ON THE WELL-POSEDNESS OF ENTROPY SOLUTIONS TO CONSERVATION LAWS WITH A ZERO-FLUX BOUNDARY CONDITION

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ABSTRACT. We study a zero-flux type initial-boundary value problem for scalar conservation laws with a genuinely nonlinear flux. We suggest a notion of entropy solution for this problem and prove its well-posedness. The asymptotic behavior of entropy solutions is also discussed.

1. INTRODUCTION

In recent years significant advances have been made in the analysis of initial-boundary value problems for multi-dimensional scalar conservation laws of the type

$$\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = 0, \quad (\mathbf{x}, t) \in Q_T := \Omega \times (0, T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded spatial domain, $T > 0$, and the flux vector $\mathbf{f}$ is a smooth function of the unknown u. Moreover, (1.1) is supplemented with an initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{1.2}$$

It is well known that solutions of nonlinear conservation laws may become discontinuous as time evolves, even for smooth initial data, such that (1.1) has to be understood in the distributional sense. This in turn requires an entropy condition to select the physically relevant discontinuous solution, called the *entropy solution*.

A well-studied boundary condition for (1.1), (1.2) is the Dirichlet boundary condition

$$u(\mathbf{x}, t) = \phi(\mathbf{x}) \quad \text{for } (\mathbf{x}, t) \in \partial\Omega \times (0, T), \text{ e.g. } \phi \in L^\infty(\partial\Omega). \tag{1.3}$$

However, the boundary datum (1.3) may not always provide the most natural setting for conservation laws on bounded domains. For example, assume that u is the local density of a continuous phase that assumes values from a finite interval $[0, u_{\max}]$ only, and is associated with a kinematic flow velocity $\mathbf{v}(u)$. Then a bounded domain Ω typically corresponds to a closed container with impermeable rigid walls that induce the zero-flux boundary condition $(u\mathbf{v}(u)) \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outer normal vector to the boundary $\partial\Omega$ of Ω. This suggests the alternative zero-flux boundary condition

$$\mathbf{f}(u) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{1.4}$$

Published applications of scalar conservation laws that explicitly use zero-flux boundary conditions include, for example, the sedimentation of suspensions in closed vessels [2, 3, 4] and the dispersal of a single species of animals in a finite territory [18]. However, the boundary condition (1.4) is physically reasonable also in other applications, for example when (1.1) appears as the vanishing viscosity limit of a multi-dimensional model of turbulence [5] or of a simple model of two-phase flow in porous media [21].

To put the paper in the proper perspective, let us first recall some previous treatments of the Dirichlet problem (1.1), (1.2), (1.3). One major difficulty associated with this problem is due to the well-known propagation of solution values of (1.1) along characteristics, which may intersect

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$\partial\Omega$ from the interior of Ω, such that (1.3) does not hold in a pointwise sense for all times. The well-posedness of (1.1), (1.2), (1.3) in this situation has been recovered by the use of (for example, set-valued) entropy boundary conditions. The first existence and uniqueness analysis for BV solutions of (1.1), (1.2), (1.3) is due to Bardos et al. [1]. The BV property, which is established in [1] by deriving uniform BV estimates for the solutions of a regularized (uniformly parabolic) problem, ensures the existence of boundary traces, which is crucial for the uniqueness result. It was only later that Otto [15, 19, 20] was able to study the same problem in the less restrictive L^∞ setting, for which boundary traces do not exist in general, a fact that complicates significantly the notion of solution and the proofs. See also Chen and Frid [7, 8, 9] for formulations of boundary conditions in terms of divergence-measure fields. Finally, we mention that the recent results in the L^∞ setting were extended to strongly degenerate parabolic equations by Carrillo [6], Mascia et al. [16], and Michel and Vovelle [17], while the BV approach of [1] had been transferred to this type of equations much earlier [23]. Recently Vasseur [22] showed that L^∞ entropy solutions to (1.1) always have traces at the boundaries of Q_T. This result holds for genuinely nonlinear fluxes $\mathbf{f}(u)$ (in the sense of [14]), on domains Q_T whose boundaries satisfy a mild regularity assumption, and is independent of the initial and boundary conditions. Consequently, the L^∞ case for genuinely nonlinear fluxes can be treated as in Bardos et al. [1], i.e., the more complicated notion of entropy solution used by Otto can be avoided.

Karlsen, Lie and Risebro [11] showed that a front tracking method [10] converges to a weak solution of (1.1), (1.2), (1.4) if this problem is studied in one spatial dimension. This weak solution is unique in the class of functions that can be constructed as the L^1 limit of front tracking approximations. Moreover, they present numerical results for the case of two spatial dimensions. However, for none of these cases they present a notion of entropy solution for which existence *and* uniqueness is proved.

In this paper, we suggest a notion of L^∞ entropy solutions to the zero-flux problem (1.1), (1.2), (1.4) and prove its well-posedness (existence and uniqueness) in arbitrary space dimensions. Our notion of entropy solution involves a certain boundary term in the entropy integral inequality. In fact, we can show that this entropy formulation implies that the zero-flux boundary condition is satisfied in an almost everywhere sense. The new results are valid if the flux vector satisfies the genuine nonlinearity condition of [14]. This condition is imposed to ensure the existence of boundary traces via the result of Vasseur [22]. Vasseur's result is used herein as a main tool for establishing the equivalence of two alternative definitions of entropy solutions. One of them (Definition 3) consists of the above-mentioned entropy integral inequality that incorporates the boundary term, while the other (Definition 4) states the entropy inequality in the interior of the domain and the initial and boundary conditions as separate ingredients. We mention that the fluxes used in [11] satisfy the genuine nonlinearity condition used of [14].

