From hyperbolicity to dominated splitting

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1 Introduction

In the theory of differentiable dynamics, one may say that a fundamental problem is to describe the dynamics of "open sets" in the space of all dynamical systems, hoping that these systems can be interesting either in terms of their own rich mathematical structure or in terms of the meaning to other areas of sciences.

In this sense, the theory of the hyperbolic dynamics has been extremely successful: hyperbolicity is the cornerstone of uniform and robust chaotic dynamics; it characterizes the structural stable systems; it provides the structure underlying the presence of homoclinic points; a large category of rich dynamics are hyperbolic (geodesic flows in negative curvature, billiards with negative curvature, linear automorphisms, mechanical systems, etc.); the hyperbolic theory has been fruitful in developing a geometrical approach to dynamical systems; and, under the assumption of hyperbolicity one obtains a satisfactory (complete) description of the dynamics of the system from a topological and statistical point of view. Moreover, hyperbolicity has provided paradigms or models of behavior than can be expected to be obtained in specific problems.

A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$||Df^n_{/E^s(x)}|| \leq C\lambda^n, \quad \|Df^{-n}_{/E^u(x)}| \leq C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n. A hyperbolic diffeomorphism means a diffeomorphism such that its limit set (noted as $L(f)$) is hyperbolic (where the limit set is the closure of the forward and backward accumulation points of all orbits).

A diffeomorphism is called Axiom A if its the non-wandering set is hyperbolic and it is the closure of the periodic points. It is called an Anosov diffeomorphisms if f is hyperbolic on the whole manifold.

Nevertheless, hyperbolicity was soon realized to be a property less universal than it was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic and there are many dynamical phenomena coming from applications that cannot be described in the framework of hyperbolicity. To overcome these difficulties, the theory moved in different directions; one being to develop weaker or relaxed forms of hyperbolicity, hoping to include a larger class of dynamics.

There is an easy way to relax hyperbolicity, called partial hyperbolicity, which allows the tangent bundle to split into Df-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, such that the behavior of vectors in E^s, E^u is similar to the hyperbolic case, but vectors in E^c may be neutral for the action of the tangent map. This notion arose in a natural way in the context of time one maps of Anosov flows, frame flows or group extensions. See [Sh], [M2], [BD], [BV] for examples of these systems and [HP], [PS3] for an overview.

There is also another category which includes the partially hyperbolic system: *dominated splitting*. An f-invariant set Λ is said to have dominated splitting if we can decompose its tangent bundle in two invariant subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\|\|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

with $C > 0$ and $0 < \lambda < 1$.

Of course, it is assumed that neither of the subbbundless is trivial (otherwise, the other one has a uniform hyperbolic behavior: contracting or expanding). Also observe that any hyperbolic splitting is a dominated one.

Let us explain briefly the meaning of the above definition: it says that, for n large, the "greatest expansion" of Df^n on E is less than the "greatest contraction" of Df^n on F and by a factor that becomes exponentially small with n. In other words, every direction not belonging to E must converge exponentially fast under iteration of Df to the direction F. This notion was first introduced independently by Mañé, Liao, and Pliss, as a first step in the attempt to prove that structurally stable systems satisfy a hyperbolic condition on the tangent map. Simple examples of invariant sets exhibiting dominated splitting which are not hyperbolic splitting are normally hyperbolic closed invariant curves with dynamics conjugate to irrational rotations and homoclinic classes associated to non-hyperbolic fixed points. Later we shall expose more elaborate examples.

In this survey, we show a series of dynamical contexts where the notion of dominated splitting appears naturally. In particular, we show how it extends the notion of hyperbolicity and explains robust dynamical phenomena that are not hyperbolic. On the other hand, we expose how the dominated splitting helps us to find hyperbolic structures: in fact, we show that in certain cases, the dominated splitting structure arises naturally and from there hyperbolicity is concluded. To attain this last kind of result, we expose a series of theorems that explain the dynamical consequences that follow from a dominated splitting.

However, we have to make clear that the theory of dominated splitting is far from being successful as the theory of hyperbolicity. Moreover, it is not complete and many of the examples that are known could be considered extremely limited.

2 Hyperbolicity and stability. Robust transitivity and dominated splitting.

In loose terms, robustness means that some main feature of a dynamical system (an attractor, a given geometric configuration, or some form of recurrence, to name a few possibilities) is shared by all nearby systems. If we consider that the mathematical formulation of natural phenomena always involves simplifications of the physical laws, features specific to the model may have nothing to do with the actual phenomena. Therefore, real significance of a model may be accorded only to those properties that are robust, or even stable, under perturbations. The typical models showing robust properties are the well known Anosov maps (as examples of global hyperbolic dynamics) and the Smale's horseshoes (as examples of local hyperbolic ones).

