LYAPUNOV EXPONENTS OF TEICHMÜLLER FLOWS

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ABSTRACT. We study the dynamical properties of Teichmüller flows and renormalization operators in the moduli space of Abelian differentials, and use the conclusions to analyze the quantitative behavior of geodesics on typical translation surfaces. Three main results are reviewed: existence of asymptotic cycles, the asymptotic flag theorem, and simplicity of the Lyapunov spectrum. We give complete proofs of the first two and an outline for the last one.

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Contents

0. Introduction

A detailed introduction to the study of interval exchange maps, translation surfaces, renormalization operators, and Teichmüller flows, starting from the basic notions and including complete proofs, was given in [21]. Here we start from where that article left, to review three main results:

0.1. **Asymptotic cycles.** Let α be an Abelian differential, that is, a non-zero holomorphic complex 1-form on some compact Riemann surface M. We denote by $g \geq 1$ the genus of the surface. Let γ be any vertical geodesic in M. We denote by $\gamma(p,l)$ the vertical segment of length $l > 0$ starting from any point $p \in \gamma$ in the upward direction. If the geodesic γ hits a singularity then, by convention, we extend it along the next separatrix in the clockwise orientation. Let $[\gamma(p,l)] \in H_1(M,\mathbb{R})$ represent the homology class of the closed curve obtained when one connects p to the other endpoint p' of $\gamma(p,l)$ by some curve segment with uniformly bounded length. See Figure 1. The particular choice of the connecting segment is not relevant here: different choices give rise to homology classes whose difference is uniformly bounded, and this does not affect any of our statements.

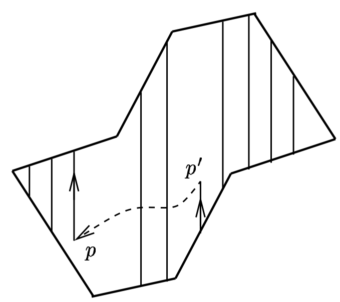

Figure 1.

The first result is that these homology classes have a well defined asymptotic direction in homology space, if the vertical flow is uniquely ergodic. This assumption is very general: by Kerckhoff, Masur, Smillie [12] the vertical geodesic flow of $e^{i\theta}\alpha$ is uniquely ergodic for every Abelian differential α and almost every $\theta \in S^1$.

Theorem A. *Let α be an Abelian differential such that its vertical geodesic flow is uniquely ergodic. Then there exists $c_1 = c_1(\alpha)$ in $H_1(M,\mathbb{R})$ such that*

$$\frac{1}{l}[\gamma(p,l)] \to c_1 \text{ when } l \to \infty, \quad \text{uniformly in } p \in M.$$

This fact was first observed by Zorich [25]. A proof is presented in Section 3. The notion of asymptotic cycle had been introduced earlier by Schwartzmann [17].

0.2. **Asymptotic flag.** The second result gives a much more detailed description of the asymptotic behavior of long geodesic segments, also for almost all Abelian differentials. Indeed, we are going to see that the component of $[\gamma(p,l)]$ orthogonal to the line $L_1 = \mathbb{R}c_1$ is asymptotic to some $c_2 \in H_1(M,\mathbb{R})$ and its norm

$$\operatorname{dist}([\gamma(p,l)], L_1) \lesssim l^{\nu_2}$$

(meaning ν_2 is the infimum of all ν such that the left hand side is less than l^ν for every large l) for some constant $\nu_2 < 1$. See Figure 2, where we represent possible values of this orthogonal component for different values of the length. More generally, for any $j = 2, \dots, g$, the component of $[\gamma(p,l)]$ orthogonal to the subspace $L_{j-1} = \mathbb{R}c_1 \oplus \dots \oplus \mathbb{R}c_{j-1}$ is asymptotic to some $c_j \in H_1(M,\mathbb{R})$ and its norm

$$\operatorname{dist}([\gamma(p,l)], L_{j-1}) \lesssim l^{\nu_j}$$

for some constant $\nu_j < \nu_{j-1}$. Finally, the component of $[\gamma(p,l)]$ orthogonal to $L_g = \mathbb{R}c_1 \oplus \dots \oplus \mathbb{R}c_g$ is uniformly bounded in norm. Let us state these facts more precisely.

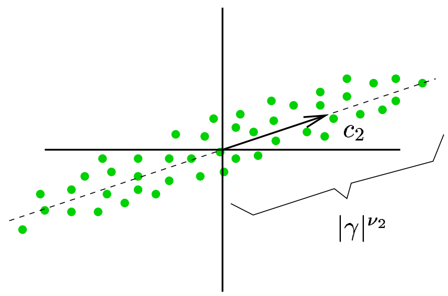

FIGURE 2.

Let $\mathcal{A}_g(m_1, \dots, m_\kappa)$ denote the *stratum* of Abelian differentials on a surface of genus g having exactly κ zeroes, with multiplicities $m_1, \dots, m_\kappa$. The connected components of these strata have been catalogued by Kontsevich, Zorich [14]: each stratum has at most 3 connected components.

Theorem B. *For any connected component $\mathcal{C}$ of $\mathcal{A}_g(m_1, \dots, m_\kappa)$ there exist numbers $1 > \nu_2 > \dots > \nu_g > 0$ and for almost every Abelian differential $\alpha \in \mathcal{C}$ there exist subspaces $L_1 \subset L_2 \subset \dots \subset L_g$ of the homology $H_1(M,\mathbb{R})$ such that $\dim L_i = i$ for $i = 1, 2, \dots, g$ and*

(1) *for all $i < g$, the deviation of $[\gamma(p,l)]$ from L_i has amplitude $l^{\nu_{i+1}}$:*

$$\limsup_{l\to\infty} \frac{\log \operatorname{dist}([\gamma(p,l)], L_i)}{\log l} = \nu_{i+1} \quad \textit{uniformly in } p \in M$$

(2) *the deviation $\operatorname{dist}([\gamma(p,l)], L_g)$ is bounded, by some constant that depends only on α and the choice of the norm.*

This remarkable statement was discovered by Zorich in the early nineties, from computer calculations of $[\gamma(p,l)]$ for various translation surfaces. An explanation was provided by Zorich and Kontsevich, in terms of the Lyapunov spectrum of the Teichmüller flow in the connected component $\mathcal{C}$, restricted to the (invariant) hypersurface of Abelian differentials with unit area. Indeed, $\mathcal{C}$ admits a natural volume measure which is finite, invariant, and ergodic under the Teichmüller flow. Thus, we may use the Oseledets theorem (Section 2) to conclude that the Teichmüller flow has a well-defined Lyapunov spectrum with respect to this measure. It is not difficult to show (Section 6) that this spectrum has the form

$$2 \ge 1 + \nu_2 \ge \dots \ge 1 + \nu_g \ge 1 = \dots = 1 \ge 1 - \nu_g \ge \dots \ge 1 - \nu_2 \ge 0 \ge$$
$$-1 + \nu_2 \ge \dots \ge -1 + \nu_g \ge -1 = \dots = -1 \ge -1 - \nu_g \ge \dots \ge -1 - \nu_2 \ge -2,$$

where the so-called *trivial exponents* ± 1 have multiplicity $\kappa - 1$. It was observed by Veech [19, 20] that the Teichmüller flow is non-uniformly hyperbolic, which amounts to saying that $\nu_2 < 1$ (Sections 5 and 6). Zorich and Kontsevich [13, 22, 24, 25] conjectured that all the inequalities in the previous formula are strict, and proved that Theorem B would follow from this conjecture (Sections 7 and 8).

0.3. **Lyapunov exponents.** In this direction, Forni [6] proved that one always has $\nu_g > 0$. This result implies the Zorich-Kontsevich conjecture in genus 2 and has also been used to obtain other properties of geodesic flows on translation surfaces, such as the weak mixing theorem of Avila, Forni [1]. The full statement of the Zorich-Kontsevich conjecture was proved by Avila, Viana [3]:

Theorem C. *For each connected component $\mathcal{C}$ of any stratum $\mathcal{A}_g(m_1, \ldots, m_\kappa)$ the non-trivial Lyapunov exponents of the Teichmüller flow are all distinct:*

$$\begin{aligned} 2 > & 1 + \nu_2 > \cdots > 1 + \nu_g > 1 - \nu_g > \cdots > 1 - \nu_2 > \\ & > -1 + \nu_2 > \cdots > -1 + \nu_g > -1 - \nu_g > \cdots > -1 - \nu_2 > -2. \end{aligned}$$

The connection between the Teichmüller flow on the connected component $\mathcal{C}$ and the geodesic flow of typical Abelian differentials $\alpha \in \mathcal{C}$ is made through another object, the Zorich linear cocycle, that we recall and analyze in Sections 4 and 5. We make the connection precise in Sections 6 through 8, where we also use it to prove Theorem B from Theorem C. In the rest of the chapter we describe the key ingredients in the proof of Theorem C. There are two main parts.

The first part corresponds to Theorem 9.2 (Sections 9 and 10), where we give general sufficient conditions for the Lyapunov spectrum of a locally constant cocycle to be simple. A simplicity criterium for Lyapunov spectra of independent random matrices was first given by Guivarc'h, Raugi [10], and their condition was improved by Gol'dsheid, Margulis [9]. Theorem 9.2 is due to Bonatti, Viana [5] and Avila, Viana [3, Appendix]. In fact, [5] contains a version that applies to non-locally constant cocycles, and this has been improved by Avila, Viana [2]. The second part corresponds to Theorem 11.1 (Sections 11 and 12), where we check that those sufficient conditions are fulfilled by the Zorich cocycles. This theorem is due to Avila, Viana [3].

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1. Preliminaries

Here we recall a number of notions and facts in the theory of interval exchange maps and translation surfaces, which also allows us to introduce several notations to be used in the rest of the text. Much more information can be found in [21].

1.1. **Interval exchange maps.** A bijective transformation $f : I \to I$ on an interval $I \subset \mathbb{R}$ is called an *interval exchange map* if there exists a finite partition $\{I_\alpha : \alpha \in \mathcal{A}\}$ of I such that the restriction of f to each I_α is a translation. We always take intervals to be closed on the left and open on the right. The left endpoint of an interval J is denoted ∂J. For definiteness, we take ∂I to coincide with the origin. Let d denote the number of symbols in the alphabet $\mathcal{A}$.

Thus, an interval exchange transformation is characterized by two sets of data:

- A pair $\pi = (\pi_0, \pi_1)$ of bijections $\pi_i : \mathcal{A} \to \{1, \ldots, d\}$, where π_0 determines the ordering of the subintervals I_α and π_1 determines the ordering of their images $f(I_\alpha)$, relative to the usual order in I. Sometimes we write

$$\pi = \begin{pmatrix} \alpha_1^0 & \alpha_2^0 & \cdots & \alpha_d^0 \\ \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_d^1 \end{pmatrix} \quad \text{where } \alpha_j^\varepsilon = \pi_\varepsilon^{-1}(j). \tag{1}$$

- A vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ where λ_α is the length of the subinterval I_α.

To each (π, λ) one associates the *translation vector* $w = (w_\alpha)_{\alpha \in \mathcal{A}}$ defined by

$$w = \Omega_\pi(\lambda), \tag{2}$$

where $\Omega_\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ is the linear operator whose matrix $(\Omega_{\alpha,\beta})_{\alpha,\beta \in \mathcal{A}}$ relative to the canonical basis $\{e_\alpha : \alpha \in \mathcal{A}\}$ of $\mathbb{R}^{\mathcal{A}}$ is given by

$$\Omega_{\alpha,\beta} = \begin{cases} +1 & \text{if } \pi_0(\alpha) < \pi_0(\beta) \text{ and } \pi_1(\alpha) > \pi_1(\beta) \\ -1 & \text{if } \pi_0(\alpha) > \pi_0(\beta) \text{ and } \pi_1(\alpha) < \pi_1(\beta) \\ 0 & \text{in all other cases.} \end{cases} \tag{3}$$

Then the interval exchange map is given by $f(x) = x + w_\alpha$ for every $x \in I_\alpha$ and $\alpha \in \mathcal{A}$. We always assume the pair π to be *irreducible*, meaning that

$$\{\alpha_1^0, \ldots, \alpha_k^0\} \neq \{\alpha_1^1, \ldots, \alpha_k^1\} \quad \text{for any } 1 \le k < d \tag{4}$$

(otherwise I may be decomposed into f-invariant subintervals restricted to which f is again an interval exchange map). Let $\Pi_{\mathcal{A}}$ denote the set of irreducible pairs π on the alphabet $\mathcal{A}$.

1.1.1. *Induction and renormalization.* The idea of induction is to associate to each interval exchange transformation f its return map to some chosen subinterval. This is again an interval exchange transformation. The Rauzy-Veech induction operator is designed in such a way that the return map has exactly the same alphabet $\mathcal{A}$. It is defined as follows.

Let $\alpha(\varepsilon) = \alpha_d^\varepsilon$ for $\varepsilon = 0, 1$. In other words, $I_{\alpha(0)}$ is the rightmost partition subinterval and $f(I_{\alpha(1)})$ is the rightmost subinterval image. Assume these subintervals have different lengths. We say that (π, λ) has *type* 0 (or *top type*) if $I_{\alpha(0)}$ is longer that $f(I_{\alpha(1)})$, otherwise we say it has *type* 1 (or *bottom type*). In either case, we call *winner* the longest of the two subintervals, as well as the corresponding symbol $\alpha(0)$ or $\alpha(1)$, and we call *loser* the shortest of the two subintervals, as well as the corresponding symbol.

The *Rauzy-Veech induction* $\hat{R}(f)$ is the first return map to the interval $I' \subset I$ obtained when one removes the loser subinterval from I. It corresponds to data $(\pi', \lambda') = \hat{R}(\pi, \lambda)$ as follows. Let $\varepsilon \in \{0, 1\}$ be the type of (π, λ) and $k = \pi_{1-\varepsilon}^{-1}(\alpha(\varepsilon))$ be the position occupied by the winner in the opposite row of π. Then

- $\pi'_\varepsilon = \pi_\varepsilon$ and $\pi'_{1-\varepsilon}(\alpha) = \begin{cases} \pi_{1-\varepsilon}(\alpha) & \text{if } \pi_{1-\varepsilon}(\alpha) \le k \\ \pi_{1-\varepsilon}(\alpha) + 1 & \text{if } k < \pi_{1-\varepsilon}(\alpha) < d \\ k + 1 & \text{if } \pi_{1-\varepsilon}(\alpha) = d \end{cases}$
  In other words, the ε-row of π is left unchanged and, as for the $(1-\varepsilon)$-row, the first k symbols are also not affected, whereas the remaining ones are rotated cyclically to the right.
- $\lambda'_\alpha = \lambda_\alpha$ for all $\alpha \neq \alpha(\varepsilon)$ and $\lambda'_{\alpha(\varepsilon)} = \lambda_{\alpha(\varepsilon)} - \lambda_{\alpha(1-\varepsilon)}$. In other words, all subinterval lengths remaining unchanged, except that the length of the loser is deducted from the length of the winner.

