THE RIEMANN SOLUTION FOR THE INJECTION OF STEAM AND NITROGEN IN A POROUS MEDIUM

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Abstract. Conservation laws have been used to model a variety of physical phenomena and therefore the theory for this class of equations is well developed. However, in many problems, such as transport of hot fluids with mass interchange between phases balance laws are required to describe the flow.

Using the model proposed by Bruining et. al. [4], we formulate a system that describes the flow of nitrogen, steam and water in a porous medium. We neglect compressibility, heat conductivity and capillarity effects and present a physical model for steam injection based on the mass balance and energy conservation equations.

We describe a formalism for solving Riemann problem associated to this class of balance equations for a 3×3 system. It is a step towards obtaining a general method for a wide class of balance equations with phase changes (see [10]).

1. Introduction

The frequent and widespread occurrence of contamination due to spills and leaks of organic materials, such as petroleum products, that occur during their transport, storage and disposal constitute a menace to our high-quality groundwater resources. In spite of increased awareness of the environmental impacts of oil spills, it appears to be impossible to avoid these accidents, so it is necessary to develop techniques for groundwater remediation.

Steam injection is widely studied in Petroleum Engineering, see [3]. In [9], we study the steam and water injection in several proportions into a porous medium saturated with a different mixture of steam and water. A disadvantage of pure steam injection is the ecological impact of high temperatures. This can be alleviated if we co-inject nitrogen leading to a lower temperatures. Within of such a mixture, the boiling temperature of water depends on nitrogen concentration and it is lower than water regular boiling temperature.

Another application is the recovery of geothermal energy. Injecting water and nitrogen into a hot porous rock, the water in the mixture evaporates and rock thermal heat can be recovered even if the rock is not very hot.

We present a physical model for steam, nitrogen and water flow based on mass balance and energy conservation equations. We present the main physical definitions and equations of the model. We study the three possible physical phase mixture situations. For each physical situation, we reduce the original four balance equations system to be presented in Section 2

Date: 5th April 2006.

Key words and phrases. Porous medium, steamdrive, distillation, multiphase flow.

This work was supported in part by: BOLSA DOUTORADO CNPq under grant 141573/2002-3, ANP/PRH-32, CNPq under Grant 301532/2003-6, FAPERJ under Grant E-26/152.163/2002, FINEP under CTPETRO Grant 21.01.0248.00, PETROBRAS under CTPETRO Grant 650.4.039.01.0, Brazil; NWO under Fellowship Grant R 75-389, The Netherlands.

to systems of conservation laws of type:

$$\frac{\partial}{\partial t}G(V) + \frac{\partial}{\partial x}uF(V) = 0, \tag{1.1}$$

supplemented by appropriate thermodynamic constraints between variables, such as Raoult's law and others described in Appendix A. Here $V = (V_1, V_2) : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{D} \subset \mathbb{R}^3$ is a subset of the variables: gas saturation s_g, steam composition ψ_{gw} and temperature T and it represents the unknowns in each physical situation; $G = (G_1, G_2, G_3) : \mathbb{D} \longrightarrow \mathbb{R}^3$ and $F = (F_1, F_2, F_3) : \mathbb{D} \longrightarrow \mathbb{R}^3$ are the accumulation vector and the flux vector, respectively; u is a total velocity.

The state of the general system is represented by (s_g, ψ_{gw}, T, u). Systems of conservation equations of the type (1.1) have an important feature: the variable u does not appear in the accumulation terms, it appears isolated in the flux terms, thus such systems have an infinite speed mode associated to u; despite this fact and the presence of source terms in the original balance equations we are able to solve the associated Riemann problem, which is a mathematical novelty. Another surprise in this model is the presence of a rarefaction wave associated to evaporation in the two-phase situation.

In Section 2, we present the physical model that describes the injection of steam and nitrogen in a one-dimensional horizontal porous rock initially filled with water liquid. In Section 3, we present the model equations for balance of water, steam, nitrogen and energy. The full governing equations are not a system of conservation laws, since there is a water mass transfer term, which is the condensation or evaporation of water between the liquid and gaseous phases. In Section 4, we describe three physical situations in which the balance system is rewritten as conservation systems of type (1.1). In Section 5, we study the type of waves that appear in the solution and some general results for 3×3 systems; a more complete theory is established in [10]. In Secs. 6, 7 and 8, we obtain the waves in each physical situation, it is remarkable that there is an evaporation rarefaction wave in the two-phase situation. In Section 9, we study the shocks between different regions. In Section 10, we show an example of the Riemann solution for a mixture of steam and nitrogen in the *spg* injected into a rock initially filled with water.

2. Physical model

We consider the injection of steam and nitrogen in a one-dimensional horizontal porous rock core initially filled with water at temperature T^0 . The core consists of the rock with constant porosity φ and absolute permeability k (see Appendix A). We assume that the fluids are incompressible and that the pressure variations along the core are so small that they do not affect the physical properties of the fluids.

Darcy's law for multiphase flow relates the pressure gradient in each fluid phase with its seepage speed:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p_w}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p_g}{\partial x}. \tag{2.1}$$

The water and gas relative permeability functions k_{rw} and k_{rg} are considered to be functions of their respective saturations (see Appendix A); μ_w and μ_g are the viscosities of liquid and gaseous phases; p_w and p_g are the pressures in the liquid and gaseous phases. The capillary pressure P_c and the capillary diffusion coefficient Ω are:

$$P_c = P_c(s_w) = p_g - p_w \qquad \text{and} \qquad \Omega = -f_w \frac{k k_{rg}}{\mu_g} \frac{dP_c}{ds_w} \geq 0. \tag{2.2}$$

The fractional flow functions for water and steam are defined by:

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{rg}/\mu_g}, \qquad f_g = \frac{k_{rg}/\mu_g}{k_{rw}/\mu_w + k_{rg}/\mu_g}. \tag{2.3}$$

Using Darcy's law (2.1) and the definition of P_c in Eq. (2.2.a), Eqs. (2.2) and (2.3) yield:

$$u_w = u f_w - \Omega \frac{\partial s_w}{\partial x}, \qquad u_g = u f_g - \Omega \frac{\partial s_g}{\partial x}, \quad \text{where } u = u_w + u_g \tag{2.4}$$

is the *total* or *Darcy velocity*. We will see that Ω acts as a capillary diffusion coefficient; s_w is the water saturation, i.e., the pore fraction filled with water; similarly, s_g is the gas saturation.

3. The model equations

Neglecting molecular diffusion effects, we write the equations of mass balance for water, steam and nitrogen as:

$$\frac{\partial}{\partial t}\varphi \rho_W s_w + \frac{\partial}{\partial x} u_w \rho_W = +q_{g\longrightarrow a,w}, \tag{3.1}$$

$$\frac{\partial}{\partial t}\varphi \rho_{gw} s_g + \frac{\partial}{\partial x} u_{gw} \rho_{gw} = -q_{g\longrightarrow a,w}, \tag{3.2}$$

$$\frac{\partial}{\partial t}\varphi \rho_{gn} s_g + \frac{\partial}{\partial x} u_{gn} \rho_{gn} = 0, \tag{3.3}$$

where $q_{g\longrightarrow a,w}$ is the water mass source term (i.e., the condensation rate between the steam and water phases); here ρ_W is the water density, which is assumed to be constant, ρ_{gw} (ρ_{gn}) denote the concentration of steam (nitrogen) in the gaseous phase (mass per unit gas volume); the concentration dependence on temperature is specified by Raoult's laws and other thermodynamical relationships given in Appendix A. The saturations s_w and s_g add to 1. By (2.3), the same is true for f_w and f_g .

Since there are 4 unknowns (T, s_g, ψ_{gw} and u) and 3 equations, it is necessary to specify another equation, representing energy conservation, based on an enthalpy formulation, see [1, 2]. We include longitudinal heat conduction, but neglect heat losses to the surrounding rock. We ignore adiabatic compression and decompression effects. Thus the energy conservation is given by:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \rho_W h_W s_w + (\rho_{gw} h_{gW} + \rho_{gn} h_{gN}) s_g\right) + \\ \frac{\partial}{\partial x}\left(u_w \rho_W h_W + u_g(\rho_{gw} h_{gW} + \rho_{gn} h_{gN})\right) = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right), \tag{3.4}$$

here H_r is the rock enthalpy per unit volume and $\hat{H}_r = H_r/\varphi$; h_W, h_{gW} and h_{gN} are the enthalpies per unit mass of water in the liquid aqueous phase, water and nitrogen in the gaseous phase; these enthalpies depend on temperature as described by Eqs. (A.3) and (A.4). The composite conductivity of the rock/fluid system $\kappa > \kappa_r > 0$ is given by:

$$\kappa = \kappa_r + \varphi(s_w \kappa_w + s_g \kappa_g). \tag{3.5}$$

We define H_W and H_g, the effective water and gas enthalpies per unit volume as:

$$H_W = \rho_W h_W \qquad \text{and} \qquad H_g = \rho_{gw} h_{gW} + \rho_{gn} h_{gN}, \tag{3.6}$$

so from (2.4), (3.6), we can rewrite (3.1)-(3.4) as:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = \frac{\partial}{\partial x}\left(\rho_W \Omega \frac{\partial s_w}{\partial x}\right) + q_{g\longrightarrow a,w}, \tag{3.7}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = \frac{\partial}{\partial x}\left(\rho_{gw} \Omega \frac{\partial s_g}{\partial x}\right) - q_{g\longrightarrow a,w}, \tag{3.8}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u\rho_{gn} f_g = \frac{\partial}{\partial x}\left(\rho_{gn} \Omega \frac{\partial s_g}{\partial x}\right), \tag{3.9}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\Big(H_w f_w + H_g f_g\Big) = \\ = \frac{\partial}{\partial x}\left(\left(H_g - H_w\right)\Omega\frac{\partial s_g}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right). \tag{3.10}$$

We are interested in scales dictated by field reservoirs. The effect of spatial second derivative terms (capillary pressure, heat conductivity, etc) is to widen the heat condensation front as well as other shocks, while the convergence of the characteristics tries to sharpen them. The balance of these effects yields the width of these fronts. In the field this width is typically a few tenth of centimeters; on the other hand, the distance between injection and production wells is of the order of 1000 meters. Thus this width is negligible, so we can set it to zero and simplify our analysis with no error of practical importance. From now on we disregard the diffusive terms in (3.7)-(3.10), resulting in the following balance system:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u\rho_W f_w = +q_{g\longrightarrow a,w}, \tag{3.11}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u\rho_{gw} f_g = -q_{g\longrightarrow a,w}, \tag{3.12}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u\rho_{gn} f_g = 0, \tag{3.13}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\Big(H_w f_w + H_g f_g\Big) = 0. \tag{3.14}$$

4. Physical situations

There are three regions in different physical situations. Initially, there is only water in the core; we call this situation the single-phase liquid situation, *spl*. We can inject a superheated gas to form a region containing only gaseous steam and nitrogen; we call this the single-phase gaseous situation, *spg*. There is also the two-phase situation, *tp*, where there is a mixture of liquid water, gaseous nitrogen and steam at boiling temperature. The latter depends on the concentration of nitrogen in the gas. We assume that each situation is in local thermodynamical equilibrium, so we can use Gibbs' phase rule:

$$f = c - p + 2, \tag{4.1}$$

where f represents the number of thermodynamical degrees of freedom, c is the number of components and p is the number of phases.

4.1. **Single-phase gaseous situation -** ***spg*****.** In this situation there are two components and one gaseous phase, i.e., $c = 2$ and $p = 1$, so from Gibbs' phase rule (4.1) there are three thermodynamical degrees of freedom. Since in our thermodynamical model the pressure is fixed, there are two unknown thermodynamical variables: temperature and gas composition.

We assume that nitrogen and steam in the gaseous phase behave as ideal gases with densities depending on T denoted by ρ_{gN} and ρ_{gW}, see (A.9) in Appendix A. We use Raoult's law (see [14]) and assume that there are no volume effects due to mixing, so that the volumes of the components are additive:

$$\rho_{gw}/\rho_{gW}(T) + \rho_{gn}/\rho_{gN}(T) = 1. \tag{4.2}$$

This fact allows to define the steam and nitrogen gas composition, respectively, as:

$$\psi_{gw} = \rho_{gw}/\rho_{gW}(T) \qquad \text{and} \qquad \psi_{gn} = \rho_{gn}/\rho_{gN}(T), \quad \text{so} \quad \psi_{gw} + \psi_{gn} = 1, \tag{4.3}$$

thus when the steam or nitrogen composition is known, the other composition is obtained trivially from (4.3.c). We use ψ_{gw} as the basic state variable. The compositions ρ_{gw} and ρ_{gn} are functions of temperature and composition; they are obtained from (4.3.a) and (4.3.b).

Therefore, there are three unknowns to be determined: temperature T, gas composition ψ_{gw} and speed u. The saturations are trivial, $s_g = 1$ and $s_w = 0$.

We rewrite equations (3.11)-(3.14) using Eqs. (4.3.a) and (4.3.b). Since $s_w = 0$, $f_w = 0$, so Eq. (3.11) disappears and we obtain that $q_{g\longrightarrow a,w}$ vanishes; thus the remaining system becomes:

$$\frac{\partial}{\partial t}\varphi\theta_W\psi_{gw}T^{-1} + \frac{\partial}{\partial x}u\theta_W\psi_{gw}T^{-1} = 0, \tag{4.4}$$

$$\frac{\partial}{\partial t}\varphi\theta_N\psi_{gn}T^{-1} + \frac{\partial}{\partial x}u\theta_N\psi_{gn}T^{-1} = 0, \tag{4.5}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}H_{gW} + \psi_{gn}H_{gN}\right) + \frac{\partial}{\partial x}u\Big(\psi_{gw}H_{gW} + \psi_{gn}H_{gN}\Big) = 0; \tag{4.6}$$

we have substituted ρ_{gW} and ρ_{gN} from Eq. (A.9), and defined θ_W, θ_N, $H_{gW}(T)$ and $H_{gN}(T)$:

$$\theta_W = \frac{M_W p_{at}}{R}, \quad \theta_N = \frac{M_N p_{at}}{R}, \quad H_{gW}(T) = \frac{\theta_W h_{gW}}{T} \quad \text{and} \quad H_{gN}(T) = \frac{\theta_N h_{gN}}{T}, \tag{4.7}$$

where M_N, M_W are the molar masses of nitrogen and water; R is the universal gas constant; p_{at} is the atmospheric pressure; h_{gW} and h_{gN} are functions of T given in Appendix A.

Remark 4.1. *The spg does not exist for any pair (T, ψ_{gw}), because of thermodynamic constraints. The tp situation occurs for states where the composition is given in terms of the temperature T as:*

$$\psi_{gw}(T) = \rho_{gw}(T)/\rho_{gW}(T), \tag{4.8}$$

where ρ_{gw} and ρ_{gW} given by Eqs. (A.8.a), (A.9.a). The composition $\psi_{gw}(T)$ is the projection of the tp on the plane $\{T, \psi_{gw}\}$. As we assume thermodynamical equilibrium, the solution of Eq. (4.8) *defines the boundary of the physical region in the plane $\{T, \psi_{gw}\}$, see Fig.* (4.1.a)*.*

We define the spg physical region as:

$$\Gamma = \{(T, \psi_{gw}) \in \ spg \text{ that satisfy } \psi_{gw} \leq \rho_{gw}(T)/\rho_{gW}(T)\}\,. \tag{4.9}$$

For each ψ_{gw}, the temperature that satisfies (4.8) *is called* ***saturation temperature****. The curve $\partial\Gamma$ is formed by the pairs $(T, \psi_{gw}(T))$ that satisfy* (4.8)*.*

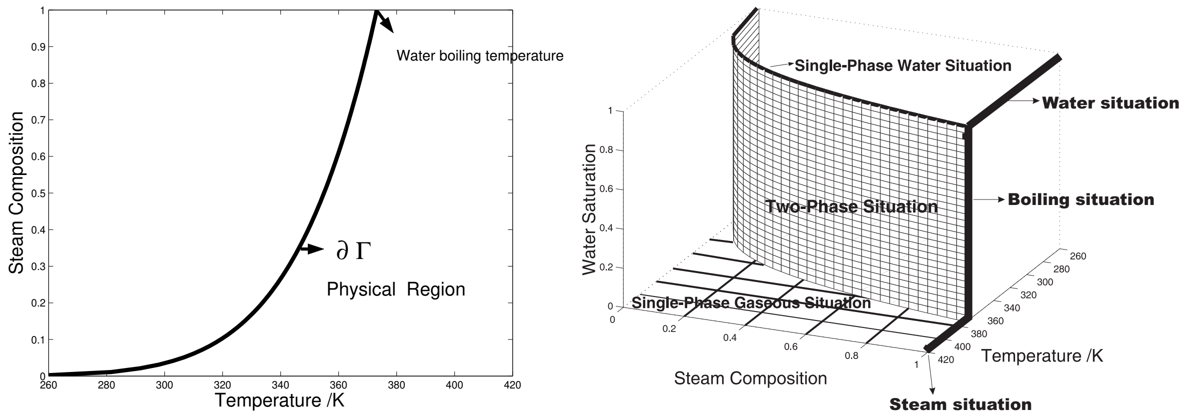

FIGURE 4.1. *a) Left:* The physical region lies to the right of the curve defined by (4.8). The continuous graph represents the saturation temperature for the mixture of water and nitrogen. *b) Right:* Phase space for $V = (s_W, \psi_{gw}, T)$ and physical situations. The bold lines represent the physical regions for water and steam without nitrogen, see [9].