In contrast to the instability of their orbits, the hyperbolic dynamics are stable in the sense that any perturbation of the system is conjugated to the initial one, meaning that the relevant dynamical behavior is actually the same, in some appropriate sense, again for all nearby systems. In particular, this shows that transitive hyperbolic systems (hyperbolic systems that have a dense trajectory) are in fact C^r-robust transitive ones; i.e: *any C^r small perturbation of the initial system remains transitive.* Moreover, hyperbolic dynamics have provided the first examples of robust dynamics showing chaotic or mixing properties.

As stated above, there are open sets in the space of dynamics which are nonhyperbolic. Indeed, in [Sh], an open set of non-hyperbolic transitive diffeomorphisms on $\mathbb{T}^4$ were exhibited (open sets of diffeomorphisms exhibiting hyperbolic periodic points of different indices inside a transitive set). This example motivated other different types of constructions of robustly transitive systems: skew products of a hyperbolic system with a non-hyperbolic one; derived from Anosov or bifurcation of Anosov maps; time one maps of an Anosov flow. These examples were a serious blow at the time, since they meant that even such seemingly simple situations, exhibiting a unique dynamical piece, cannot be understood within the framework of hyperbolicity. In the late 90s, examples of robust transitive dynamics without any hyperbolic subbundles were shown. More precisely, it was shown that *there are robustly transitive diffeomorphisms in the 4-torus which have no expanding or contracting invariant sub-bundle; therefore, they are neither hyperbolic nor partially hyperbolic; they only have a truly dominated splitting.* In fact, those examples exhibit a splitting $TM = E^{cs} \oplus E^{cu}$ which is dominated splitting, where E^{cs} and E^{cu} are indecomposable and nonhyperbolic (see [BV]).

The construction follows closely the example by Mañé and it can be summarized in the following way.

Consider a linear Anosov map A of the torus $\mathbb{T}^4$ having four real eigenvalues, $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4$. Then fix $A-$invariant cone fields C^{cu} corresponding to the expanding eigenvalues λ_3 and λ_4 and a cone field C^{cs} around the contracting eigenspaces. Now, take two small boxes C_1 and C_2, and consider a diffeomorphism f coinciding with A outside the boxes C_1, and C_2 and verifying the following:

1. f contains in C_1 a fixed point p (of index 2) with a contractive complex eigenvalue with eigenspace inside C^{cs} and a fixed point q, of index 1 with eigenspace also in C^{cs}. This implies that C^{cs} does not contain a hyperbolic stable direction.

2. f in C_2 contains a fixed point p_2 (of index 2) with an expanding complex eigenvalue and having a fixed point q_2 of index 1. These properties prevent the existence of a hyperbolic unstable subbundle.

Therefore, it follows that the hyperbolic splitting $E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$ of A was deformed into a dominated splitting $E \oplus F$ such that $dim(E) = 2$.

To assure that f is robust transitive it is required that Df (Df^{-1}) preserves the cone field C^{cu} (C^{cs}, respectively) and uniformly expands the area in this cone field; the restriction of f to the complement of C_1 uniformly expands the vectors in C^{cu} and the restriction of f^{-1} to the complement of C_2 uniformly expands the vectors in C^{cs}

Arguing as in [M], using the uniform expansion of the area in C^{cu}, it is shown that every center unstable disk D contains a point whose forward orbit remains in the complement of C_1 and this allows one to show that, for every large $n > 0$, $f^n(D)$ contains a center-unstable disk of radius bigger than L (where L is the maximum of the radius of the boxes C_1 and C_2). The same argument shows that the large negative iterates of any center-stable disk D contain a center stable disk of radius bigger that L and this implies the transitivity of f.

This example causes us to propose the general principle: *robust dynamical phenomena reflect some robust structure of the tangent map.*

In fact, in [M2] for surface diffeomorphisms, in [DPU] for dimension three, and in [BDP] for any dimension it is shown that this is the main characteristic of C^1-robust transitivity (C^1-nearby systems are transitive). In fact, the following was proved:

Theorem 2.0.1. *Any C^1-robust transitive diffeomorphism exhibits a dominated splitting such that its extremal bundles are uniformly volume contracted or expanded.*

This last theorem has other formulations in terms of certain generic dichotomy and also in conservative terms: see [Ab], [B], [BoV], [BB], [BFP], [AM].

