# Global hyperbolicity of renormalization for $C^r$ unimodal mappings

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**Abstract**

In this paper we extend M. Lyubich's recent results on the global hyperbolicity of renormalization of quadratic-like germs to the space of $C^r$ unimodal maps with quadratic critical point. We show that in this space the bounded-type limit sets of the renormalization operator have an invariant hyperbolic structure provided $r \geq 2+\alpha$ with $\alpha$ close to one. As an intermediate step between Lyubich's results and ours, we prove that the renormalization operator is hyperbolic in a Banach space of real analytic maps. We construct the local stable manifolds and prove that they form a continuous lamination whose leaves are $C^1$ codimension one Banach submanifolds of the ambient space, and whose holonomy is $C^{1+\beta}$ for some $\beta > 0$. We also prove that the global stable sets are $C^1$ immersed (codimension one) submanifolds as well, provided $r \geq 3+\alpha$ with $\alpha$ close to one. As a corollary, we deduce that in generic one parameter families of $C^r$ unimodal maps, the set of parameters corresponding to infinitely renormalizable maps of bounded combinatorial type is a Cantor set with Hausdorff dimension less than one.

## 1. Introduction

In 1978, M. Feigenbaum [10] and independently P. Coullet and C. Tresser [4] made a startling discovery concerning certain rigidity properties in one-dimensional dynamics. While analysing the transition between simple and "chaotic" dynamical behavior in "typical" one-parameter families of unimodal maps – such as the quadratic family $x \mapsto \lambda x(1-x)$ – they recorded the parameter values $\lambda_n$ at which successive period-doubling bifurcations ocurred in the family and found a remarkable universal scaling law, namely

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \;\to\; 4.669\ldots\ .$$

They also found universal scalings within the geometry of the post-critical set of the limiting map corresponding to the parameter $\lambda_\infty = \lim \lambda_n$ (*cf.* the work of E. Vul, Ya. Sinai and K. Khanin [29]). In an attempt to explain these phenomena, they introduced a certain non-linear operator acting on the space of unimodal maps – the so-called *period doubling operator*. They conjectured that the period-doubling operator has a unique fixed-point which is hyperbolic with a one-dimensional unstable direction. They also conjectured that the universal constants they found in their experiments are the eigenvalues of the derivative of the operator at the fixed point.

A few years later (1982) this conjecture was confirmed by O. Lanford [18] through a computer assisted proof. Working in a cleverly defined Banach space of real analytic maps and using rigorous numerical analysis on the computer, Lanford established at once the existence and hyperbolicity of the fixed point of the period-doubling operator. Subsequent work by M. Campanino and H. Epstein [2] (also Campanino et al. [3] and Epstein [9]) established the existence (but neither uniqueness nor hyperbolicity) of the fixed point without essential help from the computer.

It was soon realized by Lanford and others that the period-doubling operator was just a restriction of another operator acting on the space of unimodal maps – the *renormalization operator* – whose dynamical behavior is much richer. The hopes were high that the iterates of this operator would reveal the small scale geometric properties of the critical orbits of many interesting one-dimensional systems. Hence, the initial conjecture was generalized to the following.

**Renormalization Conjecture.** *The limit set of the renormalization operator in the space of maps of bounded combinatorial type is a hyperbolic Cantor set where the operator acts as the full shift in a finite number of symbols.*

(For a precise formulation of what is meant by bounded combinatorial type, see §2.2 below.)

In the path towards a proof of this conjecture, several new ideas were developed in the last 20 years by a number of mathematicians, especially D. Sullivan, C. McMullen and M. Lyubich. Among the deepest in Dynamical Systems, these ideas have the complex dynamics of quadratic-like maps (in the sense of Douady and Hubbard [6]) as a common thread. Sullivan proved in [28] that all limits of renormalization are quadratic-like maps with a definite modulus. Then, constructing certain Teichmüller spaces from quadratic-like maps and using a substitute of Schwarz's lemma in these spaces, Sullivan established the existence of horseshoe-like limit sets for renormalization. Later, using a different approach based on Mostow rigidity, McMullen [23] gave another proof of this result and went further by showing that the convergence (in the $C^0$ sense) towards the limit set is exponential.

The final breakthrough came with the work of Lyubich [20]. He endowed the space of *germs* of quadratic-like maps (modulo affine conjugacies) with a very subtle complex structure, showing that the renormalization operator is complex-analytic with respect to such structure. In Lyubich's space, the stable sets of maps in the limit set of renormalization coincide with the very *hybrid classes* of such maps, and inherit a natural structure making them (complex codimension one) analytic submanifolds. Combining McMullen's rigidity of towers with Schwarz's lemma in Banach spaces, Lyubich proved exponential contraction along such stable leaves. To obtain expansion in the transversal directions to such leaves at points of the limit set, Lyubich argued by contradiction: if expansion fails, then one can find a map in the limit set whose orbit under renormalization is slowly shadowed by another orbit (the *small orbits theorem*, page 323 of [20]). This however contradicts another theorem of his, namely the combinatorial rigidity theorem of [21]. It follows that the limit set is indeed hyperbolic in the space of germs. Based on this result of Lyubich and using the real and complex bounds given by Sullivan, we prove in Theorem 2.4 that the attractor (for bounded combinatorics) is hyperbolic in a Banach space of real analytic maps.

In the present paper, we give the last step in the proof of the above renormalization conjecture in the (much larger) space of $C^r$ smooth unimodal maps with $r$ sufficiently large. The very formulation of the conjecture in this setting requires some care, because the renormalization operator is *not* differentiable in $C^r$. For the correct formulation, see Theorem 2.5 below. To prove the conjecture, we combine Theorem 2.4 with some non-linear functional analysis inspired by the work of A. Davie [5]. In that work, Davie constructs the stable manifold of the fixed point of the period doubling operator in the space of $C^{2+\epsilon}$ maps "by hand", showing it to be a $C^1$ codimension one submanifold of the ambient space, even though the operator is not differentiable. To do this, he first extends the hyperbolic splitting of the derivative at the fixed point from Lanford's Banach space of real-analytic maps to the larger space of $C^{2+\epsilon}$ maps (to which the derivative extends as a bounded linear operator). This gives him an extended codimension one stable subspace in $C^{2+\epsilon}$ to work with, and he views the local stable set in $C^{2+\epsilon}$ as the graph of a function over the extended stable subspace. In attempting to prove that such function is $C^1$, he goes around the inherent loss of diferentiability of renormalization by first noting that the local unstable manifold coming from Lanford's theorem is still there (and is still smooth in $C^{2+\epsilon}$) and then showing that there is afterall a contraction in $C^{2+\epsilon}$ towards that unstable manifold, whose elements are *analytic* maps. Thus, the loss of differentiability is somehow compensated by the contraction towards the unstable manifold. Davie's crucial estimates show that the renormalization operator in $C^{2+\epsilon}$ is sufficiently well-approximated by the extension of its derivative in Lanford's space to a bounded linear operator in $C^{2+\epsilon}$.

Our approach is based on the idea that whatever Davie can do with Lanford's Banach space relative to the fixed point, we can do with the Banach space obtained in Theorem 2.4 relative to the whole limit set. There is one fundamental difference, however. The linear and non-linear estimates carried out by Davie rely on the special fact that the period-doubling fixed point is *concave*. This allows him to prove his main theorems in $C^{2+\epsilon}$ for all $\epsilon > 0$. By contrast, we cannot – and do not – rely on any such convexity assumptions. We derive our estimates (in §5 and §8) directly from the geometric properties of the postcritical set of maps in the limit set (these properties – proved in §5.2 – are a consequence of the real a-priori bounds). As a result, our local stable manifold theorem in $C^r$ requires $r \geq 2 + \alpha$ with $\alpha$ close to one.

We go beyond the conjecture in at least three respects. First, we show that the local stable manifolds form a $C^0$ lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$. In particular, every smooth curve which is transversal to such lamination intersects it at a set of constant Hausdorff dimension less than one. Second, we prove that the global stable sets are $C^1$ (immersed) codimension one submanifolds in $C^r$ provided $r \geq 3 + \alpha$ with $\alpha$ close to one (we globalize the local stable manifolds via the implicit function theorem, hence the further loss of one degree of differentiability). Third, we prove that in an open and dense set of $C^k$ one-parameter families of $C^r$ unimodal maps (for any $k \geq 2$), each family intersects the global stable lamination transversally at a Cantor set of parameters and the small-scale geometry of this intersection is the same for all nearby families. In particular, its Hausdorff dimension is strictly smaller than one.

In the path towards these results, we have made an attempt to abstract out the more general features of the renormalization operator in the form of a few properties or "axioms" – the notion of a robust operator introduced in §6. We prove a general local stable manifold theorem for robust operators in §6. It is our hope that this might be useful in other renormalization problems. For example in the case of critical circle maps (see [7] and [8]).

## 2. Preliminaries and statements of results

In this section, we introduce the basic notions of the theory of renormalization of unimodal maps. Then we state Sullivan's theorem on the existence of topological limit sets for the renormalization operator, the exponential convergence results of McMullen, and Lyubich's theorem showing the full hyperbolicity of such limit sets in the space of germs of quadratic-like maps. Finally, we state our main results extending Lyubich's hyperbolicity theorem to the space of $C^r$ unimodal maps with $r$ sufficiently large.

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## 2.1. Quadratic unimodal maps

We describe here two types of ambient spaces of $C^r$ unimodal maps. These will be determined by two families of Banach spaces, denoted $\mathbb{A}^r$ and $\mathbb{B}^r$.

### 2.1.1. The Banach spaces $\mathbb{A}^r$

Let $I = [-1, 1]$ and for all $r \geq 0$ let $C^r(I)$ be the Banach space of $C^r$ real-valued functions on $I$. Here $r$ can be either a non-negative real number, say $r = k + \alpha$ with $k \in \mathbb{N}$ and $0 \leq \alpha < 1$, in which case $C^r(I)$ is the space of $C^k$ functions whose $k$-th derivative is $\alpha$-Hölder, or else $r = k + \mathrm{Lip}$, in which case $C^r(I)$ means the space of $C^k$ functions whose $k$-th derivative is Lipschitz (so whenever we say that $r$ is not an integer, we include the Lipschitz cases). Let us denote by $\mathbb{A}^r$ the space $C^r_e(I)$ consisting of all $C^r$ functions on $I$ which are *even* and vanish at the origin, in other words

$$\mathbb{A}^r = \{v \in C^r(I) : v \text{ is even and } v(0) = 0\} \ .$$

Then $\mathbb{A}^r$ is a closed linear subspace of $C^r(I)$ and therefore also a Banach space under the $C^r$ norm. Now, for each $r \geq 2$, define

$$\mathbb{U}^r \subset 1 + \mathbb{A}^r \subset C^r(I)$$

to be the set of all maps $f : I \to I$ of the form $f(x) = 1 + v(x)$, where $v \in \mathbb{A}^r$ satisfies $v''(0) < 0$, which are unimodal. Then $\mathbb{U}^r$ is a Banach manifold; indeed it is an open subset of the affine space $1 + \mathbb{A}^r$. Note that for all $f \in \mathbb{U}^r$ the tangent space $T_f\mathbb{U}^r$ is naturally identified with $\mathbb{A}^r$. The elements of $\mathbb{U}^r$ are called $C^r$ *unimodal maps with a quadratic critical point.*

### 2.1.2. The Banach spaces $\mathbb{B}^r$

We define $\mathbb{B}^r$ to be the space of functions $v : I \to \mathbb{R}$ of the form $v = \varphi \circ q$ where $q(x) = x^2$ and $\varphi \in C^r([0, 1])$ vanishes at the origin. The norm of $v$ in this space is given by the $C^r$ norm of $\varphi$. This makes $\mathbb{B}^r$ into a Banach space. Note that for each $s \leq r$ the inclusion map $j : \mathbb{B}^r \to \mathbb{A}^s$ is linear and continuous (hence $C^1$). Now, for each $r \geq 1$, let

$$\mathbb{V}^r \subset 1 + \mathbb{B}^r$$

be the open subset of the affine space $1 + \mathbb{B}^r$ consisting of those $f = \phi \circ q$ such that $\phi([0, 1]) \subseteq (-1, 1]$, $\phi(0) = 1$ and $\phi'(x) < 0$ for all $0 \leq x \leq 1$. Just as before, $\mathbb{V}^r$ is a Banach manifold. Note that each $f \in \mathbb{V}^r$ is a unimodal map belonging to $\mathbb{U}^r$ when $r \geq 2$. Moreover, the inclusion of $\mathbb{V}^r$ in $\mathbb{U}^r$ is strict (for each $r \geq 2$).

## 2.2. Renormalization operator

A map $f \in \mathbb{U}^r$ is said to be *renormalizable* if there exist $p = p(f) > 1$ and $\lambda = \lambda(f) = f^p(0)$ such that $f^p|[-|\lambda|, |\lambda|]$ is unimodal and maps $[-|\lambda|, |\lambda|]$ into itself. In this case, taking the smallest possible value of $p$, the map $Rf : [-1, 1] \to [-1, 1]$ given by

$$Rf(x) \;=\; \frac{1}{\lambda}\, f^p(\lambda x) \tag{2.2.1}$$

is called the *first renormalization* of $f$. We have $Rf \in \mathbb{U}^r$. The intervals $f^j([-|\lambda|, |\lambda|])$, for $0 \le j \le p - 1$, are pairwise disjoint and their relative order inside $[-1, 1]$ determines a *unimodal* permutation $\theta$ of $\{0, 1, \ldots, p - 1\}$. The set of all unimodal permutations is denoted $\mathbf{P}$. The set of $f \in \mathbb{U}^r$ that are renormalizable with the same unimodal permutation $\theta \in \mathbf{P}$ is a connected subset of $\mathbb{U}^r$ denoted $\mathbb{U}^r_\theta$. Hence we have an operator

$$R : \bigcup_{\theta \in \mathbf{P}} \mathbb{U}^r_\theta \to \mathbb{U}^r \;, \tag{2.2.2}$$

the so-called *renormalization operator*.

Now let us fix a finite subset $\Theta \subseteq \mathbf{P}$. Given an infinite sequence of unimodal permutations $\theta_0, \theta_1, \ldots, \theta_n, \ldots \in \Theta$, write

$$\mathbb{U}^r_{\theta_0,\theta_1,\cdots,\theta_n,\cdots} \;=\; \mathbb{U}^r_{\theta_0} \cap R^{-1}\mathbb{U}^r_{\theta_1} \cap \cdots \cap R^{-n}\mathbb{U}^r_{\theta_n} \cap \cdots \;,$$

and define

$$\mathcal{D}^r_\Theta \;=\; \bigcup_{(\theta_0,\theta_1,\cdots,\theta_n,\cdots)\in\Theta^{\mathbb{N}}} \mathbb{U}^r_{\theta_0,\theta_1,\cdots,\theta_n,\cdots} \;.$$

The maps in $\mathcal{D}^r_\Theta$ are *infinitely renormalizable* maps with (bounded) combinatorics belonging to $\Theta$. Note that $R(\mathcal{D}^r_\Theta) \subseteq \mathcal{D}^r_\Theta$, in fact

$$R(\mathbb{U}^r_{\theta_0,\theta_1,\cdots,\theta_n,\cdots}) \subseteq \mathbb{U}^r_{\theta_1,\theta_2,\cdots,\theta_{n+1},\cdots} \;. \tag{2.2.3}$$

We note that if $f$ is a renormalizable map in $\mathbb{V}^r$, then $R(f)$ belongs to $\mathbb{V}^r$ also. Hence, taking $\mathbb{V}^r_\theta = \mathbb{U}^r_\theta \cap \mathbb{V}^r$, the restriction of the renormalization operator

$$R : \bigcup_{\theta \in \mathbf{P}} \mathbb{V}^r_\theta \to \mathbb{V}^r \tag{2.2.4}$$

is well-defined.

## 2.3. The limit sets of renormalization

In [28], Sullivan established the existence of horseshoe-like invariant sets for the renormalization operator, showing that they all consist of real analytic maps of a special kind, namely, restrictions to $[-1, 1]$ of *quadratic-like maps* in the sense of Douady-Hubbard. We remind the reader that a quadratic-like map $f : V \to W$ is a holomorphic map with the property that $V$ and $W$ are topological disks with $V$ compactly contained in $W$, and $f$ is a proper degree two branched covering map with a continuous extension to the boundary of $V$. The *conformal modulus of* $f$ is the modulus of the annulus $W \setminus \overline{V}$.

We are interested only in quadratic-like maps that commute with complex conjugation, for which $V$ is symmetric about the real axis. Consider the real Banach space $\mathcal{H}_0(V)$ of holomorphic functions which commute with complex conjugation and are continuous up to the boundary of $V$, with the $C^0$ norm. Let $\mathbb{A}_V \subset \mathcal{H}_0(V)$ be the closed linear subspace of functions of the form $\varphi = \phi \circ q$, where $q(z) = z^2$ and $\phi : q(V) \to \mathbb{C}$ is holomorphic with $\phi(0) = 0$. Also, let $\mathbb{U}_V$ be the set of functions of the form $f = 1 + \varphi$, where $\varphi = \phi \circ q \in \mathbb{A}_V$ and $\phi$ is *univalent* on some neighborhood of $[-1, 1]$ contained in $V$, such that the restriction of $f$ to $[-1, 1]$ is unimodal. Then $\mathbb{U}_V$ is an open subset of the affine space $1 + \mathbb{A}_V$, which is linearly isomorphic to $\mathbb{A}_V$ via the translation by 1, and we shall regard $\mathbb{U}_V$ as an open subset of $\mathbb{A}_V$ itself via this identification. For each $a > 0$, let us denote by $\Omega_a$ the set of points in the complex plane whose distance from the interval $[-1, 1]$ is smaller than $a$. We may now state Sullivan's theorem as follows.

Theorem 2.1. *Let $\Theta \subseteq \mathbf{P}$ be a non-empty finite set. Then there exist $a > 0$, a compact subset $\mathbb{K} = \mathbb{K}_\Theta \subseteq \mathbb{A}_{\Omega_a} \cap \mathcal{D}^\omega_\Theta$ and $\mu > 0$ with the following properties.*

*(i) Each $f \in \mathbb{K}$ has a quadratic-like extension with conformal modulus bounded from below by $\mu$.*

*(ii) We have $R(\mathbb{K}) \subseteq \mathbb{K}$, and the restriction of $R$ to $\mathbb{K}$ is a homeomorphism which is topologically conjugate to the two-sided shift $\sigma : \Theta^{\mathbb{Z}} \to \Theta^{\mathbb{Z}}$: in other words, there exists a homeomorphism $H : \mathbb{K} \to \Theta^{\mathbb{Z}}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{R} & \mathbb{K} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \Theta^{\mathbb{Z}} & \xrightarrow[\sigma]{} & \Theta^{\mathbb{Z}} \end{array}$$

*commutes.*

*(iii) For all $g \in \mathcal{D}^r_\Theta \cap \mathbb{V}^r$, with $r \geq 2$, there exists $f \in \mathbb{K}$ with the property that $||R^n(g) - R^n(f)||_{C^0(I)} \to 0$ as $n \to \infty$.*

For a detailed exposition of this theorem, see Chapter VI of [26].

Later, in [23], C. McMullen established the exponential convergence of renormalization for bounded combinatorics (using rigidity of towers). His theorem forms the basis for the contracting part of Lyubich's hyperbolicity theorem in [20].

THEOREM 2.2. *If $f$ and $g$ are infinitely renormalizable quadratic-like maps with the same bounded combinatorial type in $\Theta \subset P$, and with conformal moduli greater than or equal to $\mu$, we have*

$$\|R^n f - R^n g\|_{C^0(I)} \leq C\lambda^n$$

*for all $n \geq 0$ where $C = C(\mu, \Theta) > 0$ and $0 < \lambda = \lambda(\Theta) < 1$.*

The above result was extended by Lyubich to all combinatorics. In particular it follows, in the case of bounded combinatorics, that the exponent $\lambda$ and the constant $C$ in Theorem 2 do not depend on $\Theta$. The conclusion of the above theorem can also be improved in bounded combinatorics: for $r \geq 3$, the exponential convergence holds in the $C^r$ topology if the maps are in $\mathbb{V}^r$ (see [24] and [25]).

In [20], Lyubich considered the space of quadratic-like germs modulo affine conjugacies in which the limit set $\mathbb{K}$ is naturally embedded. This space is a manifold modeled on a complex topological vector space (arising as a direct limit of Banach spaces of holomorphic maps). In this setting, Lyubich established in [8] the full hyperbolicity of the renormalization operator. With the help of Sullivan's real and complex bounds and Lyubich's theorem we prove the hyperbolicity of some iterate of the renormalization operator acting on a space $\mathbb{A}_{\Omega_a}$ for some $a > 0$ (see Theorem 2.4 in §2.5). Then we extend Davie's analysis for the Feigenbaum fixed point to the context of bounded combinatorics to conclude that the hyperbolic picture also holds true in the much larger space $\mathbb{U}^r$ (see Theorem 2.5 in §2.5).

## 2.4. Hyperbolic basic sets

We need to introduce the well-known concept of hyperbolic basic set for non-linear operators acting on Banach spaces. Let us consider a Banach space $\mathcal{A}$, and an open subset $\mathcal{O} \subseteq \mathcal{A}$.

DEFINITION 2.1. *Let $T : \mathcal{O} \to \mathcal{A}$ be a smooth non-linear operator. A hyperbolic basic set of $T$ is a compact subset $\mathbb{K} \subset \mathcal{O}$ with the following properties.*

*(i) $\mathbb{K}$ is $T$-invariant and $T|\mathbb{K}$ is a topologically transitive homeomorphism whose periodic points are dense.*

*(ii) If $y \in \mathcal{O}$ and all $T$-iterates of $y$ are defined, then $T^n(y)$ converges to $\mathbb{K}$.*

*(iii) There exist a continuous, $DT$-invariant splitting $\mathcal{A} = E_x^s \bigoplus E_x^u$, for $x \in \mathbb{K}$, and uniform constants $C > 0$ and $0 < \theta < 1$ such that*

$$\|DT^n(x)\, v\| \leq C\theta^n \|v\|$$

*for all $v \in E^s_x$, as well as*

$$\|DT^n(x)\, v\| \geq C\theta^{-n}\|v\| \tag{2.4.1}$$

*for all $v \in E^u_x$.*

*(iv) The dimension of $E^u_x$ is finite and constant.*

The following notions are also standard. Let $\mathcal{A}(x,\epsilon)$ be the ball in $\mathcal{A}$ with center $x$ and radius $\epsilon$. The *local stable manifold $W^s_\varepsilon(x)$ of $T$ at $x$* consists of all points $y \in \mathcal{A}(x,\varepsilon)$ such that, for all $n > 0$, we have $T^n(y) \in \mathcal{A}(T^n(x),\varepsilon)$ and

$$\|T^n(y) - T^n(x)\| \to 0 \text{ when } n \to \infty \ .$$

The *local unstable manifold $W^u_\varepsilon(x)$ of $T$ at $x$* consists of all points $y \in \mathcal{A}(x,\varepsilon)$ such that, setting $y_0 = y$, for all $n \geq 1$ there exists $y_n \in \mathcal{A}(T^{-n}(x),\varepsilon)$ such that $y_{n-1} = T(y_n)$ and

$$\|T^{-n}(x) - y_n\| \to 0 \text{ when } n \to \infty \ .$$

Finally the *global stable set of $T$ at $x$* is defined as

$$W^s(x) = \{y \in \mathcal{O} \ : \ \|T^n(y) - T^n(x)\| \to 0 \text{ when } n \to \infty\} \ .$$

The question arises as to whether these sets have smooth manifold structures. We have the following general result.

Theorem 2.3. *If $\mathbb{K}$ is a hyperbolic basic set of a $C^1$ operator $T : \mathcal{O} \to \mathcal{A}$ then*

*(i) the local stable (resp. unstable) set at $x \in \mathbb{K}$ is a $C^1$ Banach submanifold of $\mathcal{A}$ which is tangent to $E^s_x$ (resp. $E^u_x$) at $x$.*

*(ii) If $y \in W^s(x)$ then*

$$\|T^n(x) - T^n(y)\| \leq C\theta^n\|x - y\| \ .$$

*Moreover, $T(W^u_\epsilon(x)) \supseteq W^u_\epsilon(T(x))$, the restriction of $T$ to $W^u_\epsilon(x)$ is one-to-one and for all $y \in W^u_\epsilon(x)$ we have*

$$\left\|T^{-n}(x) - T^{-n}(y)\right\| \leq C\theta^n\|x - y\| \ .$$

*(iii) If $y \in \mathcal{A}(x,\epsilon)$ is such that $T^i(y) \in \mathcal{A}(T^i(x),\epsilon)$ for $i \leq n$ then*

$$\mathit{dist}\,(T^n(y), W^u_\epsilon(T^n(x))) \leq C\theta^n, \text{ as well as } \mathit{dist}\,(y, W^s_\epsilon(x)) \leq C\theta^n \ .$$

*(iv) The family of local stable manifolds (and also the family of local unstable manifolds) form a $C^0$ lamination: the tangent spaces to the leaves vary continuously.*

We do not prove this theorem here since we will not use it, but instead make the following comments. Using the arguments of Hirsch-Pugh in [14], we can prove that the local unstable set is a smooth manifold. The local stable set is also a smooth manifold, but a different proof is needed: one can use the ideas of Irwin in [16]. See also Theorem 2.1 in page 375 of [27]. In both cases the smoothness can be improved to $C^k$ if the operator $T$ is $C^k$.

For invertible operators the global stable set is also a smooth submanifold. In the non-invertible situation, this is not always true. However, we will prove in §9.1 that this is the case for the renormalization operator acting on $\mathbb{V}^r$, provided $r \geq 3 + \alpha$ and $\alpha > 0$ is close to one.

## 2.5. Hyperbolicity of renormalization

In the present paper we prove three main theorems. The first main theorem shows that there exists a real Banach space of analytic maps, containing the topological limit set $\mathbb{K}$ of renormalization, on which the renormalization operator $R$ acts as a real-analytic operator and has $\mathbb{K}$ as a hyperbolic basic set. More precisely, we have the following result.

THEOREM 2.4. *(Hyperbolicity in a real Banach space) There exist $a > 0$, an open set $\mathbb{O} \subset \mathbb{A} = \mathbb{A}_{\Omega_a}$ containing $\mathbb{K} = \mathbb{K}_\Theta$ and a positive integer $N$ with the following property. There exists a real analytic operator $T : \mathbb{O} \to \mathbb{A}$ having $\mathbb{K}$ as a hyperbolic basic set with co-dimension one stable manifolds at each point, such that $T(f)|[-1,1] = R^N(f|[-1,1])$, for all $f \in \mathbb{O}$, is the $N$-th iterate of the renormalization operator.*

The proof of this theorem, presented in §3 (see Theorem 3.9), combines Lyubich's hyperbolicity results with Sullivan's real and complex bounds.

The second main theorem establishes the "hyperbolicity" of renormalization in $\mathbb{U}^r$. As we have mentioned before, the renormalization operator is not smooth in $\mathbb{U}^r$, so the definition of hyperbolicity of an invariant set does not even make sense. However, the hyperbolic picture holds in this situation. More precisely, we have the following theorem.

THEOREM 2.5. *(Hyperbolic Picture in $\mathbb{U}^r$) If $r \geq 2 + \alpha$, where $\alpha > 0$ is close to one, then statements (i), (ii), (iii) and (iv) of Theorem 2.3 hold true for the renormalization operator acting on $\mathbb{U}^r$. Furthermore,*

*(i) the local unstable manifolds are real analytic curves;*

*(ii) the local stable manifolds are of class $C^1$, and together they form a continuous lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$;*

The main difficulty behind the proof of this theorem is the fact the operator $T$ is not Fréchet differentiable in $C^r$ (in fact it is only continuous in a dense subset of $\mathbb{U}^r$). However, as we shall see in §8.2, it is a $C^1$ mapping from its domain in $\mathbb{U}^r$ into $\mathbb{U}^s$ if $s < r - 1$ (even for $s = r - 1$ if $r$ is an integer). Hence its tangent map defines a continuous map $L\colon \mathbb{K} \times \mathbb{A}^r \to \mathbb{K} \times \mathbb{A}^s$ by $L(g, v) = (T(g), DT(g)(v)) = (T(g), L_g(v))$. The bounded linear mappings $L_g\colon \mathbb{A}^r \to \mathbb{A}^s$ extend to bounded linear operators $L_g\colon \mathbb{A}^t \to \mathbb{A}^t$ for all $0 \le t \le r$. Although $L_g$ is not the derivative of $T$ at $g$ in $C^r$, it is nevertheless a sufficiently good linear approximation to $T$ near $g$ (see the properties of Definition 6.1, checked in §8).

Corollary 2.6. *(Hyperbolic Picture in $\mathbb{V}^r$) If $r \ge 2 + \alpha$, where $\alpha > 0$ is close to one, then statements (i), (ii), (iii) and (iv) of Theorem 2.3 hold true for the renormalization operator acting on $\mathbb{V}^r$. Furthermore,*

*(i) the local unstable manifolds are real analytic curves;*

*(ii) the local stable manifolds are of class $C^1$, and together they form a continuous lamination whose holonomy is $C^{1+\beta}$ for some $\beta > 0$.*

For the proofs of Theorem 2.5 and Corollary 2.6, see §8.

By an argument using the implicit function theorem and the results in [24], which *a priori* are valid only in $\mathbb{V}^r$, we shall prove in §9 our third main theorem, which we state as follows.

Theorem 2.7. *If $r \ge 3+\alpha$, where $\alpha > 0$ is close to one, then the following assertions hold true for the renormalization operator acting in $\mathbb{V}^r$:*

*(i) The global stable sets are $C^1$ immersed submanifolds.*

*(ii) For each integer $2 \le k \le r$, there exists an open dense set of $C^k$ one-parameter families of maps in $\mathbb{V}^r$ all of whose elements intersect the global stable lamination of $(T, K_\Theta)$ transversally.*

*(iii) In each such family, the set of parameters where the intersections occur is a Cantor set which is locally $C^{1+\beta}$ diffeomorphic to the corresponding Cantor set of the quadratic family. In particular, its Hausdorff dimension is a universal number depending only on $\Theta$ which lies strictly between zero and one if $\Theta$ has more than one element.*

It is worth emphasizing that when a generic family (in the sense of the above corollary) intersects the stable lamination at a point, then any neighborhood of this point in parameter space contains a renormalization window that is mapped under a suitable power of the renormalization operator onto a *full* transversal family.

## 3. Hyperbolicity in a Banach space of real analytic maps

In this section we give a proof of Theorem 2.4. Using the real and complex bounds given by Sullivan in [28], we prove in §3.1 that there is an iterate of the renormalization operator which extends as a real analytic map $T$ to an open set $\mathbb{O}_{\Omega_a}$ of the Banach space $\mathbb{A}_{\Omega_a}$ consisting of real analytic maps whose domain is an $a$-neighborhood of the interval $[-1,1]$, for a suitable $a > 0$. The maps $g \in \mathbb{K}$ have unique extensions belonging to $\mathbb{O}_{\Omega_a}$. In §3.2, using lemmas 4.16 and 4.17 in Lyubich's paper [20], we show that the hybrid conjugacy classes of the maps $g \in \mathbb{K}$ form a continuous lamination of codimension one real analytic manifolds. Then in §3.4 we construct a skew-product renormalization operator that satisfies properties (W1) to (W4) in page 395 of [20] in the real analytic case (restated in §3.5). By theorems 8.2 and 8.8 in [20] the skew-product renormalization operator will have fiberwise stable and unstable leaves (as defined in §3.3). The local stable leaf at $g \in \mathbb{K}$ is a relatively open set of the hybrid conjugacy class of $g$. Then using the skew-product renormalization operator, we prove in §3.5 that $\mathbb{K}$ is a basic set for the real analytic renormalization operator $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$.

### 3.1. Real analyticity of the renormalization operator

Using Sullivan's real and complex bounds in [28], we will show that there exists $a > 0$ such that some iterate $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$ of the renormalization operator is a (well-defined) real analytic operator with a compact derivative.

For each $f \in \mathbb{K}$, let $\mathcal{I}_f \subseteq [-1,1]$ be the postcritical set of $f$ (the Cantor attractor of $f$). For each $k \geq 0$, we can write

$$R^k f(x) = \Lambda_k^{-1} \circ f^{p_k} \circ \Lambda_k(x)$$

where

$$\begin{aligned} p_k &= p(f,k) = \prod_{i=0}^{k-1} p(R^i f) \ , \\ \lambda_k &= \lambda(f,k) = \prod_{i=0}^{k-1} \lambda(R^i f) \ , \\ \Lambda_k(x) &= \Lambda(f,k)(x) = \lambda_k \cdot x \ , \end{aligned} \qquad (3.1.1)$$

with $p(\cdot)$ and $\lambda(\cdot)$ as defined in §2.2. Consider the renormalization intervals $\Delta_{0,k} = \Delta_{0,k}(f) = [-|\lambda_k|, |\lambda_k|] \subset [-1,1]$, and define $\Delta_{i,k} = \Delta_{i,k}(f) = f^i(\Delta_{0,k})$ for $i = 0, 1, \ldots, p_k - 1$. The collection $\mathbf{C}_k = \{\Delta_{0,k}, \ldots, \Delta_{p_k-1,k}\}$ consists of pairwise disjoint intervals at level $k$. Moreover, $\bigcup\{\Delta : \Delta \in \mathbf{C}_{k+1}\} \subseteq \bigcup\{\Delta :$

$\Delta \in \mathbf{C}_k\}$ for all $k \geq 0$ and we have

$$\mathcal{I}_f = \bigcap_{k=0}^{\infty} \bigcup_{i=0}^{p_k - 1} \Delta_{i,k} \ .$$

DEFINITION 3.1. *The set $\mathcal{I}_f$ has geometry bounded by $0 < \tau < 1$ with respect to $(\mathbf{C}_k)_{k\in\mathbb{N}}$ if the following conditions are met for $k \geq 1$.*

*(i) If $\Delta_{j,k+1} \subset \Delta_{i,k}$ then $\tau < |\Delta_{j,k+1}| \,/\, |\Delta_{i,k}| < 1 - \tau$.*

*(ii) If $I$ is a connected component of $\Delta_{i,k} \setminus \bigcup_j \Delta_{j,k+1}$ then $\tau < |I| \,/\, |\Delta_{i,k}| < 1 - \tau$.*

By Sullivan's real bounds (see [28] and Section VI.2 in page 453 of [26]), there exists $\alpha > 0$, such that for every $g \in \mathbb{K}$ the set $\mathcal{I}_g$ has geometry bounded by $\alpha$ with respect to $(\mathbf{C}_k)_{k\in\mathbb{N}}$.

The following result is a consequence of Sullivan's complex bounds (see [28] and Section VI.5 in page 483 of [26]).

THEOREM 3.1. *There exist $\mu > 0$, $N_0 > 0$ and a neighborhood $V$ of the dynamics with the following properties. Every $g \in \mathbb{K}$ extends to a holomorphic map $g : V \to \mathbb{C}$ and for every $N \geq N_0$ there exists a symmetric neighborhood $O_{g,N}$ of the interval $\Delta_{0,N}(g)$ such that*

*(i) the diameter of the set $g^i(O_{g,N}) \subset V$ is comparable to the length $|\Delta_{i,N}(g)|$ of the interval $\Delta_{i,N}(g)$ for every $0 \leq i \leq p = p(N, g)$;*

*(ii) the map $g^p : O_{g,N} \to g^p(O_{g,N})$ is a quadratic-like map with conformal modulus greater than $\mu > 0$.*

Applying Theorem 3.1 (ii) to $g \in \mathbb{K}$, we see that $R^N(g)$ has a quadratic-like extension to

$$U_{g,N} = \Lambda_g^{-1}(O_{g,N}) \tag{3.1.2}$$

(where $\Lambda_g = \Lambda(g, N)$) and such extension has conformal modulus greater than $\mu > 0$.

Recall that the *filled-in Julia set $\mathcal{K}_f$ of a quadratic-like map $f : U \to U'$* is the set $\{z : f^n z \in U, n = 0, 1, \ldots\}$, and its boundary is the Julia set $\mathcal{J}_f$ of $f$. Since all maps in $\mathbb{K}$ have conformal modulus greater than or equal to $\mu > 0$, we deduce from Proposition 4.8 in page 83 of McMullen's book [23] that there exists $b > 0$ such that for every $g \in \mathbb{K}$ we have

$$\Omega_b(\mathcal{K}_{R^N(g)}) \subset U_{g,N} \ . \tag{3.1.3}$$

Here the notation $\Omega_\varepsilon(K)$ means the set of all points whose distance from $K$ is less than $\varepsilon/2$ times the diameter of $K$.

For each neighborhood $U$ of $[-1,1]$ in $\mathbb{C}$, symmetric about the real axis, we consider the real Banach space $\mathbb{A}_U$ of holomorphic functions defined earlier. We denote by $\mathbb{A}_U(g,\delta)$ the open ball of radius $\delta$ around $g$. By (3.1.3), the inclusion map $i_{g,N}:\mathbb{A}_{U_{g,N}}\to\mathbb{A}_{\Omega_\alpha}$ is a well-defined compact linear operator for every $0<\alpha<b$.

Lemma 3.2. *Let $\mu>0$ and $N_0>0$ be as in Theorem 3.1 and $b>0$ as in* (3.1.3). *For every $0<\alpha<b$ there exist $N>N_0$ and $\delta_0>0$ such that*

*(i) for every $g\in\mathbb{K}$, the operator $T_{g,N}:\mathbb{A}_{\Omega_\alpha}(g,\delta_0)\to\mathbb{A}_{U_{g,N}}$ is well-defined if we set*

$$T_{g,N}(f)={\Lambda_f}^{-1}\circ f^p\circ\Lambda_f:U_{g,N}\to\mathbb{C}\ ,$$

*where $p=p(f,N)=p(g,N)$, $\Lambda_f=\Lambda(f,N)$, and $T_{g,N}(f)$ is a quadratic-like map with conformal modulus greater than $\mu/2$;*

*(ii) the operator $T:\mathbb{O}_{\Omega_\alpha}\to\mathbb{A}_{\Omega_\alpha}$ given by $T=i_{g,N}\circ T_{g,N}$ is real analytic with a compact derivative, where*

$$\mathbb{O}_{\Omega_\alpha}=\bigcup_{g\in\mathbb{K}}\mathbb{A}_{\Omega_\alpha}(g,\delta_0)\ .$$

*Proof.* By Sullivan's real bounds, there exist $C_1>1$ and $0<\nu_1<\nu_2<1$ such that for all $g\in\mathbb{K}$, all $k\in\mathbb{N}$ and all $0\le j\le p(k,g)-1$, we have $C_1^{-1}\nu_1^k<|\Delta_{j,k}(g)|<C_1\nu_2^k$ . Thus, by property (i) in Theorem 3.1, for every $\alpha>0$ there is $N>0$ so large that the open sets $g^j(O_{g,N})$ have diameter smaller than $\alpha/3$ for all $0\le j\le p(N,g)$. Recall that $O_{g,N}=\Lambda_g(U_{g,N})$. By a continuity argument, there is $\delta_g>0$ such that for every $f\in\mathbb{A}_{\Omega_\alpha}(g,\delta_g)$, the restriction $f|[-1,1]$ is $N$-times renormalizable, $f^j(\Lambda_f(U_{g,N}))\subset\Omega_{\alpha/2}$ for every $0\le j\le p=p(N,f)$, and moreover $f^p:\Lambda_f(U_{g,N})\to f^p(\Lambda_f(U_{g,N}))$ is a quadratic-like map with conformal modulus greater than $\mu/2$. By compactness of $\mathbb{K}$ in $\mathbb{A}_{\Omega_\alpha}$, there is a finite set $\{g_i:i=1,\ldots,l\}$ such that

$$\mathbb{K}\subset\bigcup_{i=1}^{l}\mathbb{A}_{\Omega_\alpha}(g_i,\delta_{g_i}/2)\ .$$

Set $\delta_0=\min_{i=1,\ldots,l}\{\delta_{g_i}/2\}$. Then, for every $g\in\mathbb{K}$ there exists $i=i(g)$ such that $\mathbb{A}_{\Omega_\alpha}(g,\delta_0)\subset\mathbb{A}_{\Omega_\alpha}(g_i,\delta_{g_i})$. Hence $T_{g,N}(f)$ is well-defined, and it is a quadratic-like map with conformal modulus greater than $\mu/2$, for every $f\in\mathbb{A}_{\Omega_\alpha}(g,\delta_0)$ which proves (i).

Note that the real Banach space $\mathbb{A}_{\Omega_\alpha}$ is naturally embedded in the complex Banach space $\mathbb{A}_{\Omega_\alpha,\mathbb{C}}$ of maps $f:\Omega_\alpha\to\mathbb{C}$ which are holomorphic and continuous up to the boundary and that $T_{g,N}$ extends to an operator $T^{\mathbb{C}}_{g,N}$ in an open set of $\mathbb{A}_{\Omega_\alpha,\mathbb{C}}$, given by the same expression. Applying Cauchy's integral formula, we see that $T^{\mathbb{C}}_{g,N}$ is complex-analytic, and so $T_{g,N}$ is real analytic. Since by

Montel's theorem the inclusion $i_{g,N}$ is a compact linear operator, we deduce that $T : \mathbb{O}_{\Omega_\alpha} \to \mathbb{A}_{\Omega_\alpha}$ is a real-analytic operator with a compact derivative, which proves (ii). □

## 3.2. Real analytic hybrid conjugacy classes

We will introduce later (in §3.4) a skew-product renormalization operator. The fiberwise local stable manifolds of such skew-product – which will be used to determine the stable manifolds of the real-analytic operator $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$, for some suitable $a > 0$ – turn out to be openly contained in the hybrid conjugacy classes of the maps in the limit set $\mathbb{K}$. Here we analyze the manifold structure of hybrid classes in more detail.

A homeomorphism $h : U \to V$, where $U$ and $V$ are contained in $\mathbb{C}$ or $\overline{\mathbb{C}}$, is *quasiconformal* if it has locally square integrable distributional derivatives $\partial h$, $\overline{\partial} h$, and there exists $\epsilon < 1$ with the property that $\left|\overline{\partial} h / \partial h\right| \leq \epsilon$ almost everywhere. The Beltrami differential $\mu_h$ of $h$ is given by $\mu_h = \overline{\partial} h / \partial h$. A quasiconformal map $h$ is $K$ quasiconformal if $K \geq (1 + ||\mu_h||_\infty)/(1 - ||\mu_h||_\infty)$.

Two quadratic-like maps $f$ and $g$ are *hybrid conjugate* if there is a quasiconformal conjugacy $h$ between $f$ and $g$ with the property that $\overline{\partial} h(z) = 0$ for almost every $z \in \mathcal{K}_f$. Let us denote by $\mathcal{H}(f)$ the hybrid conjugacy class of $f$.

