{g}$.*

Proof: Let us consider a homotopy $\mathcal{F} = \{g_\eta\}_{0 \leq \eta \leq 1}$ such that $g_\eta \in \mathcal{U}$ for any η, $g_0 = f$ and g_1 is a diffeomorphism that satisfies the hypothesis of the present proposition. For each $g \in \mathcal{F}$ let us take $W_\epsilon^{cu}(y_g^-)$ and the connected components of $W_\epsilon^{cu}(y_g^-) \setminus W_\epsilon^u(y_g^-)$ that we note as $L^+(y_g^-)$ and $L^-(y_g^-)$.

By hypothesis, we are assuming that $y_{g_1}^- \notin W_\epsilon^{ss}(x_{g_1}^-)$ and we can suppose that

$$\Pi_{g_1}^{ss}(x_{g_1}^-) \in L_{g_1}^+(y_{g_1}^-).$$

Recalling that there are periodic points in $Per_{\lambda_c}(f/V_1)$ accumulating on x and y from the right, we can choose a periodic point q_x and q_y such that

$$\Pi_{f,x}^{ss}(q_x) \in L^+(x) \text{ and } \Pi_{f,y}^{ss}(q_y) \in L^+(y).$$

For these points and its continuation, and for any $g \in \mathcal{U}$ we take the points

$$q_x(g)^- = W_\epsilon^u(q_x(g)) \cap D \text{ and } q_y(g)^- = W_\epsilon^u(q_y(g)) \cap D.$$

In particular it holds that

$$\Pi^{ss}(q_x^-) \in L^+(y^-) \text{ and } \Pi^{ss}(q_y^-) \in L^+(y^-).$$

We take the point q_y such that for any $g \in \mathcal{F}$ holds that

$$dist(\Pi_g^{ss}(q_y^-(g)), y_g^-) < \frac{1}{2} dist(\Pi_{g_1}^{ss}(x_{g_1}^-), y_{g_1}^-). \tag{16}$$

We chose the periodic points q_x such that verifies

$$dist(\Pi_f^{ss}(q_x^-), \Pi_f^{ss}(x^-)) < \frac{1}{2} dist(\Pi_f^{ss}(q_y^-), y^-) \quad \text{and,} \tag{17}$$

$$dist(\Pi_{g_1}^{ss}(q_x(g_1)^-), \Pi_{g_1}^{ss}(x_{g_1}^-)) < \frac{1}{2} dist(\Pi_{g_1}^{ss}(y_{g_1}^-), x_{g_1}^-). \tag{18}$$

For each $g \in \mathcal{F}$, we consider the connected components of $W_\epsilon^{cu}(y_g^-) \setminus \Pi_g^{ss}(W_\epsilon^u(q_y(g)^-)))$.
We note this components as $L^+(\Pi_g^{ss}(q_y(g)^-)))$ and $L^-(\Pi_g^{ss}(q_y(g)^-)))$ and we can choose them such that they moves continuously with g and such that,

$$L^+(\Pi_g^{ss}(q_y(g)^-))) \subset L^+(y^-).$$

By (17), follows that

$$\Pi^{ss}(q_x^-) \in L^-(\Pi^{ss}(q_y^-)).$$

From (16) and (18), follows that

$$\Pi^{ss}_{g_1}(q_x(g_1)^-) \in L^+(\Pi^{ss}_{g_1}(q_y(g_1)^-)).$$

From the continuity of $g \to \Pi^{ss}_g$ follows that there is another $\hat{g}$ such that

$$\Pi^{ss}_{\hat{g}}(q_x(\hat{g})^-) = q_y(\hat{g})^-.$$

This implies that

$$q_y(\hat{g})^- \in W^{ss}_\epsilon(q_x(\hat{g})^-).$$

Therefore, we get that there exists $\hat{g}$ such that the local unstable manifold of the periodic points $q_x(\hat{g})$ and $q_y(\hat{g})$ s-intersect each other (at $q_y(\hat{g})^-$ and $q_x(\hat{g})^-$) and these points of s-intersection belong to the stable manifold of a periodic point. Therefore, we can apply the lemma 3.1.3 to finish the proof of the proposition 3.7 is the present case.

In the case that

$$\Pi^{ss}_{g_1}(x^-_{g_1}) \in L^-_{g_1}(y^-_{g_1}),$$

it follows that

$$\Pi^{ss}_{g_1}(y^-_{g_1}) \in L^+_{g_1}(x^-_{g_1})$$

and therefore, interchanging the role of x and y we can repeat the same argument.

∎

2.2. For every $g \in \mathcal{U}$ holds that $y^-_g \in W^{ss}_\epsilon(x^-_g)$.

To deal with the case that for every $g \in \mathcal{U}$ follows that $y^-_g \in W^{ss}_\epsilon(x^-_g)$ we introduce a special perturbations such that the strong manifolds associated to x^-_g and y^-_g are not jointly integrable. In other words, in the present case we prove that there is g C^1-close to f such that

$$\Pi^{ss}_g(W^u_\epsilon(g(y^-_g))) \text{ does not coincide with } W^u_\epsilon(g(x^-_g)).$$

After that, arguing in a similar way as in proposition 6.1 we conclude the proposition 3.7.

Remark 6.5 *Assuming that x does not belong to the unstable manifold of a periodic points, follows that there exist a neighborhood R_0 of $f(x^-)$ and $z \in W^s(p_0)$ such that $\mathcal{O}(z) \cap R_0 = \emptyset$.*

Proposition 6.2 *Let us assume that for every $g \in \mathcal{U}$ holds that $y^-_g \in W^{ss}_\epsilon(x^-_g)$. Let us assume that for any $\hat{\mathcal{U}} \subset \mathcal{U}$ there exists a one parameter family of diffeomorphisms $\mathcal{F}$ such that*

1. *for any $g \in \mathcal{F}$ follows that $g\hat{\mathcal{U}}$,*
2. *$g_{|R_0{}^c} = f_{|R_0{}^c}$ where R_0 is a neighborhood as in remark 6.5,*
3. *there exist $r > 0$ and $g \in \mathcal{F}$ such that for any $z \in W^u_r(g(y^-_g)) \setminus \{g(y^-_g)\}$ follows that $W^{ss}_\epsilon(z) \cap W^u_r(g(x^-_g)) = \emptyset$.*

Then, the proposition 3.7 follows.

Proof:

Observe that the previous proposition implies that the local unstable manifold of $g(y_g)$ and $g(x_g)$ are not jointly integrable. To precise, the proposition 6.2 implies the next assertion (recall that we can assume that for any g close to f we can assume that p_0 is a fixed point and $W^s_\epsilon(p_0, g) = W^s_\epsilon(p_0, f)$):

Claim 9 *There is a compact disk D^* contained in the stable manifold of p_0 for any $g \in \mathcal{F}$ such that there exist x^*_g and y^*_g verifying:*

1. $x^*_g \in W^u_\epsilon(g(x_g)) \cap D^*$, $y^*_g \in W^u_\epsilon(g(y_g)) \cap D^*$
2. $y^*_g \notin W^{ss}_\epsilon(x^*_g)$.

Proof of the claim: Let z and R_0 be the point and neighborhood considered in remark 6.5. Let $k > 0$ such that $f^{-k}(z) \in W^u_\epsilon(p_0)$. Then, we can take a small disk D_0 containing z and contained in the stable manifold of p_0 such that $\cup_{i>0}^k f^{-i}(D_0) \cap R_0 = \emptyset$. In particular, for any $g \in \mathcal{F}$ follows that $g_{|\cup_{i>0}^k f^{-i}(D_0)} = f_{|\cup_{i>0}^k f^{-i}(D_0)}$. We can take $n > 0$ and $D^* \subset f^{-n}(f^{-k}(D_0))$ close enough to a disk in $W^s_\epsilon(p_0)$ such that

1. $dist(D^*, W^s_\epsilon(p_0))) < r$,
2. $W^u_r(z) \cap D^* \neq \emptyset$ for any $z \in R_0 \cap W^s_\epsilon(p_0)$,
3. $f^j(D^*) \cap R_0 = \emptyset$ for $1 \leq j \leq n$.

From these properties follows that $g_{|\cup_{i>0}^{k+n} f^i(D^*)} = f_{|\cup_{i>0}^{k+n} f^i(D^*)}$, and therefore D^* is contained in the stable manifold of p_0 and verifies the thesis of the claim.

■Taking the disk D^* follows that for any $g \in \mathcal{U}$ holds that the disk D^* intersects the unstable manifold of size r of the points x^-_g and y^-_g. To conclude the proof of the proposition, we repeat the proof of lemma 6.1 changing the points x^-_g, y^-_g by x^*_g, y^*_g and the disc D by D^*

■

Now, we have to show that there exists a diffeomorphisms a one parameter family of diffeomorphisms C^1-close to f verifying the hypothesis of proposition 6.2.

To do that, first we introduce some coordinates nearby the point p_0. In particular, recall the lemma 2.2.1 and corollary 2.2 that state that the strong stable foliation is C^1.

1. Local coordinates. First we introduce in the next remark a system of local coordinates used to perform the perturbations of f that verify the hypothesis of proposition 6.2.

Remark 6.6 *Let us take the point p_0 such that $x^-, y^- \in D \subset W^s_\epsilon(p_0)$. We can assume that there is a C^1-map H from a neighborhood B_0 of p_0 to a neighborhood $\tilde{B}_0$ of $(0,0,0)$ in $\mathbb{R}^3$ such that if we denote with $(\bar{x}, \bar{y}, \bar{z})$ the coordinates of a point in $\tilde{B}_0$ and with $(\bar{x}(z), \bar{y}(z), \bar{z}(z))$ the coordinates of a point $H(z)$ follows that*

1. $H(W^s_\epsilon(p_0)) = \{\bar{z} = 0\}$,

2. $H(W^u_\epsilon(p_0)) = \{\bar{x} = 0, \bar{y} = 0\}$,

3. $H(W^u_\epsilon(f(x^-))) = \{z \in B_0 : \bar{x}(z) = \bar{x}(f(x^-)), \bar{y}(z) = \bar{y}(f(x))\}$,

4. $H(W^u_\epsilon(f(y^-))) = \{z \in B_0 : \bar{x}(z) = \bar{x}(f(y^-)), \bar{y}(z) = \bar{y}(f(y))\}$.

5. *given a point z in R, follows that*

$$DH[E_1(z,f)] = (1,0,0) \quad \textit{and} \quad H(W^{ss}_\epsilon(z)) = \{\bar{x} = \bar{x}(z)\}.$$

Moreover, we can take the neighborhood B_0 in such a way that $W^u_\epsilon(x) \cap f(B_0) = \emptyset$ and $W^u_\epsilon(y) \cap f(B_0) = \emptyset$.

