Killing graphs with prescribed mean curvature

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Abstract

It is proved the existence and uniqueness of Killing graphs with prescribed mean curvature in a large class of Riemannian manifolds.

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1 Introduction

A basic strategy to obtain hypersurfaces with prescribed mean curvature in Euclidean space is to describe them non-parametrically as solutions of a Dirichlet problem for a certain quasilinear elliptic PDE. The solutions are then graphs over domains in totally geodesic hypersurfaces of the ambient space. Classical references on the subject are [7], [9] and [16]. Existence results were obtained also for curved space forms in several formulations as one may consult [4], [8], [11], [12], [13] and [14]. Recently, the cases of Riemannian and warped products also deserved research efforts. For instance, see [1], [6] and [17].

The mere possibility of applying similar analytical and geometrical tools in all contexts seen above indicates that one must search for general existence results in a large class of Riemannian spaces. The presence of an isometric or conformal Killing vector field playing a major role is

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certainly the distinguishing feature for all the particular ambient spaces aforementioned.

In this paper, we deal with Riemannian spaces endowed with a Killing vector field and define a notion of Killing graph in these general ambients. We are then able to solve the corresponding Dirichlet problem for prescribed mean curvature under hypothesis involving domain data and the Ricci curvature. Since the ambient metric is indeed warped in a sense we made precise later, the present article should be considered as an extension and generalization of results proved in [6]. In that sense, it is worth to mention that key arguments in [6] do not longer work in the more general framework considered here.

The Ricci curvature naturally arises in all *apriori* estimates we made since they are based on comparison of geometric data. By its turn, geometric comparison results follow from the classical Jacobi and Ricatti equations. The last one is used in Section 3 for controlling the extrinsic geometry of the barriers.

We now explain more precisely the framework we are considering. Let M be a $(n+1)-$dimensional Riemannian manifold endowed with a non-singular Killing vector field Y. We assume that the distribution orthogonal to Y is integrable. Then, the leaves are easily seen to be totally geodesic hypersurfaces. Let $\mathbb{P}$ be a fixed integral leaf and assume that the flow lines of the flux $\Psi : \mathbb{R} \times \mathbb{P} \to M$ generated by Y are complete. Given a bounded domain Ω in $\mathbb{P}$, the Killing graph Σ associated to a function u on $\bar{\Omega}$ is the hypersurface

$$\Sigma = \{q = \Psi(u(p), p) : p \in \bar{\Omega}\}.$$

Our results assure the existence of Killing graphs with prescribed mean curvature H and boundary data ϕ. Here, the functions H and ϕ are defined respectively on $\bar{\Omega}$ and Γ where $\Gamma = \partial\Omega$. The problem of existence of such Killing graphs is formulated in terms of a Dirichlet problem for a divergence form elliptic PDE. The barriers we used for estimating height and gradient of u are the Killing cylinders. The Killing cylinder K over Γ is ruled by the flow lines of Y through Γ. Therefore, we have

$$K = \{q = \Psi(s, p) : p \in \Gamma\}.$$

In the sequel, the mean curvature of K pointing inward is denoted by H_{cyl}.

We are now in condition to state the theorems proved in this paper. We refer to Section 2 for the convention we used for the Ricci tensor.

Theorem 1. *Let $\Omega \subset \mathbb{P}$ be a bounded domain with $C^{2,\alpha}$ boundary Γ. Suppose that $H_{\rm cyl} \geq 0$ and that*

$$\inf_M \mathrm{Ric}_M \geq -n \inf_\Gamma H_{\rm cyl}^2.$$

Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\Gamma)$ be given. If $\sup_\Omega |H| \leq \inf_\Gamma H_{\rm cyl}$, then there exists a unique function $u \in C^{2,\alpha}(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose Killing graph has mean curvature H.

If the Ricci tensor does not satisfy the assumption given in the above result, then we may use comparison theorems with geodesic spheres as barriers in order to prove the following result. In this situation, we must impose certain condition either on the function H or in the size of the domain Ω.

Theorem 2. *Let $\Omega \subset \mathbb{P}$ be a bounded domain with $C^{2,\alpha}$ boundary Γ contained in a normal geodesic disk with radius r_0. Suppose that $\inf_M \mathrm{Ric}_M \geq -(n-1)k$ for some positive constant k. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\partial\Omega)$ be given. Assume that $H_{\rm cyl} \geq 0$, that $\sup_\Omega |H| \leq \inf_\Gamma H_{\rm cyl}$ and that*

$$r_0 \leq \frac{1}{\sqrt{k}} \coth^{-1} \frac{\sup_\Omega |H|}{\sqrt{k}}.$$

Then there exists a unique function $u \in C^{2,\alpha}(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose Killing graph has mean curvature H.

If the metric induced on $\mathbb{P}$ is rotationally symmetric then we may take certain constant mean curvature spheres as barriers. These spheres are rotationally invariant hypersurfaces and their qualitative aspect is described by a flux formula. In what follows, we denote $r = \mathrm{dist}(p_0, \cdot)$ for some point $p_0 \in \mathbb{P}$.

Theorem 3. *Suppose that the induced metric in $\mathbb{P}$ is of the form*

$$\mathrm{d}r^2 + \xi^2(r)\mathrm{d}\theta^2$$

for a given function ξ, where $\mathrm{d}\theta^2$ is the canonical metric on the unit sphere $\mathbb{S}^{n-1}$. Let $\Omega \subset \mathbb{P}$ be a bounded domain with $C^{2,\alpha}$ boundary Γ contained in a normal geodesic disk with radius r_0. Let $H \in C^\alpha(\Omega)$ and $\phi \in C^{2,\alpha}(\partial\Omega)$ be given. Assume that $H_{\rm cyl} \geq 0$, that $\sup_\Omega |H| \leq \inf_\Gamma H_{\rm cyl}$ and that

$$\sup_\Omega |H| \leq -\frac{\varrho(r_0)\xi^{n-1}(r_0)}{\int_0^{r_0} \varrho(r)\xi^{n-1}(r)\mathrm{d}r}$$

where $\varrho = |Y|^2$. Then there exists a unique function $u \in C^{2,\alpha}(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose Killing graph has mean curvature H.

We emphasize that in the statements and proofs of the theorems we may replace M (after passing to the universal cover if necessary) by the solid cylinder $\Psi(\mathbb{R} \times \bar{\Omega})$ whose boundary is K. Moreover, the hypothesis in Theorems 2 and 3 may be understood either as restrictions on the size of the domains for arbitrary mean curvature functions, or as upper bounds on $|H|$ in the case of arbitrarily large domains. Finally, we remark that standard regularity theorems imply that the results remain true for continuous boundary data.

If the assumption $\sup_\Omega |H| \le \inf_\Gamma H_{\rm cyl}$ fails at some point, we do not show here that the our results are no longer true in the sense that there is a boundary data ϕ for which no solution exists. Nevertheless, this is well known in some cases, namely, the Euclidean space for standard graphs and has also been proved for two types of Killing graphs in the hyperbolic space in [8] and [14].

This paper is organized as follows. In Section 2, we present the basic geometric structure of the ambient spaces we are dealing with, including calculations concerning the mean curvature of the Killing cylinders and of the Killing graphs. In Section 3, we deduce the height estimates. Sections 4 and 5 are devoted, respectively, to boundary and interior gradient estimates. The last section presents the proof of the theorems following the well-known continuity method. An appendix contains the sketched proof of the flux formula.

We point out that ambient spaces with a Killing vector field correspond in Lorentzian setting to the important notion of stationary space-times on which the metric tensor is time-independent. Prescribed mean curvature hypersurfaces work in this context as Cauchy hypersurfaces for the initial value formulation for the Cauchy problem in General Relativity. Thus, it seems interesting to investigate the Lorentzian analogues to our existence results.

