Integrability on codimension one dominated splitting

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Abstract

We study the unique integrability of the center unstable subbundle of a codimension one dominated splitting

1 Introduction

The problem of (unique) integrability of a one dimensional distribution (or vector field) is an old problem that goes back to the 19th century. For a non-one dimensional distribution the problem has been solved by Frobenius who gave a necessary and sufficient condition for the integrability (see [AM] and [L]). Nevertheless this conditions is not easy to check. When the distribution is related to a dynamical systems $f : M \to M$ the integrability has been solved under some dynamic assumptions like hyperbolicity by many authors and proofs along the 20th century (see for instance [HPS]). More precisely, if $TM = E \oplus F$ is an invariant dominated decomposition under the tangent map Df and F has a uniform expanding behavior it follows that F is uniquely integrable (this is the so called strong stable manifold theorem for f^{-1}.) The problem of the integrability come out when the we consider the "central"distribution. In other words no condition for the unique integrability is known when F has not (a priori) a uniform hyperbolic behavior, and moreover, there exist examples where it fails to be integrable.

In this paper we deal with the case that the distribution F is one-dimensional. By Peano's Theorem (see [KF]) it is integrable, but we shall be concerned with the unique integrability (and, as it is well known, we can not expect the central distribution to be smooth, even Lipchitz.)

Before state our result let us recall some definitions. Let $f : M \to M$ be a diffeomorphisms. An f-invariant set Λ is said to have *dominated splitting* if we can decompose its tangent bundle in two invariant subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\|\|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0,$$

with $C > 0$ and $0 < \lambda < 1$.

We say that the dominated splitting is a *codimension one dominated splitting* if the dimension of F is one and we shall say that it is a *contractive* if E is a contractive subbundle, i.e., there exists $C > 0$ and $0 < \lambda < 1$ such that for any x and any n holds that $|Df^n_{|E_x}| < C\lambda^n$.

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A periodic point p is *semi-attractor* or *attractor* provided that the set of points y that verifies that $dist(f^n(p), f^n(y)) \to 0$ contains an open set in M.

Maim Theorem: *Let $f : M \to M$ be a C^r diffeomorphisms, $r \geqslant 1$, exhibiting a codimension one dominated splitting $TM = E \oplus F$ over the whole manifold. Then F is uniquely integrable provided one of the following conditions hold:*

1. *$\Omega(f) = M$ (where $\Omega(f)$ denote the non-wandering set of f).*
2. *The dominated splitting is contractive.*
3. *There are neither semi-attracting or attracting periodic points and f is C^r with $r \geqslant 2$.*

The paper is organized as follows: in section 2 we state a series of results prove somewhere else. In section 3 we prove a codimension one Denjoy Property regarding the existence of wandering intervals (a similar result has already been proved in dimension two and with some adjustments the proof works in the codimension one case). In the same section, we derive some consequences regarding the central unstable invariant manifolds. In the last section we conclude the proof of the maim theorem.

2 Preliminaries

Let $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^1(I_1, M)$ the set of C^1-embedding of I_1 on M, and denote by $Emb^1(I_1^{n-1}, M)$ the set of C^1-embedding of I_1^{n-1} on M, where n is the dimension of M.

Recall by [HPS] that codimension one dominated splitting implies the next lemma:

Lemma 2.1. *There exist two continuous functions $\varphi^{cs} : \Lambda \to Emb^1(I_1^{n-1}, M)$ and $\varphi^{cu} : \Lambda \to Emb^1(I_1, M)$ such that if define $W^{cs}_\epsilon(x) = \varphi^{cs}(x)I_\epsilon^{n-1}$ and $W^{cu}_\epsilon(x) = \varphi^{cu}(x)I_\epsilon$ the following properties holds:*

1. *$T_xW^{cs}_\epsilon(x) = E(x)$ and $T_xW^{cu}_\epsilon(x) = F(x)$,*
2. *for all $0 < \epsilon_1 < 1$ there exist ϵ_2 such that and*
$$f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x)).$$
3. *for all $0 < \epsilon_1 < 1$ there exist ϵ_2 such that and*
$$f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x)).$$

In particular, there exists $\delta = \delta(\epsilon_1)$ such that if $y \in W^{cu}_{\epsilon_1}(x)$ and $dist(f^{-j}(y), f^{-j}(x)) < \delta$ for $0 \leq j \leq n$ then $f^{-j}(y) \in W^{cu}_{\epsilon_1}(f^{-j}(x))$ for $0 \leq j \leq n$.

Corolary 2.0.1. *For any $0 < \gamma < 1$, there exists $\epsilon = \epsilon(\gamma)$ such that for $x \in \Lambda$ holds that*
$$\|Df^n_{/E(x)}\| \leq \gamma^n, \forall n \geq 0,$$
then follows that
$$W^{cs}_\epsilon(x) \subset W^s_\epsilon(x) = \{y : dist(f^n(x), f^n(y)) < \epsilon\, dist(f^n(x), f^n(y)) \to 0\}$$
i.e., the central stable manifold of size ϵ is in fact a stable manifold.