Besides, the situation of hyperbolicity for continuous-time systems was no better. Geometric models for the Lorenz equations had just been proposed [ABS], [GW] which showed, in particular, that robust attractors of 3-dimensional flows need not be hyperbolic either. Diverse results that characterize robustly transitive flows with an equilibrium point have been developed. We refer to [MPP], [AP], [AR], [BDU], [PS5] for further reading.

3 Dominated splittings versus homoclinic tangencies.

In all the above examples of open sets of non-hyperbolic dynamics the underlying manifolds must be of dimension at least three and so the case of surfaces is not covered. It was through the seminal works of Newhouse (see [N1], [N2], [N3]) that hyperbolicity was shown to be not dense in the space of C^r diffeomorphisms ($r \geqslant 2$) of compact surfaces. The underlying mechanism here was the presence of a *homoclinic tangency* leading nowadays to the so-called "Newhouse phenomena", i.e., residual subsets of diffeomorphisms displaying infinitely many periodic attractors (this conclusion implies the existence of an open set of non-hyperbolic surfaces maps).

To be precise, first we recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \; dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \; dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f.

Definition 3.0.1. *Let $f : M \to M$ be a diffeomorphism. We say that f exhibits a homoclinic tangency if there is a hyperbolic periodic point p of f such that the stable and unstable manifolds of p have a non-transverse intersection.*

After the works of Newhouse, many other results were obtained in the direction of understanding the dynamics induced by unfolding homoclinic tangencies, especially in the case of one-parameter families. Many fundamental dynamical prototypes were found in the context of this bifurcation, namely the so called cascade of bifurcations, the Hénon-like strange attractor ([BC], [MV]) and infinitely many coexisting ones [C]. All the previous results hold for diffeomorphisms of at least class C^2. However, in [BD2] and [DNP] the Newhouse's phenomenon was obtained for C^1-diffeomorphisms on three dimensional manifolds.

These results suggest that one seek a characterization of universal mechanisms that lead to robustly (meaning any perturbation of the initial system) nonhyperbolic behavior.

From the works of [N1] and [D1], two basic mechanisms were found to the obstruction of hyperbolicity, namely *heterodimensional cycles* and *homolicinic tangencies.* In the early 80's Palis conjectured that these are very common in the complement of the hyperbolic systems: *Every C^r diffeomorphism of a compact manifold M can be C^r approximated by one which is hyperbolic, or by one exhibiting a heterodimensional cycle, or by one exhibiting a homoclinic tangency.*

The presence of homoclinic tangencies has many analogies with the presence of critical points for one-dimensional endomorphisms. On one hand, homoclinic tangencies correspond in the one-dimensional setting to preperiodic critical points and it is known that their bifurcation leads to complex dynamics. On the other hand, Mañé (see [M1]) showed that for a regular, generic one-dimensional endomorphism, the absence of critical points is enough to guarantee hyperbolicity. This result raises the question about the dynamical properties of surface maps exhibiting no homoclinic tangencies. As a dominated splitting prevents the presence of tangencies, we could say that *domination plays for surface diffeomorphisms the role that the non critical behavior does for one-dimensional endomorphisms.*

The above conjecture was proven to be true [PS1] for the case of surfaces and the C^1 topology: *Systems that are C^1-far from tangencies (meaning systems that can not be C^1-approximated by another one that exhibits a homoclinic tangency) are hyperbolic.* This result is false in higher dimensions. In fact there are open sets of non-hyperbolic dynamics which are far from tangencies.

However, at least partial results have been obtained in the direction of understanding systems that remain far from tangencies: *The limit set of a system that is C^1-far from tangencies has a dominated splitting.*

The next theorem says that the lack (in a robust way) of homoclinic tangency guarantees the existence of a dominated splitting. It was originally proven for surface diffeomorphisms in [PS1] and extended to higher dimensions by L. Wen in [We].

Theorem 3.0.2. *Let $f : M \to M$ be a diffeomorphism which is C^1 far from tangencies. Then $\overline{Per_i(f)}$ (the closure of the set of hyperbolic periodic points of index i) has a dominated splitting of index i, where $i = 1, ..., dim(M) - 1$.*

It is important to remark that dominated splitting of index i can coexists with tangencies. In fact, examples like the one shown in section 2 are not approximated by diffeomorphisms exhibiting a tangency of index 2 although they can be approximated by one having tangencies associated to periodic points of index different from 2.

4 Dominated splittings and dynamical consequences for surfaces maps.

One of the main goals in dynamics is to understand how the dynamics of the tangent map Df controls or determines the underlying dynamics of f. Actually, this paradigm is motivated after the success of the hyperbolic theory.