The case of non-integrable orthogonal distributions corresponds to Riemannian submersions other than the simple ones associated to warped products. The particular situation of the three-dimensional Heisenberg Lie group with a left-invariant metric was treated in [2]. Two of the authors of the pressent paper address in [5] the question for Riemannian submersions with uni-dimensional totally geodesic fibers. The case of conformal vector fields was studied by one of the authors in joint work with F. Andrade in [3].

2 Killing graphs

Let M be a connected $(n+1)-$dimensional Riemannian manifold endowed with a non-singular Killing vector field Y. We denote the metric and the Riemannian connection in M by $\langle\cdot\,,\cdot\rangle$ and $\bar\nabla$, respectively. We assume that the flow lines of Y are complete and that the distribution

$$p\in M\mapsto\{v\in T_pM:\langle v,Y\rangle=0\}$$

is integrable. Then, it is easy to verify that the integral leaves are totally geodesic hypersurfaces.

Let $\mathbb{P}$ be such an integral leaf. The flux $\Psi\colon\ \mathbb{R}\times\mathbb{P}\to M$ generated by Y takes isometrically $\mathbb{P}$ to the leaves $\mathbb{P}_s=\Psi_s(\mathbb{P})$, where $\Psi_s=\Psi(s,\,\cdot\,)$. Given local coordinates $x^1,\ldots,x^n$ for $\mathbb{P}$, then $s,x^1,\ldots,x^n$ are local coordinates for M defined by

$$q\in M\mapsto(s,x^1,\ldots,x^n)\quad\text{ if }\quad q=\Psi(s,p),$$

where $p\in\mathbb{P}$ is the point with coordinates $x^1,\ldots,x^n$. The corresponding coordinate vector fields are

$$\partial_s=\frac{\mathrm{d}}{\mathrm{d}s}\,\Psi(s,p)=Y(\Psi(s,p))$$

and

$$\partial_i(q)=\frac{\partial}{\partial x^i}\Psi(s,p)=\Psi_{s_*}(p)\partial_i(p).$$

The ambient metric in terms of these coordinates has components

$$g_{00}=\langle\partial_s,\partial_s\rangle=\varrho^2,\quad g_{0i}=\langle\partial_s,\partial_i\rangle=0$$

and

$$g_{ij}=\langle\Psi_{s_*}\partial_i,\Psi_{s_*}\partial_j\rangle=\langle\partial_i,\partial_j\rangle=\sigma_{ij},$$

where σ_{ij} are the components of the metric in $\mathbb{P}$ in terms of the coordinates x^i. Observe that the components of the metric do not depend on s. The gradient of the function s is

$$\bar\nabla s=g^{00}\,\partial_s=|Y|^{-2}\,Y=:f\,Y.$$

Since the flow lines of Y have constant geodesic curvature and parallel curvature vector it is a standard fact that the solid cylinder $\Psi(\mathbb{R}\times\bar{\Omega})$ has a warped product structure as

$$\bar{\Omega}\times_{\varrho}\mathbb{R}$$

whose metric is

$$\sigma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j+\varrho^2\mathrm{d}s^2.$$

In fact, this is the setting considered in [6]. Notice that the relation between the Ricci curvatures of M and $\mathbb{P}$ is determined by $\varrho=|Y|$.

Given a bounded $C^{2,\alpha}$ domain Ω on $\mathbb{P}$ and a smooth function u on Ω, we define the associated *Killing graph* Σ by

$$\Sigma=\{q=\Psi(u(p),p):p\in\Omega\}.$$

We may think of Σ as the locus

$$\Phi(s,p)=s-u(p)=0,$$

where $u(p)=u(s,p)$. An orientation for Σ is given at $q=\Psi(u(p),p)$ by

$$\begin{aligned}\bar{\nabla}\Phi(q)&=g^{00}\,\partial_s-g^{ij}(p)\,u_i\,\partial_j(q)=f\partial_s-\sigma^{ij}(p)\,u_i\,\Psi_{u(p)*}\partial_j(p)\\&=f\,\partial_s-\Psi_{u(p)*}\nabla u(p),\end{aligned}$$

where ∇ denotes the Riemannian connection in $\mathbb{P}$ and

$$\nabla u=\sigma^{ij}u_i\partial_j=u^j\partial_j$$

is the gradient relatively to $\mathbb{P}$. Then

$$N=\frac{1}{W}\,\bar{\nabla}\Phi=\frac{1}{W}(f\,\partial_s-\Psi_*\nabla u)\qquad(1)$$

defines a unit normal vector field along Σ, where

$$W^2=f+|\nabla u|^2$$

and Ψ_* is a shorthand notation for $\Psi_{u(p)*}$.

2.1 Killing cylinder

The *Killing cylinder* over $\Gamma = \partial\Omega$ is the surface ruled by the flow lines of Y given by

$$K = \{\Psi(s,p) : s \in \mathbb{R},\ p \in \Gamma\}.$$

If $s^1, \ldots, s^{n-1}$ are local coordinates for Γ, then $s, s^1, \ldots, s^{n-1}$ are local coordinates for K. Let η be the unit inward normal vector along Γ as a submanifold of $\mathbb{P}$. We equally denote by η the unit normal vector field $\Psi_{s_*}\eta$ along K. Thus, we have

$$\langle \eta, \partial_s \rangle = 0 = \langle \eta, \partial_i \rangle.$$

Since η and ∂_i are tangent to the totally geodesic leaves $\mathbb{P}_s$, it results that

$$\langle \bar\nabla_{\partial_i} \partial_s, \eta \rangle = 0.$$

Hence ∂_s is a principal direction of K, and the corresponding principal curvature is the geodesic curvature

$$\kappa = f \langle \bar\nabla_{\partial_s} \partial_s, \eta \rangle$$

of the flow lines through Γ.

In the sequel, we deduce some useful properties of the distance function $d = \mathrm{dist}(\,\cdot\,, K)$ from K. We denote by Γ_ϵ and K_ϵ the level sets $d = \epsilon$ in $\mathbb{P}$ and M, respectively. Thus, these level sets are equidistant respectively from Γ and K. It is immediate that K_ϵ is the Killing cylinder over Γ_ϵ. Since Γ is assumed to be $C^{2,\alpha}$, the function d is also $C^{2,\alpha}$ at points of $\Psi(\mathbb{R} \times \Omega_\epsilon)$, where $\Omega_\epsilon \subset \Omega$ is a small tubular neighborhood of Γ. Thus, we may define Fermi coordinates on $\Psi(\mathbb{R} \times \Omega_\epsilon)$ as follows: for $q \in \Psi(\mathbb{R} \times \Omega_\epsilon)$ we associate coordinates s^i, d by $q = \exp_p d\,\eta$ when $p = p(s, s^1, \ldots, s^{n-1})$ in K. Then

$$|\bar\nabla d| = 1. \tag{2}$$

From this it follows that

$$d^i d_{i;j} = 0, \tag{3}$$

where $d^i = g^{ij} d_j$ as usual. We also have

$$\langle \bar\nabla_{\partial_d} \bar\nabla d, \partial_d \rangle = \frac{1}{2} \partial_d |\bar\nabla d|^2 = 0.$$

Therefore,

$$\begin{aligned}\Delta d|_{d=\epsilon} &= \langle \bar{\nabla}_{\partial_d} \bar{\nabla} d, \partial_d \rangle + f \langle \bar{\nabla}_{\partial_s} \bar{\nabla} d, \partial_s \rangle + \sigma^{ij} \langle \bar{\nabla}_{\partial_i} \bar{\nabla} d, \partial_j \rangle \\ &= f \langle \bar{\nabla}_{\partial_s} \bar{\nabla} d, \partial_s \rangle + \sigma^{ij} \langle \bar{\nabla}_{\partial_i} \bar{\nabla} d, \partial_j \rangle .\end{aligned}$$

However $\bar{\nabla} d|_{d=\epsilon} = \partial_d$ is the unit inward normal vector field η_ϵ to the equidistant cylinders K_ϵ. Denoting

$$\kappa_\epsilon = f \langle \bar{\nabla}_{\partial_s} \partial_s, \bar{\nabla} d \rangle ,$$

we have at points of K_ϵ that

$$\Delta d = -\kappa_\epsilon - \sigma^{ij} b_{ij}(\epsilon) = -\kappa_\epsilon - (n-1) h_\epsilon , \tag{4}$$

where $b_{ij}(\epsilon)$ and h_ϵ are the second fundamental form and the mean curvature, respectively, of the hypersurface Γ_ϵ relatively to the unit inward normal η_ϵ. It follows that

$$\Delta d|_{K_\epsilon} = -n H_{\text{cyl}}(\epsilon),$$

where $H_{\text{cyl}}(\epsilon)$ is the mean curvature of K_ϵ with respect to η_ϵ. Its Weingarten operator is denoted by A_ϵ. The mean curvature of Γ and K are denoted, respectively, by h and H_{cyl}.