This correspondence $\lambda \mapsto \lambda'$ may be rewritten as

$$\lambda = \Theta^*(\lambda') \tag{5}$$

where $\Theta = \Theta_{\pi,\lambda} : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ is the linear operator whose matrix $(\Theta_{\alpha,\beta})_{\alpha,\beta\in\mathcal{A}}$ with respect to the canonical basis $\{e_\alpha : \alpha \in \mathcal{A}\}$ of $\mathbb{R}^{\mathcal{A}}$ is

$$\Theta_{\alpha,\beta} = \begin{cases} 1 & \text{if either } \alpha = \beta \text{ or } (\alpha,\beta) = (\text{loser,winner}) \\ 0 & \text{in all other cases} \end{cases} \tag{6}$$

and $\Theta^* = \Theta^*_{\pi,\lambda}$ denotes the adjoint linear operator. Moreover, the translation vector of $\hat{R}(f)$ is

$$w' = \Theta(w). \tag{7}$$

In this way, we may view the Rauzy-Veech induction operator $\hat{R}$ as a map from a full measure subset of $\Pi_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_+$ back to $\Pi_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_+$: the domain is the subset of (π, λ) for which the two rightmost subintervals have different lengths.

The *Rauzy-Veech renormalization* $R(f) : I \to I$ is defined as $h \circ \hat{R}(f) \circ h^{-1}$ where h is the linear map that rescales the subinterval I' to the length of I. Since the renormalization operator commutes with any rescaling, we may consider it to act on maps defined on the unit interval. In other words, we may view R as a map from a full measure subset of $\Pi_{\mathcal{A}} \times \Lambda_{\mathcal{A}}$ back to $\Pi_{\mathcal{A}} \times \Lambda_{\mathcal{A}}$, where

$$\Lambda_\alpha = \{\lambda \in \mathbb{R}^{\mathcal{A}}_+ : \sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1\}. \tag{8}$$

is the unit simplex in $\mathbb{R}^{\mathcal{A}}_+$: again, the domain is the subset of (π, λ) for which the rightmost subintervals have different lengths.

1.1.2. *Keane condition and minimality.* It is clear that if λ is rationally independent then $\hat{R}^n(\pi, \lambda)$ is defined for every integer n. However, rational independence is too strong. We say that (π, λ) satisfies the *Keane condition* if

$$f^m(\partial I_\alpha) \neq \partial I_\beta \quad \text{for every } m \geq 1 \text{ and any } \alpha, \beta \in \mathcal{A} \text{ with } \pi_0(\beta) \neq 1. \tag{9}$$

In other words, the orbits of the left endpoints ∂I_α are pairwise disjoint, except for the unavoidable fact that $f(\partial I_\alpha) = 0 = \partial I_\beta$ when $\pi_1(\alpha) = 1 = \pi_0(\beta)$. This condition is optimal: the iterates $\hat{R}^n(\pi, \lambda)$ are defined for every integer n if and only if (π, λ) satisfies the Keane condition.

Another important property is that if (π, λ) satisfies the Keane condition then the map f is *minimal*, meaning that every orbit is dense in the interval I. In particular, f has no periodic orbits. If λ is rationally independent then (π, λ) satisfies the Keane condition for every π. Thus, almost every interval exchange map is minimal. It was conjectured by Keane [11], and proved by Masur [15] and Veech [18], that almost every interval exchange map is even *uniquely ergodic*: the normalized Lebesgue measure is the unique invariant probability measure.

1.1.3. *Symplectic structure.* The linear map Ω_π is usually not surjective. We denote

$$H_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}}). \tag{10}$$

Observe that Ω_π is anti-symmetric. Thus, H_π is the orthogonal complement of $\ker \Omega_\pi$ relative to the usual inner product in $\mathbb{R}^{\mathcal{A}}$. Moreover,

$$\mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}, \quad (u, v) \mapsto -u \cdot \Omega_\pi(v)$$

defines an alternate bilinear form on $\mathbb{R}^{\mathcal{A}}$. This induces symplectic forms (*non-degenerate* alternate bilinear forms)

$$\omega'_\pi : \mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi \times \mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi \to \mathbb{R}, \quad \omega'_\pi([u],[v]) = -u \cdot \Omega_\pi(v), \tag{11}$$

where $[u]$ denotes the class represented by a vector $u \in \mathbb{R}^{\mathcal{A}}$, and

$$\omega_\pi : H_\pi \times H_\pi \to \mathbb{R}, \quad \omega_\pi(\Omega_\pi(u), \Omega_\pi(v)) = -u \cdot \Omega_\pi(v) \tag{12}$$

(the minus sign in the definition is somewhat unusual in the literature, but it becomes natural in the context of (35) below).

The fact that H_π admits a symplectic form implies that its dimension is even: we denote it is as $2g(\pi)$ and call $g(\pi)$ the *genus*, because it coincides with the genus of the suspension surface to be introduced in the next section. From the definitions one obtains ([21, Lemma 10.2])

$$\Theta_{\pi,\lambda} \Omega_\pi \Theta^*_{\pi,\lambda} = \Omega_{\pi'}. \tag{13}$$

This means that the conjugacy diagram

$$\begin{array}{ccc} \mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi & \xrightarrow{\Theta^{-1*}_{\pi,\lambda}} & \mathbb{R}^{\mathcal{A}} / \ker \Omega_{\pi'} \\ \Omega_\pi \downarrow & & \downarrow \Omega_{\pi'} \\ H_\pi & \xrightarrow{\Theta_{\pi,\lambda}} & H_{\pi'} \end{array} \tag{14}$$

commutes, and it also implies that the actions defined by $\Theta_{\pi,\lambda}$ on H_π and by $\Theta^*_{\pi,\lambda}$ on $\mathbb{R}/\ker \Omega_\pi$ preserve the corresponding symplectic forms ω_π and ω'_π.

1.1.4. *Rauzy classes and strata.* Associated to the Rauzy-Veech induction operator we have the following binary relation in the set $\Pi_{\mathcal{A}}$ of irreducible pairs $\pi = (\pi_0, \pi_1)$. We say that $\pi' \in \Pi_{\mathcal{A}}$ is a *successor* of $\pi \in \Pi_{\mathcal{A}}$ if there exist $\lambda, \lambda' \in \mathbb{R}^{\mathcal{A}}_+$ such that $\hat{R}(\pi, \lambda) = (\pi', \lambda')$. Each π has exactly two successors, corresponding to type 0 and type 1 induction, respectively. Dually, each π' is a successor to exactly two pairs π. This relation may be represented as a directed graph, that one calls *Rauzy diagram*: the vertices are the elements of $\Pi_{\mathcal{A}}$ and there is an arrow from vertex π to vertex π' if and only if π' is a successor of π. The *Rauzy classes* are the connected components of the Rauzy diagram.

The *extended Rauzy diagram* is obtained from the Rauzy diagram by adding arrows between vertices π and π' whenever one is obtained from the other by reversing the order of the symbols in both rows in (1). The *extended Rauzy classes* are the connected components of the extended Rauzy diagram. One reason why this notion is important is that the connected components of strata of the moduli space of Abelian differentials are in 1-to-1 correspondence to certain extended Rauzy classes, that one calls non-degenerate. We shall further comment on this in a while,

1.1.5. *Invariant measures.* It is clear from the definition that, for any Rauzy class $C \subset \Pi_{\mathcal{A}}$, the domain $C \times \mathbb{R}^{\mathcal{A}}_+$ is invariant under the induction operator $\hat{R}$ and the domain $C \times \Lambda_{\mathcal{A}}$ is invariant under the renormalization operator R. Masur [15] and Veech [18] proved that there exists a measure ν on $C \times \Lambda_{\mathcal{A}}$ which is absolutely continuous with respect to Lebesgue measure along the simplex $\Lambda_{\mathcal{A}}$ and invariant under R. For each Rauzy class, this measure is unique up to multiplication by a constant, but it is usually infinite.

The *Zorich induction* operator is an acceleration of the Rauzy-Veech induction operator, defined on a full measure subset of $C \times \mathbb{R}_+^{\mathcal{A}}$ by

$$\hat{Z}^n(\pi, \lambda) = \hat{R}^{n(\pi,\lambda)}(\pi, \lambda) \tag{15}$$

where $n(\pi, \lambda)$ is the smallest positive integer for which the type of $\hat{R}^n(\pi, \lambda)$ is different from the type of (π, λ). We also define the *Zorich renormalization* operator

$$Z^n(\pi, \lambda) = R^{n(\pi,\lambda)}(\pi, \lambda). \tag{16}$$

It was observed by Zorich [23] that, for any Rauzy class C, the accelerated renormalization operator Z admits a unique invariant probability measure μ on $C \times \Lambda_{\mathcal{A}}$. Moreover, μ is ergodic. This measure plays an important part in what follows.

1.2. **Translation surfaces.** By *translation surface* we mean a compact Riemann surface M endowed with an Abelian differential, that is, a holomorphic complex 1-form which is not identically zero. Near any point where the Abelian differential α does not vanish, one can always find *adapted* local coordinates ζ that trivialize the 1-form:

$$\alpha_\zeta = d\zeta. \tag{17}$$

The family of such adapted coordinates is a *translation atlas*: all coordinate changes are translations in the complex plane. Thus, we may use it to transport the usual metric of $\mathbb{C}$ to a flat metric on M, defined on the complement of the zeros of α. Similarly, the translation atlas transports the constant vector fields $(1, 0)$ and $(0, 1)$ on the plane to unit parallel vector fields on the complement of the zeros of α, that we call *horizontal vector field* and *vertical vector field*, respectively.

We refer to the zeros $z_1, \ldots, z_\kappa$ of the Abelian differential as *singularities*. Indeed, they correspond to conical singularities of the flat metric: the conical angle at each z_i is equal to $2\pi(m_i + 1)$, where m_i denotes the multiplicity of the zero. The horizontal and vertical vector fields extend continuously to each z_i but these extensions are $(m_i + 1)-$valued. The multiplicities of the zeros are related to the genus $g = g(M)$ of the surface through

$$\sum_{i=1}^{\kappa} m_i = 2g - 2. \tag{18}$$

In particular, the genus g must be positive, and the number κ of singularities is zero if and only if the genus is equal to 1.

For any $g \geq 1$, we denote by $\mathcal{A}_g$ the moduli space of all Abelian differentials on a compact Riemann surface of genus g. It is naturally stratified as

$$\mathcal{A}_g = \bigcup \mathcal{A}_g(m_1, \ldots, m_\kappa),$$

where $\mathcal{A}_g(m_1, \ldots, m_\kappa)$ is the moduli space of Abelian differentials having exactly κ singularities, with multiplicities $m_1, \ldots, m_\kappa$, and the union is over all choices of κ and the m_i compatible with (18). All these moduli spaces are complex orbifolds whose dimensions can be computed explicitly from g and κ.

1.2.1. *Suspension of an interval exchange.* Every interval exchange map may be realized as the Poincaré return map to a convenient cross-section of the vertical flow in some translation surface. One way to construct such a surface is as a planar polygon with pairs of sides identified by translation, as we now explain.

Let T_π^+ denote the cone of vectors $\tau = (\tau_\delta)_{\delta\in\mathcal{A}}$ such that

$$\sum_{\pi_0(\delta)\le k} \tau_\delta > 0 \quad \text{and} \quad \sum_{\pi_1(\delta)\le k} \tau_\delta < 0 \quad \text{for all } 1 \le k < d. \tag{19}$$

For each $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ and $\tau \in T_\pi^+$ consider the polygon in the plane bounded by the line segments connecting

- $\sum_{\pi_0(\delta)<k}(\lambda_\delta,\tau_\delta)$ to $\sum_{\pi_0(\delta)\le k}(\lambda_\delta,\tau_\delta)$: denote it ζ_β where $\beta = \pi_0^{-1}(k)$

and

- $\sum_{\pi_1(\delta)<k}(\lambda_\delta,\tau_\delta)$ to $\sum_{\pi_1(\delta)\le k}(\lambda_\delta,\tau_\delta)$: denote it ζ'_β where $\beta = \pi_1^{-1}(k)$

for $1 \le k \le d$. See Figure 3. Identifying the sides ζ_β and ζ'_β by translation, for every

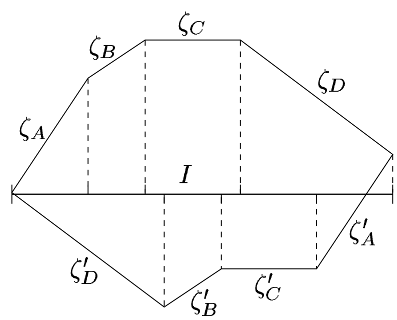

FIGURE 3.

$\beta \in \mathcal{A}$, we obtain a translation surface $M = M(\pi,\lambda,\tau)$: the conformal structure is inherited from the planar polygon and the Abelian differential corresponds to the canonical 1-form dz on the plane. The interval $I = [0, \sum_{\delta\in\mathcal{A}} \lambda_\delta)$ embeds as a horizontal cross-section to the vertical flow in M, and the corresponding Poincaré return map is just the original interval exchange map f.