Let us remark that it is unclear whether the BV approach may be applied at all to the zero-flux problem. In particular, the estimating techniques of Bardos et al. [1] cannot be applied here. The difficulty is that the regularized zero-flux boundary condition does not permit control over the tangential derivatives (with respect to $\partial\Omega$) of the solution. Thus, boundary traces of solutions to (1.1), (1.2), (1.4) seem hard to obtain via BV estimates, and this has motivated the approach taken in the present paper.

The remainder of this paper is organized as follows. In Section 2 we state some technical assumptions, introduce the concepts of domains with Lipschitz deformable boundaries and traces, and recall Vasseur's result from [22]. In Section 3 we present two alternative definitions of entropy solutions to (1.1), (1.2), (1.4), and prove their equivalence by using Vasseur's result. In particular, it turns out that these entropy solutions are characterized by pointwise satisfaction of the boundary condition (1.4), in contrast to what is known for the Dirichlet problem. In Sections 4 and 5 we prove the existence and uniqueness of entropy solutions, respectively. Finally, in Section 6 we study the asymptotic behavior (for $t \to \infty$) of the entropy solutions under some additional assumptions on Ω and $\mathbf{f}(u)$.

2. Assumptions and preliminaries

We ask that the flux vector $\mathbf{f}(u)$ depends smoothly on u for $u \in [0, u_{\max}]$, for some fixed $u_{\max} > 0$. To ensure an L^∞ bound on the solutions, we assume that

$$\mathbf{f}(0) = 0, \quad \mathbf{f}(u_{\max}) = 0. \tag{2.1}$$

To ensure the existence of boundary traces, we assume that the flux $\mathbf{f}(u)$ is genuinely nonlinear in the following sense [14]:

$$\forall (\tau, \boldsymbol{\zeta}) \in \mathbb{R} \times \mathbb{R}^N, \ \tau^2 + |\boldsymbol{\zeta}|^2 = 1 : \quad \mathcal{L}\big(\big\{u \in [0, u_{\max}] \,|\, \tau + \boldsymbol{\zeta} \cdot \mathbf{f}'(u) = 0\big\}\big) = 0, \tag{2.2}$$

where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure. This condition is satisfied if (see [9])

$$\mathcal{L}\big(\{u \in [0, u_{\max}] \,|\, \boldsymbol{\zeta} \cdot \mathbf{f}''(u) = 0\}\big) = 0 \quad \text{for all } \boldsymbol{\zeta} \in \mathbb{R}^N \text{ with } |\boldsymbol{\zeta}| = 1.$$

We adopt the usual entropy criterion, namely we consider only those weak solutions u that satisfy the inequality

$$\partial_t \eta(u) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(u) \leqslant 0 \quad \text{on } Q_T \text{ in the sense of distributions,}$$

for every entropy pair $(\eta, \mathbf{q})$ consisting of a convex entropy function $\eta = \eta(u)$ and a corresponding entropy flux defined by $\mathbf{q}'(u) = \mathbf{f}'(u)\eta'(u)$. It is sufficient to consider the Kružkov entropy functions $|u - k|$ along with the associated entropy fluxes $\operatorname{sgn}(u-k)(\mathbf{f}(u) - \mathbf{f}(k))$, $k \in \mathbb{R}$.

To state Vasseur's result [22], we introduce the concept of sets with Lipschitz deformable boundaries [7]. To this end, consider an open subset $\Omega \subset \mathbb{R}^N$ with a boundary $\partial\Omega$.

Definition 1. *We say that $\partial\Omega$ is a* deformable Lipschitz boundary *provided that the following hold:*

(a) *For all $\mathbf{x} \in \partial\Omega$ there exists a number $r > 0$ and a Lipschitz map $h : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,*

$$\Omega \cap \mathcal{Q}(\mathbf{x}, r) = \big\{\mathbf{y} \in \mathbb{R}^N : \, h(y_1, \dots, y_{N-1}) < y_N \big\} \cap \mathcal{Q}(\mathbf{x}, r),$$

where $\mathcal{Q}(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^N : |x_i - y_i| \le r, \ i = 1, \dots, N\}$. We denote by $\tilde{h}$ the map $(y_1, \dots, y_{N-1}) =: \tilde{y} \mapsto (\tilde{\mathbf{y}}, h(\tilde{y}))$.

(b) *There exists a mapping $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism that bi-Lipschitz over its image with $\Psi(\boldsymbol{\omega}, 0) = \boldsymbol{\omega}$ for all $\boldsymbol{\omega} \in \partial\Omega$. The map Ψ is called a* Lipschitz deformation *of the boundary $\partial\Omega$. We denote $\Psi_s(\boldsymbol{\omega}) = \Psi(\boldsymbol{\omega}, s)$ and $\partial\Omega_s = \Psi_s(\partial\Omega)$. We also denote by Ω_s the bounded open set whose boundary is $\partial\Omega_s$.*

Moreover, the Lipschitz deformation is said to be regular *if*

$$\lim_{s \to 0+} \nabla\Psi_s \circ \tilde{h} = \nabla\tilde{h} \quad \text{in } L^1_{\mathrm{loc}}(B), \tag{2.3}$$

where B denotes the greatest open set such that $\tilde{h}(B) \subset \partial\Omega$.

Obviously, if $\Omega \subset \mathbb{R}^N$ is an open set with a deformable Lipschitz boundary, then $Q_T = \Omega \times (0,T)$ is also an open set with deformable Lipschitz boundary in $\mathbb{R}^{N+1}$.