By a slight abuse of notation, we will denote by $\mathbb{K} \cap \mathbb{A}_V(g, \delta)$ the set of maps $f \in \mathbb{A}_V(g, \delta)$ with the property that $f|[-1, 1]$ belongs to $\mathbb{K}$.

In the proof of the following theorem, we will need to work with the complexification of $\mathbb{A}_V$. Let $\mathbb{A}_{V,\mathbb{C}}$ be the complex Banach space of all holomorphic maps $f : V \to \mathbb{C}$ with a continuous extension to the boundary of $V$. Let $\mathbb{A}_{V,\mathbb{C}}(f, \delta)$ be the open ball in $\mathbb{A}_{V,\mathbb{C}}$ centered in $f$ and with radius $\delta > 0$. Let $C : \mathbb{A}_{V,\mathbb{C}} \to \mathbb{A}_{V,\mathbb{C}}$ be the conjugation operator given by $C(f) = c \circ f \circ c$, where $c(z) = \overline{z} \in \mathbb{C}$. We note that $f \in \mathbb{A}_V$ if and only if $f \in \mathbb{A}_{V,\mathbb{C}}$ and $C(f) = f$.

Theorem 3.3. *For every $g \in \mathbb{K}$, there exists a symmetric neighborhood $\hat{V}_g$ of the reals such that $g$ has a quadratic-like extension to $\hat{V}_g$ (which we also denote by $g$), $\hat{V}_g$ contains a definite neighborhood of $\mathcal{K}_g$ and for every neighborhood $V \subset \hat{V}_g$ symmetric with respect to $\mathbb{R}$ and with the property that $g|V$ is a quadratic-like map, there is $\delta_{g,V} > 0$ such that for all $f \in \mathbb{K} \cap \mathbb{A}_V(g, \delta_{g,V})$,*

$$\mathcal{H}_V(f) = \mathcal{H}(f) \cap \mathbb{A}_V(g, \delta_{g,V})$$

*are codimension one real analytic leaves varying continuously with $f$.*

*Proof.* By lemmas 4.16 and 4.17 in page 354 of Lyubich's paper [20], we obtain that for all $f \in \mathbb{K} \cap \mathbb{A}_{V,\mathbb{C}}(g, \delta_{g,V})$, $\mathcal{H}_{V,\mathbb{C}}(f) = \mathcal{H}(f) \cap \mathbb{A}_{V,\mathbb{C}}(g, \delta_{g,V})$ are codimension one complex analytic leaves varying continuously with $f$. If $f \in \mathbb{A}_V(g, \delta_{g,V})$ then the hybrid conjugacy class of $f$ in $\mathbb{A}_{V,\mathbb{C}}(g, \delta_{g,V})$ is invariant under the conjugation operator $C$. Hence, the tangent space $T_f\mathcal{H}_{V,\mathbb{C}}(f)$ at $f$ to its hybrid conjugacy class is invariant under the conjugation operator $C$, and there is a one dimensional transversal $E_f$ to $T_f\mathcal{H}_{V,\mathbb{C}}(f)$ which is also invariant under the conjugation operator $C$. Locally $\mathcal{H}_{V,\mathbb{C}}(f)$ is a graph of $\mathcal{G} : Z \subset T_f\mathcal{H}_{V,\mathbb{C}}(f) \to E_f$ with the property that if $h = v + \mathcal{G}(v)$ then $C(h) = C(v) + \mathcal{G}(C(v))$. Thus, locally $\mathcal{H}_V(f)$ is also the graph of $\mathcal{G}|Z \cap \mathbb{A}_V(g, \delta_{g,V})$, and so it is a codimension one real analytic leaf. Since the complex analytic leaves $\mathcal{H}_{V,\mathbb{C}}(f)$ vary continuously with $f$, we deduce that the real analytic leaves $\mathcal{H}_V(f)$ also vary continuously with $f$. □

### 3.3. Hyperbolic skew-products

Before going further, we pause for a moment to introduce the elementary concept of hyperbolic skew product in an abstract setting. Let $\mathbb{K}$ be a compact metric space and assume that $\mathbb{K}$ is totally disconnected. Let $\mathcal{F}$ be a finite collection of (real) Banach spaces, say $\mathcal{F} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N\}$, and assume we have a locally constant map $\varphi : \mathbb{K} \to \mathcal{F}$. We write $\mathcal{A}_x = \varphi(x) \in \mathcal{F}$, for all $x \in \mathbb{K}$. Let $E = \cup_{x\in\mathbb{K}}\{x\} \times \mathcal{A}_x$. We endow $E$ with a topology as follows. If $\mathbb{K}_i = \varphi^{-1}(\mathcal{A}_i)$, then $\mathbb{K}_i$ is an open and closed set in $\mathbb{K}$, for each $i = 1, 2, \ldots, N$. Note that $E$ is the disjoint union of $\mathbb{K}_i \times \mathcal{A}_i$, $i = 1, 2, \ldots, N$. Hence endow each factor $\mathbb{K}_i \times \mathcal{A}_i$ with the product topology and then $E$ with the union topology. It is clear that $E$ is metrizable also. The natural projection $E \to \mathbb{K}$ is open and continuous. We shall assume that there exists a continuous injection $\mathbb{K}_i \to \mathcal{A}_i$ for each $i$, and will accordingly identify each $x \in \mathbb{K}_i$ with its image in $\mathcal{A}_i$.

Now suppose $T : \mathbb{K} \to \mathbb{K}$ is a homeomorphism (in the case we are interested, $T$ is transitive), and also that for each $x \in \mathbb{K}$ we have a real-analytic map $S_x : \mathcal{A}_x(x, \delta) \to \mathcal{A}_{T(x)}$, where $\mathcal{A}_x(x, \delta) = \{x + v \in \mathcal{A}_x : \|v\|_{\mathcal{A}_x} < \delta\}$. We define a skew-product operator $S : E(\delta) \to E$ over $T$, where

$$E(\delta) = \{(x, y) : x \in \mathbb{K}, y \in \mathcal{A}_x, \|y - x\|_{\mathcal{A}_x} < \delta\},$$

by $S(x, y) = (T(x), S_x(y))$.

Definition 3.2. *We say that $S$ is fiberwise hyperbolic if there exists a continuous spliting $\mathcal{A}_x = E^s_x \bigoplus E^u_x$ with dim $E^u_x = 1$ which is invariant in the sense that $DS_x(E^s_x) \subseteq E^s_{T(x)}$ and $DS_x(E^u_x) \subseteq E^u_{T(x)}$, satisfying for all $v^s \in E^s_x$ and all $v^u \in E^u_x$ the inequalities*

$$\left\|D\left(S_{T^{n-1}(x)} \circ \ldots S_x\right)(x)v^s\right\|_{\mathcal{A}_{T^n(x)}} \le C\theta^n\|v^s\|_{\mathcal{A}_x}$$
$$\left\|D\left(S_{T^{n-1}(x)} \circ \ldots S_x\right)(x)v^u\right\|_{\mathcal{A}_{T^n(x)}} \ge C^{-1}\theta^{-n}\|v^u\|_{\mathcal{A}_x}\ ,$$

*where $C > 1$ and $0 < \theta < 1$ are uniform constants on $g$.*

Definition 3.3. *The fiberwise local stable manifold $W^s_\beta(x)$ of $S$ at $x$ consists of all points $y \in \mathcal{A}_x(x, \beta)$ such that for all $n \ge 1$, we have $S_{T^{n-1}(x)} \circ \ldots \circ$*

$S_x(y) \in \mathcal{A}_{T^n(x)}(T^n(x), \beta)$ *and*

$$\left\| S_{T^{n-1}(x)} \circ \ldots \circ S_x(y) - S_{T^{n-1}(x)} \circ \ldots \circ S_x(x) \right\|_{\mathcal{A}_{T^n(x)}} \leq C\theta^n$$

*where* $C > 0$ *and* $0 < \theta < 1$ *are uniform constants on* $x \in \mathbb{K}$. *The fiberwise local unstable manifold* $W^u_\beta(x)$ *of* $S$ *at* $x$ *consists of all points* $y \in \mathcal{A}_x(x, \beta)$ *such that setting* $y_0 = y$, *for each* $n \geq 1$ *there exists* $y_n \in \mathcal{A}_{T^{-n}(x)}$ *such that* $y_{n-1} = S_{T^{-n}(x)}(y_n)$ *and* $\|T^{-n}(x) - y_n\|_{\mathcal{A}_{T^{-n}(x)}} \leq C\theta^n$.

## 3.4. Skew-product renormalization operator

Our goal in this section is to build a skew-product renormalization operator that will play a central role in the proof that $\mathbb{K}$ is a basic set for $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$, for a suitable $a > 0$. Our skew-product is constructed so as to satisfy properties (W1) to (W4) in page 395 of [20] in the real analytic case – restated in §3.5 – and therefore will have fiberwise stable and unstable manifolds, as we will explain in that section.

Using Theorem 3.1 and (3.1.3), we know that for every $0 < \alpha < b$, $\mathbb{K}$ injects continuously into $\mathbb{A}_{\Omega_\alpha}$. Hence for $f, g \in \mathbb{K}$ we define $\mathrm{dist}_{\mathbb{K}}(f, g) = \|f - g\|_{\mathbb{A}_{\Omega_\alpha}}$. We also denote by $\mathbb{K}(g, \varepsilon)$ the ball of radius $\varepsilon$ centered at $g$ in this metric. The metric is compatible with the natural topology of $\mathbb{K}$, independently of which $\alpha$ we take.

LEMMA 3.4. *The filled-in Julia set* $\mathcal{K}_g$ *varies continuously in the Hausdorff metric with respect to* $g \in \mathbb{K}$.

*Proof.* We need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathrm{dist}_{\mathbb{K}}(f, g) < \delta$ then (a) $\mathcal{K}_g \subset \Omega_\varepsilon(\mathcal{K}_f)$ and (b) $\mathcal{K}_f \subset \Omega_\varepsilon(\mathcal{K}_g)$. Let $U = U_{R^{-N_0}(g), N_0} \subset \mathbb{C}$ be the symmetric neighborhood of $[-1, 1]$ given by Lemma 3.2. Since the operator $T_{R^{-N_0}(g), N_0}$ is continuous, every $f \in \mathbb{K}$ sufficiently close to $g$ in $\mathbb{A}_{\Omega_\varepsilon}$ is quadratic-like on $U$ ($f = T_{R^{-N_0}(g), N_0}(T^{-1}(f)) : U \to \mathbb{C}$ ) and is also close to $g$ in $\mathbb{A}_U$.

To prove (a), cover $\mathcal{K}_g$ by finitely many disks $D(z_i(g), \varepsilon/2)$, $i = 1, 2, \ldots, m$, where each $z_i(g)$ is an expanding periodic point of $g$. For $f$ sufficiently close to $g$, the corresponding periodic points $z_i(f) \in D(z_i(g), \varepsilon/2)$. Hence each $z \in \mathcal{K}_g$ is at distance at most $\varepsilon$ from some $z_i(f)$, which proves (a).

To prove (b), let $n > 0$ be so large that $W = g^{-n}(U) \subset \Omega_\varepsilon(\mathcal{K}_g)$. Since $f$ is close to $g$ and $W \subseteq U$ is symmetric $f : W \to f(W)$ is quadratic-like also, whence $\mathcal{K}_f \subset W \subset \Omega_\varepsilon(\mathcal{K}_g)$ and so (b) is proved. □

LEMMA 3.5. *Let* $g \in \mathbb{K}$ *and let* $V \subset \mathbb{C}$ *be a symmetric neighborhood of* $[-1, 1]$ *which is compactly contained in* $\Omega_{b/2}(\mathcal{K}_g)$, *where* $b$ *is given by* (3.1.3). *Then for all* $\varepsilon > 0$ *sufficiently small* $\mathbb{K} \cap \mathbb{A}_V(g, \varepsilon)$ *is an open subset of* $\mathbb{K}$.

*Proof.* Take $0 < \alpha < b$ sufficiently small such that $\Omega_\alpha$ is compactly contained in $V$. By Theorem 3.1 and (3.1.3), every $f \in \mathbb{K}$ is well-defined on $\Omega_b(\mathcal{K}_f)$. Since by Lemma 3.4 the map $f \mapsto \mathcal{K}_f$ is continuous in the Hausdorff metric, there exists $\varepsilon_0 > 0$ such that if $f \in \mathbb{K}$ is such that $\mathrm{dist}_{\mathbb{K}}(f, g) < \varepsilon_0$ then $\Omega_{b/2}(\mathcal{K}_g) \subset \Omega_b(\mathcal{K}_f)$. Since $\overline{V} \subset \Omega_{b/2}(\mathcal{K}_g)$, it follows that $f$ is well-defined on $V$, that is $f \in \mathbb{A}_V$. Hence there is a well-defined injection $\mathbb{K}(g, \varepsilon_0) \to \mathbb{A}_V$. Such injection is continuous. Indeed, for $f \in \mathbb{K}(g, \varepsilon_0)$, the $C^0$ norm of $f$ in $\Omega_{b/2}(\mathcal{K}_g)$ is uniformly bounded, while $\|f\|_{\mathbb{A}_{\Omega_\alpha}}$ varies continuously with $f$. Since $\overline{\Omega_\alpha} \subset V \subset \overline{V} \subset \Omega_{b/2}(\mathcal{K}_g)$, we deduce from Hadamard's three circles theorem (see Lemma 11.5 in page 415 of [20]) that $\|f\|_{\mathbb{A}_V}$ varies continuously with $f$ also. Therefore the map $\mathbb{K}(g, \varepsilon_0) \to \mathbb{A}_V$ is continuous as asserted. Now let $f \in \mathbb{K} \cap \mathbb{A}_V(g, \varepsilon_0)$. Since the inclusion $\mathbb{A}_V \to \mathbb{A}_{\Omega_\alpha}$ has Lipschitz constant one, we have that $f \in \mathbb{K}(g, \varepsilon_0)$. Hence, by continuity of the map $\mathbb{K}(g, \varepsilon_0) \to \mathbb{A}_V$, there exists $\varepsilon_1 > 0$ such that $\mathbb{K}(f, \varepsilon_1) \subseteq \mathbb{K} \cap \mathbb{A}_V(g, \varepsilon_0)$, which shows that this last set is open in $\mathbb{K}$. This completes the proof. □

Lemma 3.6. *Let $b > 0$ be as defined in* (3.1.3) *and $\delta_0 > 0$ as in Lemma 3.2. There exist $\nu > 0$, $0 < \delta < \delta_0$, a finite set $\mathcal{V}$ of symmetric neighborhoods of $[-1, 1]$ and a locally constant map $\mathbb{K} \ni g \mapsto V_g \in \mathcal{V}$ with the following properties:*

*(i) The neighborhood $V_g$ is compactly contained in $\Omega_{b/2}(\mathcal{K}_g)$;*

*(ii) Every $f \in \mathbb{A}_{V_g}(g, \delta)$ is a quadratic-like map with conformal modulus larger than $\nu$;*

*(iii) If $f \in \mathbb{K} \cap \mathbb{A}_{V_g}(g, \delta)$ then $\mathcal{H}(f) \cap \mathbb{A}_{V_g}(g, \delta)$ is a codimension one real analytic submanifold varying continuously with $f$.*

*Proof.* For every $g \in \mathbb{K}$, let $U_g \subset \mathbb{C}$ be a symmetric neighborood of $[-1, 1]$ where $g$ is quadratic-like, and take $n_g > 0$ so large that $V'_g = g^{-n_g}(U_g) \subset \Omega_{b/3}(\mathcal{K}_g)$ and $V'_g \subseteq \hat{V}_g$, where $\hat{V}_g$ is as given in Theorem 3.3.

Let $\delta_g > 0$ be so small that each $f \in \mathbb{A}_{V'_g}(g, \delta_g)$ is quadratic-like in $V'_g$ with conformal modulus greater than $\nu_g > 0$ and also so that Theorem 3.3 holds true (for $V'_g$ and $\delta_g$). By Lemma 3.4, making $\delta_g$ smaller if necessary, we see that $V'_g = g^{-n_g}(U_g) \subset \Omega_{b/2}(\mathcal{K}_f)$ for all $f \in \mathbb{K} \cap \mathbb{A}_{V'_g}(g, \delta_g)$.

By Lemma 3.5, each set $\mathbb{K} \cap \mathbb{A}_{V'_g}(g, \delta_g/2)$ is open in $\mathbb{K}$. Since $\mathbb{K}$ is compact, there exists a finite set $\{g_i : i = 1, \ldots, l\}$ such that

$$\mathbb{K} \subset \bigcup_{i=1}^{l} \mathbb{A}_{V'_{g_i}}(g_i, \delta_{g_i}/2) \ .$$

Thus we can set

$$\mathcal{V} = \left\{V'_{g_i} : i = 1, \ldots, l\right\}, \ \delta = \min_{i=1,\ldots,l} \{\delta_{g_i}/2\} \quad \text{and} \quad \nu = \min_{i=1,\ldots,l} \{\nu_{g_i}\} \ .$$

Therefore, since $\mathbb{K}$ is totally disconnected, there exists a locally constant map $\mathbb{K} \ni g \mapsto V_g \in \mathcal{V}$ so that properties (i), (ii) and (iii) are satisfied. □

We are now in a position to define the skew-product renormalization operator. This is accomplished in our next lemma. Let us define first its range and domain, respectively, as follows

$$E = \{(g, f) : g \in \mathbb{K} \text{ and } f \in \mathbb{A}_{V_g}\}$$
$$E(\delta) = \{(g, f) \in E : f \in \mathbb{A}_{V_g}(g, \delta)\} \ .$$

Let us now fix once and for all $a > 0$ so small that $\overline{\Omega_a} \subset V_g$ for every $g \in \mathbb{K}$ (this is possible because $\mathcal{V}$ in Lemma 3.6 is a finite set). The inclusion $k_g : \mathbb{A}_{V_g} \to \mathbb{A}_{\Omega_a}$ is a well-defined compact linear operator. By (3.1.3) and Lemma 3.6 (i) we also have

$$V_{R^N(g)} \subset \Omega_{b/2}(\mathcal{K}_{R^N(g)}) \subset \Omega_b(\mathcal{K}_{R^N(g)}) \subset U_{g,N} \ .$$

Therefore the inclusion $j_{g,N} : \mathbb{A}_{U_{g,N}} \to \mathbb{A}_{V_{R^N(g)}}$ is also a well-defined compact linear operator.

LEMMA 3.7. *Let $\delta > 0$ and $V_g \in \mathcal{V}$ be as in Lemma 3.6. Let $N = N(a) > 0$, $T_{g,N}$ and $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$ be as in Lemma 3.2.*

*(i) For every $g \in \mathbb{K}$, the operator $T_g : \mathbb{A}_{\Omega_a}(g, \delta) \to \mathbb{A}_{V_{R^N(g)}}$ given by $T_g = j_{g,N} \circ T_{g,N}$ is real analytic with a compact derivative.*

*(ii) The skew-product renormalization operator $S : E(\delta) \to E$ given by $S(g, f) = (T(g), S_g(f))$, where $S_g = T_g \circ k_g : \mathbb{A}_{V_g}(g, \delta) \to \mathbb{A}_{V_{T(g)}}$, is well-defined. Furthermore,*

$$k_{T(g)} \circ S_g = T \circ k_g \ . \tag{3.4.1}$$

*Proof.* The proof is similar to the proof of Lemma 3.2 (ii). □

## 3.5. Hyperbolicity of the renormalization operator

The purpose of this section is to show that $\mathbb{K}$ is a hyperbolic basic set for the operator $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$. This will follow from the fact (Lemma 3.8 below) that the skew-product renormalization operator has fiberwise real analytic stable manifolds and fiberwise one dimensional real analytic unstable manifolds.

We start by noting that our skew-product operator satisfies the conditions W1-W4 in page 395 of Lyubich [20] in the real analytic case. Namely, we have

W1. The conformal modulus of each $g \in \mathbb{K}$ is larger than an uniform constant $\mu > 0$.

W2. There exists $\eta > 0$ such that if $\mathrm{dist}_{\mathbb{K}}(f, g) < \eta$ for some $f, g \in \mathbb{K}$, then $\mathbb{A}_{V_f} = \mathbb{A}_{V_g}$.

W3. There exists $\delta > 0$ such that $S_g(\mathbb{A}_{V_g}(g,\delta)) \subseteq \mathbb{A}_{V_{T(g)}}$.

W4. The vertical fibers $\mathcal{Z}_g$ (consisting of those normalized symmetric quadratic-like germs whose external class is the same as that of $g$) sit locally in $\mathbb{A}_{V_g}$ for each $g \in \mathbb{K}$.

Condition W1 is satisfied because of the complex bounds (Theorem 3.1). Condition W2 follows from Lemma 3.6. Condition W3 holds by the construction of $S_g$ in Lemma 3.7. Condition W4 is a consequence of Lemma 3.6 (iii).

Now we have the following result.

LEMMA 3.8. *The skew-product renormalization operator $S : E(\delta) \to E$ defined in Lemma 3.7 is fiberwise hyperbolic. Moreover*

*(i) The local stable set $W^s_\delta(g)$ of $S$ at $g$ is a co-dimension one sub-manifold of $\mathbb{A}_{V_g}$ which is relatively open in $\mathcal{H}(g) \cap \mathbb{A}_{V_g}(g,\delta)$, and $W^s_\delta(g)$ is tangent to $E^s_g$ at $g$.*

*(ii) The local unstable set $W^u_\delta(g) \subset \mathbb{A}_{V_g}$ of $S$ at $g$ is a one-dimensional real analytic manifold, and $\{g\} \times W^u_\delta(g)$ varies continuously with $g \in \mathbb{A}$ in $E$.*

*Proof.* Since the operator $S$ satisfies Lyubich's conditions W1-W4 stated above, part (i) follows from Theorem 8.2 in page 392 of [20] and Theorem 3.3, and part (ii) follows from Theorem 8.8 in page 398 of Lyubich's paper [20]. □

THEOREM 3.9. *Let $T : \mathbb{O}_{\Omega_a} \to \mathbb{A}_{\Omega_a}$ be the real analytic operator defined in Lemma 3.7. Then there is a continuous, $DT$-invariant splitting $\mathbb{A}_{\Omega_a} = E^s_g \bigoplus E^u_g$, for $g \in \mathbb{K}$, such that if $v^u \in E^u_g$ and $v^s \in E^s_g$ then*

$$\|DT^n(g)v^u\|_{\mathbb{A}_{\Omega_a}} \geq C^{-1}\theta^{-n}\|v^u\|_{\mathbb{A}_{\Omega_a}} \tag{3.5.1}$$

$$\|DT^n(g)v^s\|_{\mathbb{A}_{\Omega_a}} \leq C\theta^{n}\|v^s\|_{\mathbb{A}_{\Omega_a}} , \tag{3.5.2}$$

*where $C > 1$ and $0 < \theta < 1$ are uniform constants on $g$.*

*Proof.* Since for every $g \in \mathbb{K}$ the map $k_g : \mathbb{A}_{V_g} \to \mathbb{A}_{\Omega_a}$ is linear and injective, it follows from Lemma 3.8 (ii) that $Z^u_g = k_g(W^u_\delta(g))$ is a real analytic one dimensional manifold varying continuously with $g$. Let $w_g$ be the unitary vector tangent to $W^u_\delta(g)$ at $g$. Then $v_g = k_g(w_g)$ is a vector tangent to $Z^u_g$ at $g$ and also varies continuously with $g$. Since $k_g$ and $k_{T(g)}$ are linear maps we see from (3.4.1) that if $\lambda_g$ is such that $DS_g(g)w_g = \lambda_g w_{T(g)}$ then $DT(g)v_g = \lambda_g v_{T(g)}$. Thus a natural candidate for $E^u_g$ is the one dimensional linear subspace generated by $v_g$. In particular, (3.5.1) is satisfied.

Let us find the natural candidate for $E^s_g$. We have that $DT_g(g)v_g = w_{T(g)}$ and by hypothesis $w_{T(g)}$ is transversal to the tangent space of $W^s_\delta(T(g))$. Thus, by the implicit function theorem $Z^s_g = T_g^{-1}(W^s_\delta(Tg))$ is a codimension one manifold transversal to $Z^u_g$. Taking $E^s_g$ equal to the tangent space of $Z^s_g$, we obtain that $E^s_g \bigoplus E^u_g = \mathbb{A}_{\Omega_a}$. By (3.4.1), we have that a neighborhood of $T(g)$ intersected with $Z^u_{T(g)}$ is contained in $T(Z^u_g)$ and a neighborhood of $T(g)$ intersected with $T(Z^s_g)$ is contained in $Z^s_{T(g)}$, which implies that the spliting $E^s_g \bigoplus E^u_g$ is invariant under $DT$. From assertion (i) in Lemma 3.8, we obtain that $E^s_g$ varies continuously with $g$ and so the spliting $E^s_g \bigoplus E^u_g$ also varies continuously with $g$.

Finally, let $M > 0$ be such that $\|DT_g(g)\|_{\mathbb{A}_{\Omega_a}} \leq M$ and note that $\|k_g\|_{\mathbb{A}_{\Omega_a}} \leq 1$ for all $g \in \mathbb{K}$. For all $v^s \in E^s_g$ with unit norm, let $u^s = DT_g(g)v^s \in \mathbb{A}_{V_{T(g)}}$. By Lemma 3.8 (i) and (3.4.1) there exists $C_1 > 1$ and $0 < \theta < 1$ such that

$$\begin{aligned}\|DT^n(g)v^s\|_{\mathbb{A}_{\Omega_a}} &= \|k_{T^n(g)} \circ DS_{T^{n-1}(g)}(T^{n-1}(g)) \circ \ldots \circ DS_{T(g)}(T(g))u^s\|_{\mathbb{A}_{\Omega_a}} \\ &\leq \|DS_{T^{n-1}(g)}(T^{n-1}(g)) \circ \ldots \circ DS_{T(g)}(T(g))u^s\|_{\mathbb{A}_{V_{T^n(g)}}} \\ &\leq C_1 M \theta^{n-1} ,\end{aligned}$$

which shows that (3.5.2) is satisfied. This completes the proof. □

With the above results, we have therefore established Theorem 2.4, to the effect that a suitable power of the renormalization operator is indeed hyperbolic in a suitable (real) Banach space of real analytic mappings. From now on, we shall concentrate on the problem of extending such hyperbolicity to larger ambient spaces of smooth mappings. Our journey will take us far into the wilderness of non-linear functional analysis.

## 4. Extending invariant splittings

In this section we prove a certain result from functional analysis (Theorem 4.1 below) that is absolutely crucial for the stable manifold theorem that we shall prove later. This result deals with the notion of compatibility presented below and is a strong generalization of a key idea of Davie in [5]. In §5, we shall use the results presented here to show that the invariant splitting for the renormalization operator $T$ in $\mathbb{A}_{\Omega_a}$ of §3 extends to an invariant splitting for the action of $T$ in the larger spaces $\mathbb{A}^r$ of $C^r$ maps.

### 4.1. Compatibility

We are interested in the answer to the following question. Given a smooth operator $T : \mathcal{O} \to \mathcal{A}$ having a hyperbolic basic set $\mathbb{K}$, and given a larger ambient space $\mathcal{B} \supseteq \mathcal{A}$ to which $T$ extends (not necessarily smoothly), under which conditions does $\mathbb{K}$ have a hyperbolic structure in $\mathcal{B}$? To give a precise meaning to this question (and then answer it!) we introduce the following notion.

We have a natural continuous map $L : \mathbb{K} \to \mathcal{L}(\mathcal{A}, \mathcal{A})$ given by

$$\begin{aligned} \mathbb{K} \ni x \ \mapsto \ & L_x : \mathcal{A} \to \mathcal{A} \\ & L_x(v) = DT(x)\, v \ . \end{aligned}$$

We will also assume that for every $x \in \mathbb{K}$, $E^u_x$ is a one-dimensional subspace and that we can chose a unit vector $\mathbf{u}_x \in E^u_x$ varying continuously with $x$ so that $L_x(\mathbf{u}_x) = \delta_x \cdot \mathbf{u}_{T(x)}$ with $\delta_x > 0$. In the case of the renormalization operator there is a natural choice for the vectors $\mathbf{u}_x$: choose the unit vector pointing in the direction of increasing topological entropy.

For every $x \in \mathbb{K}$, we denote $DT^n(x) = L_{T^{n-1}(x)} \circ \cdots \circ L_x$ by $L^{(n)}_x$ and $\delta_{T^{n-1}(x)} \cdots \delta_x$ by $\delta^{(n)}_x$. By hyperbolicity of $\mathbb{K}$, there exist $C_0 > 0$ and $\lambda > 1$ such that for every $x \in \mathbb{K}$ and every $n \geq 1$ we have

$$\delta^{(n)}_x > C_0 \lambda^n \ . \tag{4.1.1}$$

We denote by $\mathcal{X}(r)$ the open ball in the Banach space $\mathcal{X}$ centered at the origin and with radius $r > 0$.

Definition 4.1. *Let $\theta < \rho < \lambda$ where $\theta$ is the contraction exponent of the hyperbolic basic set $\mathbb{K}$ of the operator $T$ and $\lambda$ is as in* (4.1.1)*. The pair $(\mathcal{B}, \mathcal{C})$ is $\rho$-compatible with $(T, \mathbb{K})$ if the following conditions are satisfied.*

**A1**. *The inclusions $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ are compact operators.*

**A2**. *There exists $M > 0$ such that each linear operator $L_x = DT(x)$ extends to a linear operator $\hat{L}_x : \mathcal{C} \to \mathcal{C}$ with*

$$\begin{aligned} \left\| \hat{L}_x \right\|_{\mathcal{C}} &< M \\ \hat{L}_x(\mathcal{B}) &\subset \mathcal{B} \\ \left\| \hat{L}_x(v) \right\|_{\mathcal{B}} &< M \|v\|_{\mathcal{B}} \end{aligned}$$

**A3**. *The map $\tilde{L} : \mathbb{K} \to \mathcal{L}(\mathcal{B}, \mathcal{C})$ given by $\tilde{L}_x = \hat{L}_x | \mathcal{B}$ is continuous.*

**A4**. *There exists $\Delta > 1$ such that $\mathcal{B}(\Delta) \cap \mathcal{A}$ is $\mathcal{C}$-dense in $\mathcal{B}(1)$.*

**A5**. *There exist $K > 1$ and a positive integer $m$ such that*

$$\left\| \hat{L}^{(m)}_x(v) \right\|_{\mathcal{B}} \leq \max \left\{ \frac{\rho^m}{2(1+\Delta)} \|v\|_{\mathcal{B}}, K \|v\|_{\mathcal{C}} \right\} \ .$$

Remark 4.1. *Note that neither the map $\hat{L} : \mathbb{K} \times \mathcal{C} \to \mathbb{K} \times \mathcal{C}$ given by $\hat{L}(x, v) = (T(x), \hat{L}_x(v))$ nor its restriction from $\mathbb{K} \times \mathcal{B}$ to $\mathbb{K} \times \mathcal{B}$ are necessarily continuous.*

EXAMPLE 4.1. *As we know from Theorem 2.4, $\mathbb{K}$ is a hyperbolic basic set of the renormalization operator $T = R^N : \mathbb{O} \to \mathbb{A}$. In §5 (see Theorem 5.1), we will show that the pair $(\mathbb{A}^r, \mathbb{A}^0)$ is $\rho$-compatible for $r$ sufficiently close to 2 and 1-compatible for $r > 2$ non-integer.*

Let $\pi_x^u : \mathcal{A} \to E_x^u$ and $\pi_x^s : \mathcal{A} \to E_x^s$ be the canonical projections. We define $P_x = \pi_{T(x)}^u \circ L_x$ and $Q_x = \pi_{T(x)}^s \circ L_x$ which have the property that $L_x = P_x + Q_x$ and that $P_{T(x)}Q_x = Q_{T(x)}P_x = 0$. We also define the linear functional $\sigma_x : \mathcal{A} \to \mathbb{R}$ by $\pi_x^u(v) = \sigma_x(v)\mathbf{u}_x$, and observe that $P_x(v) = \delta_x\sigma_x(v)\mathbf{u}_{T(x)}$. We note that the map $\sigma : \mathbb{K} \to \mathcal{L}(\mathcal{A}, \mathbb{R})$ which associates to each $x$ the linear functional $\sigma_x$ is continuous.

THEOREM 4.1. *If $(\mathcal{B}, \mathcal{C})$ is $\rho$-compatible with $(T, \mathbb{K})$ then each stable functional $\sigma_x$ extends to a unique linear functional $\hat{\sigma}_x \in \mathcal{B}^*$ satisfying*

$$\left\| \hat{L}_x^{(n)}(v) - \delta_x^{(n)}\hat{\sigma}_x(v)\mathbf{u}_{T^n(x)} \right\|_{\mathcal{B}} \leq C\hat{\theta}^n \|v\|_{\mathcal{B}} \tag{4.1.2}$$

*for some $C > 0$ and $0 < \hat{\theta} < \rho$. Furthermore, the map $\hat{\sigma} : \mathbb{K} \to \mathcal{L}(\mathcal{B}, \mathbb{R})$ which associates to each $x$ the linear functional $\hat{\sigma}_x$ is continuous.*

*Proof.* Let $m$ and $M$ be as given in Definition 4.1. Since by property **A1** the $\mathcal{C}$-closure of $\mathcal{B}(1)$ is compact, and since by property **A4** the intersection $\mathcal{A} \cap \mathcal{B}(\Delta)$ is $\mathcal{C}$-dense in $\mathcal{B}(1)$ we can find a finite set

$$\Phi \subseteq \mathcal{A} \cap \mathcal{B}(\Delta)$$

such that for each $w \in \mathcal{B}(1)$ there exists $w' \in \Phi$ such that

$$\|w - w'\|_{\mathcal{C}} < \frac{\rho^m}{4K} .$$

Now let $v \in \mathcal{B}(1)$, and let $v_0 \in \Phi$ be such that

$$\|v - v_0\|_{\mathcal{C}} < \frac{\rho^m}{2K} .$$

Since $\|v - v_0\|_{\mathcal{B}} < 1 + \Delta$, applying the inequality of property **A5** to $v - v_0$ yields

$$\begin{aligned} \left\| \hat{L}_x^{(m)}(v - v_0) \right\|_{\mathcal{B}} &\leq \max\left\{ \frac{\rho^m}{2(1+\Delta)} \|v - v_0\|_{\mathcal{B}}, K\|v - v_0\|_{\mathcal{C}} \right\} \\ &< \rho^m/2 . \end{aligned}$$

Therefore $\hat{L}_x^{(m)}(v) = \hat{L}_x^{(m)}(v_0) + (\rho^m/2)w_1$ for some $w_1 \in \mathcal{B}(1)$. Repeating the argument with $w_1$ replacing $v$ and proceeding inductively in this fashion, we get after $k$ steps

$$\hat{L}_x^{(km)}(v) = \sum_{j=0}^{k-1} \frac{\rho^{jm}}{2^j} L_{T^{jm}(x)}^{((k-j)m)}(v_j) + \frac{\rho^{km}}{2^k} w_k$$

for some $w_k \in \mathcal{B}(1)$ and $v_j \in \Phi$. Now recall that

$$\begin{aligned} L^{((k-j)m)}_{T^{jm}(x)}(v_j) &= P^{((k-j)m)}_{T^{jm}(x)}(v_j) + Q^{((k-j)m)}_{T^{jm}(x)}(v_j) \\ &= \delta^{((k-j)m)}_{T^{jm}(x)} \sigma_{T^{jm}(x)}(v_j) u_{T^{km}(x)} + Q^{((k-j)m)}_{T^{jm}(x)}(v_j) \ . \end{aligned}$$

Hence we can write

$$\begin{aligned} \hat{L}^{(km)}_x(v) &= \delta^{(km)}_x \left( \sum_{j=0}^{k-1} \frac{1}{2^j} \frac{\rho^{jm}}{\delta^{(jm)}_x} \sigma_{T^{jm}(x)}(v_j) \right) \mathbf{u}_{T^{km}(x)} \\ &+ \sum_{j=0}^{k-1} \left(\frac{\rho^m}{2}\right)^j Q^{((k-j)m)}_{T^{jm}(x)}(v_j) + \frac{\rho^{km}}{2^k} w_k \end{aligned} \tag{4.1.3}$$

The first summation in parentheses converges to a limit because, by (4.1.1), $\left|\delta^{(jm)}_x\right| \geq C_0 \lambda^{jm} > C_0 \rho^{jm}$ and $\left\{\sigma_{T^{jm}(x)}(v_j)\right\}$ is bounded, as the $v_j$ run through finitely many values and $\left\|\sigma_{T^i(x)}\right\| \leq M$ for all $i$. We therefore define

$$\hat{\sigma}_x(v) \ = \ \lim_{k\to\infty} \sum_{j=0}^{k-1} \frac{1}{2^j} \frac{\rho^{jm}}{\delta^{(jm)}_x} \sigma_{T^{jm}(x)}(v_j) \ . \tag{4.1.4}$$

It will be clear in a moment that this extension of $\sigma_x$ is independent of the choices of approximants $v_j$ performed above, linear, continuous, and the unique extension satisfying (4.1.2). We know that

$$\left\| Q^{((k-j)m)}_{T^{jm}(x)} \right\|_{\mathcal{A}} \leq C_1 \theta^{m(k-j)}$$

for all $j < k$. Thus the second summation plus the last term in (4.1.3) add up to a vector with $\mathcal{B}$-norm bounded by

$$C_2 \left[ \sum_{j=0}^{k-1} \frac{1}{2^j} \left(\frac{\theta}{\rho}\right)^{m(k-j)} + \frac{1}{2^k} \right] \rho^{km} \ .$$

This gives

$$\left\| \hat{L}^{(km)}_x(v) - \delta^{(km)}_x \hat{\sigma}_x(v) \mathbf{u}_{T^{km}(x)} \right\|_{\mathcal{B}} \leq \ C_3(k+1)\beta^k \rho^{km} \ , \tag{4.1.5}$$

where $\beta = \max\{1/2, \theta/\rho\} < 1$. Now choose $0 < \hat{\theta} < \rho$ so that $(k+1)\beta^k < C_4(\frac{\hat{\theta}}{\rho})^{km}$ for all $k$. Since by property **A2** for all $x \in \mathbb{K}$ and for all $v \in \mathcal{B}$ we have $\|\hat{L}_x(v)\|_{\mathcal{B}} < M\|v\|_{\mathcal{B}}$, writing $n = km + r$ and using the above estimates we obtain the desired inequality (4.1.2).

Let us now verify that $\hat{\sigma}_x(v)$ is the unique value satisfying (4.1.2). In particular, it does not depend on the choices of approximants $v_j$ taken in (4.1.4). To do this we represent by $\sigma^*_x(v)$ a value satisfying (4.1.2), for instance obtained in (4.1.4) by taking another choice of approximants. Therefore we have

$$\begin{aligned}\left\|\hat\sigma_x(v)\mathbf{u}_{T^{km}(x)} - \sigma^*_x(v)\mathbf{u}_{T^{km}(x)}\right\|_{\mathcal{B}} \leq & \left\|\hat\sigma_x(v)\mathbf{u}_{T^{km}(x)} - \left(\delta_x^{(km)}\right)^{-1}\hat L_x^{(km)}(v)\right\|_{\mathcal{B}} \\ & + \left\|\left(\delta_x^{(km)}\right)^{-1}\hat L_x^{(km)}(v) - \sigma^*_x(v)\mathbf{u}_{T^{km}(x)}\right\|_{\mathcal{B}} \\ \leq & \, 2C\hat\theta^{km}\left(\delta_x^{(km)}\right)^{-1} .\end{aligned}$$

Letting $k \to \infty$ in this inequality we deduce that $\hat\sigma_x(v) = \sigma^*_x(v)$. A similar argument shows that $\hat\sigma_x$ is linear. Using inequality (4.1.5) with $k = 1$, we obtain that $\|\hat\sigma_x\|_{\mathcal{B}}$ is bounded. Finally, the fact that $\hat\sigma_x$ is continuous in $x$ can be deduced from (4.1.4) using property **A3**. □

Corollary 4.2. *Let $(\mathcal{B},\mathcal{C})$ be $\rho$-compatible with $(T,\mathcal{A})$. Let the linear functional $\hat\sigma_x \in \mathcal{B}^*$ be the extension of the stable functional $\sigma_x$ satisfying inequality* (4.1.2) *for all $x \in \mathbb{K}$. Then, there exists a continuous splitting $\mathcal{B} = \hat E^s_x \bigoplus \hat E^u_x$ with the following properties:*

*(i) $\hat E^u_x$ is the inclusion in $\mathcal{B}$ of the unstable linear space $E^u_x \subset \mathcal{A}$;*

*(ii) $\hat E^s_x = \mathrm{Ker}(\hat\sigma_x)$;*

*(iii) the splitting is invariant by $\hat L_x$;*

*(iv) there exist a constant $C > 0$ such that*

$$\left\|\hat L^m_x(v)\right\|_{\mathcal{B}} \geq C\lambda^m\|v\|_{\mathcal{B}}$$

*for all $x \in \mathbb{K}$, for all $v \in \hat E^u_x$, and for all $m \in \mathbb{N}$ (where $\lambda > 1$ is the same as in* (4.1.1)*);*

*(v) there exist constants $C > 0$ and $0 < \hat\theta < \rho < \lambda$ such that*

$$\left\|\hat L^m_x(v)\right\|_{\mathcal{B}} \leq C\hat\theta^m\|v\|_{\mathcal{B}}$$

*for all $x \in \mathbb{K}$, for all $v \in \hat E^s_x$, and for all $m \in \mathbb{N}$. In particular, if $\rho \leq 1$ then $\hat\theta < 1$.*

*(vi) Let $\hat\pi^s_x : \mathcal{B} \to \hat E^s_x$ and $\hat\pi^u_x : \mathcal{B} \to \hat E^u_x$ be the natural projections such that*

$$\hat\pi^s_x \circ \hat\pi^s_x = \hat\pi^s_x,\ \hat\pi^u_x \circ \hat\pi^u_x = \hat\pi^u_x \text{ and } \hat\pi^s_x \circ \hat\pi^u_x = \hat\pi^u_x \circ \hat\pi^s_x = 0 .$$

*Let us define the operators $\hat{Q}_x : \mathcal{B} \to \mathcal{B}$ and $\hat{P}_x : \mathcal{B} \to \mathcal{B}$ by $\hat{Q}_x^m = \hat{L}_x^m \circ \hat{\pi}_x^s$ and by $\hat{P}_x^m = \hat{L}_x^{\circ} \hat{\pi}_x^u$. Then, there exists $C > 1$ such that*

$$\left\| \hat{Q}_x^m \right\|_{\mathcal{B}} \leq C \hat{\theta}^m$$
$$\left\| \hat{P}^m \right\|_{\mathcal{B}} \geq C^{-1} \lambda^m$$

*for all $x \in \mathbb{K}$ and for all $m \in \mathbb{N}$.*

*Proof.* First, we observe that for all $x \in \mathbb{K}$ and for all $v \in \mathcal{B}$, we can write $v = (v - \hat{\sigma}_x(v)\mathbf{u}_x) + \hat{\sigma}_x(v)\mathbf{u}_x$, where $v - \hat{\sigma}_x(v)\mathbf{u}_x \in \operatorname{Ker}(\hat{\sigma}_x)$ and $\hat{\sigma}_x(v)\mathbf{u}_x \in \hat{E}_x^u$. Since $\hat{\sigma}_x(\mathbf{u}_x) = 1 \neq 0$, we obtain that $\mathcal{B} = \hat{E}_x^s \bigoplus \hat{E}_x^u$.