4.2. **Two-phase situation -** ***tp*****.** In the *tp* there are two components, $c = 2$, and two-phases, $p = 2$; using Gibbs' phase rule (4.1), there are two thermodynamical degrees of freedom, temperature and pressure (which is fixed). The compositions in Eqs. (4.10) and (4.11) depend on temperature, and vice-versa inverting Eq. (4.8). As the pressure is given, the boiling temperature of water is specified by these compositions. When pure steam is injected the condensation temperature is constant $T^b = 373.15$, see Appendix A. We have three variables to be determined: temperature, saturation and total Darcy velocity.

There is a linearly dependent equation in the system (3.11)-(3.14). We add (3.11) and (3.12) and replace (3.11)-(3.14) by:

$$\frac{\partial}{\partial t}\varphi\left(\rho_W s_w + \rho_{gw} s_g\right) + \frac{\partial}{\partial x}u\left(\rho_W f_w + \rho_{gw} f_g\right) = 0, \tag{4.10}$$

$$\frac{\partial}{\partial t}\varphi\left(\rho_{gn} s_g\right) + \frac{\partial}{\partial x}u(\rho_{gn} f_g) = 0, \tag{4.11}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_W s_w + H_g s_g\right) + \frac{\partial}{\partial x}u(H_W f_w + H_g f_g) = 0, \tag{4.12}$$

The compositions ρ_{gw} and ρ_{gn} are functions of temperature only that can be obtained from Eqs. $(A.8.a)$ and $(A.8.b)$ in Appendix A.

4.3. **Single-phase liquid situation -** ***spl*****.** In the *spl* there is one component and one phase, so from Gibbs' phase rule (4.1), there are two thermodynamical degrees of freedom, temperature and pressure (which is fixed). Since $s_w = 1$ and $s_g = 0$, we have $f_w = 1$ and $f_g = 0$. From (3.8), $q_{g\longrightarrow a,w}$ vanishes. One can prove that the total Darcy velocity u is independent of position, so (3.11)-(3.14) become:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w\right) + u_W\frac{\partial H_w}{\partial x} = 0, \tag{4.13}$$

where we use u_W to indicate that the velocity is spatially constant in the *spl*; V is just the temperature T. We assume that rock and water enthalpy depend linearly on temperature, so we can rewrite (4.13) as:

$$\frac{\partial}{\partial t}T + \lambda_T^W \frac{\partial T}{\partial x} = 0, \quad \text{where} \quad \lambda_T^W = \frac{u_W}{\varphi} \frac{C_W}{C_W + \hat{C}_r}, \tag{4.14}$$

and C_W is the water rock heat capacity and $\hat{C}_r$ is the rock heat capacity C_r divided by φ. All quantities are given in Appendix A.

4.4. **Primary, secondary and trivial variables.** Fig. 4.1.b shows the 3 physical situations in the variables $V = (T, \psi_{gw}, s_w)$. As basic variable, we choose s_w instead of s_g for convenience.

For each physical situation there are three groups of variables. One group encompasses the variables V in the system (1.1). We call the variables in V "primary variables" or basic variables. The variable u is called "secondary variable" because we will see that it is obtained from the primary variables. The trivial variables are the variables that depend on primary variables in a simple way (for example, in the *tp* the steam composition depends on temperature) or are obtained trivially (for example, in the *spg* the gas saturation is trivial and its value is 1). In Table 4.4 we summarize these variables.

Physical situation \ Variable types	Primary	Secondary	Trivial
Single-phase gaseous situation	T, ψ_{gw}	u	$s_g = 1$
Two-phase situation	s_g, T	u	$\psi_{gw}(T)$
Single-phase liquid situation	T	u	$s_g = 0$, ψ_{gw}

Table 1: Classification according to situation.

5. General theory of Riemann Solutions

We are interested in the Riemann problem associated to (3.11)-(3.14), that is the solution of these equations with initial data

$$\begin{cases} (s_g, \psi_{gw}, T, u)_L & \text{if } x > 0 \\ (s_g, \psi_{gw}, T, \cdot)_R & \text{if } x < 0, \end{cases} \tag{5.1}$$

The speed $u_L > 0$ is specified at the injection point. In the next sections we show that u_R can be obtained as function of u_L and the primary variables.

The general solution of the Riemann problem associated to Eq. (1.1) consists of a sequence of elementary waves, rarefactions and shocks; they are studied in the sections that follow, [13]. We assume that the cumulative and flux functions have continuous second derivative ($\mathcal{C}^2$) within each physical situation and that are only continuous between the physical situations.

5.1. **Characteristic speeds in each physical situation.** System of conservation laws in different forms must be used to find the characteristic speeds. If we assume that the solution is sufficiently smooth, we differentiate all equations in (1.1) with respect to their variables, obtaining a system of the form:

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} V \\ u \end{pmatrix} = 0, \tag{5.2}$$

where the matrices B and A (which depend on V) are the derivatives of $G(V)$ and $uF(V)$ with respect to the variables V and u. Since $G(V)$ does not depend on u, the last column in the matrix B is zero. The characteristic values $\lambda^i := \lambda^i(V, u)$ and vectors $\vec{r}^i := \vec{r}^i(V, u)$, (where i is the label of each eigenvector) for the following system are the rarefaction wave speeds and directions:

$$A\vec{r^i} = \lambda^i B\vec{r^i} \quad \text{where } \lambda^i \text{ is obtained by solving} \quad det(A - \lambda^i B) = 0. \tag{5.3}$$

Similarly the left eigenvectors $\vec{\ell^i} = (\ell_1^i, \ell_2^i, \ell_3^i)$ satisfy:

$$\vec{\ell^i} A = \lambda^i \vec{\ell^i} B, \quad \text{for the same } \lambda^i. \tag{5.4}$$

Remark 5.1. *We consider here a general* 3×3 *system because it is the simplest non-trivial system of type* (1.1). *Generically, the right and left eigenvectors have three components: two primary variables and one secondary variable. However, in the spl situation we do not utilize this formalism because the system* (1.1) *reduces to a single equation.*

Hereafter , the word "eigenvectors" represents "right eigenvectors". For brevity, we will write the right and left eigenvectors without the upper arrow, i.e., we replace $\vec{r}$ by r and $\vec{\ell}$ by ℓ.

Remark 5.2. *In* [10], *we prove that the left eigenvectors* ℓ^i *do not depend on the Darcy speed* u.

Definition 5.1. *For each* i, *the* ***integral curves*** *in the* (V, u) *plane are solutions of*

$$\left(\frac{dV}{d\xi}, \frac{du}{d\xi}\right) = r, \quad i.e, \quad \frac{dV_1}{d\xi} = r_1, \quad \frac{dV_2}{d\xi} = r_2 \quad and \quad \frac{du}{d\xi} = r_3. \tag{5.5}$$

When ξ *satisfies:*

$$\xi = \lambda(V(\xi), u(\xi)), \tag{5.6}$$

the integral curves define rarefaction curves. If in addition to (5.6), *the variable* ξ *increases and satisfies* $x = \xi t$ *the integral curves define rarefaction waves in the* (x, t) *plane.*

Remark 5.3. *In Secs. 6 and 8, we calculate the eigenvalues and eigenvectors for the nitrogen-steam-water model and show that they have the following form:*

$$\lambda = u(\xi)\hat{\lambda}(V(\xi)) \qquad and \qquad r = (\eta_1(V(\xi)), \eta_2(V(\xi)), u(\xi)\eta_3(V(\xi))\,, \tag{5.7}$$

where $\hat{\lambda}$ *and* η_j *for* $j = 1, 2, 3$ *do not depend on* u. *It is useful to define:*

$$\hat{\lambda}^{i,-}(V) := \lambda^i/u^-, \quad \hat{\lambda}^{i,+}(V) := \lambda^i/u^+ \quad and \quad \hat{r}^i = (r_1^i, r_2^i) \quad for\ i = 1, 2, \tag{5.8}$$

where u^- *is the Darcy speed for* V^- *and* u^+ *is the Darcy speed for* V^+. *The primary variables* V^- *and* V^+ *are the first and last value of* V *in the rarefaction wave. Then we can see from Eq.* (5.5) *that* (V, u) *along the rarefaction wave satisfy:*

$$\left(\frac{dV_1}{d\xi}, \frac{dV_2}{d\xi}\right) = \Big(\eta_1(V(\xi)), \eta_2(V(\xi))\Big) \qquad and \qquad \frac{du}{d\xi} = u\eta_3(V(\xi)). \tag{5.9}$$

Since $V = (V_1, V_2)$ *does not depend on* u *we can solve first* $(5.9.a)$ *obtaining* $V(\xi)$ *and then substitute it in* $(5.9.b)$ *to obtain* $u(\xi)$*:*

$$u(\xi) = u^- exp\left(\int_{\xi^-}^{\xi} \eta_3(V(\varsigma))d\varsigma\right). \tag{5.10}$$

Here u^- is the first value for u on this integral curve, and ξ^- is chosen such that $\xi^- = \lambda(V^-, u^-)$. Notice that ξ is given implicitly by (5.6)*; using* (5.10) *this implicit relation is:*

$$\xi = u^- \hat{\lambda}(V(\xi)) exp\left(\int_{\xi^-}^{\xi} \eta_3(V(\varsigma))d\varsigma\right). \tag{5.11}$$

ξ depends on u^- and V, but it does not depend on the Darcy speed in the rarefaction curve. When $V = V^+$, i.e., the last V in the rarefaction wave, we denote $\xi = \xi^+$.

Remark 5.4. *Assume that $u^- > 0$, so we perform the change of variable:*

$$\xi = \frac{x}{t} \longrightarrow \hat{\xi} = \frac{x}{u^- t}, \tag{5.12}$$

The system (5.2) *can be written in terms of the independent variables $(x, T) = (x, u^- t)$. In this space, one can prove that the eigenvalues and eigenvectors have the form $\lambda^{i,-}(V)$; r_1^i and r_2^i are the first two components of the vector r^i in Eq.* (5.7.*b*)*. From Eqs.* (5.6) *and* (5.11)*, it follows that $\hat{\xi}^-$ and $\hat{\xi}$ satisfy:*

$$\hat{\xi}^- = \hat{\lambda}(V^-), \quad and \quad \hat{\xi} = \hat{\lambda}(V(\hat{\xi})) exp\left(\int_{\hat{\xi}^-}^{\hat{\xi}} \eta_3(V(\varsigma))d\varsigma\right). \tag{5.13}$$

It follows that in (x, T) the variables V and $\hat{\xi}$ do not depend on u, so the rarefaction curves too do not depend on u. The Darcy speed keeps the form (5.10)*.*

5.2. **Shock waves.** Frequently, discontinuities or shocks appear in solution of the Riemann (or Cauchy) problems of non-linear hyperbolic or hyperbolic-elliptic equations (see [13]). The Rankine-Hugoniot condition for shocks (RH) needs to be satisfied.

The main feature of this class of equations is the existence of separated regions where the system of balance equations (3.11)-(3.14) reduces to different systems of conservation equations (1.1). In the nitrogen/steam/water flow problem considered in this paper, there are three different situations: the *spg*, the *tp* and the *spl*. To obtain the complete Riemann problem solution it is necessary to link these regions. As the water mass source term acts in the infinitesimal space between physical situations, we propose the existence of shocks linking these different regions. Thus we divide the study of the shocks into different groups: shocks within a physical situation and shocks between different physical situations.

For a fixed $W^- = (V^-, u^-)$ the **Rankine-Hugoniot Locus**, RH locus, parametrizes the discontinuous solutions of Eq. (3.11)-(3.14), ie., it consists of the $W^+ = (V^+, u^+)$ that satisfy the following RH condition and it is denoted by $\mathcal{RH}(W^-)$, see [10]:

$$v^s \left(G^+(V^+) - G^-(V^-)\right) = u^+ F^+(V^+) - u^- F^-(V^-), \tag{5.14}$$

where (V^+, u^+) is the state at the right of the shock and (V^-, u^-) is the state at the left of the shock; v^s is the shock speed; G^+ (G^-) and F^+ (F^-) are the accumulation and flux terms at the right (left) of the shock, which in general have different expressions. We specify the state (V^-, u^-) on the left hand side, but at the right u^+ is not specified, it is obtained from the RH condition (5.15). From the continuity of F and G at the interface between different physical situations is clear that $W^+ = W^-$ is always a solution of (5.15), i.e., the $(-)$ state always belongs to a branch of RH locus.

We define the function $\mathcal{H} = \mathcal{H}(W^-; W^+)$ as:

$$\mathcal{H} := v^s \left(G^+(V^+) - G^-(V^-)\right) - u^+ F^+(V^+) + u^- F^-(V^-), \tag{5.15}$$

notice that the RH locus of W^- are the states W that satisfy $\mathcal{H}(W^-;W^+):=0$.

Since in this work we are interested in connected branches (i.e., that contain the $(-)$ state), we use the following criterion for admissibility of shocks instead of the viscosity profile criterion:

Definition 5.2. *(T.P. Liu. [11, 12]) We call **shock curve** the parts V of Rankine-Hugoniot curve where the shock speed decreases when V moves away from V^-. When we consider the waves in (x,t) also, each point of the shock curve represents **shock wave**. The shock curve parametrizes the $(+)$ states of admissible shocks waves $(-)$, $(+)$.*

5.3. **Wave Sequences and Riemann Solution.** A Riemann solution is a sequence of elementary waves w_k (shocks and rarefactions) for $k=1,2,\cdots,m$ and constant states W_k for $k=1,2,\cdots,m$.

$$W_L \equiv W_0 \xrightarrow{w_1} W_1 \xrightarrow{w_2} \cdots \xrightarrow{w_m} W_m \equiv W_R. \tag{5.16}$$

In (5.16), we represent any state (s_w,ψ_{gw},T,u) by W. The wave w_k has left and right states W_{k-1} and W_k and speeds $\xi_k^- < \xi_k^+$ in case of rarefaction waves and $v=\xi_k^-=\xi_k^+$ in case of shock waves. The left state of the first wave w_1 is (s_L,ψ_L,T_L,u_L) and the right state of w_m is (s_R,ψ_R,T_R,u_R), where u_R needs to be found. In the Riemann solution it is necessary that $\xi_k^+ \le \xi_{k+1}^-$; this inequality is called *geometrical compatibility.* When $\xi_k^+ < \xi_{k+1}^-$ there is a separating constant state W_{k+1} between w_k and w_{k+1}; in this sequence the wave w_k is indicated by $\mapsto$. If $\xi_k^+ = \xi_{k+1}^-$ there is no actual constant state in physical space, so the wave w_k is a composite with w_{k+1} in this sequence; it is indicated by $\twoheadrightarrow$.

Different physical situations are separated by shocks respecting the geometrical compatibility. When it is useful to emphasize the waves in the sequence (5.16) rather than the states, we use the notation:

$$w_1 \rightarrowtail w_2 \rightarrowtail \cdots \rightarrowtail w_m, \tag{5.17}$$

where for each w_k, $\rightarrowtail$ represents $\mapsto$ for the case $\xi_k^+ < \xi_{k+1}^-$ and $\twoheadrightarrow$ for the case $\xi_k^+ = \xi_{k+1}^-$.

5.3.1. *Rankine-Hugoniot locus.* For a fixed $W^-=(V^-,u^-)$, we obtain the RH locus, or $\mathcal{RH}(W^-)$, from Eq. (5.15), which can be written as:

$$v^s[G_1] = u^+F_1^+ - u^-F_1^-, \tag{5.18}$$

$$v^s[G_2] = u^+F_2^+ - u^-F_2^-, \tag{5.19}$$

$$v^s[G_3] = u^+F_3^+ - u^-F_3^-, \tag{5.20}$$

where $[G_i]=G_i^+-G_i^-$, $G_i^\pm = G_i^\pm(V^\pm)$ and $F_i^\pm = F_i^\pm(V^\pm)$ for $i=1,2,3$. We rewrite the system (5.18)-(5.20) as:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} \begin{pmatrix} v^s \\ u^+ \\ u^- \end{pmatrix} = 0, \quad \text{which requires} \quad det \begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} = 0 \tag{5.21}$$

to have a non-trivial solution. Eq. (5.21.b) yields the implicit expression $\mathcal{H}_e=0$ with:

$$\mathcal{H}_e := [G_1](F_3^+F_2^- - F_3^-F_2^+) + [G_2](F_1^+F_3^- - F_1^-F_3^+) + [G_3](F_2^+F_1^- - F_2^-F_1^+). \tag{5.22}$$

Since we can obtain the RH locus in V, it more useful denote it by $\mathcal{RH}(V^-)$ instead of $\mathcal{RH}(W^-)$. Generically, the expression $\mathcal{H}_e(V^-;V^+)=0$ defines a 1-dimensional structure, see Section 5.3.2. Notice that there are only 2 primary variables in V^+ for the *spg* and the *tp* (see the following Section for definition). In the *spl* there is only one scalar equation with the temperature as the primary variable, so $\mathcal{RH}(T^-)$ is the whole temperature physical range.