1.2.2. *Zippered rectangles.* Let us mention a useful alternative way to describe the suspension of an interval exchange transformation. Let $\tau \in T_\pi^+$ be as before and

$$h = -\Omega_\pi(\tau). \tag{20}$$

In other words,

$$h_\beta = \sum_{\pi_0(\delta)<\pi_0(\beta)} \tau_\delta - \sum_{\pi_1(\delta)<\pi_1(\beta)} \tau_\delta \quad \text{for every } \beta \in \mathcal{A}.$$

The definition (19) implies that $h_\beta > 0$ for every $\beta \in \mathcal{A}$. For each $\beta \in \mathcal{A}$, consider the rectangles of width λ_β and height h_β defined by (see Figure 4)

$$R^0_\beta = \left(\sum_{\pi_0(\delta)<\pi_0(\beta)} \lambda_\delta, \sum_{\pi_0(\delta)\le\pi_0(\beta)} \lambda_\delta \right) \times [0, h_\beta]$$
$$R^1_\beta = \left(\sum_{\pi_1(\delta)<\pi_1(\beta)} \lambda_\delta, \sum_{\pi_1(\delta)\le\pi_1(\beta)} \lambda_\delta \right) \times [-h_\beta, 0]$$

and consider also the vertical segments

$$S^0_\beta = \left\{ \sum_{\pi_0(\delta)\le\pi_0(\beta)} \lambda_\delta \right\} \times \left[0, \sum_{\pi_0(\delta)\le\pi_0(\beta)} \tau_\delta \right]$$
$$S^1_\beta = \left\{ \sum_{\pi_1(\delta)\le\pi_1(\beta)} \lambda_\delta \right\} \times \left[\sum_{\pi_1(\delta)\le\pi_1(\beta)} \tau_\delta, 0 \right].$$

We may think of these vertical segments S^ε_β as "zipping" adjacent rectangles together up to a certain height, which is determined by the vector τ. Notice that

$$S^0_{\alpha(0)} = S^1_{\alpha(1)} = \left\{ \sum_{\delta\in\mathcal{A}} \lambda_\delta \right\} \times \left[0, \sum_{\delta\in\mathcal{A}} \tau_\delta \right]$$

and it is above or below the horizontal axis depending on wether $\sum_{\beta\in\mathcal{A}} \tau_\beta$ is positive or negative. The two possibilities are illustrated in Figure 4.

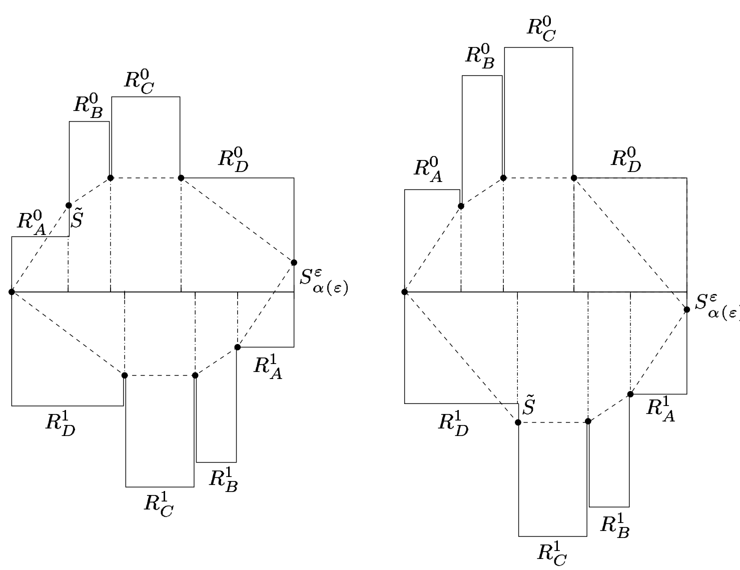

FIGURE 4.

The suspension surface $M = M(\pi, \lambda, \tau, h)$ is the quotient of the union

$$\bigcup_{\beta\in\mathcal{A}} \bigcup_{\varepsilon=0,1} R^\varepsilon_\beta \cup S^\varepsilon_\beta$$

by the following identifications. First of all, we identify each R^0_β to R^1_β through the translation

$$(x, z) \mapsto (x + w_\beta, z - h_\beta).$$

Note that this is just the same map we used before to identify the sides ζ_β and ζ'_β of the polygon in the previous construction. Secondly, let

$$\tilde{S} = \left\{ \sum_{\pi_0(\delta) \le \pi_0(\beta)} \lambda_\delta \right\} \times \left[h_\beta, \sum_{\pi_0(\delta) \le \pi_0(\beta)} \tau_\delta \right], \quad \beta = \alpha(1)$$

if the sum of all τ_δ is positive, and let

$$\tilde{S} = \left\{ \sum_{\pi_0(\delta) \le \pi_0(\beta)} \lambda_\delta \right\} \times \left[\sum_{\pi_1(\delta) \le \pi_1(\beta)} \tau_\delta, -h_\beta \right], \quad \beta = \alpha(0)$$

if the sum of all τ_δ is negative. In both cases, we identify $\tilde{S}$ with the vertical segment $S^0_{\alpha(0)} = S^1_{\alpha(1)}$ by translation. Compare Figure 4. This ends the zippered rectangles construction of the suspension surface.

Notice that in either representation, polygon or zippered rectangles, the area of the suspension surface is given by

$$\text{area}\,(M) = \lambda \cdot h = -\lambda \cdot \Omega_\pi(\tau). \tag{21}$$

1.2.3. *Representation of translation surfaces.* Conversely, most translation surfaces may be represented as zippered rectangles (or as polygons with identifications): it suffices that the vertical and horizontal vector fields have no *saddle-connections*, that is, no trajectories going from one singularity to another. This property is indeed typical among translation surfaces: given any Abelian differential α then $e^{2\pi i\theta}\alpha$ has no saddle-connections for all but countably many values of $\theta \in \mathbb{R}/\mathbb{Z}$.

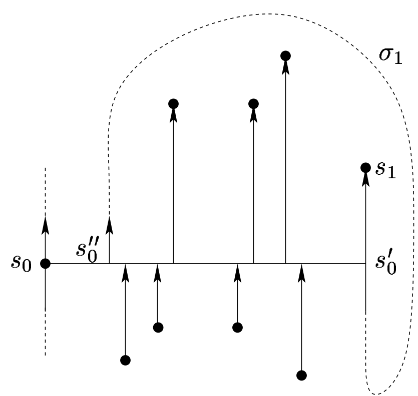

FIGURE 5.

Such a representation may be obtained as follows (Veech [19]). Pick any outgoing horizontal separatrix σ_0 of (that is, any of the trajectories of the horizontal flow starting at) a singularity s_0. The assumption that there are no horizontal saddle-connections ensures that σ_0 contains no other singularities. Let s_0' be a point of intersection between σ_0 and any vertical separatrix σ_1 of a singularity s_1 (possibly,

$s_1 = s_0$). Denote by $I \subset \sigma_0$ the segment bounded by s_0 and s_0' and consider the finite subsets ∂ and ∂', consisting of the following points:

(i) s_0 and s_0' belong to both ∂ and ∂'

(ii) for each vertical separatrix incoming to every singularity, its last point of intersection with I is in ∂

(iii) for each vertical separatrix outgoing from every singularity, its first point of intersection with I is in ∂'

(iv) the next point s_0'' of intersection between σ_1 and I (beyond s_0') is in ∂ if σ_1 is an incoming separatrix, and it is in ∂' if σ_1 is an outgoing separatrix.

Figure 5 describes one example where σ_1 is incoming. The assumption that there are no vertical saddle-connections is crucial for this definition: it implies that the vertical foliation is minimal and, in particular, the vertical separatrices do intersect I. Notice that ∂ (respectively, ∂') contains one point for each incoming (respectively, outgoing) vertical separatrix, plus two additional points that are introduced in steps (i) and (iv) of the definition. In other words, both ∂ and ∂' have $d+1$ elements, where (recall (18) above)

$$d = 1 + \sum_{i=1}^{\kappa}(m_i + 1) = 2g - 1 + \kappa, \tag{22}$$

and so each of these sets defines a partition of I into d subintervals. The Poincaré return map of the vertical flow is smooth on each connected component I_α of $I \setminus \partial$ and the image is some connected component I'_α of $I \setminus \partial'$. Moreover, the vertical trajectories connecting I_α to I'_α fill-in a rectangle R_α in M. This defines the zippered rectangle structure $M(\pi, \lambda, \tau, h)$ on the surface.

Different choices of the singularity s_0, the separatrix σ_0, and the endpoint s_0' give rise to different zippered rectangles representations of the surface. However, the extended Rauzy class of the pair π is independent of all the choices. Then, by continuity, it is constant on each connected component of any stratum $\mathcal{A}_g(m_1, \ldots, m_\kappa)$. Thus, we have a well defined map

$$\{\text{connected components of strata}\} \to \{\text{extended Rauzy classes}\}. \tag{23}$$

This map is injective, because the combinatorial data π alone determines the genus g, the number κ and multiplicities m_i of the singularities, and a couple more invariants (hyperellipticity and spin parity), that characterize the connected components of strata completely (Kontsevich-Zorich [14]). Moreover, the image of this map coincides with the subset of non-degenerate extended Rauzy classes. An extended Rauzy class is *non-degenerate* if the multiplicities m_i determined by any pair π in it are strictly positive. This means that no removable singularities (i.e. with conical angle 2π) are introduced by the representation of the translation surface in terms of data (π, λ, τ).

1.2.4. *Induction and renormalization revisited.* The Rauzy-Veech induction operator has an extension to the level of the suspension surface of an interval exchange map f, that corresponds to replacing the horizontal cross-section I by a smaller one I', so that the Poincaré return map of the vertical flow to the new cross-section is just $\hat{R}(f)$. Thus, the translation surface remains the same, but its representation in terms of the parameters π, λ, τ, and h, does change. In the polygon representation of the suspension surface this change is described by the *invertible Rauzy-Veech* *induction operator* $\hat{\mathcal{R}}(\pi,\lambda,\tau) = (\pi',\lambda',\tau')$ given by $(\pi',\lambda') = \hat{R}(\pi,\lambda)$ and

$$\tau = \Theta^*_{\pi,\lambda}(\tau). \tag{24}$$

In the zippered rectangles representation we also need the transformation rule for the *height vector* h:

$$h' = \Theta_{\pi,\lambda}(h). \tag{25}$$

The *invertible Rauzy-Veech renormalization operator* $\mathcal{R}$ is defined by

$$\mathcal{R}(\pi,\lambda,\tau) = (\pi', e^t\lambda', e^{-t}\tau')$$

where $(\pi',\lambda',\tau') = \hat{\mathcal{R}}(\pi,\lambda,\tau)$ and $t = -\log\sum_{\delta\in\mathcal{A}}\lambda'_\delta$. In other words, t is determined by $R(\pi,\lambda) = (\pi', e^t\lambda')$. We also define *invertible Zorich induction and renormalization* operators

$$\hat{\mathcal{Z}}(\pi,\lambda) = \hat{\mathcal{R}}^{n(\pi,\lambda)}(\pi,\lambda) \quad \text{and} \quad \mathcal{Z}(\pi,\lambda) = \mathcal{R}^{n(\pi,\lambda)}(\pi,\lambda), \tag{26}$$

where, as before, $n(\pi,\lambda)$ is the smallest positive integer for which the type of $\mathcal{R}(\pi,\lambda)$ is different from the type of (π,λ).

These operators $\hat{\mathcal{R}}$, $\mathcal{R}$, $\hat{\mathcal{Z}}$, $\mathcal{Z}$ may be viewed as natural extensions (inverse limits) of the induction and renormalization operators $\hat{R}$, R, $\hat{Z}$, Z that were introduced in the previous section. In particular, there exists a unique $\mathcal{Z}$-invariant probability measure that projects down to the Z-invariant probability measure μ (Section 1.1) under $(\pi,\lambda,\tau)\mapsto(\pi,\lambda)$.

1.2.5. *Poincaré duality.* The vector spaces H_π and $\mathbb{R}^{\mathcal{A}}/\ker\Omega_\pi$ and the symplectic forms ω_π and ω'_π may be interpreted in terms of the homology and cohomology of the suspension surface M. To explain this, let us consider the zippered rectangles representation. For each symbol $\beta\in\mathcal{A}$, let $[v_\beta]\in H_1(M,\mathbb{R})$ be the homology class represented by a vertical segment crossing from bottom to top the rectangle $R^0_\beta = R^1_\beta$, with its endpoints joined by a horizontal segment inside I.

The $\{[v_\beta]:\beta\in\mathcal{A}\}$ generate the homology and so the map $\Phi:\mathbb{R}^{\mathcal{A}}\to H_1(M,\mathbb{R})$ defined by $\Phi(\tau)=\sum_{\beta\in\mathcal{A}}\tau_\beta[v_\beta]$ is surjective. Moreover, it induces an isomorphism from $\mathbb{R}^{\mathcal{A}}/\ker\Omega_\pi$ to the homology group:

$$\begin{array}{ccc} \mathbb{R}^{\mathcal{A}} & \xrightarrow{\ \Phi\ } & H_1(M,\mathbb{R}) \\ {\scriptstyle quotient}\downarrow & \nearrow_{\cong} & \\ \mathbb{R}^{\mathcal{A}}/\ker\Omega_\pi & & \end{array} \tag{27}$$

Similarly, the map $\Psi: H^1(M,\mathbb{R})\to\mathbb{R}^{\mathcal{A}}$ defined by $\Psi([\phi]) = \left(\int_{v_\beta}\phi\right)_{\beta\in\mathcal{A}}$ sends the cohomology space $H^1(M,\mathbb{R})$ isomorphically onto H_π:

$$\begin{array}{ccc} H^1(M,\mathbb{R}) & \xrightarrow{\ \Psi\ } & \mathbb{R}^{\mathcal{A}} \\ & \searrow_{\cong} & \uparrow{\scriptstyle inclusion} \\ & & H_\pi \end{array} \tag{28}$$

Identifying $\mathbb{R}^{\mathcal{A}}/\ker\Omega_\pi$ with $H_1(M,\mathbb{R})$ through (27), we may think of Θ^{-1*} as acting on the homology space $H_1(M,\mathbb{R})$. Analogously, identifying $H^1(M,\mathbb{R})$ with

H_π through (28), we may think of Θ as acting on the cohomology space $H^1(M,\mathbb{R})$. Then the diagram (14) becomes

$$\begin{array}{ccc} H_1(M,\mathbb{R}) & \xrightarrow{\Theta^{-1*}} & H_1(M,\mathbb{R}) \\ \mathcal{P}\downarrow & & \downarrow\mathcal{P} \\ H^1(M,\mathbb{R}) & \xrightarrow{\Theta} & H^1(M,\mathbb{R}), \end{array} \tag{29}$$

where the isomorphism $\mathcal{P}$ is the Poincaré duality $[c] \mapsto [\phi_c]$, that we recall next.