By inequality (4.1.2), there exists $C > 0$ such that

$$\left\| \hat{L}_x^{(m)}(v) \right\|_{\mathcal{B}} \leq C \hat{\theta}^m \|v\|_{\mathcal{B}} \tag{4.1.6}$$

for all $x \in \mathbb{K}$, for all $v \in \operatorname{Ker}(\hat{\sigma}_x)$, and for all $m \in \mathbb{N}$; if $v \in \mathcal{B} \setminus \operatorname{Ker}(\hat{\sigma}_x)$ then there exists $C_v > 0$ such that

$$\left\| \hat{L}_x^{(m)}(v) \right\|_{\mathcal{B}} \geq C_v \lambda^m \ .$$

Therefore, $\hat{L}_x(\operatorname{Ker}(\hat{\sigma}_x)) \subset \operatorname{Ker}(\hat{\sigma}_{T(x)})$. Since $L_x(E_x^u) = E_{T(x)}^u$ implies that $\hat{L}_x(\hat{E}_x^u) = \hat{E}_{T(x)}^u$, the splitting is invariant by $\hat{L}_x$.

By inequality (4.1.6), we obtain that property (v) and the first inequality in property (vi) are satisfied. Since $\mathbb{K}$ is compact and the map $\mathbb{K} \to \mathbb{R}^+$ which associates $\|\mathbf{u}_x\|_{\mathcal{B}}$ to each $x$ is continuous, there is $C > 1$ such that

$$C^{-1} \|v\|_{\mathcal{A}} < \|v\|_{\mathcal{B}} < C \|v\|_{\mathcal{A}} \tag{4.1.7}$$

for all $x \in \mathbb{K}$ and for all $v \in E_x^u$. Thus, property (iv) and the second inequality in property (vi) follow from (4.1.1) and (4.1.7). □

## 5. Extending the invariant splitting for renormalization

Our aim in this section is to show that the invariant splitting on the limit set $\mathbb{K}$ of the operator $T$ given by Theorem 2.4, which is an iterate of the renormalization operator, can be extended to an invariant splitting of the same operator acting in the space of $C^r$ unimodal maps. Given the abstract results of the previous section, namely Theorem 4.1 and Corollary 4.2, all we have to do is find the appropriate compatible spaces and the corresponding compatibility constants. More precisely, we shall prove the following theorem.

THEOREM 5.1. *Let $T$ and $\mathbb{K}$ be as above, and let $\lambda$ be the expansion constant satisfying* (4.1.1).

*(i) For all* $\alpha > 0$ *the pair of spaces* $(\mathbb{A}^{2+\alpha}, \mathbb{A}^0)$ *is* 1*-compatible with* $(T, \mathbb{K})$*.*

*(ii) For all* $1 < \rho < \lambda$ *there exists* $\alpha > 0$ *sufficiently small such that* $(\mathbb{A}^{2-\alpha}, \mathbb{A}^0)$ *is* $\rho$*-compatible with* $(T, \mathbb{K})$*.*

The path towards the proof of this theorem (presented in §5.3) leads us to perform what amounts to a *spectral analysis* of the formal derivative of the renormalization operator, which in turn call for certain estimates on the geometry of the post-critical set of each map in the limit set of renormalization. We have the following explicit formula for the derivative $L_f = DT(f)$ of $T$ at $f \in \mathbb{K}$:

$$
\begin{aligned}
DT(f)v = & \frac{1}{\lambda_f} \sum_{j=0}^{p-1} Df^j(f^{p-j}(\lambda_f x))v(f^{p-j-1}(\lambda_f x)) \\
& + \frac{1}{\lambda_f}[x(Tf)'(x) - Tf(x)] \sum_{j=0}^{p-1} Df^j(f^{p-j}(0))v(f^{p-j-1}(0)) \ ,
\end{aligned}
$$

where as before $\lambda_f = f^p(0)$ for some positive integer $p = p(f, N)$. We observe that the operator $L_f$ extends naturally to each of the spaces $\mathbb{A}^\gamma$ for $\gamma \geq 0$.

Properties **A1**, **A2** and **A3** of Definition 4.1 are easily verified in our setting. Property **A4** follows from a general result of Hölder spaces that can be proved via smoothing operators. Hence, the heart of the matter is verifying property **A5**. This is where the geometric scaling properties of the invariant Cantor set of a map in $\mathbb{K}$ become important – see §5.2. We follow Davie's observation that $L_f^{(m)}$ is a special sort of operator – what we call an $L$-operator – which is amenable to analysis. The verification of the fifth property (with $(\mathcal{B}, \mathcal{C}) = (\mathbb{A}^\gamma, \mathbb{A}^0)$) – presented in §5.3 – consists in controlling the norm of a certain positive linear operator $L_{f,\gamma}^{(m)} : \mathbb{A}^0 \to \mathbb{A}^0$ associated to $L_f^{(m)}$ (see Lemma 5.3). Using the bounded distortion properties of $f \in \mathbb{K}$ and the geometry of the invariant set of $f$, we show that the exponential growth rate of the $C^0$ norm of $L_{f,\gamma}^{(m)}$ is bounded by some $\mu < \lambda$ if $\gamma = 2 - \alpha$ with $\alpha > 0$ small enough and is bounded by some $\mu < 1$ if $\gamma = 2 + \alpha$ with $\alpha > 0$.

## 5.1. Hölder norms and $L$-operators

First we define what we mean by an $L$-operator, and to each such operator $L$ we associate another operator $L_\gamma$, acting on continuous functions. Then, we use local Hölder estimates to control the norm of compositions $L_m \circ \cdots \circ L_2 \circ L_1$ of $L$-operators $L_i$ by the norm of $(L_m \circ \cdots \circ L_2 \circ L_1)_\gamma$.

Definition 5.1. *An $L$-operator is a bounded linear operator $L : C^\gamma(I) \to C^\gamma(I)$ that can be written in the form*

$$Lv(x) = \sum_{i=1}^{n} \phi_i(x) v(\psi_i(x)),$$

*where $\phi_i \in C^{\gamma_1}(I)$ and $\psi_i \in C^{\gamma_2}(I)$ are maps such that $\psi_i(I) \subset I$ for $i = 1, \ldots, n$, and where $\gamma_1 > 0$ and $\gamma_2 \geq 1$ are such that $0 < \gamma < \gamma_1, \gamma_2$.*

Example 5.1. *For all $f \in \mathbb{K}$ and all $i \geq 0$, the formal derivative $L_f = DT(T^i(f))$ is an $L$-operator.*

An $L$-operator $L$ as above yields a positive, bounded linear operator $L_\gamma : C^0(I) \to C^0(I)$ defined by

$$L_\gamma v(x) = \sum_{i=1}^{n} |\phi_i(x)| |D\psi_i(x)|^\gamma v(\psi_i(x)) \ .$$

A straightforward computation yields the following result.

Lemma 5.2. *If $L_1, L_2 : C^\gamma(I) \to C^\gamma(I)$ are $L$-operators, then $(L_1 \circ L_2)_\gamma = L_{1,\gamma} \circ L_{2,\gamma}$.*

We remind the reader that a function $\varphi : I \to I$ is *$\alpha$-Hölder continuous*, for a fixed $0 < \alpha < 1$, if there is $c > 0$ such that $|\varphi(x) - \varphi(y)| \leq c|x - y|^\alpha$ for all $x, y \in I$. Let $C^\alpha(I)$ be the Banach space of all $\alpha$-Hölder continuous real functions on $I$, with norm

$$\|\varphi\|_\alpha = \max\left\{ \|\varphi\|_0 \, , \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} \right\} \ .$$

Let $C^{k+\alpha}(I)$ be the Banach space of all real functions on $I$ for which the $k$-th derivative is $\alpha$-Hölder continuous, with norm

$$\|\varphi\|_{k+\alpha} = \max\{\|\varphi\|_0, \|D^k\varphi\|_\alpha\}.$$

Lemma 5.3. *Let $L_i : C^\gamma(I) \to C^\gamma(I)$ be a sequence of $L$-operators, and assume that there exist constants $\mu > 0$ and $C > 0$ such that for all $n$ we have*

$$\left\|(L_n \circ \cdots \circ L_2 \circ L_1)_\gamma\right\|_0 \leq C\mu^n \ . \tag{5.1.1}$$

*Then for all $\rho > \mu$ and all $\varepsilon > 0$ there exist $m > 0$ and $K > 0$ such that for all $v \in C^\gamma(I)$ we have*

$$\left\|L_m \circ \cdots \circ L_2 \circ L_1(v)\right\|_\gamma \leq \max\left\{\epsilon\rho^m \|v\|_\gamma, K\|v\|_0\right\} \ .$$

To prove the above proposition, we will use local Hölder estimates for $L$-operators given in our next lemma. For each $\eta > 0$ and each $\varphi \in C^\alpha(I)$, we consider an associated semi-norm

$$\|\varphi\|_{\alpha,\eta} = \sup_{0<|x-y|<\eta} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} \ .$$

The corresponding semi-norm of $\varphi \in C^{k+\alpha}(I)$ for $k > 0$ is $\|\varphi\|_{k+\alpha,\eta} = \|\varphi^{(k)}\|_{\alpha,\eta}$.

LEMMA 5.4. *Let $L : C^\gamma(I) \to C^\gamma(I)$ be an $L$-operator as defined above.*

*(i) For every $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\|Lv\|_{\gamma,\eta} \le (\varepsilon + \|L_\gamma\|_{C^0(I)})\|v\|_\gamma.$$

*(ii) For every $\varepsilon > 0$ and $0 < \xi < \gamma$, there is $\eta > 0$ such that*

$$\|Lv\|_{\xi,\eta} \le \varepsilon\|v\|_\gamma.$$

*Proof.* See Lemmas 1 and 2 in [5]. □

*Proof of Lemma 5.3.* Choosing $m$ such that $C\mu^m < \epsilon\rho^m/8$, we have

$$M = \|L_{m,\gamma} \circ \cdots \circ L_{1,\gamma}\|_0 < \frac{\varepsilon\rho^m}{8} \ .$$

By Lemma 5.4, given $\varepsilon' = \varepsilon\rho^m/8$, there exists $\eta > 0$ such that

$$\|L_m \circ \cdots \circ L_1(v)\|_{\gamma,\eta} \le \left(\varepsilon' + M\right) \|v\|_\gamma \le \frac{\varepsilon\rho^m}{4} \|v\|_\gamma \ .$$

Taking $K = 8k!\|L_m \circ \cdots \circ L_1\|_0/\eta^\gamma$, writing $\gamma = k + \alpha$, where $k$ is an integer and $0 < \alpha < 1$, and using interpolation of norms (see Lemma 4 in [5]), we deduce that

$$\begin{aligned}\|L_m \circ \cdots \circ L_1(v)\|_\gamma &\le 4 \max\left\{ \|L_m \circ \cdots \circ L_1(v)\|_{\gamma,\eta} , \frac{2k!}{\eta^\gamma} \|L_m \circ \cdots \circ L_1(v)\|_0 \right\} \\ &\le \max\left\{\epsilon\rho^m\|v\|_\gamma, K\|v\|_0\right\} \ .\square\end{aligned}$$

## 5.2. Bounded geometry

Our aim in this section is to prove two crucial propositions concerning the geometry of the invariant Cantor set of an infinitely renormalizable map in the limit set of renormalization. They are important not only in the proof of Theorem 5.1, but also in the proof (presented in §8) that the renormalization operator is robust (in the sense of §6).

We recall our notation. For each $f \in \mathbb{K}$, let $\mathcal{I}_f \subseteq I$ be the closure of the postcritical set of $f$ (the Cantor attractor of $f$). For each $k \ge 0$, we can write

$$R^k f(x) = \frac{1}{\lambda_k} \cdot f^{p_k}(\lambda_k x)$$

where $p_k = \prod_{i=0}^{k-1} p(R^i f)$ and $\lambda_k = \prod_{i=0}^{k-1} \lambda(R^i f)$. Recall that the renormalization intervals $\Delta_{0,k} = [-|\lambda_k|, |\lambda_k|] \subset [-1,1]$, and $\Delta_{i,k} = f^i(\Delta_{0,k})$ for $i = 0, 1, \dots, p_k - 1$. The collection $\mathbf{C}_k = \{\Delta_{0,k}, \dots, \Delta_{p_k-1,k}\}$ consists of pairwise disjoint intervals. Moreover, $\bigcup\{\Delta : \Delta \in \mathbf{C}_{k+1}\} \subseteq \bigcup\{\Delta : \Delta \in \mathbf{C}_k\}$ for all $k \geq 0$ and we have

$$\mathcal{I}_f = \bigcap_{k=0}^{\infty} \bigcup_{i=0}^{p_k-1} \Delta_{i,k} \ .$$

In our first proposition, $f$ is a normalized, symmetric quadratic unimodal map, infinitely renormalizable, sufficiently smooth (say $C^2$) for Sullivan's real bounds to be true for $f$. But there are no restrictions on the combinatorics. We shall use the general fact, due to Guckenheimer [12], that among those renormalization intervals at the $k$-th level the one that contains the critical point of $f$ (namely, $\Delta_{0,k}$) is the largest (up to multiplication by a constant). This can be seen as follows. First suppose that $f$ is also $S$-unimodal. If $n > 0$ is such that $f^n(x)$ belongs to the interval with endpoints $-x, x$ but $f^j(x)$ does not, for all $1 \leq j < n$, then $|Df^n(x)| > 1$ – this uses the fact that $f$ has negative Schwarzian. From this it follows that if $J \subset [-1,1]$ is an interval that does not contain the critical point, whose iterates $f^j(J)$ are pairwise disjoint for $0 \leq j \leq n$, such that $f^n(J)$ lies in the convex-hull of $J$ union its symmetric while the previous iterates $f^j(J)$, $1 \leq j < n$, do not, then $|f^n(J)| > |J|$. Hence, if $f$ is renormalizable, symmetric and $S$-unimodal then at each renormalization level the interval that contains the critical point is the largest. If we drop the negative Schwarzian hypothesis, the same is true up to a multiplicative constant. This is because every sufficiently deep renormalization of $f$ already has negative Schwarzian derivative.

Proposition 5.5. *For each $\alpha > 0$ there exist constants $C_0$ and $0 < \mu < 1$ such that*

$$\sum_{i=0}^{p_k-1} \frac{|\Delta_{i,k}|^{2+\alpha}}{|\Delta_{i+1,k}|} \leq C_0 \mu^k \ . \tag{5.2.1}$$

*Proof.* Let $\ell(\Delta_{i,k})$ be the level of $\Delta_{i,k}$, i.e., the largest integer $j$ such that $\Delta_{i,k} \subseteq \Delta_{0,j} \setminus \Delta_{0,j+1}$. Let $d_{i,k}$ be the distance from $\Delta_{i,k}$ to zero (the critical point). Using that $\Delta_{i,k}$ has space around itself we see that for all $i \neq 0$ and all $x \in \Delta_{i,k}$ we have $d_{i,k} \leq |x| \leq K d_{i,k}$, where $K > 1$ is a constant that depends only on the real bounds. Hence $K^{-1} \leq |x|/|y| \leq K$ whenever $x, y \in \Delta_{i,k}$. These facts are implicitly used in the estimates below.

Now, we have $|\Delta_{i,k}|/|\Delta_{i+1,k}| = 1/|f'(x_{i,k})|$ for some $x_{i,k} \in \Delta_{i,k}$. Since the critical point is quadratic, we have $|f'(x_{i,k})| \geq C_1 |x_{i,k}|$, and so

$$\frac{|\Delta_{i,k}|}{|\Delta_{i+1,k}|} \leq \frac{1}{C_1 |x_{i,k}|} \ . \tag{5.2.2}$$

Therefore, for all $0 \leq j \leq k-1$ we have

$$\sum_{\ell(\Delta_{i,k})=j} \frac{|\Delta_{i,k}|^2}{|\Delta_{i+1,k}|} \leq C_1^{-1} \sum_{\ell(\Delta_{i,k})=j} \frac{|\Delta_{i,k}|}{|x_{i,k}|} \leq C_2 \sum_{\ell(\Delta_{i,k})=j} \int_{\Delta_{i,k}} \frac{dx}{|x|}$$
$$\leq C_3 \int_{\Delta_{0,j}\setminus\Delta_{0,j+1}} \frac{dx}{|x|} \leq C_4 \log \frac{|\Delta_{0,j}|}{|\Delta_{0,j+1}|} .$$

With these estimates, and using the fact proved above that $|\Delta_{0,k}| \geq C_5|\Delta_{i,k}|$ for all $0 \leq i \leq p_k - 1$, we see that

$$\sum_{i=0}^{p_k-1} \frac{|\Delta_{i,k}|^{2+\alpha}}{|\Delta_{i+1,k}|} \leq C_6 \max|\Delta_{i,k}|^\alpha \left(1 + \sum_{j=0}^{k-1} \log \frac{|\Delta_{0,j}|}{|\Delta_{0,j+1}|}\right)$$
$$\leq C_6|\Delta_{0,k}|^\alpha \left(1 + \log \frac{1}{|\Delta_{0,k}|}\right)$$
$$\leq K_\alpha |\Delta_{0,k}|^{\alpha/2} ,$$

where $K_\alpha$ is a positive constant depending on $\alpha$. This proves (5.2.1) because $|\Delta_{0,k}|$ decays exponentially with $k$ with uniform rate depending only on the real bounds. □

In addition to Proposition 5.5 – valid for maps with arbitrary combinatorial type – we shall need also an estimate that seems specific for maps with *bounded* combinatorial type, namely Proposition 5.8 below. First, a couple of lemmas.

For each $f \in \mathbb{K}$, let $d_f$ be the infimum of all positive numbers $s$ such that

$$\sum_{j=0}^{p_k-1} |\Delta_{j,k}|^s \to 0 \quad \text{as} \quad k \to \infty .$$

It is possible to prove, using some thermodynamic formalism, that $d_f$ agrees with the Hausdorff dimension of $\mathcal{I}_f$, but we will not need this fact. Let $0 < D < 1$ be the supremum of $d_f$ as $f$ ranges through $\mathbb{K}$.

Lemma 5.6. *For each $s > D$ there exist $C_s > 0$ and $0 < \eta_s < 1$ such that for all $f \in \mathbb{K}$ we have*

$$\sum_{j=0}^{p_k-1} |\Delta_{j,k}|^s < C_s \eta_s^k .$$

*Proof.* Apply bounded geometry and the compactness of $\mathbb{K}$. □

Next, let us define

$$S_{j,k}(f;s) = \sum_{\ell(\Delta_{i,k})=j} |\Delta_{i,k}|^s ,$$

for $j = 0, 1, \ldots, k-1$, all $k \geq 0$, and all $f \in \mathbb{K}$.

LEMMA 5.7. *For each $s > D$ and each $f \in \mathbb{K}$ we have $S_{j,k}(f;s) \leq C_s \lambda_j^s \eta_s^{k-j}$, where $C_s > 0$ and $0 < \eta_s < 1$ are the constants of Lemma 5.6, and $\lambda_j = \lambda(f,j)$ is as in* (3.1.1).

*Proof.* Using renormalization, we see that $S_{j,k}(f;s) = \lambda_j^s S_{0,k-j}(R^j(f);s)$. From Lemma 5.6, we know that

$$S_{0,k-j}(R^j(f);s) \leq C_s \eta_s^{k-j} \ .$$

The result follows. □

PROPOSITION 5.8. *For each $\mu > 1$ close to one, there exist $0 < \alpha < 1 - D$ close to zero and $C > 0$ such that for all $f \in \mathbb{K}$ we have*

$$\sum_{i=0}^{p_k - 1} \frac{|\Delta_{i,k}|^{2-\alpha}}{|\Delta_{i+1,k}|} \leq C\mu^k \ .$$

*Proof.* Using (5.2.2) and the fact that $|x_{i,k}| \geq \lambda_{j+1}$ when $\ell(\Delta_{i,k}) = j$, we have

$$\begin{aligned} \sum_{i=0}^{p_k - 1} \frac{|\Delta_{i,k}|^{2-\alpha}}{|\Delta_{i+1,k}|} &= \sum_{i=0}^{p_k - 1} \frac{|\Delta_{i,k}|}{|\Delta_{i+1,k}|} \, |\Delta_{i,k}|^{1-\alpha} \\ &\leq C \left( |\Delta_{0,k}|^{-\alpha} + \sum_{j=0}^{k-1} \lambda_j^{-1} S_{j,k}(f; 1-\alpha) \right) \ . \end{aligned}$$

If $1 - \alpha > D$ then, applying Lemma 5.7 with $s = 1 - \alpha$, we get

$$\sum_{i=0}^{p_k - 1} \frac{|\Delta_{i,k}|^{2-\alpha}}{|\Delta_{i+1,k}|} \leq C C_{1-\alpha} \sum_{j=0}^{k} \lambda_j^{-\alpha} \eta_{1-\alpha}^{k-j} \leq K_\alpha \mu_\alpha^k \ ,$$

where $K_\alpha > 0$ and $\mu_\alpha = \max\{\lambda(f,1)^{-\alpha} : f \in \mathbb{K}\}$ depend on $\alpha$. But if $\alpha$ is small enough we will have $\mu_\alpha < \mu$, and this completes the proof. □

REMARK 5.1. *By a continuity argument and the real bounds, we can prove that propositions 5.5 and 5.8 remain true for maps $\tilde{f} \in \mathbb{U}^4$ sufficiently close to $\mathbb{K}$. More precisely, for each $k > 0$ there exists $\varepsilon_k > 0$ such that for all $f \in \mathbb{K}$ and all $\tilde{f} \in \mathbb{U}^4$ with $\left\| \tilde{f} - f \right\|_{C^4(I)} < \varepsilon_k$, the map $\tilde{f}$ is $k$-times renormalizable, and the statements of both propositions hold for $\tilde{f}$. This will be used in §8.4 only for real analytic maps in an open neighborhood of $\mathbb{K}$ in $\mathbb{A}$ .*

## 5.3. Spectral estimates

In this section we prove Theorem 5.1. Fixing $f \in \mathbb{K}$ and considering the Banach space $\mathcal{A}$ given by Theorem 2.4, we recall that the Fréchet derivative

$L_f = DT(f) : \mathcal{A} \to \mathcal{A}$ is given by formula (5.0.1). It is clear from that formula that $L_f$ extends to a bounded linear operator $\hat{L}_f : \mathbb{A}^0 \to \mathbb{A}^0$, and moreover $\hat{L}_f(\mathbb{A}^s) \subseteq \mathbb{A}^s$ for all $s \geq 0$ (because $f$ is analytic).

We want to verify the compatibility properties of Definition 4.1 for the spaces $\mathcal{B} = \mathbb{A}^s$ and $\mathcal{C} = \mathbb{A}^0$ when $s$ is close to (but different from) 2. Properties **A1** and **A2** are clearly satisfied, while **A3** follows from Lemma 8.13. Property **A4** is a consequence of the following simple fact about Hölder spaces (see [15]).

LEMMA 5.9. *There exists $\Delta > 1$ such that $\mathbb{A}\cap\mathbb{A}^s(\Delta)$ is $C^0$-dense in $\mathbb{A}^s(1)$.*

*Proof.* By Theorem A.10 in page 43 of [15], there exists a family $S_t$, $t > 0$, of smoothing operators preserving even functions and $C \geq 1$ such that, for all $v \in \mathbb{A}^s(1)$, we have $\|S_t v\|_{C^s} \leq C$ and $\|v - S_t v\|_{C^0} \leq Ct^s$. By the Stone-Weierstrass theorem, for all small $0 < \varepsilon < C$ there is a polynomial $w_t$ with real coefficients and vanishing at zero such that $\|w_t - S_t v\|_{C^s} < \varepsilon$. Now let $v_t(x) = \frac{1}{2}(w_t(x) + w_t(-x))$, so that $v_t \in \mathbb{A}$ and we still have $\|v_t - S_t v\|_{C^s} < \varepsilon$. Then $\|v_t\|_{C^s} < \|S_t(v)\|_{C^s} + \varepsilon < 2C$ on one hand, while $\|v_t - v\|_{C^0} < \varepsilon + Ct^s$ on the other hand. For $t$ small enough, this gives $\|v_t - v\|_{C^0} < 2\varepsilon$ with $v_t \in \mathbb{A}^s(2C)$. □

Hence all that remains is to check that property **A5** is satisfied. By Lemma 5.3, this will be the case provided we can control the $C^0$ norms of $\hat{L}_{f,s}^{(m)}$. We shall prove this now, with the help of Propositions 5.5 and 5.8.

Recall that for each $m \geq 1$ the operator $\hat{L}_f^{(m)}$ is an $L$-operator and its associated positive, bounded linear operator $\hat{L}_{f,s}^{(m)} : \mathbb{A}^0 \to \mathbb{A}^0$ is given by

$$\hat{L}_{f,s}^{(m)}(v) \;=\; \frac{1}{\lambda_k}\sum_{j=0}^{p_k-1} |Df^j(f^{p_k-j}(\lambda_k x))||\lambda_k Df^{p_k-j-1}(\lambda_k x)|^s v(f^{p_k-j-1}(\lambda_k x)) \;, \tag{5.3.1}$$

where $k = mN$ (recall that $T = R^N$). Now we have the following fact coming from bounded geometry

$$|Df^j(f^{p_k-j}(\lambda_k x))| \asymp \frac{|\Delta_{0,k}|}{|\Delta_{p_k-j,k}|} \;, \tag{5.3.2}$$

for all $0 \leq j \leq p_k - 1$. Since $|Df(\lambda_k x)| \leq C\lambda_k$ for some constant $C > 0$ independent of $k$ and uniform in $f \in \mathbb{K}$, and $|\Delta_{0,k}| = 2\lambda_k$ we have

$$|Df^{p_k-j-1}(\lambda_k x)| \leq C|\Delta_{0,k}||Df^{p_k-j-2}(f(\lambda_k x))| \;.$$

Again, by bounded geometry, for all $0 \leq j \leq p_k - 2$

$$|Df^{p_k-j-2}(f(\lambda_k x))| \asymp \frac{|\Delta_{p_k-j-1,k}|}{|\Delta_{1,k}|} \;,$$

and so

$$|Df^{p_k-j-1}(\lambda_k x)| \leq C|\Delta_{0,k}|\frac{|\Delta_{p_k-j-1,k}|}{|\Delta_{1,k}|} \;. \tag{5.3.3}$$

Using (5.3.2) and (5.3.3) in (5.3.1), we see that

$$\|\hat{L}_{f,s}^{(m)}\| \leq \frac{C}{\lambda_k}\left(\frac{\lambda_k^s|\Delta_{0,k}|}{|\Delta_{1,k}|}+\sum_{j=0}^{p_k-2}\lambda_k^{2s}\frac{|\Delta_{0,k}|}{|\Delta_{p_k-j,k}|}\frac{|\Delta_{p_k-j-1,k}|^s}{|\Delta_{1,k}|^s}\right).$$

But $|\Delta_{0,k}| = 2\lambda_k$ and since the critical point of $f$ is quadratic, $|\Delta_{1,k}| \asymp |\Delta_{0,k}|^2 \asymp \lambda_k^2$. Therefore, we arrive at

$$\|\hat{L}_{f,s}^{(m)}\| \leq C_1\sum_{j=0}^{p_k-1}\frac{|\Delta_{p_k-j-1,k}|^s}{|\Delta_{p_k-j,k}|}. \tag{5.3.4}$$

The proof of part (i) of Theorem 5.1 now follows from Proposition 5.5, while the proof of part (ii) is a consequence of Proposition 5.8. This ends the proof of Theorem 5.1.

## 6. The local stable manifold theorem

In this section we isolate those features of the renormalization operator that are essential for the promotion of "hyperbolicity" from the Banach space $\mathbb{A}$ of Theorem 2.4 to the space $\mathbb{U}^r$. This leads us to the definition of a robust operator (see §6.1). Such definition is necessarily rather technical, since it has to account for the fact that the renormalization operator is not Fréchet differentiable in $\mathbb{U}^r$. In particular a robust operator acts simultaneously on four different Banach spaces (corresponding in the case of renormalization to the space $\mathbb{A}$ given by Theorem 2.4, $\mathbb{A}^r$, $\mathbb{A}^s$ and $\mathbb{A}^0$, where $r > 1+s$ and $s$ is close to 2), and satisfies several properties. The major goal of this section is to prove a local stable manifold theorem for robust operators.

### 6.1. Robust operators

Before moving on to a precise definition of a robust operator, we give the following informal description. A robust operator acts on four Banach spaces $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C} \subset \mathcal{D}$. In the smaller space $\mathcal{A}$ it acts smoothly and has a hyperbolic basic set $\mathbb{K}$. The pairs of spaces $(\mathcal{B},\mathcal{D})$ and $(\mathcal{C},\mathcal{D})$ are compatible with $(T,\mathbb{K})$, and in particular the invariant hyperbolic splitting for $\mathbb{K}$ in $\mathcal{A}$ extends to an invariant hyperbolic splitting for $\mathbb{K}$ in $\mathcal{B}$. Viewed as a map from $\mathcal{B}$ into $\mathcal{C}$, a robust operator is $C^1$. It also satisfies a uniform Gateaux differentiability condition in $\mathcal{C}$ for points and directions in $\mathcal{B}$. Finally, as an operator in $\mathcal{B}$, it is reasonably well-approximated by the extension of its derivative at a point of $\mathbb{K}$ in $\mathcal{A}$ to a bounded linear operator in $\mathcal{B}$. It will take us considerable effort (see §8) to verify that the renormalization operator indeed satisfies all these conditions.

Let $T : \mathcal{O} \to \mathcal{A}$ be a $C^2$ operator having a compact hyperbolic basic set $\mathbb{K}$ with the property that the unstable subspace of the $DT$-invariant splitting of the tangent space at each point of $\mathbb{K}$ is one-dimensional. By standard invariant manifold theory (see [14]), we know that for all $g \in \mathbb{K}$ the local unstable manifold $W^u(g)$ of $T$ at $g$ exists and is $C^2$. In particular, we can find a $C^2$ parametrization

$$t \mapsto u_g(t) \in W^u(g) \subseteq \mathcal{A}$$

varying continuously with $g$ such that $\mathbf{u}_g = u_g'(0)$ is a unit vector. We define a $C^2$ function $t \mapsto \hat{\delta}_g(t)$ by

$$T(u_g(t)) \;=\; u_{T(g)}(\hat{\delta}_g(t)) \;.$$

This function also varies continuously with $g$ and $\hat{\delta}_g(t) = \delta_g t + O(t^2)$ for some $\delta_g > 0$. Recall that by hyperbolicity of $\mathbb{K}$, there exist $C_0 > 0$ and $\lambda > 1$ such that for every $g \in \mathbb{K}$ and every $m \geq 1$ we have

$$\delta_{T^{m-1}(g)} \cdots \delta_g > C_0 \lambda^m \;. \tag{6.1.1}$$

Definition 6.1. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{D}$ be Banach spaces, where each inclusion is a compact linear operator. Let $\mathcal{O}_\mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{O}_\mathcal{B} \subseteq \mathcal{B}$ be open sets in their respective spaces such that $\mathcal{O}_\mathcal{A} \subseteq \mathcal{O}_\mathcal{B}$. Let $\mathbb{K} \subset \mathcal{O}_\mathcal{A}$ be a hyperbolic basic set of a $C^2$ operator $T : \mathcal{O}_\mathcal{A} \to \mathcal{A}$. We say that $T$ is* robust *with respect to $(\mathcal{B}, \mathcal{C}, \mathcal{D})$ if it has an extension to an operator $T : \mathcal{O}_\mathcal{B} \to \mathcal{B}$ that satisfies the following conditions.*

**B1**. *The pair $(\mathcal{B}, \mathcal{D})$ is 1-compatible with $T$, while the pair $(\mathcal{C}, \mathcal{D})$ is $\rho_\mathcal{C}$-compatible with $(T, \mathbb{K})$ for some $\rho_\mathcal{C} < \lambda$ (where $\lambda$ is as in* (6.1.1)*).*

**B2**. *For each $m > 0$, the interior $\mathcal{O}_\mathcal{B}^{(m)}$ of the set $\{f \in \mathcal{O}_\mathcal{B} : T^i(f) \in \mathcal{O}_\mathcal{B},\ \forall i < m\}$ contains $\mathbb{K}$, and $T^m : \mathcal{O}_\mathcal{B}^{(m)} \to \mathcal{C}$ is $C^1$ and its derivative is uniformly continuous in some neighbourhood of $\mathbb{K}$. Furthermore, for all $f \in \mathcal{A} \cap \mathcal{O}_\mathcal{B}^{(m)}$ the linear map*

$$DT^m(f) : \mathcal{B} \to \mathcal{C}$$

*extends to a continuous linear operator $L_m : \mathcal{D} \to \mathcal{D}$ that satisfies $L_m(\mathcal{X}) \subseteq \mathcal{X}$, for $\mathcal{X} = \mathcal{B}, \mathcal{C}$.*

**B3**. *For every $m$ there exists $C_{m,1} > 1$ with the property that for each $g \in \mathbb{K}$ there is an open set $\mathcal{V}_g \subseteq \mathcal{O}_\mathcal{B}$ containing $g$ such that for all $f \in \mathcal{V}_g$ we have*

$$\|DT^m(f)\mathbf{u}_g - DT^m(g)\mathbf{u}_g\|_\mathcal{C} \leq C_{m,1}\|f - g\|_\mathcal{B} \;.$$

**B4**. *There exist $C_1 > 1$ and $\rho > 1$ with the property that for each $g \in \mathbb{K}$ there is an open set $\mathcal{V}_g \subseteq \mathcal{O}_\mathcal{B}$ containing $g$ such that for all $f_1, f_2 \in \mathcal{V}_g$ we have*

$$\|T(f_1) - T(f_2) - DT(f_2)(f_1 - f_2)\|_\mathcal{C} \leq C_1 \|f_1 - f_2\|_\mathcal{C}^\rho \;.$$

**B5**. *For all $m > 0$, there exists $C_{m,2} > 0$, and there exists $\nu_m > 0$ such that for all $g \in \mathbb{K}$ and for all $f \in \mathcal{B}$ with $\|f - g\|_{\mathcal{B}} < \nu_m$ we have*

$$\|DT^m(f) - DT^m(g)\|_{\mathcal{C}} \leq C_{m,2}\lambda^m \ .$$

*Moreover, there exists $m_0 > 0$ such that for all $m > m_0$ we have $C_{m,2} < C_0/8$ (where $C_0$ and $\lambda$ are as in (6.1.1)).*

**B6**. *For all $m > 0$, there exists $C_{m,3} > 0$ such that for all $g \in \mathbb{K}$, for all $f \in \mathcal{A}$ with $\|f - g\|_{\mathcal{A}} < \nu_m$ and for all $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} < \nu_m$, we have*

$$\|T^m(f + v) - T^m(f) - DT^m(g)v\|_{\mathcal{B}} \leq C_{m,3}\|v\|_{\mathcal{B}} \ .$$

*Moreover, there exists $m_0 > 0$ such that for all $m > m_0$ we have $C_{m,3} < 1/4$.*

Example 6.1. *As one might expect, the main example of a robust operator is provided by renormalization. We know from Theorem 2.4 that the renormalization operator $T = R^N : \mathbb{O} \to \mathbb{A}$ is hyperbolic over $\mathbb{K}$. We also know that this renormalization operator is well-defined as a map from an open set of $\mathbb{U}^\gamma$ containing $\mathbb{K}$ into $\mathbb{U}^\gamma \subset 1 + \mathbb{A}^\gamma \cong \mathbb{A}^\gamma$ for all $\gamma \geq 2$. We will show in §8 that $T$ is robust with respect to the spaces $\mathcal{A} = \mathbb{A}$, $\mathcal{B} = \mathbb{A}^r$, $\mathcal{C} = \mathbb{A}^s$ and $\mathcal{D} = \mathbb{A}^0$ whenever $s < 2$ is close to 2 and $r > s + 1$ is not an integer.*

## 6.2. Stable manifolds for robust operators

We can now formulate a general local stable manifold theorem for robust operators.

Theorem 6.1. *Let $T : \mathcal{O}_{\mathcal{A}} \to \mathcal{A}$ be a $C^k$ with $k \geq 2$ (or real analytic) hyperbolic operator over $\mathbb{K} \subset \mathcal{O}_{\mathcal{A}}$, and robust with respect to $(\mathcal{B}, \mathcal{C}, \mathcal{D})$. Then conditions (i), (ii), (iii) and (iv) of Theorem 2.3 hold true for the operator $T$ acting on $\mathcal{B}$. The local unstable manifolds are $C^k$ with $k \geq 2$ (or real analytic) curves, and the local stable manifolds are of class $C^1$ and form a $C^0$ lamination.*

The proof of this theorem will occupy the rest of §6. In the end, the theorem will follow by putting together Corollary 4.2, Proposition 6.13 and Theorem 6.15.

## 6.3. Uniform bounds

Before proceeding we prove the following simple bounds that we will use quite often.

Lemma 6.2. *There exist $\mu_0 > 0$ and $1 < \lambda < M$ such that for all $g \in \mathbb{K}$ and all $t \in \mathbb{R}$ with $|t| < \mu_0$, $u_g(t)$ and $\hat{\delta}_g(t)$ are well-defined and* *(i)* $M^{-1}\lambda^n < \delta_{T^{n-1}(g)} \cdots \delta_g < M^n$ *and* $\left|\hat{\delta}_g(t)\right| < M|t|$*;*

*(ii)* $M^{-1} < \|\mathbf{u}_g\|_{\mathcal{B}} < M$ *and* $M^{-1} < \|\mathbf{u}_g\|_{\mathcal{C}} < M$*;*

*(iii)* $M^{-1} < \|\sigma_g\|_{\mathcal{B}} < M$ *and* $M^{-1} < \|\sigma_g\|_{\mathcal{C}} < M$*;*

*(iv)* $M^{-1}|t| < \|u_g(t) - g\|_{\mathcal{C}} < M|t|$ *and* $M^{-1}|t| < \|u_g(t) - g\|_{\mathcal{B}} < M|t|$*;*

*(v)* $|\sigma_g(u_g(t) - g)| > \frac{1}{2}|t|$.

*Proof.* (i) By Definition 2.1 and (6.1.1), there exist $1 < \lambda < M_1$ such that for all $g \in \mathbb{K}$ and all $n \geq 1$ we have $M_1^{-1}\lambda^n < \delta_{T^{n-1}(g)} \cdots \delta_g < M_1^n$ and also $\left|\hat{\delta}_g(t)\right| < M_1|t|$ for all $|t| \leq \mu_1$ (where $\mu_1 > 0$ is a uniform constant).

(ii) For $\mathcal{X}$ equal to $\mathcal{B}$ and $\mathcal{C}$, we have that $g \mapsto \mathbf{u}_g$ as a map $\mathbb{K} \to \mathcal{X}$ is continuous and does not vanish. Hence, by compactness of $\mathbb{K}$ there is $M_2 > 1$ such that $M_2^{-1} < \|\mathbf{u}_g\|_{\mathcal{X}} < M_2$.

(iii) Since $\sigma_g(\mathbf{u}_g) = 1$ and by property **B1** in Definition 6.1, the functional $\sigma_g$ extends continuously to $\mathcal{X}$ and there is $M_3 > 1$ such that $M_3^{-1} < \|\sigma_g\|_{\mathcal{X}} < M_3$.

(iv) Since $t \to u_g(t)$ as a map $\mathbb{R} \to \mathcal{X}$ is $C^1$ and varies continuously with $g \in \mathbb{K}$, there is $M_4 > 0$ and $\mu_2 > 0$ such that $\|u_g(t) - u_g(s)\|_{\mathcal{X}} \leq M_4|t - s|$ for all $g \in \mathbb{K}$ and all $t, s$ with $|t| < \mu_2$ and $|s| < \mu_2$. Moreover, since

$$\left.\frac{d}{dt}u_g(t)\right|_{t=0} = \mathbf{u}_g \neq 0$$

there exists $M_5 > 0$ and $\mu_3 > 0$ such that $|t - s| \leq M_5\|u_g(t) - u_g(s)\|_{\mathcal{X}}$ for all $g \in \mathbb{K}$ and all $|t| < \mu_3$. Hence (iv) follows by taking $s = 0$ and noting that $u_g(0) = g$.

(v) This follows from (iv) and the fact that $\sigma_g(\mathbf{u}_g) = 1$. □

## 6.4. Contraction towards the unstable manifolds

The one-dimensional unstable manifolds of $T$ in $\mathcal{A}$ are embedded in $\mathcal{B}$, and remain invariant. The first important estimate given by the following lemma shows that in $\mathcal{B}$ the operator $T$ contracts towards such manifolds. Therefore, if $T$ is to have unstable manifolds in $\mathcal{B}$, these have to coincide with unstable manifolds in $\mathcal{A}$. In what follows, we fix $g \in \mathbb{K}$ and for simplicity of notation we write

$$\sigma_i = \sigma_{T^i(g)} \ , \ u_i = u_{T^i(g)} \ , \ \mathbf{u}_i = \mathbf{u}_{T^i(g)} \ ,$$

and

$$\delta_i^m = \prod_{j=i}^{m-1} \delta_{T^j(g)}, \ \hat{\delta}_i^m = \hat{\delta}_{T^{m-1}(g)} \circ \cdots \circ \hat{\delta}_{T^i(g)} \ .$$

Set $\mu_0 > 0$ as in Lemma 6.2.

