Operating enlargements of monotone operators: new connections with convex functions

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Abstract

Given a maximal monotone operator T in a Banach space, a family of enlargements $\mathbb{E}(T)$ of T has been introduced by Svaiter. He also defined a sum and a positive scalar multiplication of enlargements. The first aim of this work is to further study the properties of these operations. Burachik and Svaiter studied a family of convex functions $\mathcal{H}(T)$ which is in a one to one correspondence with $\mathbb{E}(T)$. The second

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aim of this work is to prove that this bijection is in fact an isomorphism, for suitable operations in $\mathcal{H}(T)$. Additionally, we prove that both spaces are convex with respect to these operations.

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1 Introduction and motivation

Let A and B be arbitrary sets and $F : A \rightrightarrows B$ a multifunction. By an *enlargement* or *extension* of F we mean a multifunction $E : \mathbb{R}_+ \times A \rightrightarrows B$ such that

$$F(x) \subseteq E(b, x) \quad \forall\, b \geq 0,\, x \in A.$$

We will be concerned with the study of those examples of E such that F is a maximal monotone multifunction. Many results appeared recently in connection with enlargements of maximal monotone operators (see e. g., [8, 21, 16, 11]).

A well-known and most important example of extension of a maximal monotone multifunction is the ε-subdifferential. Given a lower semicontinuous proper convex function f on a Banach space X, $f : X \to \mathbb{R} \cup \{+\infty\}$, the *subdifferential* of f at x, i.e., the set of *subgradients* of f at x, denoted by $\partial f(x)$, is given by

$$\partial f(x) = \{u \in X^* : f(y) - f(x) - \langle u, y - x\rangle \geq 0 \text{ for all } y \in X\}\,.$$

It is well known (see [19]) that ∂f is maximal monotone. The ε-subdifferential enlargement (of ∂f) was introduced in [3]. It is defined as

$$\partial_\varepsilon f(x) := \{u \in X^* : f(y) - f(x) - \langle u, y - x\rangle \geq -\varepsilon \text{ for all } y \in X\}\,,$$

for any $\varepsilon \geq 0$, $x \in X$. This enlargement of ∂f has been extremely useful both for theoretical and practical applications. For an arbitrary maximal monotone operator T, the following enlargement can be defined [4, 5, 7, 6]: given $\varepsilon \geq 0$, $x \in X$,

$$T^\varepsilon(x) = \{u \in X^* \mid \langle v - u, y - x\rangle \geq -\varepsilon,\, \forall y \in X, v \in T(y)\}. \tag{1}$$

The above enlargement of T has some useful applications, similar to those of the ε-subdifferential. The study of the above mentioned examples of enlargements allowed to identify some basic properties, in such a way that the ε-subdifferential and T^ε can be seen as members of a *family* of enlargements of ∂f and T, respectively. This fact was established in [21], where these characterizing properties were identified and lead to the definition of $\mathbb{E}_c(T)$, the whole family of closed enlargements which satisfy these properties. In fact, all enlargements belonging to this family share some crucial properties with T^ε and the ε-subdifferential (local boundedness, demiclosed graph, Lipschitz continuity, Brøndsted & Rockafellar property)[21, 8].

Once the family $\mathbb{E}_c(T)$ has been found for a fixed T, the next step is to see how the set of enlargements is modified when we "operate" the monotone multifunction T. This question is addressed in [21], where definitions of sum and scalar multiplication for enlargements were given.

Recall that a positive multiple of a maximal monotone operator is always a maximal monotone operator. Summing maximal monotone operators, instead, requires extra assumptions for guaranteeing maximality. Indeed, it has been proved in [18] that the sum of two maximal monotone operators is maximal when the Banach space is reflexive and $D(T_1)^0 \cap D(T_2) \neq \emptyset$.

Given T maximal monotone and $a > 0$. For E an enlargement of T, define the scalar multiplication of E by a as:

$$a \odot E(b, x) := aE(a^{-1}b, x), \tag{2}$$

where $b \geq 0$ and $x \in X$. It has been proved in [21, Proposition 7.1] that $a \odot E \in \mathbb{E}_c(aT)$. Moreover, we prove here (see Proposition 4.11) that every enlargement of aT can be obtained in this way. The sum of enlargements, instead, is not closed in general. This is in correspondence with the above-mentioned fact that the sum of two maximal monotone operators is not maximal in general. Take enlargements $E_1 \in \mathbb{E}_c(T_1)$ and $E_2 \in \mathbb{E}_c(T_2)$, define[21, Proposition 7.2]:

$$E_1 \oplus E_2(b, x) := \bigcup_{\substack{b_1, b_2 \geqslant 0 \\ b_1 + b_2 = b}} E_1(b_1, x) + E_2(b_2, x), \tag{3}$$

where $b \geq 0$ and $x \in X$. A motivation for the expression above comes from the formula[10, Theorem 2.1]

$$\partial_b(f_1 + f_2)(x) = \bigcup_{\substack{b_1, b_2 \geqslant 0 \\ b_1 + b_2 = b}} \partial_{b_1} f_1(x) + \partial_{b_2} f_2(x), \tag{4}$$

where $f_1, f_2 : X \to \overline{\mathbb{R}}$ are proper convex functions such that the following condition holds:

$$(f_1 + f_2)^*(v) = \min\{f_1^*(v_1) + f_2^*(v_2) \mid v_1 + v_2 = v\}, \qquad (5)$$

for all $v \in X^*$.

We prove in this paper that the enlargement $E_1 \oplus E_2$ is closed when X is reflexive and $D(T_1)^0 \cap D(T_2) \neq \emptyset$. We also prove that, with respect to these operations, the family $\mathbb{E}_c(T)$ is a convex set.

The second part of this work is devoted to study new connections between maximal monotone operators and convex functions defined in $X \times X^*$. This important question has been addressed by Fitzpatrick, who obtained in [9] representations of monotone operators by subdifferentials of convex functions defined in $X \times X^*$. In his work, he defined a family of convex functions associated to T. We denote this family by $\mathcal{H}(T)$. Fitzpatrick's family $\mathcal{H}(T)$ happens to be closely related with a scalar function introduced in [2] by Brézis and Haraux. This function, associated to a monotone operator T is defined as:

$$\bar{\varphi}_T(x, v) := \sup_{y \in X, u \in Ty} \langle x - y, u - v \rangle.$$

It is proved in [9, Theorem 3.10] that $\bar{\varphi}_T + \langle \cdot, \cdot \rangle$ is a minimal element in $\mathcal{H}(T)$. A remarkable fact is that the study of enlargements of monotone operators can lead independently to Fitzpatrick's family $\mathcal{H}(T)$. This is proved in [8]. More precisely, [8, Theorem 3.6] proves that the family $\mathbb{E}_c(T)$ is in a one-to-one correspondence with $\mathcal{H}(T)$. This one-to-one correspondence becomes a natural way for defining the operations in $\mathcal{H}(T)$ which correspond to (2) and (3) in $\mathbb{E}_c(T)$. This is the subject of Section 6. In the same way as in $\mathbb{E}_c(T)$, we prove that the family $\mathcal{H}(T)$ is convex with respect to these operations. Again, the sum in $\mathcal{H}(T)$ deserves special attention, and we prove that, when X is reflexive and $D(T_1)^0 \cap D(T_2) \neq \emptyset$, the result of the sum belongs to $\mathcal{H}(T)$.

The paper is organized as follows. In Section 2 we give the theoretical preliminaries related to the family $\mathbb{E}(T)$, while in Section 3 we list known results of the family $\mathcal{H}(T)$. The basic properties of the operations in $\mathbb{E}(\cdot)$ are established in Section 4, while the whole Section 5 is devoted to prove closedness of the sum of enlargements. The operations in $\mathcal{H}(\cdot)$ are introduced and studied in Section 6. We finish this section with an application of this theory to convex analysis. More precisely, we use this new theory for proving that condition (5) is not only sufficient, but also necessary for the formula (4) to hold.

2 Basic Definitions and Preliminary Results on $\mathbb{E}(T)$

From now on X is a real Banach space and X^* its dual. Given $x \in X$ and $v \in X^*$, $v(x)$ will be denoted by $\langle v, x\rangle$. In products of Banach spaces (e.g., $X \times X^*$, $\mathbb{R} \times X \times X^*$) we shall consider the canonical product topology. Given a multifunction $T : X \rightrightarrows X^*$, recall that the *graph* of T, $G(T)$ is

$$G(T) := \{(x, v) \,|\, x \in X \text{ and } v \in T(x)\}.$$

Based on Minkowski's definition of scalar multiples and sums of sets [14], the corresponding operations are defined for multifunctions. Consider $T_1, T_2 : X \rightrightarrows X^*$ and define $T_1 + T_2 : X \rightrightarrows X^*$ as

$$(T_1 + T_2)(x) := \{v_1 + v_2 \;: v_1 \in T_1(x) \text{ and } v_2 \in T_2(x)\}. \tag{6}$$

For any $a \in \mathbb{R}$, define $aT : X \rightrightarrows X^*$ as

$$aT(x) := \{av \;: v \in T(x)\}. \tag{7}$$

The operator T is called *monotone* if $\langle u - v, x - y\rangle \geq 0$ for all $u \in T(x)$ and $v \in T(y)$, for all $x, y \in X$. It is called *maximal monotone* if it is monotone and its graph is maximal with respect to this property, i.e., it is not properly contained in the graph of any other monotone operator.