Remark 1. All of the above calculations on the distance function remain valid if we replace Ω_ϵ by the larger subset Ω_0 in Ω consisting of the points which can be joined to Γ by a *unique* minimizing geodesic. It was shown in [10] that in this set the function d has the same regularity as Γ.

Throughout this paper, the ambient Ricci tensor in a given direction v is defined by

$$\text{Ric}_M(v) = \sum_{i=1}^{n} \langle \bar{R}(e_i, v)v, e_i \rangle ,$$

where $\bar{R}$ is the curvature tensor in M and $e_1, \ldots, e_n, v$ is an orthonormal basis. We follow [7] or [17] and use the fact referred to in Remark 1 on the distance function in Ω_0 for proving in terms of the notation we fixed above the following result.

Lemma 1. *Assume that the Ricci curvature satisfies* $\mathrm{Ric}_M \geq -n \inf_\Gamma H^2_{\mathrm{cyl}}$. *Let* $y_0 \in \Gamma$ *be the closest point to a given point* $x_0 \in \Gamma_\epsilon \subset \Omega_0$. *Then, we have*

$$H_{\mathrm{cyl}}(\epsilon)|_{x_0} \geq H_{\mathrm{cyl}}|_{y_0}.$$

Proof: We use local coordinates $s^0 = s, s^1, \ldots, s^{n-1}$ as defined above. At $d = \epsilon$ and since ∂_d is the unit speed of a geodesic, we have

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle A_\epsilon \partial_i, \partial_j\rangle &= \partial_d|_{d=\epsilon}\langle \bar\nabla_{\partial_i}\partial_d, \partial_j\rangle = \langle \bar\nabla_{\partial_d}\bar\nabla_{\partial_i}\partial_d, \partial_j\rangle + \langle \bar\nabla_{\partial_i}\partial_d, \bar\nabla_{\partial_d}\partial_j\rangle \\ &= -\langle \bar R(\partial_i, \partial_d)\partial_d, \partial_j\rangle + \langle A_\epsilon\partial_i, A_\epsilon\partial_j\rangle \end{aligned} \tag{5}$$

where $\bar R$ is the curvature tensor of M. On the other hand,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle A_\epsilon \partial_i, \partial_j\rangle &= \langle \bar\nabla_{\partial_d} A_\epsilon\partial_i, \partial_j\rangle + \langle A_\epsilon\partial_i, \bar\nabla_{\partial_d}\partial_j\rangle \\ &= \langle (\bar\nabla_{\partial_d} A_\epsilon)\partial_i, \partial_j\rangle + \langle A_\epsilon\partial_j, \bar\nabla_{\partial_d}\partial_i\rangle + \langle A_\epsilon\partial_i, \bar\nabla_{\partial_j}\partial_d\rangle \\ &= \langle A'_\epsilon\partial_i, \partial_j\rangle - 2\langle A_\epsilon\partial_i, A_\epsilon\partial_j\rangle. \end{aligned} \tag{6}$$

From (5) and (6) we obtain the well-known Ricatti equation

$$A'_\epsilon - A^2_\epsilon - \bar R_\epsilon = 0$$

where $\bar R_\epsilon = \langle R(\cdot, \partial_d)\partial_d, \cdot\rangle|_{d=\epsilon}$. Taking traces we obtain

$$n\frac{\mathrm{d}}{\mathrm{d}\epsilon}H_{\mathrm{cyl}}(\epsilon) = \partial_d \mathrm{tr} A_\epsilon = \mathrm{tr}\bar\nabla_{\partial_d}A_\epsilon = \mathrm{tr}\big(A^2_\epsilon + \bar R_\epsilon\big) \geq nH^2_{\mathrm{cyl}}(\epsilon) + \mathrm{Ric}_M(\partial_d).$$

From our hypothesis on Ric_M we have that $z = H_{\mathrm{cyl}}(d) - \inf_\Gamma H_{\mathrm{cyl}}$ satisfies

$$z'(d) \geq z^2(d) - z^2(0) = (z(d) + z(0))(z(d) - z(0)).$$

Thus $z'(d) \geq c(z(d) - z(0))$ in some interval $d \in [0, d_0 > 0]$ for a constant $c > 0$. It follows easily that $H_{\mathrm{cyl}}(\epsilon)$ does not decrease with increasing d. ∎

2.2 The mean curvature equation

In what follows, we assume that the mean curvature H of the Killing graph Σ is a function on $\bar\Omega$. Computing at $q = \Psi(u(p), p) \in \Sigma$, we have

$$nH(p) = -\mathrm{tr}_\Sigma \bar\nabla N = -\mathrm{div}_\Sigma N.$$

Hence, if $e_1, \dots, e_n$ is an orthonormal tangent frame at q in Σ, then

$$-nH = \sum_i \langle \bar\nabla_{e_i} N, e_i\rangle = \sum_i \langle \bar\nabla_{e_i} N, e_i\rangle + \langle \bar\nabla_N N, N\rangle = \mathrm{div}_M \frac{\bar\nabla \Phi}{W}(q).$$

Consider a normal coordinate frame $\partial_1, \dots, \partial_n$ with $\sigma_{ij} = \delta_{ij}$ at $p \in \mathbb{P}$. Then, we have an orthonormal frame $E_0 = f^{1/2}\partial_s$, $E_i(q) = \Psi_*\,\partial_i(p)$ at q. Using this frame and (1), the divergence in the formula above becomes

$$\mathrm{div}_M \frac{\bar\nabla\Phi}{W} = \langle \bar\nabla\big(\frac{f}{W}\big), \partial_s\rangle + \frac{f}{W}\mathrm{div}_M \partial_s - f\langle \bar\nabla_{\partial_s} \frac{\Psi_*\nabla u}{W}, \partial_s\rangle - \langle \bar\nabla_{E_i}\frac{\Psi_*\nabla u}{W}, E_i\rangle.$$

The Killing equation implies that $Y = \partial_s$ is divergence-free, and that f and W do not depend on s. Thus,

$$\begin{aligned}\mathrm{div}_M \frac{\bar\nabla\Phi}{W} &= -f\langle \bar\nabla_{\partial_s}\frac{\Psi_*\nabla u}{W}, \partial_s\rangle - \sum_i \langle \bar\nabla_{E_i}\frac{\Psi_*\nabla u}{W}, E_i\rangle \\ &= f\langle \frac{\Psi_*\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle - \sum_i \langle \bar\nabla_{\Psi_*\partial_i}\Psi_*\frac{\nabla u}{W}, \Psi_*\partial_i\rangle \\ &= f\langle \frac{\Psi_*\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle - \sum_i \langle \bar\nabla_{\partial_i}\frac{\nabla u}{W}, \partial_i\rangle.\end{aligned}$$

Since Ψ_* preserves the field ∂_s and f is constant along the flow lines, we have

$$f(q)\langle \frac{\Psi_*\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle(q) = f(p)\langle \frac{\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle(p).$$

From these calculations it results that

$$nH(p) = \sum_i \langle \bar\nabla_{\partial_i}\frac{\nabla u}{W}, \partial_i\rangle - f\langle \frac{\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle,$$

where the expressions on both sides are now evaluated at $p \in \mathbb{P}$. Since the ∂_i's are orthonormal at p and $\mathbb{P}$ is totally geodesic, we may write

$$\mathrm{div}_{\mathbb{P}}\big(\frac{\nabla u}{W}\big) - f\langle \frac{\nabla u}{W}, \bar\nabla_{\partial_s}\partial_s\rangle - nH = 0. \tag{7}$$

The Killing equation implies that the field $\bar\nabla_{\partial_s}\partial_s$ is tangent to the leaf $\mathbb{P}$.