LEMMA 6.3. *For every $m > 0$ there exist $0 < \eta_m < \mu_0$ and $B_m > 0$ such that for every $g \in \mathbb{K}$ and every $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} < \eta_m$ and $t \in \mathbb{R}$ with $|t| < \eta_m$, we have $\hat{\delta}_0^m(t + \sigma_0(v)) < \mu_0$, $u_0(t) + v \in \mathcal{O}_{\mathcal{B}}^{(m)}$ and*

$$\left\| T^m(u_0(t) + v) - u_m\left(\hat{\delta}_0^m(t + \sigma_0(v))\right) \right\|_{\mathcal{B}} < B_m \|v\|_{\mathcal{B}} .$$

*Furthermore, there is $m_1 > m_0$ (where $m_0$ is as in* **B6***) such that for all $m > m_1$ we have $B_m < 1/2$.*

*Proof.* We prove the second inequality only. The first is proven in the same way. By property **B6** in Definition 6.1, there is $m_0 > 0$ such that for all $m > m_0$, all $v$ with $\|v\|_{\mathcal{B}} < \nu_m$, and all $t \in \mathbb{R}$ with $|t| < \nu_m$ we have

$$\|T^m(u_0(t) + v) - T^m(u_0(t)) - DT^m(g)v\|_{\mathcal{B}} \leq \frac{1}{4}\|v\|_{\mathcal{B}} . \tag{6.4.1}$$

By property **B1** in Definition 6.1, $(\mathcal{B}, \mathcal{D})$ is 1-compatible with $(T, \mathbb{K})$. Hence, by Corollary 4.2, there exists $m_1 > m_0$ such that for all $m > m_1$ we have

$$\|DT^m(g)v - \delta_0^m \sigma_0(v)\mathbf{u}_m\|_{\mathcal{B}} \leq \frac{1}{8}\|v\|_{\mathcal{B}} . \tag{6.4.2}$$

Putting (6.4.1) and (6.4.2) together we get

$$\|T^m(u_0(t) + v) - T^m(u_0(t)) - \delta_0^m \sigma_0(v)\mathbf{u}_m\|_{\mathcal{B}} \leq \frac{3}{8}\|v\|_{\mathcal{B}} . \tag{6.4.3}$$

Now, we know that $T^m(u_0(t)) = u_m(\hat{\delta}_0^m(t))$ and $t \to u_m \circ \hat{\delta}_0^m(t)$ is $C^2$. Hence,

$$\begin{aligned} \Big\| u_m \circ \hat{\delta}_0^m(t + \sigma_0(v)) - u_m \circ \hat{\delta}_0^m(t) - \delta_0^m \sigma_0(v)\mathbf{u}_m\|_{\mathcal{B}} & \\ \leq c_1 \left( (\sigma_0(v))^2 + |t|\sigma_0(v) \right) & \\ \leq c_2 \left( \|v\|_{\mathcal{B}} + |t| \right) \|v\|_{\mathcal{B}} . & \end{aligned} \tag{6.4.4}$$

Therefore, choosing $\eta_m < \nu_m$ so small that $C_2 \eta_m < 1/16$ and putting 6.4.3 and 6.4.4 together, we see that if $|t| < \eta_m$ and $\|v\|_{\mathcal{B}} < \eta_m$ then

$$\left\| T^m(u_0(t) + v) - u_m \circ \hat{\delta}_0^m(t + \sigma_0(v)) \right\|_{\mathcal{B}} \leq \frac{1}{2}\|v\|_{\mathcal{B}}$$

as desired. □

LEMMA 6.4. *Let $m_1 > 0$ be as in Lemma 6.3. For all $m > m_1$ there exist small constants $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$ such that the following holds for every $\varepsilon < \varepsilon_2$. For every $g \in \mathbb{K}$ and every $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} < \varepsilon$, the recursive scheme given by $f_0 = g + v$, $t_0 = 0$, $v_0 = v$ and*

$$\begin{aligned} f_{k+1} &= T^m(f_k) \\ t_{k+1} &= \hat{\delta}_{km}^{(k+1)m}(t_k + \sigma_{km}(v_k)) \\ v_{k+1} &= f_{k+1} - u_{(k+1)m}(t_{k+1}) \end{aligned} \tag{6.4.5}$$

*is well-defined for all* $k = 0, \ldots, k_0 - 1$ *where* $k_0 = k_0(g, f_0) = \min\{j : |t_j| \geq \varepsilon_1\}$. *For all* $k \leq k_0$ *we have*

$$\left\|T^{km}(g+v) - u_{km}(t_k)\right\|_{\mathcal{B}} < 2^{-k}\|v\|_{\mathcal{B}} \text{ and } \left\|T^{km}(g+v) - T^{km}(g)\right\|_{\mathcal{B}} < \varepsilon_0/M_0 \ , \tag{6.4.6}$$

*where* $M_0 = M^{m+2} + B_1 + \ldots + B_m$, $M$ *is as in Lemma 6.2 and* $B_1, \ldots, B_m$ *are as in Lemma 6.3. Furthermore,*

(i) $\varepsilon_1 \leq |t_{k_0}| < \varepsilon_0$;

(ii) $\left\|T^{k_0 m}(g+v) - T^{k_0 m}(g)\right\|_{\mathcal{B}} > \varepsilon_2$.

(iii) $\left|\sigma_{k_0 m}\left(T^{k_0 m}(g+v) - T^{k_0 m}(g)\right)\right| > \varepsilon_2$;

(iv) $\left\|T^{km+i}(g+v) - T^{km+i}(g)\right\|_{\mathcal{B}} < M_0 \left\|T^{km}(g+v) - T^{km}(g)\right\|_{\mathcal{B}}$ *(which is less than* $\varepsilon_0$*) for all* $k \leq k_0$ *and all* $i = 0, \ldots, m$.

*Proof.* For every $g \in \mathbb{K}$, let $\mathcal{B}(g, \varepsilon)$ be the open ball in $\mathcal{B}$ of radius $\varepsilon$ centered at $g$. Let us fix $m > m_1$ and choose $\varepsilon_0 < \min\{\mu_0, \eta_1, \ldots, \eta_m\}$ such that all the properties **B1** to **B6** of Definition 6.1 are satisfied in $\bigcup_{g \in \mathbb{K}} \mathcal{B}(g, \varepsilon_0) \subset \mathcal{O}_{\mathcal{B}}^{(m)}$, where $\mu_0$ is as in Lemma 6.2 and $\eta_1, \ldots, \eta_m$ are as in Lemma 6.3. Since $m > m_1$, we have that $B_m < 1/2$ where $B_m$ is as in Lemma 6.3. Let us take $M > 1$ as in Lemma 6.2. We choose $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0 < \mu$ such that

$$\begin{aligned} \varepsilon_1 &< \varepsilon_0 / \left(3 M_0 M^{m+2}\right) \ , \\ \varepsilon_2 &< \varepsilon_1 / (2 + 2M) \ . \end{aligned} \tag{6.4.7}$$

Now we work by induction on $k$. Let us assume that $f_k$, $t_k$, and $v_k$ have been defined so that (6.4.6) holds. Hence $\left\|f_k - T^{mk}(g)\right\|_{\mathcal{B}} \leq \varepsilon_0 \leq \mu$, and so $f_k \in \mathcal{O}_{\mathcal{B}}^{(m)}$ and $f_{k+1} = T^m(f_k)$ is well-defined. Since $|t_k| \leq \varepsilon_1$ and $2M^{m+1}\varepsilon_1 \leq \varepsilon_0 \leq \mu$, by Lemma 6.2 and (6.2), and by (6.4.5) and (6.4.7), we have that $t_{k+1}$ is well-defined and

$$\begin{aligned} |t_{k+1}| &= \left|\hat{\delta}_{km}^{(k+1)m}(t_k + \sigma_{km}(v_k))\right| \leq M^m(|t_k| + |\sigma_{km}(v_k)|) \\ &\leq M^m \left(\varepsilon_1 + M\frac{\varepsilon}{2^k}\right) < 2M^{m+1}\varepsilon_1 < \varepsilon_0 \ . \end{aligned} \tag{6.4.8}$$

Thus, by Lemma 6.2, $u_{(k+1)m}(t_{k+1})$ and $v_{k+1} = f_{k+1} - u_{(k+1)m}(t_{k+1})$ are also well-defined. By Lemma 6.3 and by (6.4.5), we get

$$\begin{aligned} \|v_{k+1}\|_{\mathcal{B}} &= \left\|T^m(f_k) - u_{(k+1)m}(t_{k+1})\right\|_{\mathcal{B}} \\ &= \left\|T^m(v_k + u_{km}(t_k)) - u_{(k+1)m}\left(\hat{\delta}_{km}^{(k+1)m}(t_k + \sigma_{km}(v_k))\right)\right\|_{\mathcal{B}} \\ &\leq 2^{-1}\|v_k\|_{\mathcal{B}} \leq 2^{k+1}\|v_0\|_{\mathcal{B}} \ . \end{aligned} \tag{6.4.9}$$

Now, let us estimate $\left\|f_{k+1} - T^{(k+1)m}(g)\right\|_{\mathcal{B}}$. From (6.4.5) and (6.4.9), we get

$$\left\|f_{k+1} - u_{(k+1)m}(t_{k+1})\right\|_{\mathcal{B}} \leq \|v_{k+1}\|_{\mathcal{B}} \leq \frac{\varepsilon}{2^k} \ . \tag{6.4.10}$$

From Lemma 6.2 and by (6.4.8), we obtain

$$\left\|u_{(k+1)m}(t_{k+1}) - T^{(k+1)m}(g)\right\|_{\mathcal{B}} \leq M\,|t_{k+1}| \leq 2M^{m+2}\varepsilon_1 \ . \tag{6.4.11}$$

Thus, by (6.4.7), (6.4.10) and (6.4.11) we have

$$\begin{aligned}\left\|f_{k+1} - T^{(k+1)m}(g)\right\|_{\mathcal{B}} &\leq \|f_{k+1} - u_{(k+1)m}(t_{k+1})\|_{\mathcal{B}} \\ &\quad + \left\|u_{(k+1)m}(t_{k+1}) - T^{(k+1)m}(g)\right\|_{\mathcal{B}} \\ &\leq \frac{\varepsilon}{2^k} + 2M^{m+2}\varepsilon_1 \\ &\leq 3M^{m+2}\varepsilon_1 < \varepsilon_0/M_0 \ .\end{aligned}$$

This completes the induction.

Now, we must prove (i), (ii), (iii) and (iv). Property (i) follows from (6.4.8). Let us prove (ii). By property (i) and Lemma 6.2,

$$\left\|u_{k_0m}(t_{k_0}) - T^{k_0m}(g)\right\|_{\mathcal{B}} \geq M^{-1}\varepsilon_1 \ ,$$

By (6.4.9), we get $\|v_{k_0}\|_{\mathcal{B}} \leq \varepsilon/2^{k_0}$. Thus, by (6.4.7), we obtain

$$\begin{aligned}\left\|T^{k_0m}(g+v) - T^{k_0m}(g)\right\|_{\mathcal{B}} &= \left\|u_{k_0m}(t_{k_0}) + v_{k_0} - T^{k_0m}(g)\right\|_{\mathcal{B}} \\ &\geq \left|\left\|u_{k_0m}(t_{k_0}) - T^{k_0m}(g)\right\|_{\mathcal{B}} - \|v_{k_0}\|_{\mathcal{B}}\right| \\ &\geq M^{-1}\varepsilon_1 - \frac{\varepsilon}{2^{k_0}} \\ &\geq \varepsilon_2 \ .\end{aligned}$$

Let us prove (iii). Using property (i) and Lemma 6.2, we have

$$\left|\sigma_{k_0m}(u_{k_0m}(t_{k_0} - T^{k_0m}g))\right| \geq \varepsilon_1/2 \ .$$

Using Lemma 6.2 yet again and (6.4.9), we have $|\sigma_{k_0m}(v_{k_0})| \leq M\varepsilon/2^{k_0}$. Thus, by (6.4.7),we get

$$\begin{aligned}\left|\sigma_{k_0m}\left(T^{k_0m}(g+v) - T^{k_0m}g\right)\right| &\geq \big|\ |\sigma_{k_0m}(u_{k_0m}(t_{k_0}))| - |\sigma_{k_0m}(v_{k_0})|\ \big| \\ &\geq \frac{1}{2}\varepsilon_1 - M\frac{\varepsilon_2}{2^{k_0}} \\ &\geq \varepsilon_2 \ .\end{aligned}$$

Finally, let us prove (iv). Fix $0 \leq k \leq k_0$ and $0 \leq i \leq m$. Setting $w_k = T^{km}(f) - T^{km}(g)$ we have by (6.4.6) that $\|w_k\|_{\mathcal{B}} < \varepsilon_0/M_0 < \eta_i$ where $\eta_i$ is as in Lemma 6.3. Hence $T^{km}(g) + w_k \in \mathcal{O}_{\mathcal{B}}^{(i)}$ and by Lemma 6.3 we have

$$\left\|T^i(T^{km}(g) + w_k) - u_{km+i}\left(\hat{\delta}_{km}^{km+i}(\sigma_{km}(w_k))\right)\right\|_{\mathcal{B}} \leq B_i\|w_k\|_{\mathcal{B}} \ .$$

On the other hand, by Lemma 6.2, we have

$$\left\|u_{km+i}\left(\hat{\delta}_{km}^{km+i}(\sigma_{km}(w_k))\right) - T^{km+i}(g)\right\|_{\mathcal{B}} \leq M^{m+2}\|w_k\|_{\mathcal{B}} \ .$$

Therefore,

$$\begin{aligned}\|T^{km+i}(f) - T^{km+i}(g)\|_{\mathcal{B}} \leq & \left\|T^i(T^{km}(g) + w_k) - u_{km+i}\left(\hat{\delta}_{km}^{km+i}(\sigma_{km}(w_k))\right)\right\|_{\mathcal{B}} \\ & + \|u_j\left(\hat{\delta}_{km}^{km+i}(\sigma_{km}(w_k))\right) - T^{km+i}(g)\|_{\mathcal{B}} \\ \leq & \, (B_i + M^{m+2})\|w_k\|_{\mathcal{B}} \leq \varepsilon_0 \ ,\end{aligned}$$

which ends the proof. □

## 6.5. Local stable sets

Let us now consider the local stable set $W^s_\varepsilon(g)$ of $T$ at $g$ in $\mathcal{B}$ which consists of all points $f \in \mathcal{B}(g,\varepsilon)$ such that for all $n > 0$, we have $T^n(f) \in \mathcal{B}(T^n(g),\varepsilon)$ and

$$\|T^n(f) - T^n(g)\|_{\mathcal{B}} \to 0 \text{ when } n \to \infty \ .$$

Our aim in this section is to give a finite characterization of $W^s_\varepsilon(g)$ and prove that $T$ contracts in the $\mathcal{B}$-norm exponentially along $W^s_\varepsilon(g)$. This is done in Lemma 6.5 below (see also Remark 6.1).

From now on in this section, we let $m_1$ and $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \varepsilon$ be as in Lemma 6.4. For all sufficiently small $0 < \varepsilon < \varepsilon_2$ and for all $f \in \mathcal{B}(g,\varepsilon)$, we let $k_0 = k_0(g,f)$ and $t_k = t_k(g,f)$ for $k = 0,\ldots,k_0$ be as in Lemma 6.4. We write $\mathcal{B}(g,\varepsilon) = V^-_\varepsilon(g) \cup V^0_\varepsilon(g) \cup V^+_\varepsilon(g)$ where

$$\begin{aligned} V^-_\varepsilon(g) &= \{f \in \mathcal{B}(g,\varepsilon) : -\varepsilon_0 < t_{k_0}(g,f) < -\varepsilon_1\} \ , \\ V^+_\varepsilon(g) &= \{f \in \mathcal{B}(g,\varepsilon) : \varepsilon_1 < t_{k_0}(g,f) < \varepsilon_0\} \ , \\ V^0_\varepsilon(g) &= \mathcal{B}(g,\varepsilon) \setminus \left(V^-_\varepsilon(g) \cup V^+_\varepsilon(g)\right) \ . \end{aligned}$$

LEMMA 6.5. *There exist an integer $m$ and a positive constant $C_2$ with the following properties. For all $\varepsilon > 0$ sufficiently small and for all $g \in \mathbb{K}$, the sets $V^-_\varepsilon(g)$ and $V^+_\varepsilon(g)$ are open subsets of $\mathcal{B}(g,\varepsilon)$ (and so $V^0_\varepsilon(g)$ is relatively closed in $\mathcal{B}(g,\varepsilon)$), and for all $f \in V^0_\varepsilon(g)$*

$$\left\|T^j(f) - T^j(g)\right\|_{\mathcal{B}} \leq \varepsilon C_2 2^{-j/m} \ . \tag{6.5.1}$$

*Furthermore, the local stable set $W^s_\varepsilon(g)$ is a relatively open subset of $V^0_\varepsilon(g)$ and*

$$W^s_\varepsilon(g) = \left\{f \in V^0_\varepsilon(g) : \left\|T^j(f) - T^j(g)\right\|_{\mathcal{B}} < \varepsilon, \text{ for all } 0 \leq j \leq m\log C_2/\log 2\right\} \ . \tag{6.5.2}$$

*Proof.* The first assertion is a consequence of the definitions of $V^-_\varepsilon(g)$ and $V^+_\varepsilon(g)$ and Lemma 6.4. It follows from property (i) of Lemma 6.4 that

$$V^0_\varepsilon(g) = \{f \in \mathcal{B}(g,\varepsilon) : |t_k(g,f)| < \varepsilon_1, \text{ for all } k \geq 0\} \ .$$

It also follows from property (ii) of Lemma 6.4 that if $f \in \mathcal{B}(g,\varepsilon)$ and $|t_{k_0}(g,f)| \geq \varepsilon_1$ then $\left\|T^{k_0 m}f - T^{k_0 m}g\right\|_{\mathcal{B}} > \varepsilon$ where $k_0 = k_0(g,f)$. This shows that $W^s_\varepsilon(g) \subset$

$V_\varepsilon^0(g)$, and therefore (6.5.1) implies (6.5.2). Furthermore, $W_\varepsilon^s(g)$ is a relatively open subset of $V_\varepsilon^0(g)$.

It remains to show that if $f \in V_\varepsilon^0(g)$ then (6.5.1) holds. Set $1 < \lambda < M$ as in Lemma 6.2. Fixing $\beta > 2$, by Lemma 6.2, there is $m$ large enough such that $\delta_{km}^{(k+1)m} \geq M^{-1}\lambda^m > \beta > 2$ for every $k \geq 0$. By Lemma 6.4, for all $k \geq 0$, we know that $t_k = t_k(g,f)$ and $v_k = v_k(g,f)$ are well-defined, and satisfy $|t_k| < \varepsilon_1$ and $\|v_k\|_{\mathcal{B}} \leq \varepsilon 2^{-k}$. Furthermore, $t_{k+1} = \hat{\delta}_{km}^{(k+1)m}(t_k + \sigma_{km}(v_k))$. Since $\delta_{km}^{(k+1)m}$ is $C^2$ and $\|\sigma_{km}\|_{\mathcal{B}} < M$ (see Lemma 6.2), there is $c_0 > 1$ so that

$$\begin{aligned}\left|t_{k+1} - \delta_{km}^{(k+1)m} t_k\right| &\leq \left|\hat{\delta}_{km}^{(k+1)m}(t_k + \sigma_{km}(v_k)) - \hat{\delta}_{km}^{(k+1)m}(t_k)\right| \\ &\quad + \left|\hat{\delta}_{km}^{(k+1)m}(t_k) - \delta_{km}^{(k+1)m} t_k\right| \\ &\leq c_0\left(\|v_k\|_{\mathcal{B}} + |t_k|^2\right) \\ &\leq c_0\left(\varepsilon 2^{-k} + |t_k|^2\right) .\end{aligned}$$

Hence (6.5.3) gives us

$$|t_k| \leq \varepsilon c_0 \beta^{-1} 2^{-k} + \beta^{-1}|t_{k+1}| + c_0\beta^{-1}|t_k|^2 .$$

Taking $\varepsilon$ (in Lemma 6.4) so small that $c_0\beta^{-1}\varepsilon < 1/2$, we get

$$|t_k| \leq 2\left(|t_k| - c_0\beta^{-1}|t_k|^2\right) \leq \varepsilon 2 c_0 \beta^{-1} 2^{-k} + 2\beta^{-1}|t_{k+1}| , \tag{6.5.3}$$

for all $k \geq 0$. Since $2\beta^{-1} < 1$, using induction in (6.5.3) and the fact that $t_k$ is bounded, we get $|t_k| \leq \varepsilon c_1 2^{-k}$ with $c_1 = 2c_0\beta^{-1}/\left(1 - 2\beta^{-1}\right)$ for all $k \geq 0$. Now this estimate together with Lemma 6.2 gives us

$$\|u_{km}(t_k) - T^{km}(g)\|_{\mathcal{B}} \leq M|t_k| \leq \varepsilon c_1 M 2^{-k} .$$

Hence, using Lemma 6.4 again, we get

$$\begin{aligned}\left\|T^{km}(f) - T^{km}(g)\right\|_{\mathcal{B}} &\leq \|v_k\|_{\mathcal{B}} + \left\|u_{km}(t_k) - T^{km}(g)\right\|_{\mathcal{B}} \\ &\leq \varepsilon 2^{-k} + \varepsilon c_1 M 2^{-k} = \varepsilon c_2 2^{-k} .\end{aligned}$$

Therefore, by (iv) in Lemma 6.4, for all $i \in \{1, \ldots, m-1\}$ we have

$$\left\|T^{km+i}(f) - T^{km+i}(g)\right\|_{\mathcal{B}} \leq M_0 \left\|T^{km}(f) - T^{km}(g)\right\|_{\mathcal{B}} \leq \varepsilon c_3 2^{-k} ,$$

which ends the proof. □

REMARK 6.1. *Note that since the constant $C_2$ is uniform (independent of $\varepsilon$) in the above Lemma, inequality* (6.5.1) *can be improved to*

$$\left\|T^j(f) - T^j(g)\right\|_{\mathcal{B}} \leq C' 2^{-j/m} \|f - g\|_{\mathcal{B}} ,$$

*where $C' = 2C_2$. Therefore, we have exponential contraction in $\mathcal{B}$ (along the local stable sets) in the strong sense.*

### 6.6. Tangent spaces

Our next goal is to show that $V_\varepsilon^0(g)$ is a $C^1$ manifold provided $\varepsilon$ is sufficiently small. The first step towards this goal is to find the natural candidate for the tangent space at every point $f \in V_\varepsilon^0(g)$. This will be accomplished in Lemma 6.7 below. The proof will require the following elementary bootstrapping result.

LEMMA 6.6. *Let $(a_n)$ be a sequence of real numbers such that, for some $c_0 > 0$ and all $n \geq 1$,*

$$|a_{n+1}| \leq \frac{1}{4}|a_n| + \frac{c_0}{2^n}\sum_{j=1}^{n-1}|a_j| \ . \tag{6.6.1}$$

*Then $|a_n| \leq c_1 2^{-n}$ for some $c_1 > 0$ and all $n \geq 1$.*

*Proof.* We may assume that $c_0 \geq 1$. Let $n_0 > 0$ be such that $c_0 n_0/2^{n_0} < 1/2$, and set $b = \max_{1\leq j\leq n_0}\{|a_j|\}$. Then we see by induction from (6.6.1) that $|a_n| \leq b$ for all $n \geq 1$, and so

$$\begin{aligned}|a_{n+1}| &\leq \frac{1}{4}|a_n| + \frac{nbc_0}{2^n}\\ &\leq \frac{1}{4}|a_n| + bc_0\left(\frac{3}{4}\right)^n \ .\end{aligned}$$

By induction, this yields $|a_n| \leq (2bc_0)(\frac{3}{4})^n$ for all $n \geq 1$. Hence $\sum_{n=1}^{\infty}|a_n| \leq 6bc_0$. Using (6.6.1) once more, we deduce that

$$|a_{n+1}| \leq \frac{1}{4}|a_n| + \frac{6bc_0^2}{2^n} \ ,$$

for all $n \geq 1$. Again by induction, this gives us $|a_n| \leq (24bc_0^2)2^{-n}$ for all $n \geq 1$, which is the desired result. □

LEMMA 6.7. *There exist an integer $m$, constants $C_3, C_4 > 0$ and $\varepsilon > 0$ small enough with the following properties. For every $g \in \mathbb{K}$ and for every $f \in V_\varepsilon^0(g)$ there exists a linear functional $\theta_{f,g} \in \mathcal{C}^*$ with norm bounded from above by $C_3$ and with the property that*

$$\left\|DT^j(f)v - \delta_0^j\theta_{f,g}(v)\mathbf{u}_j\right\|_{\mathcal{C}} \leq C_4\delta_0^j 2^{-j/m}\|v\|_{\mathcal{C}} \ , \tag{6.6.2}$$

*for all $v \in \mathcal{C}$ and all $j \geq 1$. If $g_0, g_1 \in \mathbb{K}$ and $f \in V_\varepsilon^0(g_1) \cap V_\varepsilon^0(g_2)$ then*

$$\theta_{f,g_1}|\mathcal{B} = \theta_{f,g_2}|\mathcal{B}.$$

*Furthermore, the map $\Psi : \bigcup_{g\in\mathbb{K}} V_\varepsilon^0(g) \to \mathcal{B}^*$ given by $\Psi(f) = \theta_f = \theta_{f,g}|\mathcal{B}$ (where $g$ is any point of $\mathbb{K}$ such that $f \in V_\varepsilon^0(g)$) is well-defined and uniformly continuous.*

Remark 6.2. *Condition* (6.6.2) *entails that for every* $g \in \mathbb{K}$, $\mathcal{B}$ *is the direct sum of the one dimensional unstable subspace* $E^u_g$ *with the kernel of* $\theta_f$*, i.e.* $\mathcal{B} = E^u_g \bigoplus \ker(\theta_f)$*, provided* $f$ *is sufficiently close to* $g$*. To see this, note that we can write*

$$v = \theta_f(v)(\theta_f(\mathbf{u}_g))^{-1}\mathbf{u}_g + \left(v - \theta_f(v)(\theta_f(\mathbf{u}_g))^{-1}\mathbf{u}_g\right) \ .$$

*Thus, from the continuity of* $f \mapsto \theta_f$ *plus the fact that* $\theta_g(\mathbf{u}_g) \neq 0$ *it follows that if* $f$ *is close to* $g$ *then* $\mathbf{u}_g$ *is transversal to* $\ker(\theta_f)$*. The hyperplane* $\ker(\theta_f)$ *is the natural candidate to be the tangent space of* $V^0_\varepsilon(f)$ *at* $f$ *since it corresponds to all vectors which expand under* $DT^j(f)$ *by a factor less than* $\delta_0^j$*.*

*Proof.* Let $\varepsilon > 0$ be small enough such that Lemma 6.5 is satisfied and $\varepsilon C_2 < \nu_m$ (where $\nu_m$ is as in property **B5** in Definition 6.1 and $C_2$ is as in Lemma 6.5). Let $R_k = R_{f,k} = \left(\delta_0^{km}\right)^{-1} DT^{mk}(f)$ and write $f_k = T^{km}(f)$, and $g_k = T^{km}(g)$ for all $k \geq 0$. Then we have

$$R_{k+1}(v) = \left(\delta_{km}^{(k+1)m}\right)^{-1} DT^m(f_k) R_k(v) \ . \tag{6.6.3}$$

Let us take $v \in \mathcal{C}$ with $\|v\|_{\mathcal{C}} = 1$. We can write $R_k(v) = \alpha_k u_{km} + w_k$, where $\alpha_k \in \mathbb{R}$ and $w_k \in \mathcal{C}$ are defined recursively by $\alpha_0 = 0$, $w_0 = v$ and

$$\begin{aligned} \alpha_{k+1} &= \alpha_k + \sigma_{km}(w_k) \\ w_{k+1} &= \alpha_k \left(\delta_{km}^{(k+1)m}\right)^{-1} (DT^m(f_k) - DT^m(g_k))\mathbf{u}_{km} \\ &\quad + \left(\delta_{km}^{(k+1)m}\right)^{-1} (DT^m(f_k) - DT^m(g_k)) w_k \\ &\quad + \left(\delta_{km}^{(k+1)m}\right)^{-1} Q_{km}^{(k+1)m-1}(w_k) \ . \end{aligned} \tag{6.6.4}$$

Now, by Lemma 6.5, we know that

$$\|f_k - g_k\|_{\mathcal{B}} \leq \varepsilon C_2 2^{-k} \ . \tag{6.6.5}$$

Since, by property **B3** of Definition 6.1, the map $f \to DT^m(f)\mathbf{u}_{km}$ is Lipschitz at $f = g_k$ (as a map from $\mathcal{B}$ to $\mathcal{C}$), we have that for all $k$ large enough

$$\|DT^m(f_k)\mathbf{u}_{km} - DT^m(g_k)\mathbf{u}_{km}\|_{\mathcal{C}} \leq c_1 \|f_k - g_k\|_{\mathcal{B}} \leq c_2 2^{-k} \ . \tag{6.6.6}$$

By property **B5** in Definition 6.1 and (6.6.5), for all $m$ large enough we also have

$$\|DT^m(f_k) - DT^m(g_k)\|_{\mathcal{C}} \leq \frac{\delta_{km}^{(k+1)m}}{8} \ . \tag{6.6.7}$$

Since, by property **B1** of Definition 6.1, $(\mathcal{C}, \mathcal{D})$ is $\rho$-compatible with $(T, \mathbb{K})$, by Corollary 4.2, for all $m$ large enough we have

$$\left\|Q_{km}^{(k+1)m-1}\right\|_{\mathcal{C}} \leq \frac{\delta_{km}^{(k+1)m}}{8} \ . \tag{6.6.8}$$

Using Lemma 6.2 and putting (6.6.6), (6.6.7) and (6.6.8) in (6.6.4) we get

$$\begin{aligned}\|w_{k+1}\|_{\mathcal{C}} &\le \frac{1}{4}\|w_k\|_{\mathcal{C}} + c_3 2^{-k}|\alpha_k| \\ &\le \frac{1}{4}\|w_k\|_{\mathcal{C}} + \frac{c_3 M}{2^k}\sum_{j=0}^{k-1}\|w_j\|_{\mathcal{C}} \ . \end{aligned} \tag{6.6.9}$$

From (6.6.9) and Lemma 6.6, we deduce that $\|w_k\|_{\mathcal{C}} < c_4 2^{-k}$. Thus, by (6.6.4) we obtain $|\alpha_{k+1} - \alpha_k| \le c_5 2^{-k}$ for all $k \ge 0$. Therefore, $\theta_{f,g}(v) = \lim \alpha_k$ exists and

$$\|R_k(v) - \theta_{f,g}(v)\mathbf{u}_{km}\|_{\mathcal{C}} \le c_6 2^{-k} \ , \tag{6.6.10}$$

for all $v \in \mathcal{C}$ with $\|v\|_{\mathcal{C}} = 1$. If $v \in \mathcal{C}$ and $\|v\|_{\mathcal{C}} \ne 1$ then we define $\theta_{f,g}(v) = \|v\|_{\mathcal{C}}\theta_{f,g}(v/\|v\|_{\mathcal{C}})$. By (6.6.10) and by Lemma 6.2, for all $v, w \in \mathcal{C}$ we have

$$\begin{aligned}&|\theta_{f,g}(v) + \theta_{f,g}(w) - \theta_{f,g}(v+w)| \\ &\le M\,\|\theta_{f,g}(v)\mathbf{u}_{km} + \theta_{f,g}(w)\mathbf{u}_{km} - \theta_{f,g}(v+w)\mathbf{u}_{km}\|_{\mathcal{C}} \\ &\le M\,\|\theta_{f,g}(v)\mathbf{u}_{km} - R_k(v)\|_{\mathcal{C}} + M\,\|\theta_{f,g}(w)\mathbf{u}_{km} - R_k(w)\|_{\mathcal{C}} \\ &\quad + M\,\|\theta_{f,g}(v+w)\mathbf{u}_{km} - R_k(v+w)\|_{\mathcal{C}} \\ &\le c_7 2^{-k}\left(\|v\|_{\mathcal{C}} + \|w\|_{\mathcal{C}}\right) \ .\end{aligned}$$

Hence, letting $k$ go to infinity we deduce that $\theta_{f,g}$ is a linear functional in $\mathcal{C}^*$. Again by (6.6.10), $\|\theta_{f,g}\|_{\mathcal{C}}$ is uniformly bounded and inequality (6.6.2) is satisfied for $j = km$. By (6.6.10) and by property **B3** in Definition 6.1, for $j = km + i$ with $i \in \{1, \ldots, m-1\}$, we get

$$\begin{aligned}\|R_j(v) - \theta_{f,g}(v)\mathbf{u}_j\|_{\mathcal{C}} &\le \left\| \left(\delta_{km}^{km+i}\right)^{-1} DT^i(T^{km}f)(R_k(v) - \theta_{f,g}(v)\mathbf{u}_{km}) \right\|_{\mathcal{C}} \\ &+ \left\| \left(\delta_{km}^{km+i}\right)^{-1} \left(DT^i(T^{km}f) - DT^i(T^{km}g)\right)\theta_{f,g}(v)\mathbf{u}_{km} \right\|_{\mathcal{C}} \\ &\le c_8 2^{-k}\,\|v\|_{\mathcal{C}} \ ,\end{aligned} \tag{6.6.11}$$

which proves (6.6.2). In particular, there is $M_0 > 0$ such that

$$\|R_{k,f}(v)\|_{\mathcal{C}} \le M_0 \ , \tag{6.6.12}$$

for all $g \in \mathbb{K}$, all $f \in V_\varepsilon^0(g)$ and all $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} = 1$.

Let us prove that the map $f \mapsto \theta_{f,g}|\mathcal{B}$ is continuous from $V_\varepsilon^0(g)$ into $\mathcal{B}^*$ for every $g \in \mathbb{K}$. By property **B1** of Definition 6.1, for every $k \ge 1$ the functional $\sigma_{km}$ is continuous on $\mathcal{C}$ and its norm is uniformly bounded. By property **B2** in Definition 6.1, the map $f \mapsto R_{k,f}$ is continuous from $\mathcal{B}$ into $\mathcal{C}$. Hence, the mapping $V_\varepsilon^0(g) \to \mathcal{B}^*$ given by $f \mapsto \sigma_{km} \circ R_{k,f}$ is also continuous. By (6.6.11), we obtain

$$\begin{aligned}|\sigma_{km} \circ R_{k,f}(v) - \theta_{f,g}(v)| &= |\sigma_{km}\left(R_k(v) - \theta_{f,g}(v)\mathbf{u}_{km}\right)| \\ &\le c_9 2^{-k}\,\|v\|_{\mathcal{B}} \ .\end{aligned} \tag{6.6.13}$$

Therefore, the continuous maps $f \mapsto \sigma_{km} \circ R_{k,f}$ converge uniformly to $f \mapsto \theta_{f,g}$, which implies that $f \to \theta_{f,g}$ is also a continuous map from $V^0_\varepsilon(g)$ to $\mathcal{B}^*$.

Let us prove that $\theta_{f,g}|\mathcal{B}$ for $f \in \bigcup_{g\in\mathbb{K}} V^0_\varepsilon(g)$ does not depend on $g \in \mathbb{K}$. Let us take $f \in \bigcup_{g\in\mathbb{K}} V^0_\varepsilon(g)$ and $g_0, g_1 \in \mathbb{K}$ such that $f \in V^0_\varepsilon(g_0)$ and $f \in V^0_\varepsilon(g_1)$. By Lemma 6.5, for every $k \geq 1$ we get

$$\begin{aligned}\left\|T^{km}(g_1) - T^{km}(g_0)\right\|_{\mathcal{B}} &\leq \left\|T^{km}(g_1) - T^{km}(f)\right\|_{\mathcal{B}} + \left\|T^{km}(f) - T^{km}(g_0)\right\|_{\mathcal{B}} \\ &\leq c_{10}\varepsilon 2^{-k} \ .\end{aligned}$$

By property **B1** of Definition 6.1, the map $g \mapsto \sigma_g$ from $\mathbb{K}$ into $\mathcal{C}^*$ is uniformly continuous. Hence, for every $\varepsilon > 0$ there is $k_0 > 0$ large enough such that for all $k > k_0$ and all $w \in \mathcal{C}$ with $\|w\|_{\mathcal{C}} \leq M_0$ we have

$$\left|\sigma_{T^{km}(g_1)}(w) - \sigma_{T^{km}(g_0)}(w)\right| \leq \varepsilon/2 \ . \tag{6.6.14}$$

By (6.6.12), (6.6.13) and (6.6.14) and taking $k$ large enough, we get

$$\begin{aligned}|\theta_{f,g_1}(v) - \theta_{f,g_0}(v)| &\leq \left|\theta_{f,g_1}(v) - \sigma_{T^{km}(g_1)} \circ R_{k,f}(v)\right| \\ &\quad + \left|\sigma_{T^{km}(g_1)} \circ R_{k,f}(v) - \sigma_{T^{km}(g_0)} \circ R_{k,f}(v)\right| \\ &\quad + \left|\sigma_{T^{km}(g_0)} \circ R_{k,f}(v) - \theta_{f,g_0}(v)\right| \\ &\leq 2c_9 2^{-k} + \varepsilon/2 \leq \varepsilon\end{aligned}$$

for all $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} = 1$. Thus, $\theta_{f,g_1}(v) = \theta_{f,g_0}(v)$ and so the map $\Psi$ is well-defined.

Let us prove that the map $\Psi$ is uniformly continuous. For every $\alpha_0 > 0$, let us choose $k_0 > 0$ large enough such that $2c_9 2^{-k_0} \leq \alpha_0/3$. Since the map $g \to \sigma_g$ is uniformly continuous, there is $\alpha_1 > 0$ small enough such that for all $g_0, g_1 \in \mathbb{K}$ with $\|g_1 - g_0\|_{\mathcal{C}} < \alpha_1$ and all $w \in \mathcal{B}$ with $\|w\|_{\mathcal{C}} \leq M_0$ we get

$$|\sigma_{g_1}(w) - \sigma_{g_0}(w)| \leq \alpha_0/3 \ . \tag{6.6.15}$$

Let us choose $k_1 > k_0$ large enough such that $\varepsilon C_2 2^{-k_1} \leq \alpha_1/3$ where $C_2 > 0$ is the constant of Lemma 6.5. Since $T : \mathcal{O}_B \to \mathcal{C}$ is a $C^1$ operator, by property **B2** of Definition 6.1, (and compactness of $\mathbb{K}$), there is $\alpha_2 > 0$ small enough such that for all $f_0 \in V^0_\varepsilon(g_0)$ and $f_1 \in V^0_\varepsilon(g_1)$ with $\|f_1 - f_0\|_{\mathcal{B}} < \alpha_2$ we obtain that $\|T^{k_1m}(f_1) - T^{k_1m}(f_0)\|_{\mathcal{C}} \leq \alpha_1/3$. Hence, by Lemma 6.5, we get

$$\begin{aligned}\left\|T^{k_1m}(g_1) - T^{k_1m}(g_0)\right\|_{\mathcal{C}} &\leq \left\|T^{k_1m}(g_1) - T^{k_1m}(f_1)\right\|_{\mathcal{C}} + \left\|T^{k_1m}(f_1) - T^{k_1m}(f_0)\right\|_{\mathcal{C}} \\ &\quad + \left\|T^{k_1m}(f_0) - T^{k_1m}(g_0)\right\|_{\mathcal{C}} \\ &\leq 2\varepsilon C_2 2^{-k_1} + \alpha_1/3 \leq \alpha_1 \ .\end{aligned} \tag{6.6.16}$$

By (6.6.15) and (6.6.16), we get

$$\left|\sigma_{T^{k_1m}(g_1)} \circ R_{k_1,f_1}(v) - \sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_1}(v)\right| \leq \alpha_0/3 \ . \tag{6.6.17}$$

Using property **B2** of Definition 6.1, choose $0 < \alpha_3 < \alpha_2$ small enough such that for all $f_1 \in \bigcup_{g\in\mathbb{K}} V_\varepsilon^0(g)$ with $\|f_1 - f_0\|_{\mathcal{B}} < \alpha_3$ we have $\|R_{k_1,f_1}(v) - R_{k_1,f_0}(v)\|_{\mathcal{C}} \leq \alpha_0/(3M)$, where $v \in \mathcal{B}$ with $\|v\|_{\mathcal{B}} = 1$ and $M$ is as in Lemma 6.2. Hence,

$$\left|\sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_1}(v) - \sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_0}(v)\right| \leq \alpha_0/3 \tag{6.6.18}$$

By (6.6.13), (6.6.17) and (6.6.18), we obtain that

$$\begin{aligned}|\theta_{f_1}(v) - \theta_{f_0}(v)| \leq &\left|\theta_{f_1}(v) - \sigma_{T^{k_1m}(g_1)} \circ R_{k_1,f_1}(v)\right| \\ &+ \left|\sigma_{T^{k_1m}(g_1)} \circ R_{k_1,f_1}(v) - \sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_1}(v)\right| \\ &+ \left|\sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_1}(v) - \sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_0}(v)\right| \\ &+ \left|\sigma_{T^{k_1m}(g_0)} \circ R_{k_1,f_0}(v) - \theta_{f_0}(v)\right| \\ \leq & 2c_9 2^{-k_1} + 2\alpha_0/3 \leq \alpha_0 \ .\end{aligned}$$

Therefore, the map $\Psi$ is uniformly continuous. □

## 6.7. The main estimates

Besides aiming at proving that the local stable set is a $C^1$ manifold, we want to show that the local hyperbolicity picture holds (in $\mathcal{B}$) near $\mathbb{K}$. In other words we want to show that if the iterates $T^{km}(f_1)$ of a point $f_1 \in \mathcal{B}(g,\varepsilon)$ remain in $\mathcal{B}(T^{km}(g),\varepsilon)$ for a long time, that is for $k = 0, 1, \ldots, N$ with $N$ large, then $f_1$ has to be very close to a point $f_0$ on the stable set $W_\varepsilon^s(g)$ at the outset, and in the end $T^{Nm}(f_1)$ has to be very close to the unstable manifold $W_\varepsilon^u(T^{Nm}(g))$.

To prove these facts, we consider in this section (see Lemma 6.11) an intermediate time $l$ for which we can find a good quantitative estimate for the point on the unstable manifold $W_\varepsilon^u(T^{lm}(g))$ that best approximates $T^{lm}(f_1)$. This estimate is provided by the value of $\theta_{f_0}(f_1 - f_0)$, and its most important consequence is obtained when $f_1$ also belongs to the local stable set $W_\varepsilon^s(g)$. In this case we prove an inequality of the form $|\theta_{f_0}(f_1 - f_0)| \leq C\|f_1 - f_0\|_{\mathcal{B}}^{1+\tau}$ (see Lemma 6.12). As we shall see in §6.8, this is precisely what we need to show that the tangent space to the stable set at $f_0$ varies continuously with $f_0$.

In this section we will fix $m$ large enough and $\varepsilon_0 > \varepsilon_1 > \varepsilon_2$ small enough such that lemmas 6.4, 6.5 and 6.7 are satisfied for all $\varepsilon < \varepsilon_2$ suficiently small.