Our concept of trace is stated in the following definition.

Definition 2. *Let $Q \subset \mathbb{R}^{N+1}$ have a regular deformable Lipschitz boundary. We say that a given function $u \in L^\infty(Q)$ possesses a strong trace u^τ at ∂Q if $u^\tau \in L^\infty(\partial Q)$ has the property that for every regular (with respect to ∂Q) Lipschitz deformation ψ and every compact set $K \subset \partial Q$,*

$$\operatorname*{ess\,lim}_{s \to 0} \int_K \big|u\big(\psi(s, \mathbf{x})\big) - u^\tau(\mathbf{x})\big| \, d\mathcal{H}^N(\mathbf{x}) = 0, \tag{2.4}$$

where $\mathcal{H}^N$ is the N-dimensional Hausdorff measure.

The following result is proved in [22].

Theorem 1. *Let $Q \subset \mathbb{R}^{N+1}$ have a regular deformable Lipschitz boundary, and assume that $\mathbf{f}(u)$ satisfies the genuine nonlinearity condition (2.2). Then for every function $u \in L^\infty$ satisfying the conservation law $\partial_t u + \nabla_{\mathbf{x}} \cdot \mathbf{f}(u) = 0$ in Q and the entropy inequality $\partial_t \eta(u) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(u) \leqslant 0$ in Q for every entropy pair $(\eta, \mathbf{q})$, the trace $u^\tau \in L^\infty(\partial Q)$ exists. In particular, $(G(u))^\tau = G(u^\tau)$ for every smooth function G.*

3. Definition of entropy solutions

From now on in this paper, it is always understood that $\Omega \subset \mathbb{R}^N$ in (1.1) is a bounded open set with a deformable Lipschitz boundary. Moreover, for each $T > 0$, we shall use the notation

$$Q_T := \Omega \times (0,T), \quad \underline{\Pi}_T = \mathbb{R}^N \times [0,T).$$

We denote by $C_0^\infty(Q_T)$ (resp., $C_0^\infty(\underline{\Pi}_T)$) the set of all infinitely smooth functions on Q_T (resp., $\underline{\Pi}_T$), with compact support.

Definition 3. *A function $u \in L^\infty(Q_T)$ is called an entropy solution of the initial-boundary value problem (1.1), (1.2), (1.4) if the following entropy inequality holds:*

$$\begin{aligned}
&\forall k \in \mathbb{R},\ \forall \varphi \in C_0^\infty(\underline{\Pi}_T) \ \text{with } \varphi \geqslant 0: \\
&\int_0^T \int_\Omega \Big\{ |u-k| \partial_t \varphi + \operatorname{sgn}(u-k)\big(\mathbf{f}(u) - \mathbf{f}(k)\big) \cdot \nabla \varphi \Big\} \, d\mathbf{x} \, dt \\
&+ \int_\Omega \big| u_0(\mathbf{x}) - k \big| \varphi(\mathbf{x},0) \, dx \\
&+ \int_0^T \int_{\partial\Omega} \operatorname{sgn}(u^\tau - k) \mathbf{f}(k) \cdot \mathbf{n} \, \varphi(\mathbf{x},t) \, d\mathcal{H}^{N-1} \, dt \geqslant 0.
\end{aligned} \tag{3.1}$$

The following definition presents an alternative solution concept.

Definition 4. *A function $u \in L^\infty(Q_T)$ is called an entropy solution of the initial-boundary value problem (1.1), (1.2), (1.4) if the following conditions are satisfied:*

(1) *The following entropy inequality is satisfied:*

$$\begin{aligned}
&\forall k \in \mathbb{R},\ \forall \varphi \in C_0^\infty(Q_T),\ \varphi \geqslant 0: \\
&\int_0^T \int_\Omega \Big\{ |u-k| \partial_t \varphi + \operatorname{sgn}(u-k)\big(\mathbf{f}(u) - \mathbf{f}(k)\big) \cdot \nabla \varphi \Big\} \, d\mathbf{x} \, dt \geqslant 0.
\end{aligned} \tag{3.2}$$

(2) *The initial condition is satisfied as a limit in the following L^1 sense:*

$$\operatorname*{ess\,lim}_{t \to 0^+} \int_\Omega \big| u(\mathbf{x},t) - u_0(\mathbf{x}) \big| \, d\mathbf{x} = 0. \tag{3.3}$$

(3) *The boundary condition (1.4) is satisfied in the following pointwise sense:*

$$\mathbf{f}\big(u^\tau(\mathbf{x},t)\big) \cdot \mathbf{n} = 0 \quad \text{a.e. on } \partial\Omega \times (0,T), \tag{3.4}$$

where u^τ is the trace of u, which exists thanks to Theorem 1.

Before we show that both definitions are equivalent, as is stated in Lemma 1 below, let us mention that Definition 4 will be used for the existence proof, while both Definition 3 and Definition 4 will be used for proving uniqueness.