In fact, assuming that the limit set $L(f)$ (the minimum closed invariant set that contains the ω and α limit set of all orbits) splits into two subbundles, $T_{L(f)}M = E^s \oplus E^u$, invariant under Df and vectors in E^s are contracted by positive iteration of the tangent map (the same holding for E^u but under negative iteration), Smale [S] proved that $L(f)$ *can be decomposed into the disjoint union of finitely compact maximal invariant and transitive sets. Moreover, the periodic points are dense in $L(f)$ and the asymptotic behavior of any point in the manifold is represented by an orbit in $L(f)$.*

A natural question arises: is it possible to describe the dynamics of a system having dominated splitting? Moreover, observing that the examples of open sets of non-hyperbolic diffeomorphisms that have a dominated splitting exist in dimension larger than two, it is natural to ask if under the assumption of dominated splitting is it possible to conclude hyperbolicity in dimension two?

In fact, a similar spectral decomposition theorem as the one stated for hyperbolic dynamics holds for smooth surface diffeomorphisms exhibiting a dominated splitting. We can summarize these results by saying that *generically, invariant compact sets with dominated splitting of a smooth surface diffeomorphism are hyperbolic sets.*

Theorem 4.0.3. *([PS1]) Let $f \in \mathrm{Diff}^2(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting such that every periodic point in Λ is hyperbolic. Then $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ..., \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugate to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

One may ask whether a set having a dominated splitting is hyperbolic. Two necessary conditions follows trivially: all the periodic points in the set must be hyperbolic and no attracting (repelling) closed invariant (periodic) curve supporting an irrational rotation is in the set.

The next result is the analog of a one-dimensional theorem by Mañe (see [M1]). A similar description can be obtained for C^2 surface diffeomorphisms having dominated splitting over the limit set $L(f)$:

Theorem 4.0.4. *([PS3]) Let $f \in \mathrm{Diff}^2(M^2)$ and assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{\mathcal{L}}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is a set of periodic points with bounded periods and contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*
2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*
3. *$\tilde{\mathcal{L}}(f)$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets. The periodic points are dense in $\tilde{\mathcal{L}}(f)$ and at most finitely many of them are non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f/\tilde{\mathcal{L}}(f)$ is expansive.*

Roughly speaking, the above theorem says that the dynamics of a C^2 surface diffeomorphism having a dominated splitting can be decomposed into two parts: one where the dynamics consists of periodic and almost periodic motions ($\mathcal{I}$, $\mathcal{R}$) with the diffeomorphism acting equicontinuously; and another, where the dynamics are expansive and similar to the hyperbolic case.

Two immediate consequences follow from the previous theorem. First, any C^2 surface diffeomorphism with a dominated splitting over $L(f)$ which has a sequence of periodic points with unbounded periods must exhibit a nontrivial homoclinic class and hence:

Corolary 4.0.1. *The topological entropy of a C^2 diffeomorphism of a compact surface having dominated splitting over $L(f)$ and having a sequence of periodic points with unbounded periods is positive.*

Second, using Theorem 3.0.2 and the one above, it can be proved that:

Corolary 4.0.2. *Let $f \in \mathrm{Diff}^2(M^2)$ be C^1-far from tangencies. Then, f can be C^1-approximated by an Axiom A diffeomorphism.*

Moreover,

Corolary 4.0.3. *Let $f \in \mathrm{Diff}^2(M^2)$ have infinitely many sinks or sources with unbounded period. Then, f can be C^1-approximated by a diffeomorphism exhibiting a homoclinic tangency.*

5 Geodesic flows and hyperbolicity. Weyl manifolds and dominated splitting. Mechanical examples.

The study of hyperbolicity goes back to the work of Hadamard in 1898 concerning geodesic flows for surfaces with negative curvature, showing the density of closed geodesics and the instability of the flow with respect to initial conditions. In the 20s and 30s, Hedlund and Hopf showed that these flows are topologically mixing and that they are ergodic with respect to the Liouville measure. Later, in [A] it was shown that geodesic flows for compact manifolds with negative sectional curvature are hyperbolic (Anosov) flows. To be precise, we need to introduce the notion of hyperbolicity for flows: Given a compact invariant set $\Lambda \subset M$ of a flow Φ_t associated to a vector field X, one says that Λ *is a hyperbolic set of Φ_t if the tangent bundle of Λ splits into three invariant sub-bundles: $T_\Lambda M = E^s \oplus [X] \oplus E^u$, such that $[X]$ is the subbundle induced by the vector field, and there are two constants $\lambda < 0$ and $c > 0$ such that the following properties hold:*