Lemma 6.8. *There exist constants $C_5, C_6, \varepsilon > 0$ with the following property. For all $g \in \mathbb{K}$, all $f_0 \in V_\varepsilon^0(g)$, and all $f_1 \in \mathcal{B}(g,\varepsilon)$ such that $\|f_1 - f_0\|_{\mathcal{C}} \leq C_5\,(\delta_0^n)^{-1}$, we have*

$$\left\|T^k(f_1) - T^k(f_0) - DT^k(f_0)\,(f_1 - f_0)\right\|_{\mathcal{C}} \leq C_6\,(\delta_k^n)^{-\rho} \ , \tag{6.7.1}$$

*for all $0 \leq k \leq \hat{k}_0(g, f_1)$, where*

$$\hat{k}_0(g, f_1) = \min\left\{j \in \{0, \ldots, n\} : \|T^j(f_1) - T^j(g)\|_{\mathcal{B}} \geq \varepsilon_0\right\}$$

*and $\rho > 1$ is as in property* **B4** *of Definition 6.1.*

*Proof.* By lemmas 6.2 and 6.7, there are $c_0, c_1 > 0$ and $\lambda > 1$ such that

$$\left\| DT^{k-i}\left(T^i f_0\right)\right\|_{\mathcal{C}} \leq c_0 \delta_i^k \text{ and } \delta_i^k > c_1 \lambda^{k-i} \tag{6.7.2}$$

for all $0 \leq i < k$. Define a sequence $v_i \in \mathcal{C}$ as follows: $v_0 = T(f_1) - T(f_0)$ and

$$v_i = T^i(f_1) - T^i(f_0) - DT\left(T^{i-1}(f_0)\right)\left(T^{i-1}(f_1) - T^{i-1}(f_0)\right) ,$$

for all $0 < i \leq k$. Hence,

$$T^i(f_1) - T^i(f_0) - DT^i(f_0)\,(f_1 - f_0) = \sum_{j=1}^{i} DT^{i-j}\left(T^j f_0\right) v_j . \tag{6.7.3}$$

Applying property **B4** of Definition 6.1, we get

$$\begin{aligned} \|v_{i+1}\|_{\mathcal{C}} &\leq c_2 \left\|T^i(f_1) - T^i(f_0)\right\|_{\mathcal{C}}^{\rho} \\ &\leq c_2 \left\| DT^i(f_0)\,(f_1 - f_0) + \sum_{j=1}^{i} DT^{i-j}(T^j f) v_j \right\|_{\mathcal{C}}^{\rho} . \end{aligned} \tag{6.7.4}$$

Let us first choose $C_6 > 0$ such that

$$C_6^{1-\rho} > c_2 \left( \frac{2 c_0 c_1^{1-\rho}}{(\delta_i)^{\rho}(1 - \lambda^{1-\rho})} \right)^{\rho} , \tag{6.7.5}$$

for all $0 < i \leq k$, and then choose $C_5 > 0$ such that

$$\begin{aligned} C_5^{\rho} &< \frac{C_6 (\delta_0)^{\rho}}{c_2} , \\ C_5 &< \frac{c_1^{1-\rho} C_6 (\delta_i)^{1-\rho}}{1 - \lambda^{1-\rho}} . \end{aligned} \tag{6.7.6}$$

Let us prove inductively for $i = 1, 2, \ldots, k$ that $\|v_i\|_{\mathcal{C}} \leq C_6\,(\delta_i^n)^{-\rho}$. Using inequality (6.7.4) and (6.7.6), we get

$$\|v_1\|_{\mathcal{C}} \leq c_2 \,\|f_1 - f_0\|_{\mathcal{C}}^{\rho} \leq \frac{c_2 C_5^{\rho}}{(\delta_0)^{\rho}} (\delta_1^n)^{-\rho} \leq C_6 (\delta_1^n)^{-\rho} .$$

Using the inequalities (6.7.2), (6.7.4), (6.7.5) and (6.7.6), we get

$$\begin{aligned} \|v_{i+1}\|_{\mathcal{C}} &\leq c_2 \left( C_5 c_0 \delta_0^i \,(\delta_0^n)^{-1} + \sum_{j=1}^{i} c_0 \delta_j^i C_6 \left(\delta_j^n\right)^{-\rho} \right)^{\rho} \\ &\leq c_2 c_0^{\rho} \left( \frac{C_5}{\delta_i} + \frac{C_6}{(\delta_i)^{\rho}} \sum_{j=1}^{i} \left(\delta_j^i\right)^{1-\rho} \right)^{\rho} \left(\delta_{i+1}^n\right)^{-\rho} \\ &\leq c_2 c_0^{\rho} \left( \frac{C_5}{\delta_i} + \frac{c_1^{1-\rho} C_6}{(\delta_i)^{\rho}\,(1 - \lambda^{1-\rho})} \right)^{\rho} \left(\delta_{i+1}^n\right)^{-\rho} \\ &\leq C_6 \left(\delta_{i+1}^n\right)^{-\rho} , \end{aligned} \tag{6.7.7}$$

which ends the induction. Thus, using (6.7.2) and (6.7.7) in (6.7.3), we get

$$\left\| T^k(f_1) - T^k(f) - DT^k(f)\,(f_1 - f_0) \right\|_{\mathcal{C}} \le \sum_{i=1}^{k} c_0 \delta_i^k C_6\, (\delta_i^n)^{-\rho}$$

$$\le c_0 C_6\, (\delta_k^n)^{-\rho} \sum_{i=1}^{k} \left(\delta_i^k\right)^{1-\rho} \le \frac{c_0 c_1^{1-\rho} C_6}{1-\lambda^{1-\rho}}\, (\delta_i^n)^{-\rho} \ .$$

This proves the Lemma. □

LEMMA 6.9. *Let $C_5, C_6, \varepsilon > 0$, $\rho > 1$ and $\hat{k}_0(g, f_1)$ be as in Lemma 6.8. There exist $C_7, C_8 > 0$ such that for all $g \in \mathbb{K}$, all $f_0 \in V_\varepsilon^0(g)$ and all $f_1 \in \mathcal{B}(g, \varepsilon)$ such that $\|f_1 - f_0\|_{\mathcal{C}} \le C_5\, (\delta_0^n)^{-1}$, we have*

$$\left\| T^k(f_1) - T^k(g) - \delta_0^k \theta_{f_0}\,(f_1 - f_0)\,\mathbf{u}_k \right\|_{\mathcal{C}}$$

$$\le C_6\, (\delta_k^n)^{-\rho} + \varepsilon C_7 2^{-k/m} + C_8 2^{-k/m}\, (\delta_k^n)^{-1} \ , \qquad (6.7.8)$$

*for all $k \le \hat{k}_0(g, f_1)$.*

*Proof.* By Lemma 6.7, we get

$$\left\| DT^k(f_0)\,(f_1 - f_0) - \delta_0^k \theta_{f_0}\,(f_1 - f_0)\,\mathbf{u}_k \right\|_{\mathcal{C}} \le C_4 2^{-k/m}\, (\delta_k^n)^{-1} \ . \qquad (6.7.9)$$

By Lemma 6.5, we obtain that

$$\left\| T^k(f_0) - T^k(g) \right\|_{\mathcal{C}} \le c_0 \left\| T^k(f_0) - T^k(g) \right\|_{\mathcal{B}} \le \varepsilon c_0 C_2 2^{-k/m} \ . \qquad (6.7.10)$$

Combining (6.7.1), (6.7.9) and (6.7.10), we get (6.7.8). □

DEFINITION 6.2. *Given $g \in \mathbb{K}$ and $p \ge 1$ we denote by $l = l(g, p)$ the smallest integer such that*

$$\left(\delta_{lm}^{pm}\right)^{\rho} \le 2^l \ , \qquad (6.7.11)$$

*where $\rho > 1$ is as in property* **B4** *of Definition 6.1.*

LEMMA 6.10. *We have the following assertions:*

*(i) There exist $0 < \mu_0 < \mu_1 < 1$ with the property that $\mu_0 p \le l = l(g, p) \le \mu_1 p$ for all $g \in \mathbb{K}$ and all $p \ge 1$.*

*(ii) There exists $0 < \tau_1 < 1$ such that for all $g \in \mathbb{K}$ and all $f_0, f_1 \in \mathcal{B}(g, \varepsilon)$, if $\|f_0 - f_1\|_{\mathcal{C}} \ge C_5 \left(\delta_0^{(p+1)m}\right)^{-1}$ then $\left(\delta_{lm}^{pm}\right)^{-1} \le C_9 \|f_0 - f_1\|_{\mathcal{C}}^{\tau_1}$, where $C_9$ depends only upon $C_5 > 0$.*

*Proof.* Let us prove part (i). Set $1 < \lambda < M$ as in Lemma 6.2. Then, by (6.7.11), we have

$$\rho \log M^{-1} + (pm - lm)\rho \log \lambda \ < \ \rho \log \delta_{lm}^{pm} \ < \ l \log 2 \ < \ lm \log 2 \ .$$

Hence we get

$$\left(1 + \frac{\log 2}{\rho \log \lambda}\right) lm \ \geq \ \frac{\log M^{-1}}{\log \lambda} + pm \ > \ \frac{pm}{2} \ ,$$

for all $p$ such that $pm > N_0 = \max\{2m, \left|2 \log M^{-1}/ \log \lambda\right|\}$. Thus, taking

$$\mu_0' = 2^{-1} \left(1 + \frac{\log 2}{\rho \log \lambda}\right)^{-1} > 0 \ ,$$

we get $\mu_0' p \leq l$ for all such values of $p$. By (6.7.11) and by Lemma 6.2 there exists a uniform constant $0 < c_0 \leq 1$ such that $c_0 2^l \leq \left(\delta_{lm}^{pm}\right)^\rho$ and so

$$\log c_0 + l \log 2 < \rho \log \delta_{lm}^{pm} \leq \rho(pm - lm) \log M \ .$$

Letting $\alpha = \log 2/\left(\rho \log M\right) > 0$ and $\beta = \log c_0/\left(\rho \log M\right)$ we get

$$lm \left(1 + \frac{\alpha}{m}\right) \leq pm - \beta \leq pm \left(1 + \frac{\alpha}{2m}\right)$$

for all $p > -2\beta/\alpha$. Thus, taking

$$0 < \mu_1' = \frac{2m + \alpha}{2(m + \alpha)} < 1$$

we obtain that $lm \leq \mu_1' pm$ for all such values of $p$. Since $\delta_g$ varies continuously with $g$ in the compact set $\mathbb{K}$, we can extend the previous results to all $p \geq 0$ for some $\mu_0 \leq \mu_0'$ and $\mu_1 \geq \mu_1'$.

Let us prove part (ii). Take $0 < \tau_1 = (1 - \mu_1) \log M/ \log \lambda < 1$. Then, by Lemma 6.2, we have

$$\begin{aligned}
\left(\delta_{lm}^{pm}\right)^{-1} &\leq c_0 \lambda^{-(p-l)m} \leq c_0 \lambda^{-(1-\mu_1)pm} \\
&\leq c_0 M^{-\tau_1 pm} \leq c_1 \omega \left(\delta_0^{(p+1)m}\right)^{-\tau_1} \\
&\leq c_1 \|f_0 - f_1\|_{\mathcal{C}}^{\tau_1} \ .
\end{aligned}$$

□

Lemma 6.11. *There exist $\varepsilon > 0$ sufficiently small and $C_{10} > 0$ such that the following holds for $g \in \mathbb{K}$, $f_0 \in V_\varepsilon^0(g)$ and $f_1 \in \mathcal{B}(g, \varepsilon)$. If $p$ is the largest integer such that*

$$\|f_1 - f_0\|_{\mathcal{C}} \leq C_5 \left(\delta_0^{pm}\right)^{-1}$$

*then $l = l(g, p) \leq k_0 = k_0(g, f_1)$ and so $t_l = t_l(g, f_1)$ is well-defined (where $l$ is as in Lemma 6.10, $k_0$ and $t_l$ are as in Lemma 6.4, and $C_5 > 0$ is as in Lemma 6.9). Furthermore,*

$$\left|t_l - \delta_0^{lm} \theta_{f_0}(f_1 - f_0)\right| \leq C_{10} \left(\delta_{lm}^{pm}\right)^{-\rho} \ . \tag{6.7.12}$$

*where $\rho > 1$ is as in Lemma 6.9.*

*Proof.* Let $\hat{k}_0 = \hat{k}_0(g, f_1)$ be as in Lemma 6.9. Let us prove that $l \le k_0$. By (iv) in Lemma 6.4, $mk_0 < \hat{k}_0$. Hence it is enough to prove that $\min\left\{lm, \hat{k}_0\right\} \le mk_0$. Let $\varepsilon > 0$ be small enough such that lemmas 6.8 and 6.9 are satisfied. Let us show that $lm \le \hat{k}_0$. By inequality (6.7.8), for all $k$ such that $mk \le \hat{k}_0$ we have

$$\left|\sigma_{km}\left(T^{km}(f_1) - T^{km}(g)\right)\right| \le \|\sigma_{km}\|_{\mathcal{C}}\left(\delta_0^{km}\left|\theta_f\left(f_1 - f_0\right)\right| + c_0\left(\delta_{km}^{pm}\right)^{-1} + \varepsilon c_1 2^{-k}\right) \ . \quad (6.7.13)$$

By Lemma 6.2 and Remark 6.2, there is $M_1 > 1$ such that $M_1^{-1} \le \|\sigma_{km}\|_{\mathcal{C}} \le M_1$ and $M_1^{-1} \le \|\theta_{f_0}\|_{\mathcal{C}} \le M_1$. Since by Lemma 6.2, we have $\left(\delta_{km}^{pm}\right)^{-1} \le M\lambda^{-(p-k)m}$ we deduce that

$$\delta_0^{km}|\theta_f\left(f_1 - f_0\right)| \le \left(\delta_{km}^{pm}\right)^{-1}\|\theta_f\|_{\mathcal{C}} \le MM_1\lambda^{-(p-k)pm} \ . \quad (6.7.14)$$

By Lemma 6.10, there is $0 < \mu_1 < 1$ such that for all $p > 0$ and all $k \le l$ we have $p - k \ge p - l \ge (1 - \mu_1)p$. Now, we make $\varepsilon > 0$ small enough (and so $p$ large enough) such that the following inequalities are satisfied

$$(c_0 + M_1)M\lambda^{-(1-\mu_1)pm} < \frac{\varepsilon_2}{2\|\sigma_{km}\|_{\mathcal{C}}} \ ,$$
$$\varepsilon c_1 2^{-k} < \frac{\varepsilon_2}{4\|\sigma_{km}\|_{\mathcal{C}}} \ ,$$

for all $k$ such that $km \le \min\{lm, \hat{k}_0\}$. Therefore, for all such $k$, combining (6.7.13) and (6.7.14) we deduce that

$$|\sigma_{km}\left(T^{km}(f_1) - T^{km}(g)\right)| < \varepsilon_2 \ . \quad (6.7.15)$$

Since $f_1 \in \mathcal{B}(g, \varepsilon)$ and (6.7.15) reverses the inequality (iii) in Lemma 6.4, we obtain that $\min\{lm, \hat{k}_0\} \le mk_0$, and so $l \le k_0$.

Now, let us prove (6.7.12). Since $l \le k_0$, by (6.4.5) and (6.4.6) in Lemma 6.4, there is $t_l = t_l(g, f_1)$ such that

$$\|T^{lm}(f_1) - u_{lm}(t_l)\|_{\mathcal{B}} \le \varepsilon 2^{-l} \le \varepsilon\left(\delta_{lm}^{pm}\right)^{-\rho} \ . \quad (6.7.16)$$

Since $lm < \hat{k}_0$, by lemmas 6.9 and 6.10 we get

$$\|T^{lm}(f_1) - T^{lm}(g) - s_l\mathbf{u}_{lm}\|_{\mathcal{C}} \le c_2\left(\delta_{lm}^{pm}\right)^{-\rho} \ ,$$

where $s_l = \delta_0^{lm}\theta_{f_0}(f_1 - f_0)$. Thus, using (6.7.16), we obtain that

$$\|u_{lm}(t_l) - T^{lm}(g) - s_l\mathbf{u}_{lm}\|_{\mathcal{C}} \le c_3\left(\delta_{lm}^{pm}\right)^{-\rho} \ .$$

Since $t \to u_{lm}(t)$ is $C^2$ as a map $\mathbb{R} \to \mathcal{C}$, we have

$$\begin{aligned}\|u_{lm}(s_l) - T^{lm}(g) - s_l \mathbf{u}_{lm}\|_{\mathcal{C}} &\leq c_4 s_l^2 \\ &= c_4 \left|\delta_0^{lm} \theta_{f_0}(f_1 - f_0)\right|^2 \\ &\leq c_5 \left(\delta_{lm}^{pm}\right)^{-2} \ .\end{aligned}$$

Therefore,

$$\begin{aligned}&\|u_{lm}(t_l) - u_{lm}(s_l)\|_{\mathcal{C}} \\ &\quad \leq \|u_{lm}(t_l) - T^{lm}(g) - s_l \mathbf{u}_{lm}\|_{\mathcal{C}} + \|u_{lm}(s_l) - T^{lm}(g) - s_l \mathbf{u}_{lm}\|_{\mathcal{C}} \\ &\quad \leq c_3 \left(\delta_{lm}^{pm}\right)^{-\rho} + c_5 \left(\delta_{lm}^{pm}\right)^{-2} \\ &\quad \leq c_6 \left(\delta_{lm}^{pm}\right)^{-\rho} \ ,\end{aligned}$$

because $1 < \rho < 2$. Hence, applying Lemma 6.2, we get

$$|t_l - s_l| \leq M^{-1} \|u_{lm}(t_l) - u_{lm}(s_l)\|_{\mathcal{C}} \leq c_7 \left(\delta_{lm}^{pm}\right)^{-\rho} \ .$$

□

LEMMA 6.12. *There exist constants $\tau, \varepsilon, C > 0$ with the following properties: for all $g \in \mathbb{K}$ and all $f_0, f_1 \in V_\varepsilon^0(g)$ we have*

$$|\theta_{f_0}(f_1 - f_0)| \leq C \|f_1 - f_0\|_{\mathcal{B}}^{1+\tau} \ .$$

*Proof.* We shall in fact prove a stronger inequality, with the $\mathcal{C}$-norm replacing the $\mathcal{B}$-norm. Let $\varepsilon > 0$ be so small that lemmas 6.8 to 6.11 are satisfied ($\varepsilon > 0$ will be made even smaller in the course of the argument). Let $p$ be such that $C_5 \delta_0^{-(p+1)m} < \|f_0 - f_1\|_{\mathcal{C}} \leq C_5 \delta_0^{-pm}$ where $C_5 > 0$ is as in Lemma 6.8. As in Lemma 6.4, set $k_0 = k_0(g, f_1)$, $t_j = t_j(g, f_1)$ and $v_j = v_j(g, f_1)$ for all $0 \leq j \leq k_0$. Also, let $l = l(g, p)$ be as in Lemma 6.10. By Lemma 6.11, we have $l \leq k_0$ and so $t_l$ is well-defined. Thus, applying lemmas 6.4, 6.5 and 6.10, we get

$$\begin{aligned}\|u_{lm}(t_l) - u_{lm}(0)\|_{\mathcal{B}} &\leq \left\|u_{lm}(t_l) - T^{lm}(f_1)\right\|_{\mathcal{B}} + \left\|T^{lm}(f_1) - T^{lm}(g)\right\|_{\mathcal{B}} \\ &\leq \varepsilon c_0 2^{-l} \leq \varepsilon c_0 \left(\delta_{lm}^{pm}\right)^{-\rho} \ .\end{aligned}$$

Hence, by Lemma 6.2 we see that $|t_l| \leq c_1 \left(\delta_{lm}^{pm}\right)^{-\rho}$. Let us write $t_j = \alpha_j \left(\delta_{jm}^{pm}\right)^{-1}$ for $l \leq j \leq k_0(g, f_1)$. Recalling that $\delta_{jm}^{(j+1)m} > \beta > 2$ for all $j$ and using Lemma 6.10, we have

$$\left(\delta_{lm}^{pm}\right)^{-1} \leq \beta^{-(1-\mu_1)p} \text{ and } \left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} \leq \beta^{-\tau_2 p} \tag{6.7.17}$$

where $\tau_2 = (1 - \mu_1)(\rho - 1)/2$. Hence, making $\varepsilon > 0$ smaller if necessary (and so $p$ large enough), we get

$$\alpha_l < 4^{-1} \left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} < 4^{-1} \beta^{-\tau_2 p} < \varepsilon_1/2 \ . \tag{6.7.18}$$

By Lemma 6.4, we have $\|v_j\|_{\mathcal{B}} \le \varepsilon 2^{-j}$ and $t_{j+1} = \hat{\delta}_{jm}^{(j+1)m}(t_j + \sigma_{jm}(v_j))$. Since $t \mapsto \hat{\delta}_{jm}^{(j+1)m}(t)$ is $C^2$ as a map $\mathbb{R} \to \mathcal{C}$, we deduce that

$$\begin{aligned}\left|t_{j+1} - \delta_{jm}^{(j+1)m} t_j\right| &\le c_2\left(|t_j|^2 + \|v_j\|_{\mathcal{B}}\right)\\ &\le c_2\left(|t_j|^2 + \varepsilon 2^{-j}\right)\\ &\le c_2\left(|t_j|^2 + \varepsilon 2^{-(j-l)}\left(\delta_{lm}^{pm}\right)^{-\rho}\right) .\end{aligned}$$

Therefore, we get

$$|\alpha_{j+1} - \alpha_j| \le c_2\left(\alpha_j^2 \beta^{-(p-j+1)} + \varepsilon (2\beta)^{-(j-l)}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)}\right) . \tag{6.7.19}$$

Let us prove that $k_0(g, f_1) > p$. To do this, we need to show that $|t_j| < \varepsilon_1$ for all $j \le p$. Let us prove by induction a slightly stonger statement, namely, that $\alpha_j \le 2^{-1}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} < \varepsilon_1$ for all $j = l, \ldots, p$. This is certainly satisfied for $j = l$, as we can see from (6.7.18). Suppose it is satisfied for $\alpha_i$ for all $i = l, \ldots, j$. Using (6.7.17) and (6.7.19), and making $\varepsilon > 0$ even smaller (and thus $p$ large enough), we get

$$\begin{aligned}|\alpha_{j+1} - \alpha_l| &\le \sum_{i=l}^{j} |\alpha_{i+1} - \alpha_i|\\ &\le \frac{1}{4}\left(\sum_{i=l}^{j}\left(c_2 \beta^{-(p-i+1)} + 4c_2\varepsilon(2\beta)^{-(i-l)}\right)\right)\left(\delta_{lm}^{pm}\right)^{-(\rho-1)}\\ &\le \frac{1}{4}\left(\frac{c_2\beta^{-1}}{1-\beta^{-1}} + \frac{4c_2\varepsilon}{1-(2\beta)^{-1}}\right)\beta^{-\tau_2 p}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2}\\ &\le \frac{1}{4}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} .\end{aligned} \tag{6.7.20}$$

Since $\alpha_l \le 4^{-1}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2}$, we deduce that $\alpha_{j+1} \le 2^{-1}\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} < \varepsilon_1$ (in particular $j + 1 \le k_0(g, f_1)$) which ends the induction.

Now set $s_j = s_j(g, f_0, f_1) = \delta_0^{jm}\theta_{f_0}(f_1 - f_0)$ for all $j$. Let us estimate $|t_p - s_p|$. By Lemma 6.11 and the above estimates on $\alpha_j$'s, we have

$$|t_p - s_p| \le |\alpha_p - \alpha_l| + \delta_{lm}^{pm}|t_l - s_l| \le c_3\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2} . \tag{6.7.21}$$

On the other hand, from lemmas 6.4, 6.5 and 6.10, we also know that

$$\begin{aligned}\|u_{pm}(t_p) - u_{pm}(0)\|_{\mathcal{B}} &\le \|u_{pm}(t_p) - T^{pm}(f_1)\|_{\mathcal{B}} + \|T^{pm}(f_1) - T^{pm}(g)\|_{\mathcal{B}}\\ &\le \varepsilon c_4 2^{-p}\end{aligned}$$

Hence, again by Lemma 6.2, we have $|t_p| \le \varepsilon c_5 2^{-p}$. Since $p \ge l$, we deduce from Lemma 6.10 that

$$|t_p| \le \varepsilon c_5 2^{-l} \le \varepsilon c_5\left(\delta_{lm}^{pm}\right)^{-\rho} . \tag{6.7.22}$$

But Lemma 6.10, also tells us that there exists $\tau_1 > 0$ such that $\left(\delta_{lm}^{pm}\right)^{-1} \leq c_6\|f_0 - f_1\|_{\mathcal{C}}^{\tau_1}$. Moreover, $(\delta_0^{pm})^{-1} \leq c_7\|f_0 - f_1\|_{\mathcal{C}}$ by hypothesis. Therefore, combining these facts with (6.7.21) and (6.7.22), we get at last

$$\begin{aligned}|\theta_{f_0}(f_1 - f_0)| &\leq (\delta_0^{pm})^{-1}\,|s_p| \\ &\leq (\delta_0^{pm})^{-1}\,(|t_p| + |t_p - s_p|) \\ &\leq (\delta_0^{pm})^{-1}\left(\varepsilon c_5\left(\delta_{lm}^{pm}\right)^{-\rho} + c_3\left(\delta_{lm}^{pm}\right)^{-(\rho-1)/2}\right) \\ &\leq c_8\|f_0 - f_1\|_{\mathcal{C}}^{1+\tau_1(\rho-1)/2}\,,\end{aligned}$$

which finishes the proof. □

## 6.8. The local stable sets are graphs

We shall prove now that the local stable set of every $g_0 \in \mathbb{K}$ in a sufficiently small neighborhood of $g$ is the graph of a function defined over $\mathrm{Ker}\theta_g \cap \mathcal{B}$ (and taking values on the one-dimensional subspace $\mathbb{R}\mathbf{u}_g \subset \mathcal{B}$). The idea is to show that every "vertical line" of the form $f + \mathbb{R}\mathbf{u}_g$ with $f$ close to $g$ cuts the local stable set $W_\varepsilon^s(g_0)$ exactly at one point. All other points in the same vertical line escape exponentially fast away from $W_\varepsilon^s(g_0)$ under iteration by $T$ and the time $k_0 m$ each such point $f$ takes to escape is logarithmic on the reciprocal of its distance to $W_\varepsilon^s(g_0)$. Moreover, $T^{k_0 m}(f)$ will be exponentially close (in $k_0$) to $W_\varepsilon^u(T^{k_0 m}(g_0))$.

Proposition 6.13. *There exist $0 < \alpha_0, \alpha_1, \alpha_2 < \varepsilon$, $0 < \mu_0 < \mu_1$ and $M_0 > 1$ with the following properties. If $g_0 \in \mathbb{K}$ and $g \in \mathbb{K}$ is such that $\|g - g_0\|_{\mathcal{B}} < \alpha_0$, then for every $v \in \mathrm{Ker}\ \theta_{g_0} \cap \mathcal{B}$ with $\|v\|_{\mathcal{B}} < \alpha_1$, there exists $-\alpha_2/2 < \tau(g,v) < \alpha_2/2$ such that*

*(i) $f_{\tau(g,v)} = g_0 + v + \tau(g,v)\mathbf{u}_{g_0} \in W_\varepsilon^s(g) \subset V_\varepsilon^0(g)$;*

*(ii) $f_t = g_0 + v + t\mathbf{u}_{g_0} \in V_\varepsilon^+(g)$ for all $\tau(g,v) < t < \alpha_2$;*

*(iii) $f_t = g_0 + v + t\mathbf{u}_{g_0} \in V_\varepsilon^-(g)$ for all $-\alpha_2 < t < \tau(g,v)$;*

*(iv) $-\mu_0 \log(|t - \tau(g,v)|) \leq k_0(g, f_t) \leq -\mu_1 \log(|t - \tau(g,v)|)$, where $k_0(g,\phi_t)$ is as in Lemma 6.4.*

*Proof.* Let $\varepsilon > 0$ be sufficiently small such that lemmas 6.8 to 6.11 are satisfied and $0 < \varepsilon' < \varepsilon$ such that Lemma 6.5 is satisfied. Let $M > 0$ be as in Lemma 6.2 and take positive numbers $\alpha_1$ and $\alpha_2$ such that

$$0 < 8\alpha_1 M < \alpha_2 \text{ and } \alpha_1 + 2\alpha_2 M < \varepsilon'/2\,. \qquad (6.8.1)$$

Take $g \in \mathbb{K}$ and $f \in V_\varepsilon(g_0)$ with $\|f - g\|_{\mathcal{B}} < \varepsilon'/2$. Let $v \in \mathrm{Ker}\ \theta_{g_0} \cap \mathcal{B}$ with $\|v\|_{\mathcal{B}} < \alpha_1$, and $t \in \mathbb{R}$ with $2M\|v\|_{\mathcal{B}} < |t| < 2\alpha_2$. By the second inequality in

(6.8.1), we have $\phi_t = f + v + t\mathbf{u}_{g_0} \in \mathcal{B}(g, \varepsilon)$ and $\|\phi_t - g\|_{\mathcal{B}} < \varepsilon'$ for all $|t| < 2\alpha_2$. Now, we have the following claim.

*Claim.* The family $\phi_t$ satisfies the following property

$$\begin{cases} \phi_t \in V_\varepsilon^+(g), & \text{if } \ 2M\|v\|_{\mathcal{B}} < t < 2\alpha_2 \\ \phi_t \in V_\varepsilon^-(g), & \text{if } \ -2\alpha_2 < t < -2M\|v\|_{\mathcal{B}} \ . \end{cases} \tag{6.8.2}$$

To prove this claim, let $C_5 > 0$ be as in Lemma 6.8 and let $p$ be such that $C_5\delta_0^{(p+1)m} < \|\phi_t - f\|_{\mathcal{C}} \leq C_5\delta_0^{pm}$. Set $k_0 = k_0(g, \phi_t)$, $t_j = t_j(g, \phi_t)$ and $v_j = v_j(g, \phi_t)$ for all $0 \leq j \leq k_0$ as in Lemma 6.4. Set $s_j = s_j(g, f, \phi_t) = \delta_0^{jm}\theta_f(\phi_t - f)$. Set $l = l(g, p)$ as in Lemma 6.10. Using Lemma 6.2 and Remark 6.2, there exist $c_0 > 1$ and $\alpha_0 > 0$ sufficiently small such that if $\|g - g_0\|_{\mathcal{B}} < \alpha_0$ then

$$c_0^{-1}|t| \leq |\theta_f(\phi_t - f)| \leq c_0|t| \ , \tag{6.8.3}$$

(noting that $\|f - g\|_{\mathcal{B}} < \varepsilon'/2$ and making $\varepsilon' > 0$ smaller if necessary). Since $t\mathbf{u}_{g_0} = \phi_t - f + v$ and $2M\|v\|_{\mathcal{B}} < |t|$, by Lemma 6.2 there is $c_1 > 1$ such that

$$c_1^{-1}\left(\delta_0^{pm}\right)^{-1} \leq |t| \leq c_1\left(\delta_0^{pm}\right)^{-1} \ . \tag{6.8.4}$$

Hence, by (6.8.3), we obtain that

$$c_2^{-1}\left(\delta_0^{pm}\right)^{-1} \leq |\theta_f(\phi_t - f)| \leq c_2\left(\delta_0^{pm}\right)^{-1} \ .$$

Thus,

$$c_3^{-1}\left(\delta_{lm}^{pm}\right)^{-1} \leq |s_l| \leq c_3\left(\delta_{lm}^{pm}\right)^{-1} \ . \tag{6.8.5}$$

Recall that $\delta_{jm}^{(j+1)m} > \beta > 2$. By Lemma 6.10 we get $\left(\delta_{lm}^{pm}\right)^{-1} \leq \beta^{-(1-\mu_1)pm}$. Let us suppose from now on that $\theta_f(\phi_t - f)$ is positive and so $s_l > 0$. Hence, by Lemma 6.11, by (6.8.5) and making $\alpha_1$ and $\alpha_2$ smaller if necessary (and so $p$ large enough), we obtain that

$$\begin{aligned} t_l &\geq s_l - |t_l - s_l| \geq s_l\left(1 - c_4\left(\delta_{lm}^{pm}\right)^{-(\rho-1)}\right) \\ &\geq s_l\left(1 - c_4\beta^{-(\rho-1)(1-\mu_1)pm}\right) \\ &> s_l/2 > 0 \ . \end{aligned} \tag{6.8.6}$$

Thus, $t_l$ is positive and so it has the same sign as $\theta_f(\phi_t - f)$. By induction on $j = l, \ldots, k_0(g_0, \phi_t)$ let us show that $t_{j+1} \geq t_j$ and so that each $t_j$ is positive as well. By Lemma 6.4, $\|v_j\|_{\mathcal{B}} \leq \varepsilon 2^{-j}$ and $t_{j+1} = \hat{\delta}_{jm}^{(j+1)m}(t_j + \sigma_{jm}(v_j))$. Since $t \mapsto \hat{\delta}_{jm}^{(j+1)m}(t)$ is $C^2$ as a map $\mathbb{R} \to \mathcal{C}$, we obtain that $\left|t_{j+1} - \delta_{jm}^{(j+1)m}t_j\right| \leq c_5\left(|t_j|^2 + \|v_j\|_{\mathcal{B}}\right)$. Thus,

$$\begin{aligned} t_{j+1} &\geq \beta t_j - c_5|t_j|^2 - c_6\varepsilon 2^{-j} \\ &\geq t_j(\beta - c_5|t_j|) - c_6\varepsilon 2^{-j} \ . \end{aligned} \tag{6.8.7}$$

Let $\varepsilon_1 > 0$ as given by Lemma 6.4 and recall that $|t_j| < \varepsilon_1$ and $\varepsilon < \varepsilon_1$. Since $\beta > 1$ and by taking $\varepsilon_1 > 0$ sufficiently small, there is $\tau' > 0$ with the property that $\beta - c_5|t_j| > \beta - c_5\varepsilon_1 > 1 + 2\tau'$. By (6.8.5) and (6.8.6), we get

$$t_j(\beta - c_5|t_j| - 1 - \tau') > t_j\tau' \geq t_l\tau' > s_l\tau'/2 > c_7 \left(\delta_{lm}^{pm}\right)^{-1} . \quad (6.8.8)$$

By Lemma 6.10, we obtain that

$$2^{-j} \leq 2^{-l} \leq \left(\delta_{lm}^{pm}\right)^{-\rho} . \quad (6.8.9)$$

Putting together (6.8.7), (6.8.8) and (6.8.9), we deduce that

$$t_{j+1} \geq (1+\tau')t_j + c_7 \left(\delta_{lm}^{pm}\right)^{-1} - c_6\varepsilon \left(\delta_{lm}^{pm}\right)^{-\rho} . \quad (6.8.10)$$

Making $\alpha_2$ sufficiently small (and so $p$ large enough) and recalling from Lemma 6.10 that $l$ is a fraction of $p$, we obtain that $c_7 \left(\delta_{lm}^{pm}\right)^{-1} - c_6\varepsilon \left(\delta_{lm}^{pm}\right)^{-\rho} \geq 0$. Thus, by (6.8.10), we get

$$t_{j+1} > (1+\tau')t_j \quad (6.8.11)$$

which implies that $t_{j+1}$ has the same sign as $t_j$ and that $\phi_t \in V_\varepsilon^+(g)$. If we suppose that $\theta_{f_0}(\phi_t - f)$ is negative, the proof that $t_l$ is negative and that $t_{j+1} < (1+\tau')t_j$ follows in the same way for all $j = l, \ldots, k_0(g_0, \phi_t)$ and so $\phi_t \in V_\varepsilon^-(g)$. Therefore (6.8.2) is satisfied and the claim is proved.

Let us now prove the assertions of the lemma. Take $f = g_0$ and consider the family $\phi_t = g_0 + v + t\mathbf{u}_{g_0}$. Since $2M\|v\|_{\mathcal{B}} < 2M\alpha_1 < \alpha_2/4$, the claim tell us that

$$\begin{cases} \phi_t \in V_\varepsilon^+(g), & \text{if} \quad \alpha_2/4 \leq t < 2\alpha_2 \\ \phi_t \in V_\varepsilon^-(g), & \text{if} \quad -\alpha_2 < t \leq -\alpha_2/4 . \end{cases}$$

Thus, by Lemma 6.5, there is at least one value $-\alpha_2/4 < \tau(g,v) < \alpha_2/4$ such that $\phi_{\tau(g,v)} = g_0 + v + \tau(g,v)\mathbf{u}_{g_0} \in V_\varepsilon^0(g)$.

Next, take $f = \phi_{\tau(g,v)}$, and define a new family $\psi_t = \phi_{\tau(g,v)} + t\mathbf{u}_{g_0}$. Using the claim again, this time to the family $\psi_t$ (for which $v = 0$), we obtain that

$$\begin{cases} \psi_t \in V_\varepsilon^+(g), & \text{if} \quad 0 < t < \alpha_2/2 \\ \psi_t \in V_\varepsilon^-(g), & \text{if} \quad -\alpha_2/2 < t < 0 . \end{cases}$$

Therefore, $\tau(g,v)$ is the only value of $t \in \mathbb{R}$ between $-2\alpha_2$ and $2\alpha_2$ such that $\phi_t \in V_\varepsilon^0(g)$. Since $\|\phi_{\tau(g,v)} - g\|_{\mathcal{B}} < \varepsilon'$ we deduce from Lemma 6.5 that $\phi_{\tau(g,v)} \in W_\varepsilon^s(g)$. This proves assertions (i), (ii) and (iii).

Let us now prove assertion (iv). Set $k_0 = k_0(g, \psi_t)$. Using (6.8.11) and then (6.8.6), we have

$$|t_{k_0}| \geq (1+\tau')^{k_0-l}|t_l| \geq (1+\tau')^{k_0-l}|s_l|/2 . \quad (6.8.12)$$

Combining (6.8.4) and (6.8.5), we see that

$$|s_l| \geq c_3^{-1}\delta_0^{lm}\left(\delta_0^{pm}\right)^{-1} \geq c_8\beta^l|t| . \quad (6.8.13)$$

Taking $\tau'' = \min\{1+\tau', \beta\}$ and putting (6.8.13) back into 6.8.12 we get

$$|t_{k_0}| \geq c_9(\tau'')^{k_0}|t| \ . \tag{6.8.14}$$

By Lemma 6.4, there are $0 < \varepsilon_1 < \varepsilon_0$ such that $\varepsilon_1 \leq |t_{k_0}| \leq \varepsilon_0$. Thus, by (6.8.14), there is $\mu_0 > 0$ such that $k_0 \geq -\mu_0 \log(t)$. By Lemma 6.4, $|t_{k_0}| \leq c_9\beta^{k_0}|t|$ and so there is $\mu_1 \geq \mu_0$ such that $k_0 \leq -\mu_1 \log(t)$, which proves assertion (iv). □

## 6.9. Proof of the local stable manifold theorem

It will follow from Theorem 6.15 in this section that for every $g \in \mathbb{K}$ the local stable manifold at $g$ is a $C^1$ submanifold varying continuously with $g$. In the proof of this theorem will use the following basic fact of calculus.

LEMMA 6.14. *Let $X, Y$ be Banach spaces, let $x_0 \in X$ and consider a map $\xi : B_X(x_0, \varepsilon) \to Y$ whose image in $Y$ falls within $B_Y(\xi(x_0), \varepsilon)$. Suppose we have a bounded linear operator $L : X \to Y$ such that for all $v \in X$ with $\|v\|_X \leq \varepsilon$ we have*

$$\|\xi(x_0+v) - \xi(x_0) - L(v)\|_Y \leq c_0 \left(\|v\|_X + \|\xi(x_0+v) - \xi(x_0)\|_Y\right)^{1+\tau} \tag{6.9.1}$$

*where $c_0 > 0$ and $\tau > 0$. If $c_0(2\varepsilon)^\tau < 1$ then $\xi$ is differentiable at $x_0$ and $D\xi(x_0) = L$.*

*Proof.* Say $\|L(v)\|_Y \leq a\|v\|_X$ for some $a > 0$. Noting that $\|v\|_X + \|\xi(x_0 + v) - \xi(x_0)\|_Y < 2\varepsilon$, we have from (6.9.1) that

$$\|\xi(x_0+v) - \xi(x_0)\|_Y \leq (a + c_0(2\varepsilon)^\tau)\,\|v\|_X + c_0(2\varepsilon)^\tau \|\xi(x_0+v) - \xi(x_0)\|_Y$$

whence

$$\|\xi(x_0+v) - \xi(x_0)\|_Y \leq \frac{a + c_0(2\varepsilon)^\tau}{1 - c_0(2\varepsilon)^\tau}\|v\|_X = c_1\|v\|_X \ .$$

Putting this back into the right-hand side of (6.9.1) we get

$$\|\xi(x_0+v) - \xi(x_0) - L(v)\|_Y \leq c_2 \left(\|v\|_X\right)^{1+\tau}$$

and therefore $D\xi(x_0)$ exists and equals $L$. □

For every $g \in \mathbb{K}$ and $\alpha_1 > 0$, let us consider the following sets

$$\begin{aligned} E_{g,\alpha_1} &= \{v \in \ker\theta_g : \|v\|_{\mathcal{B}} < \alpha_1\} \ , \\ F_g &= \{g + t\mathbf{u}_g : t \in \mathbb{R}\} \ , \\ G_{g,\alpha_1} &= \{g + v + t\mathbf{u}_g : v \in E_{g,\alpha_1} \quad \text{and} \quad t \in \mathbb{R}\} \ . \end{aligned}$$

THEOREM 6.15. *Set $0 < \alpha_0 < \alpha_1 < \varepsilon$ and $\tau(g,v)$ as in Proposition 6.13. For every $g_0 \in \mathbb{K}$ and every $g \in \mathbb{K}$ with $\|g - g_0\|_{\mathcal{B}} < \alpha_0$, the map $\xi_g : E_{g_0,\alpha_1} \to F_{g_0}$ given by $\xi_g(v) = g_0 + \tau(g,v)\mathbf{u}_{g_0}$ is well-defined and has the following properties:*

*(i) The graph of* $\xi_g$ *is equal to* $W^s_\varepsilon(g) \cap G_{g_0,\alpha_1}$*;*

*(ii)* $\xi_g$ *is* $C^1$ *and varies continuously with* $g$.