2. Selection of rectangles around $f(x^-)$.

Using lemma 4.5.5 and the coordinates introduced above, we can take constants

$$c_1 < c_2 < c_3 < c_4 < \bar{y}(H(f(x^-))) < d_4 < d_3 < d_2 < d_1,$$

$$a_1 < a_2 < a_3 < a_4 < \bar{x}(H(f(x^-))) < b_4 < b_3 < b_2 < b_1,$$

such that for each $\eta^u_0 > 0$ if we consider the rectangles $R_i(\eta^u_0)$ $i = 1,2,3,4$, defined by

$$R_i(\eta^u_0) = \{(\bar{x},\bar{y},\bar{z}) : a_i < \bar{x} < b_i; c_i < \bar{y} < d_i; |\bar{z}| < \eta^u_0\},$$

they verify that

1− $H(f(x^-)) \in R_4(\eta^u_0) \subset R_3(\eta^u_0) \subset R_2(\eta^u_0) \subset R_1(\eta^u_0) \subset B$
2− $H(f(y^-)) \notin R_4$,
3− $[R_1(\eta^u_0) \setminus R_4(\eta^u_0)] \cap H_p = \emptyset$ and
4− $R_1(\eta^u_0) \subset R_0$

where R_0 is the neighborhood given in remark 6.5. To avoid notation, we also note the rectangles $H^{-1}(R_i(\eta^u_0))$ and $H^{-1}(\hat{R}_i(\eta^u_0))$ with $R_i(\eta^u_0)$ and $\hat{R}_i(\eta^u_0)$.

Remark 6.7 *For any g C^1-close enough to f follows that*

$$\cap_{\{n>0\}} g^n(U) \cap [R_1(\eta^u_0) \setminus R_4(\eta^u_0)] = \emptyset.$$

3. One parameter family of perturbation of the map f.

Now, given η_0 small, it is constructed a one parameter family of C^1-perturbation g of f with the property that $|f-g|_1 < \eta_0$ and the local unstable manifold of $g(x_g)$ and $g(y_g)$ are not jointly integrable.

Lemma 6.1.2 *Given $\eta_0 > 0$, follows that for any $\eta^u_0 > 0$ there exists a C^1-diffeomorphism $g = g(\eta_0, \eta^u_0)$ such that the following properties hold:*

1. *$|g-f|_1 \leq \eta_0$ where $|.|_1$ is the C^1-norm,*

2. $g_{|R_1(\eta_0^u)^c} = f$,

3. *for every* $z \in R_4(\eta_0^u) \cap D$ *follows that* $DH^{-1} \circ D_z g \circ DH[(0,0,1)]$ *is collinear to the vector* $(0, \eta_0, 1)$,

4. *for every* $z \in \cap_{\{n>0\}} g^n(U)$ *follows that* $E_1(z,g) = E_1(z,f)$,

5. *there exists a neighborhood* $B(f(y^-))$ *of* $W_\epsilon^{ss}(f(y^-))$ *such that* $f(x^-) \in B(f(y^-))$ *and for every* $z \in B(f(y^-)) \cap \Lambda_g(U)$ *follows that*

$$W_\epsilon^{ss}(z,g) = W_\epsilon^{ss}(z,f).$$

Proof:

Let us consider the map

$$H : B_0 \to \tilde{B}_0$$

given by remark 6.6. First, we consider a perturbation of the identity map in $\tilde{B}_0$. We take the map $T : \tilde{B}_0 \to \tilde{B}_0$ defined as

$$T(\bar{x},\bar{y},\bar{z}) = (\bar{x}, \bar{y} + T_1(\bar{x})T_2(\bar{y})T_3(\bar{z}), \bar{z}) \text{ if } (\bar{x},\bar{y},\bar{z}) \in R_1(\eta_0^u),$$
$$T(\bar{x},\bar{y},\bar{z}) = (\bar{x},\bar{y},\bar{z}) \text{ if } (\bar{x},\bar{y},\bar{z}) \in R_1(\eta_0^u)^c.$$

for some appropriate C^1-maps T_1, T_2 and T_3 defined over the real line. We assume that T_1 and T_2 verify:

1. $T_1(\bar{x}) = 0$ for all $\bar{x} \notin [a_2, b_2]$, $T_2(\bar{y}) = 0$ for all $\bar{y} \notin [c_2, d_2]$;

2. $T_1(\bar{x}) \leq 1$ for all $x \in [a_2, a_3] \cup [b_3, b_2]$, $T_2(\bar{y}) = 1$ for all $\bar{y} \in [c_2, c_3] \cup [d_3, d_2]$;

3. $T_1(\bar{x}) = 1$ for all $x \in [a_3, b_3]$, $T_2(\bar{y}) = 1$ for all $\bar{y} \in [c_3, d_3]$;

4. $|T_1'(\bar{x})| \leq \min\{\frac{1}{a_3-a_2}, \frac{1}{b_2-b_3}\}$ for any $\bar{x}$, $|T_2'(\bar{y})| \leq \min\{\frac{1}{c_3-c_2}, \frac{1}{d_2-d_3}\}$ for any $\bar{y}$.

We assume that T_3 verifies:

1. $T_3'(0) = \eta_0$,

2. $|T_3'(\bar{z})| < \eta_0$ for any $\bar{z}$,

3. $T_3(\bar{z}) = 0$ for all $\bar{z} \in [-\eta_0^u, \eta_0^u]^c$,

4. $|T_3(\bar{z})| \leq \eta_0^u \eta_0$ for any $\bar{z}$,

Observe that

$$DT = \begin{bmatrix} 1 & 0 & 0 \\ T_1'(\bar{x})T_2(\bar{y})T_3(\bar{z}) & 1 + T_1(\bar{x})T_2'(\bar{y})T_3(\bar{z}) & T_1(\bar{x})T_2(\bar{y})T_3'(\bar{z}) \\ 0 & 0 & 1 \end{bmatrix}.$$

So, taking η_0^u small enough, follows that

$$|T - I|_1 < \eta_0.$$

Moreover, we can assume that

$$|H^{-1} \circ T \circ H - I| < \eta_0.$$

Then we get the map g equal to f in the complement of $\tilde{B}_0$ and inside $\tilde{B}_0$ it is taken the map

$$g = H^{-1} \circ T \circ H \circ f.$$

To conclude the third item, observe that for any $(x, y, 0) \in R_4(\eta_0^u)$ follows that

$$T_1(x)T_2(y)T_3'(0) = \eta_0.$$

To conclude the fourth item, first observe that for any $(\bar{x}, \bar{y}, \bar{z}) \in [R_1 \setminus R_2(\eta_0^u)] \cup R_3(\eta_0^u)]$ follows that $T_1'(\bar{x})T_2(\bar{y})T_3(\bar{z}) = 0$. Therefore, if $\mathcal{O}(z) \cap [[R_1 \setminus R_2(\eta_0^u)] \cup R_3(\eta_0^u)]] = \emptyset$ then follows that $E_1(z, g) = E_1(z, f)$. Since by remark 6.7 follows that $\cap_{\{n>0\}} g^n(U) \cap [R_1(\eta_0^u) \setminus R_4(\eta_0^u)] = \emptyset$ follows that for any $z \in \cap_{\{n>0\}} g^n(U)$ then $E_1(z, g) = E_1(z, f)$.
To conclude the last item, first observe that if we take

$$\epsilon_1 < \min\{a_2 - a_1, a_4 - a_3, b_1 - b_2, b_3 - b_4\}$$

follows that for any $z \in \Lambda_g(U) = \cap_{\{n>0\}} g^n(U)$ and$z' \in W_{\epsilon_1}^{ss}(z)$ then $\mathcal{O}(z') \cap [[R_1 \setminus R_2(\eta_0^u)] \cup R_3(\eta_0^u)]] = \emptyset$ and so, for any $z' \in W_{\epsilon_1}^{ss}(z)$ holds that $E_1(z', g) = E_1(z', f)$ and so

$$W_{\epsilon_1}^{ss}(z, g) = W_{\epsilon_1}^{ss}(z, f).$$

To conclude that also holds $W_{\epsilon}^{ss}(z, g) = W_{\epsilon}^{ss}(z, f)$ we need the next claim.

Lemma 6.1.3 *For any positive integer M there exists $\hat{\eta}_0^u$ such that for $\eta_0^u < \hat{\eta}_0^u$ and $g = g(\eta_0, \eta_0^u)$ as in lemma 6.1.2 follows that if $z \in H_p$, $\hat{z} \in h_g^{-1}(z) \cap R(\eta_0^u)$ and $g^n(\hat{z}) \in R(\eta_0^u)$ then $|n| > M$.*

Proof of the lemma 6.1.3: Wedeal first wit forward iterates. Let us fix $\eta_0 > 0$. Now, for each η_0^u and $z \in R_1(\eta_0, \eta_0^u)$ follows that there exists $n = n(z, \eta_0^u)$ (n could be equal to $+\infty$) such $f^k(z) \notin R_1(\eta_0, \eta_0^u)$ for any $1 \leq k < n$. From the fact that if η_0^u is small, then the points in $R_1(\eta_0, \eta_0^u)$ are close to the local stable manifold of p_0 follows that there exists $\hat{\eta}_0^u$ such that for any $\eta_0^u < \hat{\eta}_0^u$ and $z \in R_1(\eta_0, \eta_0^u)$ if $f^n(\hat{z}) \in R_1(\eta_0, \eta_0^u)$ then $|n| > M$. In other words, if $N(\eta_0^u) = \min\{n(z, \eta_0^u) : z \in R_1(\eta_0, \eta_0^u)\}$ follows that $N(\eta_0^u) \to \infty$ as $\eta_0^u \to 0$.

Observe also that there exists $\beta_0 > 0$ and $\hat{\eta}_0^u$ such that for any $\eta_0^u < \hat{\eta}_0^u$ such that for any $\eta_0^u < \hat{\eta}_0^u$ follows that if $z \in R_1(\eta_0, \eta_0^u)$ then

$$dist(f^k(z), R_1(\eta_0, \eta_0^u)) > \beta_0, \;\; 1 \leq k < n = n(z, \eta_0^u). \tag{19}$$

Now we take a neighborhood $\mathcal{U}$ of f such that if $g \in \mathcal{U}$ then for any $z \in H_p$ and $\hat{z} \in h_g^{-1}(z)$ follows that

$$dist(g^n(\hat{z}), f^n(z)) < \frac{\beta_0}{2}. \tag{20}$$

Now we take η_0 and η_0^u small that if $g = g(\eta_0, \eta_0^u)$ as in lemma 6.1.2 then $g \in \mathcal{U}$.

Therefore, if η_0^u is small enough such that $N(\eta_0^u) > M$ from (19) and (20) follows the lemma for positive iterates. The proof is similar for backwards iterates.