Sometimes, one needs the central manifold to be of class C^2. This is guaranteed, for C^2-diffeomorphisms, by the so called 2-domination: the splitting $E \oplus F$ is 2-dominated if there exists $0 < \sigma < 1$ such that

$$\|Df^n_{/E(x)}\| \|Df^{-1}_{/F(f^n(x))}\|^2 \leq C\sigma^n, \; n \geq 0.$$

Remark 2.0.1. *It follows that if f is a C^2 diffeomorphisms and Λ is a compact invariant manifold exhibiting a codimension one dominated splitting which is also 2-dominated then the map φ^{cu} in Lemma 2.1 is indeed a map $\varphi^{cu} : \Lambda \to Emb^2(I_1, M)$ (see [HPS] for details).*

The following result in [PS1] guarantee that a codimension one dominated splitting is 2-dominated:

Lemma 2.2. *Let f be a C^2 diffeomorphisms and let Λ be a compact invariant manifold exhibiting a codimension one dominated splitting. Then, there exists at most finitely many periodic attractors (sinks) in Λ such that any compact invariant set $\Lambda_0 \subset \Lambda$ and disjoint from these periodic attractors is 2-dominated.*

We will need also the following beautiful result form Pliss:

Lemma 2.3. Pliss's Lemma ([Pl]): *Given a diffeomorphisms f and $0 < \gamma_1 < \gamma_2$ there exist $N = N(\gamma_1, \gamma_2, f)$ and $c = c(\gamma_1, \gamma_2, f) > 0$ with the following property: given $x \in M$, a subspace $S \subset T_xM$ such that for some $n \geq N$ we have (denoting $S_i = Df^i(S)$)*

$$\prod_{i=0}^{n} \|Df_{/S_i}\| \leq \gamma_1^n$$

then there exist $0 \leq n_1 < n_2 < .... < n_l \leq n$ such that

$$\prod_{i=n_r}^{j} \|Df_{/S_i}\| \leq \gamma_2^{j-n_r}; \; r = 1, ..., l; \; n_r \leq j \leq n.$$

Moreover, $l \geq cn$.

The next lemma is a classical one about the existences of admissible neighborhood for sets having dominated splitting.

Lemma 2.4. *Let Λ be a set with dominated splitting. Then there exists a neighborhood V of Λ such that any compact invariant set in V has dominated splitting. This type of neighborhood is called an admissible neighborhood of Λ.*

3 Denjoy's Property

A C^r-arc is a C^r embedding of the interval $(-1, 1)$. We denote by $\ell(I)$ the length of a C^r-arc I.

Definition 3.1. *Let $f : M \to M$ be a C^r diffeomorphisms and let Λ be a compact invariant set having dominated splitting and let V be an admissible neighborhood (see lemma 2.4). Let U be an open set containing Λ such that $\overline{U} \subset V$. We say that a C^r-arc I in M is a δ-**E-arc** provided the next two conditions holds:*

1. $f^n(I) \subset U$, $n \geq 0$ *and* $\ell(f^n(I)) \leq \delta$ *for all* $n \geq 0$.

2. $f^n(I)$ *is always transverse to the* E*-direction.*

In other words, a δ-E-arc is an arc that does not growth in length in the future and always remains transversal to the E subbundle.

Related to a δ-E-arc I we can obtain the following result. Before to state it, choose $\lambda_2, \lambda_3;\ \lambda < \lambda^{\frac{1}{2}} < \lambda_2 < \lambda_3 < 1$.

Lemma 3.1. *There exists* $\delta > 0$ *such that given a* δ*-*E*-arc* I *follows that there exists a sequence of integers* $n_i \to \infty$ *such that*

$$\|Df^j_{/E(x)}\| < \lambda_2^j \text{ for all } j \geq 0\ x \in f^{n_i}(I). \tag{1}$$

Proof. First, we take n_i such that

$$\ell(f^{n_i}(I)) \geqslant \ell(f^j(I)),\ \ \forall j > n_i.$$

To avoid notation, we note the arcs

$$I_{n_i} = f^{n_i}(I).$$

Observe that this implies that for any $n_i > 0$ there is some $x_i \in I_{n_i}$ such that $\|Df^k_{/F(x_i)}\| \leq 1$ and since the iterates of I_{n_i} remain small (less than δ) it follows that there is β small such that for any $z \in f^{n_i}(I)$ then

$$||Df^j_{/F(z)}|| < (1+\beta)^k.$$

Using the domination property and β small, the thesis of the lemma holds (see page 987-988 of [PS1] for details).

$\square$

Lemma 3.2. *For any point* $x \in I_{n_i}$ *there is an stable manifold* $W^s_\epsilon(x)$ *of uniform size.*

The proof follows from corollary 2.0.1. This implies that we can consider the box

$$W^s_\epsilon(I_{n_i}) = \cup_{x \in I_{n_i}} W^s_\epsilon(x).$$

Definition 3.2. *We say that a* δ*-*E*-arc* I *is wandering if for any* n_i, n_j *satisfying (1) follows that*

$$W^s_\epsilon(I_{n_i}) \cap W^s_\epsilon(I_{n_j}) = \emptyset.$$

The next theorem characterize the dynamic of a δ-E-arc. More precisely, characterize the $\omega-$limit of I (noted with $\omega(I)$). The theorem is a more general version of proposition 3.1 in [PS1] and theorem 4.1.3 in [PS2] where the results are stated for surfaces diffeomorphisms. The proof has some similarities and here it is adapted to the case of codimension one dominated splitting.