1. *$||D\Phi_t|_{E^s}|| < c\exp(\lambda t)$ for $t > 0$; that is, E^s is uniformly contracted in the future.*
2. *$||D\Phi_{-t}|_{E^u}|| < c\exp(\lambda t)$ for $t > 0$; that is, E^u is uniformly contracted in the past.*

The results for geodesic flows have been extended to the case of Hamiltonian dynamics. A long standing question has been whether there is a physical example of a Hamiltonian system with Anosov energy levels – i.e., the Hamiltonian flow is Anosov on some energy level sets. A positive answer to this question was given in the remarkable paper of [HM] where the dynamics of a triple linkage is studied: three disks in a plane, free to rotate about pivots fixed in a triangle, but constrained by three rods connecting one point of each disk to a pivot x. For its free frictionless motion Hunt and Mackay proved existence of an open set of three linkage configurations for which the dynamics in each positive energy level set is a geodesic flow arising from negative curvature.

A natural generalization of a Riemann manifold is a Weyl structure. It is a torsion free connection whose parallel transport preserves a given conformal class of metrics. We follow the work and exposition of Maciej P. Wojtkowski [Wo] to describe the interplay between a Weyl manifold, a Weyl flow, a Gaussian thermostat, and a dominated splitting.

Fix a Riemannian metric $<,>$, let ∇ be the Levi-Civita connection, and let E be a vector field. Define the connection $\hat{\nabla}$ as

$$\hat{\nabla}_X Y = \nabla_X Y + < X, E > Y + < Y, E > X - < X, Y > E;$$

where X, Y denote arbitrary vector fields. The geodesics of the Weyl connection are given by the equations in TM

$$\frac{\partial q}{\partial s} = w, \ \frac{\hat{D} w}{\partial s} = 0$$

where $\frac{\hat{D}w}{\partial s}$ denotes the covariant derivative $\hat{\nabla}_w$. These equations provide geodesics with a distinguished parameter s, unique up to scale. The W-flow $\Phi_t : SM \to SM$ is obtained by parameterizing the geodesics of the Weyl connection with the arc length of g.

Example: ([Wo]) Let $\mathbb{T}^2$ be the flat torus with coordinates $(x; y) \in \mathbb{R}^2$ and $E = (a; 0)$ be the constant vector field on $\mathbb{T}^2$ then the equations of the W-flow given by $x'' = ay'$, $y'' = -ax'y'$ can be integrated and we obtain as trajectories translations of the curve $ax = \ln(\cos(ay))$ or the horizontal lines. Assuming that E has irrational direction on $\mathbb{T}^2$ one obtains the following global phase portrait for the W-flow. In the unit tangent bundle $S\mathbb{T}^2 = \mathbb{T}^3$ there are two invariant tori A and R with minimal quasiperiodic motions, A contains the unit vectors in the direction of E and it is a global attractor, while R contains the unit vectors opposite to E and is a global repeller. This example reveals a major departure from geodesic flows and Hamiltonian dynamics. W-flows may contract phase volume and they may have no absolutely continuous invariant measure.

The curvature tensor can be defined as $\hat{R}(X, Y) = \hat{\nabla}_X \hat{\nabla}_Y - \hat{\nabla}_Y \hat{\nabla}_X - \hat{\nabla}_{[X,Y]}$ and therefore the sectional curvature of a plane Π spanned by the vectors X and Y, $K(\Pi) =< \hat{R}(X, Y)Y, X >$ can be introduced. The next theorem is similar to the one obtained for geodesic flows on manifolds of negative curvature, replacing hyperbolicity by a dominated splitting:

Theorem 5.0.5. *If the sectional curvatures of the Weyl structure are negative everywhere in M then the W-flow has a dominated splitting $E \oplus F$ such that the flow shows exponential growth of volume in F and exponential decay of volume in E.*

In this paper, Wojtkowski conjectured that there are three-dimensional manifolds and vector fields such that the sectional curvatures of the corresponding Weyl structure are negative but the W-flow is not Anosov.

Such W-flows turn out to have a natural physical interpretation: they are identical to Gaussian thermostats, or isokinetic dynamics, introduced by Hoover in [H]. Isokinetic dynamics provides useful models in nonequilibrium statistical mechanics, discussed in the papers of Gallavotti and Ruelle, [G],[R], [GR].

6 Billiards and hyperbolicity. Pin-ball billiards and dominated splitting.

Billiards are mathematical models for many physical phenomena where one or more particles move in a container and collide with its walls and/or with each other. The dynamical properties of such models are determined by the shape of the walls of the container, and they may vary from completely regular (integrable) to fully chaotic. The most intriguing are chaotic billiards. This is the case of the dispersing billiard tables due to Ya. Sinai, introduced as a model of hard balls studied by L. Boltzmann in the XIX century and the Lorentz gas introduced to describe electricity in 1905. In his paper [Sin70] showed that *billiards with dispersive walls (billiards such that the walls have negative curvature) are prototypes of hyperbolic dynamics.* In contrast, billiards induced by polygonal tables are integrable and so they are non-hyperbolic.