■

Now we take η_0^u small enough such that the previous claim follows with M the positive integer such that

$$\lambda_s^{-M}\epsilon_1 > \epsilon.$$

Taking a small neighborhood $B(f(y^-))$ of $W_\epsilon^{ss}(f(y^-))$, follows that if $z \in B(f(y^-))$ and $g^n(z)$ is the first forward iterate that $g^n(z) \in R_1(\eta_0^u)$ then $f^i(z') = g^i(z')$ for any $z' \in W_\epsilon^{ss}(z)$ and $0 \le i \le n$. Therefore,

$$g^n(W_\epsilon^{ss}(z,g)) \subset W_{\epsilon_1}^{ss}(g^n(z,g)).$$

Since $g^i(W_\epsilon^{ss}(z,g)) = f^i(W_\epsilon^{ss}(z,g))$ for $0 \le i \le n$ and $W_{\epsilon_1}^{ss}(z,g) = W_{\epsilon_1}^{ss}(z,f)$, then the lemma follows. ■

Remark 6.8 *Let $g = g(\eta_0, \eta_0^u)$ as in the previous lemma. Then*

$$dist(g(z), f(z)) < \eta_0\eta_0^u.$$

Remark 6.9 *Let $g = g(\eta_0, \eta_0^u)$ as in the previous lemma and let $z \in B(f(y^-))$. Let*

$$\Pi_f^{ss}, \Pi_g^{ss} : B(f(y^-)) \to W_\epsilon^{cu}(x^-).$$

Then

$$\Pi_f^{ss}{}_{|B(f(y^-))} = \Pi_g^{ss}{}_{|B(f(y^-))}.$$

In particular, follows that

$$D\Pi_f^{ss}{}_{|B(f(y^-))} = D\Pi_g^{ss}{}_{|B(f(y^-))}.$$

Moreover, using the linear coordinates introduced in R follows that

$$\Pi_f^{ss}{}_{|B(f(y^-))} = Id.$$

34. One parameter family that verifies the hypothesis of proposition 6.2.

Now, we introduce a proposition that implies the proposition 6.2.

Proposition 6.3 *Given $\eta_0 > 0$ there exists η_0^u a diffeomorphisms $g = g(\eta_0, \eta_0^u)$ as in lemma 6.1.2 and $r > 0$ such that for every $z \in W_r^u(g(y_g^-)) \setminus \{g(y_g^-)\}$ follows that $W_\epsilon^{ss}(z) \cap W_r^u(g(x_g^-)) = \emptyset$.*

Observe that the proposition 6.3 implies immediately the proposition 6.2. In fact, we take η_0^u small enough and we take the family

$$\mathcal{F} = \{g(\eta_0, \eta_0^u)\}_{\eta_0}.$$

To finish, we have to prove proposition 6.3. In this direction, first we need to compute how the strong stable manifold and unstable manifold changes for the perturbed maps $g = g(\eta_0, \eta_0^u)$ as the one introduced in lemma 6.1.2. This is the goal of the next proposition. It states that the angle between the local unstable manifold of $f(x^-)$ and $g(x_g^-)$ is much larger than the angle between the local unstable manifold of $f(y^-)$ and $g(y_g^-)$.

Proposition 6.4 *Given η_0 small, follows that for any $\theta_1 > 0$, $\theta_2 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ as in lemma 6.1.2 such that:*

1. $dist(y_g^-, y^-) < \theta_1$,
2. $SL(E_3(g(y_g^-), g), E_3(f(y^-), f)) < \theta_2$,
3. $Sl(E_3(g(x_g^-), g), E_3(f(x^-), f)) \geq \frac{\eta_0}{2}$.

Remark 6.10 *Let $\Pi_g^{ss} : W_\epsilon^{cu}(y) \to W_\epsilon^{cu}(x)$ be the strong stable holonomy induced by the subbundle $E_1(.,g)$. If there is $\delta > 0$ such that $\Pi_g^{ss}(W_\delta^u(y)) \subset W_\epsilon^u(x)$ then follows that*

$$D\Pi_g^{ss}(E_3(y,g)) = E_3(x,g).$$

Proposition 6.4 implies proposition 6.3:

We want to show that

$$D\Pi_g^{ss}(E_3(g(y_g^-), g)) \neq E_3(g(x_g^-), g); \tag{21}$$

in fact, if the inequality holds then by remark 6.10 the local unstable manifold can not be jointly integrable and so the proposition 6.3 follows.

From the fact that for f the strong foliation are jointly integrable, follows that

$$D\Pi_f^{ss}(E_3(f(y^-), f) = E_3(f(x^-), f).$$

Using that θ_1 is small, follows that $g(y_g^-) \in B(f(y^-))$. From the fact that inside $B(f(y^-))$ the strong stable holonomy map $D\Pi_g^{ss}$ is the identity and that $D\Pi_f^{ss}(E_3(f(y^-), f) = E_3(f(x^-), f)$, it follows that

$$Sl(D\Pi_g^{ss}(E_3(g(y_g^-), g)), E_3(f(x^-), f)) < \theta_2.$$

Observe that

$$\begin{aligned} &Sl(E_3(g(x_g^-), g), D\Pi_g^{ss}(E_3(g(y_g^-), g))) > \\ &Sl(E_3(g(x^-), g), E_3(f(x^-), f))) - Sl(D\Pi_g^{ss}(E_3(f(x^-), f), E_3(g(y^-), g))) > \\ &\frac{\eta_0}{2} - \theta_2. \end{aligned}$$

Taking θ_2 sufficiently small follows that $\frac{\eta_0}{2} - \theta_2 > 0$ and so the inequality (21) holds.

■

To finish, we have to give the proof of proposition 6.4.

Proof of Proposition 6.4:

To prove the first item, we prove a more general statement that estimates the distance between a point z and $\hat{z} \in h_g^{-1}(z)$ for $z \in \Lambda$ and g as in lemma 6.1.2.

Lemma 6.1.4 *Given η_0 follows that for any $\gamma_0 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ as in lemma 6.1.2 such that if $z \in \Lambda$ then for any $\hat{z} \in h_g^{-1}(z)$ follows that*

$$dist(\hat{z}, z) < \gamma_0.$$

In few words, the previous lemma states that if the vertical size of the support of the perturbation is made extremely small (i.e.: η_0^u small), then the map h_g is extremely close to the identity, despite the fact the perturbation twist the vertical vector in a fix quantity (i.e.: η_0).
Proof of lemma 6.1.4: We start with a claim that follows from the fact that f is expansive and topologically hyperbolic.

Claim 10 *For any $\gamma_1 > 0$ there exists $N = N(\gamma_1)$ such that for any $z \in H_p$ and $z' \in U$ follows that:*

1. *if $dist(f^n(z), f^n(z')) < \epsilon$ for all $0 \le n \le N$ then $dist(z', W_\epsilon^{cs}(z)) < \gamma_1$;*
2. *if $dist(f^n(z), f^n(z')) < \epsilon$ for all $-N \le n \le 0$ then $dist(z', W_\epsilon^{u}(z)) < \gamma_1$.*

The next claim follows from the fact that the local unstable and local stable manifold are transversal:

Claim 11 *There is a constant c such that if $dist(z', W_\epsilon^{cs}(z)) < r$ and $dist(z', W_\epsilon^{u}(z)) < r$ with r small, then $dist(z, z') < c.r$. In what follows, Without loss of generality we assume that $c = 1$*

Now we continue with the proof of the lemma 6.1.4.
First, we take $\gamma_1 > 0$ smaller than γ_0. Let $N(\gamma_1)$ be the positive integer given by the first claim. Now we choose η_0^u such that

1. $\gamma_1 + \eta_0 \eta_0^u < \gamma_0$,
2. if $\hat{z} h_g^{-1}(z) \cap R(\eta_0^u)$ and $g^n(\hat{z}) \in R(\eta_0^u)$ then $|n| > N(\gamma_1)$ (recall lemma 6.1.3 that guarantee the election of η_0^u).

Let $z \in \Lambda_f$ and $\hat{z} \in h_g^{-1}(z) \cap R(\eta_0^u)$, then if $n_{\hat{z}}^+$ and $n_{\hat{z}}^-$ are the positive integer such that

$$g^{n_{\hat{z}}^+}(\hat{z}) \in R(\eta_0^u) \quad \text{and} \quad g^{-n_{\hat{z}}^-}(\hat{z}) \in R(\eta_0^u)$$

follows that either $n_{\hat{z}}^+ > N(\gamma_1)$ or $n_{\hat{z}}^- > N(\gamma_1)$.
Let us suppose that $n_{\hat{z}}^+ > N(\gamma_1)$. Observe that $g^i(\hat{z}) \notin R(\eta_0^u)$ for $0 \le i < n_{\hat{z}}^+$ therefore $g^i(\hat{z}) = f^i(\hat{z})$ and $dist(f^i(\hat{z}), f^i(z)) < \epsilon$ for $0 \le i < n_{\hat{z}}^+$. Then,

$$dist(\hat{z}, W_\epsilon^{cs}(z)) < \gamma_1 < \gamma_0.$$

Now, let us consider the points $f^{-n_{\hat{z}}^-}(z)$ and $g^{-n_{\hat{z}}^-}(\hat{z})$. Observe that from the fact $g^{-n_{\hat{z}}^-}(\hat{z}) \in R(\eta_0^u)$ follows that the number of backward iterates to visit again $R(\eta_0^u)$ is larger than $N(\gamma_1)$ and therefore

$$dist(g^{-n_{\hat{z}}^-}(\hat{z}), W_\epsilon^u(f^{-n_{\hat{z}}^-}(z))) < \gamma_1.$$

By remark 6.8 follows that

$$dist(g(g^{-n_{\hat{z}}^-}(\hat{z})), W^u_\epsilon(f(f^{-n_{\hat{z}}^-}(z))) < \gamma_1 + \eta_0\eta_0^u.$$

So,

$$dist(\hat{z}, W^u_\epsilon(z)) < \gamma_1 + \eta_0\eta_0^u < \gamma_0.$$

Therefore, we conclude that the distance $dist(\hat{z}, W^{cs}_\epsilon(z)) < \gamma_0$ and $dist(\hat{z}, W^u_\epsilon(z)) < \gamma_0$, so by claim 11 follows that

$$dist(\hat{z}, z) < \gamma_0.$$

■

The next lemma states that for points that do not belong to $f(B_0) \cap B_0$ then the variation in direction E_3 for the perturbed map is small:

To prove it, we need a series of lemmas. The first one, it is a folklore results and it states that the strong unstable foliation is Holder (see [HPS]).

Lemma 6.1.5 *There exists $\alpha > 0$, a neighborhood V of Λ_f and a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, $z \in \Lambda_f(V)$ and $z' \in \Lambda_g(V)$ follows that*

$$Sl(E_3(z, f), E_3(z', g)) < dist(z, z')^\alpha + |g - f|_1^\alpha.$$

Lemma 6.1.6 *Given η_0 follows that for any $\gamma_1 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ such that if $z \notin f(B_0) \cap B_0$ and $\hat{z} \in h_g^{-1}(z)$ then*

$$Sl(E_3(z, f), E_3(\hat{z}, g)) < \gamma_1.$$

Proof: First observe that the splitting in H_p can be extended continuously to the neighborhood U of H_p. Using this, observe that

$$\begin{aligned} Sl(E_3(\hat{z}, g), E_3(z, f)) < {} & Sl(E_3(\hat{z}, g), E_3(\hat{z}, f)) + \\ & + Sl(E_3(\hat{z}, f), E_3(z, f)) \end{aligned}$$

and by lemma 6.1.5 follows that

$$Sl(E_3(\hat{z}, f), E_3(z, f)) < dist(\hat{z}, z)^\alpha$$

From lemma 6.1.4 follows $dist(\hat{z}, z)$ can be taken arbitrarily small if η_0^u is sufficiently small; therefore, to conclude the proof we only need to bound

$$Sl(E_3(\hat{z}, f), E_3(\hat{z}, f)).$$

Let N_0 be the minimum positive integer such that $g^{-N_0}(z) \in R(\eta_0^u)$. Observe that if η_0^u is small, by the fact that $z \notin f(B_0) \cap B_0$ follows that N_0 is large (see lemma 6.1.3).