Theorem 3.1. *Let* f *be a* C^r *diffeomorphisms,* $r \geq 1$*, and let* Λ *be a compact invariant set exhibiting a codimension one dominated splitting. There exists* δ_0 *such that if* I *is a* C^r δ*-*E*-interval with* $\delta \leq \delta_0$*, then one of the following properties holds:*

1. *$\omega(I) \subset \mathcal{C}$ where $\mathcal{C}$ is a periodic simple closed curve normally attracting and $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (where m is the period of $\mathcal{C}$) has irrational rotation number.*

2. *There exists a normally attracting periodic arc J such that $I \subset W^s(J)$ and f^k restricted to J (k being the period of J) is the identity map on J.*

3. *$\omega(I) \subset Per(f_{/V})$ where $Per(f_{/V})$ is the set of the periodic points of f in V. Moreover, one of the periodic points is either a semi-attracting periodic point or a attracting one.*

4. *Neither of the above and I is wandering.*

Proof. To conclude the proof it is enough to show that if there exist $n_i < n_j$ verifying (1) such that

$$W^s_\epsilon(I_{n_i}) \cap W^s_\epsilon(f^{n_j-n_i}(I_{n_i})) \neq \emptyset. \tag{2}$$

then either (1), (2) or (3) of theorem 3.1 hold.

Let $m = n_j - n_i$. If $\ell(f^{km}(I_{n_i})) \to 0$ as $k \to \infty$, then $\omega(I_{n_i})$ consist of a periodic orbit. Indeed, if $\ell(f^{km}(I_{n_i})) \to 0$, then $\ell(f^k(I_{n_i})) \to 0$ as $k \to \infty$. Let p be an accumulation point of $f^k(I_{n_i})$, that is, $f^{k_j}(I_{n_i}) \to p$ for some $k_j \to \infty$, and so, $f^{k_j+m}(I_{n_i}) \to f^m(p)$. But by the property we are assuming, i.e., $W^s_\epsilon(I_{n_i}) \cap W^s_\epsilon(f^{n_j-n_i}(I_{n_i})) \neq \emptyset$, we have $f^{k_j+m}(I_{n_i}) \to p$, implying that p is a periodic point. Thus, for any $x \in I_{n_i}$ we have that $\omega(x)$ consists only of periodic orbits, and so $\omega(x)$ is single periodic orbit p. Since $\ell(f^k(I_{n_i})) \to 0$ we conclude that $\omega(I_{n_i})$ is the orbit of the periodic point p. By the way we choose I_{n_i}, we have $f^{n_i}(I) \subset I_{n_i}$ and so $\omega(I)$ consists of a periodic orbit, as the thesis of the theorem requires.

On the other hand, if $\ell(f^{km}(I_{n_i}))$ does not goes to zero, we take a sequence k_j such that $f^{k_jm}(I_{n_i}) \to L$ for some arc L (which is at least C^1, and have F as its tangent direction). Now $f^{(k_j+1)m}(I_{n_i}) \to L'$ and $f^m(L) = L'$. Moreover, $L \cup L'$ is an interval (with F as its tangent direction). Let

$$J = \cup_{n \geq 0} f^{nm}(L).$$

We claim that there are only two possibilities: either J is an arc or a simple closed curve. To prove this, notice that $f^{nm}(L)$ is a δ-E-interval for any $n \geq 0$. In particular, for any $x \in J$ there exists $\epsilon(x)$ such that $W^{cs}_{\epsilon(x)}(x)$ is stable manifold for x, and so

$$W(J) = \bigcup_{x \in J} W^{cs}_{\epsilon(x)}(x)$$

is a neighborhood of J. $\Omega(f) = M$ We only have to show that, given $x \in J$, there exists a neighborhood $U(x)$ such that $U(x) \cap J$ is an arc. This implies that J is a simple closed curve or an interval. Thus, take $x \in J$, in particular $x \in f^{n_1m}(L)$. Take U an open interval, $x \in U \subset f^{n_1m}(L)$ and let $U(x)$ be a neighborhood of x such that $U(x) \subset W(J)$ and such $U(x) \cap L_1 \subset U$ where L_1 is any interval containing $f^{n_1m}(L)$, transverse to the E-direction and $|L_1| \leq 2\delta_0$ (this is always possible if δ_0 is small). Now let $y \in J \cap U(x)$. We have to prove that $y \in U$. There is n_2 such that $y \in f^{n_2m}(L)$. Since

$$f^{n_1m}(L) = \lim_j f^{k_jm+n_1m}(I_{n_i})$$

$$f^{n_2 m}(L) = \lim_j f^{k_j m + n_2 m}(I_{n_i})$$

and both have nonempty intersection with $U(x)$, we conclude that for some j follows that $f^{k_j m+n_1 m}(I_{n_i})$ and $f^{k_j m+n_2 m}(I_{n_i})$ are linked by a local stable manifold. Hence $f^{n_1 m}(L) \cup f^{n_2 m}(L)$ is an arc L_1 transverse to the E-direction with $\ell(L_1) \leq 2\delta_0$. Therefore $y \in U(x) \cap L_1 \subset U$ as we wish, completing the proof that J is an arc or a simple closed curve.

In case J is an arc, since $f^m(J) \subset J$, it follows that for any $x \in I$, $\omega(x)$ is a ω-limit point of a point in J, hence either (2) or (3) holds, completing the proof in this case. On the other hand, if J is a simple closed curve, which is of class C^1 because is normally hyperbolic (attractive), then we have two possibilities. If $f^m_{/J} : J \to J$ has rational rotation number, then we can see that $\omega(I_{n_i})$ consist of a union of periodic points, and the same happens to I. If $f^m_{/J} : J \to J$ has an irrational rotation number, then it is semiconjugated to an irrational rotation. Since we are assuming that there is not wandering interval, it follows that it is conjugated. Denoting $\mathcal{C} = J$, we have that $\omega(I)$ is as in the first property of the thesis of the theorem.