Solving the system (5.21), we obtain v^s and u^+ as functions of V^-, V^+ and u^-:

$$v^s = u^- \frac{F_2^+ F_1^- - F_1^+ F_2^-}{F_1^+[G_2] - F_2^+[G_1]} = u^- \frac{F_3^+ F_2^- - F_2^+ F_3^-}{F_2^+[G_3] - F_3^+[G_2^+]} = u^- \frac{F_1^+ F_3^- - F_3^+ F_1^-}{F_3^+[G_1] - F_1^+[G_3]}, \tag{5.23}$$

$$u^+ = u^- \frac{F_1^-[G_2] - F_2^-[G_1]}{F_1^+[G_2] - F_2^+[G_1]} = u^- \frac{F_2^-[G_3] - F_3^-[G_2]}{F_2^+[G_3] - F_3^+[G_2]} = u^- \frac{F_3^-[G_1] - F_1^-[G_3]}{F_3^+[G_1] - F_1^+[G_3]}. \tag{5.24}$$

Of course, Eqs. (5.23) and (5.24) are valid if each denominator is non-zero.

Remark 5.5. *In the definitions that follow, all wave structures can be obtained in the space of primary variables V. Using $u^+ = u^+(V^-, u^-; V^+)$ and $v^s := v^s(V^-, u^-; V^+)$, we define:*

$$\frac{u^+}{u^-} = \chi(V^-; V^+), \quad \hat{v}^-(V^-; V^+) := \frac{v^s}{u^-} \quad \textit{and} \quad \hat{v}^+(V^-; V^+) := \frac{v^s}{u^- \chi(V^-; V+)}. \tag{5.25}$$

We define the set of unordered index pairs for Eqs. (5.23) and (5.24) as:

$$\mathbb{P} = \{\{1,2\}, \{3,1\}, \{2,3\}\}. \tag{5.26}$$

From this Section 5.3.1 and Rems. 5.3 and 5.4, we have proved the following:

Proposition 5.1. *Let u_L be positive. The primary variables V in the shock and rarefaction curves do not depend on the left Darcy speed $u_L > 0$. If a sequence of waves and states solve the Riemann problem in the primary variables, for a given $u_L > 0$ (or $u_R > 0$), then it is also a solution for any other $u_L > 0$ (or $u_R > 0$):*

$$\begin{cases} (V_L, \cdot) & \textit{if } x < 0 \\ (V_R, \cdot) & \textit{if } x > 0. \end{cases} \tag{5.27}$$

Moreover, assume that for each m there is $\{p, q\} \in \mathbb{P}$ such that the following inequality is satisfied: $F^+_{q,m}[G_{p,m}] - F^+_{p,m}[G_{q,m}] \neq 0$. Then u_R is given by:

$$u_R = u_L \prod_{l=1}^{\varrho_1} exp\left(\int_{\xi^{-,l}}^{\xi^{+,l}} \eta_3^l(V(\varsigma))d\varsigma\right) \prod_{m=1}^{\varrho_2} \frac{F^-_{q,m}[G_{p,m}] - F^-_{p,m}[G_{q,m}]}{F^+_{q,m}[G_{p,m}] - F^+_{p,m}[G_{q,m}]}, \tag{5.28}$$

where η_3^l is the third component of eigenvector r, $\xi^{-,l}$ and $\xi^{+,l}$ are the first and the last values of ξ^l associated to the l-th rarefaction wave.

In the Proposition above, $G_{j,m}$ and $F_{j,m}$ represent the j-th components ($j = \{1, 2, 3\}$) of G and F on the m-th shock wave. Similarly $exp\left(\int_{\xi^{-,l}}^{\xi^{+,l}} \eta_3^l(V(\varsigma))d\varsigma\right)$ is computed along the l-rarefaction curve.

Proposition 5.2. *Assume that $u_L > 0$. If the Darcy speed u_L in the initial data is modified while V_L and V_R are kept fixed, the Darcy speed u_R as well as the Riemann solutions are rescaled in the plane (x, t), while the wave sequences and the values of V are kept unchanged.*

Proof: The rarefaction case was proved in Remark 5.4. In the shock case, performing the change of variable (5.12) with $u^- = u_L$ and using (5.15):

$$\frac{v^s}{u_L}(G^+ - G^-) = \frac{u^+}{u_L}F^+ - \frac{u^-}{u_L}F^-. \tag{5.29}$$

Defining $U := u/u_L$ and $\hat{v}^L = v^s/u_L$, we have:

(1) If there exists a rarefaction before u^-, from Remark 5.4, we see that $u^- = u_L\mathcal{T}(V)$, where $\mathcal{T}(V) := exp(\gamma(V(\xi)))$ depends only on V.

(2) If there is no rarefaction with $u^- = u_L$, we conclude that (5.29) can be rewritten as:

$$\hat{v}^L(G^+ - G^-) = U^+(V)F^+ - U^-(V)F^-, \tag{5.30}$$

where $U^-(V) = \mathcal{T}(V)$ if (1) is satisfied or $U^-(V) = 1$ if (2) is satisfied. Since $U^+(V)$ depends only on V, using Remark 5.4 we obtain the result. □

A very important result follows from this Proposition:

Corollary 5.1. *Assume the same hypothesis of Proposition* 5.2*, then the Riemann solution can be obtained in each physical situation first in the primary variables V. Then the Darcy speed can be obtained at any point of the space* (V, u) *in terms of* V *and* u_L *by an equation analogous to Eq.* (5.28)*.*

Remark 5.6. *Corollary* 5.1 *states that the secondary variable* u *can be obtained from the primary variables, so the former usually will not appear in figures.*

Proposition 5.3. *(Quadruple Shock Rule.) In the tp and the spg situations, consider four primary points and three Darcy speeds, determining four states:* (V^-, u^-) *in the tp,* (V^+, u^+) *in the spg and* (V^M, u^M) *and* (V^*, u^*) *both in the tp or the spg. Assume that there are shocks between the following pairs of states:*

(i) (V^-, u^-) *and* (V^+, u^+) *with speed* $v^{-,+}$*,*

(ii) (V^-, u^-) *and* (V^M, u^M) *with speed* $v^{-,M}$*,*

(iii) (V^M, u^M) *and* (V^*, u^*) *with speed* $v^{M,*}$*,*

such that two speeds coincide, i.e., at least one of the following equalities is satisfied:

$$\textit{either} \quad v^{-,+} = v^{-,M} \quad \textit{or} \quad v^{-,+} = v^{M,*} \quad \textit{or} \quad v^{-,M} = v^{-,+}. \tag{5.31}$$

If the following conditions (I) *to* (III) *are satisfied:*

(I) $G(V^+) - G(V^-)$ *and* $G(V^*) - G(V^M)$ *are LI;*

(II) V^+ *and* V^* *have one coordinate* V_i *with coinciding values;*

(III) $\partial\mathcal{H}_e/\partial V_j \neq 0$ *for* $j \neq i$ *for all* $V \in \mathcal{RH}(V^-)$*,*

then

(1) $V^*=V^+$*;*

(2) $u^* = u^+$*;*

(3) all the three speeds are equal:

$$v^{-,+} = v^{-,M} = v^{M,*}. \tag{5.32}$$

Proof: Disregarding the index in accumulation and flux terms, the RH conditions for (V^-, u^-)-(V^+, u^+), (V^-, u^-)-(V^M, u^M) and (V^M, u^M)-(V^*, u^*) are respectively:

$$v^{-,+}(G(V^+) - G(V^-)) = u^+F(V^+) - u^-F(V^-), \tag{5.33}$$

$$v^{-,M}(G(V^M) - G(V^-)) = u^MF(V^M) - u^-F(V^-), \tag{5.34}$$

$$v^{M,*}(G(V^*) - G(V^M)) = u^*F(V^*) - u^MF(V^M). \tag{5.35}$$

Assume that now Eq. (5.31.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (5.33) and (5.34) and subtracting Eq. (5.33) from (5.34), we obtain:

$$v(G(V^+) - G(V^M)) = u^+F(V^+) - u^MF(V^M). \tag{5.36}$$

Notice that Eqs. (5.36) and (5.35) define implicitly the RH locus by $\mathcal{H}_e(V^M; V^+) = 0$ in the variables V^M and V^+. Since the RH locus depends solely on V^M and the accumulation and flux functions, we obtain that both RH locus defined by Eqs. (5.36) and (5.35) coincide. Since (I) to (III) are satisfied, then $V^* = V^+$.

Now from Eq. (5.24), we notice for a fixed u^- that the Darcy and shock speeds depend solely on V^- and V^+. From Eqs. (5.35) and (5.36), we can see that the $(-)$ and $(+)$ states are the same for each expression and that they define the same RH locus, so $u^* = u^+$ and Eq. (5.32) is satisfied.

The other cases are proved similarly. □

5.3.2. *Degeneracies of RH Locus.* We utilize the notation:

$$\mathcal{D}_{ij}(V^-, V^+) = F_i^+[G_j] - F_j^+[G_i]. \tag{5.37}$$

Proposition 5.1 is valid if the denominator of Eq. (5.28) is non zero for some $\{i, j\} \in \mathbb{P}$. So it is necessary to study the behavior of the solution when $\mathcal{D}_{ij}(V^-, V^+) = 0$ for all $\{i, j\} \in \mathbb{P}$. For a fixed pair $\{i, j\} \in \mathbb{P}$, it easy to prove that:

Lemma 5.1. *Let $\{V^-, V^+\}$ satisfy $\mathcal{H}_e(V^-; V^+) = 0$, where $\mathcal{H}_e$ is given by Eq. (5.22). If $\mathcal{D}_{ij}(V^-, V^+) = 0$, then one of following conditions is satisfied:*

$$(i) \quad F_i^+ F_j^- - F_i^+ F_j^- = 0, \tag{5.38}$$

or

$$(ii) \quad \mathcal{D}_{12}(V^-, V^+) = \mathcal{D}_{31}(V^-, V^+) = \mathcal{D}_{23}(V^-, V^+) = 0. \tag{5.39}$$

From this Lemma it follows immediately that:

Corollary 5.2. *Let $\{V^-, V^+\}$ satisfy $\mathcal{H}_e(V^-; V^+) = 0$. If $\mathcal{D}_{ij}(V^-, V^+)$ vanishes for two index pairs $\{i, j\} \in \mathbb{P}$, then it vanishes for all pairs.*

Proposition 5.4. *If $\mathcal{D}_{ij}(V^-, V^+) = 0$ for all $\{i, j\} \in \mathbb{P}$ and $(F_1^+, F_2^+, F_3^+) \neq 0$, we obtain:*

$$[G_k] = \varrho_1 F_k^+ \qquad \text{and} \qquad F_k^- = \varrho_2 F_k^+, \quad \text{for } k = 1, 2, 3. \tag{5.40}$$

where ϱ_1 and ϱ_2 are constants depending on $[G]$, F^- and F^+. Moreover, for χ defined in $(5.25.a)$, the shock speed v^s satisfies:

$$v^s = u^- \frac{\chi(V^-; V^+) - \varrho_2}{\varrho_1}. \tag{5.41}$$

Proof: Let $u^- > 0$. Since $\mathcal{D}_{ij}(V^-, V^+) = 0 \ \forall \ \{i, j\} \in \mathbb{P}$, it follows that:

$$(F_2^+[G_3] - F_3^+[G_2])e_1 + (F_3^+[G_1] - F_1^+[G_3])e_2 + (F_1^+[G_2] - F_2^+[G_1])e_3 = 0, \tag{5.42}$$

where e_i for $i = 1, 2, 3$ is the canonical basis for $\mathbb{R}^3$. Eq. (5.42) can be written as:

$$(F_1^+, F_2^+, F_3^+) \times ([G_1], [G_2], [G_3]) = 0, \tag{5.43}$$

where $\times$ represents the curl. Since Eq. (5.43) is satisfied, it follows that $[G]$ is parallel to F^+, so there is a constant ϱ_1 so Eq. $(5.40.a)$ is satisfied. Substituting $[G] = \varrho_1 F^+$ into the RH condition (5.15), we obtain:

$$v^s \varrho_1 F_1^+ = u^+ F_1^+ - u^- F_1^-, \tag{5.44}$$

$$v^s \varrho_1 F_2^+ = u^+ F_2^+ - u^- F_2^-, \tag{5.45}$$

$$v^s \varrho_1 F_3^+ = u^+ F_3^+ - u^- F_3^-, \tag{5.46}$$

If $F_i^- = 0$ for some $i = 1, 2, 3$, it follows that:

$$v^s = u^+ / \varrho_1. \tag{5.47}$$

If $F_i^- \neq 0$ for all $i = 1, 2, 3$, multiplying Eq. (5.44) by F_2^- and (5.45) by $-F_1^-$ and adding the results, we obtain:

$$v^s \varphi \varrho_1 (F_1^+ F_2^- - F_2^+ F_1^-) = u^+ (F_1^+ F_2^- - F_2^+ F_1^-). \tag{5.48}$$

Let us assume temporarily that $F_1^+ F_2^- - F_2^+ F_1^- \neq 0$, so Eq. (5.47) is satisfied. Substituting v^s given by Eq. (5.47) into Eq. (5.44) we obtain $u^- F_1^- = 0$. Since $F_1^- \neq 0$, generically, it follows that $u^- = 0$, which is false. So $F_1^+ F_2^- - F_2^+ F_1^- = 0$.

Similar calculations for Eqs. (5.45), (5.46) and (5.44), (5.46) show that:

$$F_1^+ F_2^- - F_2^+ F_1^- = F_2^+ F_3^- - F_3^+ F_2^- = F_1^+ F_3^- - F_3^+ F_1^- = 0. \tag{5.49}$$

Since Eq. (5.49) is satisfied, there exists a constant ϱ_2 such that $F^- = \varrho_2 F^+$. Eq. (5.41) can be obtained by substituting $[G] = \varrho_1 F^+$ and $F^- = \varrho_2 F^+$ in the RH condition (5.15). □

Remark 5.7. *If* $(F_1^+, F_2^+, F_3^+) = 0$ *and* $([G_1], [G_2], [G_3]) \neq 0$*, it is easy to prove that:*

$$F_k^- = \rho_3 [G_k] \quad \textit{and} \quad v^s = u^- \rho_3, \tag{5.50}$$

where ρ_3 *is a constant that depends on* $[G]$ *and* F^-*.*

Corollary 5.3. *The states* $\{V^-, V^+\}$ *satisfying the RH condition* (5.15)*, for which* $\mathcal{D}_{ij}(V^-, V^+) = 0$ *for all* $\{i, j\} \in \mathbb{P}$*, satisfy also:*

$$F_i^+ F_j^- - F_j^+ F_i^- = 0, \quad \forall \{i, j\} \in \mathbb{P}. \tag{5.51}$$

We notice that the system (5.51) has always the trivial solution $V^+ = V^-$.

5.4. **Bifurcation loci.** Assume that there are no degeneracies any RH locus. For standard conservation laws, there are loci where the solutions change topology such as: secondary bifurcation, coincidence, double contact, inflection, hysteresis and interior boundary contact. The proof of this change depends on the Bethe-Wendroff Theorem. We extend this theorem for the balance system (1.1) so as to include shocks between regions in different situations; see [10] for the proof.

Proposition 5.5. *Assume that* F *and* G *are* $\mathcal{C}^2$*. Let* $v^s(W^+; W^-)$ *be the shock speed between different physical situations. Assume that* $\ell^i(V^+) \cdot (G^+(V^+) - G^-(V^-)) \neq 0$*. Then* v^s *has a critical point at* W^+ *(and* $\hat{v}^+(V^-; V^+)$ *has a critical point at* V^+*), if and only if:*

$$\hat{v}^+(V^-; V^+) = \lambda^{i,+}(V^+) \qquad \textit{for} \quad i = 1 \textit{ or } 2, \tag{5.52}$$

where $\lambda^+(V)$ *is given by Eq. (*5.8*) and* $v^+(V^-; V^+)$ *is given by (*5.25*).*

5.4.1. *Secondary bifurcation manifold.* The RH locus for a fixed V^- is obtained implicitly by $\mathcal{H}_e(V^-; V^+) = 0$, where $\mathcal{H}_e : \mathbb{R}^4 \longrightarrow \mathbb{R}$, and V^+ are the primary variables. At some points, these implicit expression fail to define a curve for V^+ in the space of primary variables. We call the set of these points the **secondary bifurcation locus**. From the implicit function theorem, they are the "+" states for which there exists a "−" state such that the following equalities are satisfied:

$$\mathcal{H}_e(V^-; V^+) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{H}_e}{\partial V_j^+} = 0, \quad \text{for} \quad j = 1 \text{ and } 2, \tag{5.53}$$

where $\mathcal{H}_e$ is defined by Eq. (5.22) and $\partial\mathcal{H}_e/\partial V_j^+$ for $j=1,2$ are:

$$\frac{\partial \mathcal{H}_e}{\partial V_j^+} = \left(\frac{\partial F_3^+}{\partial V_j^+}F_2^- - F_3^-\frac{\partial F_2^+}{\partial V_j^+}, \frac{\partial F_1^+}{\partial V_j^+}F_3^- - F_1^-\frac{\partial F_3^+}{\partial V_j^+}, \frac{\partial F_2^+}{\partial V_j^+}F_1^- - F_2^-\frac{\partial F_1^+}{\partial V_j^+}\right)[G]^T + $$
$$+\left(F_3^+F_2^- - F_3^-F_2^+, F_1^+F_3^- - F_1^-F_3^+, F_2^+F_1^- - F_2^-F_1^+\right)\left(\frac{\partial G^+}{\partial V_j^+}\right)^T, \qquad (5.54)$$

where $[G] = ([G_1],[G_2],[G_3])$ and $\partial G/\partial V = (\partial G_1/\partial V, \partial G_2/\partial V, \partial G_3/\partial V)$.

Equation (5.53) yields an equivalent expression for the secondary bifurcation.