Let us take $E_3(g^{-N_0}(\hat{z}), g)$ and $E_3(g^{-N_0}(\hat{z}), f)$. Observe that if $x \notin R(\eta_0^u)$ follows that $D_x g = D_x f$. Then, by the domination property and previous observation follows that

$$\begin{aligned}
&Sl(E_3(\hat{z}, g), E_3(\hat{z}, f)) = \\
&Sl(Dg^{N_0}(E_3(g^{-N_0}(\hat{z}), g)), Df^{N_0}(E_3(g^{-N_0}(\hat{z}), f))) < \\
&< \lambda^{N_0} Sl(E_3(g^{-N_0}(\hat{z}), g), E_3(g^{-N_0}(\hat{z}), f))
\end{aligned}$$

where λ is the constant of domination.
By lemma 6.1.5 follows that $Sl(E_3(g^{-N_0}(\hat{z}), g), E_3(g^{-N_0}(\hat{z}), f)) < |g - f|_1$. Then,

$$Sl(E_3(\hat{z}, g) E_3(z, f)) < dist(\hat{z}, z)^\alpha + |g - f|_1 \lambda^{N_0}.$$

So, taking η_0^u such that N_0 is sufficiently large and $dist(\hat{z}, z)$ is sufficiently small, it is concluded the proof.

■

The next corollary applies the previous lemma to $W_\epsilon^u(x)$ and $W_\epsilon^u(y)$.

Corollary 6.1 *Given η_0 follows that for any $\gamma_1 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ such that if $z \in W_\epsilon^u(x_g, g)$ and $\hat{z} \in h_g^{-1}(z)$ then*

$$Sl(E_3(z, f), E_3(\hat{z}, g)) < \gamma_1.$$

In particular follows that

$$Sl(DH[E_3(z, g)], (0, 0, 1)) < \gamma_1.$$

The same result follows replacing x_g by y_g.

Proof:
It follows from the previous lemma after checking that $W_\epsilon^u(x_g, g)$ is equal to $h_g^{-1}(W_\epsilon^u(x, f))$ and that $W_\epsilon^u(x, f) \cap f(B_0) \cap B_0 = \emptyset$ (see remark 6.6).

■

End of proof of proposition 6.4:
Now, we are in condition to finish the proof of proposition 6.4:
Given η_0, we take γ_1 such that $2\gamma_1 + \gamma_1^\alpha < \frac{\eta_0}{2}$. By corollary 6.1 we can choose η_0^u and $g = g(\eta_0, \eta_0^u)$ such that

$$Sl(DH[E_3(x_g^-, g)], (0, 0, 1)) < \gamma_1.$$

Observe that $x_g^- \in R_4(\eta_0^u)$. In fact, taking η_0^u small by corollary $W_\epsilon^u(x_g, g)$ is close to $W_\epsilon^u(x, f)$ and so $x_g^- \in R_4(\eta_0^u)$. By the construction of g follows that

$$Sl(DH[E_3(g(x_g^-), g)], (0, 0, 1)) > \eta_0 - \gamma_1 > \frac{\eta_0}{2}.$$

Again by corollary 6.1 follows that

$$Sl(DH[E_3(y_g^-, g)], (0, 0, 1)) < \gamma_1$$

and therefore

$$Sl(DH[Dg(E_3(g(y_g^-),g))],(0,0,1)) < \gamma_1$$

Taking $\theta_1 = \gamma_1$ and $\theta_2 = \gamma_1 + \gamma_1^\alpha$ the proposition follows.

∎

7 Proof of theorem 3.1

The proof of the theorem is based on the proof of the theorem B in the paper [PS1].

We give the steps of the proof of theorem 3.1, we make the references to the lemmas of the cited paper and we give the proof of the lemma and definition which are different to the one given in [PS1].

To prove that there is a transitive invariant compact subset Λ such that $Df_{|E_3}$ restricted to Λ is expansive and $\mathcal{T}_\Lambda \neq \emptyset$, we take a compact invariant subset $\Lambda \subset H_p$ which is the minimal set, in the Zorn's lemma sense, such that Λ is not uniform hyperbolic. To prove the existence of this set, it is enough to show that given a sequences of nonhyperbolic compacts invariant sets $\{\Lambda_\alpha\}_{\alpha\in\mathcal{A}}$ ordered by inclusion follows that $\cap_{\alpha\in\mathcal{A}}\Lambda_\alpha$ is a nonhyperbolic compact invariant set.
Related to this set, we prove the following:

Proposition 7.1 *Let H_p be a maximal invariant topologically hyperbolic homoclinic class exhibiting a splitting $E_1 \oplus E_2 \oplus E_3$, such that it is not hyperbolic, $\mathcal{T} \neq \emptyset$ and the interior of $\mathcal{T}$ is empty. Then, the minimal nonhyperbolic set Λ is a compact invariant set, such that verifies:*

1. *it is transitive,*
2. *$Df_{|E_3}$ is expansive restricted to Λ.*

Before to give the proposition 7.1, we show that this proposition implies theorem 3.1.
Proof of theorem 3.1. Proposition 7.1 implies theorem 3.1:
We take a small neighborhood W of Λ and then we take

$$\hat{\Lambda} = Closure(\cap_{\{n\in\mathbb{Z}\}} f^n(W)).$$

It follows that:

1. $\hat{\Lambda}$ is transitive,
2. there is a pair of points $x, y \in \hat{\Lambda}$ such that $y \in W^{ss}_\epsilon(x)$,
3. $Df_{|E_3}$ is expansive restricted to Λ.

The first item follows from the fact that Λ is transitive and H_p is topologically hyperbolic. The last item is straightforward. To prove the second item of proposition 7.1, observe that from theorem 3.1 and theorem 3.3 of [Pu] and also stated in section 2 follows that there is a pair of points $x, y \in \hat{\Lambda}$ such that $y \in W^{ss}_\epsilon(x)$. In fact, if it is not the case, follows that the set $\hat{\Lambda}$ is hyperbolic and so Λ is hyperbolic, which is a contradiction.

■

Proof of proposition 7.1:
The first item is easy to prove. In fact, since any proper compact subset of Λ is hyperbolic, it follows that Λ is transitive. In fact, if it is not the case, follows that for any $x \in \Lambda$ then $\alpha(x)$ and $\omega(x)$ are properly contained in Λ; so it follows that both sets are hyperbolic. This implies that for any $x \in \Lambda$

follows that $|Df^{-n}_{|E_3(x)}| \to 0$ as $n \to +\infty$ and $|Df^{n}_{|E_1 \oplus E_2(x)}| \to 0$ as $n \to +\infty$ and therefore, Λ is hyperbolic, a contradiction.

So, it only remains to prove the last item of previous proposition 7.1. To do that, we find a set R such that for any $x \in R \cap \Lambda$ follows that $|Df^{-n}_{|E_3(x)}| \to 0$ as $n \to +\infty$, which would imply that E_3 is uniformly expansive in Λ. In fact, if $x \in \Lambda$ and $\alpha(x)$ (the $\alpha-$limit of x) is a proper subset of Λ, follows that $|Df^{-n}_{|E_3(x)}| \to 0$. If $\alpha(x)$ coincides with Λ, there is $k > 0$ such that $f^{-k}(x) \in R$ and so again $|Df^{-n}_{|E_3(x)}| \to 0$. Then, for any $x \in \Lambda$ follows that $|Df^{-n}_{|E_3(x)}| \to 0$. The set R is some kind of rectangle in terms of the splitting $E_1 \oplus E_2 \oplus E_3$. Recall the definition of rectangle given in 14 and the definition of stable, unstable and center boundary of a rectangle given in 16.

It remains the question if assuming that the interior of $\mathcal{T}$ is empty is possible to prove that E_2 is hyperbolic.

Recall that given an open set R and point $x \in R$ we denote with $W^u_R(x)$ ($W^{ss}_R(x)$) be the connected component of $W^u_\epsilon(x)$ contained in R (the connected component of $W^{ss}_\epsilon(x)$ contained in R). Moreover, given an unstable segment J we define J_R as the connected component of J contained in R.

Definition 16 *Given a rectangle R, we define*

1. *the unstable boundary of R as $\partial^u R = h([-1,1]^2 \times \{-1\}) \cup h([-1,1]^2 \times \{1\})$ of R.*
2. *the stable boundary as $\partial^{cs} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{cs}_R(x))$;*
3. *the strong stable boundary as $\partial^{ss} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{ss}_R(x))$;*
4. *the center boundary as $\partial^{c} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{c}_R(x))$.*

Definition 17 Adapted rectangle *Given a rectangle R we say that it is an adapted rectangle if for any $z \in \Lambda \cap R$ then*

1. *$W^u_R(z)$ is a connected component of $W^u_\epsilon(z)$ that intersects the two components of the unstable boundary of R;*
2. *for any positive integer n one of the following holds:*
   (a) *$f^{-n}(W^u_R(z)) \subset R$;*
   (b) *$f^{-n}(W^u_R(z)) \cap R = \emptyset$.*

Related to the notion of rectangle we define the notion of return maps.

Definition 18 Returns.

Let R be an adapted rectangle. A map $\psi : S \to R$ (where $S \subset R$) is called a return of R associated to Λ if:

- *$S \cap \Lambda \neq \emptyset$*
- *there exist $k > 0$ such that $\psi = f^{-k}_{/S}$*

- *$\psi(S) = f^{-k}(S)$ is a connected component of $f^{-k}(R) \cap R$*
- *$f^{-i}(S) \cap R = \emptyset$ for $1 \le i < k$*

We denote the set of returns of R associated to Λ by $\mathcal{R}(R, \Lambda)$. Moreover, we define with R_ψ the image of Ψ and we say that a return $\psi \in \mathcal{R}(R, \Lambda)$ have $|\psi'| < \xi < 1$ if

$$|D^f - k_{|E_3} y| < \xi \text{ for all } y \in W^u_R(z), z \in dom(\psi) \cap \Lambda,$$

where $\psi = f^{-k}_{/dom(\psi)}$.

We prove that there exists a rectangle R such that if $x \in R$ then $|Df^{-n}_{|E_3(x)}| \to 0$. In this direction, we prove the following propositions.

Proposition 7.2 *Let R be an adapted rectangle and assume that for every return $\psi \in \mathcal{R}(R, \Lambda)$ we have $|\psi'| < \xi < 1$ for some ξ. Then for all $y \in R \cap \Lambda$ the following holds:*

$$\sum_{n \ge 0} \ell(f^{-n}(J(y))) < \infty, \quad \text{and } |Df^{-k}_{|E_3(y)}| \to_{n \to \infty} 0.$$

Following this strategy, to conclude the theorem 3.1, it is enough to prove the following proposition.