The homology and cohomology spaces are dual to each other through

$$H_1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \to \mathbb{R}, \quad ([c],[\phi]) \mapsto [c]\cdot[\phi] = \int_c \phi \tag{30}$$

(the integral is independent of the choices of representatives of $[c]$ and $[\phi]$). Moreover, $H^1(M,\mathbb{R})$ comes with the *intersection form*

$$H^1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \to \mathbb{R}, \quad ([\phi_1],[\phi_2]) \mapsto [\phi_1]\wedge[\phi_2] = \int_M \phi_1 \wedge \phi_2. \tag{31}$$

The Poincaré duality theorem (see Chapters 18 and 24 of Fulton [7]) states that (31) is a *perfect pairing*: for every linear map $L : H^1(M,\mathbb{R}) \to \mathbb{R}$ there exists a unique $[\phi_L] \in H^1(M,\mathbb{R})$ such that $L([\phi]) = [\phi]\wedge[\phi_L]$ for every $[\phi] \in H^1(M,\mathbb{R})$. On the other hand, (30) associates to any $[c] \in H_1(M,\mathbb{R})$ the linear map $L([\phi]) = \int_c \phi$. Then there exists a unique cohomology class $[\phi_c] = [\phi_L]$ such that

$$\int_c \phi = L(\phi) = [\phi]\wedge[\phi_c] = \int_M \phi\wedge\phi_c \quad \text{for all } [\phi] \in H^1(M,\mathbb{R}). \tag{32}$$

We say $[c] \in H_1(M,\mathbb{R})$ and $[\phi_c] \in H^1(M,\mathbb{R})$ are *Poincaré dual* to each other.

The *intersection form* in homology is defined by

$$H_1(M,\mathbb{R}) \times H_1(M,\mathbb{R}) \to \mathbb{R}, \quad ([c_1],[c_2]) \mapsto [c_1]\wedge[c_2] = [\phi_{c_1}]\wedge[\phi_{c_2}]. \tag{33}$$

It has the following geometric interpretation. Consider $[c_1]$ and $[c_2]$ in $H_1(M,\mathbb{Z})$. Choose representatives c_1 and c_2 that intersect transversely. Let $\iota_j \in \{+1,-1\}$ be the *intersection sign* at each intersection point p_j: the sign is positive if the tangent vectors to c_1 and c_2 form a positive basis, relative to the orientation of M, and it is negative otherwise. Then

$$[c_1]\wedge[c_2] = \sum_j \iota_j\,. \tag{34}$$

This intersection form in homology corresponds to the symplectic form ω'_π under the identification between $\mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi$ and the homology $H_1(M,\mathbb{R})$ defined by (27). Indeed, applying (34) to appropriate representatives, we find that

$$[v_\beta]\wedge[v_\delta] = \begin{cases} -1 & \text{if } \pi_0(\beta) < \pi_0(\delta) \text{ and } \pi_1(\beta) > \pi_1(\delta) \\ +1 & \text{if } \pi_0(\beta) > \pi_0(\delta) \text{ and } \pi_1(\beta) < \pi_1(\delta) \\ 0 & \text{in all other cases,} \end{cases}$$

that is to say,

$$[v_\beta]\wedge[v_\delta] = -\Omega_{\beta,\delta} = -e_\beta\cdot\Omega_\pi(e_\delta) = \omega'_\pi([e_\beta],[e_\delta]) \tag{35}$$

for every $\beta,\delta \in \mathcal{A}$. Analogously, the intersection form in cohomology corresponds to the symplectic form ω_π, under the identification between H_π and the cohomology $H^1(M,\mathbb{R})$ defined by (28).

2. Oseledets theorem

Now we recall some basic facts and terminology relative to linear cocycles and the multiplicative ergodic theorem of Oseledets [16].

2.1. **Cocycles over maps.** Let μ be a probability measure on some space M and $f : M \to M$ be a measurable transformation that preserves μ. Let $\pi : \mathcal{E} \to M$ be a finite-dimensional vector bundle endowed with a Riemannian norm $\|\cdot\|$. A *linear cocycle* (or vector bundle morphism) over f is a map $F : \mathcal{E} \to \mathcal{E}$ such that

$$\pi \circ F = f \circ \pi$$

and the action $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ of F on each fiber is a linear isomorphism. It is often possible to assume that the vector bundle is trivial, meaning that $\mathcal{E} = M \times \mathbb{R}^d$, restricting to some full μ-measure subset of M if necessary. Then $A(\cdot)$ takes values in the linear group $\mathrm{GL}(d, \mathbb{R})$ of invertible $d \times d$ matrices. Notice that, in general, the action of the nth iterate is given by $A^n(x) = A(f^{n-1}(x)) \cdots A(f(x)) \cdot A(x)$, for every $n \geq 1$. Given any $y > 0$, we denote $\log^+ y = \max\{\log y, 0\}$.

Theorem 2.1. *Assume the function* $\log^+ \|A(x)\|$ *is μ-integrable. Then, for μ-almost every $x \in M$, there exists $k = k(x)$, numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, and a filtration $\mathcal{E}_x = F_x^1 > \cdots > F_x^k > \{0\} = F_x^{k+1}$ of the fiber, such that*

(1) $k(f(x)) = k(x)$ *and* $\lambda_i(f(x)) = \lambda_i(x)$ *and* $A(x) \cdot F_x^i = F_{f(x)}^i$ *and*
(2) $\displaystyle\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x)$ *for all* $v \in F_x^i \setminus F_x^{i+1}$ *and all* $i = 1, \ldots, k$.

The *Lyapunov exponents* λ_i and the subspaces F^i depend in a measurable (but usually not continuous) fashion on the base point. The statement of the theorem, including the values of $k(x)$, the $\lambda_i(x)$, and the $F^i(x)$, is not affected if one replaces $\|\cdot\|$ by any other Riemann norm $|||\cdot|||$ equivalent to it in the sense that there exists some μ-integrable function $c(\cdot)$ such that

$$e^{-c(x)}\|v\| \leq |||v||| \leq e^{c(x)}\|v\| \quad \text{for all } v \in T_x M. \tag{36}$$

When the measure μ is ergodic, the values of $k(x)$ and of each of the $\lambda_i(x)$ are constant on a full measure subset, and so are the dimensions of the subspaces F_x^i. We call $\dim F_x^i - \dim F_x^{i+1}$ the *multiplicity* of the corresponding Lyapunov exponent $\lambda_i(x)$. The *Lyapunov spectrum* of F is the set of all Lyapunov exponents, each counted with multiplicity. The Lyapunov spectrum is *simple* if all Lyapunov exponents have multiplicity 1.

2.2. **The invertible case.** If the transformation f is invertible then so is the cocycle F. Applying Theorem 2.1 also to the inverse F^{-1} and combining the invariant filtrations of the two cocycles, one gets a stronger conclusion than in the general non-invertible case:

Theorem 2.2. *Let $f : M \to M$ be invertible and both functions* $\log^+ \|A(x)\|$ *and* $\log^+ \|A^{-1}(x)\|$ *be μ-integrable. Then, for μ-almost every point $x \in M$, there exists $k = k(x)$, numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, and a decomposition $\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^k$ of the fiber, such that*

(1) $A(x) \cdot E_x^i = E_{f(x)}^i$ *and* $F_x^i = \oplus_{j=i}^k E_x^j$ *and*
(2) $\displaystyle\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_i(x)$ *for all non-zero* $v \in E_x^i$ *and*

(3) $\lim_{n\to\pm\infty} \frac{1}{n} \log \angle\big(E^i_{f^n(x)}, E^j_{f^n(x)}\big) = 0$ *for all* $i, j = 1, \ldots, k$.

Note that the multiplicity of each Lyapunov exponent λ_i coincides with the dimension $\dim E^i_x = \dim F^i_x - \dim F^{i+1}_x$ of the associated *Oseledets subspace* E^i_x. From the conclusion of the theorem one easily gets that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\det A^n(x)| = \sum_{i=1}^{k} \lambda_i(x) \dim E^i_x. \tag{37}$$

In most cases we deal with, the determinant is constant equal to 1. Then the sum of all Lyapunov exponents, counted with multiplicity, is identically zero.

Remark 2.3. The *natural extension* of a (non-invertible) map $f : M \to M$ is defined on the space $\hat{M}$ of sequences $(x_n)_{n\le 0}$ with $f(x_n) = x_{n+1}$ for $n < 0$, by

$$\hat{f} : \hat{M} \to \hat{M}, \quad (\ldots, x_n, \ldots, x_0) \mapsto (\ldots, x_n, \ldots, x_0, f(x_0)).$$

Let $P : \hat{M} \to M$ be the canonical projection assigning to each sequence $(x_n)_{n\le 0}$ the term x_0. It is clear that $\hat{f}$ is invertible and $P \circ \hat{f} = f \circ P$. Every f-invariant probability μ lifts to a unique $\hat{f}$-invariant probability $\hat{\mu}$ such that $P_*\hat{\mu} = \mu$. Every cocycle $F : \mathcal{E} \to \mathcal{E}$ over f extends to a cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ over $\hat{f}$, as follows: $\hat{\mathcal{E}}_{\hat{x}} = \mathcal{E}_{P(\hat{x})}$ and $\hat{A}(\hat{x}) = A(P(\hat{x}))$, where $\hat{A}(\hat{x})$ denotes the action of $\hat{F}$ on the fiber $\hat{\mathcal{E}}_{\hat{x}}$. Clearly, $\int \log^+ \|\hat{A}\| \, d\hat{\mu} = \int \log^+ \|A\| \, d\mu$ and, assuming the integrals are finite, the two cocycles F and $\hat{F}$ have the same Lyapunov spectrum and the same Oseledets filtration. Moreover, $\int \log^+ \|\hat{A}^{-1}\| \, d\hat{\mu} = \int \log^+ \|A^{-1}\| \, d\mu$ and when the integrals are finite we may apply Theorem 2.2 to the cocycle $\hat{F}$.

Remark 2.4. Any sum $F^i_x = \oplus_{j=i}^{k} E^j_x$ of Oseledets subspaces corresponding to the smallest Lyapunov exponents depends only on the forward iterates of the cocycle. Analogously, any sum of Oseledets subspaces corresponding to the largest Lyapunov exponents depends only on the backward iterates.

2.3. **Symplectic cocycles.** Suppose there exists some symplectic form, that is, some non-degenerate alternate 2-form ω_x on each fiber $\mathcal{E}_x$, which is preserved by the linear cocycle F:

$$\omega_{f(x)}(A(x)u, A(x)v) = \omega_x(u, v) \quad \text{for all } x \in M \text{ and } u, v \in \mathcal{E}_x.$$

Assume the symplectic form is integrable, in the sense that there exists a μ-integrable function $x \mapsto c(x)$ such that

$$|\omega_x(u, v)| \le e^{c(x)} \|u\| \, \|v\| \quad \text{for all } x \in M \text{ and } u, v \in \mathcal{E}_x.$$

Remark 2.5. We are going to use the following easy observation. Let μ be an invariant ergodic probability for a transformation $f : M \to M$, and let $\phi : M \to \mathbb{R}$ be a μ-integrable function. Then

$$\lim_{n\to\infty} \frac{1}{n} \phi(f^n(x)) = 0 \quad \mu\text{-almost everywhere}.$$

This follows from the Birkhoff ergodic theorem applied to $\psi(x) = \phi(f(x)) - \phi(x)$. Note that the argument remains valid under the weaker hypothesis that the function ψ be integrable.

Proposition 2.6. *If F preserves an integrable symplectic form then its Lyapunov spectrum is symmetric: if λ is a Lyapunov exponent at some point x then so is $-\lambda$, with the same multiplicity.*

This statement can be justified as follows. Consider any i and j such that $\lambda_i(x) + \lambda_j(x) \neq 0$. For all $v^i \in E_x^i$ and $v^j \in E_x^j$,

$$|\omega_x(v^i, v^j)| = |\omega_{f^n(x)}(A^n(x)v^i, A^n(x)v^j)| \leq e^{c(f^n(x))} \|A^n(x)v^i\| \|A^n(x)v^j\|$$

for all $n \in \mathbb{Z}$. Since $c(x)$ is integrable the first factor has no exponential growth: by Remark 2.5,

$$\lim_{n \to \pm\infty} \frac{1}{n} c(f^n(x)) = 0 \quad \text{almost everywhere.}$$

The assumption implies that $\|A^n(x)v^i\| \|A^n(x)v^j\|$ goes to zero exponentially fast, either when $n \to +\infty$ or when $n \to -\infty$. So, the right hand of the previous inequality goes to zero either when $n \to +\infty$ or when $n \to -\infty$. Therefore, in either case, the left hand side must vanish. This proves that

$$\lambda_i(x) + \lambda_j(x) \neq 0 \quad \Rightarrow \quad \omega_x(v^i, v^j) = 0 \;\text{ for all } v^i \in E_x^i \text{ and } v^j \in E_x^j.$$

Since the symplectic form is non-degenerate, it follows that for every i there exists j such that $\lambda_i(x) + \lambda_j(x) = 0$. We are left to check that the multiplicities of such symmetric exponents coincide. We may suppose $\lambda_i(x) \neq 0$, of course. Let s be the dimension of E_x^i. Using a Gram-Schmidt argument, one constructs a basis $v_1^i, \ldots, v_s^i$ of E_x^i and a family of vectors $v_1^j, \ldots, v_s^j$ in E_x^j such that

$$\omega_x(v_p^i, v_q^j) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Notice that $\omega_x(v_p^i, v_q^i) = 0 = \omega_x(v_p^j, v_q^j)$ for all p and q, since $\lambda_i(x) = -\lambda_j(x)$ is non-zero. The relations (38) imply that the $v_1^j, \ldots, v_s^j$ are linearly independent, and so $\dim E_x^j \geq \dim E_x^i$. The converse inequality is proved in the same way.

2.4. **Adjoint linear cocycle.** Let $\pi^* : \mathcal{E}^* \to M$ be another vector bundle which is *dual* to $\pi : \mathcal{E} \to M$, in the sense that there exists a nondegenerate bilinear form

$$\mathcal{E}_x^* \times \mathcal{E}_x \ni (u, v) \mapsto u \cdot v \in \mathbb{R}, \quad \text{for each } x \in M.$$

The *annihilator* of a subspace $E^* \subset \mathcal{E}_x^*$ is the subspace $E \subset \mathcal{E}_x$ of all $v \in \mathcal{E}_x$ such that $u \cdot v = 0$ for all $u \in E^*$. We also say that E^* is the annihilator of E. Notice that $\dim E + \dim E^* = \dim \mathcal{E}_x = \dim \mathcal{E}_x^*$. The norm $\| \cdot \|$ may be transported from $\mathcal{E}$ to $\mathcal{E}^*$ through the duality:

$$\|u\| = \sup\{|u \cdot v| : v \in \mathcal{E}_x \text{ with } \|v\| = 1\} \quad \text{for } u \in \mathcal{E}_x^*. \tag{39}$$

For $x \in M$, the *adjoint* of $A(x)$ is the linear map $A^*(x) : \mathcal{E}_{f(x)}^* \to \mathcal{E}_x^*$ defined by

$$A^*(x)u \cdot v = u \cdot A(x)v \quad \text{for every } u \in \mathcal{E}_{f(x)}^* \text{ and } v \in \mathcal{E}_x. \tag{40}$$

The inverses $A^{-1*}(x) : \mathcal{E}_x^* \mapsto \mathcal{E}_{f(x)}^*$ define a linear cocycle $F^{-1*} : \mathcal{E}^* \to \mathcal{E}^*$ over f.