Lemma 1. *A function $u \in L^\infty(Q_T)$ is an entropy solution in the sense of Definition 3 if and only if it is an entropy solution in the sense of Definition 4.*

Proof. We first prove that Definition 3 implies Definition 4. It is obvious that (3.1) implies (3.2). To show that (3.3) is satisfied, we choose in (3.1) the test function $\varphi(\mathbf{x},t) = \zeta(t)\xi(\mathbf{x})$, where $\zeta \in C_c^\infty(-\infty,\delta)$, $\delta > 0$, $\xi \in C_0^\infty(\Omega)$, $\zeta \geqslant 0$, $\xi \geqslant 0$, which implies

$$\forall k \in \mathbb{R}: \quad \int_0^\delta \zeta'(t) \int_\Omega |u-k| \xi(\mathbf{x}) \, d\mathbf{x} \, dt + \int_\Omega \big| u_0(\mathbf{x}) - k \big| \xi(\mathbf{x}) \, d\mathbf{x} + C \int_0^\delta \zeta(t) \, dt \geqslant 0.$$

Choosing $\zeta(t) = \chi_{(-\delta,\delta)}(t)$ (after mollifying and passing to the limit), we get for $\delta \to 0$

$$\forall k \in \mathbb{R}: \quad -\operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega |u-k|\xi(\mathbf{x})\,d\mathbf{x} + \int_\Omega \big|u_0(\mathbf{x}) - k\big|\xi(\mathbf{x})\,d\mathbf{x} \geqslant 0. \tag{3.5}$$

The limit on the left-hand side exists due to Theorem 1. The initial condition (3.3) follows by taking $k < 0$ and $k > u_{\max}$ in (3.5). Inequality (3.1) implies

$$\forall \varphi \in C_0^\infty(Q_T): \quad \int_0^T \int_\Omega \big\{u\partial_t\varphi + \mathbf{f}(u)\cdot\nabla\varphi\big\}\,d\mathbf{x}\,dt = 0. \tag{3.6}$$

Indeed, it suffices to take $k = u_{\max}$ and $k = 0$ in (3.1), where we recall (2.1). Now we use in (3.6) the test function $\varphi(x,t) = \Phi(t)\xi(\mathbf{x})(1-\mu_h(\mathbf{x}))$, where $\Phi \in C_0^\infty(0,T)$, $\xi \in C_0^\infty\left(\overline{\Omega}\right)$, and $\{\mu_h\}_{h>0}$ is a sequence of functions in $C^2(\Omega)\cap C\left(\overline{\Omega}\right)$ such that

$$\lim_{h\to 0}\mu_h = 1 \ \text{ pointwise in } \Omega, \quad 0 \leqslant \mu_h \leqslant 1, \quad \mu_h = 0 \ \text{ on } \partial\Omega. \tag{3.7}$$

Taking the limit $h \to 0$ in the equation

$$\int_0^T \int_\Omega \Big\{u\Phi'(t)\xi(\mathbf{x})\big(1-\mu_h(\mathbf{x})\big) + \Phi(t)\big(1-\mu_h(\mathbf{x})\big)\mathbf{f}(u)\cdot\nabla\xi(\mathbf{x}) - \Phi(t)\xi(\mathbf{x})\mathbf{f}(u)\cdot\nabla\mu_h\Big\}d\mathbf{x}\,dt = 0,$$

and using that the function $\xi(\mathbf{x})$ may be chosen arbitrarily, we obtain (3.4).

As for the converse, let

$$\chi^h(t) := \begin{cases} t/h & \text{for } 0 \le t \le h,\\ 1 & \text{for } h < t \le T-h,\\ (T-t)/h & \text{for } T-h < t < T,\\ 0 & \text{for } t \notin (0,T).\end{cases}$$

Also, for $s \in [0,1]$ let the function $\zeta_s \in \operatorname{Lip}(\mathbb{R}^N)$ be defined by

$$\zeta_s(\mathbf{x}) := \begin{cases} 1 & \text{for } \mathbf{x} \in \Omega_s,\\ r/s & \text{for } \mathbf{x} \in \partial\Omega_r,\ 0 \le r \le s,\\ 0 & \text{for } \mathbf{x} \notin \Omega,\end{cases}$$

where $\partial\Omega_s$ is the image of $\partial\Omega$ under the Lipschitz deformation $\Psi(\boldsymbol{\omega}, s)$ with $\partial\Omega_0 = \partial\Omega$, and Ω_s is the bounded open set whose boundary is $\partial\Omega_s$. For notational convenience, we also introduce the function $\mathbf{F}(u,k) := \operatorname{sgn}(u-k)(\mathbf{f}(u) - \mathbf{f}(k))$.

Now let us define the function $\varphi(\mathbf{x},t) = \chi^h(t)\zeta_s(\mathbf{x})\tilde{\varphi}(\mathbf{x},t)$, where $\tilde{\varphi} \in C^\infty(\underline{\Pi}_T)$. An approximation argument reveals that we may use φ as a test function for (3.2). Then we obtain

$$\begin{aligned}\forall k \in \mathbb{R}: \quad & \int_0^T \int_\Omega \big\{|u-k|\partial_t\tilde{\varphi} + \mathbf{F}(u,k)\cdot\nabla\tilde{\varphi}\big\}\,d\mathbf{x}\,dt \\ & - \int_0^T \int_\Omega \big\{|u-k|\partial_t\tilde{\varphi} + \mathbf{F}(u,k)\cdot\nabla\tilde{\varphi}\big\}\big(1-\chi^h(t)\zeta_s(\mathbf{x})\big)\,d\mathbf{x}\,dt \\ & + \int_0^T \int_\Omega |u-k|\zeta_s(\mathbf{x})(\chi^h)'(t)\tilde{\varphi}\,d\mathbf{x}\,dt + \int_0^T \int_\Omega \chi^h(t)\tilde{\varphi}\mathbf{F}(u,k)\cdot\nabla\zeta_s(\mathbf{x})\,d\mathbf{x}\,dt \geqslant 0.\end{aligned} \tag{3.8}$$