2.1 Enlargements of maximal monotone operators

We need some additional notation, to deal with enlargements. Given $E : \mathbb{R}_+ \times X \rightrightarrows X^*$

- The *graph* of E, $G(E)$ is

  $$G(E) := \{(b, x, v) \in \mathbb{R} \times X \times X^* \mid v \in E(b, x)\}.$$

  Define also

  $$\Gamma(E) := \{(x, v, b) \in X \times X^* \times \mathbb{R} \mid v \in E(b, x)\}.$$

- The *closure* of E, $\overline{E} : \mathbb{R}_+ \times X \rightrightarrows X^*$ is defined as

$$\begin{aligned} \overline{E}(b,x) &:= \{v \in X^* \mid (b,x,v) \in \overline{G(E)}\}, \\ &= \{v \in X^* \mid (x,v,b) \in \overline{\Gamma(E)}\}, \end{aligned}$$

  where $\overline{A}$ stands for the closure of $A \subset \mathbb{R} \times X \times X^*$ with respect to the strong topology in X and X^*. We say that E is *closed* if $E = \overline{E}$. So, E is closed if and only if $\Gamma(E)$ is closed.

- We say that E is an *enlargement* of $T : X \rightrightarrows X^*$ if for any $x \in X$, $b \geq 0$,

$$T(x) \subseteq E(b,x).$$

- We say that $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ is *non-decreasing* if for each $x \in X$, the mapping $\mathbb{R}_+ \ni \varepsilon \mapsto E(\varepsilon, x)$ is non-decreasing with respect to the inclusion, i.e.,

$$0 \leq \varepsilon \leq \varepsilon' \Rightarrow E(\varepsilon,x) \subseteq E(\varepsilon',x), \forall x \in X. \tag{8}$$

- We say that $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ is *additive* if for all $(x^1, v^1, b_1), (x^2, v^2, b_2) \in G(E)$, it holds that

$$\langle v^1 - v^2, x^1 - x^2 \rangle \geq -(b_1 + b_2).$$

We mention bellow two important examples.

Let f be a proper convex function on X and consider the ε-subdifferential enlargement of ∂f. As we work with enlargements as multifunctions on $\mathbb{R}_+ \times X$, we must have a different notation for the multifunction $(\varepsilon, x) \mapsto \partial_\varepsilon f(x)$. It will be denoted by $\breve{\partial} f$. So,

$$\begin{aligned} &\breve{\partial} f : \mathbb{R}_+ \times X \rightrightarrows X^*, \\ &\breve{\partial} f(\varepsilon, x) = \partial_\varepsilon f(x). \end{aligned} \tag{9}$$

We will refer to $\breve{\partial} f$, after its authors, as the *Brøndsted-Rockafellar enlargement of* ∂f. The enlargement $\breve{\partial} f$ is closed, nondecreasing and additive.

Given $T : X \rightrightarrows X^*$ *monotone*, the multifunction $(\varepsilon, x) \mapsto T^\varepsilon(x)$, defined in $\mathbb{R}_+ \times X$ according to (1), will be denoted by B^T.

Definition 2.1 *Let $T : X \rightrightarrows X^*$ be monotone. Then B^T is defined as*

$$\begin{aligned} &B^T : \mathbb{R}_+ \times X \rightrightarrows X^* \\ &B^T(b,x) := \{v \in X^* \mid \langle v-u, x-y\rangle \geq -b, \forall u \in T(y), y \in X\} \end{aligned}$$

Trivially, B^T is nondecreasing. Monotonicity of T implies in that B^T is an enlargement (of T). It is also a closed enlargement of T.

The following "transportation formula" is satisfied by $\breve{\partial} f$ [13] and by B^T, when T is maximal monotone [6, 7].

Definition 2.2 *We say that $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ satisfies the* transportation formula *if for every pair*

$$(x^1, v^1, \varepsilon_1), (x^2, v^2, \varepsilon_2) \in \Gamma(E), \tag{10}$$

(or equivalently, $v^i \in E(\varepsilon_i, x^i), i = 1, 2$) and every $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, it holds that the triplet (x, v, ε) defined as

$$x := \alpha_1 x^1 + \alpha_2 x^2, \quad v := \alpha_1 v^1 + \alpha_2 v^2, \tag{11}$$

$$\varepsilon := \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_1 \alpha_2 \langle v^1 - v^2, x^1 - x^2\rangle\,, \tag{12}$$

satisfies $(x, v, \varepsilon) \in \Gamma(E)$. In other words, $\varepsilon \geq 0$ and $v \in E(\varepsilon, x)$.

Now we recall the definition of a family of enlargements, introduced in [21], which will be central in our analysis.

Definition 2.3 *Let $T : X \rightrightarrows X^*$ be a monotone multifunction. Define $\mathbb{E}(T)$ as the family of multifunctions $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ satisfying the following properties:*

(r1) *E is an enlargement of T, i.e.:*

$$E(\varepsilon, x) \supseteq T(x), \quad \forall \varepsilon \geq 0, x \in X.$$

(r2) *E is non-decreasing, in the sense of Definition 8.*

(r3) *E satisfies the transportation formula of Definition 2.2.*

The subfamily of those $E \in \mathbb{E}(T)$ which are closed will be denoted by $\mathbb{E}_c(T)$, and the subfamily of those $E \in \mathbb{E}(T)$ which are additive will be denoted by $\mathbb{E}_a(T)$.

By the remarks above,

$$\breve{\partial} f \in \mathbb{E}_c(\partial f) \cap \mathbb{E}_a(\partial f). \tag{13}$$

The proposition below, proved in [21, Lemma 4.1], will be useful.

Proposition 2.4 *Let T be maximal monotone. For every $E \in \mathbb{E}(T)$, it holds that*

$$E(0, x) = \cap_{b>0} E(b, x) = T(x).$$

From now on, $T : X \rightrightarrows X^*$ is an *arbitrary* maximal monotone multifunction. In [7, 21] it was proved that $B^T \in \mathbb{E}_c(T)$. In particular, $\mathbb{E}(T)$ is nonempty. The multifunction $S^T : \mathbb{R}_+ \times X \rightrightarrows X^*$,

$$S^T(\varepsilon, x) := \bigcap_{E \in \mathbb{E}(T)} E(\varepsilon, x). \tag{14}$$

is well defined. In [21] it was proved that $S^T \in \mathbb{E}(T)$ and that B^T, S^T are respectively the biggest and the smallest elements of $\mathbb{E}(T)$, with respect to the (partial) order of the inclusion between the graphs, i.e.,

$$E \in \mathbb{E}(T) \Rightarrow \Gamma(S^T) \subseteq \Gamma(E) \subseteq \Gamma(B^T). \tag{15}$$

Observe that x, v, ε as in (11), (12) depend continuously on the points (x^i, v^i, b_i), $i = 1, 2$. Hence, if E satisfies the transportation formula, then $\overline{E}$ likewise. Note also that if E is nondecreasing/an enlargement of T, then $\overline{E}$ is nondecreasing/an enlargement of T. Therefore, if $E \in \mathbb{E}(T)$, $\overline{E} \in \mathbb{E}_c(T)$. The family $\mathbb{E}_c(T)$ will play a central role in the next section.

3 The family of convex functions $\mathcal{H}(T)$

Any enlargement E can be studied via its graph, or equivalently, via the set $\Gamma(E) \subseteq X \times X^* \times \mathbb{R}_+$. A usual way to associate a scalar function with to set $A \subseteq X \times X^* \times \mathbb{R}$ is to consider the *lower envelope* of A [1], defined as $\alpha : X \times X^* \to \overline{\mathbb{R}}$,

$$\alpha(x, v) := \inf\{b \in \mathbb{R} \mid (x, v, b) \in A\}.$$

The graph of α is the "lower boundary" of A (see [17, Sec. 5]). Trivially, $A \subseteq \text{Epi}(\alpha)$. Additionally, $A = \text{Epi}(\alpha)$ if A is closed and has an "epigraphical structure":

$$(x, v, b) \in A \Longrightarrow (x, v, b') \in A, \ \forall b' \geq b. \tag{16}$$

In this way, we associate an enlargement E with the lower envelope of $\Gamma(E)$. Namely, define $\lambda_E : X \times X^* \to \overline{\mathbb{R}}$ as

$$\lambda_E(x, v) := \inf\{\varepsilon \geq 0 \mid v \in E(\varepsilon, x)\}. \tag{17}$$

Observe that

$$\lambda_E(x, v) = \inf\{b \mid (x, v, b) \in \Gamma(E)\}.$$

As pointed out above, $\Gamma(E) \subseteq \text{Epi}(\lambda_E)$. The conditions of $\Gamma(E)$ being closed and having an epigraphical structure are respectively equivalent to E being closed and nondecreasing. Enlargements satisfying these conditions are fully characterized by their lower envelope.