On the other hand, it is easy to see that

$$\langle \bar\nabla f, \nabla u\rangle = 2f^2\langle \bar\nabla_{\partial_s}\partial_s, \nabla u\rangle.$$

Using this expression and after some manipulation, one proves that another way to write out (7) is

$$\frac{1}{W}\big(\sigma^{ij}-\frac{u^iu^j}{W^2}\big)u_{i;j}-\frac{1}{W^3}(f+W^2)\langle\nabla u,\bar{\nabla}_{\partial_s}\partial_s\rangle-nH=0 \tag{8}$$

where $u_{i;j}$ is the Hessian of u in terms if the coordinates x^i in $\mathbb{P}$. Denoting

$$a^{ij}(x,\nabla u)=\frac{1}{W}\big(\sigma^{ij}-\frac{u^iu^j}{W^2}\big) \tag{9}$$

and

$$b(x,\nabla u)=-\frac{1}{W^3}(f+W^2)\langle\nabla u,f\bar{\nabla}_{\partial_s}\partial_s\rangle, \tag{10}$$

the mean curvature equation becomes

$$\mathcal{Q}[u]=a^{ij}u_{i;j}+b-nH=0.$$

The matrix a^{ij} is positive-definite with eigenvalues

$$\lambda=\frac{f}{W^3}\quad\text{and}\quad\Lambda=\frac{1}{W}$$

with multiplicities 1 and $n-1$ corresponding to the directions parallel and orthogonal to ∇u, respectively. Notice that $\lambda\le\Lambda$ since $f\le W^2$ by definition.

Let ϕ be a $C^{2,\alpha}$ function on Γ. The Killing graph of ϕ is a codimension two submanifold of M. Thus, Σ is a Killing graph with prescribed mean curvature H and prescribed boundary given by the graph of ϕ if and only if u solves the Dirichlet problem

$$\mathcal{Q}[u]=0,\quad u|_\Gamma=\phi \tag{11}$$

for a quasilinear elliptic PDE. We may apply maximum and comparison principles to (11). Indeed, this follows from our hypothesis that the function H does not depend on u (see, e.g., [7], Ch. 10).

3 Height estimates

In this section, we obtain *apriori* C^0 estimates for solutions of the Dirichlet problem (11). We divided the exposition in two cases concerning different assumptions on the ambient Ricci curvature.

From now on, the distance function d is regarded as the distance from Γ on the totally geodesic hypersurface $\mathbb{P}$.

3.1 Killing cylinders as barriers

We assume that the ambient Ricci curvature satisfies

$$\mathrm{Ric}_M \geq -n \inf_\Gamma H_{\mathrm{cyl}}^2,$$

i.e., the hypothesis of Lemma 1. In this case, we construct barriers for u in (11) of the form

$$\varphi(x) = \sup_\Gamma \phi + h(d(x)) \tag{12}$$

where the real function h will be chosen later. Along Ω_0 we have

$$\varphi_i = h' d_i \quad \text{and} \quad \varphi_{i;j} = h'' d_i d_j + h' d_{i;j}. \tag{13}$$

As in (2) and (3) above we have $|\nabla d|^2 = d^i d_i = 1$ and $d^i d_{i;j} = 0$. Moreover, (4) now reads as

$$d^i_{\ ;i} = \sigma^{ij} d_{i;j} = -(n-1)h_\epsilon.$$

It is convenient to write $W^2 = f + h'^2$. Then (8) yields

$$\begin{aligned}
\mathfrak{Q}[\varphi] + nH &= \frac{1}{W}\Big(\varphi^i_{\ ;i} - \frac{\varphi^i\varphi^j\varphi_{i;j}}{W^2}\Big) - \frac{1}{W^3}(f+W^2)\langle f\bar\nabla_{\partial_s}\partial_s, \nabla\varphi\rangle \\
&= \frac{1}{W}\Big(-h'(n-1)h_\epsilon + h'' - \frac{h'^2 h''}{W^2}\Big) - \frac{h'}{W^3}(f+W^2)\langle f\bar\nabla_{\partial_s}\partial_s, \eta_\epsilon\rangle \\
&= \frac{f}{W^3}\big(h'' - h'\langle f\bar\nabla_{\partial_s}\partial_s, \eta_\epsilon\rangle\big) - \frac{h'}{W}\big((n-1)h_\epsilon + \langle f\bar\nabla_{\partial_s}\partial_s, \eta_\epsilon\rangle\big).
\end{aligned}$$

We choose for (12) the test function

$$h = \frac{e^{CA}}{C}\big(1 - e^{-Cd}\big)$$

where $A > \mathrm{diam}(\Omega)$ and C is a positive constant to be chosen later. Then,

$$h' = e^{C(A-d)} \quad \text{and} \quad h'' = -Ch'.$$

Since the mean curvature of the equidistant cylinder K_ϵ is given by

$$nH_{\mathrm{cyl}}(\epsilon) = (n-1)h_\epsilon + \kappa_\epsilon,$$

we get

$$\mathfrak{Q}[\varphi] + nH = -\frac{fh'}{W^3}(C+\kappa_\epsilon) - \frac{h'}{W}nH_{\mathrm{cyl}}(\epsilon).$$

Assuming $\sup_\Omega |H| \leq \inf_\Gamma H_{\text{cyl}}$ and using Lemma 1, we obtain

$$\mathcal{Q}[\varphi] + nH \leq -\frac{fh'}{W^3}(C + \kappa_\epsilon) - \frac{h'}{W}n|H|.$$

Observe that $f/W^2 \leq 1$. Moreover, as $C \to \infty$ we have that

$$\frac{h'}{W} = \frac{h'}{\sqrt{f + h'^2}} \to 1.$$

Choosing $C \gg 0$ such that $C + \kappa_\epsilon > 0$, we obtain

$$\mathcal{Q}[\varphi] < -n(H + |H|) \leq 0.$$

We conclude that at points of Ω_0 it holds that

$$\begin{aligned} &\mathcal{Q}[\varphi] < \mathcal{Q}[u] = 0, \\ &\varphi|_\Gamma \geq u|_\Gamma. \end{aligned}$$

We now prove that $\varphi \geq u$ on $\bar{\Omega}$. By contradiction, assume that there exist points for which the continuous function $\hat{u} := u - \varphi$ satisfies $\hat{u} > 0$. Hence $m := \hat{u}(y) > 0$ at a maximum point $y \in \bar{\Omega}$ of $\hat{u}$. Choose a minimizing geodesic γ joining y to Γ for which the distance $d = d(y, \Gamma)$ is attained. Thus, $\gamma(t) = \exp_{y_0} t\eta,\ 0 \leq t \leq d$, starts from a point $y_0 \in \Gamma$ with unit speed η. Since γ is minimizing, we have $d(\gamma(t), \Gamma) = t$ and the function φ restricted to γ is differentiable with $\varphi'(\gamma(t)) = e^{C(A-t)}$. Since the maximum of $\hat{u}$ restricted to γ occurs at $t = d$, i.e., at the point y, one has that

$$u'(\gamma(d)) - \varphi'(\gamma(d)) = \hat{u}'(\gamma(d)) \geq 0.$$

This implies that

$$\langle \nabla u(y), \gamma'(d) \rangle \geq \varphi'(\gamma(d)) = e^{C(A-d)} > 0.$$