Proposition 5.6. *A state V^- belongs to the bifurcation manifold for the family i when there exists a state V^+ such that:*

$$V^+ \in \mathcal{RH}(V^-) \quad \textit{with} \quad \hat{v}^+(V^-;V^+) = \hat{\lambda}^{i,+}(V^+) \quad \textit{and} \quad \ell^i(V^+)\cdot[G]=0. \qquad (5.55)$$

Proof: We drop the family index i. Assume that (5.53) is satisfied. Rearranging the terms, we can rewrite $\partial\mathcal{H}_e/\partial V_j^+$ for $j=1,2$:

$$\frac{\partial \mathcal{H}_e}{\partial V_j^+} = \left(F_3^-[G_2]-F_2^-[G_3]\right)\frac{\partial F_1^+}{\partial V_j^+} - \left(F_3^-F_2^+ - F_3^+F_2^-\right)\frac{\partial G_1^+}{\partial V_j^+} + \left(F_1^-[G_3]-F_3^-[G_1]\right)\frac{\partial F_2^+}{\partial V_j^+}$$
$$-\left(F_3^+F_1^- - F_3^-F_1^+\right)\frac{\partial G_2^+}{\partial V_j^+} + \left(F_2^-[G_1]-F_1^-[G_2]\right)\frac{\partial F_3^+}{\partial V_j^+} - \left(F_1^+F_2^- - F_1^-F_2^+\right)\frac{\partial G_3^+}{\partial V_j^+}. \qquad (5.56)$$

Notice that $\lambda^+ = u^+\hat{\lambda}^+$, so the matrix $\partial_W(uF) - \lambda\partial_W G$, is written at (V^+,u^+) as:

$$\begin{pmatrix} u^+\left(\frac{\partial F_1^+}{\partial V_1^+} - \hat{\lambda}^+\frac{\partial G_1^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_1^+}{\partial V_2^+} - \hat{\lambda}^+\frac{\partial G_1^+}{\partial V_2^+}\right) & F_1^+ \\ u^+\left(\frac{\partial F_2^+}{\partial V_1^+} - \hat{\lambda}^+\frac{\partial G_2^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_2^+}{\partial V_2^+} - \hat{\lambda}^+\frac{\partial G_2^+}{\partial V_2^+}\right) & F_2^+ \\ u^+\left(\frac{\partial F_3^+}{\partial V_1^+} - \hat{\lambda}^+\frac{\partial G_3^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_3^+}{\partial V_2^+} - \hat{\lambda}^+\frac{\partial G_3^+}{\partial V_2^+}\right) & F_3^+ \end{pmatrix} \qquad (5.57)$$

Setting $\hat{v}^+(V^-;V^+) = \hat{\lambda}^+(V^+)$ in (5.57) and assuming that $\mathcal{D}_{ij}(V^-,V^+) = F_i^-[G_j] - F_j^-[G_i] \neq 0$ for all $\{i,j\}\in\mathbb{P}$, we use Eqs. (5.23) and (5.25) to write an equivalent but convenient expression for $\hat{v}^+$ in each matrix element, obtaining:

$$\begin{pmatrix} u^+\left(\frac{\partial F_1^+}{\partial V_1^+} - \frac{F_3^-F_2^+-F_3^+F_2^-}{F_3^-[G_2]-F_2^-[G_3]}\frac{\partial G_1^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_1^+}{\partial V_2^+} - \frac{F_3^-F_2^+-F_3^+F_2^-}{F_3^-[G_2]-F_2^-[G_3]}\frac{\partial G_1^+}{\partial V_2^+}\right) & F_1^+ \\ u^+\left(\frac{\partial F_2^+}{\partial V_1^+} - \frac{F_3^+F_1^--F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_2^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_2^+}{\partial V_2^+} - \frac{F_3^+F_1^--F_3^-F_1^+}{F_1^-[G_3]-F_3^-[G_1]}\frac{\partial G_2^+}{\partial V_2^+}\right) & F_2^+ \\ u^+\left(\frac{\partial F_3^+}{\partial V_1^+} - \frac{F_1^+F_2^--F_1^-F_2^+}{F_2^-[G_1]-F_1^-[G_2]}\frac{\partial G_3^+}{\partial V_1^+}\right) & u^+\left(\frac{\partial F_3^+}{\partial V_2^+} - \frac{F_1^+F_2^--F_1^-F_2^+}{F_2^-[G_1]-F_1^-[G_2]}\frac{\partial G_3^+}{\partial V_2^+}\right) & F_3^+ \end{pmatrix} \qquad (5.58)$$

Since $\partial\mathcal{H}_e/\partial V_j^+ = 0$ for $j=1,2$, from Eq (5.56), it follows that for ℓ given by

$$\ell = \left(F_3^-[G_2]-F_2^-[G_3],\;\; F_1^-[G_3]-F_3^-[G_1],\;\; F_2^-[G_1]-F_1^-[G_2]\right), \qquad (5.59)$$

the inner products of columns 1 and 2 by ℓ are zero. Since at (V^-,V^+) the expression $\mathcal{H}_e(V^-)$ vanishes, after some calculations, we obtain:

$$\ell\cdot(F_1^+,F_2^+,F_3^+) = 0.$$

First let us consider the case $\mathcal{D}_{ij}(V^-,V^+)\neq 0$ for all $\{i,j\}\in\mathbb{P}$, so it is clear that ℓ is a left eigenvector of the system. One can prove that all eigenvectors for $\lambda(V^+) = v^s(V^-,V^+)$ are parallel to ℓ. Notice that $\ell \cdot ([G_1], [G_2], [G_3] = 0$, because this equality satisfies the RH locus $\mathcal{H}_e = 0$, with $\mathcal{H}_e$ given by (5.22).

If $\mathcal{D}_{ij}(V^-, V^+) = 0$ for a pair $\{i, j\} \in \mathbb{P}$, using Lemma 5.1, the relationships $F_i^+ F_j^- - F_j^+ F_i^- = 0$ or $\mathcal{D}_{ij}(V^-, V^+) = 0$ for all $\{i, j\} \in \mathbb{P}$ are satisfied.

Assume that $\mathcal{D}_{ij}(V^-, V^+) \neq 0$ for some $\{i, j\} \in \mathbb{P}$. For concreteness we set $i = 3$ and $j = 2$; the other cases can be proved similarly. Since $F_1^-[G_3] - F_3^-[G_1] \neq 0$ and the matrix (5.58) has the form (a_{ij}) for $i, j = 1, 2, 3$, and we can write a_{11}, a_{12} and a_{13}, respectively as:

$$u^+ \left(\frac{\partial F_1^+}{\partial V_1^+} - \frac{F_3^+ F_1^- - F_3^- F_1^+}{F_1^-[G_3] - F_3^-[G_1]} \frac{\partial G_1^+}{\partial V_1^+} \right), \qquad u^+ \left(\frac{\partial F_1^+}{\partial V_2^+} - \frac{F_3^+ F_1^- - F_3^- F_1^+}{F_1^-[G_3] - F_3^-[G_1]} \frac{\partial G_1^+}{\partial V_2^+} \right) \quad \text{and} \quad F_1^+.$$

Substituting ℓ in (5.59) by the the following vector:

$$\ell = \big(0, \;\; F_1^-[G_3] - F_3^-[G_1], \;\; F_2^-[G_1] - F_1^-[G_2]\big)\,, \tag{5.60}$$

it is easy to prove that this vector ℓ is in the kernel of the transpose of the matrix (5.58).

Finally assume that $\mathcal{D}_{ij} = 0$ for all $\{i, j\} \in \mathbb{P}$. Since ℓ is a left eigenvector of the matrix (5.58) it follows that $\ell \cdot (F_1^+, F_2^+, F_3^+) = 0$. From Eq. (5.40.$a$) in the Proposition 5.4, we know that $([G_1], [G_2], [G_3]) = (\rho_1 F_1^+, \rho_1 F_2^+, \rho_1 F_3^+)$ for any constant $\rho_1 \in \mathbb{R}$, so:

$$\ell \cdot [G] = \ell \cdot (\rho_1 F^+) = \rho_1 \ell \cdot F^+ = 0.$$

The converse can be proved similarly by reversing the calculations. $\square$

5.4.2. *Coincidence loci.* There are two important types of speed coincidence: coincidence between eigenvalues and coincidence between eigenvalues and shock speeds. These structures are important because the Riemann solution changes when the L (or R) data is prescribed in different regions relatively to these curves.

Coincidence between eigenvalues in each physical situation. Notice that it is possible to analyze the system in the variables V without taking into account the Darcy speed u. At the coincidence locus between eigenvalues the Darcy speeds also coincide, so the coincidence in the V space consists of the states in a certain physical situation such that

$$\hat{\lambda}^1(V) = \hat{\lambda}^2(V).$$

5.4.3. *Double contact curves.* It consists of the states V^- for which there is a state V^+ such that the shock joining V^- and V^+ has speed coinciding with the characteristic speed of family i at $(-)$ and of family j at $(+)$:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \hat{\lambda}^{i,-}(V^-) = \hat{v}^-(V^-, V^+) \quad \text{and} \quad \hat{v}^+(V^-, V^+) = \hat{\lambda}^{j,+}(V^+).$$

Notice that V^- and V^+ can be in the same or in different physical situations.

5.4.4. *Inflection curves.* The rarefaction curves are useful to construct rarefaction waves where the eigenvalue varies monotonically; the inflection curve is the curve where the monotonicity fails, thus rarefaction curves stop at this curve. Since u varies along the rarefaction wave, it is impossible to disregard the effect of u. Any state $W = (V, u)$ on the inflection curve satisfies:

$$\nabla \lambda^i(W) \cdot r^i(W) = 0. \tag{5.61}$$

However, the Darcy speed can be isolated in Eq. (5.61), so it is easy to prove that in the space of primary variables, the inflection curve of family i, for $i = 1, 2$, consists of the states V satisfying the equation:

$$\nabla_V \hat{\lambda}^i \cdot \hat{r} = -\hat{\lambda}^i \eta_3, \tag{5.62}$$

where $\eta_i(V)$ for $i = 1, 2, 3$ are the coordinates for the eigenvector r given in Eq. (5.7.b).

5.4.5. *Interior boundary contact (extension of the boundary).* Because of the presence of physical region boundaries it is important to obtain the states V joined to points V' on the physical boundary by shock waves that are characteristic at V for the family i. These states satisfy:

$$V \in \mathcal{RH}(V') \quad \text{with } V' \text{ on the boundary and } \hat{\lambda}^i(V) = \hat{v}^-(V, V').$$

5.5. **Admissibility of shocks.** Frequently, for a fixed state V^- there are points in the RH curve of V^- that separate potentially admissible shocks from non-admissible shocks. At these points the shock speed coincides with the characteristic speed and the shock speed has an extremum, see Proposition 5.5, so the Riemann solution usually changes. There are two type of such shocks: left characteristic and right characteristic shocks. In this structure, the shock has speed equal to the characteristic speed at the left or at the right state.

5.5.1. *Left characteristic shock.* For a fixed state V^-, these points are the V^+ such that:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \hat{v}^+(V^-; V^+) = \hat{\lambda}^{j,+}(V^+) \quad \text{for family } j = 1 \text{ or } 2.$$

Notice that V^- and V^+ can be in the same or different physical situations.

5.5.2. *Right characteristic shock.* For a fixed V^+, these points are the V^- such that:

$$V^- \in \mathcal{RH}(V^+) \quad \text{with} \quad \hat{v}^-(V^-, V^+) = \hat{\lambda}^{j,+}(V^+) \quad \text{for family } j = 1 \text{ or } 2.$$

Notice that V^- and V^+ can be in the same or in different physical situations.

6. Elementary waves in the single-phase gaseous situation

6.1. **Characteristic speed analysis.**

We simplify the system (4.4)-(4.6) by substituting (4.4) by (4.4) divided by θ_W and (4.4) by a certain linear combination of (4.4) with (4.5):

$$\frac{\partial}{\partial t}\varphi\psi_{gw}T^{-1} + \frac{\partial}{\partial x}u\psi_{gw}T^{-1} = 0, \tag{6.1}$$

$$\frac{\partial}{\partial t}\varphi T^{-1} + \frac{\partial}{\partial x}uT^{-1} = 0, \tag{6.2}$$

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + \psi_{gw}H_{gW} + \psi_{gn}H_{gN}\right) + \frac{\partial}{\partial x}u(\psi_{gw}H_{gW} + \psi_{gn}H_{gN}) = 0, \tag{6.3}$$

where we have used the equality $\psi_{gn} = 1 - \psi_{gw}$.

We differentiate Eqs. (6.1)-(6.3) and we rewrite it in the form (5.2) with $V = (\psi_{gw}, T)$ as:

$$B\frac{\partial}{\partial t}\begin{pmatrix} \psi_{gw} \\ T \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} \psi_{gw} \\ T \\ u \end{pmatrix} = 0, \tag{6.4}$$

where (indicating the differentiation relative to temperature by $'$):

$$B = \varphi\begin{pmatrix} T^{-1} & -\psi_{gw}T^{-2} & 0 \\ 0 & -T^{-2} & 0 \\ (H_{gW} - H_{gN}) & \hat{C}_r + \psi_{gw}H'_{gW} + \psi_{gn}H'_{gN} & 0 \end{pmatrix}, \tag{6.5}$$

$$A = \begin{pmatrix} uT^{-1} & -u\psi_{gw}T^{-2} & \psi_{gw}T^{-1} \\ 0 & -uT^{-2} & T^{-1} \\ u(H_{gW}-H_{gN}) & u(\psi_{gw}H'_{gW}+\psi_{gn}H'_{gN}) & \psi_{gw}H_{gW}+\psi_{gn}H_{gN} \end{pmatrix}; \tag{6.6}$$

in (6.5) the constant $\hat{C}_r$ is the rock heat capacity divided by φ, see Appendix A.

Using (6.5) and (6.6), we notice that $(u-\varphi\lambda)$ is repeated in the first column of $A-\lambda B$. Factoring this term, the determinant of $A-\lambda B$ is $(u-\varphi\lambda)$ times the determinant of

$$\begin{pmatrix} T^{-1} & -\psi_{gw}T^{-2}(u-\varphi\lambda) & \psi_{gw}T^{-1} \\ 0 & -(u-\varphi\lambda)T^{-2} & T^{-1} \\ (H_{gW}-H_{gN}) & (u-\varphi\lambda)(\psi_{gw}H'_{gW}+\psi_{gn}H'_{gN})-\varphi\lambda\hat{C}_r & \psi_{gw}H_{gW}+\psi_{gn}H_{gN} \end{pmatrix}, \tag{6.7}$$

so an eigenvalue and eigenvector for the system (6.4) are

$$\lambda_c = u/\varphi, \qquad r_c = (1,0,0)^T, \tag{6.8}$$

which correspond to fluid transport and the wave for this eigenvalue is a contact discontinuity. The composition ψ_{gw} changes but the speed and the temperature are constant. There is a single vector for λ_c, because the matrix (6.7) with $\lambda=\lambda_c$ has rank 3, generically. We denote this eigenvalue by λ_c to indicate that in the associated wave only the composition changes.

We find the other characteristic speed λ in (6.7). If $\psi_{gw}\neq 0$, we add the first line to the second line multiplied by $-\psi_{gw}$ obtaining:

$$\begin{pmatrix} T^{-1} & 0 & 0 \\ T^{-1} & -(u-\varphi\lambda)T^{-2} & T^{-1} \\ (H_{gW}-H_{gN}) & (u-\varphi\lambda)(\psi_{gw}H'_{gW}+\psi_{gn}H'_{gN})-\varphi\lambda\hat{C}_r & \psi_{gw}H_{gW}+\psi_{gn}H_{gN} \end{pmatrix}. \tag{6.9}$$

The determinant of (6.9) is $-T^{-2}$ times

$$\Big[(u-\varphi\lambda)(\psi_{gw}H'_{gW}+\psi_{gn}H'_{gN})-\varphi\lambda\hat{C}_r+(u-\varphi\lambda)T^{-1}(\psi_{gw}H_{gW}+\psi_{gn})H_{gN}\Big]. \tag{6.10}$$

Setting (6.10) to zero we obtain:

$$\lambda_T = \frac{u}{\varphi}\left(1-\frac{\hat{C}_r T}{\mathbb{F}}\right), \tag{6.11}$$

where using (4.7), $\mathbb{F} := \mathbb{F}(T)$ is given by:

$$\mathbb{F}(T) = \Big(\psi_{gw}\rho_{gW}(T)h'_{gW}(T)+\psi_{gn}\rho_{gN}(T)h'_{gN}(T)+\hat{C}_r\Big)\,T. \tag{6.12}$$

As ρ_{gW}, ρ_{gN}, h'_{gW}, h'_{gN} and $\hat{C}_r$ are positive and ψ_{gw} and ψ_{gn} are non-negative, $\mathbb{F}(T)$ is always positive. The eigenvector associated to λ_T is

$$r_T = (0,\mathbb{F},u\hat{C}_r). \tag{6.13}$$

This eigenvector is unique, because substituting λ_T in (6.9) we see that the resulting matrix has rank 2. The rarefaction wave has constant composition ψ_{gw}. Notice that $\lambda_T < \lambda_c$ in the physical range, so we can order these waves: the slowest wave is a thermal wave, with speed λ_T; the fastest is a compositional wave, with speed λ_c.