Proposition 3.1 *If $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ is closed and nondecreasing then*

1. $\Gamma(E) = \text{Epi}(\lambda_E)$,
2. λ_E *is (strongly) l.s.c.,*
3. $\lambda_E \geq 0$,
4. $E(b, x) = \{v \in X^* \mid \lambda_E(x, v) \leq b\}, \quad \forall b \in \mathbb{R}_+, x \in X.$

Furthermore, λ_E is the unique function from $X \times X^*$ to $\overline{\mathbb{R}}$ satisfying items 3, 4.

Proof. (see [8, Proposition 3.1]). ☐

However, the function λ_E is not necessarily convex. To associate a convex function to the enlargement E, we shall add to λ_E the duality product. More precisely, given $E : \mathbb{R}_+ \times X \rightrightarrows X^*$, define $\Lambda_E : X \times X^* \to \overline{\mathbb{R}}$ by

$$\Lambda_E(x, v) := \lambda_E(x, v) + \langle v, x\rangle. \tag{18}$$

In order to study the epigraph of the function above, let $\psi : X \times X^* \times \mathbb{R}_+ \to X \times X^* \times \mathbb{R}_+$, be defined as

$$(x, v, \varepsilon) \mapsto (x, v, \varepsilon + \langle x, v\rangle). \tag{19}$$

The following result has been proved in [8, Corollary 3.2]. It establishes an important connection between enlargements and convexity.

Theorem 3.2 *Let E be a closed, non-decreasing enlargement of T. Then $E \in \mathbb{E}(T)$ if and only if Λ_E is convex. Moreover, $\psi(\Gamma(E)) = \mathrm{Epi}(\Lambda_E)$.*

The above result, combined with Proposition 3.1, yields the following proposition.

Proposition 3.3 *Take $E \in \mathbb{E}_c(T)$. Then Λ_E is convex, l.s.c. and*

$$\begin{array}{l} \Lambda_E(x,v) \geq \langle v, x\rangle, \forall x \in X, v \in X^*, \\ v \in T(x) \Rightarrow \Lambda_E(x,v) = \langle v, x\rangle. \end{array} \tag{20}$$

Furthermore, the application $E \mapsto \Lambda_E$ is one-to-one in $\mathbb{E}_c(T)$.

Proposition 3.3 is essential for defining a family of convex functions associated to T.

Definition 3.4 *Define $\mathcal{H}(T)$ as the family of l.s.c. convex functions $h : X \times X^* \to \overline{\mathbb{R}}$ such that*

$$h(x,v) \geq \langle v, x\rangle, \forall x \in X, v \in X^*, \tag{21}$$

$$v \in T(x) \Rightarrow h(x,v) = \langle v, x\rangle. \tag{22}$$

We have seen that, associated to any enlargement $E \in \mathbb{E}(T)$, there is an element Λ_E in $\mathcal{H}(T)$. Conversely, if we have an element $h \in \mathcal{H}(T)$, the enlargement $L^h : \mathbb{R}_+ \times X \rightrightarrows X^*$, defined by

$$L^h(\varepsilon, x) := \{v \in X^* \mid h(x,v) \leq \varepsilon + \langle v, x\rangle\}, \tag{23}$$

is a closed enlargement of T. More precisely, the following fact has been established in [8, Theorem 3.6].

Theorem 3.5 *The map*

$$\begin{array}{rcl} \mathbb{E}_c(T) & \to & \mathcal{H}(T) \\ E & \mapsto & \Lambda_E \end{array} \tag{24}$$

is a bijection, with inverse given by

$$\begin{array}{rcl} \mathcal{H}(T) & \to & \mathbb{E}_c(T) \\ h & \mapsto & L^h. \end{array} \tag{25}$$

As a consequence of the theorem above, we see that each element of $\mathcal{H}(T)$ fully characterizes the operator T.

Corollary 3.6 *Take $h \in \mathcal{H}(T)$. Then for any $(x, v) \in X \times X^*$*

$$v \in T(x) \iff h(x, v) = \langle v, x \rangle.$$

Based in (15), the following result has been established in [8, Corollary 4.1].

Corollary 3.7 *The functions Λ_{B^T}, $\Lambda_{\overline{S^T}}$ belong to $\mathcal{H}(T)$ and are respectively the minimum and maximum of this family, i.e.,*

$$\Lambda_{B^T} \leq h \leq \Lambda_{\overline{S^T}}, \ \forall h \in \mathcal{H}(T).$$

If f is a proper l.s.c. convex function, we have mentioned above that $\breve{\partial} f \in \mathbb{E}_c(\partial f)$. It has been proved in [8, Eq(42)] that the corresponding convex function in $\mathcal{H}(\breve{\partial} f)$ is given by

$$\Lambda_{\breve{\partial} f}(x, v) = f(x) + f^*(v), \text{ for all } (x, v) \in X \times X^*, \tag{26}$$

where $f^* : x^* \to \overline{\mathbb{R}}$ is the convex conjugate of f.

4 Operations with Enlargements

Given $a > 0$ and T monotone, aT is monotone. Monotonicity is preserved by the positive scalar multiplication. Monotone operators can also be added. If T_1, T_2 are monotone, then $T_1 + T_2$ is monotone. For enlargements, scalar multiplication and addition were defined as follows [21].

Definition 4.1 *Let $E_1, E_2 : \mathbb{R}_+ \times X \rightrightarrows X^*$. Define $E_1 \oplus E_2 : \mathbb{R}_+ \times X \rightrightarrows X^*$ by*

$$E_1 \oplus E_2(b, x) := \bigcup_{\substack{b_1, b_2 \geqslant 0 \\ b_1 + b_2 = b}} E_1(b_1, x) + E_2(b_2, x). \tag{27}$$

Definition 4.2 *Let $a > 0$ and $E : \mathbb{R}_+ \times X \rightrightarrows X^*$. Define $a \odot E : \mathbb{R}_+ \times X \rightrightarrows X^*$ by*

$$a \odot E(b, x) := aE(a^{-1}b, x). \tag{28}$$

Setting $b = 0$ in (27) and (28), and considering $E(0, \cdot)$, $E_1(0, \cdot)$ and $E_2(0, \cdot)$ as multifunctions defined on X, we retrieve operations (6)-(7). In this sense, we can say that $\odot$ and $\oplus$ extend these usual operations between multifunctions defined on X. More similarities with (6)-(7) are described in Lemma 4.3 below. Define $\mathbf{0} : \mathbb{R}_+ \times X \rightrightarrows X^*$ by

$$\mathbf{0}(b, x) = \{0\}.$$

Lemma 4.3 *Take a, b positive, $E, E_1, E_2 : \mathbb{R}_+ \times X \rightrightarrows X^*$. Then*

(1) $1 \odot E = E$,

(2) $(ab) \odot E = a \odot (b \odot E)$,

(3) $a \odot (E_1 \oplus E_2) = (a \odot E_1) \oplus (a \odot E_2)$,

(4) $E \oplus (E_1 \oplus E_2) = (E \oplus E_1) \oplus E_2$,

(5) $E \subseteq \mathbf{0} \oplus E$,

(6) $(a + b) \odot E \subseteq (a \odot E) \oplus (b \odot E)$.

Inclusion (5) holds as an equation for E non-decreasing in the sense of Definition 8. The last inclusion holds as an equation when E satisfies the transportation formula of Definition 2.2.

Proof. Properties (1)-(5) are direct consequences from the definitions. We proceed to prove (6). Take $v \in (a + b) \odot E(\beta, x)$. By definition of $\odot$, this means that there exists $w \in E(\beta/(a + b), x)$ such that $v = (a + b)w$. It follows from the definitions that

$$aw \in aE(\beta/(a + b), x) = (a \odot E)(a\beta/(a + b), x), \text{ and}$$

$$bw \in bE(\beta/(a + b), x) = (b \odot E)(b\beta/(a + b), x).$$

Use now Definition 4.1 to conclude that $v = aw + bw \in (a \odot E) \oplus (b \odot E)(\beta, x)$. For the converse inclusion, assume that E satisfies the transportation formula. Take $v \in [(a \odot E) \oplus (b \odot E)]\,(\beta, x)$. By (27), there exist β_1, $\beta_2 \geq 0$ and $v_1, v_2 \in X^*$ such that $\beta_1 + \beta_2 = \beta$ and $v = a\,v_1 + b\,v_2$, where $v_1 \in E(\beta_1/a, x)$ and $v_2 \in E(\beta_2/b, x)$. On the other hand, by (11)-(12), and the fact that E verifies **r3**:

$$\begin{aligned} \tfrac{1}{a+b}\, v = \tfrac{a}{a+b} v_1 + \tfrac{b}{a+b} v_2 &\in E([\tfrac{a}{a+b}]\beta_1/a + [\tfrac{b}{a+b}]\beta_2/b, x) \\ &= E((\beta_1 + \beta_2)/(a + b), x) = E(\beta/(a + b), x), \end{aligned}$$

and hence $v \in (a + b) \odot E(\beta, x)$. $\square$

Remark 4.4 *The scalar product and the sum of definitions 4.2 and 4.1 are compatible in the following sense. By Lemma 4.3(6), if E satisfies the transportation formula, then*

$$k \odot E = \underbrace{E \oplus E \oplus \cdots \oplus E}_{k \text{ times}},$$

for all $k \in \mathbb{N}$.

