Minimal convex functions bounded below by the duality product*

J.-E. Martínez-Legaz†‡ B. F. Svaiter§¶

1 Introduction

Convex analysis is involved in many branches of mathematics, from functional analysis to optimization. Maximal monotone operators appear naturally in convex analysis, but their study is in many parts performed outside convex analysis. This statement may undergo a drastic change. In a 1988 paper, Fitzpatrick [4] proved that any maximal monotone operator can be represented by convex functions in a special class. Moreover, he explicitly defined a convex representation of a given maximal monotone operator and proved this function to be minimal in the class. These results were recently rediscovered [6, 2] and since then Fitzpatrick's results have been the subject of intense research [9, 3, 12, 7, 10, 5, 11, 8, 1, 13] .

The aim of this paper is to provide a partial converse of one of Fitzpatrick's results. Namely, we will provide a partial characterization of the minimal elements of the family of convex functions bounded below by the duality product. In the special setting of a reflexive Banach space, we will show that the minimal elements are the Fitzpatrick functions associated to maximal monotone operators.

This paper is organized as follows. In Section 2 we introduce the notation and some basic definitions and recall some auxiliary results. In Section 3 the main theorem is proved and some open questions are discussed.

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†Department d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain. E-mail address: JuanEnrique.Martinez@uab.es

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§Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorinha 110, Jardim Botânico, Rio de Janeiro, CEP 22460-320, Brazil. E-mail address: benar@impa.br

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2 Basic definitions and auxiliary results

Let X be a real Banach space and X^* be its dual. For $x \in X$, $x^* \in X^*$ we use the notation

$$\langle x, x^* \rangle = x^*(x).$$

We will identify a point-to-set operator $A : X \rightrightarrows X^*$ with its graph,

$$A \triangleq \{(x, x^*) \mid x^* \in A(x)\}.$$

The operator $A : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \qquad \forall (x, x^*), (y, y^*) \in A.$$

A monotone operator is *maximal monotone* if it is not properly contained in any other monotone operator.

The *conjugate* of $f : X \to \bar{R}$ is $f^* : X^* \to \bar{R}$,

$$f^*(x^*) := \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

If f is an (extended) real valued function defined on $X \times X^*$, then f^* is defined on $X^* \times X^{**}$. We will identify an element $x \in X$ with its image under the canonical injection of X into X^{**},

$$x : X^* \to R, \qquad x(x^*) = x^*(x) = \langle x, x^* \rangle.$$

We use this convention whenever we write $f^*(x^*, x)$ for $f : X \times X^* \to \bar{R}$, $x \in X$ and $x^* \in X^*$.

The Fitzpatrick function associated to an operator $A : X \rightrightarrows X^*$ is defined as

$$\begin{aligned} \varphi_A(x, x^*) = \quad & \sup_{(y,y^*) \in A} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \\ & \sup_{(y,y^*) \in A} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle. \end{aligned}$$

This function has some nice properties. It is convex and l.s.c., and if A is maximal monotone, then φ_A characterizes A:

Theorem 1 *[4, Cor. 3.9, Prop. 3.2 and Thm. 3.7] Let $A \subset X \times X^*$ be maximal monotone. Then, for any $(x, x^*) \in X \times X^*$,*

$$\begin{aligned} \varphi_A(x, x^*) &\geq \langle x, x^* \rangle, \\ \varphi_A(x, x^*) &= \langle x, x^* \rangle \iff (x, x^*) \in A \end{aligned} \tag{1}$$

and

$$\begin{aligned} (\varphi_A)^*(x^*, x) &\geq \langle x, x^* \rangle, \\ (\varphi_A)^*(x^*, x) &= \langle x, x^* \rangle \iff (x, x^*) \in A. \end{aligned}$$

Moreover, φ_A is minimal in the family of convex functions defined on $X \times X^*$ that are bounded below by the duality product.

Proof. The only part of the statement that cannot be found in [4] is the minimality of φ_A, though it easily follows from the main results of that paper. Indeed, suppose that $f : X \times X^* \to \overline{R}$ is convex, bounded below by the duality product and $f \leq \varphi_A$; then, by (1), $f(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in A$ and hence, in view of [4, Thm. 3.7], $\varphi_A \leq f$. ■

Let us denote by π the duality product,

$$\pi : X \times X^* \to R, \qquad \pi(x, x^*) = \langle x, x^* \rangle ,$$

and let $\mathcal{F}$ be the family of convex functions on $X \times X^*$ that are bounded below by the duality product,

$$\mathcal{F} = \left\{ f : X \times X^* \to \overline{R} \mid f \text{ is convex and proper, } f \geq \pi \right\} .$$

As we already mentioned, Fitzpatrick functions of maximal monotone operators are minimal in the family $\mathcal{F}$. It is natural to ask which are the minimal elements of this family. We will partially characterize the minimal elements of $\mathcal{F}$ in a generic Banach space, and for reflexive spaces we will show that these minimal elements are Fitzpatrick functions.