6.1.1. *Behavior of the gaseous thermal inflection curve.* To obtain the rarefaction curves in the *spg*, we need to study the sign of $\nabla\lambda_T \cdot r_T$. After a lengthly calculation, we obtain from (6.11) and (6.13)

$$\nabla\lambda_T \cdot r_T = \frac{uC_rT^2}{\varphi}\Xi(\psi_{gw},T), \quad \text{where} \quad \Xi = \psi_{gw}\rho_{gW}h''_{gw} + (1-\psi_{gw})\rho_{gN}h''_{gn}. \tag{6.14}$$

As u, $\hat{C}_r$, T and φ are positive, we need to study the sign of Ξ.

The gaseous thermal inflection curve is denoted by $\mathcal{I}_T$; for Γ given by (4.9) it is defined by:

$$\mathcal{I}_T = \{(T,\psi_{gw}) \in \Gamma \text{ that satisfy } \nabla\lambda_T \cdot r_T = 0\}\,. \tag{6.15}$$

We plot the physical region and $\mathcal{I}_T$ in Figure (6.1.*a*), showing the signs of Ξ. In Figure 6.1.*b*, we plot the horizontal rarefaction lines associated to λ_c and the vertical rarefaction lines associated to λ_T, see (6.13). The waves associated to λ_c are contact discontinuities.

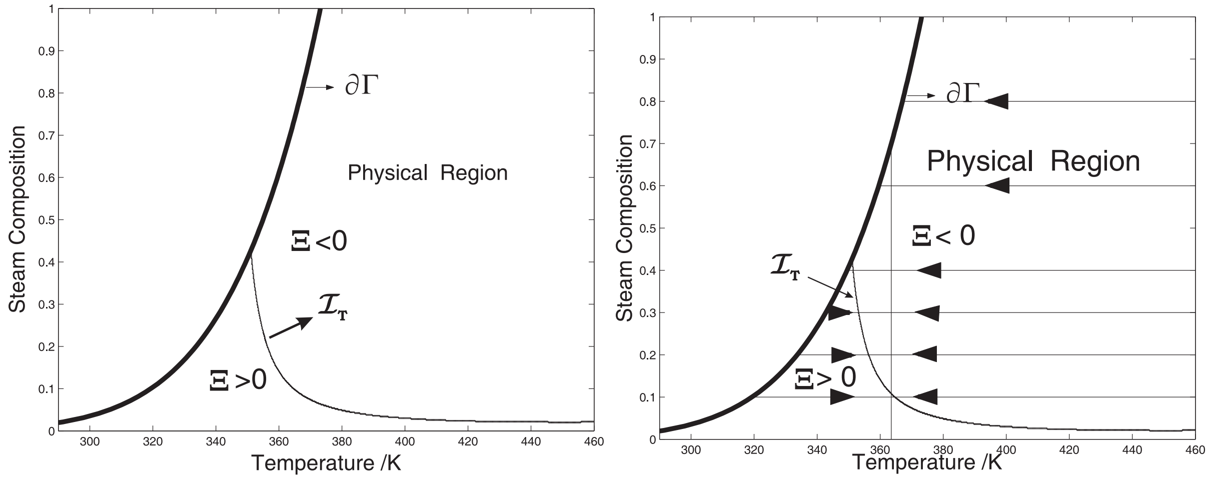

FIGURE 6.1. *a)-left:* The single-phase gaseous physical region Γ, and inflection locus. *b)-right:* Rarefaction curves. The horizontal rarefaction curves are associated to λ_T; we indicate with an arrow the direction of increasing speed. The vertical lines are contact discontinuity curves associated to λ_c, in which ψ_{gw} changes, T and u are constant.

6.2. **Shocks and contact discontinuities.** Using (5.15) in Eqs. (6.1)-(6.3), the RH condition is:

$$v^s\varphi\left(\psi^+_{gw}/T^+ - \psi^-_{gw}/T^-\right) = u^+\psi^+_{gw}/T^+ - u^-\psi^-_{gw}/T^-, \tag{6.16}$$

$$v^s\varphi\left(1/T^+ - 1/T^-\right) = u^+/T^+ - u^-/T^-, \tag{6.17}$$

$$\begin{aligned} v^s\varphi\big(H^+_r + \psi^+_{gw}H^+_{gW} + (1-\psi^+_{gw})H^+_{gN} - (H^-_r + \psi^-_{gw}H^-_{gW} + (1-\psi^-_{gw})H^-_{gN})\big) = \\ u^+(\psi^+_{gw}H^+_{gW} + (1-\psi^+_{gw})H^+_{gN}) - u^-(\psi^-_{gw}H^-_{gW} + (1-\psi^-_{gw})H^-_{gN}), \end{aligned} \tag{6.18}$$

where $H^\pm_{gW} = \theta_W h_{gW}(T^\pm)/T^\pm$ and $H^\pm_{gN} = \theta_N h_{gN}(T^\pm)/T^\pm$.

In (6.16)-(6.18), there is a contact discontinuity with speed (6.8); the Darcy speed and the temperature are constant on this wave, i.e., $u = u^-$ and $T = T^-$.

The other shock occurs with ψ_{gw} constant; it is a thermal shock with speed v^s and Darcy speed u^+ on the right side given by:

$$v^s = \frac{u^-}{T^-}\left\{\Upsilon(T^-;T^+)(T^+ - T^-) + T^+\right\} \qquad \text{and} \qquad u^+ = \frac{u^-}{\varphi}\Upsilon(T^-;T^+),$$

where Υ is defined as follows, with θ_W, θ_N, H_{gW} and H_{gN} given by Eq. (4.7):

$$\Upsilon = \frac{\psi_{gw}\theta_W\left(h^+_{gW} - h^-_{gW}\right) + (1-\psi_{gw})\theta_N\left(h^+_{gN} - h^-_{gN}\right)}{T^-\left(\hat{H}^+_r - \hat{H}^-_r\right) + \psi_{gw}\theta_W\left(h^+_{gW} - h^-_{gW}\right) + (1-\psi_{gw})\theta_N\left(h^+_{gN} - h^-_{gN}\right)}. \tag{6.19}$$

The RH and rarefaction curves are contained in the horizontal lines in Figure (6.1.b).

7. Contact discontinuities in the single-phase liquid situation

Eq. (4.14) is linear in the *spl*, so a unique wave is associated to λ^W_T, given by (4.14.b). This wave is a contact discontinuity and there is no genuine rarefaction wave.

Remark 7.1. *In the spl there exists a cooling discontinuity between a state with temperature T^- and another state with temperature T^+. For the Riemann data: T^- if $x < 0$ and T^+ if $x > 0$, the solution is T^- if $x/t < \lambda^W_T$ and T^+ if $x/t > \lambda^W_T$.*

8. Elementary waves in the two-phase situation

8.1. **Characteristic speed analysis.** By means of a procedure similar to that in Section 5.1, we write the system (4.10)-(4.12) in the form (5.2), with $V = (s_g, T)$, as:

$$B\frac{\partial}{\partial t}\begin{pmatrix} s_g \\ T \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} s_g \\ T \\ u \end{pmatrix} = 0, \tag{8.1}$$

with

$$B = \begin{pmatrix} \varphi(\rho_{gw} - \rho_W) & \varphi\rho'_{gw}s_g & 0 \\ \varphi\rho_{gn} & \varphi\rho'_{gn}s_g & 0 \\ \varphi(H_g - H_W) & \varphi(\hat{C}_r + C_W + s_g(C_g - C_W)) & 0 \end{pmatrix}, \tag{8.2}$$

$$A = \begin{pmatrix} u(\rho_{gw} - \rho_W)\frac{\partial f_g}{\partial s_g} & u\left((\rho_{gw} - \rho_W)\frac{\partial f_g}{\partial T} + \rho'_{gw}f_g\right) & \rho_W + f_g(\rho_{gw} - \rho_W) \\ u\rho_{gn}\frac{\partial f_g}{\partial s_g} & u\left(\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}f_g\right) & f_g\rho_{gn} \\ u(H_g - H_W)\frac{\partial f_g}{\partial s_g} & u\left(C_W + f_g(C_g - C_W) + (H_g - H_W)\frac{\partial f_g}{\partial T}\right) & f_g(H_g - H_W) + H_w \end{pmatrix}, \tag{8.3}$$

where $C_g = \partial H_g/\partial T$ and the constant C_W are the gas and water heat capacities given in Appendix A.

In order to use Eq. (5.3.b) for system (8.1), we calculate $A - \lambda B$ as:

$$\begin{pmatrix} \gamma_2\gamma_1 & u\gamma_2\frac{\partial f_g}{\partial T} + \rho'_{gw}\gamma_4 & \rho_W + f_g\gamma_2 \\ \rho_{gn}\gamma_1 & u\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}\gamma_4 & f_g\rho_{gn} \\ \gamma_3\gamma_1 & C_W(u - \lambda\varphi) + (C_g - C_W)\gamma_4 - \varphi\lambda\hat{C}_r + u\gamma_3\frac{\partial f_g}{\partial T} & f_g\gamma_3 + H_w \end{pmatrix}, \tag{8.4}$$

where we have defined

$$\gamma_1 = \gamma_1(s, T, u) = u\frac{\partial f_g}{\partial s_g} - \lambda\varphi \qquad \text{and} \qquad \gamma_4 := \gamma_4(s, T, u) = uf_g - \lambda\varphi s_g, \tag{8.5}$$

$$\gamma_2 := \gamma_2(T) = \rho_{gw} - \rho_W, \qquad \text{and} \qquad \gamma_3 := \gamma_3(T) = H_g - H_W. \tag{8.6}$$

Setting $\gamma_1 = 0$ in (8.4), its determinant vanishes, and we find the Buckley-Leverett characteristic speed and characteristic vector:

$$\lambda_s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}, \qquad r_s = (1,0,0)^T, \tag{8.7}$$

On this rarefaction curve T and u are constant, only s_g changes, hence the subscript s in λ_s and r_s is of a saturation wave. The eigenvector associated to the eigenvalue (8.7) is unique: substituting λ_s in (8.4) leads to a matrix with rank 2, generically.

We factor γ_1 in first column, for $\gamma_1 \neq 0$. The other eigenvalue of (8.4) is the root of

$$det\begin{pmatrix} \gamma_2 & u\frac{\partial f_g}{\partial T}\gamma_2 + \rho'_{gw}\gamma_4 & \rho_W + f_g\gamma_2 \\ \rho_{gn} & u\rho_{gn}\frac{\partial f_g}{\partial T} + \rho'_{gn}\gamma_4 & f_g\rho_{gn} \\ \gamma_3 & C_W(u-\lambda\varphi) + \gamma'_3\gamma_4 - \varphi\lambda\hat{C}_r + u\frac{\partial f_g}{\partial T}\gamma_3 & f_g\gamma_3 + H_W \end{pmatrix} = 0. \tag{8.8}$$

We consider first the case $\rho_{gn} \neq 0$, the case $\rho_{gn} = 0$ is considered later. We divide the second row by ρ_{gn} and we obtain the eigenvalue:

$$\lambda_e = \frac{u}{\varphi}\frac{f_g\left[(\gamma'_3 - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_3)\rho_W - H_W(\rho'_{gw} - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_2)\right] + C_W\rho_W}{\rho_W(C_W + \hat{C}_r) + s_g\left[(\gamma'_3 - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_3)\rho_W - H_W(\rho'_{gw} - \left(\rho'_{gn}/\rho_{gn}\right)\gamma_2)\right]}, \tag{8.9}$$

with eigenvector

$$r_e = (-\varpi_1, \bar{\gamma}_1, -u\varpi_3)^T, \tag{8.10}$$

where

$$\bar{\gamma}_1 = \frac{\partial f_g}{\partial s_g} - \frac{f_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right] + C_W\rho_W\rho_{gn}}{\rho_W\rho_{gn}(C_W + \hat{C}_r) + s_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right]}, \tag{8.11}$$

$$\bar{\gamma}_4 = \frac{C_W(f_g - s_g)\rho_W\rho_{gn} + \hat{C}_r f_g\rho_W\rho_{gn}}{\rho_W\rho_{gn}(C_W + \hat{C}_r) + s_g\left[(\gamma'_3\rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right]}, \tag{8.12}$$

$$\varpi_1 = \bar{\gamma}_4\left(\frac{\left(\rho'_{gn}\right)\rho_W - f_g(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)}{\rho_{gn}\rho_W}\right) + \frac{\partial f_g}{\partial T}, \tag{8.13}$$

$$\varpi_3 = \frac{\bar{\gamma}_1\bar{\gamma}_4\left(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2\right)}{\rho_W\rho_{gn}}, \tag{8.14}$$

and γ_2 is given in (8.6) .

In Figs. (8.3.a) and (8.3.b) we can see that in the region where $\lambda_s > \lambda_e$, along the rarefaction waves the temperature, gaseous water saturation and u increase; and in the region where $\lambda_s < \lambda_e$ along the rarefaction waves the temperature and u increase and the gaseous water saturation decreases. We utilize the subscript e in λ_e, r_e to indicate that this eigenpair is associated to an evaporation wave.

8.1.1. *Coincidence curve.* The coincidence curve between λ_e and λ_s is:

$$\mathbb{C}_{s,e} = \{(T, s_g) \in tp \text{ such that } \lambda_s(T, s_g) = \lambda_e(T, s_g)\}\,. \tag{8.15}$$

Differentiating λ_e in (8.9) with respect to s_g, equating it to zero and isolating $\partial f_g/\partial s_g$, we obtain:

Lemma 8.1. *On the coincidence curve $\mathbb{C}_{s,e}$, the derivative $\partial\lambda_e/\partial s_g$ vanishes.*

Thus coincidence between eigenvalues occurs where λ_e is a stationary. As in Prop. 6.4 in [9], we can prove the following Proposition:

Proposition 8.1. *Lines with fixed T intercept the coincidence curve $\mathbb{C}_{s,e}$ twice. At the lowest value of s_g, λ_e has a minimum and at the higher value λ_e has a maximum.*

Thus, in the tp, when the temperature T varies two branches of the coincidence curve are generated as sketched in Figure (8.2.a).

Proof: We verify this proposition geometrically. First, we define

$$\mathbb{A} = -\left(\left((H_W - H_g)'\rho_{gn} - \rho'_{gn}(H_W - H_g)\right)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}(\rho_{gw} - \rho_W))\right). \tag{8.16}$$

One can verify that $\mathbb{A} > 0$ for all T in the tp physical range. We multiply and divide λ_e from Eq. (8.9) by $\mathbb{A}$ obtaining:

$$\lambda_e = \frac{u}{\varphi}\frac{f_g - f_e}{s_g - s_e}, \quad \text{where} \quad f_e \equiv \frac{\rho_W\rho_{gn}C_W}{\mathbb{A}} \quad \text{and} \quad s_e \equiv \frac{\rho_W\rho_{gn}\left(C_W + \hat{C}_r\right)}{\mathbb{A}}. \tag{8.17}$$

For the coincidence points with fixed u, it follows from (8.7):

$$\frac{\partial f_g}{\partial s_g} = \frac{f_g - f_e}{s_g - s_e}. \tag{8.18}$$

Eq. (8.18) defines the tangency points of the secant from the point (s_e, f_e) to the graph of f_g. One can prove that $s_e > 1$ and $f_e > 1$. In Figure 8.1, we plot a possible point (s_e, f_e). At the first point where λ_e and λ_T coincide the speed is a minimum and at the second point the speed is maximum. $\square$

8.1.2. *Behavior of the two-phase evaporation inflection curve.* To obtain the evaporation rarefaction curves we need to study the sign of $\nabla\lambda_e \cdot r_e$. After a lengthly calculation, we get

$$\nabla\lambda_e \cdot r_e = \frac{u}{\varphi}\frac{\bar{\gamma}_1\bar{\gamma}_4}{s_g - s_e}\left(\mathbb{A}' - \mathbb{A}\left(\frac{\rho'_{gn}}{\rho_{gn}}\right) - C_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)\right), \tag{8.19}$$

where $\bar{\gamma}_1$, $\bar{\gamma}_4$ are defined in Eqs. (8.11) and (8.12). The expression for $\mathbb{A}$ is given by (8.16) and $\mathbb{A}'$ is its derivative with respect to temperature.

One can verify that $\mathbb{A}' - \mathbb{A}\rho'_{gn}/\rho_{gn} - C_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2) > 0$, so the points where $\nabla\lambda_e \cdot r_e$ vanishes satisfy:

$$\lambda_e = \lambda_s \qquad \text{or} \qquad \frac{f_g}{s_g} = \frac{C_W}{C_W + \hat{C}_r}. \tag{8.20}$$

We define the curve $\mathbb{I}_e$ as:

$$\mathbb{I}_e := \{(T, s_g) \in tp \;\text{ satisfying Eq. } (8.20.b)\}.$$

We denote the inflection curve in the tp of λ_e by $\mathcal{I}_e$; it is defined by:

$$\mathcal{I}_e = \{(T, s_g) \text{ satisfying } (8.20.a) \text{ and } (8.20.b)\} \tag{8.21}$$

The following Lemma follows from Eqs. (8.20) and (8.15).