Letting $h \to 0$ and using (3.3), we get

$$\begin{aligned}\forall k \in \mathbb{R}: \quad & \int_0^T \int_\Omega \big\{|u-k|\partial_t\tilde{\varphi} + \mathbf{F}(u,k)\cdot\nabla\tilde{\varphi}\big\}\,d\mathbf{x}\,dt \\ & - \int_0^T \int_\Omega \big\{|u-k|\partial_t\tilde{\varphi} + \mathbf{F}(u,k)\cdot\nabla\tilde{\varphi}\big\}\big(1-\zeta_s(\mathbf{x})\big)\,d\mathbf{x}\,dt \\ & + \int_\Omega \big|u_0(\mathbf{x}) - k\big|\tilde{\varphi}(\mathbf{x},0)\zeta_s(\mathbf{x})\,d\mathbf{x}\,dt\end{aligned}$$

$$+\int_0^T\int_\Omega \tilde\varphi \mathbf{F}(u,k)\cdot\nabla\zeta_s(\mathbf{x})\,d\mathbf{x}\,dt \geqslant 0.$$

Finally, sending $s\to 0$, using (3.4) and replacing $\tilde\varphi$ by the symbol φ again, we get

$$\begin{aligned}&\forall k\in\mathbb{R}:\quad \forall\varphi\in C^\infty(\mathbb{R}^{N+1}),\ \varphi\geqslant 0:\\ &\int_0^T\int_\Omega\big\{|u-k|\partial_t\varphi+\mathbf{F}(u,k)\cdot\nabla\varphi\big\}\,d\mathbf{x}\,dt\\ &+\int_\Omega\big|u_0(\mathbf{x})-k\big|\varphi(\mathbf{x},0)\,d\mathbf{x}\\ &+\int_0^T\int_{\partial\Omega}\operatorname{sgn}(u-k)\mathbf{f}(k)\cdot\mathbf{n}\,\varphi(x,t)\,d\mathcal{H}^{N-1}\,dt\geqslant 0,\end{aligned}$$

which is exactly (3.1). □

4. Existence of entropy solutions

To show the existence of an entropy solution (in the sense of the previous section), we consider as in [2] the following regularized parabolic problem for each $\varepsilon>0$:

$$\partial_t u^\varepsilon+\nabla_{\mathbf{x}}\cdot\mathbf{f}(u^\varepsilon)=\varepsilon\Delta u^\varepsilon,\quad (\mathbf{x},t)\in Q_T, \tag{4.1a}$$

$$u^\varepsilon(\mathbf{x},0)=u_0^\varepsilon(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{4.1b}$$

$$\big(\mathbf{f}(u^\varepsilon)-\varepsilon\nabla_{\mathbf{x}}u^\varepsilon\big)\cdot\mathbf{n}=0\quad\text{on }\partial\Omega, \tag{4.1c}$$

where u_0^ε is a sequence of smooth functions that converges to u_0 in $L^p(\Omega)$ for $1\le p<\infty$ assuming values in $[\varepsilon,u_{\max}-\varepsilon]$, for $\varepsilon>0$ sufficiently small. The existence and uniqueness of a classical solution to (4.1) follows from standard arguments, see e.g. [13, Ch. V]. In particular, for each fixed $\varepsilon>0$, the solution of (4.1) may be obtained as the limit when $\delta\to 0$ of a sequence of solutions to

$$\partial_t u+\nabla_{\mathbf{x}}\cdot\mathbf{f}(u)=\varepsilon\Delta u+\delta h(u),\quad (\mathbf{x},t)\in Q_T, \tag{4.2a}$$

$$u(\mathbf{x},0)=u_0^\varepsilon(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{4.2b}$$

$$\big(\mathbf{f}(u)-\varepsilon\nabla_{\mathbf{x}}u\big)\cdot\mathbf{n}=\delta(u-u^b(\mathbf{x},t))\quad\text{on }\partial\Omega, \tag{4.2c}$$

where $h(u)$ is a smooth function satisfying $h(0)>0$, $h(u_{\max})<0$, and $u^b(\mathbf{x},t)$ is a smooth function assuming values in $[\delta,u_{\max}-\delta]$, for sufficiently small $\delta>0$.

Lemma 2. *Suppose* (2.1) *holds. Then $u_0(x)\in[0,u_{\max}]$ for a.e. $x\in\Omega$ implies $u^\varepsilon(x,t)\in[0,u_{\max}]$ for every $(x,t)\in Q_T$.*

Proof. It suffices to prove the same assertion for the solution of (4.2). We will prove the latter by contradiction. So, assume the assertion does not hold. Hence, since $u_0^\varepsilon(\mathbf{x})\in[\varepsilon,u_{\max}-\varepsilon]$, there exists a $t_0\in(0,T)$ such that

$$t_0=\sup\{t\in[0,T]\,:\,u(\mathbf{x},\tau)\in[0,u_{\max}]\text{ for all }\mathbf{x}\in\Omega\text{ and }\tau\in[0,t]\}.$$

Therefore, there is a $\mathbf{x}_0\in\bar\Omega$ such that $u(\mathbf{x},t_0)\in\{0,u_{\max}\}$. Assuming $\mathbf{x}_0\in\Omega$ leads to a contradiction by using (4.2a). On the other hand, assuming $\mathbf{x}_0\in\partial\Omega$ leads to a contradiction by using (4.2c) and the resulting sign of $\partial_{\mathbf{n}}u$. This concludes the proof.