Proposition 2.7. *The Lyapunov spectra of F and F^{-1*} are symmetric to one another at each point.*

Indeed, the definitions (39) and (40) imply $\|A^*(x)\| = \|A(x)\|$ and, analogously, $\|A^{-1*}(x)\| = \|A^{-1}(x)\|$ for any $x \in M$. Thus, F^{-1*} satisfies the integrability condition in Theorem 2.2 if and only if F does. Let $\mathcal{E}_x = \oplus_{j=1}^k E_x^j$ be the Oseledets decomposition of F at each point x. For each $i = 1, \ldots, d$ define

$$E_x^{i*} = \text{ annihilator of } E_x^1 \oplus \cdots \oplus E_x^{i-1} \oplus E_x^{i+1} \oplus \cdots \oplus E_x^k. \tag{41}$$

The decomposition $\mathcal{E}_x^* = \oplus_{j=1}^k E_x^{j*}$ is invariant under F^{-1*}. Moreover, given any $u \in E_x^{i*}$ and any $n \geq 1$,

$$\|A^{-n*}(x)u\| = \max_{\|v\|=1} |A^{-n*}(x)u \cdot v| = \max_{\|v\|=1} |u \cdot A^{-n}(x)v| \,.$$

Fix any $\varepsilon > 0$. Begin by considering $v \in E_{f^n(x)}^i$. Then $A^{-n}(x)v \in E_x^i$, and so

$$|u \cdot A^{-n}(x)v| \geq c\,\|u\|\,\|A^{-n}(x)v\| \geq c\,\|u\|\,e^{-(\lambda_i(x)+\varepsilon)n}$$

for every n sufficiently large, where $c = c(E_x^i, E_x^{i*}) > 0$. Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{-n*}(x)u\| \geq -(\lambda_i(x) + \varepsilon). \tag{42}$$

Next, observe that a general unit vector $v \in \mathcal{E}_{f^n(x)}$ may be written

$$v = \sum_{j=1}^k v^j \quad \text{with } v^j \in E_{f^n(x)}^j.$$

Using part 3 of Theorem 2.2, we see that every $\|v^j\| \leq e^{\varepsilon n}$ if n is sufficiently large. Therefore, given any $u \in E_x^{i*}$,

$$|u \cdot A^{-n}(x)v| = |u \cdot A^{-n}(x)v^i| \leq \|u\|\,e^{-(\lambda_i(x)-\varepsilon)n}\,\|v^i\| \leq \|u\|\,e^{-(\lambda_i(x)-2\varepsilon)n}$$

for every unit vector $v \in \mathcal{E}_{f^n(x)}$, and so

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{-n*}(x)u\| \leq -(\lambda_i(x) - 2\varepsilon). \tag{43}$$

Since $\varepsilon > 0$ is arbitrary, the relations (42) and (43) show that the Lyapunov exponent of F^{-1*} along E_x^{i*} is precisely $-\lambda_i(x)$, for every $i = 1, \ldots, k$. Thus, $\mathcal{E}_x^* = \oplus_{j=1}^k E_x^{j*}$ must be the Oseledets decomposition of F^{-1*} at x. Observe, in addition, that $\dim E_x^{i*} = \dim E_x^i$ for all $i = 1, \ldots, k$.

2.5. **Cocycles over flows.** We call *linear cocycle over a flow* $f^t : M \to M$, $t \in \mathbb{R}$ a flow extension $F^t : \mathcal{E} \to \mathcal{E}$, $t \in \mathbb{R}$ such that $\pi \circ F^t = f^t \circ \pi$ and the action $A^t(x) : \mathcal{E}_x \to \mathcal{E}_{f^t(x)}$ of F^t on every fiber is a linear isomorphism. Notice that $A^{t+s}(x) = A^s(f^t(x)) \cdot A^t(x)$ for all $t, s \in \mathbb{R}$.

Theorem 2.8. *Assume* $\log^+ \|A^t(x)\|$ *is* μ*-integrable for all* $t \in \mathbb{R}$. *Then, for* μ*-almost every* $x \in M$, *there exists* $k = k(x) \leq d$, *numbers* $\lambda_1(x) > \cdots > \lambda_k(x)$, *and a decomposition* $\mathcal{E}_x = E_x^0 \oplus E_x^1 \oplus \cdots \oplus E_x^k$ *of the fiber, such that*

(1) $A^t(x) \cdot E_x^i = E_{f^t(x)}^i$ *and* E_x^0 *is tangent to the flow lines*
(2) $\displaystyle\lim_{t\to\pm\infty} \frac{1}{t} \log \|A^t(x)\| = \lambda_i(x)$ *for all non-zero* $v \in E_x^i$
(3) $\displaystyle\lim_{t\to\pm\infty} \frac{1}{t} \log \angle\big(E_{f^t(x)}^i, E_{f^t(x)}^j\big) = 0$ *for all* $i, j = 1, \ldots, k$.

As a consequence, the relation (37) also extends to the continuous time case, as do the observations made in the previous sections for discrete time cocycles.

An important special case is the *derivative cocycle* $Df^t : TM \to TM$ over a smooth flow $f^t : M \to M$. We call Lyapunov exponents and Oseledets subspaces of the flow the corresponding objects for this cocycle Df^t, $t \in \mathbb{R}$.

2.6. **Induced cocycle.** The following construction will be useful later. Let $f : M \to M$ be a transformation, not necessarily invertible, μ be an invariant probability measure, and D be some positive measure subset of M. Let $\rho(x) \geq 1$ be the first return time to D, defined for almost every $x \in D$. Given any cocycle $F = (f, A)$ over f, there exists a corresponding cocycle $G = (g, B)$ over the first return map $g(x) = f^{\rho(x)}(x)$, defined by $B(x)v = A^{\rho(x)}(x)v$.

Proposition 2.9. (1) *The normalized restriction μ_D of the measure μ to the domain D is invariant under the first return map g.*

(2) $\log^+ \|B^{\pm 1}\|$ *are integrable for μ_D if* $\log^+ \|A^{\pm 1}\|$ *are integrable for μ.*

(3) *For μ-almost every $x \in D$, the Lyapunov exponents of G at x are obtained multiplying the Lyapunov exponents of F at x by some constant $c(x) \geq 1$.*

Proof. First, we treat the case when the transformation f is invertible. For each $j \geq 1$, let D_j be the subset of points $x \in D$ such that $\rho(x) = j$. The $\{D_j : j \geq 1\}$ are a partition of a full measure subset of D, and so are the $\{f^j(D_j) : j \geq 1\}$. Notice also that $g \mid D_j = f^j \mid D_j$ for all $j \geq 1$. For any measurable set $E \subset D$ and any $j \geq 1$,

$$\mu\big(g^{-1}(E \cap f^j(D_j))\big) = \mu\big(f^{-j}(E \cap f^j(D_j))\big) = \mu\big(E \cap D_j\big),$$

because μ is invariant under f. It follows that

$$\mu\big(g^{-1}(E)\big) = \sum_{j=1}^{\infty} \mu\big(g^{-1}(E \cap f^j(D_j))\big) = \sum_{j=1}^{\infty} \mu\big(E \cap D_j\big) = \mu(E).$$

This implies that μ_D is invariant under g, as claimed in part (1). Next, from the definition $B(x) = A^{\rho(x)}(x)$ we conclude that

$$\int_D \log^+ \|B\| \, d\mu = \sum_{j=1}^{\infty} \int_{D_j} \log^+ \|A^j\| \, d\mu \leq \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \int_{D_j} \log^+ \|A \circ f^i\| \, d\mu.$$

Since μ is invariant under f and the domains $f^i(D_j)$ are pairwise disjoint for all $0 \leq i \leq j-1$, it follows that

$$\int_D \log^+ \|B\| \, d\mu \leq \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \int_{f^i(D_j)} \log^+ \|A\| \, d\mu \leq \int \log^+ \|A\| \, d\mu.$$

The corresponding bound for the norm of the inverse is obtained in the same way. This implies part (2) of the proposition. To prove part (3), define

$$c(x) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \rho(f^j(x)).$$

Notice that ρ is integrable relative to μ_D:

$$\int_D \rho \, d\mu = \sum_{j=1}^{\infty} j\mu(D_j) = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \mu(f^i(D_j)) \leq 1.$$

Thus, by the ergodic theorem, $c(x)$ is well defined at μ_D-almost every x. It is clear from the definition that $c(x) \geq 1$. Now, given any vector $v \in \mathcal{E}_x \setminus \{0\}$ and a generic point $x \in D$,

$$\lim_{k\to\infty} \frac{1}{k} \log \|B^k(x)v\| = c(x) \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\|$$

(we are assuming $\log^+ \|A\|$ is μ-integrable and so Theorem 2.1 ensures that both limits exist). This proves part (3) of the proposition, when f is invertible.

Finally, we extend the proposition to the non-invertible case. Let $\hat{f}$ be the natural extension of f and $\hat{\mu}$ be the lift of μ (Remark 2.3). Denote $\hat{D} = P^{-1}(D)$. It is clear that the $\hat{f}$-orbit of a point $\hat{x} \in \hat{D}$ returns to $\hat{D}$ at some time n if and only if the f-orbit of $x = P(\hat{x})$ returns to D at time n. Thus, the first return map of $\hat{f}$ to the domain $\hat{D}$ is

$$\hat{g}(x) = \hat{f}^{\rho(x)}(\hat{x}), \quad x = P(\hat{x}),$$

and so it satisfies $P \circ \hat{g} = g \circ P$. It is also clear that the normalized restriction $\hat{\mu}_D$ of $\hat{\mu}$ to the domain $\hat{D}$ satisfies $P_* \hat{\mu}_D = \mu_D$. By the invertible case, $\hat{\mu}_D$ is invariant under $\hat{g}$. It follows that μ_D is invariant under g:

$$\mu_D(g^{-1}(E)) = \hat{\mu}_D(P^{-1}g^{-1}(E)) = \hat{\mu}_D(\hat{g}^{-1}P^{-1}(E)) = \hat{\mu}_D(P^{-1}(E)) = \mu_D(E),$$

for every measurable set $E \subset D$. This settles part (1). Now let $\hat{F} = (\hat{f}, \hat{A})$ be the natural extension of the cocycle F (Remark 2.3) and $\hat{G}$ be the cocycle it induces over $\hat{g}$:

$$\hat{G}(\hat{x}, v) = (\hat{g}(\hat{x}), \hat{B}(\hat{x})v), \quad \hat{B}(\hat{x}) = \hat{A}^{\rho(x)}(\hat{x}).$$

By definition, $\hat{A}(\hat{x}) = A(x)$, and so $\hat{B}(\hat{x}) = B(x)$. Consequently,

$$\int \log^+ \|A\| \, d\mu = \int \log^+ \|\hat{A}\| \, d\hat{\mu} \quad \text{and} \quad \int \log^+ \|B\| \, d\mu_D = \int \log \|\hat{B}\| \, d\hat{\mu}_D.$$

By the invertible case, $\log^+ \|\hat{B}\|$ is $\hat{\mu}_D$-integrable if $\log^+ \|\hat{A}\|$ is $\hat{\mu}$-integrable. It follows that $\log^+ \|B\|$ is μ_D-integrable if $\log^+ \|A\|$ is μ-integrable. The same argument applies to the inverses. This settles part (2) of the proposition. Part (3) also extends easily to the non-invertible case: as observed in Remark 2.3, the Lyapunov exponents of $\hat{F}$ at $\hat{x}$ coincide with the Lyapunov exponents of F at x. For the same reasons, the Lyapunov exponents of $\hat{G}$ at $\hat{x}$ coincide with the Lyapunov exponents of G at x. By the invertible case, the exponents of $\hat{G}$ at $\hat{x}$ are obtained multiplying the exponents of $\hat{F}$ at $\hat{x}$ by some positive factor. Consequently, the exponents of G at x are obtained multiplying the exponents of F at x by that same factor. This concludes the proof of the proposition. □

3. Asymptotic cycles

Here we prove Theorem A. There are two main steps. First, we reduce the statement to the special case when the geodesic segments are taken with their endpoints in a given cross-section to the vertical flow. Then, we choose a convenient cross-section and use unique ergodicity of the Poincaré return map to prove the claim in that special case.

3.1. **Proof of Theorem A.** Let σ be some cross-section to the vertical flow. The corresponding Poincaré map f is an interval exchange transformation and our assumptions imply that it is uniquely ergodic. Given $x \in \sigma$ and $k \geq 1$, denote by $\gamma(x,k)$ the vertical geodesic segment starting at x, in the upward direction, and ending at $f^k(x)$. Then let $[\gamma(x,k)]$ be the homology class of the closed curve obtained joining the endpoints of $\gamma(x,k)$ by a line segment inside the cross-section. Let $H(x)$ be the length of $\gamma(x,1)$. Clearly, $H(\cdot)$ is positive and continuous on each partition subinterval I_α. Moreover, it has a continuous positive extension to the closure of I_α: this is because the endpoints arise from the singularities of α, which are of saddle type. Hence, the function $H(\cdot)$ is bounded from zero and infinity on its domain σ. Let $H(x,k)$ denote the length of $\gamma(x,k)$, for every $k \geq 1$. Since f is uniquely ergodic,

$$\frac{1}{k}H(x,k) = \frac{1}{k}\sum_{j=0}^{k-1} H(f^j(x)) \text{ converges uniformly to } \int_\sigma H, \tag{44}$$

where the integral is with respect to the unique invariant probability of f. The factor $\int_\sigma H$ is precisely the area of the translation surface, as we shall see.

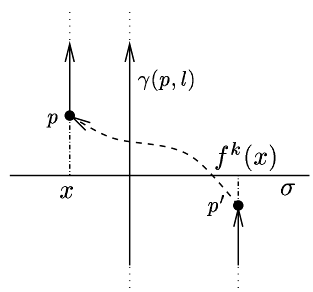

FIGURE 6.

Lemma 3.1. *We have*

$$\lim_{k\to\infty} \frac{1}{k}[\gamma(x,k)] = \lim_{l\to\infty} \frac{1}{l}[\gamma(p,l)] \int_\sigma H$$

and either limit is uniform if and only if the other one is.