Remark 4.5 *It is clear from Definitions 4.2 and 4.1, that the scalar product and the sum preserve inclusion in the following sense. Take $b \geq 0$, $x \in X$,*

- *if $E(b,x) \subset E'(b,x)$, then $a \odot E(b,x) \subset a \odot E'(b,x)$;*
- *if $E_1(b,x) \subset E_1'(b,x)$, and $E_2(b,x) \subset E_2'(b,x)$, then $(E_1 \oplus E_2)(b,x) \subset (E_1' \oplus E_2')(b,x)$.*

Properties **(r1)-(r3)** of Definition 2.3, as well as additivity and closedness have been important for studying elements $E \in \mathbb{E}(T)$. When analyzing these properties in the case of an arbitrary point-to-set mapping $E : \mathbb{R}_+ \times X \rightrightarrows X^*$, it is useful to establish whether each of them is preserved or not by the above defined operations. The following trivial lemma will be useful for analyzing this question. Before stating it, we need some definitions.

Definition 4.6 *a) Fix $a > 0$. Define the mapping $\gamma_a : X \times X^* \times \mathbb{R}_+ \to X \times X^* \times \mathbb{R}_+$ as $\gamma_a(x,v,\varepsilon) = (x, av, a\varepsilon)$.*

b) Take two arbitrary triplets with the same X-coordinate $(x,v_1,\varepsilon_1), (x,v_2,\varepsilon_2) \in X \times X^* \times \mathbb{R}_+$. We define a "partial sum" between such triplets, denoted by $\tilde{+}$, as

$$(x,v_1,\varepsilon_1) \tilde{+} (x,v_2,\varepsilon_2) := (x, v_1+v_2, \varepsilon_1+\varepsilon_2).$$

Two arbitrary sets $A, B \subset X \times X^* \times \mathbb{R}_+$ can be partially summed:

$$A \tilde{+} B := \{(x, v+v', \varepsilon+\varepsilon') \mid \text{ such that } (x,v,\varepsilon) \in A \text{ and } (x,v',\varepsilon') \in B\}.$$

Lemma 4.7 *The mapping γ_a preserves closedness and convexity of sets, as well as the property of having epigraphical structure. If A and B are convex,* *then $A\tilde{+}B$ is convex. If A or B have epigraphical structure, then $A\tilde{+}B$ has epigraphical structure. It holds that*

$$\psi(A\tilde{+}B) = \psi(A)\tilde{+}\psi(B), \tag{29}$$

where ψ is the mapping defined in (19)*. Also,*

$$\gamma_a(\Gamma(E)) = \Gamma(a \odot E), \tag{30}$$

and

$$\Gamma(E \oplus E') = \Gamma(E)\tilde{+}\Gamma(E'). \tag{31}$$

Now we can state the Lemma involving properties **(r1)**-**(r3)**. This is an extension for arbitrary mappings of [21, Proposition 7.1, Lemma 7.2].

Lemma 4.8 *Let $E, E' : \mathbb{R}_+ \times X \rightrightarrows X^*$ and $S, S' : X \rightrightarrows X^*$ be arbitrary point-to-set mappings.*

(1) Fix $a > 0$. If E is an enlargement of S then $a \odot E$ is an enlargement of aS. It holds that E is non-decreasing (closed, additive, satisfies the transportation formula) if and only $a \odot E$ is non-decreasing (closed, additive, satisfies the transportation formula).

(2) If E and E' are enlargements of S and S', respectively, then $E \oplus E'$ is an enlargement of $S + S'$. If E or E' is non-decreasing, then $E \oplus E'$ is non-decreasing. If E and E' are additive (satisfy the transportation formula), then $E \oplus E'$ is additive (satisfies the transportation formula).

Proof.

(1) Define the multifunction $\mathbf{0} \times S : \mathbb{R}_+ \times X \rightrightarrows X^*$ as $\mathbf{0} \times S(\varepsilon, x) = Sx$ for all $\varepsilon \geq 0$ and all $x \in X$. It is clear that

$$\begin{aligned} E \text{ is an enlargement of } S \quad &\longleftrightarrow \quad \Gamma(\mathbf{0} \times S) \subset \Gamma(E) \\ &\longleftrightarrow \quad \gamma_a(\mathbf{0} \times S) \subset \gamma_a(\Gamma(E)). \end{aligned}$$

The last equivalence can be rewritten as

$$E \text{ is an enlargement of } S \quad \longleftrightarrow \quad \Gamma(\mathbf{0} \times aS) \subset \Gamma(a \odot E).$$

But the right hand side means that $a \odot E$ is an enlargement of aS. We proceed to prove the equivalence regarding condition **r2**. It holds that

$$\begin{aligned} E \text{ is non-decreasing} \quad &\longleftrightarrow \quad \Gamma(E) \text{ satisfies (16)} \\ &\longleftrightarrow \quad \gamma_a(\Gamma(E)) \text{ satisfies (16),} \end{aligned}$$

where we are using the fact that γ_a preserves epigraphical structure. Combining the last equivalence with (30), we obtain the desired conclusion. The part of (1) regarding closedness follows from the identity $\gamma_a(\Gamma(E)) = \Gamma(a \odot E)$ and the fact that γ_a preserves closedness. Now we prove the statement regarding additivity of $a \odot E$. Assume that E is additive and take $v^1 \in (a \odot E)(b_1, x^1)$ and $v^2 \in (a \odot E)(b_2, x^2)$. Use now the definition of $a \odot E$ and additivity of E to conclude that

$$\langle v^1/a - v^2/a, x^1 - x^2 \rangle \geq -(b_1/a + b_2/a),$$

which readily implies additivity of $a \odot E$. The converse is identical. Let us prove now the equivalence regarding condition **r3**. By Theorem 3.2, E satisfies the transportation formula if and only if $\psi(\Gamma(E))$ is convex. The latter fact is equivalent to the convexity of the set $\gamma_a(\psi(\Gamma(E))) = \psi(\Gamma(a \odot E))$. Using the right hand side of this identity and Theorem 3.2 again, we obtain the desired equivalence.

(2) The first assertion follows directly from (27). To prove that $E \oplus E'$ is non-decreasing, it is enough to prove that $\Gamma(E \oplus E')$ has epigraphical structure. Assume E is non-decreasing. Hence $\Gamma(E)$ has epigraphical structure. By Lemma 4.7, we conclude that $\Gamma(E) \tilde{+} \Gamma(E')$ also has this property. Now (31) implies that $\Gamma(E \oplus E')$ has epigraphical structure. The statement regarding additivity is a simple consequence of the definitions. Let us prove now the statement regarding the transportation formula. By Theorem 3.2, we have that $\psi(\Gamma(E))$ and $\psi(\Gamma(E'))$ are convex. Using now (29) for $A := \Gamma(E)$ and $B := \Gamma(E')$, the fact the operation $\tilde{+}$ preserves convexity and (31), we conclude that $E \oplus E'$ satisfies the transportation formula. $\square$

In the case in which the multifunction $E : \mathbb{R}_+ \times X \rightrightarrows X^*$ is an element of $\mathbb{E}(T)$, for a given $T : X \rightrightarrows X^*$ monotone, we use Lemma 4.8 to get the following result.

Corollary 4.9 *Take $T, T' : X \rightrightarrows X^*$ monotone multifunctions It holds that*

- *(a) $E \in \mathbb{E}(T)$ ($E \in \mathbb{E}_a(T)$, $E \in \mathbb{E}_c(T)$) if and only if $a \odot E \in \mathbb{E}(aT)$ ($a \odot E \in \mathbb{E}_a(aT)$, $a \odot E \in \mathbb{E}_c(aT)$) for all $a > 0$.*
- *(b) If $E \in \mathbb{E}(T)$, and $E' \in \mathbb{E}(T')$, ($E \in \mathbb{E}_a(T)$, and $E' \in \mathbb{E}_a(T')$), then $E \oplus E' \in \mathbb{E}(T + T')$ ($E \oplus E' \in \mathbb{E}_a(T + T')$).*

Using Corollary 4.9, we conclude that if $E_1 \in \mathbb{E}(T_1)$, $E_2 \in \mathbb{E}(T_2)$, T_1, T_2 monotone and a, b positive, then

$$(a \odot E_1) \oplus (b \odot E_2) \in \mathbb{E}(aT_1 + bT_2),$$

and the same holds for the subfamily $\mathbb{E}_a$.

These inclusions have interesting consequences.

Remark 4.10 *Let T be monotone. The family $\mathbb{E}(T)$ is "convex" with respect to the operations $\odot$, $\oplus$ in the following sense:*

$$(p \odot E_1) \oplus (q \odot E_2) \in \mathbb{E}(T),$$

for any $E_1, E_2 \in \mathbb{E}(T)$, $p, q \geq 0$, $p + q = 1$. The same holds for the subfamily $\mathbb{E}_a(T)$.