□

Theorem 2. *Suppose $u_0(x)\in[0,u_{\max}]$ for a.e. $x\in\Omega$ and that conditions* (2.1), (2.2) *hold. Then there exists an entropy solution u of the zero-flux initial-boundary value problem* (1.1), (1.2), (1.4), *which moreover satisfies $u(x,t)\in[0,u_{\max}]$ for a.e. $(x,t)\in Q_T$.*

Proof. We use Definition 4. By the compactness result of Lions, Perthame and Tadmor [14], we may extract a subsequence of solutions of (4.1) which converges in $L^1_{loc}(Q_T)$ to a function $u(\mathbf{x},t)$ which assumes values in $[0,u_{\max}]$ due to Lemma 2. The verification of (3.2) is completely standard. In particular, we have that $(u,\mathbf{f}(u))$, $(|u-k|,\operatorname{sgn}(u-k)(\mathbf{f}(u)-\mathbf{f}(k)))\in\mathcal{DM}^\infty(Q_T)$ for all $k\in\mathbb{R}$, (see [7, 8]). Now, to prove (3.3) and (3.4), we argue as follows. We multiply (4.1a) by

$\varphi \in C_0^\infty(\underline{\Pi}_T)$, integrate over Q_T, use integration by parts, use (4.1b) and (4.1c), and make $\varepsilon \to 0$, to obtain

$$\int_{Q_T} \{u\varphi_t + \mathbf{f}(u) \cdot \nabla_{\mathbf{x}}\varphi\}\, d\mathbf{x}\, dt + \int_\Omega u_0(\mathbf{x})\varphi(\mathbf{x}, 0)\, d\mathbf{x} = 0. \tag{4.3}$$

Now, (4.3) tells us about the normal trace $(u, \mathbf{f}(u)) \cdot \nu$ of the $\mathcal{DM}$-field $(u, \mathbf{f}(u))$ that

$$(u, \mathbf{f}(u)) \cdot \nu\big|_{\Omega\times\{0\}} = u_0(\mathbf{x}), \qquad (u, \mathbf{f}(u)) \cdot \nu\big|_{\partial\Omega\times(0,T)} = 0.$$

Hence, Vasseur's Theorem 1 directly implies (3.3) and (3.4), and the proof is finished. □

5. Uniqueness of entropy solutions

Theorem 3. *Suppose u_0, $v_0 \in L^\infty(\Omega)$ and that conditions (2.1), (2.2) hold. Let u and v be entropy solutions of (1.1), (1.2), (1.4) with initial conditions $u|_{t=0} = u_0$ and $v|_{t=0} = v_0$, respectively. Then for any $t > 0$*

$$\int_\Omega \big|u(\mathbf{x}, t) - v(\mathbf{x}, t)\big|\, d\mathbf{x} \leqslant \int_\Omega \big|u_0(\mathbf{x}) - v_0(\mathbf{x})\big|\, d\mathbf{x}. \tag{5.1}$$

In particular, there exists at most one entropy solution to the zero-flux initial-boundary value problem (1.1), (1.2), (1.4).

Proof. We consider two entropy solutions $u = u(\mathbf{x}, t)$ and $v = v(\mathbf{y}, s)$. Then the standard "doubling of the variables" argument [12] yields that for all nonnegative functions $\varphi = \varphi(\mathbf{x}, t, \mathbf{y}, s)$ in $C^\infty(Q_T \times Q_T)$ having the property that $\varphi(\cdot, \cdot, \mathbf{y}, s), \varphi(\mathbf{x}, t, \cdot, \cdot) \in C_c^\infty(Q_T)$ for each $(\mathbf{y}, s) \in Q_T$ and $(\mathbf{x}, t) \in Q_T$, respectively, the following inequality holds:

$$\iiiint\limits_{Q_T\times Q_T} \big\{|u - v|(\partial_t\varphi + \partial_s\varphi) + \mathbf{F}(u, v) \cdot (\nabla_{\mathbf{x}}\varphi + \nabla_{\mathbf{y}}\varphi)\big\}\, ds\, d\mathbf{y}\, dt\, d\mathbf{x} \geqslant 0. \tag{5.2}$$

We pick $\theta \in C_c^\infty(0, T)$, $\theta \geqslant 0$, and choose in (5.2)

$$\varphi(\mathbf{x}, t, \mathbf{y}, s) := \mu_\delta(\mathbf{x})\mu_\eta(\mathbf{y})\rho_{l,m}(\mathbf{x}, t, \mathbf{y}, s)\theta(t), \quad \delta, \eta > 0,\ l, m \in \mathbb{N},$$

where μ_δ, μ_η are sequences of the type used in the proof of Lemma 1, see in particular (3.7), and

$$\rho_{l,m}(\mathbf{x}, t, \mathbf{y}, s) := \rho_l(t - s)\rho_m(\mathbf{x} - \mathbf{y}),$$

with $\{\rho_l\}_{l\in\mathbb{N}}$ and $\{\rho_m\}_{m\in\mathbb{N}}$ being sequences of symmetric mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. Setting $\partial_{t+s} := \partial_t + \partial_s$, $\nabla_{\mathbf{x}+\mathbf{y}} := \nabla_{\mathbf{x}} + \nabla_{\mathbf{y}}$, we obtain from (5.2)

$$\begin{aligned} &\iiiint\limits_{Q_T\times Q_T} |u - v|\mu_\delta\mu_\eta\rho_{l,m}\theta'\, ds\, d\mathbf{y}\, dt\, d\mathbf{x} \\ &+ \iiiint\limits_{Q_T\times Q_T} \mathbf{F}(u, v)\big(\nabla_{\mathbf{x}}\mu_\delta\big)\mu_\eta\rho_{l,m}\theta\, ds\, d\mathbf{y}\, dt\, d\mathbf{x} \\ &+ \iiiint\limits_{Q_T\times Q_T} \mathbf{F}(u, v)\mu_\delta\big(\nabla_{\mathbf{y}}\mu_\eta\big)\rho_{l,m}\theta\, ds\, d\mathbf{y}\, dt\, d\mathbf{x} \\ &=: I_1^{\delta,\eta,l,m} + I_2^{\delta,\eta,l,m} + I_3^{\delta,\eta,l,m} \geqslant 0. \end{aligned} \tag{5.3}$$