Proposition 7.3 *Let Λ be the minimal non-hyperbolic set associated to H_p.*
Then, there exists an adapted rectangle R such that for every return $\psi \in \mathcal{R}(R, \Lambda)$ we have $|\psi'| < \xi$ for some $\xi < 1$.

Therefore, to finish we have to prove proposition 7.2 and 7.3. This is done in the next two subsections.

■

7.1 Proofs of proposition 7.2.

First, we start establishing the relation between summability of the length of the unstable arcs and the hyperbolicity along the subbundle E_3. In other words, we show that if the sum of the length of the negative iterates of the unstable leaves is uniformly bounded then the derivative of f along the subbundle E_3 goes to zero for backward iterates. It is a general argument that follows from smoothness. In our case, since the map is C^2 and 2−domination holds, follows that the unstable discs are C^2. In fact:

Lemma 7.1.1 *Let Λ be a topologically hyperbolic set exhibiting a splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1 \oplus$ is topologically contractive. Then, there exists $\lambda < 1$ such that $\frac{|Df_{|E_1 \oplus E_2}|}{|Df_{|E_3}|^2} < \lambda$. Morover, it follows that the unstable discs $W^u_\epsilon(x)$ are C^2 for any x.*

The proof is similar to the proof of lemma 12 of [Pu]. From the fact that the unstable arc are C^2 we can get the following lemma:

Lemma 7.1.2 *There exists a constant K_0 such that if $y \in W^u_\epsilon(x)$ follows that*

$$\frac{|Df^{-n}_{E_3(x)}|}{|Df^{-n}_{E_3(y)}|} \leq \exp(K_0 \sum_{i=0}^{n_1} |f^{-i}(x) - f^{-i}(y)|).$$

Moreover

$$|Df^{-n}_{E_3(x)}| \leq \frac{\ell(f^{-n}(J_x))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n_1} \ell(f^{-1}(J)))$$

where $J \subset W^u_\epsilon(x)$.

Proof of proposition 7.2:

The proof is similar to the proof of lemma 3.7.2 given in [PS1] (page 10012) and the key argument is that in each return we have contraction along the E_3-subbundle, combined with the fact that the sum up to a return of the length of iterates of the unstable arc is uniformly bounded.

More precisely, it is necessary the following lemma which is useful also in the rest of the proof of proposition 7.3. The lemma state the uniform bound of the sum up to a return of the length of the unstable arcs. Moreover, state that the subbundle E_2 is contractive for sufficently large positive iterates.

Lemma 7.1.3 *Let R be an adapted rectangle. There exists $K_1 = K_1(R)$ such that if $x \in R$, $J = W^u_\epsilon(x) \cap R$ and $f^{-k_0}(J)$ is the first return of J to R then follows that*

$$\sum_{i}^{k_0} \ell(f^{-i}(J)) < K_1.$$

Moreover, there exist a positive integer N_0, a positive constant C_0 and $0 < \lambda_0 < 1$ such that if $k > N_0$ then

$$|Df^i_{|E_2(f^{-k}(z))}| < C_0 \lambda_0^i \quad \forall z \in J \; i > N_0.$$

The proof of this lemma is similar to the proof of lemma 3.7.1 given in [PS1] (page 1010) and the key argument is the fact that the maximal invariant subset of Λ outside R, i.e.,

$$\Lambda_1 = Closure[\bigcap_{n\in\mathbb{Z}} f^n(\Lambda - R)]$$

is a proper set of Λ and so it verifies that it is a hyperbolic set. More precisely, if the previous set is empty, follows that for any point in R the return time are uniformly bounded, and so the lemma holds immediately. If the set it is no empty, it is possible to get a neighborhood of Λ_1 such that while the iterates remain in this neighborhood follows that the subbundle E_2 and E_3 are hyperbolic; moreover, the number of iterates that an orbit remains in the complement of the mentioned neighborhood of Λ_1 and R is uniformly bounded. From these facts together follows the conclusion of the lemma.

After that, as we mentioned, to conclude proposition 7.2 we can repeat the arguments done in lemma 3.7.2 proved in [PS1] (page 1012).

■

7.2 Proof of proposition 7.3.

The proof of the proposition is done in different steps. First, we need more geometrical properties. More precisely it is introduced some kind of special type of adapted rectangle. This is shown in subsection 7.2.1 where also is proved the existences of this kind of rectangle in lemma 7.4. Latter, in subsection 7.2.2 are introduced some techniques called distortion, and which are useful to compare the volume of this rectangle to the length of the local unstable manifold. In subsection 7.2.3 it is study how the distortion changes under iterations. In subsection 7.2.4 it is ended the proof of proposition 7.3.

7.2.1 Well adapted rectangles.

Recall that we want to show the existence of a rectangle that verifies the hypothesis of proposition 7.2. To do that, we need some definitions (see figure 13).

Definition 19 Horizontal rectangle. *Given a rectangle R as the one defined in definition 14, we say that $R^h \subset R$ is an horizontal rectangle if there exists $[a, b] \subset [-1, 1]$ such that $R^h = h([-1, 1]^2 \times [a, b])$*

Definition 20 Vertical rectangle. *Given a rectangle R as the one defined in definition 14, we say that $R^v \subset R$ is a vertical rectangle if there exists $[a, b] \subset [-1, 1]$ and $[c, d] \subset [-1, 1]$ such that $R^h = h([a, b] \times [c, d] \times [-1, 1])$.*

Remark 7.1 *Given an adapted rectangle R and a return ψ observe that its domain is a vertical rectangle and its image is contained in a horizontal rectangle. Moreover, if the domain is properly contained in R follows that the image is a horizontal rectangle.*

To check the remark, observe that if $x \in S$, where S is the domain of a return associated to a rectangle R, by the definition of adapted box follows that that $W^u_R(x) \subset S$.

Lemma 7.2.1 *Let R be an adapted rectangle. Then for every $\psi \in \mathcal{R}(R, \Lambda)$ follows that $R_\psi = Image(\psi)$ is an adapted rectangle.*

Proof: Observe that by definition of R_ψ, the unstable boundary of it(we called it the bottom and the top of R_ψ) is given by the center stable manifold of some points in H_p. More precisely, the top and bottom of R_ψ are contained in a connected component of $f^{-k}(W^{cs}_\epsilon(y_1)) \cap R$ and in a connected component of $f^{-k}(W^{cs}_\epsilon(y_1)) \cap R$, where $f^{-k} = \psi$ and y_1, y_{-1} are the points such that their center stable manifolds contains the top and bottom of R. To finish, we have to check that if $x \in R_\psi$ follows that $f^{-n}(W^u_{R_\psi}(x)) \subset R_\psi$ or $f^{-n}(W^u_{R_\psi}(x)) \cap R_\psi = \emptyset$. If it is not the case, i.e.: if there is x, and a positive integer n such that $f^{-n}(W^u_{R_\psi}(x)) \cap R_\psi \neq \emptyset$ and $f^{-n}(W^u_{R_\psi}(x))$ is not contained in R_ψ, follows that $f^{-n}(W^u_{R_\psi}(x)) \cap \partial^u R_\psi \neq \emptyset$, and this implies, that $f^{k-n}(W^u_{R_\psi}(x)) \cap \partial^u R \neq \emptyset$ and since $f^k(W^u_{R_\psi}(x)) = W^u_R(f^k(x))$ follows that $f^{-n}(W^u_R(f^k(x))) \cap \partial^u R \neq \emptyset$ which is absurd since R is an adapted rectangle.

■

Definition 21 Well adapted rectangle. *Given a rectangle $R = h([-1, 1]^3)$, we say that R is a well adapted rectangle if there is a positive integer N_0 such that $f^{-N_0}(W^s_\epsilon(p)) \cap R = h([-1, 1]^2 \times \{1\}) \cup h([-1, 1]^2 \times \{-1\})$ and there exists a rectangle $\hat{R}$ contained in R such that*

1. $\hat{R} = h([-1,1] \times [a,b] \times [-1,1])$ *for* $-1 < a < b < 1$*;*
2. $[R \setminus \hat{R}] \cap H_p = \emptyset$.

Moreover, there exist two vertical rectangles R_1^v, R_2^v *such that for each* R_i^v *follows that one of the connected component of* $\partial^c(R_i^v)$ *is contained in the center boundary of* $\partial^c(R)$ *for* $i = 1,2$ *and one of the following options holds:*

1. *either* $[W_R^{ss}(R_i^v) \cup R_i^v] \cap H_p = \emptyset$
2. *or there is a horizontal rectangles* R_i^h *and a returns* ψ_i, ψ_2 *such that*
   (a) R_i^v *and* R_i^h *are the domain and image of* ψ_1*;*
   (b) $[W_R^{ss}(R_i^v) \setminus R_i^v] \cap H_p = \emptyset$ *for* $i = 1,2$*; where* $W_\epsilon^{ss}(A) = \cup_{x \in A} W_\epsilon^{ss}(x)$*.*

Observe that on one hand, if R is a well adapted rectangle then the strong stable boundary of R does not intersect Λ. On the other hand R is a well adapted rectangle if either the set Λ does not intersect the central boundaries of R or if it is not the case, the central boundary is contained in the domain of some return. See figure 14.

Lemma 7.2.2 *If* R *is a well adapted rectangle then it is an adapted rectangle.*

Proof: First, we have to check that if $x \in R \cap H_p$ then $W_R^u(x)$ is a connected component of $W_\epsilon^u(z)$ that intersects the top and the bottom of R. If $x \in R_1^v \cup R_2^v$, follows from the definition. If $x \notin R_1^v \cup R_2^v$, then $W_\epsilon^u(x) \cap R_1^v \cup R_2^v = \emptyset$. In other case, it would imply that $x \in R_1^v \cup R_2^v$. So, if $W_R^u(x)$ is not a connected component of $W_\epsilon^u(x)$ that intersects the top and the bottom of R follows that $W_\epsilon^u(x)$ intersect $[W_R^{ss}(R_i^v) \setminus R_i^v]$ (for $i = 1$ or $i = 2$) which is an absurd because $W_\epsilon^u(x) \subset H_p$ and $[W_R^{ss}(R_i^v) \setminus R_i^v] \cap H_p = \emptyset$.

To check the second items in the definition of adapted box, observe that for any $x \in R$ and any positive integer k follows that $f^{-k}(W_R^u(x)) \cap \partial^u(R) = \emptyset$. If it is not the case, then follows that $f^k(f^{-N_0}(W_\epsilon^s(p))) \cap interior(R) \neq \emptyset$. Which is an absurd because $f^k(f^{-N_0}(W_\epsilon^s(p))) \subset f^{-N_0}(W_\epsilon^s(p))$ and $f^{-N_0}(W_\epsilon^s(p)) \cap R = h([-1,1]^2 \times \{1\}) \cup h([-1,1]^2 \times \{-1\})$.

∎

Lemma 7.2.3 *Given a well adapted rectangle* R *and a return* ψ*, follows that* R_ψ *(the image of* ψ*) is a well adapted rectangle.*

Proof: It follows immediately and it is similar to the proof of lemma 7.2.1.