To be precise, let B be an open bounded and connected subset of the plane whose boundary consists of a finite number of closed C^2-curves $\Gamma_i, i \geqslant 2$. The billiard in B is the dynamical system describing the free motion of a point mass inside B with elastic reflections at the boundary $\Gamma = \cup_i \Gamma_i$. Let $n(q)$ be the unit normal of the curve Γ at the point $q \in \Gamma$ pointing toward the interior of B. The phase space of such a dynamical system is given by

$$M = \{(q, v) : \quad q \in \Gamma, \quad |v| = 1, \quad \langle v, n(q) \rangle \ > \ 0\}.$$

In this space, the set of coordinates (s, φ) is introduced on M where s is the arc length parameter along Γ and φ is the angle between v and the normal vector $n(q)$ to the boundary at q. Clearly $-\pi/2 \leqslant \varphi \leqslant \pi/2$ and $\langle n(q), v \rangle \ = \ \cos\varphi$.

The *billiard map* T is defined by $T(q_0, v_0) = (q_1, v_1)$ where q_1 is the point of Γ hit first by the oriented line through (q_0, v_0) and v_1 is the velocity vector after the reflection at q_1. Formally,

$$v_1 = v_0 - 2\langle n(q_1), v_0 \rangle n(q_1).$$

The map T is piecewise C^1. Even if the billiard map T has discontinuities it is possible to define the notion of hyperbolicity and prove that *billiard tables with dispersing walls are hyperbolic.*

The dynamics of these types of billiards are prototypes of conservative dynamics: the Liouville measure is preserved. Therefore, these billiards are not useful to model rich phenomena that could hold in regimes far from the equilibrium. In this direction, moving towards overcoming these restrictions, in [CELS] one obtains several results about nonequilibrium states in the Lorentz gas, studying the dynamics of a system defined by a single particle traveling in a billiard table (bouncing off the scatterers with elastic collisions) and such that the particle is subjected to an electric field and a momentum-dependent frictional force between collisions with the scatterer. This unusual frictional force is chosen so that the total kinetic energy of the system is conserved although the dynamics do not preserve Liouville measure. The deep study in this system depends on the rather detailed knowledge that it has properties of hyperbolic type (e.g. existence of stable and unstable manifolds and rate of decay of correlations) for billiard systems.

Other types of nonconservative billiards are the *pinball billiards* which involve a table billiard with the property that when the ball touches one of the scatters, it suddenly reacts in a way that shoots or kicks the ball radially outward. After a number of collisions this system ends up like a "*particle accelerator*". The particle moves along straight lines inside the billiard table and when it hits one of the walls with angle α with respect to the normal, it is reflected with angle $\lambda\alpha$ with respect to normal line (with $\lambda < 1$): this follows from the fact that the ball is "kicked" by the wall giving a new impulse in the direction of the normal and thereby increasing its kinetic energy. The billiard map is defined in the same way,

$$v_1 = v_0 - (2-\alpha)\langle n(q_1), v_0\rangle n(q_1), \ \ \alpha > 0,$$

so contraction or dissipation in angle occurs while kinetic energy is increased. For this type of billiards one concludes the following:

Theorem 6.0.6. *([MPS])*
The billiard map associated to a billiard table with non-positive curvature (or non-focusing walls) has a dominated splitting.

In view of theorem 4.0.4 one also concludes:

Theorem 6.0.7. *([MPS]) Given a smooth billiard table with non-focusing walls, follows that the billiard map associated to it admits a spectral decomposition (as in theorem 4.0.4) on the closure of the set of trajectories that neither hit a corner of the table nor are tangent to the boundary of the table.*

There is an extreme case of the one that we considered before: the particle moves along straight lines inside the billiard table and it reflects at the boundary along the normal line. We call these billiards, *slap billiard maps* and they induce a one-dimensional map T defined on the union of a finite number of arcs of length $|\Gamma|$ whose derivative is $\frac{t_0 K_0+1}{-\cos\eta_1}$; here η_1 is the angle of incidence of the trajectory at q_1.