*Proof.* Set $\alpha_1 < \alpha_2 < \varepsilon$ and $0 < \mu_0 < \mu_1$ as in Proposition 6.13. By Proposition 6.13, the map $\xi_g : E_{g_0,\alpha_1} \to F_{g_0}$ is well-defined and assertion (i) is satisfied. Let $\hat{\xi}_g : E_{g_0,\alpha_1} \to \mathbb{R}$ be given by $\hat{\xi}_g(v) = \tau(g, v)$ where $\tau(g, v)$ is given by (i) in Proposition 6.13. To prove assertion (ii), it is enough to show that $\hat{\xi}_g : E_{g_0,\alpha_1} \to \mathbb{R}$ is $C^1$ and varies continuously with $g$. Let $v_1, v_2 \in E_{g_0,\alpha_1}$ and set

$$f_1 = g_0 + v_1 + \hat{\xi}_g(v_1)\mathbf{u}_{g_0} \tag{6.9.2}$$

and $f_2 = g_0 + v_2 + \hat{\xi}_g(v_2)\mathbf{u}_{g_0}$. By Lemma 6.12, we get

$$\begin{aligned}&\left|\theta_{f_1}(v_2 - v_1) + \theta_{f_1}(\mathbf{u}_{g_0})(\hat{\xi}_g(v_2) - \hat{\xi}_g(v_1))\right| = |\theta_{f_1}(f_2 - f_1)| \\ &\leq c_0 \left(\|v_2 - v_1\|_{\mathcal{B}} + |\hat{\xi}_g(v_2) - \hat{\xi}_g(v_1)|\right)^{1+\tau}\end{aligned}$$

By Lemma 6.7, and taking $\varepsilon > 0$ sufficiently small, we have that $\theta_{f_1}(\mathbf{u}_{g_0})$ is uniformly bounded away from 0. Therefore, by Lemma 6.14, we deduce that $\hat{\xi}_g$ is differentiable at every $v_1$ with derivative given by

$$D\hat{\xi}_g(v_1) = -(\theta_{f_1}(\mathbf{u}_{g_0}))^{-1}\theta_{f_1} \ . \tag{6.9.3}$$

From Lemma 6.7, $\theta_{f_1}$ varies continuously with $f_1$ and so $D\hat{\xi}_g(v_1)$ also varies continuously in a neighborhood of $v_1$. Hence, $\hat{\xi}_g$ is a $C^1$ map.

Let us check that $\hat{\xi}_g$ varies continuously with $g$ in the $C^1$ sense; more precisely, that the map $\mathbb{K} \cap \mathcal{B}(g_0, \alpha_0) \to C^1(E_{g_0,\alpha_1}, \mathbb{R})$ given by $g \mapsto \hat{\xi}_g$ is continuous. Taking into account that $D\hat{\xi}_g$ is given by (6.9.3) and that $f_1$ is given by (6.9.2), and since by Lemma 6.7 the map $f_1 \mapsto \theta_{f_1}$ is uniformly continuous (as a map into $\mathcal{B}^*$), it suffices to prove that $g \mapsto \hat{\xi}_g$ is continuous as a map into $C^0(E_{g_0,\alpha_1}, \mathbb{R})$.

To do this, let $v \in E_{g_0,\alpha_1}$ be such that $g = g_0 + v + \hat{\xi}_g(v)\mathbf{u}_{g_0}$, take $g_1 \in \mathbb{K}$ with $\|g_1 - g_0\|_{\mathcal{B}} < \alpha_0$ and let $w \in E_{g_0,\alpha_1}$ be such that $g_1 = g_0 + w + \hat{\xi}_{g_1}(w)\mathbf{u}_{g_0}$. Now, we have the following claim.

*Claim.* There exist $c_1 > 0$ and $0 < \gamma < 1$ such that

$$c_1^{-1}|\hat{\xi}_{g_1}(w) - \hat{\xi}_g(w)|^{1/\gamma} \leq |\hat{\xi}_{g_1}(z) - \hat{\xi}_g(z)| \leq c_1|\hat{\xi}_{g_1}(w) - \hat{\xi}_g(w)|^{\gamma} \ , \tag{6.9.4}$$

for all $z \in E_{g_0,\alpha_1}$.

Let us assume this claim for a moment. Its geometric meaning is that the distances between corresponding points of the graphs of $\hat{\xi}_g$ and $\hat{\xi}_{g_1}$ along the vertical fibers $(\{z\} \times F_{g_0})$ are uniform. We want to control such distances in terms of $\|g_1 - g\|_{\mathcal{B}}$. The above claim tell us that it is enough to control

$|\hat{\xi}_{g_1}(w) - \hat{\xi}_g(w)|$. Hence, write

$$\begin{aligned} g - g_1 &= v - w + \left(\hat{\xi}_g(v) - \hat{\xi}_{g_1}(w)\right)\mathbf{u}_{g_0} \\ &= v - w + (a+b)\,\mathbf{u}_{g_0} \end{aligned}$$

where $a = \hat{\xi}_g(w) - \hat{\xi}_{g_1}(w)$ and $b = \hat{\xi}_g(v) - \hat{\xi}_g(w)$. Since $\hat{\xi}_g$ is $C^1$, we have $|b| \le c_2\|v-w\|_{\mathcal{B}}$. On the other hand, since $\mathcal{B} = \operatorname{Ker}\theta_{g_0} \bigoplus \mathbb{R}\mathbf{u}_{g_0}$ is a splitting into closed subspaces, there exists a constant $c_3 > 0$ such that

$$\max\{\|v-w\|_{\mathcal{B}}, |a+b|\} \le c_3\|g-g_1\|_{\mathcal{B}} \ .$$

But then

$$\begin{aligned} |a| &\le \|g-g_1\|_{\mathcal{B}} + \|v-w\|_{\mathcal{B}} + |b| \\ &\le (1 + c_3 + c_2c_3)\|g-g_1\|_{\mathcal{B}} \ . \end{aligned}$$

Hence $\left|\hat{\xi}_g(w) - \hat{\xi}_{g_1}(w)\right| \le c_4\|g-g_1\|_{\mathcal{B}}$, and given the claim this proves that $g \mapsto \hat{\xi}_g$ is indeed continuous.

Finally, let us prove the claim. For each $z \in E_{g_0,\alpha_1}$, let $h = g_0 + z + \hat{\xi}_{g_1}(z)\mathbf{u}_{g_0}$. Set

$$\begin{aligned} t'_k &= t_k(g, g_1), \qquad & t''_k &= t_k(g,h), \\ u'_k &= u_{T^{km}(g)}(t'_k), \quad & u''_k &= u_{T^{km}(g)}(t''_k) \ , \end{aligned}$$

as given by Lemma 6.4, and set (also as in that lemma)

$$\begin{aligned} k'_0 &= k_0(g,g_1) = \min\{j : |t'_j| \ge \varepsilon_1\} \\ k''_0 &= k_0(g,h) = \min\{j : |t''_j| \ge \varepsilon_1\} \ . \end{aligned} \tag{6.9.5}$$

Applying Lemma 6.4, we obtain, for all $k \le \min\{k'_0, k''_0\}$, the estimates

$$\begin{aligned} \|T^{km}(g_1) - u'_k\|_{\mathcal{B}} &\le 2^{-k}\|g_1 - g\|_{\mathcal{B}} \\ \|T^{km}(h) - u''_k\|_{\mathcal{B}} &\le 2^{-k}\|h - g\|_{\mathcal{B}} \ . \end{aligned} \tag{6.9.6}$$

Since $h \in W^s_\varepsilon(g_1)$, we also have, by Lemma 6.5,

$$\|T^{km}(h) - T^{km}(g_1)\|_{\mathcal{B}} \le \varepsilon c_5 2^{-k} \ . \tag{6.9.7}$$

Combining (6.9.6) and (6.9.7), we get

$$\|u'_k - u''_k\|_{\mathcal{B}} \le c_6 2^{-k} \ .$$

Hence, by Lemma 6.2, we get

$$|t_k(g,g_1) - t_k(g,h)| \le c_7 2^{-k} \ , \tag{6.9.8}$$

for all $k \le \min\{k'_0, k''_0\}$. Using (6.8.11) together with (6.9.8), we deduce that there exists a uniform constant $c_8 > 0$ such that

$$k'_0 - c_8 \le k''_0 \le k'_0 + c_8 \ . \tag{6.9.9}$$

On the other hand, applying (iv) in Proposition 6.13, we also have

$$
\begin{aligned}
-\mu_0 \log(|\tau(g_1,w)-\tau(g,w)|) &\le k_0' \le -\mu_1 \log(|\tau(g_1,w)-\tau(g,w)|) \\
-\mu_0 \log(|\tau(g_1,z)-\tau(g,z)|) &\le k_0'' \le -\mu_1 \log(|\tau(g_1,z)-\tau(g,z)|) \ .
\end{aligned}
\tag{6.9.10}
$$

Combining (6.9.9) and (6.9.10) and noting that

$$\tau(g_1,w)=\hat{\xi}_{g_1}(w)\ ,\ \tau(g_1,z)=\hat{\xi}_{g_1}(z)\ ,\tau(g,w)=\hat{\xi}_g(w) \text{ and } \tau(g,z)=\hat{\xi}_g(z)\ ,$$

we get at last

$$c_9^{-1}\left|\hat{\xi}_{g_1}(w)-\hat{\xi}_g(w)\right|^{\mu_0/\mu_1} \le \left|\hat{\xi}_{g_1}(z)-\hat{\xi}_g(z)\right| \le c_9\left|\hat{\xi}_{g_1}(w)-\hat{\xi}_g(w)\right|^{\mu_1/\mu_0} \ .$$

for some $c_9>1$, and this proves the claim with $\gamma=\mu_0/\mu_1<1$ and $c_1=c_9$. □

Remark 6.3. *Note that by Proposition 6.13 there exists a uniform* $0<\tilde{\varepsilon}<\varepsilon$ *such that* $W^s_{\tilde{\varepsilon}}(g)\subset G_{g_0,\alpha_1}$ *for all* $g_0\in\mathbb{K}$ *with* $\|g_0-g\|_{\mathcal{B}}<\alpha_0$.

## 7. Smooth holonomies

In the previous section we proved that a robust operator $T$ has $C^1$ local stable manifolds through each point of its hyperbolic basic set $\mathbb{K}$, and that such manifolds form a $C^0$ lamination (near each point of $\mathbb{K}$). A natural question that may be asked at this point is this: how smooth is the holonomy of this lamination? To answer this question we shall assume that there exists a homeomorphism $H:\Theta^{\mathbb{Z}}\to\mathbb{K}$ of a finite-type shift space onto $\mathbb{K}$ which conjugates the two-sided full shift $\sigma:\Theta^{\mathbb{Z}}\to\mathbb{K}$ to our robust operator $T$ restricted to $\mathbb{K}$. Under this topological assumption, and an additional geometric assumption concerning the unstable manifolds of points in the attractor –both of which are satisfied by the renormalization operator– we shall prove below that the holonomies of the local stable laminations are $C^{1+\theta}$ for some $\theta>0$.

### 7.1. Smooth holonomies for robust operators

Let $\mathbb{K}\subset\mathcal{O}_A$ be a hyperbolic basic set of a $C^2$ operator $T:\mathcal{O}_A\to\mathcal{A}$ which is topologically conjugated to a two-sided shift of finite type. For $\varepsilon_0>0$ small enough and for every $g\in\mathbb{K}$ let $t\to u_g(t)$ be a parametrization of the local unstable manifold $W^u_{\varepsilon_0}(g)$. Set

$$\mathbb{K}_g=\mathbb{K}\cap\overline{W^u_{\varepsilon_0}(g)} \text{ and } K_g=u_g^{-1}(\mathbb{K}_g)\ .$$

Let

$$\Sigma_{\dots,\theta_{k-1},\theta_k}=\left\{\left(\theta_j'\right)\in\Theta^{\mathbb{Z}}:\theta_j'=\theta_j \quad \text{for} \quad \text{all} j\le k\right\}\ .$$

If $H(\Sigma_{\dots,\theta_k}) \cap \mathbb{K}_g \neq \emptyset$ then denote by $\Delta_{\dots,\theta_k}(g)$ the smallest interval in $\mathbb{R}$ such that $u_g(\Delta_{\dots,\theta_k}(g)) \supset H(\Sigma_{\dots,\theta_k}) \cap \mathbb{K}_g$. Let $\mathbf{C}_k(g)$ be the set of all these intervals $\Delta_{\dots,\theta_k}(g)$.

Definition 7.1. *We say that the local unstable manifolds $W^u_{\varepsilon_0}(g)$ have geometry bounded by $\alpha > 0$ if for every $g \in \mathbb{K}$, $K_g$ has geometry bounded by $\alpha > 0$ with respect to the collection $(\mathbf{C}_k(g))_{k\geq 0}$ (in the sense of §3).*

Now suppose in addition that the operator $T$ is robust with respect to the Banach spaces $(\mathcal{B}, \mathcal{C}, \mathcal{D})$. By Theorem 6.1, the local stable manifolds of $T$ in $\mathcal{B}$ form a $C^0$ lamination Let $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ be a $C^2$ curve transversal to the stable lamination. Let $K_F$ be the set of all values $r \in [-\mu_0, \mu_0]$ such that

$$f_r = F(r) \in \bigcup_{g_0 \in \mathbb{K} \cap W^u_{\varepsilon_0}(g_0)} W^s_{\varepsilon_0}(g_0) \ .$$

The *holonomy map* $\phi_F : F(K_F) \to W^u_{\varepsilon_0}(g)$ associates to each $f_r$ the point $\phi_F(f_r)$ such that $f_r \in W^s_{\varepsilon_0}(\phi_F(f_r))$. In local coordinates, the holonomy map $\phi_F$ is given by $\psi_F : K_F \to K_g$ where $\psi_F(t) = u_g \circ \phi_F \circ F^{-1}$ and $K_F, K_g \subset \mathbb{R}$. The $C^2$ curve $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ is an *ordered transversal to the stable foliation* if $F$ is transversal to the stable foliation, $\phi_F : F(K_F) \to W^u_{\varepsilon_0}(g)$ extends to $F([-\mu_0, \mu_0])$ as an homeomorphism $\hat{\phi}_F$ over its image such that $\phi_F(F(K_F)) = \hat{\phi}_F(F(K_F)) \cap \mathbb{K}$.

We note that, by Remark 6.2 and by Theorem 6.15, there is $\varepsilon_1 < \varepsilon_0$ small enough such that a $C^2$ transversal to $W^s_{\varepsilon_1}(g)$ in a point $f$ is an ordered transversal to the stable foliation in a small neighborhood of $f$.

Theorem 7.1. *Let $\mathbb{K} \subset \mathcal{O}_A$ be a hyperbolic basic set of a $C^2$ operator $T : \mathcal{O}_A \to \mathcal{A}$ which is robust with respect to $(\mathcal{B}, \mathcal{C}, \mathcal{D})$. Suppose that there exits $\varepsilon_0 > 0$ such that the local unstable manifolds $W^u_{\varepsilon_0}(g)$ of $g \in \mathbb{K}$ have bounded geometry. There exists $0 < \varepsilon < \varepsilon_0$ with the property that for every $C^2$ ordered transversal $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ to the stable foliation in $\mathcal{B}$, the holonomy $\phi_F : F(K_F) \to W^u_{\varepsilon_0}(g)$ has a $C^{1+\theta}$ diffeomorphic extension to $F([-\mu_0, \mu_0])$ for some $\theta > 0$.*

Example 7.1. *As we know from Theorem 2.4, the renormalization operator $T = R^N : \mathbb{O}_{\mathbb{A}} \to \mathbb{A}$ is hyperbolic over $\mathbb{K}$. As we shall see in Theorem 8.1, $T$ is robust with respect to $(\mathbb{A}^r, \mathbb{A}^s, \mathbb{A}^0)$ provided $s > s_0$ with $s_0$ sufficiently close to 2 and $r > s+1$ is not an integer. By Theorem 2.1, there is a two-sided full shift topologically conjugated to $T|\mathbb{K}$. By lemmas 9.3 and 9.6 respectively in pages 403 and 405 of Lyubich's paper [20], there is $\alpha > 0$ such that the local unstable manifolds $W^u_{\varepsilon_0}(g)$ have geometry bounded by $\alpha$. Hence the renormalization operator $T$ satisfies the hypotheses of Theorem 7.1.*

In what follows the notation $A = \mathcal{O}(B)$ means that $\mu_1^{-1}B \le A \le \mu_1 B$ and the notation $A = B(1 \pm \mathcal{O}(C))$ means that $B(1-\mu_2 C) \le A \le B(1+\mu_2 C)$ for some constants $\mu_1 > 1$ and $\mu_2 > 0$.

The proof of Theorem 7.1 will be a consequence of the following lemmas.

LEMMA 7.2. *For every $C^2$ curve $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ transversal to the stable foliation and for all $r, t \in [-\mu_0, \mu_0]$ such that $r < t$, we have*

$$\begin{aligned} \|f_t - f_r\|_{\mathcal{X}} &= \mathcal{O}(|t-r|) \\ |\theta_{f_r}(f_t - f_r)| &= \mathcal{O}(|t-r|) \ , \end{aligned} \tag{7.1.1}$$

*and for all $s, r, t \in [-\mu_0, \mu_0]$ such that $s < r < t$,*

$$\begin{aligned} \frac{\|f_t - f_r\|_{\mathcal{X}}}{\|f_s - f_r\|_{\mathcal{X}}} &= \frac{|t-r|}{|s-r|}(1 \pm \mathcal{O}(|t-s|)) \\ \frac{\|\theta_{f_r}(f_t - f_r)\|_{\mathcal{X}}}{\|\theta_{f_r}(f_s - f_r)\|_{\mathcal{X}}} &= \frac{|t-r|}{|s-r|}(1 \pm \mathcal{O}(|t-s|)) \ , \end{aligned} \tag{7.1.2}$$

*where $\mathcal{X} \in \{B, C, D\}$.*

*Proof.* By Lemma 6.2, there are $\nu_1, \nu_2 > 0$ such that for all $r \in K_f$, $\|\mathbf{u}_r\|_{\mathcal{X}} > \nu_1$ and $|\theta_{f_r}(\mathbf{u}_r)| > \nu_2$. Since $F$ is $C^2$, we have

$$\begin{aligned} f_t - f_r &= (t-r)\mathbf{u}_r \pm \mathcal{O}(|t-r|^2) \\ \theta_{f_r}(f_t - f_r) &= (t-r)\theta_{f_r}(\mathbf{u}_r) \pm \mathcal{O}(|t-r|^2) \ , \end{aligned}$$

and so (7.1.1) follows. Taking $s < r < t$, we get

$$\begin{aligned} \frac{\|f_t - f_r\|_{\mathcal{X}}}{\|f_s - f_r\|_{\mathcal{X}}} &= \frac{\|\mathbf{u}_r\|_{\mathcal{X}}\,|t-r|(1 \pm \mathcal{O}(|t-r|))}{\|\mathbf{u}_r\|_{\mathcal{X}}\,|s-r|(1 \pm \mathcal{O}(|s-r|))} \\ &= \frac{|t-r|}{|s-r|}(1 \pm \mathcal{O}(|t-s|)) \ . \end{aligned}$$

The second estimate in (7.1.2) is obtained in similar fashion. □

In what follows, it will be more convenient to denote $\phi_F(f_r)$ by $g_{\psi_F(r)}$. We will also work with a fixed $0 < \varepsilon < \varepsilon_0$ for which Lemma 6.8 holds.

LEMMA 7.3. *Set $l = l(g_{\psi_F(r)}, p)$ as in Lemma 6.10. Let $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ be a $C^2$ curve transversal to the stable foliation. For all $p > 0$ sufficiently large and all $s, r, t \in K_F$ such that*

$$|t-r| = \mathcal{O}\left((\delta_0^{pm})^{-1}\right) \quad \text{and} \quad |s-r| = \mathcal{O}\left((\delta_0^{pm})^{-1}\right)$$

*we have*

$$\begin{aligned}
\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}} &= \mathcal{O}\left((\delta^{pm}_{lm})^{-1}\right) \\
\|T^{lm}(f_s) - T^{lm}(f_r)\|_{\mathcal{C}} &= \mathcal{O}\left((\delta^{pm}_{lm})^{-1}\right) \\
\frac{\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}}}{\|T^{lm}(f_s) - T^{lm}(f_r)\|_{\mathcal{C}}} &= \frac{|t-r|}{|s-r|}\left(1 \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-(\rho-1)}\right)\right) \ . \qquad (7.1.3)
\end{aligned}$$

*Proof.* Using Lemma 6.2 and (7.1.1), we get

$$|\theta_{f_r}(f_t - f_r)| = \mathcal{O}\left((\delta^{pm}_0)^{-1}\right) \quad \text{and} \quad |\theta_{f_r}(f_s - f_r)| = \mathcal{O}\left((\delta^{pm}_0)^{-1}\right) \ .$$

Thus, taking $p$ sufficiently large and using lemmas 6.9 and 6.10 we deduce that

$$\begin{aligned}
\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}} &= \left|\delta^{lm}_0 \theta_{f_r}(f_t - f_r)\right| \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-\rho}\right) \\
&= \mathcal{O}\left((\delta^{pm}_{lm})^{-1}\right) \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-\rho}\right) \\
&= \mathcal{O}\left((\delta^{pm}_{lm})^{-1}\right) \ . \qquad (7.1.4)
\end{aligned}$$

Similarly, $\left\|T^{lm}(f_s) - T^{lm}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left((\delta^{pm}_{lm})^{-1}\right)$. This proves the first two inequalities in (7.1.3). Combining (7.1.1) with (7.1.4), we see that

$$\begin{aligned}
\frac{\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}}}{\|T^{lm}(f_s) - T^{lm}(f_r)\|_{\mathcal{C}}} &= \frac{\left|\delta^{lm}_0 \theta_{f_r}(f_t - f_r)\right| \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-\rho}\right)}{\left|\delta^{lm}_0 \theta_{f_r}(f_s - f_r)\right| \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-\rho}\right)} \\
&= \frac{|\theta_{f_r}(f_t - f_r)|\left(1 \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-(\rho-1)}\right)\right)}{|\theta_{f_r}(f_s - f_r)|\left(1 \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-(\rho-1)}\right)\right)} \ .
\end{aligned}$$

Therefore, by Lemma 7.2, we get

$$\frac{\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}}}{\|T^{lm}(f_s) - T^{lm}(f_r)\|_{\mathcal{C}}} = \frac{|t-r|}{|s-r|}\left(1 \pm \mathcal{O}\left((\delta^{pm}_{lm})^{-(\rho-1)}\right)\right) \ ,$$

and this proves the last inequality in (7.1.3). □

LEMMA 7.4. *Set $l = l\left(g_{\psi_F(r)}, p\right)$ as in Lemma 6.10. Let $F : [-\mu_0, \mu_0] \to \mathcal{B}(g, \varepsilon)$ be a $C^2$ curve transversal to the stable foliation. For every $s \in K_F$ and $s' = \psi_F(s) \in K_g$, we have*

$$\left\|T^{lm}(f_s) - T^{lm}(g_{s'})\right\|_{\mathcal{C}} \le \mathcal{O}\left((\delta^{pm}_{lm})^{-\rho}\right) \ .$$

*Furthermore, for all $p$ large enough and all $s', r', t' \in K_g$ such that $s = \psi_F^{-1}(s'), r = \psi_F^{-1}(r'), t = \psi_F^{-1}(t') \in K_f$,*

$$|t' - r'| = \mathcal{O}\left((\delta^{pm}_0)^{-1}\right) \quad \textit{and} \quad |s' - r'| = \mathcal{O}\left((\delta^{pm}_0)^{-1}\right) \ ,$$

*we have*

$$\frac{\left\|T^{lm}(f_t)-T^{lm}(f_r)\right\|_{\mathcal{C}}}{\left\|T^{lm}(f_s)-T^{lm}(f_r)\right\|_{\mathcal{C}}}=\frac{\left\|T^{lm}(g_{t'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}{\left\|T^{lm}(g_{s'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right) .$$

*Proof.* By lemmas 6.5 and 6.10, we get

$$\left\|T^{lm}(f_s)-T^{lm}(g_{s'})\right\|_{\mathcal{C}}\le C_3\varepsilon 2^{-l}\le\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-\rho}\right) .$$

Thus, applying Lemma 7.3 to the transversal given by the local unstable manifold $\{g_t\}$ we get

$$\begin{aligned}\frac{\left\|T^{lm}(f_t)-T^{lm}(f_r)\right\|_{\mathcal{C}}}{\left\|T^{lm}(f_s)-T^{lm}(f_r)\right\|_{\mathcal{C}}}&=\frac{\left\|T^{lm}(g_{t'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right)}{\left\|T^{lm}(g_{s'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right)}\\&=\frac{\left\|T^{lm}(g_{t'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}{\left\|T^{lm}(g_{s'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right) .\end{aligned}$$

□

*Proof of Theorem 7.1.* Let $p>0$ be so large such that lemmas 7.3 and 7.4 are satisfied and let $t,s,r,t',s',r'$ be as in Lemma 7.4. First, a claim.

*Claim.* We have

$$|t-r|=\mathcal{O}\left(\left(\delta^{pm}_{0}\right)^{-1}\right)\quad\text{and}\quad|s-r|=\mathcal{O}\left(\left(\delta^{pm}_{0}\right)^{-1}\right) .$$

Assuming this claim for a moment, we finish the proof of Theorem 7.1 as follows. Set $l=l(g_{r'},p)$ as in Lemma 6.10. By lemmas 6.10 and 7.2, there is $0<\tau_1<1$ such that $\delta^{pm}_{lm}\le\mathcal{O}\left(|t'-r'|^{\tau_1}\right)$. Therefore, by lemmas 7.3 and 7.4 we deduce that

$$\begin{aligned}\frac{|t-r|}{|s-r|}&=\frac{\left\|T^{lm}(f_t)-T^{lm}(f_r)\right\|_{\mathcal{C}}}{\left\|T^{lm}(f_s)-T^{lm}(f_r)\right\|_{\mathcal{C}}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right)\\&=\frac{\left\|T^{lm}(g_{t'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}{\left\|T^{lm}(g_{s'})-T^{lm}(g_{r'})\right\|_{\mathcal{C}}}\left(1\pm\mathcal{O}\left(\left(\delta^{pm}_{lm}\right)^{-(\rho-1)}\right)\right)\\&=\frac{|t'-r'|}{|s'-r'|}\left(1\pm\mathcal{O}\left(|t'-r'|^{-(\rho-1)\tau_1}\right)\right)\end{aligned}\tag{7.1.5}$$

Since $\mathcal{K}_g$ has bounded geometry and using Theorem 9.5 in page 549 of [26], the inequalities (7.1.5) imply that the map $\psi_F$ has a $C^{1+\theta}$ diffeomorphic extension to $\mathbb{R}$ for some $0<\theta<1$.

Let us now prove the claim. Let $\hat{p}$ be such that $|t-r|=\mathcal{O}\left(\left(\delta^{\hat{p}m}_{0}\right)^{-1}\right)$. All that we have to show is that

$$|\hat{p}-p|\le\mathcal{O}(1)\tag{7.1.6}$$

Set $\hat{l} = \hat{l}(g_{r'}, \hat{p})$ as in Lemma 6.10. By lemmas 6.9 and 6.11, and the estimates in (7.1.1), for $k \leq \min\{l, \hat{l}\}$ we see that

$$\left\|T^{km}(g_{t'}) - T^{km}(g_{r'})\right\|_{\mathcal{C}} = \left|\delta_0^{km}\theta_{g_r}(g_{t'} - g_{r'})\right| \pm \mathcal{O}\left((\delta_{km}^{pm})^{-\rho} + 2^{-k}\right)$$
$$\left\|T^{km}(f_t) - T^{km}(f_r)\right\|_{\mathcal{C}} = \left|\delta_0^{km}\theta_{f_r}(f_t - f_r)\right| \pm \mathcal{O}\left(\left(\delta_{km}^{\hat{p}m}\right)^{-\rho} + 2^{-k}\right) \quad (7.1.7)$$

By Lemma 6.5, we get (for all $k \leq \min\left\{l, \hat{l}\right\}$)

$$\left\|T^{km}(f_t) - T^{km}(f_r)\right\|_{\mathcal{C}} = \left\|T^{km}(g_{t'}) - T^{km}(g_{r'})\right\|_{\mathcal{C}} \pm \mathcal{O}\left(2^{-k}\right) \ . \quad (7.1.8)$$

Let us consider separately the case (i) where $p \leq \hat{p}$ and the case (ii) where $p \geq \hat{p}$.

*Case (i).* Here $l \leq \hat{l}$ and by Lemma 6.10 we get

$$\left(\delta_{lm}^{\hat{p}m}\right)^{-1} \leq (\delta_{lm}^{pm})^{-1}\, 2^{-l} \leq \mathcal{O}\left((\delta_{lm}^{pm})^{-\rho}\right) \ . \quad (7.1.9)$$

By Lemma 7.3 applied to the transversal given by the local unstable manifold $\{g_t\}$, we have

$$\left\|T^{lm}(g_{t'}) - T^{lm}(g_{r'})\right\|_{\mathcal{C}} = \mathcal{O}\left((\delta_{lm}^{pm})^{-1}\right) \ . \quad (7.1.10)$$

On the other hand, by (7.1.8), we have

$$\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}} = \left\|T^{lm}(g_{t'}) - T^{lm}(g_{r'})\right\|_{\mathcal{C}} \pm \mathcal{O}\left(2^{-l}\right) \ .$$

But $2^{-l}$ is much smaller than $\mathcal{O}\left((\delta_{lm}^{pm})^{-1}\right)$. Hence by (7.1.10) we get

$$\left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left((\delta_{lm}^{pm})^{-1}\right) \ . \quad (7.1.11)$$

Thus, by (7.1.7) and (7.1.9), we deduce that

$$\left|\delta_0^{lm}\theta_{f_r}(f_t - f_r)\right| = \left\|T^{lm}(f_t) - T^{lm}(f_r)\right\|_{\mathcal{C}} \pm \mathcal{O}\left(\left(\delta_{lm}^{\hat{p}m}\right)^{-\rho}\right) \pm \mathcal{O}\left(2^{-l}\right)$$
$$= \mathcal{O}\left((\delta_{lm}^{pm})^{-1}\right) \pm \mathcal{O}\left(\left(\delta_{lm}^{\hat{p}m}\right)^{-\rho}\right) \ .$$

Since $(\delta_{lm}^{pm})^{-1}$ is much larger than $\left(\delta_{lm}^{\hat{p}m}\right)^{-\rho}$, it follows that

$$\left|\delta_0^{lm}\theta_{f_r}(f_t - f_r)\right| = \mathcal{O}\left((\delta_{lm}^{pm})^{-1}\right) \ .$$

This shows that

$$|\theta_{f_r}(f_t - f_r)| = \mathcal{O}\left((\delta_0^{pm})^{-1}\right) \ .$$

Therefore, by Lemma 7.2, we get $\|f_t - f_r\|_{\mathcal{C}} = \mathcal{O}\left((\delta_0^{pm})^{-1}\right)$, which in turn implies (7.1.6).

*Case (ii).* Here $\hat{l} \leq l$, and Lemma 6.10 tells us that

$$\left(\delta^{pm}_{\hat{l}m}\right)^{-1} \leq \left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-1} 2^{-\hat{l}} \leq \mathcal{O}\left(\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-\rho}\right) . \tag{7.1.12}$$

Applying Lemma 7.3, we get

$$\left\|T^{\hat{l}m}(f_t) - T^{\hat{l}m}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left(\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-1}\right) . \tag{7.1.13}$$

On the other hand, by (7.1.12) and (7.1.7), we have

$$\left\|T^{\hat{l}m}(f_t) - T^{\hat{l}m}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left(\left|\delta^{\hat{l}m}_{0}\theta_{f_r}(f_t - f_r)\right|\right) \pm \mathcal{O}\left(\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-\rho}\right) .$$

But $\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-\rho}$ is much smaller than $\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-1}$. Hence, by (7.1.13), we get

$$\left\|T^{\hat{l}m}(f_t) - T^{\hat{l}m}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left(\left|\delta^{\hat{l}m}_{0}\theta_{f_r}(f_t - f_r)\right|\right) . \tag{7.1.14}$$

By (7.1.8), we have also

$$\left\|T^{\hat{l}m}(f_t) - T^{\hat{l}m}(f_r)\right\|_{\mathcal{C}} = \left\|T^{\hat{l}m}(g_{t'}) - T^{\hat{l}m}(g_{r'})\right\|_{\mathcal{C}} \pm \mathcal{O}\left(2^{-\hat{l}}\right) .$$

But $2^{-\hat{l}} \leq \mathcal{O}\left(\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-\rho}\right)$. Hence, again by (7.1.13), we deduce that

$$\left\|T^{\hat{l}m}(f_t) - T^{\hat{l}m}(f_r)\right\|_{\mathcal{C}} = \mathcal{O}\left(\left\|T^{\hat{l}m}(g_{t'}) - T^{\hat{l}m}(g_{r'})\right\|_{\mathcal{C}}\right) . \tag{7.1.15}$$

By Lemma 7.2 and (7.1.7), we have

$$\left\|T^{\hat{l}m}(g_{t'}) - T^{\hat{l}m}(g_{r'})\right\|_{\mathcal{C}} = \mathcal{O}\left(\left(\delta^{pm}_{\hat{l}m}\right)^{-1}\right) + \mathcal{O}\left(\left(\delta^{\hat{p}m}_{\hat{l}m}\right)^{-\rho}\right) .$$

Therefore, using (7.1.13) and (7.1.15) we obtain that

$$\left\|T^{\hat{l}m}(g_{t'}) - T^{\hat{l}m}(g_{r'})\right\|_{\mathcal{C}} = \mathcal{O}\left(\left(\delta^{pm}_{\hat{l}m}\right)^{-1}\right) .$$

But then, using (7.1.14) and (7.1.15) once more, we deduce that

$$|\theta_{f_r}(f_t - f_r)| = \mathcal{O}\left((\delta^{pm}_{0})^{-1}\right) .$$

Therefore, by Lemma 7.2, we get $\|f_t - f_r\|_{\mathcal{C}} = \mathcal{O}\left((\delta^{pm}_{0})^{-1}\right)$ which in turn implies (7.1.6). □

## 8. The renormalization operator is robust

From the very beginning, our main goal is to show that the renormalization operator is "hyperbolic" in $\mathbb{U}^r$, provided $r$ is sufficiently large. More precisely, we want to establish Theorem 2.5 and Corollary 2.6. We have already at our disposal an abstract theorem (Theorem 6.1) showing that any robust operator is indeed "hyperbolic". Hence, our work has been essentially reduced to showing that the renormalization operator $T$, or any one of its powers, is robust (see Theorem 8.1 below). The proof of Theorem 2.5and the proof of Corollary 2.6 will be given in §8.6.

We emphasize the important role played by the geometric estimates of §5.2 in the verification of properties **B5** and **B6** of a robust operator (Definition 6.1) for an iterate of the renormalization operator (see §8.4). Properties **B2**, **B3**, and **B4** are relatively straightforward consequences of the properties of the composition operator studied in §8.1 and are proved in §8.2 and §8.3 .

In this section, we shall prove the following result (see §8.5).

THEOREM 8.1. *Let* $T : \mathbb{O} \to \mathbb{A}$ *be the renormalization operator given by Theorem 2.4. If* $s > s_0$ *with* $s_0 < 2$ *sufficiently close to* 2 *and* $r > s+1$ *not an integer, then* $T$ *is a robust operator with respect to* $(\mathbb{A}^r, \mathbb{A}^s, \mathbb{A}^0)$.

We shall present in the sequel complete proofs of all the estimates that are necessary for establishing the above result, carefully checking all the properties of robustness along the way.

In our estimates we will often concern ourselves with a power $T^m$ of $T$. For each $m \geq 1$, let $\mathbb{O}_m \subseteq \mathbb{O}$ be the (open) set of those $f$'s which are $mN$ times renormalizable. Then $T^m$ is well-defined in $\mathbb{O}_m$ and we can write

$$T^m(f) = \frac{1}{\lambda_f} \cdot f^p \circ \Lambda_f \ ,$$

where $p = p(f, mN)$, $\lambda_f = f^p(0)$, and $\Lambda_f : x \mapsto \lambda_f x$ is the linear scaling. Note that $p$ (and hence $\lambda_f$ and $\Lambda_f$) depends on $m$, but if $m$ is held fixed then $p$ is a locally constant function of $f \in \mathbb{O}_m$. To keep track of the dependence of constants on $m$, we shall denote by $K$ those constants that may depend on $m$, and by $c$ those that are independent of $m$.

Likewise, we define $\mathbb{O}_m^r$ to be an open set in $\mathbb{U}^r$ containing $\mathbb{K}$, all of whose elements are $mN$ times renormalizable, so that $T^m = R^{mN} : \mathbb{O}_m^r \to \mathbb{U}^r$ is well-defined.

### 8.1. A closer look at composition

From a differentiable viewpoint, composition is a notoriously ill-behaved operation. Such bad behavior is the source of most technical difficulties arising in this work. Fortunately, some positive results lie at hand. For example, it is well-known that if $r$ is a positive integer then composition, viewed as an operator from $C^r \times C^{r-1}$ into $C^{r-1}$, is a $C^1$ map (see [11]). We shall need not only this result but also a less well-known generalization of it for Hölder spaces: if $r-1 > s \geq 1$ are *real* numbers then composition, as an operator from $C^r \times C^s$ into $C^s$, is a $C^1$ map (we can say a little bit more – see Proposition 8.7 below). For related results on the smoothness of composition, see [19].

Before we can prove this fact, some auxiliary results are in order. In what follows, our definition of $C^r$ norm of $\varphi \in C^r(I)$ is this: for $r = k + \alpha$ with $k \in \mathbb{N}$ and $0 \leq \alpha < 1$ we write

$$\|\varphi\|_{C^r} = \max\{\|\varphi\|_0, \|\varphi'\|_0, \ldots, \|\varphi^{(k)}\|_0; \|\varphi^{(k)}\|_\alpha\} .$$

For $r = k + \mathrm{Lip}$ we define $\|\varphi\|_{C^r}$ as above with $\alpha = 1$. This norm is equivalent to the one introduced earlier in 5.1, and has the advantage that $\|\varphi\|_{C^r} = \max\{\|\varphi\|_{C^0}, \|\varphi'\|_{C^{r-1}}\}$ whenever $r \geq 1$. This allows us to prove certain estimates by induction on $k$, which will be very useful later.

LEMMA 8.2. *Given* $0 \leq \alpha < 1$ *and* $0 \leq \epsilon \leq 1 - \alpha$, *let* $w \in C^{\alpha+\epsilon}(I)$, $\varphi, \psi \in C^1(I, I)$ *and* $||\psi - \varphi||_{C^1} \leq 1$.

*(i) If* $\epsilon > 0$ *then there exists* $K = K(\|\psi\|_{C^1}) > 0$ *such that*

$$\|w \circ \varphi - w \circ \psi\|_{C^\alpha} \leq K\|w\|_{C^{\alpha+\epsilon}}\|\varphi - \psi\|^\epsilon_{C^1} .$$

*(ii) If* $\epsilon = 0$ *then there exist* $c > 0$ *and* $K = K(\|\psi\|_{C^1}) > 0$ *such that*

$$\|w \circ \varphi - w \circ \psi\|_{C^\alpha} \leq c\|w\|_{C^\alpha}\|\psi'\|^\alpha_{C^0} + K\|w\|_{C^\alpha}\|\varphi - \psi\|^\alpha_{C^1} .$$

*Proof.* Let us start proving part (i) of this lemma. By the mean value theorem, we obtain

$$\|w \circ \varphi - w \circ \psi\|_{C^0} \leq \|w\|_{C^{\alpha+\epsilon}}\|\varphi - \psi\|^{\alpha+\epsilon}_{C^0}. \tag{8.1.1}$$

If $|y - x| \leq \|\varphi - \psi\|_{C^\alpha}$ then

$$|w \circ \varphi(y) - w \circ \varphi(x)| \leq c_0\|w\|_{C^{\alpha+\epsilon}}\|\varphi\|^{\alpha+\epsilon}_{C^1}\|\varphi - \psi\|^\epsilon_{C^\alpha}|y - x|^\alpha$$
$$|w \circ \psi(y) - w \circ \psi(x)| \leq c_1\|w\|_{C^{\alpha+\epsilon}}\|\psi\|^{\alpha+\epsilon}_{C^1}\|\varphi - \psi\|^\epsilon_{C^\alpha}|y - x|^\alpha .$$

If $|y - x| > \|\varphi - \psi\|_{C^\alpha}$, by (8.1.1) then

$$\|w \circ \psi - w \circ \varphi\|_{C^0} \leq c_2\|w\|_{C^{\alpha+\epsilon}}\|\varphi - \psi\|^\epsilon_{C^\alpha}|y - x|^\alpha ,$$

which ends the proof of part (i) of this lemma.

Let us prove part (ii) of this lemma. By the mean value theorem, we obtain

$$\|w \circ \varphi - w \circ \psi\|_{C^0} \leq \|w\|_{C^\alpha}\|\varphi - \psi\|^\alpha_{C^0}. \tag{8.1.2}$$

Furthermore,

$$|w \circ \varphi(y) - w \circ \varphi(x)| \leq c_3 \|w\|_{C^\alpha} \|\varphi'\|_{C^0}^\alpha |y-x|^\alpha$$
$$|w \circ \psi(y) - w \circ \psi(x)| \leq c_4 \|w\|_{C^\alpha} \|\psi'\|_{C^0}^\alpha |y-x|^\alpha \ ,$$

and so

$$\|w \circ \varphi - w \circ \psi\|_{C^\alpha} \leq c_5 \|w\|_{C^\alpha} (\|\varphi' - \psi'\|_{C^0} + \|\psi'\|_{C^0})^\alpha \tag{8.1.3}$$

which ends the proof of part (ii) of this lemma. □

We shall need also some estimates on polynomial operators coming from simple algebraic considerations. For every polynomial $P$ of degree $d$ in $n$ variables $x_1, x_2, \ldots, x_n$ over $\mathbb{R}$, define $\nu(P)$ as the sum of the absolute values of the coefficients of $P$. This is a well-known valuation in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$, but all that really matters to us is that $\nu(P+Q) \leq \nu(P) + \nu(Q)$ (sub-additivity), and that $\nu(\partial_{x_i} P) \leq d\nu(P)$.