We have pointed out above that $a \odot E$ is an enlargement of $a\,T$ if E is an enlargement of T. If E is one of the extremal elements of $\mathbb{E}(T)$, then $a \odot E$ is the corresponding extremal element in $\mathbb{E}(a\,T)$.

Proposition 4.11 *Let T be maximal monotone and $a > 0$. With the same notation as in* (15)*, it holds that*

(i) $a \odot B^T = B^{aT}$,

(ii) $a \odot S^T = S^{aT}$.

(iii) $a \odot \overline{S^T} = \overline{S^{aT}}$.

Proof. (i) Using (15) and the fact that $a \odot B^T \in \mathbb{E}(aT)$, it is enough to prove that $\Gamma(a \odot B^T) \supset \Gamma(B^{aT})$. Take for this $v \in B^{aT}(b, x)$. This means that

$$\langle v - aw, x - y \rangle \geq -b,$$

for all $y \in Y$, $w \in T(y)$. Or , equivalently

$$\langle v/a - w, x - y \rangle \geq -b/a,$$

for all $y \in Y$, $w \in T(y)$. But this implies that $v/a \in B^T(b/a, x)$, which is the same as $v \in a \odot B^T(b, x)$. We prove now (ii). We claim first that any element of $E' \in \mathbb{E}(aT)$ can be written as $a \odot E$, for some $E \in \mathbb{E}(T)$. In other words, that $\mathbb{E}(aT) = \{a \odot E \mid \text{ such that } E \in \mathbb{E}(T)\}$. Indeed, given $E' \in \mathbb{E}(aT)$, define $E \in \mathbb{E}(T)$ by $E(b,x) = 1/aE'(ab,x)$. It is clear that $E \in \mathbb{E}(T)$ and $E' = a \odot E$. Hence by (14) and (30), we get

$$\begin{aligned}\Gamma(S^{aT}) &= \bigcap_{E' \in \mathbb{E}(aT)} \Gamma(E') = \bigcap_{E \in \mathbb{E}(T)} \Gamma(a \odot E) \\ &= \bigcap_{E \in \mathbb{E}(T)} \gamma_a(\Gamma(E)) = \gamma_a(\bigcap_{E \in \mathbb{E}(T)} \Gamma(E)) \\ &= \gamma_a(\Gamma(S^T)) = \Gamma(a \odot S^T).\end{aligned}$$

Part (iii) follows from (30) and the fact that the mapping $\gamma_a(\overline{A}) = \overline{\gamma_a(A)}$ for every set $A \subset X \times X^* \times \mathbb{R}_+$. $\square$

Remark 4.12 *Assume T_1 and T_2 are maximal monotone and such that $T_1 + T_2$ is also maximal monotone. Then it is clear from the Definition 4.1 and* (15) *that*

$$S^{T_1+T_2} \subset S^{T_1} \oplus S^{T_2} \subset E_1 \oplus E_2 \subset B^{T_1} \oplus B^{T_2} \subset B^{T_1+T_2},$$

for all $E_1 \in \mathbb{E}(T_1)$ and all $E_2 \in \mathbb{E}(T_2)$.

From the proof of Lemma 4.8, we see that closedness of $a \odot E$ is a direct consequence of the closedness of E. When it comes to conclude closedness of $E \oplus E'$ from closedness of E and E', we need some extra conditions. This is the subject of the next Section.

5 Closedness of Addition

The sum of two maximal monotone operators T_1 and T_2 does not preserve maximal monotonicity in general. Rockafellar[18] proved that if X is reflexive and $D(T_1)^0 \cap D(T_2) \neq \emptyset$, then $T_1 + T_2$ is maximal monotone. We will show that under the same assumptions, addition of closed enlargements in $\mathbb{E}(T_1), \mathbb{E}(T_2)$ yields a closed enlargement in $\mathbb{E}(T_1 + T_2)$.

Recall that $E \in \mathbb{E}_c(T)$ if and only if $\Gamma(E)$ is closed. However, the fact of being a closed enlargement of a monotone T allows us to establish a stronger closedness property. Namely, $\Gamma(E)$ contains not only the limits of its strongly convergent sequences, but also the limits of its sequences which converge weakly both in X and X^*. In particular, it will contain the limits of its sequences, when they converge strongly on X and weakly on X^*. A set $A \subset X \times X^* \times \mathbb{R}$ satisfying this property will be called *strong-weakly closed* ((s-w)-closed, for short). In a similar way, if we consider weak* convergence in X^*, we define *strong-weakly* closed* sets.

The closedness property announced above is connected with condition **r3**, and is proved below.

Lemma 5.1 *Assume $T : X \rightrightarrows X^*$ maximal monotone and take $E \in \mathbb{E}(T)$. Then*

$$E \in \mathbb{E}_c(T) \text{ if and only if } \Gamma(E) \text{ is (s-w)-closed.}$$

Proof. If $E \in \mathbb{E}_c(T)$, then by definition $\Gamma(E)$ is closed. By Theorem 3.2, $\Psi(\Gamma(E)) = \mathrm{Epi}(\Lambda_E)$. Since Ψ maps closed sets in closed sets and Λ_E is convex, $\Psi(\Gamma(E))$ is a closed and convex set. By convexity, this epigraph is closed w.r.t. the weak topologies both in X^* and X. In particular, this implies that $\mathrm{Epi}(\Lambda_E)$ is (s-w)-closed. Noting that Ψ^{-1} maps (s-w)-closed sets in (s-w)-closed sets, we conclude that $\Gamma(E)$ is (s-w)-closed. Conversely, if $\Gamma(E)$ is (s-w)-closed, then it is closed, which means that $E \in \mathbb{E}_c(T)$. □

For every $E \in \mathbb{E}(T)$, the set $G(T)$ has an important relationship with $\Gamma(E)$. Indeed, if $(x, v, \varepsilon) \in \Gamma(E)$ then it holds that

$$\sup_{(y,u)\in G(T)} \langle x - y, u - v\rangle \leq \varepsilon,$$

where we are using (15) and Definition 2.1. We want to express formally this relationship. Let A be a subset of $X \times X^* \times \mathbb{R}$. We say that $\tilde{A} \subset X \times X^*$ is a *core of* A when for every $(x, v, \varepsilon) \in A$ it holds that

$$\sup_{(y,u)\in \tilde{A}} \langle x - y, u - v\rangle \leq \varepsilon.$$

We point out that any subset of $G(T)$ is a core of $\Gamma(E)$, and when T is maximal, $G(T)$ is the biggest possible core of $\Gamma(E)$.

In view of (31), in order to establish closedness of $E \oplus E'$, we need to find conditions under which the partial sum of sets (see Definition 4.6(b)), preserves closedness. This is the question addressed by the following lemma. Denote by P_X the projection onto the X-coordinate, and by $B_X(\cdot, r)$, $B_{X^*}(\cdot, R)$ the closed balls of radius r and R of X and X^*, respectively.

Lemma 5.2 *Let $A, B \subset X \times X^* \times \mathbb{R}_+$ be (s-w*)-closed sets, and let $\tilde{A}$, $\tilde{B} \subset X \times X^*$ be cores of A and B, respectively. Assume also that there is an* *element $x \in P_X(\tilde{A}) \cap P_X(\tilde{B})$ for which there exist $r, R > 0$ such that the following properties hold:*

$$\begin{array}{ll} I) & B_X(x,r) \subset P_X(\tilde{A}) \cap P_X(A), \\ II) & (B_X(x,r) \times X^*) \cap \tilde{A} \subset X \times B_{X^*}(0,R). \end{array}$$

Then $A \tilde{+} B$ is (s-w*)-closed.

Proof. Take a sequence $\{(z^k, w^k, \varepsilon_k)\}_k \subset A \tilde{+} B$, converging to (z, w, ε), with respect to the strong topology in X and with respect to the weak* topology in X^*. Our aim is to prove that $(z, w, \varepsilon) \in A \tilde{+} B$. By definition of $A \tilde{+} B$, there exist sequences $\{(z^k, u^k, a_k)\}_k \subset A$ and $\{(z^k, v^k, b_k)\}_k \subset B$, such that

$$u^k + v^k = w^k, \quad a_k + b_k = \varepsilon_k. \tag{32}$$

Take $x \in P_X(\tilde{A}) \cap P_X(\tilde{B})$ satisfying the assumption of the Lemma. Then, there exist $v_1, v_2 \in X^*$ such that $(x, v_1) \in \tilde{A}$ and $(x, v_2) \in \tilde{B}$. Since $\tilde{A}$, $\tilde{B}$ are cores of A and B, respectively, we have that

$$\langle u^k - v_1, z^k - x \rangle \geq -a_k, \text{ and} \tag{33}$$

$$\langle v^k - v_2, z^k - x \rangle \geq -b_k. \tag{34}$$

Take now $r, R > 0$ as in (I)-(II). Then for every $\xi \in B_X(0,r)$, there exists $w_0 \in B_{X^*}(0,R)$ such that $(x + \xi, w_0) \in \tilde{A}$. Using again the fact that $\tilde{A}$ is a core of A, we get for every $\xi \in B_X(0,r)$,