The following remarks and questions on the one-dimensional dynamics follow immediately:

- Critical points appear if $t_0 K_0 + 1 = 0$. In this case the boundary has negative curvature (focusing components): the criticalities are intimately related with the length of the normal lines inside the billiard table. For example, they do not appear in the elliptical billiards close to the circular one, since this number is close to 1.
- If the boundary contains only dispersing components (negative curvature) the slap billiard map is an expansive map with discontinuities (due to the corner and tangent points).
- Any polygonal billiards with an odd number of sides induces an expanding slap billiard one-dimensional map.

If one considers a small perturbation of the slap billiard map, i.e., after the reflection the trajectory follows not exactly along the perpendicular line, but inside a cone centered on it, it follows that we obtain attractors. Moreover, they are expanding attractors in the case that the slap billiard map is expanding.

7 Holomorphic dynamics and dominated splitting

For one-dimensional complex dynamics, the notion of hyperbolicity is usually replaced by the notion of expanding dynamics. In fact, a rational map on $\mathbb{C}$ is called dynamically hyperbolic if f is expanding on the Julia set (the boundary of the set of orbits that do not escape to infinity): there exists a conformal metric μ defined in a neighborhood of J such that the derivative $D_z f$ at a point z of this neighborhood satisfies

$$||D_z f(v)||_\mu > ||v||_\mu$$

for every nonzero vector in $T_z\mathbb{C}$. A classical theorem states that:

Theorem 7.0.8. *A rational map of degree $d \geqslant 2$ is dynamically hyperbolic if its poscritical closure $\bar{P}$ (the closure of the forward orbits of the critical points) is disjoint from its Julia set, or if and only if the orbit of every critical point converges to an attracting periodic point. In this case, the Julia set is connected.*

The simplest example is the map $B(z) = z^2$. A similar situation holds for an open sets of parameter of the quadratic family map $B_\mu(z) = z^2 + \mu$.

The quadratic family is naturally embedded as a two-dimensional diffeomorphism in the following classical way well known as the Henon family $H_{(a,b)}(z, w) = (a - bw - z^2, z)$, where the parameter b is the Jacobian determinant of $H_{(a,b)}$. This family has appeared often both in physics and mathematics literature and in general is extensively studied with real variables, i.e., $(z, w) \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$. Virtually all interesting dynamical behavior which is known to occur for two dimensional diffeomorphisms is known to occur in this family.

Because the Hénon family is also given by polynomial equations it also has a natural complex extension, which has been extremely successful in the quadratic family. In the case of the Hénon family and the complex dynamics in several variables, this approach has been extremely well developed in the works of Hubbard, Sibony, Fornaess, Bedford, Smillie and Lyubich.

The one-dimensional notion of the Julia set is naturally extended to two-dimensional complex polynomials as follows. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial diffeomorphism with dynamical degree $d > 1$. Let K^+ be the set of points with bounded forward orbits and let K^- be the set of points with bounded backward orbits. The sets $J^{\pm}$ are defined as the boundaries of $K^{\pm}$ and the Julia set J is defined to be $J^+ \cap J^-$. In dimension two all chaotic recurrent behavior is contained in J. Thus J seems to be a good analog of the one-dimensional Julia set. For instance, if the parameters a and b are close to zero, then the two-dimensional Julia set is a hyperbolic solenoid. In particular, it is natural to ask if there are results analogous to theorem 7.0.8 about the hyperbolicity and connectivity of the Julia sets in the two-dimensional setting. In a series of works of Bedford and Smilie, many results were obtained about the connectivity of the Julia set and the locus of hyperbolicity. As we will see later, some results for the case of two-dimensional Blaschke products suggest this possibility.

Coming back to the one-dimensional setting, a typical example of systems that are satisfy the conditions of theorem 7.0.8 are the Blaschke products. They are interesting in their own right but also from the point of view of more general complex dynamics. For example, any meromorphic map such that its Julia set bounds an invariant simple connected neighborhood in $\mathbb{C}$ is conjugate to a Blaschke product which is expanding in the unit circle. A (finite) Blaschke product is a map of the form

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

where $n \geqslant 2$, $a_i \in \mathbb{C}, |a_i| < 1$, $i = 1 \dots n$ and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. B is a rational mapping of $\mathbb{C}$, it is an analytic function in a neighborhood of the unit disc $\mathbb{D}$, and B maps the unit circle $\mathbb{T}$ to itself.

In view of this remark, it is natural to study two-dimensional Blaschke products of two forms:

$$F(z, w) = (A(z)B(w), C(z)D(w)) \tag{1}$$

$$F(z, w) = (\frac{A(z)}{B(w)}, \frac{D(w)}{C(z)}) \tag{2}$$

considered as dynamical systems on $\mathbb{T} \times \mathbb{T}$ where A, B, C, D are one-dimensional Blaschke products and we allow the possibility that some of the degrees of A, B, C, D may be 1.