It is clear that

$$I_1^{\delta,\eta,l,m} \xrightarrow{\delta,\eta\to 0} \iiiint\limits_{Q_T\times Q_T} |u - v|\rho_{l,m}\theta'\, ds\, d\mathbf{y}\, dt\, d\mathbf{x} =: I_1^{0,0,l,m}. \tag{5.4}$$

By first taking the limits $\delta,\eta\to 0$ and then taking into account that $\mathbf{f}(u^\tau)\cdot\mathbf{n}=0$ a.e. on $\partial\Omega\times(0,T)$, we obtain

$$\begin{aligned} I_2^{\delta,\eta,l,m} &\overset{\delta,\eta\to 0}{\longrightarrow} \int_0^T\int_{\partial\Omega}\iint_{Q_T}\mathbf{F}\big(u^\tau, v(\mathbf{y},s)\big)\cdot\mathbf{n}\,\rho_{l,m}\theta\,ds\,d\mathbf{y}\,d\mathcal{H}^{N-1}\,dt \\ &= -\int_0^T\int_{\partial\Omega}\iint_{Q_T}\operatorname{sgn}\big(u^\tau - v(\mathbf{y},s)\big)\mathbf{f}\big(v(\mathbf{y},s)\big)\cdot\mathbf{n}\,\rho_{l,m}\theta\,ds\,d\mathbf{y}\,d\mathcal{H}^{N-1}\,dt \\ &=: I_2^{0,0,l,m}. \end{aligned} \tag{5.5}$$

In the same way, using and $\mathbf{f}(v^\tau)\cdot\mathbf{n}=0$ a.e. on $\partial\Omega\times(0,T)$, we obtain

$$\begin{aligned} I_3^{\delta,\eta,l,m} &\overset{\delta,\eta\to 0}{\longrightarrow} = -\iint_{Q_T}\int_0^T\int_{\partial\Omega}\operatorname{sgn}\big(u(\mathbf{x},t)-v^\tau\big)\mathbf{f}\big(u(\mathbf{x},t)\big)\cdot\mathbf{n}\,\rho_{l,m}\theta\,d\mathcal{H}^{N-1}\,ds\,dt\,d\mathbf{x} \\ &=: I_3^{0,0,l,m}. \end{aligned} \tag{5.6}$$

Now setting $\varphi(\mathbf{x},t)=\rho_{l,m}(\mathbf{x},t,\mathbf{y},s)\theta(t)$ in (3.1) (with $\mathbf{y}$ and s considered as parameters), we get

$$\begin{aligned} &-\int_0^T\int_{\partial\Omega}\operatorname{sgn}(u^\tau-k)\mathbf{f}(k)\cdot\mathbf{n}\,\rho_{l,m}\theta\,d\mathcal{H}^{N-1}\,dt \\ &\quad\leqslant \iint_{Q_T}|u-k|\rho_{l,m}\theta'\,dt\,d\mathbf{x} \\ &\quad\quad + \iint_{Q_T}\big\{|u-k|\partial_t\rho_{l,m}\theta + \mathbf{F}(u,k)\cdot\nabla_{\mathbf{x}}\rho_{l,m}\theta\big\}\,dt\,d\mathbf{x}, \qquad \forall k\in\mathbb{R}. \end{aligned}$$

Setting $k=v(\mathbf{y},s)$ and integrating the result over $(\mathbf{y},s)\in Q_T$, we obtain

$$\begin{aligned} I_2^{0,0,l,m} &\leqslant \iiiint_{Q_T\times Q_T}\{|u-v|\rho_{l,m}\theta'\,ds\,d\mathbf{y}\,dt\,d\mathbf{x} \\ &\quad + \iiiint_{Q_T\times Q_T}\big\{|u-v|\partial_t\rho_{l,m}\theta + \mathbf{F}(u,v)\nabla_{\mathbf{x}}\rho_{l,m}\theta\big\}\,ds\,d\mathbf{y}\,dt\,d\mathbf{x}. \end{aligned} \tag{5.7}$$

In a similar way, using the test function $\varphi(\mathbf{y},s)=\rho_{l,m}(\mathbf{x},t,\mathbf{y},s)\theta(t)$ in the analogue of (3.1) for the entropy solution $v=v(\mathbf{y},s)$ (with $\mathbf{x}$ and t considered as parameters) and taking into account that θ is a function of t only, we get

$$\begin{aligned} I_3^{0,0,l,m} &= -\iint_{Q_T}\int_0^T\int_{\partial\Omega}\operatorname{sgn}\big(v^\tau-u(\mathbf{x},t)\big)\mathbf{f}\big(u(\mathbf{x},t)\big)\cdot\mathbf{n}\rho_{l,m}\theta\,d\mathcal{H}^{N-1}\,dt\,d\mathbf{x} \\ &\leqslant I_4^{l,m} + \iiiint_{Q_T\times Q_T}\big\{|u-v|\partial_s\rho_{l,m}\theta + \mathbf{F}(u,v)\cdot\nabla_{\mathbf{y}}\rho_{l,m}\theta\big\}\,ds\,d\mathbf{y}\,dt\,d\mathbf{x}, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} I_4^{l,m} := \iint_{Q_T}\int_\Omega\Big\{&\big|v_0(\mathbf{y})-u(\mathbf{x},t)\big|\rho_l(t) \\ &-\big|v(\mathbf{y},T)-u(\mathbf{x},t)\big|\rho_l(t-T)\Big\}\rho_m(\mathbf{x}-\mathbf{y},t)\theta(t)\,d\mathbf{y}\,dt\,d\mathbf{x}. \end{aligned}$$