$$\langle u^k - w_0, z^k - (x + \xi) \rangle \geq -a_k, \tag{35}$$

We claim that there exists $K_0 > 0$ such that

$$\langle u^k, \xi \rangle \leq K_0,$$

for all k and for all $\xi \in B_X(0,r)$. This fact readily implies boundedness of $\{u^k\}$. Indeed, if the claim is true then

$$\|u^k\| = 1/r \sup_{y \in X, \, \|y\|=1} \langle u^k, ry \rangle \leq K_0/r,$$

for all k. Hence $\{u^k\}$ is bounded. We point out that this is all we need for proving that $(z, w, \varepsilon) \in A \tilde{+} B$. Indeed, using that $\{u^k + v^k\}$ converges weakly* to w, we would also get boundedness of $\{v^k\}$. As a consequence, each of these sequences have weakly* convergent subsequences, with limits u and $w-u$ respectively. Since $\{a_k\}_k, \{b_k\}_k$ are nonnegative and the sum $\{a_k+b_k\}_k$ converges, they are bounded, and hence have convergent subsequences, with limits a and $\varepsilon - a$ respectively. Using that $\{z^k\}$ converges strongly to z, and A, B are strong-weakly* closed, we conclude that $(z, u, a) \in A$ and $(z, w-u, \varepsilon - a) \in B$, yielding $(z, w, \varepsilon) \in A\tilde{+}B$. So we proceed to prove the claim. From (35) we get

$$\langle u^k, \xi\rangle \leq a_k + \langle w_0, x+\xi - z^k\rangle - \langle u^k, x - z^k\rangle.$$

We know that $\{z^k\}_k$ and $\{a_k\}_k$ are bounded, and by (II) it holds that $\|w_0\| \leq R$. Hence there exists $M_1 \geq 0$ such that

$$\langle u^k, \xi\rangle \leq M_1 - \langle u^k, x - z^k\rangle.$$

Therefore, the claim will hold if we show that $|\langle u^k, x - z^k\rangle| \leq L$, for some $L \geq 0$. Equivalently, we will show that $\langle u^k, x-z^k\rangle$ is bounded from above and from below. Boundedness from above follows from (33), since this inequality yields the existence of some $L_1 \geq 0$ such that

$$\langle u^k, x - z^k\rangle \leq a_k + \langle v_1, x - z^k\rangle \leq L_1,$$

where we used again boundedness of $\{z^k\}_k$ and $\{a_k\}_k$. To prove boundedness from below, use (34) to get (in the same way as in the expression above)

$$\langle v^k, x - z^k\rangle \leq b_k + \langle v_2, x - z^k\rangle \leq L_2,$$

for some $L_2 \geq 0$. The sequences $\{z^k\}_k$ and $\{u^k + v^k\}_k$ are bounded, so we get for some $M \geq 0$

$$-M \leq \langle u^k + v^k, x - z^k\rangle \leq M.$$

Hence

$$\langle u^k, x - z^k\rangle = \langle u^k + v^k, x - z^k\rangle - \langle v^k, x - z^k\rangle \geq -M - L_2,$$

which implies the boundedness from below. The claim is true and the lemma is proved. ◻

Note that the compacity property used in the proof above holds for any X which is a locally convex topological linear space (see, e.g. [22, Theorem 1, Appendix to Chapter V]). So the Lemma is still true for such X, of course, in this case the balls $B_X(x, r)$ and $B_{X^*}(0, R)$ should be replaced by suitable neighborhoods of $x \in X$ and $0 \in X^*$.

Now we are in conditions to state our first result on closedness of $E_1 \oplus E_2$.

Theorem 5.3 *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators such that $D(T_1)^0 \cap D(T_2) \neq \emptyset$. Take $E_1 \in \mathbb{E}(T_1)$ and $E_2 \in \mathbb{E}(T_2)$. If $\Gamma(E_1)$ and $\Gamma(E_1)$ are (s-w*)-closed, then $\Gamma(E_1 \oplus E_2)$ is (s-w*)-closed.*

Proof. By (31), $\Gamma(E_1 \oplus E_2) = \Gamma(E_1)\tilde{+}\Gamma(E_2)$. Hence it is enough to prove that the sets $A := \Gamma(E_1)$ and $B := \Gamma(E_2)$ are in conditions of Lemma 5.2. Take $\tilde{A} := G(T_1)$ and $\tilde{B} := G(T_2)$ as cores of A and B respectively. We claim that assumptions (I)-(II) of the Lemma hold for $x \in D(T_1)^0 \cap D(T_2)$. Indeed, for such an x there exists $r_0 > 0$ such that $B_X(x, r_0) \subset D(T_1) = P_X(\tilde{A}) \subset P_X(\Gamma(E_1)) = P_X(A)$, which implies condition (I). For checking (II), note that T_1 is locally bounded at such x, and then there exist $r_1, R > 0$ such that $T_1(B_X(x, r_1)) \subset B_{X^*}(0, R)$. This implies that $(B_X(x, r_1) \times X^*) \cap \tilde{A} = (B_X(x, r_1) \times X^*) \cap G(T_1) \subset X \times B_{X^*}(0, R)$. Take $r := \min\{r_0, r_1\}$. Then the assumptions of Lemma 5.2 hold for these x, r, R and hence $\Gamma(E_1 \oplus E_2)$ is (s-w*)-closed. □

Corollary 5.4 *Assume that X is reflexive and let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators such that $D(T_1)^0 \cap D(T_2) \neq \emptyset$. If $E_1 \in \mathbb{E}_c(T_1)$ and $E_2 \in \mathbb{E}_c(T_2)$, then $\Gamma(E_1 \oplus E_2)$ is (s-w)-closed. In particular, $(E_1 \oplus E_2) \in \mathbb{E}_c(T_1 + T_2)$.*

Proof. Since X is reflexive, s-w closedness and s-w* closedness are equivalent. By Lemma 5.1, we have that $\Gamma(E_1)$ and $\Gamma(E_2)$ are s-w closed. Then we are in the conditions of Theorem 5.3, which implies that $\Gamma(E_1 \oplus E_2)$ is (s-w)-closed. This readily gives the strong closedness of $\Gamma(E_1 \oplus E_2)$, and hence $(E_1 \oplus E_2) \in \mathbb{E}_c(T_1 + T_2)$. □

We establish below convexity of $\mathbb{E}_c(T)$.

Corollary 5.5 *Assume that X is reflexive and $T : X \rightrightarrows X^*$ is a maximal monotone operator such that $D(T)^0 \neq \emptyset$. Then for every E, $E' \in \mathbb{E}_c(T)$ and every $a, b > 0$, it holds*

$$(a \odot E) \oplus (b \odot E') \in \mathbb{E}_c((a + b)T).$$

As a consequence, when $a + b = 1$, we have $(a \odot E) \oplus (b \odot E') \in \mathbb{E}_c(T)$.

Proof. Take $T_1 := aT$ and $T_2 := bT$ in Corollary 5.4. □

6 Operations in $\mathcal{H}(T)$

By Theorem 3.5, if T is maximal monotone, then the family of enlargements $\mathbb{E}_c(T)$ and the family of l.s.c. convex functions $\mathcal{H}(T)$ are in a one-to-one correspondence, given by $E \mapsto \Lambda_E$. The application $\Lambda_{(\cdot)}$ maps enlargements in scalar functions. A natural question is how to express $\Lambda_{a\odot E}$ and $\Lambda_{E_1\oplus E_2}$ from Λ_E, Λ_{E_1} and Λ_{E_2}. In order to do this, we need to define in $\mathcal{H}(T)$ the operations which mirror $\odot$ and $\oplus$.

Definition 6.1 *Given $\varphi : X \times X^* \to \overline{\mathbb{R}}$, the* dual epi-multiplication *of φ by $a > 0$ is the function $a \star \varphi : X \times X^* \to \overline{\mathbb{R}}$ given by*

$$a \star \varphi(x, v) := a\varphi(x, a^{-1}v).$$

Given $\varphi_1,\ \varphi_2 : X \times X^* \to \overline{\mathbb{R}}$, the dual epi-sum *of φ_1, φ_2 is the function $\varphi_1\#\varphi_2 : X \times X^* \to \overline{\mathbb{R}}$ given by*

$$\varphi_1\#\varphi_2(x, v) := \inf_{\substack{u,w\in X^* \\ u+w=v}} \left\{\varphi_1(x, u) + \varphi_2(x, w)\right\}.$$

Let Y be a normed linear space and φ, φ_1, $\varphi_2 : Y \to \overline{\mathbb{R}}$.
The *strict epigraph* of φ, denoted by $\mathrm{Epi}'(\varphi)$, is defined as

$$\mathrm{Epi}'(\varphi) := \{(y, \alpha) \mid \varphi(y) < \alpha\}.$$

Recall that the classical *epi-sum of* φ_1 and φ_2, defined as

$$\varphi_1 \diamondsuit \varphi_2(y) := \inf_{y_1+y_2=y} \{\varphi_1(y_1) + \varphi_2(y_2)\}$$

is *exact* at $y \in Y$ provided the infimum above is attained.