Proof. Given $p \in M$ and $l > 0$, let us consider the smallest segment $\gamma(x,k)$ that contains $\gamma(p,l)$. That is, x is the last intersection of γ with the cross-section prior to p, and $f^k(x)$ is the first intersection following the other endpoint p' of $\gamma(p,l)$. See Figure 6. Then

$$\|[\gamma(p,l)] - [\gamma(x,k)]\| \leq C_0 \quad \text{for every } (p,l). \tag{45}$$

Indeed, the difference is represented by some curve whose length is uniformly bounded, namely the one obtained concatenating the vertical segment from p to x, the horizontal segment from x to $f^k(x)$, the vertical segment from $f^k(x)$ to p', and the curve segment connecting p' to p. Then,

$$\lim_{l\to\infty} \frac{1}{l}[\gamma(p,l)] = \lim_{l\to\infty} \frac{1}{l}[\gamma(x,k)] = \lim_{l\to\infty} \frac{1}{k}[\gamma(x,k)] \lim_{l\to\infty} \frac{k}{l}.$$

Since $H(\cdot)$ is bounded from zero and infinity, k goes to infinity when, and only when, l goes to infinity. Note also that $0 \le H(x,k) - l \le 2\max_z H(z)$. In view of (44), this implies that

$$\lim_{l\to\infty} \frac{k}{l} = \lim_{k\to\infty} \frac{k}{H(k,x)} = \left(\int_\sigma H\right)^{-1} \tag{46}$$

and the convergence is uniform. It follows that

$$\lim_{l\to\infty} \frac{1}{l}[\gamma(p,l)] = \left(\int_\sigma H\right)^{-1} \lim_{k\to\infty} \frac{1}{k}[\gamma(x,k)] \tag{47}$$

and either limit is uniform if and only if the other one is. □

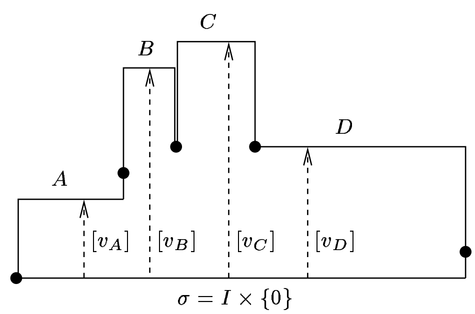

FIGURE 7.

Thus, to prove the proposition we only have to show that the limit on the left hand side of Lemma 3.2 exists and is uniform, for some choice of the cross-section. Let us fix some representation of the translation surface in the form of zippered rectangles, corresponding to data (π, λ, τ, h). As the cross-section, we choose the horizontal base segment $\sigma = I \times \{0\}$. See Figure 7. For each symbol $\beta \in \mathcal{A}$, let $[v_\beta] \in H_1(M, \mathbb{R})$ be the homology class represented by a vertical segment crossing from bottom to top the rectangle labeled by β, with its endpoints joined by a horizontal segment inside σ.

Lemma 3.2.

$$\lim_{k\to\infty} \frac{1}{k}[\gamma(x,k)] = \sum_{\beta\in\mathcal{A}} \lambda_\beta [v_\beta] \quad \textit{uniformly in } x \in \sigma.$$

Proof. For any $x \in \sigma$ and $k \ge 1$ and for each $\beta \in \mathcal{A}$, define

$$\eta_\beta(x,k) = \#\{0 \le j < k : f^j(x) \in I_\beta\}.$$

Equivalently, $\eta_\beta(x,k)$ is the number of times $\gamma(x,k)$ crosses the rectangle labeled by β. Therefore

$$[\gamma(x,k)] = \sum_{\beta\in\mathcal{A}} \eta_\beta(x,k)[v_\beta].$$

By unique ergodicity, the average $k^{-1}\eta_\beta(x,k)$ converges uniformly to the measure λ_β of the interval I_β as $k \to \infty$. It follows that

$$\frac{1}{k}[\gamma(x,k)] \to \sum_{\beta\in\mathcal{A}} \lambda_\beta [v_\beta] \quad \text{uniformly as } k \to \infty,$$

as claimed. □

In view of Lemma 3.1, this gives that the asymptotic cycle is

$$c_1 = \left(\int_\sigma H\right)^{-1} \sum_{\beta\in\mathcal{A}} \lambda_\beta [v_\beta], \tag{48}$$

and the proof of Theorem A is complete. Note that $\int_\sigma H = \sum_{\beta\in\mathcal{A}} \lambda_\beta h_\beta$ is precisely the area of the translation surface M, as defined in (21).

3.2. **Poincaré duality.** We are going to check that, up to a factor, the asymptotic cycle c_1 is the Poincaré dual of the cohomology class of the real part of the Abelian differential α. Poincaré duality and related notions were recalled in Section 1.2.

Lemma 3.3. *The multiple* $\left(\int_\sigma H\right) c_1$ *of the asymptotic cycle is the Poincaré dual of the closed* 1*-form* $\Re\alpha$.

Proof. From (35) we get that $[v_\beta] \wedge [v_\delta] = -\Omega_{\beta,\delta}$ for every $\beta, \delta \in \mathcal{A}$. Then, using (48) we get that, for any $\delta \in \mathcal{A}$,

$$\left(\int_\sigma H\right) c_1 \wedge [v_\delta] = \sum_\beta \lambda_\beta [v_\beta] \wedge [v_\delta] = -\sum_\beta \Omega_{\beta,\delta}\lambda_\beta = w_\delta$$

where $w = \Omega_\pi(\lambda)$ is the translation vector of f, defined by (2). On the other hand,

$$\int_{[v_\delta]} \Re(\alpha) = -w_\delta,$$

because $[v_\delta]$ is represented by the concatenation of a vertical line segment, on which the real part $\Re(\alpha)$ vanishes identically, and a horizontal line segment of length w_δ, with orientation opposite to the one of the translation vector. Now it suffices to note that, by (32) and (33), the Poincaré dual $[\phi_c]$ of $(\int_\sigma H)c_1$ is characterized precisely by

$$\left(\int_\sigma H\right) c_1 \wedge [v_\delta] = -\int_{[v_\delta]} [\phi_c] \quad \text{for every } \delta \in \mathcal{A}.$$

This shows that $[\phi_c] = [\Re\alpha]$, as claimed. The proof of the lemma is complete. □

4. Rauzy-Veech-Zorich cocycles

Let C be the extended Rauzy class associated to a given connected component of stratum $\mathcal{C}$. Consider $(\pi, \lambda) \in C \times \mathbb{R}_+^{\mathcal{A}}$ and let $\varepsilon \in \{0, 1\}$ be its type. Consider also the linear isomorphism $\Theta = \Theta_{\pi,\lambda}$ defined in (6): all the entries $\Theta_{\alpha,\beta}$ of the matrix of Θ are zero, except for those on the diagonal and the one where α is the loser and β is the winner of (π, λ).

We also defined the Rauzy-Veech induction $\hat{R}(f)$ of the interval exchange transformation f defined by (π, λ) to be another interval exchange transformation, corresponding to a certain partition $(I'_\alpha)_{\alpha\in\mathcal{A}}$ of the interval

$$I' = I \setminus f(I_{\alpha(1)}) \ \text{ if } \varepsilon = 0 \quad \text{and} \quad I' = I \setminus I_{\alpha(0)} \ \text{ if } \varepsilon = 1.$$

In either case, $\hat{R}(f)(x) = f^{r(x)}(x)$ where $r = r_{\pi,\lambda}$ is the first return time to I' under f, given by $r(x) = 2$ on the (loser) interval $I'_{\alpha(1-\varepsilon)}$ and $r(x) = 1$ on all the other I'_α. By construction, $f(I'_{\alpha(1-\varepsilon)}) \subset I_{\alpha(\varepsilon)}$. Thus,

$$\Theta_{\alpha,\beta} = \#\{0 \le i < r(I'_\alpha) : f^i(I'_\alpha) \subset I_\beta\} \quad \text{for all } \alpha, \beta \in \mathcal{A}. \tag{49}$$

4.1. **For interval exchange maps.** The *Rauzy-Veech cocycle* associated to the extended Rauzy class C is the linear cocycle over the Rauzy-Veech renormalization $R : C \times \Lambda_{\mathcal{A}} \to C \times \Lambda_{\mathcal{A}}$ defined by

$$F_R : C \times \Lambda_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}} \to C \times \Lambda_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}, \quad (\pi, \lambda, v) \mapsto (R(\pi, \lambda), \Theta_{\pi,v}(v)). \tag{50}$$

Note that $F_R^n(\pi, \lambda, v) = (R^n(\pi, \lambda), \Theta^n_{\pi,\lambda}(v))$ for all $n \geq 1$, where

$$\Theta^n = \Theta^n_{\pi,\lambda} = \Theta_{\pi^{n-1},\lambda^{n-1}} \cdots \Theta_{\pi',\lambda'} \Theta_{\pi,\lambda} \quad \text{and} \quad (\pi^i, \lambda^i) = R^i(\pi, \lambda).$$

In Proposition 4.3 below we obtain an important interpretation of this linear cocycle. For each $n \geq 1$, let I^n be the domain of definition of $\hat{R}^n(f)$ and $(I^n_\alpha)_{\alpha \in \mathcal{A}}$ the corresponding partition into subintervals. The proposition asserts that *each entry $\Theta^n_{\alpha,\beta}$ of the matrix of Θ^n counts the number of visits of I^n_α to the interval I_β during the induction time.* Before giving the precise statement, we need to collect a few basic facts. Notice that

$$\hat{R}^{n+1}(f)(x) = \hat{R}\big[\hat{R}^n(f)\big](x) = \begin{cases} \hat{R}^n(f)(x) & \text{if } r_{\pi^n,\lambda^n}(x) = 1 \\ \hat{R}^n(f) \circ \hat{R}^n(f)\,(x) & \text{if } r_{\pi^n,\lambda^n}(x) = 2. \end{cases}$$

Consequently, $\hat{R}^n(f)(x) = f^{r^n(x)}(x)$ where the nth Rauzy-Veech *induction time* $r^n = r^n_{\pi,\lambda}$ is defined by

$$r^1_{\pi,\lambda} = r_{\pi,\lambda} \quad \text{and} \quad r^{n+1}_{\pi,\lambda}(x) = \begin{cases} r^n_{\pi,\lambda}(x) & \text{if } r_{\pi^n,\lambda^n}(x) = 1 \\ r^n_{\pi,\lambda}(x) + r^n_{\pi,\lambda}\big(\hat{R}^n(f)(x)\big) & \text{if } r_{\pi^n,\lambda^n}(x) = 2. \end{cases}$$

We shall write $r^n(I^n_\alpha) = r^n_{\pi,\lambda}(I^n_\alpha)$ to mean $r^n(x) = r^n_{\pi,\lambda}(x)$ for any $x \in I^n_\alpha$.

Remark 4.1. If (π, λ) satisfies the Keane condition then $\min\{r^n(x) : x \in I^n\}$ goes to infinity as $n \to \infty$. Indeed, recall that $r^n(x)$ is the (first) return time of x to the interval I^n. By [21, Corollary 5.2], the interval I^n approaches the origin as $n \to \infty$. By [21, Lemma 4.4], the origin is not a periodic point of f. Thus, the return times must go to infinity, as claimed.

Lemma 4.2. *The function $r^n_{\pi,\lambda}$ is constant on I^n_α for any $n \geq 1$ and $\alpha \in \mathcal{A}$. Moreover, given any $0 \leq j < r^n(I^n_\alpha)$ there exists $\beta \in \mathcal{A}$ such that $f^j(I^n_\alpha) \subset I_\beta$.*

Proof. The case $n = 1$ is clear from the definition of the Rauzy-Veech induction. The proof proceeds by induction. Suppose first that α is not the loser of (π^n, λ^n). Then $I^{n+1}_\alpha \subset I^n_\alpha$ (they coincide unless α is the winner) and $r^{n+1}(x) = r^n(x)$ for every $x \in I^{n+1}_\alpha$. So, both claims in the lemma follow immediately from the induction hypothesis.

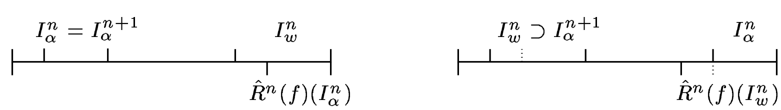

FIGURE 8.

Now take α to be the loser of (π^n, λ^n). Let $w \in \mathcal{A}$ be the winner. Suppose first that (π^n, λ^n) has type 0. Then $I^{n+1}_\alpha = I^n_\alpha$ and $\hat{R}^n(f)(I^{n+1}_\alpha) \subset I^n_w$, as shown on the left hand side of Figure 8. Hence,

$$r^{n+1}(x) = r^n(I^n_\alpha) + r^n(I^n_w) \quad \text{for all } x \in I^{n+1}_\alpha,$$

which proves the first claim. Moreover, $f^j(I_\alpha^{n+1}) = f^j(I_\alpha^n)$ for $0 \le j < r^n(I_\alpha^n)$ and $f^j(\hat{R}^n(f)(I_\alpha^{n+1})) \subset f^j(I_w^n)$ for $0 \le j < r^n(I_w^n)$. Hence, the second claim in the lemma follows directly from the induction hypothesis as well. Now suppose that (π^n, λ^n) has type 1. Then $I_\alpha^{n+1} \subset I_w^n$ and $\hat{R}^n(f)(I_\alpha^{n+1}) = I_\alpha^n$, as shown on the right hand side of Figure 8. Hence,

$$r^{n+1}(x) = r^n(I_w^n) + r^n(I_\alpha^n) \quad \text{for all } x \in I_\alpha^{n+1},$$

which proves the first claim. Moreover, $f^j(I_\alpha^{n+1}) \subset f^j(I_w^n)$ for $0 \le j < r^n(I_w^n)$ and $f^j(\hat{R}^n(f)(I_\alpha^{n+1})) \subset f^j(I_\alpha^n)$ for $0 \le j < r^n(I_\alpha^n)$. In view of the induction hypothesis, this proves the second claim in the lemma. □

In what follows we write $r_{\pi,\lambda}^n(I_\alpha^n)$ to mean the value of $r_{\pi,\lambda}^n$ at any point of I_α^n. Lemma 4.2 will be used in the proof of the next proposition, through the following immediate consequence: for any $0 \le j < r_{\pi,\lambda}^n(I_\alpha^n)$, any $J \subset I_\alpha^n$, and any $\beta \in \mathcal{A}$, we have

$$f^j(J) \subset I_\beta \quad \text{if and only if} \quad f^j(I_\alpha^n) \subset I_\beta. \tag{51}$$

Proposition 4.3. *For every $\alpha, \beta \in \mathcal{A}$ and every $n \ge 1$,*

$$\Theta_{\alpha,\beta}^n = \#\{0 \le j < r_{\pi,\lambda}^n(I_\alpha^n) : f^j(I_\alpha^n) \subset I_\beta\}.$$

Proof. The case $n = 1$ is precisely (49). The proof proceeds by induction. Let $l, w \in \mathcal{A}$ be, respectively, the loser and the winner of (π^n, λ^n). We have

$$\Theta_{\alpha,\beta}^{n+1} = \sum_{\gamma \in \mathcal{A}} (\Theta_{\pi^n,\lambda^n})_{\alpha,\gamma} \Theta_{\gamma,\beta}^n = \begin{cases} \Theta_{\alpha,\beta}^n & \text{if } \alpha \ne l \\ \Theta_{\alpha,\beta}^n + \Theta_{w,\beta}^n & \text{if } \alpha = l. \end{cases}$$

Suppose first that $\alpha \ne l$. Then $I_\alpha^{n+1} \subset I_\alpha^n$ and $r^{n+1}(I_\alpha^{n+1}) = r^n(I_\alpha^n)$. Using (51) we get that $f^j(I_\alpha^{n+1}) \subset I_\beta$ if and only if $f^j(I_\alpha^n) \subset I_\beta$, for any $0 \le j < r^n(I_\alpha^n)$. These observations show that

$$\#\{0 \le j < r^{n+1}(I_\alpha^{n+1}) : f^j(I_\alpha^{n+1}) \subset I_\beta\} = \#\{0 \le j < r^n(I_\alpha^n) : f^j(I_\alpha^n) \subset I_\beta\}.$$

By the induction hypothesis, the expression on the right hand side is equal to $\Theta_{\alpha,\beta}^n = \Theta_{\alpha,\beta}^{n+1}$ and so the statement follows in this case.