Lemma 8.3. *Let $P \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a polynomial of degree $d$, and let $\phi_1, \phi_2, \ldots, \phi_n \in C^s(I)$. Then, we have*

$$\|P(\phi_1, \phi_2, \ldots, \phi_n)\|_{C^s} \leq \nu(P) 2^{sd} M^d \ ,$$

*where $M = \max\{1, \|\phi_1\|_{C^s}, \|\phi_2\|_{C^s}, \ldots, \|\phi_n\|_{C^s}\}$. Moreover, if $\psi_1, \psi_2, \ldots, \psi_n \in C^s(I)$ also satisfy $\|\psi_i\|_{C^s} \leq M$ for all $1 \leq i \leq n$, then*

$$\|P(\phi_1, \phi_2, \ldots, \phi_n) - P(\psi_1, \psi_2, \ldots, \psi_n)\|_{C^s} \leq d\nu(P) 2^{sd} M^{d-1} \sum_{i=1}^n \|\phi_i - \psi_i\|_{C^s} \ .$$

*Proof.* The first inequality is immediate from the definition of $\nu(P)$. To prove the second, note that $P : (C^s(I))^n \to C^s(I)$ is a $C^1$ map (norm of the sum in the domain of $P$). Using the mean value inequality and the first inequality, we see that

$$\begin{aligned}
&\|P(\phi_1, \phi_2, \ldots, \phi_n) - P(\psi_1, \psi_2, \ldots, \psi_n)\|_{C^s} \\
&\leq 2^s \sup_{0 \leq t \leq 1} \max_i \|\partial_{x_i} P(t\phi_1 + (1-t)\psi_1, \ldots, t\phi_n + (1-t)\psi_n)\|_{C^s} \sum_{i=1}^n \|\phi_i - \psi_i\|_{C^s} \\
&\leq 2^s (d\nu(P) 2^{s(d-1)} M^{d-1}) \sum_{i=1}^n \|\phi_i - \psi_i\|_{C^s} \ ,
\end{aligned}$$

which is the desired result. □

We can now use the estimate given in the above lemmas to prove the following general proposition. Let $r, s \geq 1$ be real numbers and for each $w \in C^r(I)$, let

$$\Theta_w : C^s(I, I) \to C^s(I)$$

be the operator given by $\Theta_w(\varphi) = w \circ \varphi$.

PROPOSITION 8.4. *Let $r, s > 1$ be real numbers both non-integer, and let $w \in C^r(I)$, $\varphi, \psi \in C^s(I, I)$ with $\|\varphi - \psi\|_{C^s} \leq 1$.*

*(i) If $r > s$ then there exists $K = K(\|\varphi\|_{C^s}) > 0$ such that*

$$\|w \circ \varphi - w \circ \psi\|_{C^s} \leq K \|w\|_{C^r} \|\varphi - \psi\|_{C^s}^{\epsilon}$$

*where $\epsilon = \min\{1 - \{s\}, r - s\}$ ($\{s\}$ denotes the fractional part of $s$). In particular, $\Theta_w : C^s(I, I) \to C^s(I)$ is $\epsilon$-Hölder continuous.*

*(ii) If $r = s$ there exists $c > 0$ and $K = K(\|\varphi\|_{C^s}) > 0$ such that*

$$\|w \circ \varphi - w \circ \psi\|_{C^r} \leq c \|w\|_{C^r} \|\psi'\|_{C^0}^{r} + K \|w\|_{C^r} \|\varphi - \psi\|_{C^r}^{\alpha} ,$$

*where $\alpha = \{s\}$ is the fractional part of $s$.*

Part (ii) of the above proposition shows one of the main difficulties in this theory which is the fact that for $w \in C^s(I)$ the operator $\Theta_w : C^s(I, I) \to C^s(I)$ is not even $C^0$.

*Proof.* Let us write $s = k + \alpha$, with $k$ an integer and $0 < \alpha = \{s\} < 1$, and let

$$A = w \circ \varphi - w \circ \psi .$$

Since $w, \varphi, \psi \in C^1$ and $\epsilon \leq 1 - \alpha$, applying Lemma 8.2 we get

$$\|A\|_{C^\alpha} \leq K_1 \|w\|_{C^1} \|\varphi - \psi\|_{C^1}^{\epsilon} .$$

By Faa-di-Bruno's Formula (see [13], p.42), for all $1 \leq l \leq k$ we can write

$$A^{(l)} = B_l(\varphi) - B_l(\psi)$$

where

$$B_l(\phi) = \sum_{j=1}^{l} w^{(j)} \circ \phi \cdot P_{l,j}(\phi', \phi'', \ldots, \phi^{(l-j)}) ,$$

each $P_{l,j}$ being a (universal, homogeneous) polynomial of degree $j$ in $l - j$ variables (with integer coefficients explicitly computable from $l$ and $j$, see [13], p.42). We only need the expression of $P_{l,j}$ for $j = l$; it is easy to check that $P_{l,l}(\phi') = (\phi')^l$. Then, we can decompose $A^{(l)} = C_l + D_l$, where

$$C_l = \sum_{j=1}^{l} w^{(j)} \circ \varphi \cdot \left(P_{l,j}\left(\varphi', \varphi'', \ldots, \varphi^{(l-j)}\right) - P_{l,j}\left(\psi', \psi'', \ldots, \psi^{(l-j)}\right)\right)$$

$$D_l = \sum_{j=1}^{l} \left(w^{(j)} \circ \varphi - w^{(j)} \circ \psi\right) \cdot P_{l,j}\left(\psi', \psi'', \ldots, \psi^{(l-j)}\right) .$$

By Lemma 8.3 applied to each $P_{l,j}$, we have

$$\left\| P_{l,j}\left(\varphi', \varphi'', \ldots, \varphi^{(l-j)}\right) - P_{l,j}\left(\psi', \psi'', \ldots, \psi^{(l-j)}\right) \right\|_{C^\alpha} \leq K_2 \|\varphi - \psi\|_{C^s} .$$

Therefore, for all $1 \leq l \leq k$ we get

$$\|C_l\|_{C^\alpha} \leq K_3 \|w\|_{C^s} \|\varphi - \psi\|_{C^s} \ .$$

Let us now rewrite $D_l = E_l + F_l$ where,

$$E_l = \sum_{j=1}^{l-1} \left( w^{(j)} \circ \varphi - w^{(j)} \circ \psi \right) \cdot P_{l,j} \left( \psi', \psi'', \ldots, \psi^{(l-j)} \right)$$

$$F_l = \left( w^{(l)} \circ \varphi - w^{(l)} \circ \psi \right) \cdot \left( \psi' \right)^l \ .$$

In bounding the first summation in $E_l$, we apply Lemma 8.2. Since $w^{(j)}, \varphi, \psi$ is at least $C^1$ we get

$$\left\| w^{(j)} \circ \varphi - w^{(j)} \circ \psi \right\|_{C^\alpha} \leq K_4 \left\| w^{(j)} \right\|_{C^1} \|\varphi - \psi\|_{C^1}^{1-\alpha}$$

for all with $1 \leq j \leq l-1$. From this and Lemma 8.3, for all $1 \leq l \leq k$ we obtain that

$$\|E_l\|_{C^\alpha} \leq K_5 \|w\|_{C^s} \|\varphi - \psi\|_{C^s}^{1-\alpha} \ .$$

Our task has been reduced to bounding the $C^\alpha$ norm of $F_l$. Here, we will do separately the proof of part (i) and part (ii) of this Proposition.

Let us prove part (i) first. Here, for all $1 \leq l \leq k$ we have that $\varphi, \psi$ are at least $C^1$ and that $w^{(l)}$ is at least $C^{\alpha+\epsilon}$, and so by Lemma 8.2 we get

$$\left\| w^{(l)} \circ \varphi - w^{(l)} \circ \psi \right\|_{C^{\alpha+\epsilon}} \leq K_6 \|w\|_{C^{l+\alpha+\epsilon}} \|\varphi - \psi\|_{C^1}^{\epsilon} \ .$$

Thus, for all $1 \leq l \leq k$ we obtain

$$\|F_l\|_{C^\alpha} \leq K_7 \|w\|_{C^r} \|\varphi - \psi\|_{C^s}^{\epsilon} \ ,$$

which ends the proof of part (i).

Let us now prove part (ii). We know that $w^{(l)}, \varphi, \psi \in C^1$ for all $1 \leq l \leq k-1$, and so by Lemma 8.2 we have

$$\left\| w^{(l)} \circ \varphi - w^{(l)} \circ \psi \right\|_{C^{\alpha+\epsilon}} \leq K_8 \|w\|_{C^{l+\alpha+\epsilon}} \|\varphi - \psi\|_{C^1}^{\alpha} \ .$$

Thus, for all $1 \leq l \leq k-1$ we obtain

$$\|F_l\|_{C^\alpha} \leq K_9 \|w\|_{C^r} \|\varphi - \psi\|_{C^s}^{\alpha} \ .$$

Therefore, we just have to bound $\|F_k\|_{C^\alpha}$. Here, $w^{(k)}$ is only $C^\alpha$. From the inequalities (8.1.2) and (8.1.3) in the proof of Lemma 8.2, we get

$$\left\| w^{(k)} \circ \varphi - w^{(k)} \circ \psi \right\|_{C^0} \leq c_1 \|w\|_{C^r} \|\varphi - \psi\|_{C^0}^{\alpha}$$

$$\left\| w^{(k)} \circ \varphi - w^{(k)} \circ \psi \right\|_{C^\alpha} \leq c_2 \|w\|_{C^r} \left( \left\|\psi'\right\|_{C^0} + \left\|\varphi' - \psi'\right\|_{C^0} \right)^\alpha \ ,$$

and so

$$\|F_k\|_{C^\alpha} \leq c_3 \|\psi\|_{C^{1+\alpha}}^{k} \|w\|_{C^r} \|\varphi - \psi\|_{C^0}^{\alpha} + c_4 \|\psi\|_{C^1}^{k} \|w\|_{C^r} \left( \left\|\psi'\right\|_{C^0} + \left\|\varphi' - \psi'\right\|_{C^0} \right)^\alpha \ ,$$

which ends the proof of part (ii). □

LEMMA 8.5. *Given* $0 \le \alpha < 1$ *and* $0 < \epsilon \le 1 - \alpha$, *let* $f \in C^{1+\alpha+\epsilon}(I)$, $g \in C^1(I, I)$ *and* $v \in C^1(I)$ *with* $\|v\|_{C^1} \le 1$ *and* $g + v \in C^1(I, I)$. *There exists* $K = K(\|g\|_{C^1}) > 0$ *such that*

$$\left\| f \circ (g+v) - f \circ g - f' \circ g \cdot v \right\|_{C^\alpha} \le K \|f\|_{C^{1+\alpha+\epsilon}} \|v\|_{C^1}^{1+\epsilon} \ .$$

*In particular, there exists* $K = K(\|g\|_{C^1}) > 0$ *such that*

$$\|f \circ (g+v) - f \circ g\|_{C^\alpha} \le K \|f\|_{C^{1+\alpha+\epsilon}} \|v\|_{C^1} \ .$$

*Proof.* Note that we have the following identity:

$$(f \circ (g+v) - f \circ g - f' \circ g \cdot v)(x) = v(x) \int_0^1 \big[ f'(g(x) + tv(x)) - f'(g(x)) \big] \, dt \ .$$

Applying Lemma 8.2 with $w = f'$, $\varphi = g + tv$ and $\psi = g$, we can bound the $C^\alpha$ norm of the integrand by $Kt\|f'\|_{C^{\alpha+\epsilon}} \|v\|_{C^1}^{\epsilon}$. This proves the first stated estimate in slightly stronger form. The second estimate is an immediate consequence of the first. □

PROPOSITION 8.6. *Let* $2 \le s+1 < r$ *be real numbers, and let* $f \in C^r(I)$, $g \in C^s(I, I)$. *There exists* $K = K(\|g\|_{C^s}) > 0$ *such that, for all* $v \in C^s(I)$ *with* $\|v\|_{C^s} \le 1$ *and* $g + v \in C^s(I, I)$, *we have*

$$\left\| f \circ (g+v) - f \circ g - f' \circ g \cdot v \right\|_{C^s} \le K \|f\|_{C^r} \|v\|_{C^s}^{1+\theta} \ , \tag{8.1.4}$$

*where* $\theta = \min\{1 - \{s\}, r - s - 1\}$. *In particular, (a) the operator* $\Theta_f : C^s(I, I) \to C^s(I)$ *is* $C^1$ *and its derivative is given by* $D\Theta_f(g)v = f' \circ g \cdot v$, *and (b) there exists* $K = K(\|g\|_{C^s}) > 0$ *such that for all* $v$ *as above we have*

$$\|f \circ (g+v) - f \circ g\|_{C^s} \le K \|f\|_{C^r} \|v\|_{C^s} \ . \tag{8.1.5}$$

*Proof.* In this proof we use $K_1, K_2, \ldots$ to denote constants that depend only on $\|g\|_{C^s}$. Consider the remainder term

$$F = f \circ (g+v) - f \circ g - f' \circ g \cdot v \ ,$$

as well as its derivative $F' = A + B$, where

$$\begin{aligned} A &= \big( f' \circ (g+v) - f' \circ g - f'' \circ g \cdot v \big) \cdot g' \\ B &= \big( f' \circ (g+v) - f' \circ g \big) \cdot v' \ . \end{aligned}$$

We want to show that

$$\left\| F' \right\|_{C^s} \le K_1 \|f\|_{C^r} \|v\|_{C^s}^{1+\theta}$$

The proof will be by induction on the integral part of $s$. Note however that the mean value theorem already gives us $\|F\|_{C^0} \le K_2 \|f''\|_{C^0} \|v\|_{C^0}^2$ independently of $s$.

First we deal with the base of induction, namely when $1 \le s < 2$, say $s = 1 + \alpha$. By Lemma 8.5, we have

$$\|A\|_{C^\alpha} \le K_3 \|f'\|_{C^{1+\alpha+\theta}} \left(\|v\|_{C^1}\right)^{1+\theta} .$$

The same Lemma 8.5 yields

$$\|B\|_{C^\alpha} \le K_4 \|f'\|_{C^{1+\alpha+\theta}} \left(\|v\|_{C^{1+\alpha}}\right)^{2} .$$

This establishes the base of induction.

Now suppose that our lemma holds for $s > 1$. We will prove from this that it holds for $s + 1$. To do this, it suffices to show that

$$\left\|F'\right\|_{C^s} \le K_5 \|f\|_{C^r} \|v\|_{C^{s+1}}^{1+\theta} . \tag{8.1.6}$$

The proof is more of the same. By the induction hypothesis applied to $f'$, we have

$$\|A\|_{C^s} \le K_6 \|f'\|_{C^{r-1}} \left(\|v\|_{C^s}\right)^{1+\theta} . \tag{8.1.7}$$

The same fact also gives

$$\|B\|_{C^s} \le K_7 \|f'\|_{C^{r-1}} \left(\|v\|_{C^{1+s}}\right)^{2} . \tag{8.1.8}$$

Putting (8.1.7) and (8.1.8) together we get (8.1.6), and so the induction is complete. □

PROPOSITION 8.7. *Let $2 \le s + 1 < r$ be real numbers. The composition operator $\Theta : C^r(I) \times C^s(I, I) \to C^s(I)$ given by $\Theta(f, g) = f \circ g$ is $C^{1+\theta}$ and its derivative is given by $D\Theta(f, g)(u, v) = u \circ g + f' \circ g \cdot v$. In particular, there exists $K = K\left(\|f\|_{C^r}, \|g\|_{C^s}\right) > 0$ such that, for all $\|u\|_{C^r} \le 1$ and $\|v\|_{C^s} \le 1$ with $g + v \in C^s(I, I)$, we have*

$$\|\Theta(f+u, g+v) - \Theta(f, g) - D\Theta(f, g)(u, v)\|_{C^s} \le K(\|u\|_{C^r} + \|v\|_{C^s})^{1+\theta} , \tag{8.1.9}$$

*where $\theta = \min\{1 - \{s\}, r - s - 1\}$.*

*Proof.* In this proof, we denote by $K_1, K_2, \ldots$ positive constants depending only on $\|g\|_{C^s}$. Let us take $u \in C^r(I)$ and $v \in C^s(I)$ such that $\|u\|_{C^r} \le 1$ and $\|v\|_{C^s} \le 1$, respectively. We have

$$\begin{aligned} F &= \Theta(f + u, g + v) - \Theta(f, g) - u \circ g - f' \circ g \cdot v \\ &= f \circ (g + v) - f \circ g - f' \circ g \cdot v + u \circ (g + v) - u \circ g . \end{aligned}$$

Using Proposition 8.6, we see that

$$\left\|f \circ (g + v) - f \circ g - f' \circ g \cdot v\right\|_{C^s} \le K_1 \|f\|_{C^r} (\|v\|_{C^s})^{1+\theta} .$$

The same Proposition 8.6 with $u$ replacing $f$ yields

$$\begin{aligned} \|u \circ (g + v) - u \circ g\|_{C^s} &\le \|u' \circ g \cdot v\|_{C^s} + K_2 \|u\|_{C^r} (\|v\|_{C^s})^{1+\theta} \\ &\le K_3 \|u\|_{C^r} \|v\|_{C^s} . \end{aligned}$$

Therefore we get

$$\|F\|_{C^s} \leq K_1 \|f\|_{C^r} (\|v\|_{C^s})^{1+\theta} + K_3 \|u\|_{C^r} \|v\|_{C^s} ,$$

which proves that $\Theta$ is $C^1$ and that (8.1.9) is satisfied. Now, we have that

$$D\Theta(f + \phi, g + \psi)(u, v) - D\Theta(f, g)(u, v) = A + B + C$$

where

$$\begin{aligned} A &= u \circ (g + \psi) - u \circ g \\ B &= \big(f' \circ (g + \psi) - f' \circ g\big) \cdot v \\ C &= \phi' \circ (g + \psi) \cdot v \ . \end{aligned}$$

By Proposition 8.4, we obtain that

$$\begin{aligned} \|A\|_{C^s} &\leq K_4 \|u\|_{C^{r-1}} \|\psi\|^{\theta}_{C^s} \\ \|B\|_{C^s} &\leq K_5 \|f\|_{C^r} \|\psi\|^{\theta}_{C^s} \cdot \|v\|_{C^s} . \end{aligned}$$

Letting $k$ be the integer part of $s$ and $\varphi = g + \psi$, and using Faa-di-Bruno's Formula, we have

$$\big(\phi' \circ \varphi\big)^{(k)} = \sum_{j=1}^{k} \phi^{(j+1)} \circ \varphi \cdot P_{k,j}(\varphi', \varphi'', \ldots, \varphi^{(k-j)}) ,$$

each $P_{k,j}$ being a (universal, homogeneous) polynomial of degree $j$ in $k - j$ variables. Hence, using Lemma 8.3, we get that $\|C\|_{C^s} = K_6 \|\Delta f\|_{C^r} \|v\|_{C^s}$. Thus, $\Theta$ is a $C^{1+\theta}$ operator. □

Corollary 8.8. *Let $r, s > 0$ be real numbers with $r - 1 > s \geq 1$ and for each positive integer $m$, let $Q_m : C^r(I, I) \to C^s(I, I)$ be the operator given by $Q_m(f) = f^m$.*

*(i) Let $0 \leq t \leq r$ and let $U : C^t(I, I) \to C^s(I, I)$ be a $C^{1+\theta}$ operator for some $0 < \theta < 1$. Then the operator $U_m : C^r(I, I) \to C^s(I, I)$ given by $U_m(f) = Q_m \circ U(f)$ is $C^{1+\theta'}$ for some $0 < \theta' = \theta'(\theta, r, s) < 1$.*

*(ii) In particular, the operator $Q_m : C^r(I, I) \to C^s(I, I)$ is $C^{1+\theta''}$ for some $0 < \theta'' = \theta''(r, s) < 1$ and there exists $K = K(m, \|f\|_{C^r}) > 0$ such that*

$$\|Q_m(f + u) - Q_m(f) - DQ_m(f)u\|_{C^s} \leq K \|u\|^{1+\theta''}_{C^r} \ . \qquad (8.1.10)$$

*Proof.* First note that $U_{m+1}(f) = \Theta(f, U_m(f))$. The operator $U_1$ arises as the composition of the operator $C^r(I, I) \to C^r(I, I) \times C^s(I, I)$ given by $f \mapsto (f, U(f))$, which is $C^{1+\theta}$ because $U$ is $C^{1+\theta}$ (and $C^r(I, I)$ embeds in $C^t$), with the composition operator $\Theta : C^r(I, I) \times C^s(I, I) \to C^s(I, I)$, which is

$C^{1+\theta''}$ for some $0 < \theta'' = \theta''(r,s) < 1$ by Proposition 8.7. The desired result for part (i) then follows by induction. Part (ii) is a corollary of part (i), and by a computation (8.1.10) follows from (8.1.9). □

PROPOSITION 8.9. *Let $r, s, t$ be real numbers with $2 \le s+1 < r$ and $t \ge 0$. Let $U : C^t(I,I) \to C^s(I,I)$ be a $C^1$ operator. Then for each $\phi \in C^r(I)$ and each $\psi \in C^t(I,I)$ there exists a function $\sigma_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sigma_\psi(h)/h \to 0$ as $h \to 0$, varying continuously with $\psi$, such that for all $v \in C^t(I)$ with $\psi + v \in C^t(I,I)$ we have*

$$\left\|\phi \circ U(\psi+v) - \phi \circ U(\psi) - \phi' \circ U(\psi) \cdot DU(\psi)v\right\|_{C^s} \le \sigma_\psi(\|v\|_{C^t}) \ . \quad (8.1.11)$$

*Proof.* As before, we denote by $K_1, K_2, \dots$ positive constants that depend only on $\|\psi\|_{C^t}$. We have that

$$\phi \circ U(\psi+v) - \phi \circ U(\psi) - \phi' \circ U(\psi) \cdot DU(\psi)v = A + B$$

where

$$\begin{aligned} A &= \phi \circ U(\psi+v) - \phi \circ U(\psi) - \phi' \circ U(\psi) \cdot (U(\psi+v) - U(\psi)) \\ B &= \phi' \circ U(\psi) \cdot (U(\psi+v) - U(\psi) - DU(\psi)v) \ . \end{aligned}$$

Since $U$ is $C^1$, there exists a continuous function $\nu_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\nu_\psi(h)/h \to 0$ as $h \to 0$, varying continuously with $\psi$, such that

$$\|U(\psi+v) - U(\psi) - DU(\psi)v\|_{C^s} \le \nu_\psi(\|v\|_{C^t}) \ .$$

Hence, applying Proposition 8.6 with $f = \phi$ and $g = U(\psi)$ and $v$ replaced by $U(\psi+v) - U(\psi)$, we get

$$\begin{aligned} \|A\|_{C^s} &\le K_2 \|\phi\|_{C^r} \left(\|U(\psi+v) - U(\psi)\|_{C^s}\right)^{1+\theta} \\ &\le K_3 \|\phi\|_{C^r} \left(\|DU(\psi)\|^{1+\theta} \|v\|_{C^t}^{1+\theta}\right) \ , \end{aligned}$$

and

$$\|B\|_{C^s} \le K_4 \|\phi\|_{C^r} \nu_\psi(\|v\|_{C^t}) \ ,$$

where $K_3 = K_3(\|U(\psi)\|_{C^s}, \|DU(\psi)\|, \nu_\psi)$ and $K_4 = K_2(\|U(\psi)\|_{C^s})$. Therefore,

$$\|A+B\|_{C^s} \le K_3 \|\phi\|_{C^r} \|v\|_{C^t}^{1+\theta} + K_4 \|\phi\|_{C^r} \nu_\psi(\|v\|_{C^t}) \ .$$

This completes the proof. □

COROLLARY 8.10. *Let $r, s, t$ be real numbers with $r - 1 > s > 1$ and $0 \le t \le r$, and let $U : C^t(I,I) \to C^s(I,I)$ be a $C^1$ operator. For each positive integer $n$ the operator $V_n : C^r(I,I) \to C^s(I)$ given by $V_n(f) = (f^m)' \circ U(f)$ is differentiable at every $g \in C^{r+1}(I,I) \subseteq C^r(I,I)$, and as map from $C^{r+1}(I,I)$ into $\mathcal{L}(C^r(I), C^s(I))$, the derivative operator $g \mapsto DV_n(g)$ is continuous.*

*Proof.* First note that by the chain rule,

$$V_n(f) = \prod_{j=0}^{n-1} f' \circ \left(f^{(j)} \circ U(f)\right) = \prod_{j=0}^{n-1} f' \circ U_j(f) \ .$$

This reduces the problem to the case $n = 1$. We claim that the linear operator

$$L(v) = v' \circ U(g) + g'' \circ U(g) \cdot DU(g)v$$

is the derivative of $V_1$ at $g \in C^{r+1}(I, I)$. Indeed, we have

$$V_1(g + v) - V_1(g) - L(v) = A + B \ ,$$

where

$$\begin{aligned} A &= g' \circ U(g + v) - g' \circ U(g) - g'' \circ U(g) \cdot DU(g)v \\ B &= v' \circ U(g + v) - v' \circ U(g) \ . \end{aligned}$$

By Proposition 8.9 applied to $\phi = g'$ and $\psi = g$, there exists $K_1 = K_1(\|g\|_{C^{r+1}})$ such that

$$\|A\|_{C^s} \le K_1 \sigma_\psi(\|v\|_{C^r}) \ ,$$

where $\sigma_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function varying continuously with $\psi$ such that $\sigma_\psi(h)/h \to 0$ as $h \to 0$. On the other hand, by part $(i)$ of Proposition 8.4 and since $U$ is $C^1$, we have

$$\begin{aligned} \|B\|_{C^s} &\le K_2 \|v\|_{C^r} \|U(g + v) - U(g)\|_{C^s}^{\epsilon} \\ &\le K_3 \left(\|v\|_{C^r}\right)^{1+\epsilon} \ . \end{aligned}$$

where $0 < \epsilon = \min\{1 - \{s\}, r - s\} < 1$, $K_2 = K_2(\|U(g)\|_{C^s})$ and $K_3 = K_3(\|U(g)\|_{C^s}, \|DU(g)\|, \sigma_\psi(\|v\|_{C^r}))$. Combining these inequalities, we deduce that $V_1$ is differentiable at $g$ and $DV_1(g) = L$ as claimed. It is clear from the expression defining it that $L$ varies continuously with $g \in C^{r+1}(I, I)$. □

## 8.2. Checking properties B2 and B3

We now proceed to verify that the operator $T$ satisfies properties **B2** and **B3** of robustness. They will follow respectively from lemmas 8.12 and 8.13. First it is necessary to analyze the behavior of the linear scaling used in such operators. Let us fix a positive integer $p$ and for each $f \in C^r(I, I)$ let $\Lambda_f$ be the linear map $x \mapsto \lambda_f x$, where $\lambda_f = f^p(0)$.

LEMMA 8.11. *For $r > 2$, the maps $\Lambda : C^r(I, I) \to \mathcal{L}(\mathbb{R}, \mathbb{R})$ given by $\Lambda(f) = \Lambda_f$ and $\lambda : C^r(I, I) \to \mathbb{R}$ given by $\lambda(f) = \lambda_f$ are both $C^{1+\theta}$ for some $0 < \theta = \theta(r, s) < 1$. In particular, there is $K = K(p, \|f\|_{C^r}) > 0$ such that for all $v \in C^r(I)$ with $\|v\|_{C^r} \le 1$ and $f + v \in C^r(I, I)$, we have*

$$\|\lambda(f + v) - \lambda(f) - D\lambda(f)v\|_{C^r} \le K \|v\|_{C^r}^{1+\theta} \ , \tag{8.2.1}$$

*The above inequality also holds replacing $\lambda$ by $\Lambda$.*

*Proof.* Choosing $1 < s < r - 1$, we see that $\lambda = E \circ Q_p$ where $Q_p : C^r(I, I) \to C^s(I, I)$ is the operator $Q_p(f) = f^p$, which is $C^{1+\theta}$ for some $0 < \theta = \theta(r, s) < 1$, and $E : C^s(I, I) \to \mathbb{R}$ is the evaluation map $E(g) = g(0)$, which is linear. Therefore, by Corollary 8.8, $\lambda$ is $C^{1+\theta}$ and (8.2.1) follows from (8.1.10) and the linearity of $E$. The proof for $\Lambda$ is entirely analogous. □

We will also need to use the operators $U_n : C^r(I, I) \to C^s(I)$ given by $U_n(f) = f^n \circ \Lambda_f$ for all $n \geq 0$.

Property **B2** for the operator $T$ is a consequence of the following lemma (the first assertion in **B2** is actually a consequence of Lemma 8.14 below).

LEMMA 8.12. *For $2 < s + 1 < r$ and for each $n \geq 0$, the operator $U_n : C^r(I, I) \to C^s(I)$ is $C^{1+\theta}$ for some $0 < \theta = \theta(r, s) < 1$. In particular, $T^m : \mathbb{O}^r_m \to \mathbb{U}^s$ is also a $C^{1+\theta}$ operator.*

*Proof.* This follows at once from Lemma 8.11 and Corollary 8.8 applied to $U = \Lambda$. □

The following lemma is all we need to verify property **B3** for the operator $T$. In this case $g$ is a map in the limit set $\mathbb{K}$ of $T$, hence analytic, and $v = \mathbf{u}_g$ is a tangent vector to the unstable manifold of $g$, which is analytic as well.

LEMMA 8.13. *For $2 < s + 1 < r$, the map $\mathbb{O}^r_m \to \mathbb{U}^s$ given by $f \mapsto DT^m(f)v$ is differentiable at $f = g \in \mathbb{K}$. Furthermore, for every $m \geq 1$ there exist $C_m > 1$ and $\nu_m > 0$ such that for each $g \in \mathbb{K}$ and $f \in \mathbb{O}^r_m$ with $\|f - g\|_{C^r} < \nu_m$ and all $v \in \mathbb{A}^r$ with $\|v\|_{C^r} = 1$ we have*

$$\|DT^m(f)v - DT^m(g)v\|_{C^s} \leq C_m \|f - g\|_{C^r} \ . \tag{8.2.2}$$

*Proof.* Let $E : C^s(I, I) \to \mathbb{R}$ be the evaluation map $E(g) = g(0)$, which is linear. Recall that the derivative of $T^m$ is given by the expression

$$DT^m(f)v = \frac{1}{\lambda_f} \sum_{j=0}^{p-1} (f^j)' \circ U_{p-j}(f) \cdot v \circ U_{p-j-1}(f) \tag{8.2.3}$$

$$+ \frac{1}{\lambda_f} [\mathrm{id} \cdot (T^m f)' - T^m f] \sum_{j=0}^{p-1} E\left((f^j)' \circ U_{p-j}(f)\right) \cdot E\left(v \circ U_{p-j-1}(f)\right) \ ,$$

where $\lambda_f = E \circ f^p$ and $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ is the identity map. Each term of the first summation in (8.2.3) is differentiable at $f = g$. To see this apply Lemma 8.12 and Corollary 8.10 to each of the operators $f \mapsto (f^j)' \circ U_{p-j}(f)$ as well as Proposition 8.6 to each of the operators $f \mapsto v \circ U_{p-j}(f)$. On the other hand, each term of the second summation in (8.2.3) equals the corresponding term in the first summation post-composed with the evaluation map $E$ (which is linear), and is therefore differentiable at $f = g$. The analysis of the expression in square brackets in (8.2.3) is similar. By Lemma 8.11 and Corollary 8.10, the operator $f \mapsto T^m(f)' = (f^p)' \circ \Lambda_f$ is differentiable at $f = g$, and the operator $f \mapsto T^m(f) = \lambda_f \cdot f^p \circ \Lambda_f$ is also differentiable at $f = g$ by Lemma 8.11 and Corollary 8.8. From this fact and compactness of $\mathbb{K}$ the inequality (8.2.2) follows. □

## 8.3. Checking property B4

The fifth property is verified in Lemma 8.16 below. First we will need to prove the following two lemmas about the operators $U_i : C^{t+1+\varepsilon}(I,I) \to C^t(I)$ with $t \geq 1$. Recall that $U_i(f) = f^i \circ \Lambda_f$.

LEMMA 8.14. *For every $f \in C^{t+1+\epsilon}(I,I)$ and all $v \in C^t(I)$ with small norm and such that $f + v \in C^t(I,I)$, we have*

$$\|U_i(f+v) - U_i(f)\|_{C^t} \leq K\|v\|_{C^t}$$

*for all $0 \leq i \leq p$ where $K = K\left(p, \|f\|_{C^{t+1+\epsilon}}\right)$.*

*Proof.* Note that

$$U_{i+1}(f+v) - U_{i+1}(f) = f \circ U_i(f+v) - f \circ U_i(f) + v \circ U_i(f+v)$$

By Proposition 8.6, there is $K_1 = K_1\left(p, \|f\|_{C^{t+1+\epsilon}}\right)$ such that

$$\|U_{i+1}(f+v) - U_{i+1}(f)\|_{C^t} \leq K_1\|U_i(f+v) - U_i(f)\|_{C^t} + \|v \circ U_i(f+v)\|_{C^t}.$$

The required estimate now follows by induction, because $U_0$ is $C^1$. □

LEMMA 8.15. *For every $f \in C^{t+1+\epsilon}(I,I)$ and all $v \in C^{t+1+\epsilon}(I)$ with small norm and such that $f + v \in C^{t+1+\epsilon}(I,I)$, we have*

$$\|U_i(f+v) - U_i(f) - DU_i(f)v\|_{C^t} \leq K\|v\|_{C^t}^{1+\theta} \ ,$$

*for all $0 \leq i \leq p$, for some $0 < \theta = \theta(t,\epsilon) < 1$ and $K = K(p, \|f\|_{C^t}) > 0$.*

*Proof.* In this proof we denote by $K_1, K_2, \ldots$ the positive constants depending only on $m$ and $\|U_i(f)\|_{C^{t+1+\epsilon}}$. Again we use induction; the case $i = 0$ follows from the differentiability of the scaling $f \to \Lambda_f$ and inequality (8.2.1). We have

$$U_{i+1}(f+v) - U_{i+1}(f) - DU_{i+1}(f)v = A + B + C$$

where

$$\begin{aligned}
A &= f \circ U_i(f+v) - f \circ U_i(f) - f' \circ U_i(f) \cdot (U_i(f+v) - U_i(f)) \\
B &= f' \circ U_i(f) \cdot (U_i(f+v) - U_i(f) - DU_i(f)v) \\
C &= v \circ U_i(f+v) - v \circ U_i(f) \ .
\end{aligned}$$

By Proposition 8.6, we have

$$\|A\|_{C^t} \leq K_1\|U_i(f+v) - U_i(f)\|_{C^t}^{1+\epsilon}.$$

Using Lemma 8.14, we get

$$\|A\|_{C^{t-1}} \le K_2 \|v\|_{C^t}^{1+\epsilon} \ .$$

On the other hand, since $v$ is $C^{t+1+\epsilon/2}$, we know again from Proposition 8.6 that

$$\|C\|_{C^t} \le K_3 \|v\|_{C^{t+1+\epsilon/2}} \|U_i(f+v) - U_i(f)\|_{C^t} \le K_4 \|v\|_{C^{r+1+\epsilon/2}} \|v\|_{C^t} \ .$$

Since $v$ has bounded $C^{t+1+\epsilon}$ norm, by an interpolation of norms, we have $\|v\|_{C^{t+1+\epsilon/2}} \le K_5 \|v\|_{C^t}^{\theta_1}$ for some $\theta_1 > 0$. Therefore, taking $\theta = \min\{\epsilon, \theta_1\}$ we get

$$\|C\|_{C^t} \le K_6 \|v\|_{C^t}^{1+\theta} \ .$$

This allows the induction as desired. □

Property **B4** for the operator $T$ is a direct consequence of the following lemma.

LEMMA 8.16. *For every $f \in \mathbb{O}^{s+1+\epsilon}$ and all $v \in \mathbb{A}^{s+1+\epsilon}$ with small norm such that $f + v \in \mathbb{O}^{s+1+\epsilon}$, we have*

$$\|T(f+v) - T(f) - DT(f)(v)\|_{C^s} \le K \|v\|_{C^s}^{1+\tau} \ ,$$

*for some $0 < \tau = \tau(s,\epsilon) < 1$ and $K(p, \|f\|_{C^{s+1+\epsilon}}) > 0$.*

*Proof.* In this proof we denote by $K_1, K_2, \ldots$ the positive constants depending only on $m$ and $\|U_i(f)\|_{C^s}$. Start observing that since $T(f) = \lambda_f^{-1} \cdot U_p(f)$, we have

$$T(f+v) - T(f) - DT(f)v = A + B + C$$

where

$$\begin{aligned} A &= \lambda_f^{-1} \cdot (U_p(f+v) - U_p(f) - DU_p(f)v) \\ B &= \left(\lambda_{f+v}^{-1} - \lambda_f^{-1} - D\lambda_f^{-1}(v)\right) \cdot U_p(f+v) \\ C &= D\lambda_f^{-1}(v) \cdot (U_p(f+v) - U_p(f)) \ . \end{aligned}$$

Applying Lemma 8.15 with $t = s$ we get

$$\|A\|_{C^s} = K_1 \|v\|_{C^s}^{1+\theta_1} \ ,$$

for some $0 < \theta_1 = \theta_1(s,\epsilon) < 1$. By Lemma 8.11 there is $0 < \theta_2 = \{s\} < 1$ such that

$$\|B\|_{C^s} \le K_2 \|v\|_{C^s}^{1+\theta_2} \ .$$

By Lemma 8.14, we have $\|U_p(f+v) - U_p(f)\|_{C^s} \le K_3 \|v\|_{C^s}$ and so

$$\|C\|_{C^s} \le K_4 \|v\|_{C^s}^2 \ .$$

Therefore, it is enough to take $\tau = \min\{\theta_1, \theta_2\}$. □

### 8.4. Checking properties B5 and B6

We now move on to the task of proving that the operator $T = R^N$ of Theorem 2.4 satisfies properties **B5** and **B6** in the definition of robustness. Unlike the previous ones, the verification of these (last) two properties depends upon the geometry of the post-critical sets of maps near in $\mathbb{A}$ to the limit set $\mathbb{K}$ of $T$. The estimates performed here are the most delicate, and involve the results of §5.2.

Recall that $T^m$ is well-defined on an open set $\mathbb{O}_m$ in the Banach space $\mathbb{A} = \mathbb{A}_{\Omega_a}$ (see §3), which contains $\mathbb{K}$. We shall denote the renormalization intervals $\Delta_{0,mN}$, $\Delta_{1,mN}$, ..., $\Delta_{p,mN}$ simply by $\Delta_i = \Delta_{i,mN}$ (this shortened notation should cause no harm, because $N$ is fixed since Theorem 2.4, and $m$ will be fixed in the particular estimates involving these intervals).

We can write the derivative of $T^m$ in the following form

$$DT^m(f)v = A(f)\sum_{j=0}^{p-1} B_j(f)\cdot C_j(f) + A(f)\cdot D(f)\sum_{j=0}^{p-1} E\circ B_j(f)\cdot E\circ C_j(f)$$

where $E$ is the evaluation map and

$$\begin{aligned} A(f) &= (\lambda_f)^{-1} \ , \\ B_j(f) &= \left(f^j\right)'\circ U_{p-j}(f) \ , \\ C_j(f) &= v\circ U_{p-j-1}(f) \ , \\ D(f) &= \mathrm{id}\cdot (f^p)'\circ U_0(f) - \lambda_f\cdot U_p(f) \ . \end{aligned}$$

To carry out our estimates for $T^m$, we shall use the operators $U_i : f \mapsto f^i\circ\Lambda_f$ ($i \geq 0$). Note that $U_0(f) = \Lambda_f$, hence $U_0$ is $C^1$ in whichever space $C^r(I,I)$ we work in, because the scaling $f \mapsto \Lambda_f$ is $C^1$ by Lemma 8.11.

First we need some estimates for $U_i$. It is clear that $\|U_i(f)\|_{C^0} \leq 1$ always, but more is true.

Lemma 8.17. *There exists $C > 0$ with the following property. For every $m > 0$, there exists an open neigbourhood $\mathcal{O}_m \subset \mathbb{O}_m$ of $\mathbb{K}$ such that for all $f \in \mathcal{O}_m$, we have*

$$\|B_j(f)\|_{C^0} \leq C\frac{|\Delta_0|}{|\Delta_{p-j}|} \ ,$$

*for all $0 \leq j \leq p-1$. Furthermore, $\|U_i(f)'\|_{C^0} \leq C|\Delta_i|$, for all $0 \leq i \leq p$.*

*Proof.* Use bounded distortion and the real bounds (see §5.2). □

Lemma 8.18. *For all $f \in \mathcal{O}_m$ and all $v \in \mathbb{A}^r$ with small norm, we have*

$$\|U_i(f+v) - U_i(f)\|_{C^r} \leq K\|v\|_{C^r}$$

*for all $0 \leq i \leq p$, where $K = K(m) > 0$.*

*Proof.* This lemma follows from Lemma 8.14. □

Next, we show an essential result to prove that the renormalization operator satisfies properties **B5** and **B6**. Here, we use again in a crucial way the geometric properties of the postcritical set of $f \in \mathbb{O}_m$ proved in §5.

PROPOSITION 8.19. *(i) For every $t > 2$ which is not an integer there exist $0 < \mu < 1$ and $C > 0$ with the following property. For every $g \in \mathbb{K}$ and for every $m > 0$, there is an $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f - g\|_{\mathbb{A}} < \eta$ and for all $w \in \mathbb{A}^t$ with $\|w\|_{C^t} < \eta$ we have*

$$\left\| A(f) \sum_{j=0}^{p-1} B_j(f)\left(C_j(f+w) - C_j(f)\right) \right\|_{C^t} \leq C\mu^m \|v\|_{C^t} \ . \tag{8.4.1}$$

*(ii) For every $\mu > 1$ close to one, there is $s < 2$ close to two and $C > 0$ with the following property: for every $g \in \mathbb{K}$ and for every $m$, there is an $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f - g\|_{\mathbb{A}} < \eta$ and for all $w \in \mathbb{A}^t$ with $\|w\|_{C^t} < \eta$ we have that inequality* (8.4.1) *above is also satisfied.*

*Proof.* Below, the positive constants $c_1, c_2, \ldots$ depend only on $t$ (and the real bounds), while the positive constants $K_0, K_1, K_2, \ldots$ may depend also on $m$.