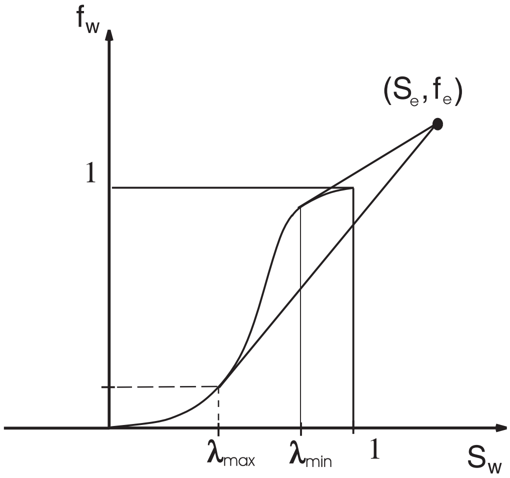

FIGURE 8.1. The coincidence between λ_e and λ_s occurs at the tangency point on the graph of f_g from (s_e, f_e). Since $\mathbb{A} > 0$, s_e and f_e are positive, the first coincidence point is a maximum while the second point is a minimum for λ_e.

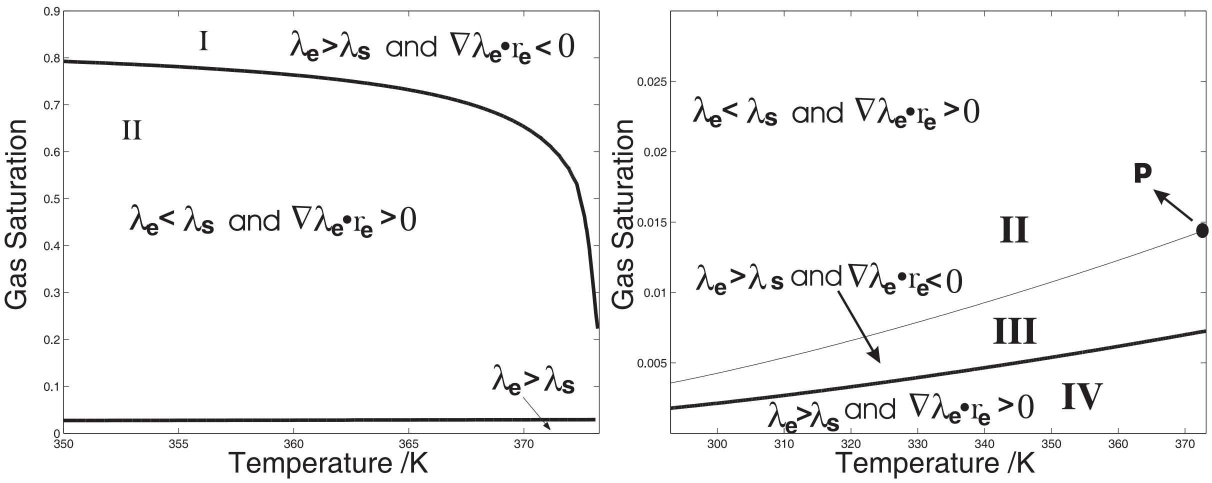

FIGURE 8.2. *a)-left:* Coincidence curve $\mathbb{C}_{s,e}$. Relative sizes of λ_s and λ_e in a piece of the *tp* state space. The almost horizontal coincidence curve $\lambda_e = \lambda_s$ is not in scale, because it is very close to the axis $S_g = 0$. *b)-right:* A zoom of the region below the coincidence curve. In both figures, all curves form the inflection curve, subdividing *tp* situation in four parts.

Lemma 8.2. *The coincidence curve* $\mathbb{C}_{s,e}$ *is contained in* $\mathcal{I}_e$*. Moreover* $\mathcal{I}_e = \mathbb{C}_{s,e} \bigcup \mathbb{I}_e$*.*

The solid curves in Fig. 8.2 are the $\mathcal{I}_e$, where we also plot the sign of $\nabla\lambda_e \cdot r_e$. In Figure 8.3, we draw rarefaction curves in the *tp*.

For the case $\rho_{gn} = 0$ but $\rho_{gn}^{'} \neq 0$, the matrix (8.8) reduces to

$$\begin{pmatrix} \gamma_2 & u\frac{\partial f_g}{\partial T}\gamma_2 + \rho_{gw}^{'}\gamma_4 & \rho_W + f_g\gamma_2 \\ 0 & \rho_{gn}^{'}\gamma_4 & 0 \\ \gamma_3 & C_W(u-\lambda\varphi) + \gamma_3^{'}\gamma_4 + \varphi\lambda\hat{C}_r + u\frac{\partial f_g}{\partial T}\gamma_3 & f_g\gamma_3 + H_W \end{pmatrix}. \tag{8.22}$$

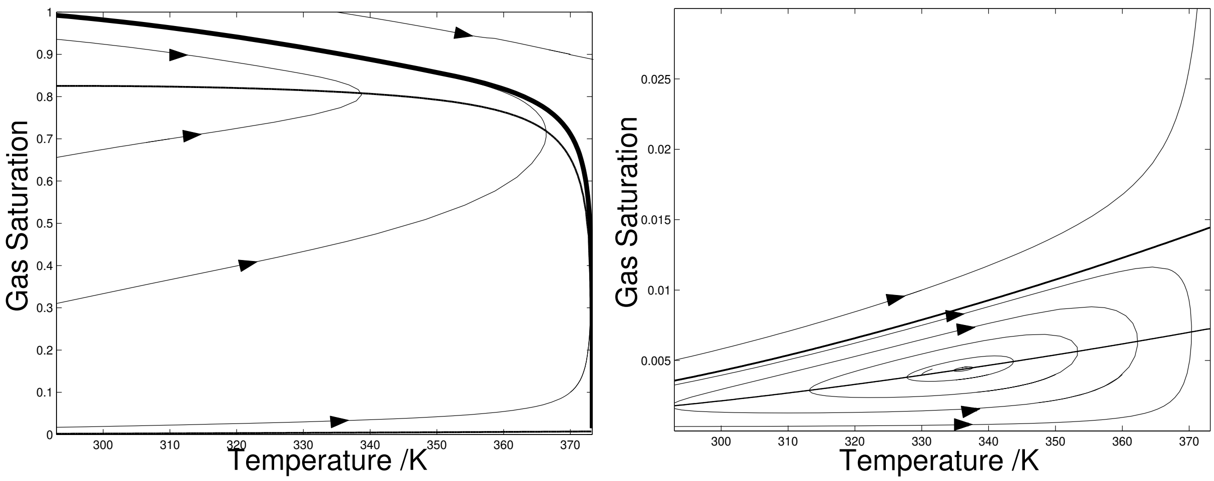

FIGURE 8.3. *a)-left:* The rarefaction curves projected in the plane T, s_g. The thin curves without arrows are $\mathbb{C}_{s,e}$. The bold curve indicates an invariant curve for the rarefaction field, which a rarefaction curve that reaches the point P in Fig. (8.2.b). *b)-right:* The rarefaction curves in the regions III and IV shown in the Figure (8.2.b). The arrows indicate the direction of increasing speed.

The eigenvalue and eigenvector are

$$\lambda_e\Big|_{\rho_{gn}=0} = \frac{u}{\varphi}\frac{f_g}{s_g}, \qquad r_e\Big|_{\rho_{gn}=0} = \left(-\bar{\gamma}_4\rho_{gn}^{'}\left(f_g\gamma_2+1\right), 1, \bar{\gamma}_4\rho_{gn}^{'}\gamma_2\right). \tag{8.23}$$

The case $\rho_{gn}=0$ and $\rho_{gn}^{'}=0$, with no nitrogen, is studied in [4, 8, 9].

8.2. **Shocks in the two-phase situation.** As in the *spg*, we use Eqs. (5.15) and (4.10)-(4.12) to obtain the RH condition:

$$v^s\varphi\left((s_W^+ - s_W^-)\rho_W + s_g^+\rho_{gw}^+ - s_g^-\rho_{gw}^-\right) = \left(u^+f_w^+ - f_w^-u^-\right)\rho_W + u^+f_g^+\rho_{gw}^+ - u^-f_g^-\rho_{gw}^-, \tag{8.24}$$

$$v^s\varphi\left(s_g^+\rho_{gn}^+ - s_g^-\rho_{gn}^-\right) = u^+f_g^+\rho_{gn}^+ - u^-f_g^-\rho_{gn}^-, \tag{8.25}$$

$$v^s\varphi\left(\hat{H}_r^+ - \hat{H}_r^- + H_W^+ - H_W^- + s_g^+(H_g^+ - H_W^+) - s_g^-(H_g^- - H_W^-)\right) = u^+(f_w^+H_W^+ + f_g^+H_g^+) - u^-(f_w^-H_W^- + f_g^-H_g^-). \tag{8.26}$$

The isothermal branch of the RH curve is obtained for constant temperature T, i.e., $T^+ = T^- = T$. In this case the shock speed is

$$v^s = \frac{u^-}{\varphi}\frac{f_w(s_w^+, T) - f_w(s_w^-, T)}{s_w^+ - s_w^-}.$$

On this wave, the Darcy speed is constant. This is the Buckley-Leverett shock; the RH and rarefaction curves associated to λ_s lie on the same straight line.

The non-isothermal wave is obtained after a lengthly calculation. We fix the left state and the temperature of the right state, with $T^+ \neq T^-$. Following [3], we can obtain the shock speed v^s and Darcy speed u^+ at the right independently of s_g^+. We rewrite (8.24)-(8.26) as:

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\left(\rho_{gw}^+ - \rho_W\right) = v^s\varphi s_g^-\left(\rho_{gw}^- - \rho_W\right) - u^-\left(f_w^-\rho_W + f_g^-\rho_{gw}^-\right) + u^+\rho_W, \tag{8.27}$$

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\rho_{gn}^+ = v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-, \tag{8.28}$$

$$\left(v^s\varphi s_g^+ - u^+ f_g^+\right)\left(H_g^+ - H_W^+\right) = v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^-\right) + u^+ H_W^+ - u^-(f_w^- H_W^- + f_g^- H_g^-), \tag{8.29}$$

where we have used $s_w = 1 - s_g$ and $f_w = 1 - f_g$.

As $\rho_W > \rho_{gw}$ and $H_W > H_g$ in the temperature physical range and we assume that $\rho_{gn} \neq 0$ (the term ρ_{gn} is zero only at the water boiling temperature), we add (8.27) divided by $\rho_W^+ - \rho_{gw}^+$ to (8.28) divided by ρ_{gn}^+. Similarly, we add (8.28) divided by ρ_{gn}^+ to (8.29) divided by $H_W^+ - H_g^+$ and we obtain:

$$\frac{v^s\varphi\left(s_g^-(\rho_{gw}^- - \rho_W)\right) - u^-\left(f_w^-\rho_W + f_g^-\rho_{gw}^-\right) + u^+\rho_W}{\rho_{gw} - \rho_W^+} = \frac{v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-}{\rho_{gn}^+}, \tag{8.30}$$

$$\frac{v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^-\right) + u^+ H_W^+ - u^-(f_w^- H_W^- + f_g^- H_g^-)}{H_g^+ - H_W^+} = \\ = \frac{v^s\varphi s_g^-\rho_{gn}^- - u^- f_g^-\rho_{gn}^-}{\rho_{gn}^+}. \tag{8.31}$$

From (8.30) and (8.31), we obtain v^s and u^+ in terms of s_g^-, T^-, u^- and T^+ as:

$$v^s = \frac{u^-}{\varphi}\frac{\rho_{gn}^+(\Pi_2 H_W^+ - \Pi_1\rho_W) - f_g^-\rho_{gn}^-(\rho_{gw}^+ H_W^+ - H_g^+\rho_W)}{\rho_{gn}^+(\Pi_4 H_W^+ - \Pi_3\rho_W) - s_g^-\rho_{gn}^-(\rho_{gw}^+ H_W^+ - H_g^+\rho_W)} \tag{8.32}$$

and

$$u^+ = u^-\left(\frac{v^s\varphi}{u^-}\left(s_g^-\frac{\rho_{gn}^-}{\rho_{gn}^+}(\rho_{gw}^+ - \rho_W) - \Pi_4\right) + \Pi_2 - f_g^-\frac{\rho_{gn}^-}{\rho_{gn}^+}(\rho_{gw}^+ - \rho_W)\right)/\rho_W, \tag{8.33}$$

where $\Pi_1(s_g^-, T^-)$, $\Pi_2(s_g^-, T^-)$, $\Pi_3(s_g^-, T^-; T^+)$ and $\Pi_4(s_g^-, T^-)$ are given by:

$$\Pi_1 = f_w^- H_W^- + f_g^- H_g^-, \qquad \Pi_2 = f_w^-\rho_W + f_g^-\rho_{gw}^-, \tag{8.34}$$

$$\Pi_3 = \hat{H}_r^- - \hat{H}_r^+ - H_W^+ + s_W^- H_W^- + s_g^- H_g^- \quad \text{and} \quad \Pi_4 = s_g^-(\rho_{gw}^- - \rho_W). \tag{8.35}$$

After finding v^s and u^+, we substitute them in (8.25) and divide the resulting equation by $u^+\rho_{gn}^+$, rearrange the terms, obtaining an implicit equation for s_g^+:

$$\frac{f_g(s_g^+, T^+) - f^*}{s_g^+ - s^*} = \hat{v}^s\varphi, \tag{8.36}$$

where

$$f^* = \hat{u}^-\left(\rho_{gn}^-/\rho_{gn}^+\right) f_g^- \quad \text{and} \quad s^* = \left(\rho_{gn}^-/\rho_{gn}^+\right) s_g^-,$$

with $\hat{v}^- = v^s/u^-$ and $\hat{u}^- = u^-/u^+$.

As v^s and u^+ do not depend on s_g^+, the solution of Eq. (8.36) is the intersection of the graph of $f_g(s_g^+)$ with a linear function of s_g^+. From the shape of the graph of f_g, we see that (8.36) can have one, two or three solutions. Each intersection point belongs to the

Rankine-Hugoniot curve $\mathcal{RH}(V^-)$. When T^+ varies each intersection generates a branch of the Rankine-Hugoniot curve.

This is called the non-isothermal Rankine-Hugoniot curve; some examples are shown in Figure 8.4.

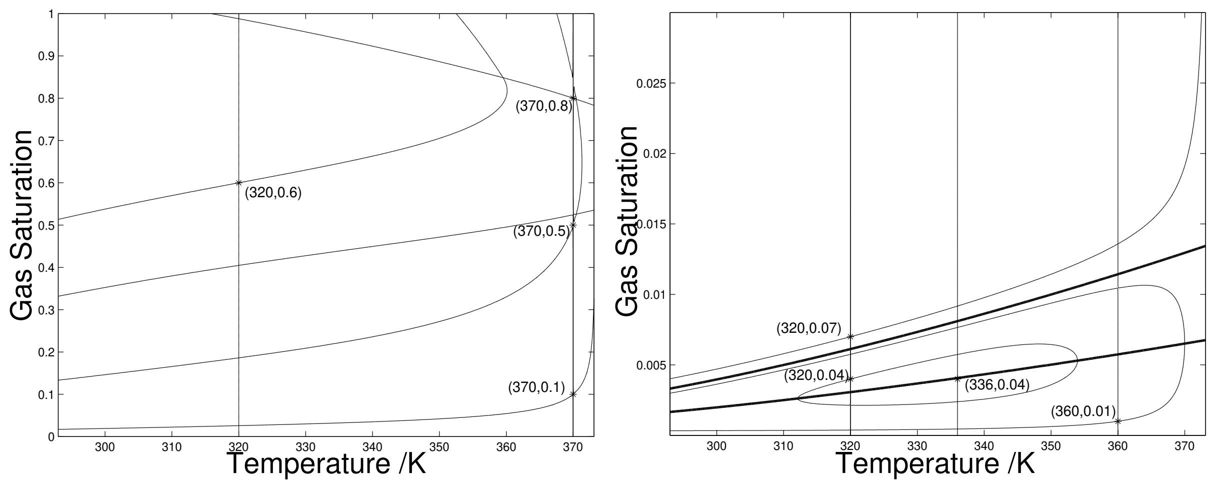

FIGURE 8.4. *a)-left:* The RH curves projected in the plane T, s_g for $(-)$ points marked. *b)- right:* The region shown in Figure (8.2.b). Each RH locus is formed by a non-isothermal curve and a vertical line which is the isothermal Buckley-Leverett RH locus. For the point $(336, 0.04)$ the Rankine-Hugoniot locus reduces to the isothermal curve only.

9. Shocks between regions

We consider now shocks that separate physical situations. In Section 5, we gave the RH condition (5.15) for shocks between regions. The main feature is that the accumulation and flux terms have different expressions at each shock side.

9.1. Shock between the gaseous and the two-phase situations.

For the left state we need to specify steam composition ψ_{gw}^-, the temperature T^- and the Darcy speed u^-. For the right state, we specify the temperature T^+; the steam composition is a function of temperature (see (4.3)) and the Darcy speed is obtained from the other variables. The RH condition (5.15) is written as:

$$v^s\varphi\left(\rho_W S_w^+ + \rho_{gw}^+ s_g^+ - \psi_{gw}^-\rho_{gW}^-\right) = u^+\left(\rho_W f_w^+ + \rho_{gw}^+ f_g^+\right) - u^-\psi_{gw}^-\rho_{gW}^-, \tag{9.1}$$
$$v^s\varphi\left(\rho_{gn}^+ s_g^+ - \psi_{gn}^-\rho_{gN}^-\right) = u^+\rho_{gn}^+ f_g^+ - u^-\psi_{gn}^-\rho_{gN}^-, \tag{9.2}$$
$$v^s\varphi\left(\hat{H}_r^+ - \hat{H}_r^- + H_w^+ s_w^+ + H_g^+ s_g^+ - \hat{H}_g^-\right) = u^+\left(H_w^+ f_w^+ + H_g^+ f_g^+\right) - u^-\hat{H}_g^-, \tag{9.3}$$

where $\hat{H}_g^- = \psi_{gw}^- H_{gW}^- + \psi_{gn}^- H_{gN}^-$ with H_{gW} and H_{gN} given by Eq. (4.7).