□

Corollary 3.1. *Let f be a C^r diffeomorphisms, $r \geq 1$, and let us assume that $\Omega(f) = M$ and there is a codimension one dominated splitting in the whole manifold. Then, there is not δ-E-interval provided δ small.*

Proof. From the fact that $\Omega(f) = M$ follows that there is not wandering δ-E-intervals. From theorem 3.1 it follows that the $\omega-$limit of a δ-E-interval it is either a periodic simple closed curve normally attracting, a semi-attracting periodic point or there exists a normally attracting periodic arc. In any case, it is contradicted that $\Omega(f) = M$.

□

Theorem 3.2. *Let f be a C^r diffeomorphisms, $r \geq 2$, and let Λ be a compact invariant set exhibiting a codimension one dominated splitting. There exists δ_0 such that if I is a C^r δ-E-interval with $\delta \leq \delta_0$, then either (1), (2) or (3) of theorem 3.1 hold.*

Proof. To prove the previous theorem, first we need a proposition that allows to compare the two dimensional volume of $W^s_\epsilon(J)$ with the one dimensional length of a δ-E-interval J that verifies that

$$\|Df^j_{/E(x)}\| < \lambda_2^j \text{ for all } j \geq 0 \ x \in J. \tag{3}$$

Proposition 3.1. *Let f be a $C^{1+\beta}$ diffeomorphisms, $\beta > 0$, and let Λ be a compact invariant set exhibiting a codimension one dominated splitting. There exists δ_0 and $K > 0$ such that if J is a C^r δ-E-interval with $\delta \leq \delta_0$ such that its $\omega-$limit is not a periodic sink and verifies (3) then*

$$K vol(W^s_\epsilon(J)) \geq \ell(J).$$

The proof of the proposition is postponed and we finish now proving theorem 3.2. In what follows we take the maximal sequences of positive integers $\{n_i\}$ such that for each n_i it is verified (1). Without loss of generality, we can assume that for each

n_i the arc I_{n_i} is the maximal δ-E-interval that contains $f^{n_i}(I)$. Let us assume that $W^s_\epsilon(I_{n_r}) \cap W^s_\epsilon(I_{n_j}) = \emptyset$ for every r, j, otherwise, arguing as in theorem 3.1 the proof is concluded.

Let λ_2 be such that $\lambda < \lambda_2 < \lambda_1 < 1$. Consider $N = N(\lambda_2, \lambda_1)$ from Pliss lemma 2.3. It follows (assuming for simplicity that $n_{i+1} - n_i \geq N = 1$) that the following holds:

$$\|Df^{n_{i+1}-j}_{|E_x}\| > \lambda_2^j \text{ for any } x \in f^j(I_{n_i}) \text{ and } 0 \leqslant j < n_{i+1} - n_i. \tag{4}$$

This implies that the derivative along the F direction behave as an expanding direction for iterates between n_i and n_{i+1}. In fact, (4) implies that given $0 \leqslant j < n_{i+1} - n_i$ then

$$\|Df^{-(n_{i+1}-j)}_{|F_x}\| < (\frac{\lambda}{\lambda_2})^j$$

for any $x \in I_{n_{i+1}}$. In particular

$$\ell(I_{-(n_{i+1}-j)}) < (\frac{\lambda}{\lambda_2})^j \ell(I_{n_{i+1}}). \tag{5}$$

Using proposition 3.1 we have that

$$\sum_{i>0}^{\infty} \ell(I_{n_i}) < \infty$$

and this together with (5) imply

$$\sum_{n>0}^{\infty} \ell(I_n) < \infty$$

and arguing as Schwartz's proof of the Denjoy Theorem for some n_i large we may find an arc J_{n_i} containing properly each I_{n_i} such that J_{n_i} is a $\delta-$interval, which is a contradiction with the maximality of I_{n_i} for every n_i. $\square$

Now we proceed to give the proof of proposition 3.1

Proof of proposition 3.1: Let us consider the box $W^s_\epsilon(J)$. To prove the proposition , it is enough to show that there is a constant C such that given two center unstable arcs $J_1,\ J_2$ in $W^s_\epsilon(J)$ transversal to the E-direction and whose endpoints are in $\partial^{cu}(W^s_\epsilon(J))$ (where $\partial^{cu}(W^s_\epsilon(J)) = W^s_\epsilon(x_1) \cup W^s_\epsilon(x_2)$ and $\{x_1, x_2\}$ are the boundary points of J) the following holds:

$$\frac{1}{C} \leq \frac{\ell(J_1)}{\ell(J_2)} \leq C.$$

To prove that, let us consider the holonomy Π induces by the stable foliation restricted to the box $W^s_\epsilon(J)$; i.e.: let $\Pi : J_1 \to J_2$ defined as $\Pi(x) = W^s_\epsilon(x) \cap J_2$. Related to it, we state the next lemma.