Combining (5.7) and (5.8) and using that $(\partial_t+\partial_s)\rho_{l,m}=0$ and $(\nabla_{\mathbf{x}}+\nabla_{\mathbf{y}})\rho_{l,m}=0$, we get

$$I_2^{0,0,l,m}+I_3^{0,0,l,m}\leqslant I_1^{0,0,l,m}+I_4^{l,m}.$$

Thus, for $\delta,\eta\to 0$ we obtain from (5.3) the inequality

$$2I_1^{0,0,l,m}+I_4^{l,m}\geqslant 0. \tag{5.9}$$

Next, we pass to the limits $l,m\to\infty$. Since $\theta(0)=\theta(T)=0$, we obtain

$$I_4^{l,m}\overset{l\to\infty}{\longrightarrow}\iint_{\Omega\times\Omega}\Big\{\big|v_0(\mathbf{y})-u_0(\mathbf{x})\big|\theta(0)-\big|v(\mathbf{y},T)-u(\mathbf{y},T)\big|\theta(T)\Big\}\rho_m(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}=0.$$

Collecting the limits, we obtain the following inequality from (5.9):

$$\iint_{Q_T} |u-v|\theta'\,dt\,d\mathbf{x} \geqslant 0, \qquad \forall \theta \in C_c^\infty(0,T). \tag{5.10}$$

Inequality (5.1) follows now from (5.10) in a standard way. □

6. Asymptotic behavior of entropy solutions

In this section, we suppose that the following additional assumptions are satisfied.

(A1) There exists a direction, given by an unity vector $\mathbf{e} \in \mathbb{R}^N$, such that the hyperplanes $\Pi_\nu = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{e}\cdot\mathbf{x} = \nu\}$ cut out Ω in open sets with Lipschitz deformable boundaries, $\Omega = \Omega^1(\nu) \cup \cdots \cup \Omega^{J_\nu}(\nu)$ for $\nu_{\min} < \nu < \nu_{\max}$, and do not intersect Ω for $\nu \notin [\nu_{\min}, \nu_{\max}]$.

(A2) $\mathbf{f}(u)\cdot\mathbf{e} > 0$, for $0 < u < u_{\max}$, and $\mathbf{f}(u_{\min})\cdot\mathbf{e} = \mathbf{f}(u_{\max})\cdot\mathbf{e} = 0$.

Without loss of generality, we assume that $\mathbf{e}$ is the unitary vector in the direction of the x_1-axis. Also, for simplicity, we assume $u_{\max} = 1$.

Before we continue, let us point out that all results obtained in the previous sections hold with $T = \infty$, in which case we use the notation Q for $\Omega \times (0,\infty)$.

Theorem 4. *Assume that* (A1) *and* (A2) *hold and let $u(\mathbf{x},t)$ be the entropy solution of* (1.1), (1.2), (1.4) *on Q. Then, for any $g \in C_{\mathrm{per}}([0,1])$ and $h > 0$, we have*

$$\lim_{t\to\infty} \int_t^{t+h} \int_\Omega g\big(u(\mathbf{x},s)\big)\,d\mathbf{x}\,ds = h|\Omega|\,g(0). \tag{6.1}$$

Here, $C_{\mathrm{per}}([0,1])$ denotes the space of the continuous periodic functions in $[0,1]$, and $|\Omega|$ denotes the measure of Ω.

Proof. Let $\Omega(\nu)$ be the union of the sets $\Omega^j(\nu)$ which lay on the left-hand (negative) side of Π_ν. Integrating (1.1) on $\Omega(\nu)\times(0,t)$, using the Gauss-Green formula [7], and (1.4), we arrive at

$$\int_{\Omega(\nu)} u(\mathbf{x},t)\,d\mathbf{x} - \int_{\Omega(\nu)} u_0(\mathbf{x})\,d\mathbf{x} + \int_0^t \int_{\Pi_\nu\cap\Omega} f_1\big(u(\mathbf{x},s)\big)\,d\mathcal{H}^{N-1}\,ds = 0. \tag{6.2}$$

Integrating (6.2) with respect to ν from $\nu_{\min}$ to $\nu_{\max}$, recalling (A2) we obtain

$$0 < \int_0^\infty \int_\Omega f_1\big(u(\mathbf{x},s)\big)\,d\mathbf{x}\,ds \le C$$

for some positive constant C. In particular, we have

$$\lim_{t\to\infty} \int_t^{t+h} \int_\Omega f_1\big(u(\mathbf{x},s)\big)\,d\mathbf{x}\,ds = 0. \tag{6.3}$$

Again recalling (A2), we immediately obtain from (6.3) that the probability measures defined by

$$\langle \mu_t, g\rangle := \frac{1}{h|\Omega|} \int_t^{t+h} \int_\Omega g\big(u(\mathbf{x},s)\big)\,d\mathbf{x}\,ds, \qquad g \in C_{\mathrm{per}}([0,1]), \tag{6.4}$$

satisfy $\mu_t \to \delta_0$ in the weak-$\star$ topology of $C_{\mathrm{per}}([0,1])^*$, and so (6.1) follows. □

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