In particular $\nabla u(y) \neq 0$, and hence the level hypersurface

$$S = \{x \in \Omega \cap B_r(y) : u(x) = u(y)\}$$

is regular for small radius r. Along S we have

$$\hat{u}(x) + \varphi(x) = \hat{u}(y) + \varphi(y) \geq \hat{u}(x) + \varphi(y),$$

and since φ is an increasing function of $d(\,\cdot\,,\Gamma)$ it follows that $d(x,\Gamma) \geq d(y,\Gamma) = d$. From this we conclude that the points in S are at a distance at least d from Γ. Since S is C^2 it satisfies the interior sphere condition: there exists a small ball $B_\varepsilon(z)$ touching S at y contained in the side to which $\nabla u(y)$ and $\gamma'(d)$ points. Thus, the points of $B_\varepsilon(z)$ satisfy $u(x) \geq u(y)$, and hence

$$\varphi(x) + m \geq u(x) \geq u(y) = \varphi(y) + m, \quad x \in B_\varepsilon(z),$$

where in the first inequality we used the definition of m. Again because φ is an increasing function of d, we have $d(x,\Gamma) \geq d$ on $B_\varepsilon(z)$ and therefore this ball is contained in the interior of Ω far away from Γ. This allows us to extend the geodesic γ through $B_\varepsilon(z)$. We claim that the center z of the ball is contained in this extension. Otherwise, the broken line consisting of γ and of the radius in $B_\varepsilon(z)$ from z to y has length smaller than *a* minimizing geodesic joining z to $y_0 \in \Gamma$ (for a suitable small ε such a geodesic must cross the level hypersurface S at a point $x \neq y$ at distance to Γ greater than d). Thus, if there exists at least two distinct minimizing geodesics joining y to Γ, then the point z is contained in the extension of both geodesics after its intersection at y. Choosing ε sufficiently small, we see that this configuration is not possible (the construction we made above applies to both geodesics). This contradiction implies that the maximum point y belongs to Ω_0. However, in this case, $\hat{u}(y) \leq 0$, a contradiction. We conclude that $u \leq \varphi$ throughout $\bar{\Omega}$ and therefore φ is a continuous super-solution for the Dirichlet problem (11).

In a similar way, we may construct lower barriers for u, that is, continuous sub-solutions for (11). It is clear that the existence of these barriers implies the desired C^0 *apriori* estimates.

3.2 Geodesic spheres as barriers

Next we assume only that the Ricci curvature has a finite lower bound, that is,

$$\mathrm{Ric}_M \geq -(n-1)k,$$

for some positive constant k. In this case, we present a strategy for obtaining height estimates for u. Our method relies in a Hessian comparison theorem and takes geodesic spheres as barriers.

We fix a point $p_0 \in \mathbb{P}$ and consider the function $r = \mathrm{dist}(p_0, \cdot)$. Let $\mathbb{H}^{n+1}(-k)$ be the hyperbolic space form with constant sectional curvature

$-k$ and $r_{\rm hyp}$ a distance function on it. The Laplacian Comparison Theorem (cf. [15], p. 5, Corollary 1.1) yields

$$\Delta r \leq \Delta_{\mathbb{H}^{n+1}(-k)} r_{\rm hyp}$$

at corresponding (equidistant) points, whenever r is differentiable. This implies that if we consider geodesic balls B in M (outside the cut locus of M) and $B_{\rm hyp}$ in $\mathbb{H}^{n+1}(-k)$ with same radius r_0, then the mean curvatures of the respective geodesic spheres calculated with respect to the gradient of the distances satisfy

$$-H_{\partial B} \leq -H_{\partial B_{\rm hyp}} = -\sqrt{k}\coth\sqrt{k}\, r_0.$$

Thus, if we assume that our function H satisfies

$$r_0 \leq \frac{1}{\sqrt{k}}\coth^{-1}\frac{\max_\Omega |H|}{\sqrt{k}},$$

then we have that

$$|H| \leq H_{\partial B}.$$

Thus, if we suppose that the domain Ω is contained in a geodesic disc of radius r_0 in $\mathbb{P}$, then the corresponding geodesic sphere in M is a barrier for C^0 estimates for the problem (11). Indeed, it suffices to move such a sphere along the flow lines of Y and then apply the maximum principle at a first tangency point between the graph Σ and the moving spheres.

Remark 2. For constant mean curvature H it was shown [6] that an apriori height estimate exists if $\mathrm{Ric}_M > -nH^2$. This is achieved by constructing a function that is subharmonic but only if H is constant. Up to this estimate, all the aforementioned results on Killing graphs in the Introduction for different ambient spaces follow from Theorems 1 and 2 in this paper.

3.3 CMC spheres as barriers

Next we assume that the induced metric in $\mathbb{P}$ is rotationally invariant. More precisely, the metric in $\mathbb{P}$ is of the form

$$\mathrm{d}r^2 + \xi^2(r)\mathrm{d}\theta^2$$

in terms of coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, where $\mathrm{d}\theta^2$ denotes the usual metric in $\mathbb{S}^{n-1}$. We also assume that $\varrho = \varrho(r)$, that is, the norm of the Killing field does not depend on θ. In this case, the ambient metric is written in terms of cylindrical coordinates s, r, θ as

$$\varrho^2(r)\mathrm{d}s^2 + \mathrm{d}r^2 + \xi^2(r)\mathrm{d}\theta^2.$$

and M is a doubly-warped product with respect to warping functions of the coordinate r.

A rotationally invariant hypersurface Σ_0 is parametrized by an immersion $\mathbb{R} \times \mathbb{S}^{n-1} \to M$ whose coordinate expression is

$$(u, \theta) \mapsto (s(u), r(u), \theta),$$

where u is the arc-length parameter of the profile curve $\theta = \theta_0$. This means that u is defined by $\varrho^2\dot{s}^2 + \dot{r}^2 = 1$ and that the induced metric in Σ_0 is

$$\mathrm{d}u^2 + \xi^2(u)\mathrm{d}\theta^2.$$

If Σ_0 has constant mean curvature H_0, then it satisfies a first order equation given by the flux formula. In the flux formula in the Appendix, we put Γ as the geodesic circle of radius $r = r(u)$ which is the intersection of Σ_0 and the leaf $\mathbb{P}_s$, where $s = s(u)$. Thus we consider D as the geodesic disc in $\mathbb{P}_s$ with radius $r = r(u)$. The co-normal ν is the unit velocity vector $\dot{s}\partial_s + \dot{r}\partial_r$. Finally, the Killing vector field Y corresponds to the coordinate vector field ∂_s. Thus, one has

$$\langle Y, \nu \rangle = \varrho^2(r(u))\dot{s} \;\;\text{and}\;\; \langle Y, N_D \rangle = \langle Y, \frac{Y}{|Y|} \rangle = \varrho(r(u)).$$

Hence, plugging these expressions in the flux formula gives

$$c \;=\; nH_0 \int_D \varrho + \int_\Gamma \dot{s}\varrho^2 = nH_0 \int_0^r \int_{\mathbb{S}^{n-1}} \varrho \xi^{n-1}\, \mathrm{d}r\mathrm{d}\theta + \int_{\mathbb{S}^{n-1}} \dot{s}\varrho^2\xi^{n-1}\, \mathrm{d}\theta.$$

Since we are integrating at a fixed value of s and therefore at a fixed value of u, we have

$$nH_0 \int_0^r \varrho\xi^{n-1}\mathrm{d}r + \dot{s}(r(u))\varrho^2(r)\xi^{n-1}(r) = \frac{c}{\omega_n}.$$

The derivative of this expression with respect to u gives a second order ODE which characterizes CMC rotationally invariant hypersurfaces in M. Compact solutions satisfy $r = 0$ at the points of maximum and minimum height, where $\dot{s} = 0$. Hence, these compact examples correspond to take $c = 0$.