Lemma 3.3. *Let f be a $C^{1+\beta}$ diffeomorphisms, $\beta > 0$, and let Λ be a compact invariant set exhibiting a codimension one dominated splitting. There exists $\delta_0 > 0$ and $C > 0$ such that if J is a C^r δ-E-interval with $\delta \leq \delta_0$ that verifies (3), it follows that the stable holonomy restricted to $W^s_\epsilon(J)$ is C^1 and*

$$\frac{1}{C} \leq \|\Pi'\| \leq C.$$

$\square$

Proof of lemma 3.3: To avoid notation, let us denote $B = W^s_\epsilon(J)$. Let us take J_1, J_2 be the center unstable arcs that bound B. In other words $J_1 \cup J_2 = \cup_{x\in J}\partial W^s_\epsilon(x)$ where $\partial W^s_\epsilon(x)$ are the boundaries of $W^s_\epsilon(x)$. For any positive integer k, let us take the set

$$B_k = f^k(W^s_\epsilon(J))$$

and let us consider a C^1 (not necessarily invariant) foliation that contains the center stable leaves of the extremal points of $J^k_1 = f^k(J_1)$ and $J^k_2 = f^k(J_2)$. Let us called this foliation $\hat{\mathcal{F}}^{cs}_k$.

Lemma 3.4. *There exists a positive constant C_1 such that for k sufficiently large, follows that there exists a C^1 foliation $\hat{\mathcal{F}}^{cs}_k$ containing the center stable leaves of the extremal points of $J^k = f^k(J)$ such that*

$$\frac{1}{C_1} \le \|\hat{\Pi}'_k\| \le C_1$$

where $\hat{\Pi}_k$ is the holonomy induced by $\hat{\mathcal{F}}^{cs}_k$ from J^k_1 to J^k_2.

Before to prove the previous lemma Let us continue with the proof lemma 3.3. Let $\mathcal{F}^{cs}_k$ be the foliation in B which is the pull-back foliation $\hat{\mathcal{F}}^{cs}$ in B_k and let us define

$$\Pi_k = f^{-k} \circ \hat{\Pi}_k \circ f^k_{/J_1}.$$

in other words Π_k is the projection along $\mathcal{F}^{cs}_k$ between J_1 and J_2. We want to prove that Π_k converge to Π in the C^1-topology. It is immediate that the convergence holds in the C^0-topology, so to conclude, we have to show that there exists C_1 such that

$$\frac{1}{C_1} \le \|\Pi'_k\| \le C_1$$

where Π_k is the projection along $\mathcal{F}^{cs}_k$ between J_1 and J_2. Notice that $J_1 = f^k(J^k_1)$ and $J_2 = f^k(J^k_2)$ are also two arcs in $B(y)$ transversal to the E-direction with endpoints in $\partial^{cu}(B(y))$. For a point $x \in f^j(J^k_i),\ i = 1,2$ set $\tilde{F}(x) = T_x f^j(J^k_i),\ 0 \le j \le k$.

By the equality

$$\Pi_k \circ f^{-k}_{/J_1} = f^{-k} \circ \hat{\Pi}_k$$

we conclude, for $z \in J_1$, that

$$\|\Pi'_k(f^{-k}(z))\|.\|Df^{-k}_{/\tilde{F}(z)}\| = \|Df^{-k}_{/\tilde{F}(\Pi(z))}\|.\|\hat{\Pi}'(z)\|$$

Hence

$$\|\Pi'_k(f^{-k}(z))\| = \frac{\|Df^{-k}_{/\tilde{F}(\Pi(z))}\|}{\|Df^{-k}_{/\tilde{F}(z)}\|}.\|\hat{\Pi}'(z)\|$$

Thus, to finish the proof of the lemma it suffices to find M such that

$$\frac{1}{M} \le \frac{\|Df^{-k}_{/\tilde{F}(\Pi(z))}\|}{\|Df^{-k}_{/\tilde{F}(z)}\|} \le M$$

which is the same, setting $x = f^{-k}(z)$, as

$$\frac{1}{M} \leq \frac{\|Df^k_{/\tilde{F}(x)}\|}{\|Df^k_{/\tilde{F}(\Pi_k(x))}\|} \leq M.$$

Observe that for any pair of point z_1, z_2 belonging to the same central leaf of $\mathcal{F}^{cs}_k$, form (3) follows that

$$dist(f^j(z_1), f^j(z_2)) \leqslant \lambda_2^j dist(z_1, z_2)$$

for $j \leq k$ and so, given some constant α, there is a constant A such that

$$\Sigma_{i=0}^k \ell(f^j(\mathcal{F}^{cs}_k(x)))^\alpha < A.$$

With the same arguments as in [Sh] pages 45-46, it is possible to prove that there exist $\tau > 0$ and $\alpha > 0$ such that

$$\left| \|Df_{/\tilde{F}(f^j(w_1))}\| - \|Df_{/\tilde{F}(f^j(w_2))}\| \right| \leq \eta^j D + dist(f^j(w_1), f^j(w_2))^\alpha$$

for some constant $0 < \eta < 1$ and D whenever $\tilde{F}$ lies in the central unstable cone and $dist(f^j(w_1), f^j(w_2)) \leq \tau,\ 0 \leq j \leq k$. (This is, roughly speaking, a consequence of the fact that the distribution F is α-holder and any other direction converges exponentially fast to F.)