Let $k$ and $0 < \alpha < 1$ be respectively the integer and the fractional part of $t$ (when $t = k + \mathrm{Lip}$ take $\alpha = 1$). We start observing that for each $j$ we have

$$\begin{aligned} \|B_j(f)\left(C_j(f+w) - C_j(f)\right)\|_{C^t} &\leq \|B_j(f)\|_{C^0}\|C_j(f+w) - C_j(f)\|_{C^t} \\ &\quad + K_0\|B_j(f)\|_{C^t}\|C_j(f+w) - C_j(f)\|_{C^k} \ . \end{aligned} \tag{8.4.2}$$

Note that in the right-hand side of (8.4.2) only the second term carries a constant $K_0$. By Lemma 8.17, there is $c_1 > 0$ such that for every integer $m$ there is an open neighbourhood $\mathcal{O}_m$ of $\mathbb{K}$ with the property that for each $f \in \mathcal{O}_m$ we have

$$\|B_j(f)\|_{C^0} \leq c_1 \frac{|\Delta_0|}{|\Delta_{p-j}|} \ . \tag{8.4.3}$$

In that neighborhood, we also have $\|B_j(f)\|_{C^t} \leq K_1$. By Proposition 8.4 and Lemma 8.18, taking $0 < \epsilon < 1$ such that $\alpha - \epsilon > 0$, we obtain

$$\begin{aligned} \|C_j(f+w) - C_j(f)\|_{C^k} &\leq \|C_j(f+w) - C_j(f)\|_{C^{t-\epsilon}} \\ &\leq K_2\|v\|_{C^t}\|U_{p-j-1}(f+w) - U_{p-j-1}(f)\|^{\epsilon}_{C^t} \\ &\leq K_3\|v\|_{C^t}\|w\|^{\epsilon}_{C^t} \ . \end{aligned} \tag{8.4.4}$$

On the other hand, putting together Proposition 8.4 with Lemma 8.17 and with Lemma 8.18, we get

$$\begin{aligned}\|C_j(f+w)-C_j(f)\|_{C^t} &\leq c_2\|U_{p-j-1}(f)'\|_{C^0}^t\|v\|_{C^t} \\ &\quad + K_4\|U_{p-j-1}(f+w)-U_{p-j-1}(f)\|_{C^t}^{\alpha}\|v\|_{C^t} \\ &\leq c_3|\Delta_{p-j-1}|^t\|v\|_{C^t} + K_5\|w\|_{C^t}^{\alpha}\|v\|_{C^t}\ . \qquad (8.4.5)\end{aligned}$$

The first term on the last line of (8.4.5) looks a bit dangerous. What saves us here is the geometric control on the post-critical set of $f$ (hence on the intervals $\Delta_i$) that we have at our disposal since §5.2. Substituting (8.4.3), (8.4.4) and (8.4.5) in (8.4.2) and adding up the terms with $j = 0, \ldots, p-1$ we arrive at

$$\begin{aligned}&\left\| A(f)\sum_{j=0}^{p-1} B_j(f)\left(C_j(f+w)-C_j(f)\right)\right\|_{C^t} \\ &\qquad \leq c_4\left\{\frac{1}{|\Delta_0|}\sum_{j=0}^{p-1}\frac{|\Delta_0|\cdot|\Delta_{p-j-1}|^t}{|\Delta_{p-j}|} + K_5\|w\|_{C^t}^{\epsilon}\right\}\|v\|_{C^t}\ ,\end{aligned}$$

But as we have seen in §5.2 :

(i) By Proposition 5.5 and Remark 5.1, if $t > 2$ there exist $0 < \gamma < 1$ and $C > 0$ with the following property. For every $g \in \mathbb{K}$ and every $m > 0$, there exists an $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f-g\|_{\mathbb{A}} < \eta$ we have

$$\sum_{j=0}^{p-1}\frac{|\Delta_j|^t}{|\Delta_{j+1}|} \leq C\gamma^{mN}\ , \qquad (8.4.6)$$

(ii) By Propostion 5.8 and Remark 5.1, for every $\gamma > 1$ close to one, there exists $t < 2$ close to two and $C > 0$ with the following property. For every $g \in \mathbb{K}$ and every $m > 0$, there exists $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f-g\|_{\mathbb{A}} < \eta$ we have that the inequality (8.4.6) above is also satisfied.

These last estimates end the proof of this proposition, provided we take $\mu = \gamma^N$ and $\eta < \mu^{m/\epsilon}$. □

We arrive at last to the main two results of this section.

THEOREM 8.20. *(i) If $t > 2$ is not an integer, there exist $0 < \mu < 1$ and $C > 0$ with the following property. For every $g \in \mathbb{K}$ and for every $m > 0$, there* *is an $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f-g\|_{\mathbb{A}} < \eta$ and for all $w \in \mathbb{A}^t$ with $\|w\|_{C^t} < \eta$ we have*

$$\|DT^m(f+w)v - DT^m(f)v\|_{C^t} \leq C\mu^m \|v\|_{C^t} \ . \tag{8.4.7}$$

*(ii) For every $\mu > 1$ close to one, there exist $t < 2$ close to 2 and $C > 0$ with the following property. For every $g \in \mathbb{K}$ and every $m > 0$, there exists $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f-g\|_{\mathbb{A}} < \eta$ and all $w \in \mathbb{A}^t$ with $\|w\|_{C^t} < \eta$, the inequality* (8.4.7) *above is also satisfied.*

Part (ii) of this theorem with $t = s$ implies property **B5** and part (i) is used later (for $t = r$) to prove property **B6**.

*Proof.* In this proof the positive constants $K_1, K_2, \ldots$ depend only on $r$ and $\mathbb{O}_m$ and also on $m$. Let $E : C^t(I,I) \to \mathbb{R}$ be the evaluation map $E(f) = f(0)$, which is linear, and let $U_n : C^s(I,I) \to C^s(I)$ be as before. Let us write $DT^m(f+w)v - DT^m(f)(v) = E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$, where

$$E_1 = (A(f+w) - A(f)) \sum_{j=0}^{p-1} B_j(f+w) \cdot C_j(f+w)$$

$$E_2 = A(f) \sum_{j=0}^{p-1} (B_j(f+w) - B_j(f)) \cdot C_j(f+w)$$

$$E_3 = A(f) \sum_{j=0}^{p-1} B_j(f) \cdot (C_j(f+w) - C_j(f))$$

$$E_4 = (A(f+w) - A(f)) \cdot D(f+w) \sum_{j=0}^{p-1} E \circ B_j(f+w) \cdot E \circ C_j(f+w)$$

$$E_5 = A(f) \cdot (D(f+w) - D(f)) \sum_{j=0}^{p-1} E \circ B_j(f+w) \cdot E \circ C_j(f+w)$$

$$E_6 = A(f) \cdot D(f) \sum_{j=0}^{p-1} (E \circ B_j(f+w) - (E \circ B_j(f)) \cdot E \circ C_j(f+w)$$

$$E_7 = A(f) \cdot D(f) \sum_{j=0}^{p-1} E \circ B_j(f) \cdot (E \circ C_j(f+w) - E \circ C_j(f)) \ .$$

By Lemma 8.11, we get

$$|A(f+w) - A(f)| = \left|\lambda_{f+w}{}^{-1} - \lambda_f{}^{-1}\right|_{C^t} \leq K_1 \|w\|_{C^t} \ . \tag{8.4.8}$$

Hence,

$$\|E_1\|_{C^t} \leq K_2 \|w\|_{C^t} \|v\|_{C^t} \qquad \text{and} \qquad \|E_4\|_{C^t} \leq K_3 \|w\|_{C^t} \|v\|_{C^t} \ .$$

By Proposition 8.6 and Lemma 8.18, we obtain

$$\begin{aligned}\|B_j(f+w)-B_j(f)\|_{C^t} &\leq K_4\|U_{p-j}(f+w)-U_{p-j}(f)\|_{C^t}\\ &\leq K_5\|w\|_{C^t}\ .\end{aligned} \tag{8.4.9}$$

Since $E$ is a bounded linear operator and from the last inequality, we obtain

$$\|E_2\|_{C^t} \leq K_6\|w\|_{C^t}\|v\|_{C^t} \quad \text{and} \quad \|E_6\|_{C^t} \leq K_7\|w\|_{C^t}\|v\|_{C^t}\ .$$

Taking $j=p$ in (8.4.9), we get

$$\|B_p(f+w)-B_p(f)\|_{C^t} \leq K_8\|w\|_{C^t}\ .$$

By Lemma 8.18 and by (8.4.8), we get

$$\|\lambda_{f+w}\cdot U_p(f+w)-\lambda_f\cdot U_p(f)\|_{C^t} \leq K_9\|w\|_{C^t}\ .$$

Combining the last two inequalities, we get $\|E_5\|_{C^t} \leq K_{10}\|w\|_{C^t}\|v\|_{C^t}$ .

Let $k$ and $0<\alpha<1$ be the integer and the fractional part of $t$, and let $0<\epsilon<1$ be such that $\alpha-\epsilon>0$. From inequality (8.4.4), and since $E$ is a bounded linear operator, we get

$$|E\left(C_j(f+w)\right)-E\left(C_j(f)\right)| \leq K_{11}\|w\|^{\epsilon}_{C^t}\|v\|_{C^t}\ .$$

Thus, $\|E_7\|_{C^t} \leq K_{12}\|w\|^{\epsilon}_{C^t}\|v\|_{C^t}$ . The only thing left to do is to bound $\|E_3\|_{C^t}$, and this follows at once from Proposition 8.19. □

THEOREM 8.21. *If $r>2$ is not an integer, there exist $0<\mu<1$ and $C>0$ with the following property. For every $g\in\mathbb{K}$ and for every $m>0$, there is an $\eta>0$ such that for all $f\in\mathbb{O}_m$ with $\|f-g\|_{\mathbb{A}}<\eta$ and for all $v\in\mathbb{A}^r$ with $\|v\|_{C^r}<\eta$ we have*

$$\|T^m(f+v)-T^m(f)-DT^m(f)v\|_{C^r} \leq C\mu^m\|v\|_{C^r}\ . \tag{8.4.10}$$

This theorem together with Theorem 8.20 (i) for $t=r$ imply that the renormalization operator satisfies property **B6**.

*Proof.* In this proof the constants $\theta,\theta_1,\theta_2,\ldots$ are greater than zero and smaller than one and just depend upon $r$. The positive constants $c,c_1,c_2,\ldots$ depend only on $r$ and $\mathbb{O}_m$, and the positive constants $K,K_1,K_2,\ldots$ depend also on $m$. Start observing that since $T^m(f)=\lambda_f^{-1}\cdot U_p(f)$, we have $T^m(f+v)-T^m(f)-DT^m(f)v=A+B+C$, where

$$\begin{aligned}A&=\lambda_f^{-1}\cdot\left(U_p(f+v)-U_p(f)-DU_p(f)v\right)\\ B&=\left(\lambda_{f+v}^{-1}-\lambda_f^{-1}-D\lambda_f^{-1}(v)\right)\cdot U_p(f+v)\\ C&=D\lambda_f^{-1}(v)\cdot\left(U_p(f+v)-U_p(f)\right)\ .\end{aligned}$$

By Lemma 8.11, we have that $f\to\lambda_f^{-1}$ is $C^1$ and that there is $\theta_1$ such that $\|B\|_{C^r}\leq K_1\|v\|^{1+\theta_1}_{C^r}$. Since $\|U_p(f+v)-U_p(f)\|_{C^r}\leq K_2\|v\|_{C^r}$, we have also

$\|C\|_{C^r} \leq K_3\|v\|^2_{C^r}$. Hence inequality (8.4.10) will be established if we prove the following claim.

*Claim.* If $r > 2$ there exist $0 < \mu < 1$ and $c_1 > 0$ with the following property: for every $g \in \mathbb{K}$ and for every $m$, there is an $\eta > 0$ such that for all $f \in \mathbb{O}_m$ with $\|f - g\|_{\mathbb{A}} < \eta$ and for all $v \in \mathbb{A}^r$ with $\|v\|_{C^r} < \eta$ we have

$$\|U_p(f+v) - U_p(f) - DU_p(f)v\|_{C^r} \leq c_1\mu^m|\lambda_f|\|v\|_{C^r} \ . \tag{8.4.11}$$

To prove this claim, we will proceed recursively. Let us write for $i = 0, \ldots, p$,

$$R_i = U_i(f+v) - U_i(f) - DU_i(f)v \ .$$

Note that $R_{i+1} = E_i + F_i + f' \circ U_i(f) \cdot R_i$, where

$$E_i = f \circ U_i(f+v) - f \circ U_i(f) - f' \circ U_i(f) \cdot (U_i(f+v) - U_i(f))$$
$$F_i = v \circ U_i(f+v) - v \circ U_i(f) \ .$$

Thus, working recursively from these expressions, we get

$$R_p = R_0 \cdot G_p + \sum_{i=0}^{p-1}(E_i \cdot G_{p-i-1} + F_i \cdot G_{p-i-1}) \ ,$$

where $G_{p-i-1} = (f^{p-i-1})' \circ U_{i+1}(f)$ and $R_0 = \Lambda_{f+v} - \Lambda_f - D\Lambda_f(v)$. Since $f \in \mathbb{O}_m$, by Proposition 8.6 and Lemma 8.18, we get

$$\|E_i\|_{C^r} \leq K_4\|U_i(f+v) - U_i(f)\|^{1+\theta_2}_{C^r} \leq K_5\|v\|^{1+\theta_2}_{C^r}$$

for $\theta_2 = 1 - \{r\}$. Therefore, $\left\|\sum_{i=0}^{p-1} E_i \cdot G_{p-i-1}\right\|_{C^r} \leq K_6\|v\|^{1+\theta_2}_{C^r}$. By Lemma 8.11, there is $\theta_3$ such that $\|R_0\|_{C^r} \leq K_7\|v\|^{1+\theta_3}_{C^r}$. Hence, $\|R_0 \cdot G_p\|_{C^r} \leq K_8\|v\|^{1+\theta_3}_{C^r}$. Finally, by Proposition 8.19, there exists $\theta_4 > 0$ such that

$$\left\|\sum_{i=0}^{p-1} F_i \cdot G_{p-i-1}\right\|_{C^r} \leq K_9\|v\|^{1+\theta_4}_{C^r} + c_2\mu^m|\lambda_f|\|v\|_{C^r} \ .$$

This proves our original claim. □

## 8.5. Proof of Theorem 8.1

All the pieces of the puzzle may now be put together. We want to check robustness of $T$ relative to the spaces $\mathcal{A} = \mathbb{A}$, $\mathcal{B} = \mathbb{A}^r$, $\mathcal{C} = \mathbb{A}^s$ and $\mathcal{D} = \mathbb{A}^0$. By Theorem 5.1, the pair $(\mathbb{A}^\gamma, \mathbb{A}^0)$ is $\rho_\gamma$-compatible with $(T, \mathbb{K})$ and $\rho_\gamma < \lambda$ for $\gamma$ sufficiently close to 2 and is 1-compatible for $\gamma > 2$. Hence property **B1** is satisfied because $s > s_0$ with $s_0 < 2$ close to 2 and $r > s+1 > 2$. Since $r > s+1$, we know from Lemma 8.12 that $T$ satisfies property **B2**. It also satisfies property **B3** by Lemma 8.13, and property **B4** by Lemma

8.16. Finally, $T$ satisfies property **B5** by Theorem 8.20, and property **B6** by Theorem 8.21. Therefore the renormalization operator $T$ is indeed robust with respect to $(\mathbb{A}^r, \mathbb{A}^s, \mathbb{A}^0)$.

## 8.6. Proof of hyperbolic picture

Having established that the renormalization operator $T$ is robust, we are now ready to show that the hyperbolic picture holds true for $T$ acting on each of the spaces $\mathbb{U}^r$ and $\mathbb{V}^r$.

### 8.6.1. Proof of Theorem 2.5

We divide the proof of Theorem 2.5 into two cases: (i) $r$ is not an integer (including the Lipschitz case $r = k + \mathrm{Lip}$), and (ii) $r = k$ is an integer.

*Proof of case (i):* Putting together Theorem 6.1 with Theorem 8.1 we deduce all the assertions of Theorem 2.5 except the fact that the holonomies are $C^{1+\beta}$ for some $\beta > 0$. This last fact follows if we combine Theorem 7.1 with Example 7.1.

*Proof of case (ii):* To prove this case, let us consider the Banach space $\mathbb{A}^{k-1+\mathrm{Lip}}$. Note that the natural inclusion $i : \mathbb{A}^k \to \mathbb{A}^{k-1+\mathrm{Lip}}$ is an isometric embedding. Indeed, for all $v \in \mathbb{A}^k$ we have $\|v\|_{C^k} = \|v\|_{\mathcal{B}}$, by the mean-value theorem. Applying case (i) to $\mathbb{A}^{k-1+\mathrm{Lip}}$, we see that for every $g \in \mathbb{K}$ the local stable set $W_\varepsilon^{s,k-1+\mathrm{Lip}}(g)$ is a codimension one $C^1$ Banach submanifold of $\mathbb{A}^{k-1+\mathrm{Lip}}$. In fact, there exists a $C^1$ function $\Phi : \mathcal{O}_0 \to \mathbb{R}$, where $\mathcal{O}_0 \subseteq \mathbb{A}^{k-1+\mathrm{Lip}}$ is an open set containing $g$, such that $0 \in \mathbb{R}$ is a regular value for $\Phi$, with

$$\Phi^{-1}(0) = \mathcal{O}_0 \cap W_\varepsilon^{s,k-1+\mathrm{Lip}}(g)$$

and such that $D\Phi(g)\mathbf{u}_g \neq 0$. Let $\mathcal{O}_1 = i^{-1}(\mathcal{O}_0) \subseteq \mathbb{A}^k$. Then $\mathcal{O}_1$ is open and $\Phi \circ i : \mathcal{O}_1 \to \mathbb{R}$ is $C^1$. Since $\mathbf{u}_g \in \mathbb{A}^k$ and $D(\Phi \circ i)(g)\mathbf{u}_g = D\Phi(g)\mathbf{u}_g \neq 0$, it follows that $0 \in \mathbb{R}$ is a regular value for $\Phi \circ i$ at $g$. Hence, by the implicit function theorem,

$$\mathcal{O}_1 \cap W_\varepsilon^{s,k}(g) = \mathcal{O}_1 \cap W_\varepsilon^{s,k-1+\mathrm{Lip}}(g) = \mathcal{O}_1 \cap (\Phi \circ i)^{-1}(0) \ ,$$

is a $C^1$, codimension one Banach submanifold of $\mathbb{A}^k$. Since by case (i) the local stable manifolds in $\mathbb{A}^{k-1+\mathrm{Lip}}$ form a continuous lamination, we deduce that the same is true for the local stable manifolds in $\mathbb{A}^k$, because $i$ is an isometric embedding. Finally, if $F$ is a $C^2$ ordered transversal (in the sense of §7) to the stable lamination in $\mathbb{U}^k$, then $i \circ F$ is a $C^2$ ordered transversal to the stable lamination in $\mathbb{A}^{k-1+\mathrm{Lip}}$, and therefore by case (i) its holonomy in $\mathbb{U}^{k-1+\mathrm{Lip}}$ is $C^{1+\theta}$ for some $\theta > 0$. But then it follows that the holonomy of the transversal $F$ in $\mathbb{U}^k$ is $C^{1+\theta}$ also.

### 8.6.2. Proof of Corollary 2.6

A similar argument to the one used in the proof of Theorem 2.5 can be used here. The map $i$ is replaced throughout by the inclusion $j : \mathbb{B}^r \to \mathbb{A}^r$, which is a bounded linear operator (see §2.1). Hence the pre-images by $j$ of the local stable leaves in $\mathbb{U}^r$ are $C^1$ manifolds and form a $C^0$ lamination in $\mathbb{V}^r$. Using [24] (see Remark 9.1 below), we see that the leaves of such lamination contain the local stable sets of each $g \in \mathbb{K}$ in $\mathbb{V}^r$.

## 9. Global stable manifolds and one parameter families

In this section, we prove Theorem 2.7. The first part will follow from Theorem 9.1 and the second part will follow from Theorem 9.2.

### 9.1. The global stable manifolds of renormalization

In this section we construct the global stable manifolds of the renormalization operator $T$ in $\mathbb{V}^r$, for all $r$ sufficiently large.

Let $g$ be an element of the (bounded-type) invariant set $\mathbb{K}$ of $T$. Recall that the global stable set $W^{s,r}(g)$ of $g \in \mathbb{V}^r$ is given by

$$W^{s,r}(g) = \{f \in \mathbb{V}^r : \|T^n(f) - T^n(g)\|_{C^r} \to 0 \quad \text{when} \;\; n \to \infty\} \; .$$

From Corollary 2.6, we know that the convergence is exponential, and the exponential rate of convergence is independent of $f$ and $g$, provided $r \geq 2 + \alpha$ with $0 < \alpha < 1$ close to one.

THEOREM 9.1. *For every $r \geq 3+\alpha$ with $\alpha < 1$ sufficiently close to 1, and every $g \in \mathbb{K}$, the global stable set $W^{s,r}(g)$ is an immersed, codimension one $C^1$ Banach submanifold of $\mathbb{V}^r$.*

REMARK 9.1. *By [24], if the invariant set $\mathbb{K}$ of the renormalization operator is of bounded type then for every $r \geq 3$ and every $g \in \mathbb{K}$ we have that $W^{s,r}(g)$ coincides with the set of all maps $f \in \mathbb{V}^r$ with the same combinatorial type of $g$.*

*Proof.* We already know that the local stable sets are $C^1$ submanifolds. The idea is to pull-back such manifold structure by $T$ using the implicit function theorem. More precisely, by Corollary 2.6 there exist $\varepsilon, \beta > 0$ so small that $W_\varepsilon^{s,r-1-\beta}(g)$ is a codimension one $C^1$ Banach submanifold of $\mathbb{V}^{r-1-\beta}$, for all $g \in \mathbb{K}$. We may assume that $\varepsilon > 0$ is so small that the vector $\mathbf{u}_g$ is transversal to the local stable set $W_\varepsilon^{s,r-1-\beta}(g)$ at each one of its points.

Now fix $g \in \mathbb{K}$ and let $f \in W^{s,r}(g)$. There exists $N = N(f) > 0$ so large that

$$T^N(f) \in W_\varepsilon^{s,r}(T^N(g)) \subset W_\varepsilon^{s,r-1-\beta}(T^N(g)) \; .$$

Since $v = \mathbf{u}_{T^N(g)}$ is transversal at $T^N(f)$ to $W_\varepsilon^{s,r-1-\beta}(T^N(g))$, There exist a small open set $\mathbb{O}_0 \subset \mathbb{V}^{r-1-\beta}$ containing $T^N(f)$ and a $C^1$ function $\Phi : \mathbb{O}_0 \to \mathbb{R}$ such that $\Phi^{-1}(0) = W_\varepsilon^{s,r-1-\beta}(T^N(g)) \subset \mathbb{O}_0$ for which $0 \in \mathbb{R}$ is a regular value and $D\Phi(T^N(f))v \neq 0$. The operator $T^N$ is $C^1$ as a map from $\mathbb{V}^r$ into $\mathbb{V}^{r-1-\beta}$. Let $\mathbb{O}_1 \subset \mathbb{V}^r$ be an open set containing $f$ such that $T^N(\mathbb{O}_1) \subset \mathbb{O}_0$. We want to show that $0 \in \mathbb{R}$ is a regular value for $\Phi \circ T^N : \mathbb{O}_1 \to \mathbb{R}$. Defining $F_t = T^N(f) + tv$ (for $|t|$ small), we get a $C^1$ family $\{F_t\}$ of maps in $\mathbb{V}^r$ which is transversal to $W_\varepsilon^{s,r-1-\beta}(T^N(g))$ at $F_0 = T^N(f)$. Now, we have the following claim.

*Claim.* There exists a $C^1$ family $\{f_t\}$ with $f_t \in \mathbb{V}^r$ such that for all small $t$ we have $T^N(f_t) = F_t$.

Let us assume this claim for a moment. Setting

$$w = \left.\frac{d}{dt}\right|_{t=0} f_t \ ,$$

we obtain that

$$D(\Phi \circ T^N)(f)w = D\Phi(F_0)v \neq 0 \ .$$

Therefore, $\Phi \circ T^N$ is a $C^1$ local submersion at $f$. By the implicit function theorem $(\Phi \circ T^N)^{-1}(0)$ is a codimension one, $C^1$ Banach submanifold of $\mathbb{O}_1$ (or $\mathbb{V}^r$). Furthermore, if $h \in (\Phi \circ T^N)^{-1}(0)$ then $T^N(h) \in W_\varepsilon^{s,r-1-\beta}(T^N(g))$, and so $h$ belongs to the global stable set $W^{s,r-1-\beta}(g)$. Using [24] (see Remark 9.1), we deduce that $h$ belongs in fact to $W^{s,r}(g)$. This proves that $W^{s,r}(g)$ is an immersed $C^1$ manifold as asserted.

It remains to prove the claim. We first note that $F_t = h_t \circ F_0$ where each $h_t \in C^r(I, I)$ is a $C^r$ diffeomorphism of $I = [-1, 1]$. Since $T^N(f) = F_0$, there exist $p > 0$ and closed, pairwise disjoint intervals $0 \in \Delta_0, \Delta_1, \ldots, \Delta_{p-1} \subseteq I$ with $f(\Delta_i) \subseteq \Delta_{i+1}$ for $0 \leq i < p-1$ and $f(\Delta_{p-1}) \subseteq \Delta_0$, such that

$$F_0 = T^N(f) = \Lambda_f^{-1} \circ f^p \circ \Lambda_f \ ,$$

where $\Lambda_f : I \to \Delta_0$ is the map $x \mapsto f^p(0)x$. Let $\overline{h}_t : \Delta_0 \to \Delta_0$ be the $C^r$ diffeomorphism given by $\overline{h}_t = \Lambda_f \circ h_t \circ \Lambda_f^{-1}$. Consider a $C^r$ extension of $\overline{h}_t$ to a diffeomorphism $H_t : I \to I$ with the property that $H_t|\Delta_i$ is the identity for all $i \neq 0$. Then let $f_t \in \mathbb{V}^r$ be the map $f_t = H_t \circ f$. Note that $f_t^i(0) = f^i(0)$ for all $0 \leq i \leq p$, that $f_t$ is $N$-times renormalizable (under $T$) and that

$$\begin{aligned}
T^N(f_t) &= \Lambda_f^{-1} \circ f_t^p \circ \Lambda_f \\
&= \Lambda_f^{-1} \circ (H_t \circ f)\,|_{\Delta_{p-1}} \circ (H_t \circ f)\,|_{\Delta_{p-2}} \circ \ldots \circ (H_t \circ f)\,|_{\Delta_0} \circ \Lambda_f \\
&= \Lambda_f^{-1} \circ \overline{h}_t \circ f^p \circ \Lambda_f \\
&= \Lambda_f^{-1} \circ \Lambda_f \circ h_t \circ \left(\Lambda_f^{-1} \circ f^p \circ \Lambda_f\right) \\
&= h_t \circ F_0 \\
&= F_t \ ,
\end{aligned}$$

which proves the claim. □

## 9.2. One-parameter families

A one-parameter family of maps is a map $\psi : [0,1] \times I \to I$ (where $I = [-1,1]$ is the phase space) such that $\psi_t = \psi(t,\cdot)$ belongs to $\mathbb{V}^r$ for all $t \in [0,1]$. If $\psi$ is a $C^k$ map, then we say that $\psi$ is a $C^k$ family (of $C^r$ unimodal maps). We often identify the family $\psi$ with the curve $\{\psi_t\}_{0\leq t\leq 1}$ of unimodal maps in $\mathbb{V}^r$. We shall denote by $\mathcal{UF}^k$ the space of all $C^k$ families with the $C^k$ topology ($\mathcal{UF}^k$ is a subset of $C^k([0,1]\times I)$).

We say that two families are $C^{1+}$ equivalent if there exist a diffeomorphism from one into the other which sends each infinitely renormalizable map (with a fixed bounded combinatorial type) to a map with the same combinatorics. We are now in a position to state the result we have in mind.

THEOREM 9.2. *Let $r \geq 3+\alpha$ with $\alpha > 0$ close to 1, and let $2 \leq k \leq r$. There exists an open and dense subset $\mathcal{O} \subseteq \mathcal{UF}^k$ of one-parameter $C^k$ families of $C^r$ unimodal maps having the following properties:*

*(i) Every family $\psi \in \mathcal{O}$ intersects the global stable lamination $\mathcal{L}^s$ of renormalization transversally.*

*(ii) For every $\psi \in \mathcal{O}$, there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ such that for each $i = 0,1,\ldots,n-1$ the sub-arc $\{\psi_t : t_i \leq t \leq t_{i+1}\}$ is $C^{1+\beta}$ diffeomorphic, via a holonomy-preserving diffeomorphism, to a corresponding sub-arc in the quadratic family. Here $\beta > 0$ is given by Corollary 2.6.*

The proof will require a few lemmas. The first Lemma says that every $C^k$ family can be approximated (in the $C^k$ sense) by a real analytic family.

LEMMA 9.3. *If $\psi \in \mathcal{UF}^k$, then for each $\varepsilon > 0$ there exists a real analytic family $f \in \mathcal{UF}^\omega$ such that $\|\psi - f\|_{C^k([0,1]\times I)} < \varepsilon$.*

*Proof.* Write each $\psi_t \in \mathbb{V}^r$ as $h_t \circ q$, where $q(x) = x^2$ and $h_t$ is a diffeomorphism, and consider the $C^k$ map $h : [0,1]\times I \to I$ given by $h(t,x) = h_t(x)$, a $C^k$ family of $C^r$ diffeomorphisms. To approximate $h$ by a real analytic family of diffeomorphisms, consider the convolution of $h$ with the heat kernel $k(t,x,\varepsilon) = e^{-(t^2+x^2)/4\varepsilon}$ for $\varepsilon > 0$ sufficiently small (see [1]). □

Given this "denseness" result, the idea will be to show that arbitrarily close to an $f$ as in Lemma 9.3 we can find a $C^k$ family which is also transversal to the global stable lamination $\mathcal{L}^s$ of renormalization, by some kind of perturbation argument, to eliminate possible tangencies between $\{f_t\}$ and $\mathcal{L}^s$.

We will reduce our problem to the following general result about laminations with complex analytic leaves, whose elegant proof is due to Douady.

LEMMA 9.4. *Let $\mathcal{L} \subseteq \mathbb{C}^2$ be a $C^0$ lamination whose leaves are complex one-manifolds, and let $F : \mathbb{D} \to \mathbb{C}$ be a holomorphic function whose graph is tangent of finite order at $(0, F(0))$ to a leaf $L_0 \in \mathcal{L}$. Then the tangency is isolated: there exists a neighborhood of $(0, F(0))$ in $\mathbb{C}^2$ on which every other intersection of the graph of $F$ with the leaves of $\mathcal{L}$ is a transversal intersection.*

*Proof.* Using a suitable chart, we may assume that the leaf $L_0$ is the horizontal plane $w = 0$ in $\mathbb{C}^2$, and that the other leaves of $\mathcal{L}$ in that chart are the graphs of holomorphic functions $\varphi_\mu : \mathbb{D} \to \mathbb{C}$ (with $\varphi_\mu(0) = \mu \in D$, where $D \subseteq \mathbb{C}$ is some open disk around zero, and $\varphi_0 \equiv 0$).

Since $\mathcal{L}$ is a $C^0$ lamination, $\varphi_\mu$ converges to 0 uniformly in $\mathbb{D}$ as $\mu$ tends to 0. Hence, for $|\mu|$ small enough, we have $\varphi_\mu(\mathbb{D}) \subset \mathbb{D}$. Moreover, $\varphi_\mu(z) \neq 0$ for all $z \in \mathbb{D}$ (leaves cannot intersect), so in fact $\varphi_\mu(\mathbb{D}) \subset \mathbb{D}^*$.

Now, we have $F(0) = F'(0) = \ldots = F^{(k-1)}(0) = 0 \neq F^{(k)}(0)$, for some $k \geq 2$. Composing the chart with a bi-holomorphic map if necessary, we may therefore assume that $F(z) = z^k$.

Let us fix $\mu \in D \setminus \{0\}$ and suppose that $z_0 \in \mathbb{D}$ is such that $\varphi_\mu(z_0) = F(z_0)$. We assume that $|z_0| < 1/2$ (taking $|\mu|$ small enough). To show that this intersection between $\varphi_\mu$ and $F$ is transversal, it suffices to show that $\varphi'_\mu(z_0) \neq F'(z_0)$. But, by Schwarz's Lemma, the derivative $\varphi'_\mu(z_0)$ measured with respect to the Poincaré metrics of domain $\mathbb{D}$ and range $\mathbb{D}^*$ must be less than or equal to 1, that is to say

$$\left\|\varphi'_\mu(z_0)\right\|_P = \frac{\left|\varphi'_\mu(z_0)\right| \left(1 - |z_0|^2\right)}{|\varphi_\mu(z_0)| \log\left(|\varphi_\mu(z_0)|^{-1}\right)} \leq 1 \ .$$

Thus, we have

$$\left|\varphi'_\mu(z_0)\right| \leq \frac{4}{3} k |z_0|^k \log\left(|z_0|^{-1}\right) \ .$$

On the other hand,

$$\left|F'(z_0)\right| = k|z_0|^{k-1} \ .$$

This shows that $\left|\varphi'_\mu(z_0)\right| / |F'(z_0)|$ converges to 0 as $\mu$ tends to 0, whence $\varphi'_\mu(z_0) \neq F'(z_0)$ for all sufficiently small $|\mu|$. Therefore $(0, F(0))$ is an isolated tangency as claimed. □

We may now state and prove the result on laminations with real analytic leaves which is needed for the proof of Theorem 9.2.

LEMMA 9.5. *Let $\mathcal{F} \subseteq [a, b] \times \mathbb{R}$ be a $C^0$ foliation whose leaves are the graphs of real analytic functions $\varphi_\mu : [a, b] \to \mathbb{R}$ with, say, $\varphi_\mu(a) = \mu \in [0, 1]$.*

*Let $\mathcal{L} \subseteq \mathcal{F}$ be a sub-lamination which is transversally totally disconnected (i.e. $K_0 = \{\mu \in [0,1] : gr(\varphi_\mu) \subseteq \mathcal{L}\}$ is a totally disconnected set). If $F : [a,b] \to \mathbb{R}$ is a real analytic function, then*

*(i) $gr(F)$ is tangent to $\mathcal{F}$ at only finitely many points;*

*(ii) for all $\varepsilon > 0$ and all $k \geq 0$, there exists a real analytic $G : [a,b] \to \mathbb{R}$ such that $\|F - G\|_{C^k} < \varepsilon$ and all tangencies of $gr(G)$ with $\mathcal{F}$ belong to $\mathcal{F} \setminus \mathcal{L}$; in particular, $gr(G)$ is transversal to $\mathcal{L}$.*

*Proof.* (i) Complexifying $\mathcal{F}$ (i.e. the leaves $\varphi_\mu$) as well as $F$, we put ourselves in the situation of Lemma 9.4. All tangencies are therefore isolated, and since $[a,b]$ is compact, there are only finitely many such, say at $x_i \in [a,b]$, $i = 1, 2, \ldots, n$.
(ii) Let $d_i$ be the order of tangency of $F$ with $\mathcal{F}$ at $(x_i, F(x_i))$. Then for every real analytic $G$ sufficiently close to $F$ in the $C^k$ topology with $k$ large ($k \geq \sum_{i=1}^n d_i$ will do), the number $n(G)$ of tangencies of $gr(G)$ with $\mathcal{F}$ – not counting multiplicies – is bounded by $\sum_{i=1}^n d_i$. Hence we can find $G_0 : [a,b] \to \mathbb{R}$ real analytic with $\|F - G_0\|_{C^k} < \varepsilon/2$ such that $n(G_0)$ is maximal. All tangencies of $G_0$ with $\mathcal{F}$ must be first-order tangencies ($d_i = 1$). Indeed, if, say, $d_1 > 1$, then adding a suitable polynomial with small $C^k$ norm to $G_0$, vanishing of very high order at $x_2, x_3, \ldots, x_{n(G_0)}$, we could unfold the tangency at $x_1$ to produce a new real analytic $G$ with $n(G) > n(G_0)$. Now we may consider $G_t : [a,b] \to \mathbb{R}$ given by $G_t(x) = G_0(x) + t$ for $|t| < \varepsilon/2$. Since first-order tangencies are persistent, each tangency $(x_i, G_0(x_i))$ of $G_0$ with $\mathcal{F}$ generates a continuous, non-constant path $(x_i(t), G_t(x_i(t))) \in gr(\varphi_{\mu_i(t)})$ of (first-order) tangencies of $G_t$ with $\mathcal{F}$. Each function $t \mapsto \mu_i(t)$, $i = 1, 2, \ldots, n(G_0)$, is continuous and non-constant. Since $K_0$ is totally disconnected, there exists $t$ (with $|t| < \varepsilon/2$) such that $\mu_i(t) \in [0,1] \setminus K_0$ for all $i$. Therefore, all tangencies of $G_t$ with $\mathcal{F}$ fall in $\mathcal{F} \setminus \mathcal{L}$, whence $G_t$ is transversal to $\mathcal{L}$. □

*Proof of Theorem 9.2.* Both properties (i) and (ii) are easily seen to be open, hence we concentrate in proving that they are dense. Let $\varepsilon > 0$.

Take any family $\psi \in \mathcal{UF}^k$. By Lemma 9.3, there exists a real analytic family $f \in \mathcal{UF}^\omega$ whose $C^k$ distance from $\psi$ is less than $\varepsilon/2$. The corresponding curve $\{f_t\}$ in $\mathbb{V}^r$ may fail to be transversal to the global stable lamination $\mathcal{L}^s$, so let us show how to perturb it locally to get a transversal family. Let $t_0 \in [0,1]$ be such that $f_{t_0} \in \mathcal{L}^s$ (and $\{f_t\}$ is tangent to $\mathcal{L}^s$ at $f_{t_0}$). Since $f_{t_0}$ is infinitely renormalizable and real analytic, there exists $N > 0$ such that $R^N(f_{t_0}) \in \mathbb{A}_{\Omega_a}$ (where $a > 0$ is the constant in Theorem 2.4). Let $J \subseteq [0,1]$ be an interval containing $t_0$ such that $R^N(f_t)$ is well-defined and belongs to $\mathbb{A}_{\Omega_a}$ for all $t \in J$. We restrict our attention to the sub-family $\{f_t\}_{t \in J}$ from now on.

First we embed $\{f_t\}_{t\in J}$ in a two-parameter family in the following way. Note that each $f_t$ belongs to $\mathbb{A}_{\Omega_\alpha}$ for some (fixed) $\alpha > 0$. As a map from (an open subset of) $\mathbb{A}_{\Omega_\alpha}$ into $\mathbb{A}_{\Omega_a}$, $R^N$ is a real analytic operator.

*Claim.* There exist analytic vectors $v \in \mathbb{A}_{\Omega_\alpha}$ and $w \in \mathbb{A}_{\Omega_a}$ with the property that $DR^N(f_{t_0})v = w$ and $w$ is transversal to $\mathcal{L}_a^s = \mathcal{L}^s \cap \mathbb{A}_{\Omega_a}$ at $R^N(f_{t_0}) \in \mathcal{L}_a^s$.

To see this, take any $w_0 \in \mathbb{A}_{\Omega_a}$ transversal to the (co-dimension one) lamination $\mathcal{L}_a^s$ at $R^N(f_{t_0})$. The same construction used in the proof of Theorem 9.1 yields a $C^\infty$ vector $v_0$ at $f_{t_0}$ such that $DR^N(f_{t_0})v_0 = w_0$. Now approximate $v_0$ by an analytic vector $v \in \mathbb{A}_{\Omega_\alpha}$ (in the $C^m$ sense for $m \geq r$). Then $w = DR^N(f_0)v$ will still be transversal to $\mathcal{L}_a^s$. Shrinking $J$ if necessary, we may in fact assume that $DR^N(f_t)v$ is transversal to $\mathcal{L}_a^s$ for all $t \in J$. Hence, let us consider the two-parameter family of maps $f_{t,s} \in \mathbb{A}_{\Omega_\alpha}$ given by $f_{t,s} = f_t + sv$ with $t \in J$ and $|s| \leq \delta$ with $\delta$ small. We have

$$W = \{f_{t,s} : \ t \in J, \ s \in [-\delta, \delta]\} \cong J \times [-\delta, \delta] \subseteq \mathbb{R}^2 \ ,$$

and $R^N|W : W \to A_{\Omega_a}$ is an injective, real analytic map. Recall now that in $\mathbb{A}_{\Omega_a}$ we have a $C^0$ foliation $\mathcal{F}$ with real analytic leaves (coming from hybrid classes, *cf.* §3) and that $\mathcal{L}_a^s \subseteq \mathcal{F}$ is the sub-lamination corresponding to the stable leaves of renormalization, which is transversally totally disconnected. Taking $\mathcal{F}_W = R^{-N}(\mathcal{F}) \subseteq W$ and $\mathcal{L}_W^s = R^{-N}(\mathcal{L}_a^s) \subseteq W$ and noting that $DR^N(f_{t,s})v = w$ is transversal to $\mathcal{L}_a^s$ for all $t \in J$, $s \in [-\delta, \delta]$ (making $\delta$ smaller if necessary) we deduce that $\mathcal{F}_W$ is a $C^0$ foliation (in $W$) by real analytic curves, and $\mathcal{L}_W^s \subseteq \mathcal{F}_W$ is a sub-lamination. Therefore we can apply Lemma 9.5 to this situation (with $\mathcal{F} = \mathcal{F}_W$ and $\mathcal{L} = \mathcal{L}_W^s$), obtaining a new analytic curve $\{g_t\}_{t\in J}$ with $\|f_t - g_t\|_{C^k} < \varepsilon/2$, transversal to $\mathcal{L}_W^s$ in $W$, and such that $\{R^N(g_t)\}$ is transversal to $\mathcal{L}_a^s$ at $R^N(g_{t_0})$. Since by Corollary 2.6 the holonomy of $\mathcal{L}_a^s$ is $C^{1+\beta}$ for some $\beta > 0$ (and the quadratic family is transversal to $\mathcal{L}_a^s$) we deduce that $\{g_t\}$ satisfies properties (a) and (b) of the statement. This completes the proof. □

## 10. A short list of symbols

For the reader's convenience, we present below a short list of symbols used in this paper.

| | |
|---|---|
| $p$ | Period of renormalization |
| $\lambda_f$ | Scaling factor $\lambda_f = f^p(0)$ |
| $\Lambda_f$ | Linear scaling $\Lambda_f : x \to f^p(0) \cdot x$ |
| $R$ | Renormalization operator $R^N f = \Lambda_f^{-1} \circ f^p \circ \Lambda_f$ |
| $\mathbb{K}$ | Bounded type limit set of $R$ |
| $p_k$ | Number of renormalization intervals at level $k$ |
| $\Delta_{j,k}(f)$ | Renormalization intervals at level $k$ $(0 \le j \le p_k - 1)$ |
| $\mathcal{I}_f$ | Post-critical set of $f$ |
| $\mathbb{A}_V$ | Real Banach space of continuous maps $\overline{V} \to \mathbb{C}$, holomorphic in $V,$ symmetric about real axis |
| $T = R^N : \mathbb{O} \to \mathbb{A}_{\Omega_a}$ | Real analytic operator for which $\mathbb{K} \subset \mathbb{O}$ is a hyperbolic basic set |
| $u_g(t)$ | Parametrization of local unstable manifold $W^u_\varepsilon(g)$ |
| $\mathbf{u}_g$ | Unit vector tangent to $W^u_\varepsilon(g)$ at $g$ |
| $\delta_g$ | Unique real number such that $DT(g)\mathbf{u}_g = \delta_g \mathbf{u}_{T(g)}$ |
| $\delta_g^{(n)}$ | The product $\delta_g \delta_{T(g)} \ldots \delta_{T^{n-1}(g)}$ |
| $L_f = DT(f)$ | Derivative of $T$ at $f$ |
| $\mathbb{V}^r$ | $C^r$ unimodal maps with quadratic critical point at 0 |
| $\mathbb{A}^r$ | Tangent space to unimodal maps contained in $\mathbb{V}^r$ |

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