On the other hand, at maximum points for r we have $\varrho^2 \dot{s}^2 = 1$ and therefore

$$nH_0 \int_0^{r_0} \varrho(r)\xi^{n-1}(r)\mathrm{d}r + \varrho(r_0)\xi^{n-1}(r_0) = 0,$$

where r_0 is the maximum value of r. Thus the mean curvature of compact rotational examples with maximum radius r_0 satisfy

$$nH_0 = -\frac{\varrho(r_0)\xi^{n-1}(r_0)}{\int_0^{r_0} \varrho(r)\xi^{n-1}(r)\mathrm{d}r} =: -F(r_0).$$

If we assume that $\Omega \subset \mathbb{P}$ is contained in a geodesic disc with radius r_0, then these CMC spheres are barriers for the height of a Killing graph with prescribed mean curvature $H(x)$ satisfying

$$n|H(x)| \leq F(r_0).$$

The value of $F(r_0)$ may be explicitly given in particular cases such as $\mathbb{H}^{n+1}(-k)$ and $\mathbb{H}^n(-k) \times \mathbb{R}$.

4 Boundary gradient estimates

Our task now is to produce *apriori* gradient estimates for the Dirichlet problem (11). In order to do that, we use barriers of the form $w + \phi$ along a tubular neighborhood Ω_ϵ of Γ as defined in Section 2.1. Here, $w = \psi(d(x))$ for some real function ψ to be chosen later and $d = \mathrm{dist}(\,\cdot\,, \Gamma)$. Moreover, the boundary data ϕ was extended to Ω_ϵ by $\phi(s^i, d) = \phi(s^i)$ for simplicity.

A simple estimate gives

$$\begin{aligned} \mathfrak{Q}[w+\phi] &= a^{ij}(x, \nabla w + \nabla\phi)(w_{i;j} + \phi_{i;j}) + b(x, \nabla w + \nabla\phi) - nH \\ &\leq a^{ij} w_{i;j} + \Lambda|\phi|_{2,\alpha} + b - nH, \end{aligned}$$

where a^{ij} and b are given by (9) and (10). Thus,

$$a^{ij} w_{i;j} = \frac{1}{W}\Delta w - \frac{1}{W^3}(w^i + \phi^i)(w^j + \phi^j) w_{i;j}.$$

On one hand, we deduce from (2) and (3) the expressions

$$w^i w^j w_{i;j} = (\psi')^2 d^i d^j(\psi'' d_i d_j + \psi' d_{i;j}) = (\psi')^2\psi''|\nabla d|^4 = (\psi')^2\psi''$$

and

$$w^i \phi^j w_{i;j} = \psi' d^i \phi^j(\psi'' d_i d_j + \psi' d_{i;j}) = \psi'\psi''|\nabla d|^2 d^j\phi_j = \psi'\psi''\langle\nabla d, \nabla\phi\rangle = 0$$

and

$$\phi^i\phi^j w_{i;j} = \phi^i\phi^j(\psi'' d_i d_j + \psi' d_{i;j}) = \psi''\langle\nabla d, \nabla\phi\rangle^2 - \psi'\phi^i\phi^j b_{ij}(\epsilon) = -\psi'\phi^i\phi^j b_{ij}(\epsilon).$$

In particular, we obtain

$$\frac{1}{W}\psi'' - \frac{1}{W^3}w^i w^j w_{i;j} = \frac{\psi''}{W}\big(1 - \frac{(\psi')^2}{W^2}\big) = \frac{\psi''}{W^3}(f + |\nabla\phi|^2)$$

since $W^2 = f + (\psi')^2 + |\nabla\phi|^2$ and

$$(w^i\phi^j + w^j\phi^i + \phi^i\phi^j)w_{i;j} = -\psi'\phi^i\phi^j b_{ij}(\epsilon).$$

On the other hand, we deduce from (4) that

$$\Delta w = \psi'' + \psi'\Delta d = \psi'' - (n-1)\psi' h_\epsilon.$$

We conclude that

$$a^{ij}w_{i;j} = -\frac{\psi'}{W}(n-1)h_\epsilon + \frac{\psi''}{W^3}(f + |\nabla\phi|^2) + \frac{\psi'}{W^3}\phi^i\phi^j b_{ij}(\epsilon).$$

A suitable expression for b is

$$\begin{aligned} b &= -\frac{1}{W^3}(f + W^2)\langle\nabla w + \nabla\phi, f\bar{\nabla}_{\partial_s}\partial_s\rangle \\ &= -\frac{\psi'}{W}\big(\frac{f}{W^2} + 1\big)\kappa_\epsilon - \frac{1}{W}\big(\frac{f}{W^2} + 1\big)\langle f\bar{\nabla}_{\partial_s}\partial_s, \nabla\phi\rangle \end{aligned}$$

since $\nabla w = \psi'\eta_\epsilon$ and $\kappa_\epsilon = f\langle\bar{\nabla}_{\partial_s}\partial_s, \eta_\epsilon\rangle$. Therefore,

$$\begin{aligned} \mathfrak{Q}[w+\phi] \le &-\tfrac{\psi'}{W}\big((n-1)h_\epsilon + \kappa_\epsilon\big) - nH + \Lambda|\phi|_{2,\alpha} - \tfrac{\psi' f\kappa_\epsilon}{W^3} \\ &-\tfrac{1}{W}\big(\tfrac{f}{W^2} + 1\big)\langle f\bar{\nabla}_{\partial_s}\partial_s, \nabla\phi\rangle + \tfrac{\psi''}{W^3}(f + |\nabla\phi|^2) + \tfrac{\psi'}{W^3}\phi^i\phi^j b_{ij}(\epsilon). \end{aligned}$$

Finally, using that $\Lambda = 1/W$, we get

$$W^3 \mathcal{Q}[w+\phi] \leq -n\psi' H_{\text{cyl}}(\epsilon) W^2 - nHW^3 + |\phi|_{2,\alpha} W^2 - \psi' f \kappa_\epsilon$$
$$- \langle f \bar\nabla_{\partial_s} \partial_s, \nabla\phi \rangle W^2 + \psi' \phi^i \phi^j b_{ij}(\epsilon) - f^2 \langle \bar\nabla_{\partial_s} \partial_s, \nabla\phi \rangle + \psi''(f + |\nabla\phi|^2).$$

Now define

$$\psi(d) = \mu \ln(1 + Kd)$$

for certain positive constants μ and K to be chosen later. We have

$$\psi' = \frac{\mu K}{1+Kd} \quad \text{and} \quad \psi'' = -\frac{1}{\mu}(\psi')^2.$$

We choose μ in such a way that $\mu \to 0$ as $K \to \infty$. It suffices to take

$$\mu = \frac{C}{\ln(1+K)}$$

for some positive constant C to be chosen later. In this case, as $K \to \infty$ one has

$$\psi'(0) = \frac{CK}{\ln(1+K)} \to +\infty.$$

It also holds that $\frac{\psi'}{W} \sim 1$ as $K \to \infty$. Thus, at points of Γ the last inequality (asymptotically) becomes

$$W^3 \mathcal{Q}[w+\phi] \leq -n(H_{\text{cyl}} + H)\psi'^3 - \frac{1}{\mu}(f + |\nabla\phi|^2)(\psi')^2$$
$$+ \big(|\phi|_{2,\alpha} - \langle f \bar\nabla_{\partial_s} \partial_s, \nabla\phi \rangle\big)(\psi')^2 - (f\kappa + \phi^i\phi^j b_{ij})\psi' - f^2 \langle \bar\nabla_{\partial_s} \partial_s, \nabla\phi \rangle$$

Therefore, assuming that $H_{\text{cyl}} + H \geq 0$ and choosing K large enough, we assure that $\mathcal{Q}[w+\phi] < 0$ on a small tubular neighborhood Ω_ϵ of Γ and that $w + \phi \geq u|_{\Gamma_\epsilon} + \phi$ on both boundary components. Therefore, $w+\phi$ is a locally defined upper barrier for the Dirichlet problem (11). A lower barrier may be constructed in a similar way.