∎

Proposition 7.4 *Let* H_p *be a topologically hyperbolic attracting homoclinic class such that the interior of* $\mathcal{T}$ *is empty. Then, there exist a well adapted rectangle associated to* Λ*.*

Proof of proposition 7.4:

Let us start taking a point $x \in H_p$. By lemma 4.5.9, for any ϵ' there exist arcs l_x^+ and l_x^- contained in opposite connected components of $W_{\epsilon'}^{ss}(x) \setminus \{x\}$ and such that $l_x^+ \cap H_p = \emptyset$ and $l_x^- \cap H_p = \emptyset$. So, there exists $\gamma_x > 0$ such that $W_{\gamma_x}^c(l_x^+) \cap H_p = \emptyset$ and $W_{\gamma_x}^c(l_x^-) \cap H_p = \emptyset$, where $W_{\gamma_x}^c(l_x^{+-}) = \cup_{\{z \in l_x^{+-}\}} W_{\gamma_x}^c(z)$. Let z_x^- and z_x^+ (they depends on the point x) in opposite connected components of $W_\epsilon^c(x)$ such that $W_\epsilon^{ss}(z_x^-) \cap W_{\gamma_x}^c(l_x^+) \neq \emptyset$, $W_\epsilon^{ss}(z_x^+) \cap W_{\gamma_x}^c(l_x^+) \neq \emptyset$ and $W_\epsilon^{ss}(z_x^-) \cap W_{\gamma_x}^c(l_x^-) \neq \emptyset$, $W_\epsilon^{ss}(z_x^+) \cap W_{\gamma_x}^c(l_x^-) \neq \emptyset$. Let us consider the region B_x in $W_\epsilon^s(x)$ bounded by $W_\epsilon^{ss}(z_x^-)$, $W_{\gamma_x}^c(l_x^+)$, $W_\epsilon^{ss}(z_x^+)$ and $W_{\gamma_x}^c(l_x^-)$. We take B_x^+ and B_x^- the two connected components of $B_x \setminus W_\epsilon^{ss}(x)$.

We consider two cases:

1. there exists x such that, there exist y^-, y^+ in opposite connected components of $W_\epsilon^c(x) \setminus \{x\}$ such that $W_\epsilon^{ss}(y^-) \cap B_x \cap H_p = \emptyset$ and $W_\epsilon^{ss}(y^+) \cap B \cap H_p = \emptyset$;

2. for every x holds that either for every $y \in B_x^+$ follows that $W_\epsilon^{ss}(y) \cap H_p \cap B_x^+ \neq \emptyset$ or for every $y \in B_x^-$ follows that $W_\epsilon^{ss}(y) \cap H_p \cap B_x^- \neq \emptyset$

First case.

In the first case, we claim that the we can built a rectangle as in the option one of lemma 7.4. To avoid notation in this part we do not write the dependence of the points on x.

In fact, let us take arcs l_{y^-} and l_{y^+} in opposite connected components of $W_\epsilon^c(x) \setminus \{x\}$ such that $W_\epsilon^{ss}(l_{y^-}) \cap B^s \cap H_p = \emptyset$ and $W_\epsilon^{ss}(l_{y^+}) \cap B^s \cap H_p = \emptyset$. Then, there are arcs l_{y^-} and l_{y^+} in opposite connected components of $W_\epsilon^c(x) \setminus \{x\}$ such that $W_\epsilon^{ss}(l_{y^-}) \cap B^s \cap H_p = \emptyset$ and $W_\epsilon^{ss}(l_{y^+}) \cap B^s \cap H_p = \emptyset$.

Now we take x_1^+, x_2^+, the boundary points of l_x^+; x_1^-, x_2^-, the boundary points of l_x^-; y_1^+, y_2^+, the boundary points of l_{y^+} and y_1^-, y_2^-, the boundary points of l_{y^-}. We order then by distances to the point x. We take the rectangle R^s bounded by $W_\epsilon^{ss}(y_1^-)$, $W_\epsilon^{ss}(y_1^+)$, $W_\epsilon^c(x_1^-)$ and $W_\epsilon^c(c_1^+)$. We take the rectangle $\hat{R}^s$ bounded by $W_\epsilon^{ss}(y_2^-)$, $W_\epsilon^{ss}(y_2^+)$, $W_\epsilon^c(x_2^-)$ and $W_\epsilon^c(c_2^+)$. Observe that $R^s \subset \hat{R}^s$. Now, we take z^+ and z^- in opposite components of $W_\epsilon^u(x) \setminus \{x\}$. For each $z \in \hat{R}^s$ we define $W_{z^+,z^-}^u(z)$ as the connected component of $W_\epsilon^u(z) \setminus \{W_\epsilon^s(z^+) \cup W_\epsilon^s(z^-)\}$ bounded by $W_\epsilon^s(z^+)$ and $W_\epsilon^s(z^-)$. Now we define

$$\hat{R} = \cup_{\{z \in \hat{R}^s\}} W_{z^+,z^-}^u(z) \quad R = \cup_{\{z \in R^s\}} W_{z^+,z^-}^u(z).$$

Observe that these rectangles verify the items 1 and 2 of the proposition. To get that the bottom and the top are contained in the stable manifold of p and that R is adapted, we use the following claim:

Claim 12 *Let p be a periodic point. For every $\delta > 0$, there exist $N_0 = N_0(\delta)$ such that for every z follows that $f^{-N_0}(W_\epsilon^s(p))$ intersects both connected components of $W_\delta^u(x) \setminus \{x\}$.*

Then, using the previous claim, we can cut the rectangle by the stable manifold of a periodic point.

Second case.

We start with the following lema:

Lemma 7.2.4 *Assuming that we are in the second case follows that given two pair of points x_1 and x_2 in H_p such that $x_1 \in W_\epsilon^{ss}(x_2)$ follows that $W_\epsilon^u(x_1)$ and $W_\epsilon^u(x_2)$ cannot s-intersect transversally.*

Proof:

Let us assume that the lemma is false. Let $x_1' \in W^u_\epsilon(x_1)$ and $x_2' \in W^u_\epsilon(x_2)$ such that $x_2' \in W^{ss}_\epsilon(x_1')$ and for any r small follows that $W^u_r(x_1')$ manifold s-intersect transversally $W^u_r(x_2')$.

Let us take $B_{x_2'}$ small enough such that $x_1' \notin W^u_\epsilon(B_{x_2'}) = \cup_{\{z\in B_{x_2'}\}} W^u_\epsilon(z)$. Let us assume that for any $y \in B^+_{x_2'}$ follows that $W^{ss}_\epsilon(y) \cap B^+_{x_2'} \cap H_p \neq \emptyset$.

Let us take $W^{cu,+}_r(x_2') = \cup_{\{z\in W^{c,+}_\epsilon(x_2')\}} W^u_r(z)$ where $W^{c,+}_\epsilon(x_2')$ is the connected component of $W^c_\epsilon(x_2') \setminus \{x_2'\}$ that intersects $B^+_{x_2'}$. Let us take

$$\Pi^{ss} : B_{x_2'} \to W^{cu}_\epsilon(x)$$

If r is small, observe that

$$W^{cu,+}_r(x_2') \subset \Pi^{ss}(W^u_\epsilon(B_{x_2'}) \cap H_p)$$

Moreover, form the fact that we are assuming that the local unstable manifold of x_2' and x_1' s-intersect transversally each other, follows that for some r' holds that

$$\Pi^{ss}(W^{u,+}_{r'}(x_1')) \subset interior(W^{cu,+}_r(x_2'))$$

where $W^{u,+}_{r'}(x_1')$ is one of the connected components of $W^u_{r'}(x_1') \setminus \{x_1'\}$.

Then, taking any point z_0 close to a point $z \in int(W^{u,+}_r(x_1'))$ such that $\Pi^{ss}(z)$ is in the interior of $W^{cu,+}_r(x_2')$ follows that $[W^{ss}_\epsilon(z_0) \setminus \{z_0\}] \cap H_p \neq \emptyset$ which implies that the interior of $\mathcal{T}$ is not empty, which is an absurd.

■

Coming back to the proof of the proposition 7.4, we take a point x such that for every $y \in B_x$ follows that $W^{ss}_\epsilon(y) \cap H_p \cap B^+_x \neq \emptyset$. Let us define

$$R_x = \cup_{\{z\in B_x\}} W^u_\epsilon(z).$$

We can find two periodic points q_1 and q_2 with large period in each side of $R_x \setminus W^{su}_\epsilon(x)$ such that for each q_i holds that $dist(q_i, x) \leq dist(f^j(q_i), x)$ for any j. Then we take the connected component of $R_x \setminus [W^{su}_\epsilon(q_1) \cup W^{su}_\epsilon(q_2)]$ that contains x. Now, we can use the claim 12 and we cut this connected component by the stable manifold of the p. We claim that the remaining rectangle is a well adapted rectangle. To check that, first observe that it is an adapted rectangle and the proof is similar to the previous case. To check that it is well adapted, it is necessary to show that associated to each q_i it can be constructed a vertical rectangle which is the domain of its associated return map. First, for each q_i, it is taken the connected component of $R_i' = f^{n_1}(W^{su}_\epsilon(q_i) \cap R) \cap R$ that contains q_i and where n_i is the period of q_i. Later, we take the connected component of $f^{-n_i}(W^c_\epsilon(R_i') \cap R) \cap R$ that contains q_i. This connected component is the horizontal rectangle R^h_i; the vertical rectangle is $f^{n_i}(R^v_i)$ and the return $\psi_i = f^{-n_i}_{|R^v_i}$. This finish the proof of the proposition.

■

7.2.2 Rectangle: volume and length. Distortions.

Now we adapt to dimension three, a series of definition given in [PS1].

Definition 22 *We say that a rectangle R has distortion (or s-distortion) C if for any $y, z \in R$*

$$\frac{1}{C} \leq \frac{\ell(W^u_R(z))}{\ell(W^u_R(y))} \leq C.$$

Remark 7.2 *If a rectangle R has distortion C then for any $z \in R$ follows that*

$$\ell(W^u_R(z)) Area(W^{cs}_R(z)) < C Volume(R).$$

Notice that, in order to guarantee distortion C on a rectangle R, it is sufficient to find a C^1 foliation $\mathcal{F}^s$ by two dimensional embedding with tangent planes close to the $E_1 \oplus E_2$-subbundle ($T_x\mathcal{F}^s$ lies in a b-cone for b small along the center-stable), such that, for any $z, y \in R$ follows that,

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

holds, where $\Pi = \Pi(W^u_R(z), W^u_R(y))$ is the projection along the foliation between these unstable arcs.

Given a point $z \in R \cap \Lambda_0$, for any positive integer n we can take the rectangle R_n around $f^{-n}(x)$ defined as the connected component of

$$R_n = f^{-n}(R) \cap B_\epsilon(f^{-n}(x))$$

that contains $f^{-n}(x)$. Observe that inside R_n we can define the foliation $\mathcal{F}^s_n$ taking the negative iterations of the foliation $\mathcal{F}^s$; i.e.: given $z \in R_n$ we take the the connected component of

$$\mathcal{F}^s_n(z) = f^{-n}(\mathcal{F}^s(f^n(z))) \cap R_n$$

that contains z.