The system (9.1)-(9.3) is similar to (8.24)-(8.26). As in that case, we eliminate the saturation and we obtain two linear equations in two unknowns v^s and u^+:

$$\frac{v^s\varphi\left(\psi_{gw}^-\rho_{gW}^- - \rho_W\right) - u^-\psi_{gw}^-\rho_{gW}^- + u^+\rho_W}{\rho_{gw}^+ - \rho_W} = \frac{\left(v^s\varphi - u^-\right)\psi_{gn}^-\rho_{gN}^-}{\rho_{gn}^+}, \tag{9.4}$$

$$\frac{v^s\varphi\left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + \hat{H}_g^-\right) + u^+H_W^+ - u^-\hat{H}_g^-}{H_g^+ - H_W^+} = \frac{\left(v^s\varphi - u^-\right)\psi_{gn}^-\rho_{gN}^-}{\rho_{gn}^+}, \tag{9.5}$$

We obtain expressions for v^s and u^+ as functions of the left state (ψ_{gw}^-, T^-, u^-) and T^+:

$$v^s = \frac{u^-}{\varphi}\frac{\psi_{gn}^-\rho_{gN}^-\left(\rho_{gw}^+H_W^+ - H_g^+\rho_W\right) - \rho_{gn}^+\left(\psi_{gw}^-\rho_{gW}^-H_W^+ - H_g^+\rho_W\right)}{\Xi_1\rho_W - \Xi_2 H_W^+}, \tag{9.6}$$

$$u^+ = \frac{u^-\left(\psi_{gw}^-\rho_{gW}^-\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-(\rho_{gw}^+ - \rho_W)\right) - v^s\varphi\Xi_2}{\rho_W\rho_{gn}^+}, \tag{9.7}$$

where Ξ_1 and Ξ_2 are given by:

$$\Xi_1 = \left(\hat{H}_r^- - \hat{H}_r^+ - H_W^+ + \hat{H}_g^-\right)\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\gamma_2^+, \quad \Xi_2 = \left(\psi_{gw}^-\rho_{gW}^- - \rho_W\right)\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\gamma_2^+.$$

After finding v^s and u^+, we use Eq. (9.2) to obtain an equation similar to (8.36) for s_g^+:

$$v^s\varphi = \frac{f_g(s_g^+, T^+) - f^\S}{s_g^+ - s^\S}, \tag{9.8}$$

where

$$s^\S = \psi_{gn}^-\rho_{gN}^-/\rho_{gn}^+ \qquad \text{and} \qquad f^\S = \hat{u}^-\psi_{gn}^-\rho_{gN}^-/\rho_{gn}^+.$$

Geometrically, Eq. (9.8) represents tangency points of the secant from the point $(s^\S, f^\S)$ to the graph f_g for fixed T^+ with slope $v^s\varphi$.

9.1.1. *The Secondary Bifurcation.* The secondary bifurcation consists of the pairs of states for which the Prop. 5.5 fails, see Sec. 5.4. The definition of secondary bifurcation is given by Eqs. (5.53)-(5.54). From Eq. (9.8), we define for T^-, ψ_{gw}^- in the *spg* and T^+, s_g^+ in the *tp*:

$$F(T^-, \psi_{gw}^-, u^-; T^+, s_g^+, u^+) = u^+\rho_{gn}^+f_g(s_g^+, T^+) - v^s\varphi\left(s_g^+\rho_{gn}^+ - \psi_{gn}^-\rho_{gN}^-\right) + u^-\psi_{gn}^-\rho_{gN}^-. \tag{9.9}$$

As v^s and u^+ do not depend on s_g^+ we obtain a simple expression for $\partial F/\partial s_g^+$; for brevity, we rewrite T^+ as T, s_g^+ as s_g and u^+ as u:

$$\frac{\partial F}{\partial s_g} = \left(v^s\varphi - u\frac{\partial f_g}{\partial s_g}\right)\rho_{gn},$$

so $\partial F/\partial s_g = 0$ yields:

$$v^s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}. \tag{9.10}$$

Since both v^s and u depend on T, the expression obtained for $\partial F/\partial T$ is complicated; a lengthly calculation yields:

$$\frac{\partial F}{\partial T} = \frac{\partial v^s}{\partial T}\varphi\left(s_g\rho_{gn} - \psi_{gn}^-\rho_{gN}^-\right) + v^s\varphi s_g\rho_{gn}' - u\frac{\partial f_g}{\partial T}\rho_{gn} - \frac{\partial u}{\partial T}f_g\rho_{gn} - uf_g\rho_{gn}'. \tag{9.11}$$

Denoting the numerator and denominator of $v^s\varphi/u^-$ in Eq. (9.6) by n^{vs} and d^{vs}, respectively, we obtain:

$$\frac{\partial v^s}{\partial T} = \frac{u^-}{\varphi}\frac{(\partial n^{vs}/\partial T)\, d^{vs} - n^{vs}\,(\partial d^{vs}/\partial T)}{(d^{vs})^2}, \tag{9.12}$$

where $\partial d^{vs}/\partial T =$:

$$\left(\hat{H}_r^- - \hat{H}_r - H_W + \hat{H}_g^-\right)\rho'_{gn} - (C_r + C_W)\rho_{gn} + \psi_{gn}^-\rho_{gN}^-\rho'_{gw}(1 - H_w) - \rho'_{gn}\psi_{gw}^-\rho_{gw}^-\rho_W - \Xi_2 C_W,$$

$$\frac{\partial n^{vs}}{\partial T} = \psi_{gn}^-\rho_{gN}^-\left(\rho'_{gw}H_W + \rho_{gw}C_W - C_g\rho_W\right) - \rho'_{gn}\left(\psi_{gw}^-\rho_{gw}^-H_W - H_g\rho_w\right) - \rho_{gn}\left(\psi_{gw}^-\rho_{gW}^-C_W - C_g\rho_W\right).$$

Denoting the numerator of u in Eq. (9.7) by N_u, we obtain that:

$$\frac{\partial u}{\partial T} = \frac{(\partial n^u/\partial T)\,\rho_{gn}\rho_W - N_u\rho'_{gn}\rho_W}{(\rho_{gn}\rho_W)^2}, \tag{9.13}$$

where

$$n^u = u^-\left(\psi_{gw}^-\rho_{gW}^-\rho'_{gn} - \psi_{gN}^-\rho'_{gw}\right) - \frac{\partial v^s}{\partial T}\varphi\Xi_2 - v^s\varphi\frac{\partial \Xi_2}{\partial T}.$$

9.2. **Shock between the two-phase and the liquid situations.** The *spl* can be considered a limit case of the *tp* situation, with $s_w = 1$ and $s_g = 0$. The left state is (s_g^-, T^-, u^-); in the right state, we specify only the temperature T^+; the Darcy speed is obtained from the other variables and the steam composition is a function of temperature (see (4.3)). The RH condition (5.15) is written as:

$$\varphi v^s\left(\rho_W - \rho_W s_W^- - \rho_{gW}^- s_g^-\right) = u^+\rho_W - u^-\rho_W f_w^- - u^-\rho_{gw}^- f_g^-, \tag{9.14}$$

$$-\varphi v^s\rho_{gn}^- s_g^- = -u^-\rho_{gn}^- f_g^-, \tag{9.15}$$

$$\varphi v^s\left(\hat{H}_r^+ - \hat{H}_r^- + H_w^+ - H_g^- s_g^- - s_w^- H_w^-\right) = u^+H_w^+ - u^-\left(H_w^- f_w^- + H_g^- f_g^-\right). \tag{9.16}$$

From (9.15), we obtain that

$$v^s = \frac{u^-}{\varphi}\frac{f_g^-}{s_g^-}, \tag{9.17}$$

Substituting v^s given by (9.17) in Eq. (9.14) we obtain :

$$u^+ = u^-. \tag{9.18}$$

There is no mass transfer between the *tp* situation and the *spl* situation, so we can consider the *spl* as a particular case of the *tp*.

Lemma 9.1. *If there is a shock between the tp "$-$" and the spl "$+$" with $T^+ \neq T^-$ (i.e., a non-isothermal shock), then*

$$\frac{f_g^-}{s_g^-} = \frac{C_W}{\hat{C}_r + C_W}. \tag{9.19}$$

Proof: We rewrite (9.16) using (9.17) and (9.18) as:

$$\left(H_r^+ + H_W^+\right)\frac{u^- f_g^-}{s_g^-} - u^-H_W^+ = \frac{u^- f_g^-}{s_g^-}\left(H_r^- + s_W^- H_W^- + s_g^- H_g^-\right) - u^-(f_w^- H_W^- + f_g^- H_g^-),$$

manipulating this equation we obtain

$$\left(H_r^+ + H_w^+\right) \frac{u^- f_g^-}{s_g^-} - u^- H_w^+ = \frac{u^- f_g^-}{s_g^-} \left(H_r^- + (1 - s_g^-) H_w^- + s_g^- H_g^-\right) - u^-(f_w^- H_w^- + f_g^- H_g^-),$$

$$\left(H_r^+ - H_r^- + H_w^+ - H_w^-\right) \frac{f_g^-}{s_g^-} = \left(H_w^+ - H_w^-\right),$$

$$\frac{f_g^-}{s_g^-} = \frac{(H_w^+ - H_w^-)}{(H_r^+ - H_r^- + H_w^+ - H_w^-)} = \frac{C_W}{\hat{C}_r + C_W},$$

where we have used that $H_w = C_W(T - T^0)$ and $\hat{H}_r = \hat{C}_r(T - T^0)$. $\square$

Remark 9.1. *We call the* $(-)$ *states that satisfy* (9.19) *the condensation curve; on this curve there is non-isothermal mass transfer between the gaseous and liquid phases. From Lemma* 9.1 *and Eq. (8.20), we obtain the following:*

Corollary 9.1. *In the tp, the inflection curve* $\mathcal{I}_e$ *is the union of coincidence locus* $\mathbb{C}_{s,e}$ *with the "condensation curve".*

Lemma 9.2. *In the set of* $(-)$ *states that satisfy Eq.* (9.19), *the evaporation characteristic speed in the tp equals the contact discontinuity speed in liquid situation, i.e.,*

$$\lambda_e = \frac{u^b}{\varphi} \frac{C_W}{C_W + \hat{C}_R}.$$

Proof: Since f_g/s_g satisfies Eq. (9.19), we substitute $f_g = s_g C_W/(C_W + \hat{C}_r)$ in Eq. (8.9). After some calculations we obtain the result.

Proposition 9.1. *The Riemann solution for any* $(-)$ *state,* $(-) = (s^-, T^-, \cdot)$ *that satisfies Eq. (*9.19*) and* $(+)$ *states,* $(+) = (0, T^+, \cdot)$ *in the spl consists of a shock between* $(-)$ *and* $(+)$.

Proof: From Lemma 9.1, if the $(-)$ belongs to the condensation curve, the shock speed between regions is

$$v^s = \frac{u^-}{\varphi} \frac{C_W}{\hat{C}_r + C_W},$$

which is the same as the the contact discontinuity speed in the *spl*. $\square$

The following corollary follows immediately:

Corollary 9.2. *The Riemann solution for left states* $L = (s_L, T_L, \cdot)$ *in regions* III *or* IV *(see Fig.* 8.2.*b) and right states* $R = (0, T_R, \cdot)$ *in the spl consists of a Buckley-Leverett shock from L to* $(0, T_L, \cdot)$ *followed by a thermal contact discontinuity in the spl with speed* λ_T^W *given by* (4.14) *and* $u_L = u_R$.

10. The Riemann solution for geothermal energy recovery

We show an example of Riemann solution. We consider the injection of a two-phase mixture of water, steam and nitrogen into a rock containing superheated steam ($\psi_R = 1$) at a temperature $T_R > T^b$:

$$\begin{cases} (s_g, \psi(T), T, u)_L & \text{if } x = 0 \text{ (the injection point)}, \qquad \text{with} \quad u_L > 0 \\ (s_g = 1, 1, T, \cdot)_R & \text{if } x > 0 \end{cases} \tag{10.1}$$

From Prop. 5.1, the projection of the solution into the primary variables does not depend on u_L. We can subdivide the tp region in 4 subregions, relatively to the left state L. For L in regions I and II of Figure (8.2.a) there are 3 subregions with the following property: for any L in a given subregion the Riemann problem with data R of form (10.1) has the same sequence of waves, see Figure 10.1. In 10.3, we conjecture that the Riemann problem with data of form (10.1) does not have a solution for states L in the regions III and IV.

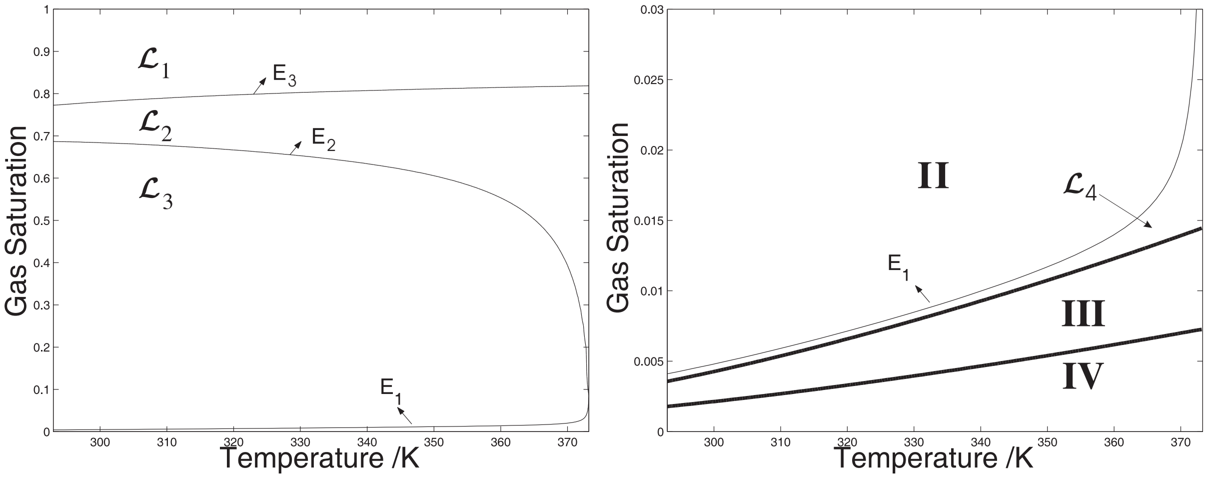

FIGURE 10.1. *a)-left* The 3 subregions in I and II for a Riemann data of form (10.1). The curves are defined in Secs. 10.1.2, 10.1.1 and 10.1.3. *b)-right* Zoom of regions II to IV. Each curve is explained in Section 10.1

10.1. **Subdivision of *tp*.** For the data (10.1) with T_R fixed, the Riemann solutions have the same wave sequence and structure in for L each subregion labelled $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ of Figure 10.1. A very important task is to obtain the curves that bound subregions: the curve E_2, see Fig. (10.1.a) and Sec. 10.1.1; the curve E_1, see Figs. (10.1.a) and (10.1.b) and Sec. 10.1.2; the curve E_3, see Fig. (10.1.a) and Sec. 10.1.3.

Remark 10.1. *The evaporation wave speed λ_e for states in $\mathcal{L}_2$ and $\mathcal{L}_3$ in tp is larger than the thermal wave speed in the spg. From geometrical compatibility, at the right of the evaporation shock there is no intermediate thermal wave (rarefaction or shock) in the spg, i.e., the shock between regions should reach a state of the form $(1, \psi_{gw}, T_R)$.*

10.1.1. *Curve E_2.* This curve is defined as the states V^- in the tp such that the evaporation rarefaction speed $\lambda_e(V^-)$ equals the speed $v^{BG}(V^-; V^+)$ of the shock to $V^+ = (1, T_R, \psi_{gw})$ in the spg. Notice that this curve is an extension of the boundary, see Section 5.4.5.

We obtain ψ_{gw} numerically. For each V^- state of the form $(S, \psi_{gw}(T), T)$ in the tp, we search the state in the RH locus starting at V^- with right temperature T_R. These restrictions yield a state in the spg with steam composition ψ_{gw} as unknown.

10.1.2. *Curve E_1.* We are interested in the Riemann solution for left states in I or II. From geometrical compatibility, to solve the Riemann problem we choose first the slowest wave whenever possible. In II, the slowest wave is the evaporation wave. Since we are interested in connecting a left state at lower temperature to a state at higher temperature, we use an evaporation rarefaction curve instead of a shock, see Section 8. We can do this until the speed of the evaporation rarefaction curve equals the speed of the shock between regions; this occurs when the rarefaction curve crosses E_2.

The integral curves starting at certain left states do not cross E_2, rather they reach directly the steam region at boiling temperature. We define the curve E_1 as the evaporation rarefaction curve that crosses E_2 at water boiling temperature. Notice that for left state in II above E_1, the evaporation rarefaction curve always crosses E_2.