5 Interior gradient estimates

The last step in providing *apriori* estimates for (11) is to verify that ∇u satisfies a kind of maximum principle for a third order equation obtained from differentiating $\mathfrak{Q}[u] = 0$ and contracting the resulting equation with the gradient itself.

Using a suitable test function taken from [19] and Ricci identities allows us to eliminate third derivatives and to obtain global estimates for $|\nabla u|$ in terms of the height and boundary C^1 estimates.

We suppose momentarily that $u \in C^3(\Omega)$. The usual regularity theorems guarantee that the estimates we will obtain are also true for a $C^{2,\alpha}$ function (see [7] and [18]). Equation (8) may be written as

$$A^{ij}\, u_{i;j} = B, \tag{14}$$

where

$$A^{ij}(x, \nabla u) = W^2\sigma^{ij} - u^i u^j \tag{15}$$

and

$$B(x, \nabla u) = (f + W^2)\langle f\bar{\nabla}_{\partial_s}\partial_s, \nabla u\rangle + nHW^3. \tag{16}$$

Differentiating covariantly with respect to the metric σ_{ij} on $\mathbb{P}$ yields

$$\nabla_k W^2 = \nabla_k f + \nabla_k(u^j u_j) = f_k + u^j_{;k}u_j + u^j u_{j;k} = f_k + 2u^j u_{j;k}.$$

We obtain from (14) that

$$\nabla_k B = ((f_k + 2u^l u_{l;k})\sigma^{ij} + W^2\nabla_k\sigma^{ij} - \nabla_k(u^i u^j))u_{i;j} + A^{ij}u_{i;jk}.$$

Contracting with u^k gives

$$u^k\nabla_k B = (f_k u^k + 2u^j u^k u_{j;k})u^i_{;i} - 2u^i u_{i;j}u^j_{;k}u^k + A^{ij}u^k u_{i;jk}. \tag{17}$$

Following [19] we define the function

$$\tau = e^{2Cu}v,$$

where $v = u^i u_i$ is the squared norm of the gradient of u and $C > 0$ a constant to be chosen later. Then,

$$\tau_i = e^{2Cu}(2Cvu_i + v_i).$$

However,

$$v_i = \nabla_i(u^j u_j) = u^j_{;i}u_j + u^j u_{j;i} = 2u^j u_{j;i}.$$

Thus,

$$\tau_i = 2e^{2Cu}(Cvu_i + u^j u_{j;i}). \tag{18}$$

If τ achieves its maximum on Γ then we have a bound for v in Ω as desired. Hence, we may assume that the maximum is attained at an interior point $p_0 \in \Omega$ where we have

$$u^j u_{j;i} = -Cvu_i. \tag{19}$$

Differentiating (18) yields

$$\tau_{i;j} = 2e^{2Cu}(2C^2 u_i u_j v + 2Cu_j u^k u_{k;i} + Cu_{i;j} v + 2Cu_i u^k u_{k;j} + u^k_{\ ;j} u_{k;i} + u^k u_{k;ij}). \tag{20}$$

From the maximum point criterion we have $\tau_{i;i} \leq 0$, and from the ellipticity of (14) that

$$A^{ij}\tau_{i;j} \leq 0. \tag{21}$$

From (19) we obtain at p_0 that $u_j u^j_{;i} = u^j u_{j;i} = -Cu_i v$. Therefore,

$$u^j u_{j;i} u^i = -Cv^2 \tag{22}$$

and

$$u^j u_{j;i} u^i_{\ ;k} u^k = C^2 v^2 u_k u^k = C^2 v^3. \tag{23}$$

We may assume that ∇u is non-singular in a neighborhood of p_0. Otherwise, we are done. Hence, we choose local coordinates at p_0 asking x^1 to parametrize the trajectories of ∇u and such that

$$\partial_1 = \nabla u / |\nabla u|$$

along the trajectory at p_0. The remaining coordinates parametrize the level sets of u and are assumed to be orthonormal and geodesic at p_0. Thus, we have at p_0 that

$$u_1 = |\nabla u| = v^{1/2} \quad \text{and} \quad u_j = 0 \ \text{ for } \ j \neq 1. \tag{24}$$

We also have $u_{j;i} = u^j_{;i} = u_{ji}$. Then (19) gives

$$u_{1;1} = u_{11} = -Cv \quad \text{and} \quad u_{1;i} = u_{1i} = 0 \ \text{ for } \ i \neq 1. \tag{25}$$

If necessary, we rotate the coordinates to assure that u_{ij} is diagonal at p_0. Then (14) at p_0 becomes

$$(f+v)\Delta u = (f+v)u_{ii} = B - vu_{11} = B - Cv^2. \tag{26}$$

We now use the Ricci identities for the Hessian $u_{i;j}$ of u

$$u_{i;jk} - u_{k;ij} = R_{ijkm} u^m,$$

where R is the curvature tensor in $\mathbb{P}$. Thus, we obtain that

$$\begin{aligned} A^{ij} u^k u_{i;jk} &= ((f+v)\sigma^{ij} - u^i u^j) u^k u_{i;jk} \\ &= ((f+v)\sigma^{ij} - u^i u^j) u^k (u_{k;ij} + R_{ijkm} u^m) \\ &= A^{ij} u^k u_{k;ij} + (f+v)\sigma^{ij} R_{ijkm} u^k u^m \end{aligned} \tag{27}$$

since $R_{ijkm} u^i u^j u^k u^m = 0$. Moreover, we have from (22) and (26) that

$$(f_k u^k + 2u^j u^k u_{j;k}) u^i_{;i} = (f_k u^k - 2Cv^2) D_1 \tag{28}$$

where $D_1 =: (B - Cv^2)/(f+v)$. From (17), (23), (27) and (28) we deduce

$$u^k \nabla_k B = (f_k u^k - 2Cv^2) D_1 - 2C^2 v^3 + A^{ij} u^k u_{k;ij} + (f+v)\sigma^{ij} R_{ijkm} u^k u^m. \tag{29}$$

We conclude that

$$A^{ij} u^k u_{k;ij} = -f_k u^k D_1 - (f+v)\sigma^{ij} \bar{R}_{ijkm} u^k u^m + 2C^2 v^2 f + 2Cv^2 D_2 + u^k \nabla_k B \tag{30}$$

where we denote $D_2 = (B - Cf^2)/(f+v)$.