Now we treat the case $\alpha = l$. Suppose first that (π^n, λ^n) has type 0. Then $I_\alpha^{n+1} = I_\alpha^n$ and $\hat{R}^n(f)(I_\alpha^{n+1}) \subset I_w^n$ (left hand side of Figure 8). Hence,

$$r^{n+1}(I_\alpha^{n+1}) = r^n(I_\alpha^n) + r^n(I_w^n).$$

Using (51) we find that $f^j(\hat{R}^n(f)(I_\alpha^{n+1})) \subset I_\beta$ if and only if $f^j(I_w^n) \subset I_\beta$, for any $0 \le j < r^n(I_w^n)$. Thus, the number of $0 \le j < r^{n+1}(I_\alpha^{n+1})$ such that $f^j(I_\alpha^{n+1}) \subset I_\beta$ is equal to

$$\#\{0 \le j < r^n(I_\alpha^n) : f^j(I_\alpha^n) \subset I_\beta\} + \#\{0 \le j < r^n(I_w^n) : f^j(I_w^n) \subset I_\beta\}.$$

By the induction hypothesis, this sum is equal to $\Theta_{\alpha,\beta}^n + \Theta_{w,\beta}^n = \Theta_{\alpha,\beta}^{n+1}$. This settles the type 0 case. Now suppose (π^n, λ^n) has type 1. Then $I_\alpha^{n+1} \subset I_w^n$ and $\hat{R}^n(f)(I_\alpha^{n+1}) = I_\alpha^n$ (right hand side of Figure 8). Hence,

$$r^{n+1}(I_\alpha^{n+1}) = r^n(I_w^n) + r^n(I_\alpha^n).$$

Using (51) once more, we find that $f^j(I_\alpha^{n+1}) \subset I_\beta$ if and only if $f^j(I_w^n) \subset I_\beta$, for $0 \le j < r^n(I_w^n)$. Moreover, $f^j(\hat{R}^n(f)(I_\alpha^{n+1})) \subset I_\beta$ if and only if $f^j(I_\alpha^n) \subset I_\beta$, for

$0 \le j < r^n(I_\alpha^n)$. Thus, the number of $0 \le j < r^{n+1}(I_\alpha^{n+1})$ such that $f^j(I_\alpha^{n+1}) \subset I_\beta$ is equal to

$$\#\{0 \le j < r^n(I_w^n) : f^j(I_w^n) \subset I_\beta\} + \#\{0 \le j < r^n(I_\alpha^n) : f^j(I_\alpha^n) \subset I_\beta\}.$$

This sum is equal to $\Theta_{w,\beta}^n + \Theta_{\alpha,\beta}^n = \Theta_{\alpha,\beta}^{n+1}$ and so the proof is complete. □

From Proposition 4.3 we also get an alternative proof of [21, Corollary 5.3]:

Corollary 4.4. *Suppose the interval exchange transformation f defined by (π, λ) is minimal. Then there is $N \ge 1$ such that $\Theta_{\alpha,\beta}^N \ge 1$ for all α, $\beta \in \mathcal{A}$.*

Proof. We use the following equivalent formulation of minimality: given any compact set $K \subset I$ and any open set $A \subset I$, there exists $N_1 \ge 1$ such that for any $x \in K$ we have $f^j(x) \in A$ for some $0 \le j < N_1$. Fix K to be the closure of the domain I' of $\hat{R}(f)$. Then there exists $N_2 \ge 1$ such that for any $x \in K$ and any $\beta \in \mathcal{A}$ we have $f^j(x) \in I_\beta$ for some $0 \le j < N_2$. By Remark 4.1, we may fix $N \ge 1$ such that $r^N(x) \ge N_2$ for all $x \in I^N$. Since $I^N \subset K$, we get that for every $\alpha, \beta \in \mathcal{A}$ and every $x \in I_\alpha^N$ there exists $0 \le j < r^N(x)$ such that $f^j(x) \in I_\beta$. Using (51) we conclude that $f^j(I_\alpha^N) \subset I_\beta$ for any such j. In view of Proposition 4.3, this means that $\Theta_{\alpha,\beta}^N \ge 1$ for every $\alpha, \beta \in \mathcal{A}$. □

4.2. **For translation surfaces.** The *invertible Rauzy-Veech cocycle* associated to an extended Rauzy class C is the linear cocycle over the invertible Rauzy-Veech renormalization $\mathcal{R} : \mathcal{H} \to \mathcal{H}$ defined by

$$F_{\mathcal{R}} : \mathcal{H} \times \mathbb{R}^{\mathcal{A}} \to \mathcal{H} \times \mathbb{R}^{\mathcal{A}}, \quad (\pi, \lambda, \tau, v) \mapsto (\mathcal{R}(\pi, \lambda, \tau), \Theta_{\pi,\lambda}(v)). \tag{52}$$

Recall $\mathcal{H} = \mathcal{H}(C)$ is the set of all (π, λ, τ) such that $\pi \in C$, $\lambda \in \Lambda_{\mathcal{A}}$, and $\tau \in T_\pi^+$.

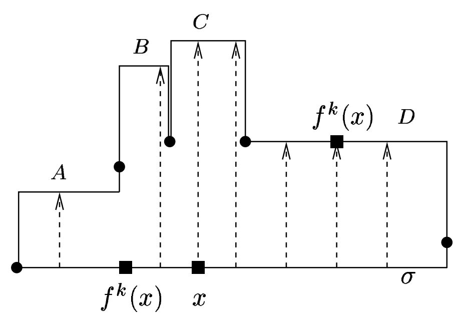

FIGURE 9.

Proposition 4.3 may be reinterpreted in terms of the suspension of the interval exchange map f defined by (π, λ), as follows. Take the suspension surface M to be represented in the form of zippered rectangles, corresponding to data (π, λ, τ, h). Let us recall some notation from Section 3. For each x in the basis horizontal segment σ and for each $k \ge 1$, let $[\gamma(x, k)] \in H_1(M, \mathbb{R})$ be the homology class represented by the vertical geodesic segment from x to $f^k(x)$, with the endpoints joined by the horizontal segment they determine inside σ. See Figure 9. Moreover, for each $\beta \in \mathcal{A}$, let $[v_\beta]$ be the homology class represented by a vertical segment crossing from bottom to top the rectangle labeled by β, with its endpoints joined by a horizontal segment.

Corollary 4.5. *For any $n \geq 1$, $\alpha \in \mathcal{A}$, and $x \in I_\alpha^n$,*

$$[\gamma(x, r^n(I_\alpha^n))] = \sum_{\beta \in \mathcal{A}} \Theta_{\alpha,\beta}^n [v_\beta].$$

Proof. Proposition 4.3 means that the vertical geodesic segment $\gamma(x, r^n(x))$ intersects each horizontal segment $I_\beta \times \{0\} \subset \sigma$ exactly $\Theta_{\alpha,\beta}^n$ times. Equivalently, $\gamma(x, r^n(x))$ crosses $\Theta_{\alpha,\beta}^n$ times the rectangle labeled by each $\beta \in \mathcal{A}$. The claim follows immediately. □

4.3. **Zorich cocycles.** Recall that $n(\pi, \lambda) \geq 1$ is the smallest integer for which the type of $R^n(\pi, \lambda)$ is different from the type of (π, λ). Let $Z(\pi, \lambda) = R^{n(\pi,\lambda)}(\pi, \lambda)$ be the Zorich renormalization and $\mathcal{Z}(\pi, \lambda, \tau) = \mathcal{R}^{n(\pi,\lambda)}(\pi, \lambda, \tau)$ be its invertible counterpart, as introduced in Section 1.1 and 1.2. Now we let

$$\Gamma = \Gamma_{\pi,\lambda} = \Theta_{\pi,\lambda}^{n(\pi,\lambda)} \tag{53}$$

and introduce the *Zorich cocycle* $F_Z(\pi, \lambda, v) = (Z(\pi, \lambda), \Gamma_{\pi,\lambda}(v))$ and the *invertible Zorich cocycle* $F_{\mathcal{Z}}(\pi, \lambda, \tau, v) = (\mathcal{Z}(\pi, \lambda, \tau), \Gamma_{\pi,\lambda}(v))$, for $v \in \mathbb{R}^{\mathcal{A}}$. According to [21, Section 30], there exists a unique ergodic Z-invariant probability measure μ absolutely continuous with respect Lebesgue measure along $\Lambda_{\mathcal{A}}$. Moreover, $\mathcal{Z}$ is equivalent to the natural extension of Z, up to zero measure sets. Hence, $F_{\mathcal{Z}}$ may be seen as the natural extension of F_Z, in the sense of Remark 2.3, and so the two cocycles have the same Lyapunov spectrum. We are going to see, in Proposition 4.7, that this Lyapunov spectrum is indeed well defined. Before that, let us translate to this setting of Zorich cocycles the properties of Rauzy-Veech cocycles we have just obtained.

By definition, the Zorich induction $\hat{Z}(f)(x) = \hat{R}^{n(\pi,\lambda)}(f)(x) = f^{z(x)}(x)$, where

$$z(x) = z_{\pi,\lambda}(x) = r_{\pi,\lambda}^{n(\pi,\lambda)}(x).$$

More generally, $\hat{Z}^m(f)(x) = \hat{R}^{n^m(\pi,\lambda)}(f)(x) = f^{z^m(x)}(x)$ for all $m \geq 1$, where

$$n^m(\pi, \lambda) = \sum_{j=0}^{m-1} n(Z^j(\pi, \lambda)) \quad \text{and} \quad z^m(x) = z_{\pi,\lambda}^m(x) = r_{\pi,\lambda}^{n^m(\pi,\lambda)}(x). \tag{54}$$

Denote by $J_\alpha^m = I_\alpha^{n^m(\pi,\lambda)}$, $\alpha \in \mathcal{A}$ the partition subintervals corresponding to the interval exchange map $Z^m(f) = R^{n^m(\pi,\lambda)}(f)$.

We shall write $z^m(J_\alpha^m) = z_{\pi,\lambda}^m(J_\alpha^m)$ to mean $z^m(x) = z_{\pi,\lambda}^m(x)$ for any $x \in J_\alpha^m$.

Corollary 4.6. *For every $\alpha, \beta \in \mathcal{A}$ and every $m \geq 1$,*

(1) $\Gamma_{\alpha,\beta}^m = \#\{0 \leq j < z_{\pi,\lambda}^m(J_\alpha^m) : f^j(J_\alpha^m) \subset I_\beta\}$ *and*
(2) $[\gamma(x, z_{\pi,\lambda}^m(J_\alpha^m))] = \sum_{\beta \in \mathcal{A}} \Gamma_{\alpha,\beta}^m [v_\beta]$ *and*
(3) $z_{\pi,\lambda}^m(J_\alpha^m) = \sum_{\beta \in \mathcal{A}} \Gamma_{\alpha,\beta}^m$.

Proof. Parts 1 and 2 follow directly from Proposition 4.3 and Corollary 4.5, respectively, simply by restricting the conclusions to appropriate subsequences. Part 3 is obtained summing the equality in part 1 over all $\beta \in \mathcal{A}$. □

Now we check that the Zorich cocycles satisfy the integrability condition in the Oseledets Theorem 2.2. For convenience, in what follows we take the norm of a vector or a matrix to be given by the largest absolute value of the coefficients.

Proposition 4.7. *The functions* $(\pi,\lambda)\mapsto\log^+\|\Gamma_{\pi,\lambda}^{\pm1}\|$ *are integrable relative to the invariant probability measure* μ *of the Zorich renormalization* Z.

Proof. Notice that $\det\Gamma_{\pi,\lambda}=1$ for all $(\pi,\lambda)\in C\times\Lambda_{\mathcal{A}}$. So, in view of our choice of the norm, $\|\Gamma_{\pi,\lambda}\|=\|\Gamma_{\pi,\lambda}^{-1}\|\geq 1$ for all (π,λ). Thus, we only have to prove that $(\pi,\lambda)\mapsto\log\|\Gamma_{\pi,\lambda}\|$ is integrable. We use

Lemma 4.8. *Let* $w=\alpha(\varepsilon)$ *be the winner of* (π,λ) *and* $s=\pi_{1-\varepsilon}^{-1}(w)$ *be its position in the other line of the pair* π. *For any integer* $L\geq 1$,

$$\max_{\alpha,\beta\in\mathcal{A}}\Gamma_{\alpha,\beta}>L \quad\Leftrightarrow\quad \lambda_{\alpha(\varepsilon)}>L\sum_{\pi_{1-\varepsilon}(\beta)>s}\lambda_\beta .$$

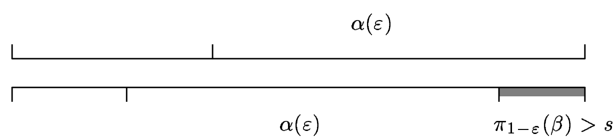

FIGURE 10.