It is clear from Definition 6.1 that the dual epi-sum has the same properties as the classical epi-sum. We need the following well-known result connected with the epi-sum. See [20, 15, 12] for more material on the classical epi-sum.

Lemma 6.2 *(i) A function φ is convex if and only if its strict epigraph is convex.*

(ii) $\mathrm{Epi}'\,\varphi_1 \diamondsuit \varphi_2 = \mathrm{Epi}'\,\varphi_1 + \mathrm{Epi}'\,\varphi_2$,

Lemma 6.3 *Let T_1, $T_2 : X \rightrightarrows X^*$ be maximal monotone operators such that $D(T_1) \cap D(T_2) \neq \emptyset$. Assume that h_1, $h_2 : X \times X^* \to \overline{\mathbb{R}}$ are convex and satisfy* (21)-(22) *for T_1 and T_2, respectively. Then $h_1 \# h_2$ is convex and verifies* (21) *and* (22) *for $T_1 + T_2$.*

Proof. Checking (21) for $h_1 \# h_2$ is straightforward from the definitions. To check (22), assume that $(x, v) \in X \times X^*$ is such that $v \in (T_1 + T_2)x$. Then there exist $\bar{u}, \bar{w} \in X^*$ such that $v = \bar{u} + \bar{w}$, with $\bar{u} \in T_1(x)$ and $\bar{w} \in T_2(x)$. Thus,

$$\begin{array}{rcl} \langle v, x \rangle & \leq & \inf\limits_{\substack{u,w \in X^* \\ u+w=v}} \{h_1(x, u) + h_2(x, w)\} \\ & \leq & h_1(x, \bar{u}) + h_2(x, \bar{w}) = \langle \bar{u}, x \rangle + \langle \bar{w}, x \rangle = \langle v, x \rangle, \end{array}$$

where we used that fact that h_1, h_2 satisfy (21)-(22). Hence $h_1 \# h_2$ verifies (22) for $T_1 + T_2$. Finally, we prove convexity of $h_1 \# h_2$. By Lemma 6.2(i), convexity of $h_1 \# h_2$ will follow from the convexity of the set $\mathrm{Epi}'\,h_1 \# h_2$. We know that the sets $\mathrm{Epi}'\,h_1$ and $\mathrm{Epi}'\,h_1$ are convex. Using now Lemma 6.2(ii), we can write

$$\mathrm{Epi}'\,h_1 \# h_2 = \mathrm{Epi}'\,h_1 \tilde{+}\, \mathrm{Epi}'\,h_1.$$

Since $\tilde{+}$ preserves convexity, we conclude that $h_1 \# h_2$ is convex. Observe that the assumption on the intersection of the domains is also used here, for $\tilde{+}$ to be well-defined. ∎

The fact that $h_1 \in \mathcal{H}(T_1)$ and $h_2 \in \mathcal{H}(T_2)$, does not necessarily imply that $h_1 \# h_2$ is lower semicontinuous. This is in correspondence with the fact that $E_1 \oplus E_2$ may not be closed, even when both enlargements are. The Lemma below gives us a basic tool for establishing lower semicontinuity of $h_1 \# h_2$.

Lemma 6.4 *Let T, $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone such that $T_1 + T_2$ is also maximal monotone. Take $E_1 \in \mathbb{E}(T_1)$ and $E_2 \in \mathbb{E}(T_2)$. If $E_1 \oplus E_2 \in \mathbb{E}_c(T_1 + T_2)$, then*

$$\mathrm{Epi}(\Lambda_{E_1} \# \Lambda_{E_2}) = \mathrm{Epi}(\Lambda_{E_1 \oplus E_2}) = \mathrm{Epi}(\Lambda_{E_1}) \tilde{+}\, \mathrm{Epi}(\Lambda_{E_2}),$$

and the set above is closed.

Proof. Using (31), (29) and Theorem 3.2 we can write

$$\begin{aligned}\mathrm{Epi}(\Lambda_{E_1\oplus E_2}) &= \psi(\Gamma(E_1 \oplus E_2)) = \psi(\Gamma(E_1)\tilde{+}\Gamma(E_2)) \\ &= \psi(\Gamma(E_1))\tilde{+}\psi(\Gamma(E_2)) = \mathrm{Epi}(\Lambda_{E_1})\tilde{+}\,\mathrm{Epi}(\Lambda_{E_2}).\end{aligned}$$

Since $E_1 \oplus E_2$ is closed, the set above is closed. It only remains to prove that $\mathrm{Epi}(\Lambda_{E_1})\tilde{+}\,\mathrm{Epi}(\Lambda_{E_2}) = \mathrm{Epi}(\Lambda_{E_1}\#\Lambda_{E_2})$. The inclusion $\mathrm{Epi}(\Lambda_{E_1})\tilde{+}\,\mathrm{Epi}(\Lambda_{E_2}) \subset \mathrm{Epi}(\Lambda_{E_1}\#\Lambda_{E_2})$ follows directly from the definitions. Let us prove the converse inclusion. Take $(x, v, \varepsilon) \in \mathrm{Epi}(\Lambda_{E_1}\#\Lambda_{E_2})$. Then $\Lambda_{E_1}\#\Lambda_{E_2}(x, v) < \varepsilon + 1/n$, for all $n \in \mathbb{N}$. This implies that there exist $u_n, w_n \in X^*$ with $u_n + w_n = v$, such that

$$\Lambda_{E_1}(x, u_n) + \Lambda_{E_2}(x, w_n) < \varepsilon + 1/n,$$

for all n. Equivalently, there exist $a_n, b_n \in \mathbb{R}$ with $a_n + b_n = \varepsilon + 1/n$, such that $\Lambda_{E_1}(x, u_n) < a_n$ and $\Lambda_{E_2}(x, w_n) < b_n$. This means that $(x, u_n, a_n) \in \mathrm{Epi}(\Lambda_{E_1})$ and $(x, w_n, b_n) \in \mathrm{Epi}(\Lambda_{E_2})$. Hence, $(x, v, \varepsilon + 1/n) = (x, u_n + w_n, a_n + b_n) \in \mathrm{Epi}(\Lambda_{E_1})\tilde{+}\,\mathrm{Epi}(\Lambda_{E_2})$. Taking limits on n and using the fact that the set in the right-hand side is closed, we conclude that $(x, v, \varepsilon) \in \mathrm{Epi}(\Lambda_{E_1})\tilde{+}\,\mathrm{Epi}(\Lambda_{E_2})$. □

We will see bellow that $h_1\#h_2$ is l.s.c. when $L^{h_1} \oplus L^{h_2} \in \mathbb{E}_c(T_1 + T_2)$, where the enlargement L^h is given by (23).

Theorem 6.5 *Let $T, T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone such that $T_1 + T_2$ is also maximal monotone.*

- *(a) For all $h \in \mathcal{H}(T)$ and all $a > 0$, the function $a \star h \in \mathcal{H}(aT)$.*
- *(b) Take $h_1 \in \mathcal{H}(T_1)$ and $h_2 \in \mathcal{H}(T_2)$. If $L^{h_1} \oplus L^{h_2} \in \mathbb{E}_c(T_1 + T_2)$, then $h_1\#h_2 \in \mathcal{H}(T_1 + T_2)$. In particular, $h_1\#h_2 \in \mathcal{H}(T_1 + T_2)$ when X is reflexive and $D(T_1)^0 \cap D(T_2) \neq \emptyset$.*

Proof. Assertion (a) follows easily from the definitions. Let us prove assertion (b). By Lemma 6.3, we only have to prove that $h_1\#h_2$ is l.s.c. Our assumption allows us to apply Lemma 6.4, for $E_1 := L^{h_1}$ and $E_2 := L^{h_2}$. On the other hand, combining Theorem 3.5 with Theorem 3.2 we get

$$\psi(\Gamma(L^{h_i})) = \mathrm{Epi}(h_i), \text{ for } i = 1, 2. \tag{36}$$

Hence, Lemma 6.4 implies that

$$\mathrm{Epi}(h_1\#h_2) = \mathrm{Epi}(\Lambda_{L^{h_1}\oplus L^{h_2}}) = \mathrm{Epi}(h_1)\tilde{+}\,\mathrm{Epi}(h_2),$$

which is a closed set. This fact readily implies that $h_1 \# h_2$ is lower semicontinuous. ∎

By Theorem 3.5, there is a bijection between the spaces $\mathbb{E}_c(T)$ and $\mathcal{H}(T)$. This bijection is in fact an isomorphism with respect to the above-defined operations. This is proved in the Theorem below.

Theorem 6.6 *Take $E, E_1, E_2 : \mathbb{R}_+ \times X \rightrightarrows X^*$, h, h_1, $h_2 : X \times X^* \to \overline{\mathbb{R}}$ and $a > 0$. Assume that $E \in \mathbb{E}_c(T)$, $E_1 \in \mathbb{E}_c(T_1)$, $E_2 \in \mathbb{E}_c(T_2)$. It holds that*

(i) $\Lambda_{a \odot E} = a \star \Lambda_E$.