Therefore, if the diameter of $B_{(\delta^s,\delta^u)}(p)$ is less than τ, it follows that

$$\frac{\|Df^n_{/\tilde{F}(x)}\|}{\|Df^n_{/\tilde{F}(\Pi_k(x))}\|} \leq \exp\left(\frac{D}{1-\eta} + \sum_{j=0}^{j=k} dist(f^j(x), f^j(\Pi_k(x)))^\alpha\right).$$

Since x and $\Pi_k(x)$ belongs to $\mathcal{F}^{cs}_k(x)$, we conclude that

$$\sum_{j=0}^{k} dist(f^j(x), f^j(\Pi_k(x)))^\alpha \leq \sum_{j=0}^{n} \ell(f^j(\mathcal{F}^{cs}_k(x)))^\alpha \leq A.$$

Thus

$$\frac{\|Df^k_{/\tilde{F}(x)}\|}{\|Df^k_{/\tilde{F}(\Pi_k(x))}\|} \leq \exp(\frac{D}{1-\eta} + A).$$

Finally, taking $M = \exp(\frac{D}{1-\eta} + A)$, we have that $C_1 = C.M$ is finished the proof of lemma

□

Proof of lemma 3.4: To prove that, we have to show that the quotient

$$\frac{\ell(J_2^k)}{\ell(J_1^k)} \tag{6}$$

is close to one. In this direction, first we establish the next assertion.

Asserts 3.0.1. *Let $x \in \Lambda$ such that does not belong to the basin of attraction of a periodic sink. Then, for any $\gamma > 0$ there exists n_0 such that if $n > n_0$ then*

$$|Df^n_{/F(x)}| > (1-\gamma)^n.$$

In fact, if it is not the case, given $\gamma > 0$ it follows from lemma 2.3 that there are two increasing sequences $\{m_k\}$ and $\{l_k\}$ such that

$$|Df^n_{/F(f^{m_k}(x))}| < (1-\gamma)^n, \;\; \forall\, 0 < n < l_k. \tag{7}$$

Without loss of generality, we can assume that $f^{m_k} \to z$ for some $z \in \Lambda$ and it is concluded that

$$|Df^n_{/F(z)}| < (1-\gamma)^n, \;\; \forall\, 0 < n. \tag{8}$$

From the domination follows that

$$||D_z f^n|| < (1-\gamma)^n, \;\; \forall\, 0 < n \tag{9}$$

and therefore there is $\epsilon = \epsilon(\gamma)$ such that

$$B_\epsilon(z) \subset W^s_\epsilon(z).$$

Since $f^{m_k}(x) \in B_\epsilon(z)$ for m_k large, it follows that $B_\epsilon(z)$ is contained in the basin of attraction of a periodic sink and therefore $\omega(x)$ is a periodic sink, which is a contradiction and so the claim follows.

Coming back to prove that (6) is close to one, observe that from the fact that $f \in C^{1+\beta}$ follows that the center stable foliation is Holder (see [HPS]) and therefore it follows that there exists $\alpha > 0$ such that

$$\ell(J_1^k) - d_k \ell(J_1^k)^\alpha < \ell(J_2^k) < \ell(J_1^k) + d_k \ell(J_1^k)^\alpha$$

where

$$d_k = \max_{x \in J_1^k} dist(x, W^{cs}_\epsilon(x) \cap J_2^k).$$

Since

$$d_k < \lambda_2^k$$

it follows that

$$1 - \lambda_2^k \ell(J_1^k)^{\alpha-1} < \frac{\ell(J_2^k)}{\ell(J_1^k)} < 1 + \lambda_2^k \ell(J_1^k)^{\alpha-1}.$$

On the other hand, from claim 3.0.1 it follows that

$$\ell(J_1^k) > (1-\gamma)^k \ell(J_1)$$

and so

$$\lambda_2^k \ell(J_1^k)^{\alpha-1} < \left[\lambda_2 (1-\gamma)^{\alpha-1}\right]^k \ell(J_1)^{\alpha-1}$$

which is small provided that k is large and γ is close enough to 0 to guarantee that $\lambda_2(1-\gamma)^{\alpha-1}$ is smaller than one. Therefore the lemma holds.

□

3.1 Denjoy's property and Lyapunov stability.

As we have mentioned, the problem of unique integrability under the hypothesis of codimension one dominated splitting, is related to the problem of characterization of the limit set of a dynamic. We want to mention here, that this characterization is useful to understand the Lyapunov stable systems (system for which the states will remain bounded for all time, see [Ly]). We say that x is *Lyapunov stable* (in the future) if given $\epsilon > 0$ there exists $\delta > 0$ such that $f^n(B_\delta(x)) \subset B_\epsilon(f^n(x))$ for any positive integer n. Under the assumption of codimension one dominated splitting it is possible to characterize the Lyapunov stable points:

Theorem 3.3. *Let $f : M \to M$ be a C^1-diffeomorphisms of a finite dimensional compact Riemannian manifold M and let Λ be a set having codimension one dominated splitting. Then there exists a neighborhood V of Λ such that if $f^n(x) \in V$ for any positive integer n and x is Lyapunov stable, one of the following holds:*

1. *$\omega(x)$ is a periodic orbit,*
2. *$\omega(x)$ is a periodic curve normally attractive supporting and irrational rotation.*
3. *Neither of the above and x is a wandering point.*

Furthermore, if f is C^2, the third option can not happen.

Proof. The proof is almost straightforward from theorem 3.1. Notice that if x is Lyapunov stable, then there is a δ-E-arc inside $B_\delta(x)$. The conclusion now follows. □

4 Proof of maim theorem

We say that F is locally uniquely integrable at x provided there exist a unique (open) arc $J(x)$ containing x such that $T_yJ(x) = F(y)$ for any $y \in J(x)$ and if for any (open) integral curve $\mathcal{C}$ contains x we have that $\mathcal{C} \cap J(x)$ is open in $J(x)$.