The following auxiliary result of Burachik and Svaiter [3] will be needed:

Theorem 2 *[3, Thm. 3.1] Let X be reflexive and $h : X \times X^* \to \bar{R}$ be a lower semicontinuous convex function. Define*

$$A := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

If

$$h(x, x^*) \geq \langle x, x^* \rangle , \quad h^*(x^*, x) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*$$

then A is maximal monotone and

$$A = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}.$$

3 Main results

The following lemma will be useful to prove the main result:

Lemma 3 *For any function $f : X \times X^* \to \overline{R}$ and any $(x, x^*), (y, y^*) \in X \times X^*$, $p, q \geq 0$, $p + q = 1$, it holds that*

$$p \max \{f(x, x^*), \langle x, x^* \rangle\} + q \max \{f^*(y^*, y), \langle y, y^* \rangle\} \geq \langle px + qy, px^* + qy^* \rangle .$$

Proof. By direct algebraic manipulation, using Fenchel inequality one gets

$$\begin{aligned}
\langle px + qy, px^* + qy^* \rangle &= p^2 \langle x, x^* \rangle + pq (\langle x, y^* \rangle + \langle y, x^* \rangle) + q^2 \langle y, y^* \rangle \\
&\leq p^2 \langle x, x^* \rangle + pq (f(x, x^*) + f^*(y^*, y)) + q^2 \langle y, y^* \rangle \\
&= p (p \langle x, x^* \rangle + q f(x, x^*)) + q (p f^*(y^*, y) + q \langle y, y^* \rangle) \\
&\leq p \max \{f(x, x^*), \langle x, x^* \rangle\} + q \max \{f^*(y^*, y), \langle y, y^* \rangle\} ,
\end{aligned}$$

which is the desired result. ■

Corollary 4 *For any $f \in \mathcal{F}$ and $(x, x^*) \in X \times X^*$ there exists $h \in \mathcal{F}$ such that*

$$f \geq h, \qquad \max \{f^*(x^*, x), \langle x, x^* \rangle\} \geq h(x, x^*).$$

Proof. Define

$$h = conv \min \left\{ f, \delta_{\{(x,x^*)\}} + \max \{f^*(x^*, x), \langle x, x^* \rangle\} \right\},$$

the largest convex minorant of f and $\delta_{\{(x,x^*)\}} + \max \{f^*(x^*, x), \langle x, x^* \rangle\}$, with δ denoting indicator function in the sense of convex analysis. Since these two functions are convex, one has

$$h(y, y^*) = \inf \{pf(z, z^*) + q \max \{f^*(x^*, x), \langle x, x^* \rangle\}\}, \qquad \forall (y, y^*) \in X \times X^*,$$

the infimum being over all $p, q \geq 0$ and $(z, z^*) \in X \times X^*$ such that

$$p + q = 1, \qquad p(z, z^*) + q(x^*, x) = (y, y^*).$$

Hence, by Lemma 3, $h \geq \pi$. From the definition of h it follows that it satisfies the required conditions. ■

Theorem 5 *For any $f \in \mathcal{F}$ there exists a minimal $g \in \mathcal{F}$ such that $f \geq g$. Moreover, any minimal element g of $\mathcal{F}$ is l.s.c., satisfies*

$$g^*(x^*, x) \geq g(x, x^*), \qquad \forall (x, x^*) \in X \times X^*, \tag{2}$$

and, if X is reflexive, then $g = \varphi_A$ for some maximal monotone $A : X \rightrightarrows X^*$.

Proof. The family $\mathcal{F}$ is (decreasingly) inductively ordered. Indeed, if $\{f_\alpha\}_{\alpha \in \Lambda}$ is a totally ordered family of functions of $\mathcal{F}$, defining

$$\hat{f} = \inf_{\alpha \in \Lambda} f_\alpha$$

we have $\hat{f}$ convex, $\hat{f} \geq \pi$ and $f_\alpha \geq \hat{f}$ for any $\alpha \in \Lambda$. Hence, using Zorn's Lemma we conclude that given any $f \in \mathcal{F}$ there exists a minimal $g \in \mathcal{F}$, $f \geq g$.