The following lemma will be useful in the sequel. The proof of this lemma is similar to the proof of lemma 3.4.1 in [PS1].

Lemma 7.2.5 *Let R, $\mathcal{F}^s$ and C be as above. If for any $z \in R_n$ follows that*

$$|Df^k_{T_z\mathcal{F}^s_n(z)}| < C_0\lambda_0^k$$

for any $N_0 \leq k \leq k_0$. Then R_n has distortion $C_1 = C_1(C, C_0, \lambda_0, N_0)$.

Applying previous lemma to lemma 7.1.3 we can conclude the following corollary:

Corollary 7.1 *Let R be an adapted box with distortion C. Then, for any $\psi \in \mathcal{R}(R, \Lambda)$ follows that R_ψ has distortion $C_1 = C_1(C, C_0, \lambda_0, N_0)$.*

7.2.3 Controling the sum up to a return.

The next lemma is similar to the lemma 3.7.3 of [PS1] (page 1014). However, in the present context the proof is simpler than the one done in [PS1] and use explicitly the properties of the adapted rectangle.

Lemma 7.2.6 *Let R be a well adapted rectangle. There exists $K = K(R)$ with the following property: for every $\psi \in \mathcal{R}(R, \Lambda)$ and $z \in B_\psi = Image(\psi)$ (denoting $J_\psi(z) = J(z) \cap B_\psi$) follows that*

$$\sum_{j=0}^{n} \ell(f^{-j}(J_\psi(z))) \leq K$$

whenever $f^{-j}(z) \notin R_\psi, 1 \leq j \leq n$.

Proof: Let R be a well adapted rectangle and let $\psi \in \mathcal{R}(R, \Lambda)$. Let $z \in R_\psi = Image(\psi)$ and a positive integer n such that $f^{-j}(z) \notin R_\psi, 1 \leq j \leq n$. Let also C_1 be as in corollary 7.1.3 and consider C_2 from corollary 7.1 (corresponding to $C_2 = C_1$). This means, that R_ψ has distortion C_2.

Let $0 < n_1 < n_2 < ... < n_s \leq n$ be the set $\{0 < j \leq n : f^{-j}(z) \in R\}$. For every n_i we have associated a return $\psi_i \in \mathcal{R}(R, \Lambda)$ such that $f^{-n_i}(z) \in R_{\psi_i}$, i.e., $f^{-n_i}(z) = \psi_i(f^{-n_{i-1}}(z))$ where $\psi = f^{-k_i}$ for some k_i.

We take positive constants λ_1 and λ_2 such that $\lambda < \lambda_1 < \lambda_2 < 1$ where λ is the constant of domination.

We consider (if exists) the sequence $0 = m_0 < m_1 < m_2 < ... < m_l \leq n$ such that

$$|Df^j_{/E_1 \oplus E_2(f^{-m_i}(z))}| < \lambda_2^j,\ 0 \leq j \leq m_i,\ \forall i = 1, ..., l.$$

We claim the following:

Claim 13 *There exists $C_4 = C_4(R)$ such that*

$$\sum_{i=0}^{l} \ell(f^{-m_i}(J_\psi(z))) \leq C_4$$

where $J_\psi(z) = W^u_\epsilon(z) \cap R_\psi$.

Proof of the claim: To show that, we construct a rectangle associated to each m_j. Recall that the rectangle R contains a sub rectangle $\hat{R}$ such that $[R \backslash \hat{R}] \cap H_p = \emptyset$. Let us take $\hat{\hat{R}} = \hat{R} \backslash [W^{ss}_\epsilon(R^v_1) \cup W^{ss}_\epsilon(R^v_2)]$. Now we select a series of constants: Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the following positive constants

$$\gamma_1 < \frac{1}{2} \min_{z \in \hat{\hat{R}}} dist(\partial^s(W^{cs}_\epsilon(z) \cap R), \partial^s(W^{cs}_\epsilon(z) \cap \hat{\hat{R}})),$$

$$\gamma_2 < \ell(f^{-n}(W^c_{\gamma_1}(z)) \cap W^c_\epsilon(f^{-n}(z)))\ \ \forall\ z \in H_p,$$

$$\gamma_3 < \min_{z \in R^h_1 \cup R^h_2} \{\ell(W^c_{R^h_1}(z)), \ell(W^c_{R^h_2}(z))\},$$

$$\gamma_4 < \ell(f^{-n}(W^c_{\gamma_3}(z)) \cap W^c_\epsilon(f^{-n}(z))) \ \forall \ z \in H_p.$$

Now, let

$$\gamma_0 = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}.$$

For each m_j, we take n_{i_j} of the sequences $\{n_1, ..., n_s\}$, such that $f^{-k}(z) \notin R$ for $n_{i_j} < k < m_j$. Given $z \in R_\psi$, we consider

$$i_0 = \min\{i_j > 0 : f^{-n_{i_j}}(z) \notin R^v_1 \cup R^v_2\}$$

and

$$j_0 = \min\{j : m_j \geq m_{i_0}\}.$$

We assert first that in this case

$$\sum_{j=j_0}^{l} l(f^{-m_j}(J_\psi(z))) < K_1.$$

To show that, we consider the rectangle $R(n_{i_j}) = R_{\psi_{i_j}}$ and we take the rectangle $R(j)$ as the connected component of

$$f^{-(m_j - n_{i_j})}(R(n_{i_j})) \cap W^{cs}_{\gamma_0}(f^{-m_j}(J_\psi(z)))$$

that contains $f^{-m_j}(z)$.

On one hand, we show that for $j_1 \neq j_2$ and larger than j_0 follows that

$$R(j_1) \cap R(j_2) = \emptyset.$$

On the other hand, from corollary 7.1 follows that R_i has distortion C_2 and so the area is compare to the length in the following way:

$$\ell(f^{-m_j}(J_\psi(z))) Area(W^{cs}_{R(j)}(f^{-m_j}(z)) < C_2 Vol(R(j)),$$

and

$$Area(W^{cs}_{R(j)}(f^{-m_j}(z)) > \gamma_0.$$

So,

$$\ell(f^{-m_j}(J_\psi(z))) < C_2 \frac{1}{\gamma_0} Vol(R(j)).$$

Therefore

$$\sum_{j=j_0}^{l} \ell(f^{-m_j}(J_\psi(z))) \leq C_2 \frac{1}{\gamma_0} \sum_{j=j_0}^{l} Vol(R(j)) \leq \frac{1}{\gamma_0} C_2 K$$

where K is such that

$$\sum_{j=0}^{j_0 - 1} Vol(R(j)) < K.$$

The constant K exists because the rectangle $R(j)$ are disjoint. So, the claim is proved.

So, to conclude that $\sum_{j=j_0}^{l} l(f^{-m_j}(J_\psi(z))) < K_1$, we have to show that the rectangles $R(j)$ are disjoint. To show that, first observe that if $f^{-n_i}(z) \in R$, then $f^{n_i}(W^s_R(f^{-n_i}(z)) \subset R_\psi$. Let us suppose that $R(j_1) \cap R(j_2) \neq \emptyset$. It follows that $W^s_{\gamma_0}(f^{-m_{j_1}}(z)) \cap f^{-m_{j_2}}(J_\psi(z)) \neq \emptyset$ so (assuming that $j_1 < j_2$) it follows that $W^{cs}_{\gamma_0}(f^{-n_{i_0}}(z)) \cap f^{k_0 - m_{j_2}}(J_\psi(z)) \neq \emptyset$ where $k_0 = m_{j_1} - n_{i_0}$. By the election of γ_0 and from the fact that $f^{-n_{i_0}}(z) \in R \backslash [R^v_1 \cup R^v_2]$ follows that $f^{k_0 - m_{j_2}}(J_\psi(z)) \subset R$. Then, $f^{k_0 - m_{j_2}}(J_\psi(z)) \cap W^{cs}_R(f^{-n_{i_0}}(z)) \neq \emptyset$ which implies that $f^{n_{i_0}}(f^{k_0 - m_{j_2}}(J_\psi(z))) \subset R_\psi$, i.e: $f^{-(m_{j_1} - m_{j_2})}(J_\psi(z)) \subset R_\psi$ which is an absurd because the first return is $f^{-n}(z)$ and $m_{j_2} - m_{j_1} < n$.

Now, to finish the proof of the claim, we have to control the sum

$$\sum_{j=0}^{j_0} l(f^{-m_j}(J_\psi(z))).$$

In this case we have that $f^{-n_i}(z) \in R^v_1 \cup R^v_2$ for any $i < i_0$. Observe that in particular $f^{-n_{i+1}}(z) \in R^v_1 \cup R^v_2$ for any $i < i_0$

Define $B(n_i)$ as the connected component of $f^{-n_i}(R_\psi) \cap R^v_l$ which contains $f^{-n_i}(z)$, and l is equal to 1 or 2 depending if $f^{n_i}(z) \in R^v_1$ or $f^{n_i}(z) \in R^v_2$. Observe that, for $B(n_i)$ follows that

$$f^{-k}(B(n_i)) \cap R = \emptyset \;\; \forall\; 0 < k < n_{i+1} - n_i. \tag{22}$$

In this case, for each m_j such that $n_{i_j} < n_{i_0}$ we consider the rectangle $R(j)$ as the connected component of

$$f^{-(m_j - n_{i_j})}(B(n_{i_j})) \cap B_{\gamma_0}(f^{-m_j}(J_\psi(z))).$$

Again, we have that for this rectangle we can uniformly compare the length with the volume. So, to conclude, we have to show that the rectangles $R(j)$ in this case are also disjoint. To show that, observe that if $R(j_1) \cap R(j_2) \neq \emptyset$ then $f^{-(m_{j_1} - n_{i_{j_1}})}(B(n_{i_{j_1}})) \cap f^{-(m_{j_2} - n_{i_{j_2}})}(B(n_{i_{j_2}})) \neq \emptyset$. Assuming that $m_{j_1} - n_{i_{j_1}} \leq m_{j_2} - n_{i_{j_2}}$ follows that $B(n_{i_{j_1}}) \cap f^{-k}(B(n_{i_{j_2}})) \neq \emptyset$ with $0 \leq k < m_{j_2} - n_{i_{j_2}} \leq n_{i_{j_2}+1} - n_{i_{j_2}}$. Which is a contradiction with (22). Then, we have concluded that

$$\Sigma^l_{j=0} l(f^{-m_j}(J_\psi(z)) < K_1.$$

■

To finish the proof of the lemma, we must control the sum between consecutive $m_i's$. To do that, we need a lemma due to Pliss:

Pliss's Lemma: Given $0 < \gamma_0 < \gamma_1$ and $a > 0$, there exist $N_1 = N_1(\gamma_0, \gamma_1, a)$ and $l = l(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $\{a_i\}_{0 \leq i \leq n}$ with $n > N_1$, $a^{-1} < a_i < a$ and $\Pi^n_{i=0} a_i < \gamma^n$ then there exist n_0 with $n_0 < ln$ such that

$$\Pi^j_{i=n_0} a_i < \gamma_1^{j-n_0} \quad n_0 < j < n.$$

Let $N = N(\lambda_1, \lambda_2)$ from Pliss's lemma and consider $K_2 = \sup\{\|Df^j\| : 1 \le j \le N\}$. There are two possibilities: $m_{i+1} - m_i < N$ or $m_{i+1} - m_i \ge N$. If $m_{i+1} - m_i < N$, then

$$\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \le NK_2 \ell(f^{m_i}(J_\psi(z))).$$

On the other hand, if $m_{i+1} - m_i \ge N$, then

$$|Df^j_{/E_1 \oplus E_2(f^{m_i - j}(z))}| \ge \lambda_1^j \text{ for } N \le j \le m_{i+1} - m_i.$$

In fact, if it is not the case, i.e.: if there exists j such that $N < j < m_{i+1} - m_i$ with

$$|Df^j_{/E_1 \oplus E_2(f^{-(m_i+j)}(z))}| < \lambda_1^j$$

follows by Pliss's lemma, there exists $m_i < \tilde{n}_i < m_{i+1}$ such that $|Df^k_{/E_1 \oplus E_2(f^{-\tilde{n}_i}(z))}| < \lambda_2^k$ for $0 < k < \tilde{n}_i - m_i$ and therefore $|Df^k_{/E_1 \oplus E_2(f^{-\tilde{n}_i}(z))}| < \lambda_2^k$ for $0 < k < \tilde{n}_i$ which is a contradiction with the election of the sequences $\{m_i\}$.