Next we write down the maximality condition (21) for τ at the point p_0. A straightforward computation using (15), (24) and (25) in (20) yields

$$\begin{aligned} 0 &\geq A^{ij}(2C^2 u_i u_j v + 2C u_j u^k u_{k;i} + C u_{i;j} v + 2C u_i u^k u_{k;j} + u^k_{;j} u_{k;i} + u^k u_{k;ij}) \\ &= (f+v)(Cv u_{ii} + u_{jj} u_{jj}) - 2C^2 v^2 f + A^{ij} v^{1/2} u_{1;ij}. \end{aligned}$$

Using that $u_{jj} u_{jj} \geq C^2 v^2$ and (26) we obtain

$$A^{ij} v^{1/2} u_{1;ij} \leq C^2 v^2 f - CBv. \tag{31}$$

Combining the expressions (30) and (31) we get

$$-f_1 v^{1/2} D_1 + (f+v)\sigma^{ij} R_{ikjm} u^k u^m + C^2 v^2 f + 2Cv^2 D_2 + CBv + v^{1/2} \nabla_1 B \leq 0.$$

Let R be the minimum eigenvalue of the Ricci tensor in $\mathbb{P}$ in the direction of ∇u. Thus, we have $Rv \leq \sigma^{ij} R_{ikjm} u^k u^m$ and multiplying the above expression by $f+v$ yields

$$Rv(f+v)^2 + C^2 v^2 f(v-f) + CBv(f+3v) + f_1 v^{1/2}(Cv^2 - B) + (f+v) v^{1/2} \nabla_1 B \leq 0. \tag{32}$$

Next we compute the last term on the left hand side of (32). From (16) we obtain

$$B(x, \nabla u, v) = (2f^2 + fv)\Gamma_{00j}u^j + nH(f+v)^{3/2} \tag{33}$$

where $\Gamma_{00j} = \sigma_{ij}\Gamma^i_{00}$. Differentiating $B = B(x, \nabla u, v)$ we have

$$\nabla_k B = B_k + B_{u^j}u^j_{;k} + B_v v_k.$$

We obtain at p_0 that

$$v^{1/2}\nabla_1 B = B_1 v^{1/2} - CB_{u^1}v^{3/2} - 2B_v C v^2.$$

Replacing this in (32) yields

$$\begin{aligned} Rv(f+v)^2 + C^2v^2 f(v-f) + (f+v)B_1 v^{1/2} + f_1 v^{1/2}(Cv^2 - B) \\ +Cv((f+3v)B - (f+v)(B_{u^1}v^{1/2} + 2B_v v)) \leq 0. \end{aligned}$$

Using (33) we have at p_0 that

$$B_1 = (4f+v)f_1\Gamma v^{1/2} + (2f^2+fv)\nabla_1\Gamma v^{1/2} + nH_1(f+v)^{3/2} + \frac{3}{2}nH(f+v)^{1/2}f_1$$

and

$$B_{u^1} = (2f^2 + fv)\Gamma$$

and

$$B_v = f\Gamma v^{1/2} + \frac{3}{2}nH(f+v)^{1/2},$$

where $\Gamma =: \Gamma_{001}$. In particular,

$$B_1 v^{1/2} = 2f(2f_1\Gamma + f\nabla_1\Gamma)v + (f_1\Gamma + f\nabla_1\Gamma)v^2 + n(f+v)^{1/2}v^{1/2}(H_1(f+v) + \frac{3}{2}Hf_1)$$

and

$$(f+3v)B - (f+v)(B_{u^1}v^{1/2} + 2B_v v) = 2f^2\Gamma v^{3/2} + nHf(f+v)^{3/2}.$$

We may assume that $\max_\Omega f \leq v$ since, otherwise, we are done. Then, we have $f + v \leq 2v$ and

$$\begin{aligned} B_1 v^{1/2} &\geq (4ff_1\Gamma + 2f^2\nabla_1\Gamma - (3/\sqrt{2})n|Hf_1|)v + (f_1\Gamma + f\nabla_1\Gamma - 2^{3/2}n|H_1|)v^2 \\ &= A_1 v + A_2 v^2. \end{aligned}$$

Moreover, we have

$$(f+3v)B-(f+v)(B_{u^1}v^{1/2}+2B_v v)\geq(2f^2\Gamma-2^{3/2}n|H|f)v^{3/2}=A_3v^{3/2}$$

and

$$\begin{aligned}f_1v^{1/2}(Cv^2-B) &= f_1v^{1/2}(Cv^2-(2f^2+fv)\Gamma v^{1/2}-nH(f+v)^{3/2})\\ &\geq -f_1(2f^2+fv)\Gamma v-2^{3/2}n|Hf_1|v^2+Cf_1v^{5/2}\\ &= C_1v+C_2v^2+Cf_1v^{5/2}.\end{aligned}$$

Replacing in (34) and grouping equal powers of v, we obtain

$$\begin{aligned}&(C^2f+R+A_2)v^3+C(A_3+f_1)v^{5/2}+(2Rf-C^2f^2+A_1+fA_2+C_2)v^2\\ &\quad+(Rf^2+A_1f+C_1)v\leq 0.\end{aligned}$$

Notice that all constants in the expression above depend only on H and Ω. We choose C large enough so that the coefficient of v^3 in the left hand side is positive. With this choice we conclude that $v\leq\tilde{C}$ at p_0 for some constant $\tilde{C}$ depending on the height estimates for u and on the data H and Ω.

6 Proof of the theorems

We apply the well-known continuity method to the family of Dirichlet problems

$$\mathcal{Q}_\sigma[u]=0,\quad u|_\Gamma=\sigma\phi,$$

where $\sigma\in[0,1]$ and

$$\mathcal{Q}_\sigma[u]=a^{ij}u_{i;j}+b-n\sigma H.$$

The subset I of $[0,1]$ consisting of values of σ for which the above Dirichlet problem has a $C^{2,\alpha}$ solution is non-empty, since $0\in I$. The hypothesis $H_u=0$ implies that I is open. This follows from a standard application of the implicit function theorem. The closedness of I follows from the *apriori* estimates we had proved. Thus, the continuity method assures that $1\in I$.

In order to prove uniqueness, it suffices to reproduce the proof presented in [6]. ∎

We point out that our existence results still hold if ϕ is only assumed continuous. We may approximate ϕ uniformly by smooth boundary data and use the interior gradient estimate to obtain strong convergence on compact subsets of Ω. A local barrier argument shows that the limiting solutions achieves the given boundary data.

7 Appendix: a flux formula

Consider a hypersurface Σ in M with boundary Γ and another hypersurface D such that $M \cup D$ bounds a domain U in M. Let H be a function that is constant along the flow lines of Y. Choose an orientation on $M \cup D$ given by unit normal vector fields N along Σ and N_D along D. By the Killing equation, we have

$$\mathrm{div}_M HY = H\mathrm{div}_M Y + \langle \bar{\nabla} H, Y\rangle = 0.$$

Thus, applying Stokes theorem to the domain U, one has

$$0 = \int_U \mathrm{div}_M HY = \int_\Sigma H\langle Y, N\rangle + \int_D H\langle Y, N_D\rangle.$$

However, if we consider an orthonormal adapted frame $N, e_1, \ldots, e_n$ to Σ we have from the Killing equation

$$0 = \sum_i \langle \bar{\nabla}_{e_i} Y, e_i\rangle = \sum_i \langle \bar{\nabla}_{e_i} Y^T, e_i\rangle + \sum_i \langle \bar{\nabla}_{e_i} N, e_i\rangle = \mathrm{div}_\Sigma Y^T - nH\langle Y, N\rangle.$$

Thus, applying Stokes theorem to Σ we obtain

$$n\int_\Sigma H\langle Y, N\rangle = \int_\Gamma \langle Y, \nu\rangle$$

where ν is the exterior unit co-normal of Σ along Γ. The two expressions we obtained above may be gathered at the *flux formula*

$$n\int_D H\langle Y, N_D\rangle + \int_\Gamma \langle Y, \nu\rangle = 0.$$

An useful variant of this reasoning is to consider a hypersurface Γ in Σ not homologous to zero in Σ such that $\Gamma = \partial D$. Then we have that

$$n\int_D H\langle Y, N_D\rangle + \int_\Gamma \langle Y, \nu\rangle = c,$$

where c is a constant depending only on the homology class of Γ.

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