10.1.3. *Curve* E_3. For states V^- above the curve E_2, the evaporation rarefaction speed is larger than $v^{BG}(V^-;V^+)$, for $V^+ = (1, T_R, \psi_{gw})$; in Figures (8.2.$a$) and (10.2.$a$) we show the relatives sizes of v^{BG}, λ_e and λ_s. When s_g tends to 1 the Buckley-Leverett shock speed is zero, so there are transition curves, $v^{BL} = v^{BG}$, $v^{BL} = v^T$ and $v^{BG} = v^T$. From the Prop. 5.3, we obtain a bifurcation curve E_3 where $v^T = v^{BL} = v^{BG}$. For states above this curve, the Buckley-Leverett shock is slower than v^{BG} and v^T, so there is no direct shock between the *tp* and the *spg* situations.

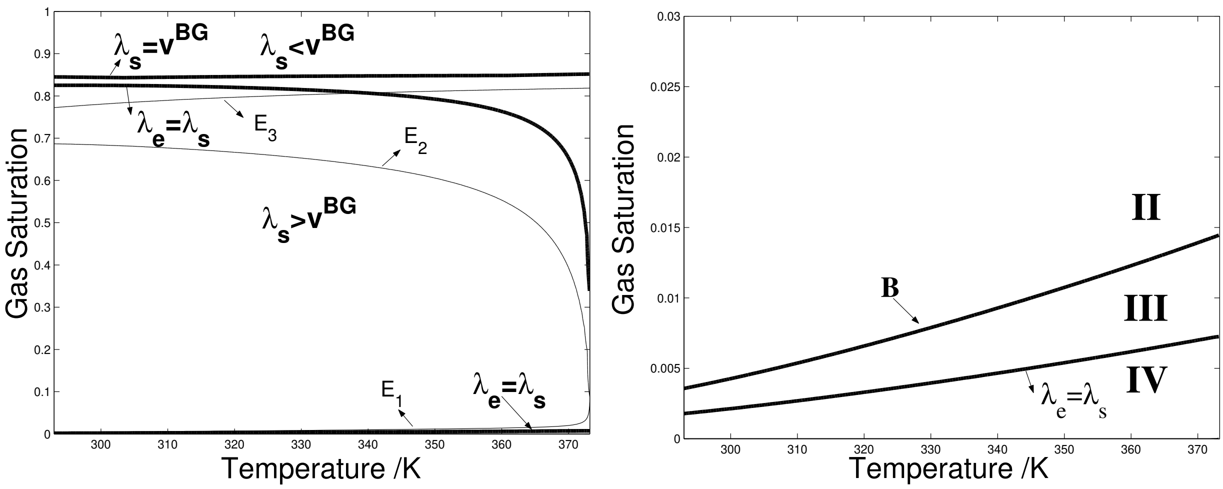

FIGURE 10.2. *a)-left)* The relevant curves for the Riemann solution for the left states in the *tp* situation. *b)-right)* The curve B is a bifurcation curve; for left states V out of this curve, the shock and rarefactions starting in V do not cross B.

10.2. **The Riemann solution.** We use the notation $V_L = (s_g, \psi_{gw}(T), T)_L$ and $V_R = (s_g = 1, 1, T)_R$. All the variables for the intermediate states are written out. The states are labelled by 1, 2, etc. We use the following nomenclature: evaporation rarefaction is R_E, the Buckley-Leverett shock and rarefaction are S_{BL} and R_{BL}, the shock between the *tp* and the *spg* is S_{BG} and the contact compositional discontinuity is S_C. We use the notation for wave sequences established in Section 5.3.

10.2.1. V_L *in* $\mathcal{L}_4$. There is an evaporation rarefaction curve from V_L up to $V_1 = (s_L, 1, T^b)$, where T^b is the boiling temperature. For this state there is no nitrogen and so $\psi_{gw} = 1$. The solution after the intermediate state $(s_M, 1, T^b)$ was found in [9]: an evaporation shock between the boiling situation to the steam situation appears, denoted by S_{VS}.

The solution consists of the waves $R_e \mapsto R_{BL} \twoheadrightarrow S_{VS}$.

$$V_L \xrightarrow{R_e} V_1 \xrightarrow{R_{BL}} V_2 \xrightarrow{S_{VS}} V_R. \tag{10.2}$$

10.2.2. V_L *in* $\mathcal{L}_3$. There is a rarefaction from V_L up to $V_1 = (s^*, \psi_{gw}(T^*), T^*)$ in *tp*; V_1 is a state where $\lambda_e(V_1) = v^{BL}(V_1; V_2)$, with $V_2 = (1, \psi_{gw}, T_R)$ in *spg*. We have described how to obtain ψ_{gw} in Section 10.1.1. Finally, there is a compositional contact discontinuity at temperature T_R with speed v^C from V_2 to V_R.

The solution consists of the waves $R_E \twoheadrightarrow S_{BG} \mapsto S_C$ with sequence:

$$V_L \xrightarrow{R_e} V_1 \xrightarrow{S_{BG}} V_2 \xrightarrow{S_C} V_R. \tag{10.3}$$

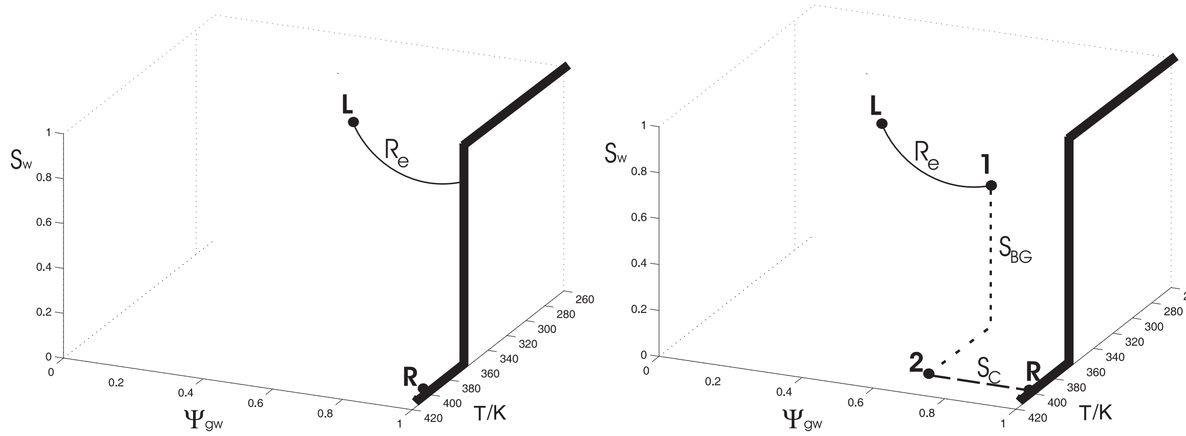

FIGURE 10.3. Riemann solutions in phase space, omitting the surface shown in Fig. 4.1. *a) Left:* Solution (10.2) for $V_L \in \mathcal{L}_4$, Sec. 10.2.1. *b) Right:* Solution (10.3) for $V_L \in \mathcal{L}_3$, Sec. 10.2.2. The numbers 1 and 2 indicate the intermediate states V in wave sequence.

10.2.3. V_L *in* $\mathcal{L}_2$. Let $V_1 = (1, \psi_{gw}, T_R)$ be a state in the *spg* situation. Since $\lambda_e(V_L) > v^{BL}(V_L; V_1)$, there is a shock between V_L and V_1 with speed v^{BG}. From the state V_1 there is compositional contact discontinuity with speed v^C to V_R.

The solution consists of the wave $S_{BG} \mapsto S_C$ with sequence:

$$V_L \xrightarrow{S_{BG}} V_1 \xrightarrow{S_C} V_R. \tag{10.4}$$

10.2.4. V_L *in* $\mathcal{L}_3$. Since v^{BG} and v^{BL} are smaller than v^T, there is a Buckley-Leverett saturation shock between V_L and $V_1 = (1, \psi_{gw}(T_L), T_L)$. From V_1 there is a thermal shock with speed v^T to $V_2 = (1, \psi_{gw}(T_L), T_R)$ and finally there is a compositional contact discontinuity with speed v^C to V_R.

The solution consists of the waves $S_{BL} \twoheadrightarrow S_T \mapsto S_C$ with sequence:

$$V_L \xrightarrow{S_{BL}} V_1 \xrightarrow{S_T} V_2 \xrightarrow{S_C} V_R. \tag{10.5}$$

10.3. **Conjecture: Absence of solution for left states V_L in III or IV.** The integral curves or RH locus starting at any point in III or IV do not reach the water boiling temperature. Moreover, the RH locus between the *tp* and the *spg* situations for points in III or IV do not reach temperatures above the temperature T_L. So the only possibility to get to the right state is by crossing the curve B. However, the evaporation wave does not cross the curve B; to cross it we need a Buckley-Leverett shock. The Buckley-Leverett shock for an initial state below B and a final state above B is faster than the evaporation wave λ_e.

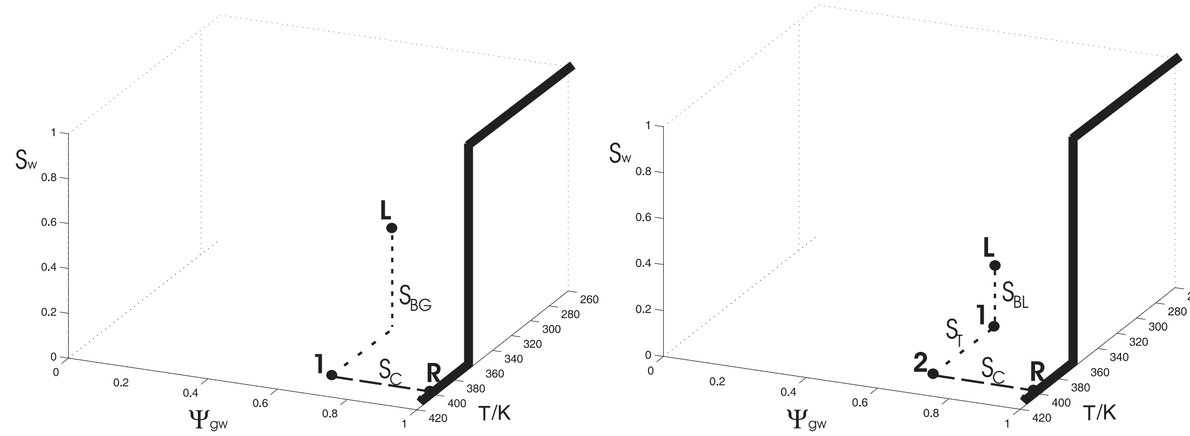

FIGURE 10.4. Riemann solutions in phase space, removing the surface shown in Fig. 4.1. *a) Left:* Solution (10.4) for $V_L \in \mathcal{L}_2$, Sec. 10.2.3. *b) Right:* Solution (10.5) for $V_L \in \mathcal{L}_1$, Sec. 10.2.4. The numbers 1 and 2 indicate the intermediate states V in wave sequence.

Moreover, it is faster than the shock between regions. So with a Buckley-Leverett shock of this type, it is only possible to reach states of form $(1, \psi_{gw}(T), T)$ with $T < T^b$. Since the thermal wave is slower than the Buckley-Leverett shock, it seems impossible to construct the Riemann solution satisfying the geometrical compatibility principle. However, since the III and IV represent a region smaller than 1 percent of the tp situation, the absence of the solution is not a worrying problem.

11. Summary and Conclusions

We have described a solution of the Riemann problem for the injection of a mixture nitrogen/steam/water into a porous rock filled with steam above boiling temperature. We show a systematic theory for the Riemann solution for a 3×3 balance system. The set of solutions depends L^1 continuously on the Riemann data. It is a step towards obtaining a general method for solving Riemann problems for a wide class of balance equations with phase changes, [10].

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Appendix A. Physical quantities; symbols and values

Table 2, Summary of physical input parameters and variables

Physical quantity	*Symbol*	*Value*	*Unit*
Water, steam fractional functions	f_w, f_g	Eq. (2.3) .	[m^3 /m^3]
Porous rock permeability	k	1.0×10^{-12}.	[m^3]
Water, steam relative permeabilities	k_{rw}, k_{rg}	Eq. (A.10).	[m^3 /m^3]
Water, steam end point permeabilities	k'_{rw}, k'_{rg}	Eq. (A.10.a), (A.10.b).	[m^3]
Pressure	p_{at}	1.0135×10^5.	[Pa]
Water Saturation Pressure	p^{sat}	Eq. (A.7).	[Pa]
Water, steam phase velocity	u_w, u_g	Eq. (2.1) .	[m^3 /(m^2 s)]
Total Darcy velocity	u	$u_w + u_g$, Eq. (2.4) .	[m^3 /(m^2 s)]
Effective rock heat capacity	C_r	2.029×10^6.	[J/(m^3 K)]
Steam and nitrogen enthalpies	h_{gW}, h_{gN}	Eqs. (A.3), (A.4).	[J/m^3]
Water enthalpy	h_W	Eq. (A.2).	[J/m^3]
Rock enthalpy	H_r	Eq. (A.1).	[J/m^3]
Water, steam saturations	s_w, s_g	Dependent variables.	[m^3 /m^3]
Connate water saturation	s_{wc}	0.15.	[m^3 /m^3]
Temperature	T	Dependent variable.	[K]
Water, steam thermal conductivity	κ_w , κ_g	0.652 , 0.0208.	[W/(mK)]
Rock, composite thermal conductivity	κ_r , κ	1.83 , Eq. (3.5) .	[W/(mK)]
Water, steam viscosity	μ_w , μ_g	Eq. (A.5) , Eq. (A.6) .	[Pa s]
Steam and nitrogen densities	ρ_{gw}, ρ_{gn}	Eq. (A.8.a), (A.8.b).	[kg/m^3]
Constant water density	ρ_W	998.2.	[kg/m^3]
Steam and nitrogen gas composition	ψ_{gw}, ψ_{gn}	Dependent variables.	[$-$]
Universal gas constant	R	8.31	[$J/mol/K$]
Nitrogen and water molar masses	M_N, M_W	0.28, 0.18	[kg/mol]
Rock porosity	φ	0.38.	[m^3 /m^3]

A.1. **Temperature dependent properties of steam and water.** The rock enthalpy H_r and $\hat{H}_r$ are given by:

$$H_r = (1-\varphi)C_r(T-\bar{T}) \qquad \text{and} \qquad \hat{H}_r = H_r/\varphi. \tag{A.1}$$

The water enthalpy per mass unit h_W is:

$$h_W = C_W T/\rho_w. \tag{A.2}$$

The steam enthalpy $h_{gW}[J/kg]$ as a function of temperature is approximated by

$$h_{gW}(T) = -2.20269 \times 10^7 + 3.65317 \times 10^5 T - 2.25837 \times 10^3 T^2 + 7.3742 T^3 - 1.33437 \times 10^{-2} T^4 + 1.26913 \times 10^{-5} T^5 - 4.9688 \times 10^{-9} T^6 - \overline{h}_w. \tag{A.3}$$

The nitrogen enthalpy $h_{gN}[J/kg]$ as a function of temperature is approximated by

$$h_{gN}(T) = 975.0T + 0.0935T^2 - 0.476 \times 10^{-7} T^3 - \overline{h}_{gN}. \tag{A.4}$$

These constants enthalpies $\overline{h}_w$ and $\overline{h}_{gN}$ are chosen such that $h_w(T)$, $h_{gN}(T)$ vanish at a reference temperature $\overline{T} = 293K$.

The temperature dependent liquid water viscosity μ_w $[Pas]$ is approximated by

$$\mu_w = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425 \times 10^7}{T^3} - \frac{2.17342 \times 10^9}{T^4} + \frac{1.86935 \times 10^{11}}{T^5}. \tag{A.5}$$

We assume that that the viscosity of the gas is independent of the composition.

$$\mu_g = 1.826\,4 \times 10^{-5} \left(\frac{T}{T^b}\right)^{0.6}. \tag{A.6}$$

The water saturation pressure as a function of temperature is given as

$$p^{sat} = 10^3(-175.776 + 2.29272T - 0.0113953T^2 + 0.000026278T^3 - 0.0000000273726T^4 + 1.13816 \times 10^{-11} T^5)^2 \tag{A.7}$$

The corresponding concentrations ρ_{gw}, ρ_{gn} are calculated with the ideal gas law:

$$\rho_{gw} = \frac{M_W\ p^{sat}}{RT},\ \rho_{gn} = \frac{M_N\ (p_{at} - p^{sat})}{RT}, \tag{A.8}$$

The pure phase densities are:

$$\rho_{gW}(T) = \frac{M_W p_{at}}{RT}, \qquad \rho_{gN}(T) = \frac{M_N p_{at}}{RT}, \tag{A.9}$$

where the gas constant $R = 8.31[J/mol/K]$, M_W and M_N are the the nitrogen and water molar mass.

For simplicity the liquid water density is assumed to be constant at $998.2.kg/m^3$.

A.2. **Constitutive relations.** The relative permeability functions k_{rw} and k_{rg} are considered to be power functions of their respective saturations, i.e.

$$k_{rw} = \begin{cases} k'_{rw}\left(\frac{s_w - s_{wc}}{1 - s_{wc} - s_{gr}}\right)^{n_w} \\ 0 \end{cases}, k_{rg} \begin{cases} k'_{rg} = \left(\frac{s_g - s_{gr}}{1 - s_{wc} - s_{gr}}\right)^{n_g} & \text{for} \quad s_{wc} \le s_w \le 1, \\ 1 & \text{for} \quad s_w < s_{wc}. \end{cases} \tag{A.10}$$

For the computations we take $n_w = 2$ and $n_g = 2$. The end point permeabilities k'_{rw}, k'_{rg} are 0.5 and 0.95 respectively. The connate water saturation s_{wc} is given in the table.

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