Thus, by the dominated splitting, there is $\lambda_3 = \frac{\lambda}{\lambda_1} < 1$ such that

$$|Df^{-j}_{/E_3(f^{m_i}(z))}| \le \frac{\lambda}{\lambda_1}^j = \lambda_3^j \text{ for } N \le j \le m_{i+1} - m_i.$$

So, for any $y \in f^{-j}(f^{-m_i}(J(z)))$ we have that

$$|Df^{-j}_{/E_3(y)}| \le \lambda_3^j \text{ for } 0 \le j \le m_{i+1} - N.$$

Hence

$$\begin{aligned}
\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) &\le \sum_{j=m_i}^{N} \ell(f^{-j}(J_\psi(z))) + \sum_{j=N}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \\
&\le NK_2 \ell(f^{-m_i}(J_\psi(z))) + \sum_{j=0}^{m_{i+1}-N} K_2 \ell(f^{-m_i}(J_\psi(z)))\lambda_3^j \\
&\le \left(NK_2 + K_2 \frac{1}{1-\lambda_3}\right) \ell(f^{-m_i}(J_\psi(z))).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j \ge 0} \ell(f^{-j}(J_\psi(z))) &= \sum_i \sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \\
&\le \left(NK_2 + K_2 \frac{1}{1-\lambda_3}\right) \sum_i \ell(f^{-m_i}(J_\psi(z))) \\
&\le \left(NK_2 + K_2 \frac{1}{1-\lambda_3}\right) K_1 = K_3.
\end{aligned}$$

Finally, if the sequence $m_i's$ does not exist, the same argument shows that

$$\begin{aligned}\sum_{j\geq 0}\ell(f^{-j}(J_\psi(z))) &\leq \left(NK_2+K_2\frac{1}{1-\lambda_3}\right)\ell(J_\psi(z))\\ &\leq \left(NK_2+K_2\frac{1}{1-\lambda_3}\right)L=K_4\end{aligned}$$

where $L=\sup\{\ell(J_\psi(z)) : z\in R\cap\Lambda\}$. Taking $K=\max\{K_3,K_4\}$ we conclude the proof.

■

7.2.4 Finishing the proof of proposition 7.3.

We shall finish the proof of the proposition 7.3 in two cases: one, when Λ is not a topological minimal set, and the other when it is. Remember that a compact invariant set is topological minimal if it has no properly compact invariant subset, or equivalently, if any orbit is dense.

Case: Λ is not a minimal set

Lemma 7.2.7 *Let R be an adapted rectangle such that $\#\mathcal{R}(R,\Lambda)=\infty$. Then there exists a return $\psi_0\in\mathcal{R}(R,\Lambda)$ such that the adapted rectangle*
$R_{\psi_0}=Image(\psi_0)$ satisfies the conditions of lemma 7.2, i.e., for every $\psi\in\mathcal{R}(R_{\psi_0},\Lambda)$, $|\psi'|<\frac{1}{2}$ holds.

The central idea of the proof of the present lemma is that there are infinitely many returns, we can get one, namely ψ_0 such that $\frac{J_{\psi_0}(z)}{J_R(z)}$ is small, so $|\psi_0'|$ is small and then it is showed that for any ψ such that $\psi\in\mathcal{R}(R_{\psi_0},\Lambda)$, follows that $|\psi'|<\frac{1}{2}$. The proof of the this lemma is similar to the proof of lemma 3.7.4 given in [PS1] (page 1016) and we give it here for completeness and to show how the lemma 7.2.6 and the corollary 7.1 are used.

Let R be box as in the hypothesis of the lemma, and let K_0,K_1,K,C_1 be as in lemmas 7.1.2, 7.1.3, 7.2.6 and corollary 7.1 respectively. Consider also $L=\min\{\ell(J(z)) : z\in B_R(J)\cap\Lambda\}$.

Let $r>0$ be such that

$$r\frac{C_1}{L}\exp(K_0K_1+K_0K)<\frac{1}{2}.$$

Since $\#\mathcal{R}(R,\Lambda)=\infty$, there exists $\psi_0\in\mathcal{R}(B_R(J),\Lambda)$ such that

$$\ell(f^j(J_{\psi_0}(z)))<r,\ \forall j\geq 0,\ \forall z\in B_{\psi_0}\cap\Lambda.$$

Let k_0 be such that $\psi_0=f^{-k_0}_{/S_0}$, where $S_0=dom(\psi_0)$.

Let us prove that the box B_{ψ_0} satisfies the thesis of the lemma. Observe that if $z\in S_0\cap\Lambda$, then for $y\in J(z)$

$$|Df^{-k_0}_{/\tilde{E}_3(y)}|\leq\frac{\ell(f^{-k_0}(J(z)))}{J(z)}\exp(K_0K_1).$$

Let now $\psi\in\mathcal{R}(R_{\psi_0},\Lambda), \psi=f^{-k}_{/S_\psi}, S_\psi=dom(\psi)$.Setting $n_0=k-k_0,\ (k\geq k_0)$ we have $f^{-n_0}(S_\psi)\subset S_0$. Then, for $y\in J_{\psi_0}(z)$, $z\in dom(\psi)$,

$$
\begin{aligned}
|\psi'(y)| &= |Df^{-k}_{/\tilde{E}_3(y)}| \leq |Df^{-k_0}_{/\tilde{E}_2(f^{-n_0}(y))}||Df^{-n_0}_{/\tilde{E}_3(y)}| \\
&\leq \frac{\ell(f^{-k_0}(J(f^{-n_0}(z))))}{\ell(J(f^{-n_0}(z)))} \exp(K_0K_1) \frac{\ell(f^{-n_0}(J_{\psi_0}(z)))}{\ell(J_{\psi_0}(z))} \exp(K_0K) \\
&= \ell(f^{-n_0}(J_{\psi_0}(z))) \frac{\ell(J_{\psi_0}(f^k(z)))}{\ell(J_{\psi_0}(z))} \frac{1}{\ell(J(f^{-n_0}(z)))} \exp(K_0K_1 + K_0K) \\
&\leq rC_1 \frac{1}{L} \exp(K_0K_1 + K_0K) < \frac{1}{2}.
\end{aligned}
$$

So, the proof is finished.

Case: Λ is a minimal set.

The proof of the this lemma is similar to the proof for the minimal case proved in [PS1] (page 10018). However, we give some overview details. We begin remarking that we cannot expect to do the same argument here as in the preceding case, due to the fact that if Λ is a minimal set, then the set of returns to R is always finite. Nevertheless we shall exploit the fact that in the case Λ is a minimal set, then there are unstable "boundary points". First, we introduce some notations. Given an unstable arc J, we order J in some way and we denote $J^+ = \{y \in J : y > x\}$, $J^- = \{y \in J : y < x\}$. Also, giving $x \in R$ we shall denote by R^+ (say the upper part of the box) the connected component of $R - W^s_{\epsilon(x)}$ which contains J^+, and by R^- (the bottom one) the one containing J^-.

Lemma 7.2.8 *Assume Λ is minimal set. Then, reducing R in the unstable subbundle such that $R^+ \cap \Lambda = \emptyset$ or $R^- \cap \Lambda = \emptyset$.*

The idea to show that is that if the lemma does not follows, we would get that there is a periodic point in Λ which is a contradiction since Λ is minimal. See the proof of lemma 3.7.5 in [PS1] (page 1018).

■

Related to this rectangle we will get the following lemma that will imply the Main Lemma when Λ is minimal:

Lemma 7.2.9 *Let R be an adapted rectangle such that $R^+ \cap \Lambda = \emptyset$. Then there exist K such that for every $y \in R \cap \Lambda$,*

$$\sum_{j \geq 0} \ell(f^{-j}(J^+(y))) < K.$$

In particular there exist $J_1(y), J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty.$$

The proof of this lemma, use the following one:

Lemma 7.2.10 *Assume that* Λ *is a minimal set and let* R *be an adapted rectangle such that* $R^+ \cap \Lambda = \emptyset$. *Then* R^+ *verifies that for all* $y \in R \cap \Lambda$,

$$f^{-n}(J^+(y)) \cap R^+ = \emptyset \text{ or } f^{-n}(J^+(y)) \subset R^+$$

where $J^+(y) = J(y) \cap R^+$. *Moreover, there exist* K_1 *such that if* $y \in R \cap \Lambda$ *and* $f^{-j}(J^+(y)) \cap R^+ = \emptyset, 1 \leq j < n$ *then*

$$\sum_{j=0}^{n} \ell(f^{-j}(J^+(y))) < K_1.$$

Again, the proof are similar to the equivalent lemmas proved in [PS1] See the lemma 3.7.7 for the first and lemma 3.7.6 for the second one in [PS1] (page 1019).

Now we can prove the proposition 7.3 when Λ is a minimal set. Take

$$R_0 = \bigcup_{y \in B \cap \Lambda} J_1(y).$$

Notice that R_0 is an open set of Λ, and for every $y \in R_0 \cap \Lambda$ (i.e. $y \in J_1(y)$), we have

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty$$

and so

$$|Df^{-n}_{/E_3(y)}| \to_{n \to \infty} 0.$$

Let z be any point in Λ. Since Λ is a minimal set there exist $m_0 = m_0(z)$ such that $f^{-m_0}(z) \in R_0$ and so

$$|Df^{-n}_{/E_3(f^{-m_0}(z))}| \to_{n \to \infty} 0$$

implying that

$$|Df^{-n}_{/E_3(z)}| \to_{n \to \infty} 0.$$

This completes the proof of the proposition 7.3.

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