A typical example of the rich dynamics associated to a two-dimensional Blaschke product is an Anosov map in complex variables. Start with a matrix $N \in SL(\mathbb{Z}, 2)$ and its inverse

$$N = \begin{bmatrix} n & m \\ k & j \end{bmatrix}, \quad N^{-1} = \begin{bmatrix} j & -m \\ -k & n \end{bmatrix}$$

with n, m, k, j positive integers. Let $F_N(z, w) = (z^n w^m, z^k w^j)$ and observe that F_N is a Blaschke product diffeomorphism, while $F_{N^{-1}}(z, w) = (z^j w^{-m}, z^{-k} w^n)$ is a quotient Blaschke product diffeomorphism. It follows that $\hat{F}_N(x, y) = N(x, y) = (nx+my, kx+jy)$, is the linear Anosov diffeomorphism induced by N on $\mathbb{T}^2$.

This is a very general phenomenon for two-dimensional Blaschke products that are diffeomorphisms of $\mathbb{T}^2$. If the periodic points of a Blaschke product are hyperbolic the diffeomorphism is hyperbolic and in some cases we can show that the Julia set is contained in $\mathbb{T}^2$. Moreover, in some cases, they are Anosov diffeomorphisms. These results could help get some insight into the dynamics of certain meromorphic maps on $\mathbb{C}^2$. To obtain these results, first it is shown that *Non trivial Blaschke products of* $\mathbb{C}^2$ *have a dominated splitting on* $\mathbb{T}^2$.

In fact, given $F(z, w) = (A(z)B(w), C(z)D(w))$, let a, b, c, d be the corresponding transformations acting on $\mathbb{R}$. Therefore

$$F(e^{2\pi i x}, e^{2\pi i y}) = (e^{2\pi i(a(x)+b(y))}, e^{2\pi i(c(x)+d(y))}).$$

So, given F we can take the map

$$\hat{F} : \mathbb{T}^2 \to \mathbb{T}^2,$$

$$\hat{F}(x, y) = (a(x) + b(y), c(x) + d(y)).$$

Observe that the positive cone field is preserved by $\hat{F}$ and if $\hat{F}$ is a diffeomorphism then *it has has a dominated splitting on all of* $\mathbb{T}^2$.

The same holds for the quotient Blaschke products introduced in (2): in this case the cone field bounded by the directions spanned by the vectors $(1, 0)$ and $(0, -1)$ is preserved by $T\hat{F}$ and if $\hat{F}$ is a diffeomorphism on $\mathbb{T}^2$ then $\hat{F}$ *has a dominated splitting on all of* $\mathbb{T}^2$.

In view of theorem 4.0.3 one concludes the following:

Theorem 7.0.9. *([PSh]) Let F be either a Blaschke product diffeomorphism or a quotient Blaschke product diffeomorphism such that all periodic points in $\mathbb{T}^2$ are hyperbolic. Then, $F_{|\mathbb{T}^2}$ is an Axiom A diffeomorphism. Moreover, one of the next options hold:*

1. *$F_{|\mathbb{T}^2}$ is Anosov and $L(F_{|\mathbb{T}^2}) = \mathbb{T}^2$,*
2. *$L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H} \cup \mathcal{S}\dashv \cup \mathcal{R}$,*

3. $L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H}$,

4. $L(F_{|\mathbb{T}^2}) = \mathcal{H} \cup \mathcal{S}\dashv \cup \mathcal{R}$;

Where $\mathcal{S}$ *is a set formed by a single attracting fixed point,* $\mathcal{R}$ *is a set formed by a finite number of repelling periodic points,* $\mathcal{S}\dashv$ *is a finite number of isolated saddles, and* $\mathcal{H}$ *is a non-trivial maximal transitive hyperbolic invariant set in* $\mathbb{T}^2$*. In the last case it follows that* $\mathcal{H}$ *is an attractor in* $\mathbb{T}^2$*. Moreover, the order relation is given by* $\mathcal{R} \to \mathcal{S}\dashv \to \mathcal{H} \to \mathcal{S}$ *(where* $A \to B$ *if* $W^u(A) \cap W^s(B) \neq \emptyset$*). In the case that* $\mathcal{S}$ *is empty,* $F|\mathbb{T}^2$ *has a unique SRB measure with positive entropy.*

F *always has an attracting or semi-attracting fixed point in* $\mathbb{D}^2$*, and any forward orbit in the interior of* $\mathbb{D}^2$ *converges to that fixed point. If this fixed point is in the interior of* $\mathbb{D}^2$ *it is attracting.*

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