To prove the maim theorem,it is enough to prove that F is locally uniquely integrable at any x in M. In each of the next subsection, it is proved the main theorem under each assumed hypothesis.

4.1 Assumption: $\Omega(f) = M$

We shall prove F is uniquely integrable at any point $x \in M$ provided $\Omega(f) = M$. The proof is based upon next lemma.

Lemma 4.1. *Let $f : M \to M$ be a C^r diffeomorphisms, $r \geq 1$ such that M has a codimension one dominated splitting $TM = E \oplus F$. Let us assume that there exists $\epsilon_1 > 0$ and that given $x \in M$ there exists $\epsilon_2 = \epsilon_2(x)$ such that $f^{-n}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-n}(x))$, and $\ell(f^{-n}(W^{cu}_{\epsilon_2}(x))) \to 0$. Then F is locally uniquely integrable at x.*

Proof. It follows immediately from the fact that in this case the center unstable manifold is dynamically defined. □

To conclude the proof of the maim theorem in the present case we use lemma 4.1. Arguing by contradiction, assume that there exist ϵ_1 such that for any ϵ_2 we have that there exists $n > 0$ such that $f^{-n}(W^{cu}_{\epsilon_2}(x))$ is not contained in $W^{cu}_{\epsilon_1}(f^{-n}(x))$. Recall that there exists of δ ($\delta < \epsilon_1$) such that if $y \in W^{cu}_{\epsilon_1}(x)$ and $dist(f^{-j}(x), f^{-j}(y)) \leq \delta$ for $0 \leq j \leq n$, then $f^{-j}(y) \in W^{cu}_{\epsilon}(f^{-j}(x))$ for $0 \leq j \leq n$.

Therefore there exist a sequence $\epsilon_n \to 0$ and $m_n \to \infty$ such that, for $0 \leq j \leq m_n$,

$$\ell(f^{-j}(W^{cu}_{\epsilon_n}(x))) \leq \delta$$

and

$$\ell(f^{-m_n}(W^{cu}_{\epsilon_n}(x))) = \delta.$$

Letting $I_n = f^{-m_n}(W^{cu}_{\epsilon_n}(x))$ we can assume (taking a subsequence if necessary) that $I_n \to I$ and $f^{-m_n}(x_n) \to z, z \in \bar{I}$ (the closure of I). Now, we have that $\ell(f^n(I)) \leq \delta$ for all positive n, and since $I \subset W^{cu}_{\epsilon}(z)$, we conclude that I is a δ-E-interval. Which is a contradiction regarding corollary 3.1.

4.2 Assumption: The dominated splitting is contractive

We shall say that I is an F-arc if for any $x \in I$ then $T_x I = F$. A simple E^s-F-loop is a loop that is the union of a E^s arc and a F-arc.

Lemma 4.2. *There is $\beta > 0$ such that there is no simple E^s-F loop inside $B_\beta(x)$ for any $x \in M$.*

Proof. It is an immediate consequence of the transversality between E^s and F. □

Lemma 4.3. *There exists ϵ_0 such that for any $\epsilon \leq \epsilon_0$ there exists $M = M(\epsilon)$ such that if I is an F-arc with $\ell(I) \leq \epsilon$ then $\ell(f^{-n}(I)) \leq M$ for any $n \geq 0$.*

Proof. Let $\epsilon_0 \leq \beta/2$ and let $\epsilon \leq \epsilon_0$ and assume that the lemma is false. Then, for every n there exists an F-arc I_n with $\ell(I_n) \leq \epsilon$ such that for some integer $m_n \geq 0$ we have $\ell(f^{-m_n}(I_n)) \geq n$. It follows that we can find two points say $x_{n_i}$and x_{n_j} in $f^{-m_n}(I_n)$ and different from the endpoints of $f^{-m_n}(I_n)$ whose distance between them is less than $\beta/2$. It follows that $W^s_\beta(x_{n_i}) \cap f^{-m_n}(I_n) \neq \{x_{n_i}\}$ and hence we may form a simple E^s-F loop, say γ, with and E^s arc inside $W^s_\beta(x_{n_i})$ and an F-arc inside $f^{-m_n}(I_n)$. It follows that $f^{m_n}(\gamma)$ is a simple E^s-F loop contained in $B_\beta(f^{m_n}(x_{n_i}))$, a contradiction. □

Now assume that F is not locally uniquely integrable at some point x. Consider J_1 and J_2 two different F-arcs whose intersection is not open in J_1. We may assume that x is at the boundary (in J_1) of this intersection, $\ell(J_1), \ell(J_2) \leq \epsilon_0$. Let $y \in J_1 \backslash J_2$ and such that $W^s_\beta(y) \cap J_2 = \{z\}$. Let $r = dist(y, z)$.

Asserts 4.2.1. *For any K there exist $n_0 = n_0(K)$ such that for any $x \in M$ follows that*

$$Radius(f^{-n}(W^s_\epsilon(x))) > K, \ \ \forall\, n > n_0,$$

where

$$Radius(B(x)) = \min_{z \in \partial B}\{dist_{B(x)}(x, z)\}$$

and $dist_{B(x)}(.,.)$ is the distance induces by the Riemannian metric restricted to $B(x)$.

With this claim in mind, we define $W^s_K(z)$ as the connected component of $W^s(x)$ that contains x and has radius equal to K.