Now, let $g \in \mathcal{F}$ be a minimal element of $\mathcal{F}$ and let $\operatorname{cl} g$ be the lower semicontinuous closure of g. Since g is convex, $g \geq \pi$ and the duality product is continuous, it follows that the lower semicontinuous closure of g is also convex and greater than π, that is, $\operatorname{cl} g \in \mathcal{F}$. Hence, g being minimal, it must be equal to its lower semicontinuous closure, i.e., g is lower semicontinuous. Take an arbitrary $(x, x^*) \in X \times X^*$. If $g^*(x^*, x) = \infty$ then, trivially, $g(x, x^*) \leq g^*(x^*, x)$. So, assume that $g^*(x^*, x) < \infty$. By Corollary 4, there exists $h \in \mathcal{F}$ such that $g \geq h$ and $h(x, x^*) \leq \max \{g^*(x^*, x), \langle x, x^* \rangle\}$. Since g is minimal, we actually have $h = g$ and hence

$$g(x, x^*) \leq \max \{g^*(x^*, x), \langle x, x^* \rangle\}.$$

In particular, $g(x, x^*)$ is also finite. Using Fenchel inequality, we obtain

$$\max \{g^*(x^*, x), \langle x, x^* \rangle\} \leq \max \left\{ g^*(x^*, x), \frac{1}{2}(g(x, x^*) + g^*(x^*, x)) \right\},$$

which, combined with the previous inequality, implies that $g(x, x^*) \leq g^*(x^*, x)$.

If X is reflexive, then, applying Theorem 2 we conclude that the set

$$A = \{(x, x^*) \in X \times X^* : g(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone. Hence $\varphi_A \in \mathcal{F}$ and, applying Theorem 1, we deduce that $g \geq \varphi_A$, which, as g is minimal, implies $g = \varphi_A$. ■

To present a variant of Theorem 5 we need the following auxiliary result.

Proposition 6 *For any $f \in \mathcal{F}$ such that*

$$f^*(x^*, x) \geq \langle x, x^* \rangle \qquad \forall\ (x, x^*) \in X \times X^*$$

there exists $h \in \mathcal{F}$ such that

$$\min\{f(x, x^*), f^*(x^*, x)\} \geq h(x, x^*) \qquad \forall\ (x, x^*) \in X \times X^*. \tag{3}$$

Proof. For $(x, x^*) \in X \times X^*$ define $h(x, x^*) = \inf\{pf(y, y^*) + qf^*(z^*, z)\}$, the infimum being over all $p, q \geq 0$ and $(y, y^*), (z, z^*) \in X \times X^*$ such that

$$p + q = 1, \qquad p(y, y^*) + q(z, z^*) = (x, x^*).$$

Clearly the function $h : X \times X^* \to \overline{R}$ so defined satisfies (3); in fact, since f and f^* are convex, it is the largest convex minorant of f satisfying this inequality. Moreover, by Lemma 3, $h \geq \pi$, so that $h \in \mathcal{F}$. ■

Since the duality product is continuous on $X \times X^*$, Proposition 6 still holds replacing $\mathcal{F}$ by $\mathcal{F}_0$

$$\mathcal{F}_0 = \{f \in \mathcal{F} : f \text{ l.s.c}\}.$$

Moreover, from Theorem 5 one concludes that the family $\mathcal{F}_0$ is also inductively ordered.

Since Fitzpatrick functions are not only lower semicontinuous in the strong topology of the space $X \times X^*$ but also in the $w \times w^*$ topology, that is, in the weak topology on $X \times X^*$ determined by the functionals in $X^* \times X$, next we will consider the family

$$\mathcal{F}_{0w} = \{f \in \mathcal{F} : f \text{ l.s.c. in the } w \times w^* \text{ topology}\}.$$

If X is reflexive, the $w \times w^*$ topology of $X \times X^*$ coincides with the weak topology. In this case, for convex functions, lower semicontinuity (in the strong topology) is equivalent to lower semicontinuity in the $w \times w^*$ topology and then $\mathcal{F}_{0w} = \mathcal{F}_0$.

If X is non-reflexive, the preceding reasoning fails and the inclusion

$$\mathcal{F}_{0w} \subset \mathcal{F}_0$$

may be strict. Moreover, in infinite dimensional Banach spaces the duality product is not continuous in the $w \times w^*$ topology of $X \times X^*$. Even though, Fitzpatrick functions of maximal monotone operators belong to $\mathcal{F}_{0w}$. This discussion raises some questions for the case where X is non-reflexive:

1) Is the family $\mathcal{F}_{0w}$ inductively ordered?

2) Does Proposition 6 hold replacing $\mathcal{F}$ by $\mathcal{F}_{0w}$?

Finally, a full characterization of the minimal elements of $\mathcal{F}$ and $\mathcal{F}_{0w}$ is missing in the non-reflexive case.

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