(ii) If $E_1 \oplus E_2$ is closed, then $\Lambda_{E_1 \oplus E_2} = \Lambda_{E_1} \# \Lambda_{E_2}$.

Proof. Consider γ_a and ψ as given in Lemma 4.7. It follows directly from the definitions that $\gamma_a(\mathrm{Epi}(\varphi)) = \mathrm{Epi}(a \star \varphi)$ and $\psi(\gamma_a(K)) = \gamma_a(\psi(K))$ for all $\varphi : X \times X^* \to \overline{\mathbb{R}}$ and all $K \subset X \times X^* \times \mathbb{R}$. Hence we can write

$$\begin{aligned} \mathrm{Epi}(\Lambda_{a \odot E}) &= \psi(\Gamma(a \odot E)) = \psi(\gamma_a(\Gamma(E)) \\ &= \gamma_a(\psi(\Gamma(E)) = \gamma_a(\mathrm{Epi}(\Lambda_E)) = \mathrm{Epi}(a \star \Lambda_E), \end{aligned}$$

where we used Theorem 3.2 in the first and next-to-last equality and (30) in the second one. Item (ii) is a direct consequence of Lemma 6.4. ∎

Using Theorem 6.5, we conclude that if $h_1 \in \mathcal{H}(T_1)$, $h_2 \in \mathcal{H}(T_2)$, T_1, T_2 maximal monotone and $L^{h_1} \oplus L^{h_2} \in \mathbb{E}_c(T_1 + T_2)$, then

$$(a \star h_1) \# (b \star h_2) \in \mathcal{H}(aT_1 + bT_2),$$

for all $a, b > 0$.

This yields the announced convexity of $\mathcal{H}(T)$.

Corollary 6.7 *Assume that X is reflexive and $T : X \rightrightarrows X^*$ is a maximal monotone operator such that $D(T)^0 \neq \emptyset$. Then for every h, $h' \in \mathcal{H}(T)$ and every $a, b > 0$, it holds*

$$(a \star h) \# (b \star h') \in \mathcal{H}((a + b)T),$$

As a consequence, when $a + b = 1$, we have $(a \star h) \# (b \star h') \in \mathcal{H}(T)$,

Proof. Take $h_1 := a \star h$, $h_2 := b \star h'$, $T_1 := aT$ and $T_2 := bT$ in Theorem 6.5(b). To apply this Theorem, we only have to check that $L^{h_1} \oplus L^{h_2}$ is closed. But this holds by Corollary 5.4 and the assumption on $D(T)$. ∎

Using Proposition 4.11 and Theorem 6.6, we can also retrieve the corresponding results for the extremal elements in $\mathcal{H}(aT)$.

Proposition 6.8 *Let T be maximal monotone and $a > 0$. With the same notation as in* (15)*, it holds that*

(i) $a \star \Lambda_{B^T} = \Lambda_{B^{aT}}$,

(ii) $a \star \Lambda_{S^T} = \Lambda_{S^{aT}}$.

(iii) $a \star \Lambda_{\overline{S^T}} = \Lambda_{\overline{S^{aT}}}$.

We finish with an application of this theory to the Brøndsted and Rockafellar enlargement of ∂f, where f is the sum of two convex and lower semicontinuous functions.

Let $f_1,\ f_2 : X \to \overline{\mathbb{R}}$ be proper, l.s.c. and convex, and such that $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$. It is a classical result (see, e.g., [10, Theorem 2.1]) that whenever

$$(f_1 + f_2)^*(v) = \min_{\substack{u,w \in X^* \\ u+w=v}} \{f_1^*(u) + f_2^*(w)\} \text{ for all } v \in X^*, \tag{37}$$

then it holds that

$$\breve{\partial}(f_1 + f_2)(b, x) = \bigcup_{\substack{b_1, b_2 \geqslant 0 \\ b_1+b_2=b}} \breve{\partial} f_1(b_1, x) + \breve{\partial} f_2(b_2, x), \tag{38}$$

for all $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ (we are using the notation of (9)).

We will prove that these conditions are equivalent.

Theorem 6.9 *Let $f_1,\ f_2 : X \to \overline{\mathbb{R}}$ be proper, l.s.c. and convex, and such that* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$*. Then* (38) *is equivalent to* (37)*.*

Proof. As stated before, under the above assumptions, (37) implies (38).

Now, assume that (38) holds. Since $f_1,\ f_2$ are proper, l.s.c. and convex, $f_1 + f_2$ is convex, proper and l.s.c. Therefore, $\partial(f_1 + f_2)$ is maximal monotone. Moreover, taking $b = 0$ in (38), we get $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$. Using (27), (38) can be rewritten as

$$\breve{\partial}(f_1 + f_2) = \breve{\partial} f_1 \oplus \breve{\partial} f_2. \tag{39}$$

Since $\partial(f_1 + f_2)$ is maximal, we conclude by (13) that the enlargement $\breve{\partial}(f_1 + f_2)$ is closed. Hence, $\breve{\partial} f_1 \oplus \breve{\partial} f_2$ is closed. Using now Theorem 6.6(ii) we get

$$\Lambda_{\breve{\partial} f_1 \oplus \breve{\partial} f_2} = \Lambda_{\breve{\partial} f_1} \# \Lambda_{\breve{\partial} f_2}.$$

Combining this equality with (39), we get

$$\Lambda_{\breve{\partial}(f_1+f_2)} = \Lambda_{\breve{\partial} f_1} \# \Lambda_{\breve{\partial} f_2}. \tag{40}$$

By (26) and the definition of $\#$,

$$\begin{aligned} \Lambda_{\breve{\partial}(f_1+f_2)}(x,v) &= (f_1+f_2)(x) + (f_1+f_2)^*(v), & (41)\\ \Lambda_{\breve{\partial} f_1} \# \Lambda_{\breve{\partial} f_2}(x,v) &= (f_1+f_2)(x) + f_1^* \diamondsuit f_2^*(v). & (42) \end{aligned}$$

Taking $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and using (40)-(42), we get

$$(f_1+f_2)^* = f_1^* \diamondsuit f_2^*. \tag{43}$$

To end the proof, we must show that this inf-convolution is exact.

Since $(f_1+f_2)^* = f_1^* \diamondsuit f_2^*$ is proper, $f_1^* \diamondsuit f_2^* > -\infty$. For those $v \in X^*$ such that $f_1^* \diamondsuit f_2^*(v) = +\infty$, the inf-convolution is trivially exact. Now take $v \in X^*$ such that

$$f_1^* \diamondsuit f_2^*(v) \in \mathbb{R}. \tag{44}$$

Let $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Then, by (42) and (44),

$$t_0 := \Lambda_{\breve{\partial} f_1} \# \Lambda_{\breve{\partial} f_2}(x_0, v) \in \mathbb{R}.$$

To show that the inf-convolution $f_1^* \diamondsuit f_2^*$ is exact at this v, we have to find $v_1, v_2 \in X^*$ such that $v_1 + v_2 = v$ and $f_1^* \diamondsuit f_2^*(v) = f_1^*(v_1) + f_2^*(v_2)$. For simplicity, call $h_1 := \Lambda_{\breve{\partial} f_1}$ and $h_2 := \Lambda_{\breve{\partial} f_2}$. Since $L^{h_1} \oplus L^{h_2} = \breve{\partial} f_1 \oplus \breve{\partial} f_2$ is closed, we can apply Lemma 6.4 with $E_1 := L^{h_1}$ and $E_2 := L^{h_2}$ to conclude that

$$\operatorname{Epi}(h_1) \tilde{+} \operatorname{Epi}(h_2) = \operatorname{Epi}(h_1 \# h_2),$$

with the set above being closed. Since $(x_0, v, t_0) \in \operatorname{Epi}(h_1 \# h_2)$, the above equality implies that there exist $v_1, v_2 \in X^*$, $a_1, a_2 \in \mathbb{R}$ such that $(x_0, v_1, a_1) \in \operatorname{Epi} h_1$, $(x_0, v_2, a_2) \in \operatorname{Epi} h_2$ with $v_1 + v_2 = v$ and $a_1 + a_2 = t_0$. Then

$$t_0 = a_1 + a_2 \geq h_1(x_0, v_1) + h_2(x_0, v_2) \geq t_0,$$

where we used the fact that $(x_0, v_i, a_i) \in \operatorname{Epi} h_i$, for $i = 1, 2$. Using the above expression and (26), we can write

$$\begin{aligned} t_0 &= (\Lambda_{\breve{\partial} f_1} \# \Lambda_{\breve{\partial} f_2})(x_0, v) = \Lambda_{\breve{\partial}(f_1+f_2)}(x_0, v) \\ &= (f_1+f_2)(x_0) + (f_1+f_2)^*(v) = h_1(x_0, v_1) + h_2(x_0, v_2) \\ &= f_1(x_0) + f_1^*(v_1) + f_2(x_0) + f_2^*(v_2), \end{aligned}$$

which readily implies $(f_1+f_2)^*(v) = f_1^*(v_1) + f_2^*(v_2)$. Combining this fact with (43), we get the exactness at v. ∎

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