Notice that for any $K > 0$ there exists n_0 such that for any $n \geq n_0$ and any $w, v \in f^{-n}(J_1)$ we have that $W^s_K(w) \cap W^s_K(v) = \emptyset$. Otherwise we can find an simple E^s-F loop such that under f^n is a simple E^s-F loop inside $B_\beta(x)$. Consider the cylinder $W^s_K(f^{-n}(J_1)) = \cup_{w \in f^{-n}(J_1)} W^s_K(w)$.

Observe that for any L there exist $K = K(L)$ such that if I is an arc joining $f^{-n}(J_1)$ with the s-boundary of the cylinder then its length must be greater than L.

Let $M = M(\epsilon_0)$ and choose $L >> M$ and set $K = K(L)$. Now, choose n large enough so that if $v \in W^s_K(w)$ then $dist(f^n(v), f^n(w)) < r/2$. Since $\ell(f^{-n}(J_2)) \leq M$ it follows that $f^{-n}(J_2)$ does not intersects the s boundary of $W^s_K(f^{-n}(J_1))$. It follows that

$$f^{-n}(J_1) \subset W^s_K(f^{-n}(J_1))$$

and so $f^{-n}(z) \in W^s_K(f^{-n}(y))$. This implies that $dist(y, z) < r/2$, a contradiction.

4.3 Assumption: f is C^2 and there is not attracting or semi-attracting periodic points.

In this section we shall prove that F is uniquely integrable provided f is C^2 and there are no semi-attracting periodic points. First we shall prove a general result regarding the dynamic of the central unstable manifolds.

Lemma 4.4. *Let $f : M \to M$ be a C^r diffeomorphisms $r \geq 1$ and let Λ be a compact invariant set having a codimension one dominated splitting. Let either I be a periodic arc such that f^k restricted to I (k being the period of I) is the identity or I be a simple closed periodic curve such that f^k restricted to I (k being the period of I) is conjugated to an irrational rotation. Then, F is uniquely integrable at any point x of I.*

Proof. It is immediate from the fact that I is attracting normally hyperbolic arc.

□

Using that the center unstable manifold of a codimension one dominated splitting are one dimensional and that they are locally invariant, it is easily concluded the next remark:

Remark 4.3.1. *Let us assume that there is a codimension one dominated splitting over M for a C^r-diffeomorphisms ($r \geqslant 1$). There exists ϵ_1 such that for any periodic point p of f follows that given a connected component of $W^{cu}_{\epsilon_1}(p) \setminus \{p\}$ either it is contained in the unstable manifold of p or the dynamic is the identity in this component or contains a semi-attracting periodic point.*

Lemma 4.5. *Let $f : M \to M$ be a C^2 diffeomorphisms and let us assume that M exhibits a codimension one dominated splitting. Let us also assume that they are not attracting or semi-attracting periodic points. Then, there exists $\epsilon_1 > 0$ such that for any $x \in \Lambda$ it follows that either*

1. *there exists $\gamma = \gamma(\epsilon_1, x)$ such that $f^{-n}(W^{cu}_\gamma(x)) \subset W^{cu}_{\epsilon_1}(f^{-n}(x))$,*

2. *x belong to a normally attracting periodic simple closed curve with dynamic conjugated to an irrational rotation,*

3. *x belong to a normally attracting periodic simple arc with dynamic (up to the period) equal to the identity on J.*

Proof. Recall from corollary 2.1 the existence of δ ($\delta < \epsilon$) such that if $y \in W^{cu}_{\epsilon}(x)$ and $dist(f^{-j}(x), f^{-j}(y)) \le \delta$ for $0 \le j \le n$, then $f^{-j}(y) \in W^{cu}_{\epsilon}(f^{-j}(x))$ for $0 \le j \le n$.

Assume that the first item conclusion of the theorem is false. Then there exist a sequence $\gamma_n \to 0$, $m_n \to \infty$ such that, for $0 \le j \le m_n$,

$$\ell(f^{-j}(W^{cu}_{\gamma_n}(x))) \le \epsilon_1$$

for some ϵ_1 (smaller than the one obtained in the previous remark and smaller than δ given by theorem 3.1) and

$$\ell(f^{-m_n}(W^{cu}_{\gamma_n}(x))) = \epsilon_1.$$

Letting $I_n = f^{-m_n}(W^{cu}_{\gamma_n}(x_n))$ we can assume (taking a subsequence if necessary) that $I_n \to I$ and $f^{-m_n}(x) \to z, z \in \Lambda, z \in \bar{I}$ (the closure of I).

Now, we have that $\ell(f^n(I)) \le \epsilon_1$ for all positive n, and since $I \subset W^{cu}_{\epsilon}(z)$, we conclude that I is a C^2 δ-E-interval. Now we apply Theorem 3.1. Since they are neither attracting or semi-attracting periodic points, then either (1) or (2) of the referred theorem happens for this arc I we conclude that x belong to a periodic invariant closed curve and so the second or third item of the present lemma holds.

□

Remark 4.3.2. *Let $f : M \to M$ be a C^r ($r \ge 1$) diffeomorphisms and let us assume that it has a codimension one dominated splitting over M. Let I be a normally attracting periodic simple arc. If they are neither semi-attracting or attracting periodic points then f to I is the identity map, where k is the period of I.*

End of proof of main theorem: To finish the proof we have to prove that F is uniquely integrable provided M has codimension one dominated splitting, f is C^2 and there are neither attracting nor semi-attracting periodic points. This is an immediate consequence of lemma 4.5, lemma 4.4 and lemma 4.1.

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