Proof. During the $n(\pi,\lambda)$ iterates that define $\Gamma_{\pi,\lambda}$ the winner does not change. Consequently, for all the matrices $\Theta_{R^i(\pi,\lambda)}$, $0\leq i<n(\pi,\lambda)$ involved, we have

$$\Theta_{\alpha,\beta}=\begin{cases}1 & \text{if either } \alpha=\beta \text{ or } \alpha=\text{loser and } \beta=w\\ 0 & \text{in all other cases.}\end{cases}$$

It follows, by induction on the iterate, that

$$\Gamma_{\alpha,\beta}=\begin{cases}1 & \text{if } \alpha=\beta\\ 0 & \text{if } \alpha\neq\beta\neq w\\ \text{number of times } \alpha \text{ is the loser} & \text{if } \alpha\neq\beta=w.\end{cases} \tag{55}$$

The losers during those iterates are $\pi_{1-\varepsilon}^{-1}(d),\ \ldots,\ \pi_{1-\varepsilon}^{-1}(s+1)$, in cyclic order. See Figure 10. Therefore,

$$\min_{\alpha\neq w}\Gamma_{\alpha,\beta}\sum_{\pi_{1-\varepsilon}(\beta)>s}\lambda_\beta<\lambda_{\alpha(\varepsilon)}<\max_{\alpha\neq w}\Gamma_{\alpha,\beta}\sum_{\pi_{1-\varepsilon}(\beta)>s}\lambda_\beta,$$

and the difference between the maximum and the minimum is at most 1. As a direct consequence, we get that for any integer $L\geq 1$,

$$\max_{\alpha,\beta\in\mathcal{A}}\Gamma_{\alpha,\beta}>L \quad\Leftrightarrow\quad \lambda_{\alpha(\varepsilon)}>L\sum_{\pi_{1-\varepsilon}(\beta)>s}\lambda_\beta ,$$

just as we claimed. The proof of Lemma 4.8 is complete. □

Let us proceed with the proof of Proposition 4.7. Let $\mathcal{N}$ denote the set of integer vectors $n=(n_\alpha)_{\alpha\in\mathcal{A}}$ such that $n_\alpha\geq 0$ for all $\alpha\in\mathcal{A}$, and the n_α are not all zero. For each $n\in\mathcal{N}$, define

$$\Lambda(n)=\{\lambda\in\Lambda_{\mathcal{A}}:2^{-n_\alpha}\leq\lambda_\alpha d<2^{-n_\alpha+1}\text{ for every }\alpha\in\mathcal{A}\},$$

except that for $n_\alpha = 0$ the second inequality is omitted. By [21, equation (111)], there exists a constant $K > 0$ such that

$$\mu(\{\pi\} \times \Lambda(n)) = \int_{\Lambda(n)} \prod_{\beta \in \mathcal{A}} \frac{1}{\lambda \cdot h^\beta} \, d_1\lambda \le K 2^{-\max_{\mathcal{A}} n_\alpha}$$

Lemma 4.8 implies that, for any integer $L \ge 1$,

$$\|F_Z\| > L \quad \Rightarrow \quad \lambda_{\alpha(\varepsilon)} > L\lambda_{\alpha(1-\varepsilon)} \quad \Rightarrow \quad \lambda_{\alpha(1-\varepsilon)} < L^{-1}.$$

Taking $L = 2^k d$, with $k \ge 0$, we find that

$$\|F_Z\| > 2^k d \quad \Rightarrow \quad \lambda_{\alpha(1-\varepsilon)} d < 2^{-k} \quad \Rightarrow \quad \lambda \in \bigcup_{\max n_\alpha \ge k} \Lambda(n).$$

For each $k \ge 0$ there are at most $(k+1)^d$ vectors $n \in \mathcal{N}$ with $\max_{\mathcal{A}} n_\alpha = k$. So, the previous observations yield

$$\mu(\{\|F_Z\| > 2^k d\}) \le \sum_{l=k}^{\infty} \sum_{\max n_\alpha = l} \mu(\Lambda(n)) \le \sum_{l=k}^{\infty} K(l+1)^d 2^{-l} \le K'(k+1)^d 2^{-k}$$

for some constant K'. This inequality implies that $\|F_Z\|^\theta$ is μ-integrable for all $\theta < 1$. In particular, $\log \|F_Z\|$ is μ-integrable. □

This proposition ensures that Zorich cocycles have well defined Lyapunov exponents which, since the measure μ is ergodic, are constant on a full μ-measure set. Next, we analyze the corresponding Lyapunov spectra.

5. Lyapunov spectra of Zorich cocycles

5.1. **Symmetry.** First, we prove that these Lyapunov spectra have a symmetric structure:

Proposition 5.1. *The Lyapunov spectrum of the Zorich cocycle F_Z corresponding to any connected component of a stratum $\mathcal{A}_g(m_1, \ldots, m_\kappa)$ has the form*

$$\theta_1 \ge \theta_2 \ge \cdots \ge \theta_g \ge 0 = \cdots = 0 \ge -\theta_g \ge \cdots \ge -\theta_2 \ge -\theta_1$$

where 0 occurs with multiplicity $\kappa - 1$.

Proof. The proof has three main steps, corresponding to Lemmas 5.2 to 5.4. First, we exhibit a $2g$-dimensional subbundle H which is invariant under F_Z. Next, we prove that the Lyapunov exponents corresponding to Oseledets subspaces transverse to H are all zero. Then, we check that the restriction of F_Z to the invariant subbundle preserves a symplectic form, and so its Lyapunov spectrum is symmetric around zero. Let us detail each of these steps.

Let $H = \{(\pi, \lambda) \times H_\pi\}$ be the subbundle of $C \times \Lambda_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ whose fiber over each $(\pi, \lambda) \in C \times \Lambda^{\mathcal{A}}$ is the subspace $H_\pi = \Omega_\pi(\mathbb{R}^{\mathcal{A}})$ defined by (10). Since Ω_π is anti-symmetric, H_π is the orthogonal complement of $\ker \Omega_\pi$. By [21, Proposition 16.1] and [21, Lemma 16.3], we have $\dim \ker \Omega_\pi = \kappa - 1$ and $\dim H_\pi = 2g$, where κ is the number of singularities and g is the genus.

Lemma 5.2. *H_π is invariant under the linear cocycles F_R and F_Z.*

Proof. Let $(\pi',\lambda') = R(\pi,\lambda)$. The relation (13) implies $\Theta^{-1*}(\ker\Omega_\pi) = \ker\Omega_{\pi'}$. In other words, the subbundle whose fiber over each $(\pi,\lambda) \in C \times \Lambda^{\mathcal{A}}$ is the subspace $\ker\Omega_\pi$ is invariant under the adjoint cocycle

$$F_R^{-1*} : (\pi,\lambda,v) \mapsto (\pi',\lambda',\Theta^{-1*}(v)).$$

Since H_π is the orthogonal complement of the kernel, it follows that the subbundle H is invariant under the Rauzy-Veech cocycle F_R. Consequently, H is invariant under the Zorich cocycle $F_Z(\pi,\lambda,v) = F_R^{n(\pi,\lambda)}(\pi,\lambda,v)$ as well. □

Let us denote $(\pi^n,\lambda^n) = R^n(\pi,\lambda)$, for generic $(\pi,\lambda) \in C \times \Lambda_{\mathcal{A}}$ and $n \geq 1$.

Lemma 5.3. *There exists $C_0 > 0$ such that the component of every $\Theta^n_{\pi,\lambda}(v)$ orthogonal to H_{π^n} is bounded by $C_0\|v\|$ for any (π,λ,v) and any $n \geq 1$.*

Proof. Let σ be the permutation of $\{0,1,\dots,d\}$ defined by

$$\sigma(j) = \begin{cases} p^{-1}(1) - 1 & \text{if } j = 0 \\ d & \text{if } j = p^{-1}(d) \\ p^{-1}\big(p(j)+1\big) - 1 & \text{otherwise.} \end{cases} \tag{56}$$

According to [21, Section 16], to each orbit $\mathcal{O}$ of σ not containing 0 one may associate a vector $\lambda(\mathcal{O}) \in \mathbb{R}^{\mathcal{A}}$ such that these $\lambda(\mathcal{O})$ form a basis of $\ker\Omega_\pi$. The dynamics of Θ^{-1*} on the invariant subbundle $\{(\pi,\lambda) \times \ker\Omega_\pi\}$ is trivial: the image of every $\lambda(\mathcal{O})$ coincides with some element $\lambda(\mathcal{O}')$ of the basis of $\ker\Omega_{\pi'}$. It follows that for every $n \geq 1$ there exists a bijection $\mathcal{O} \mapsto \mathcal{O}^n$ between the set of orbits of σ not containing 0 and the set of orbits of σ^n not containing 0, such that $\Theta^{-n*}(\lambda(\mathcal{O})) = \lambda(\mathcal{O}^n)$ for all $\mathcal{O}$. Then,

$$\lambda(\mathcal{O}^n)\cdot\Theta^n(v) = \Theta^{n*}(\lambda(\mathcal{O}^n))\cdot v = \lambda(\mathcal{O})\cdot v \quad \text{for every } v \in \ker\Omega_\pi \text{ and every } \mathcal{O}.$$

This implies that the component of $\Theta^n(v)$ in the direction of $\ker\Omega_{\pi^n}$ is bounded in norm by $C_0\|v\|$, for some constant C_0 that depends only on the choice of the norm ($C_0 = 1$ if the bases $\{\lambda(\mathcal{O})\}$ are orthonormal). □

Recall that F_Z and the invertible Zorich cocycle $F_{\mathcal{Z}}$ have the same Lyapunov spectrum. Let $E^i_{\pi,\lambda,\tau}$ be any Oseledets subspace of $F_{\mathcal{Z}}$ transverse to H. Given any non-zero $v \in E^i_{\pi,\lambda}$ and $n \geq 1$, denote $v_n = \Gamma^n_{\pi,\lambda}(v)$. Write $v_n = v_n^H + v_n^K$, where v_n^H is the projection to H and v_n^K is the projection to the orthogonal complement of H. According to Lemma 5.3, $\|v_n^K\| \leq C_0\|v\|$ for all n. Moreover, given any $\varepsilon > 0$,

$$\|v_n^K\| \geq e^{-\varepsilon n}\|v_n^H\| \quad \text{for all large } n,$$

because the angles between the iterates of $E^i_{\pi,\lambda}$ and the subbundle H decay at most sub-exponentially (part 3 of Theorem 2.2). This implies

$$e^{-2\varepsilon n}\|v\| \leq \|v_n\| \leq e^{2\varepsilon n}\|v\| \quad \text{for all large } n.$$

Thus, the Lyapunov exponent corresponding to $E^i_{\pi,\lambda}$ is smaller than 2ε in absolute value. Since ε is arbitrary, the exponent must vanish, as we claimed.

The expression (12) defines a symplectic form ω on the invariant subbundle H, and we have seen that every $\Theta_{\pi,\lambda} : H_\pi \to H_{\pi'}$ is symplectic relative to the forms ω_π and $\omega_{\pi'}$. In other words,

Lemma 5.4. *The symplectic form ω is invariant under both F_R and F_Z.*

By Proposition 2.6, this implies that the Lyapunov spectrum of F_Z restricted to H is symmetric around zero. This ends the proof of Proposition 5.1. □

5.2. **Extremal Lyapunov exponents.** The final step in Proposition 5.5 is to show that the extremal exponents have multiplicity 1:

Proposition 5.5. *The largest Lyapunov exponent θ_1 and the smallest Lyapunov exponent $-\theta_1$ of every Zorich cocycle F_Z are simple, and the same is true for the adjoint cocycle F_Z^{-1*}.*

A much stronger fact will be obtained later, in Theorem 7.1: all Lyapunov exponents $\pm\theta_j$ are distinct and non-zero.

Proof. By Proposition 5.1 the spectra of F_Z and F_Z^{-1*} are symmetric with respect to the origin. By Proposition 2.7 they are symmetric to one another. Thus, it suffices to prove that the smallest exponent of the adjoint cocycle F_Z^{-1*} is simple. This is done as follows.

By Corollary 4.4, for almost every (π, λ) we may find $N \geq 1$ such that $\Theta^N_{\alpha,\beta} \geq 1$ for all $\alpha, \beta \in \mathcal{A}$. Then the same is true for every Θ^k, $k \geq N$ and, in particular, $\Gamma^k_{\alpha,\beta} \geq 1$ for all $\alpha, \beta \in \mathcal{A}$ and $k \geq N$. Let (π, λ) and N be fixed. In view of the Markov structure of the Zorich renormalization described in [21, Section 8], λ is contained in some subsimplex D of $\Lambda_{\mathcal{A}}$ such that $Z^N \mid \{\pi\} \times D$ is the projective map defined by Γ^{-N*} and maps the domain bijectively to $\{\pi^N\} \times \Lambda_{\pi^N,1-\varepsilon}$. Since the coefficients of Γ^{N*} are all positive, D is relatively compact in $\Lambda_{\mathcal{A}}$. See Figure 11.

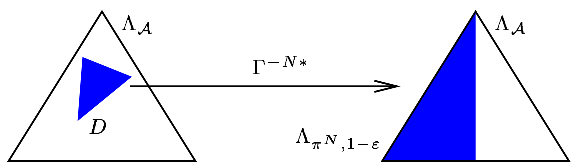

FIGURE 11.

By Poincaré recurrence, for μ-almost every $(\pi, \lambda) \in \{\pi\} \times D$ there exists a first return time $\rho(\pi, \lambda) \geq N$ to the domain $\{\pi\} \times D$ under the map Z. Note that $\mu(\{\pi\} \times D) > 0$, since μ is positive on open sets. The normalized restriction μ_D of the measure μ to the domain $\{\pi\} \times D$ is invariant and ergodic under this return map

$$\tilde{Z} : \{\pi\} \times D \to \{\pi\} \times D, \quad \tilde{Z}(\pi, \lambda) = Z^{\rho(\pi,\lambda)}(\pi, \lambda).$$

The adjoint Zorich cocycle induces a linear cocycle $\tilde{F}_Z$ over $\tilde{Z}$, given by

$$\tilde{F}_Z(\pi, \lambda, v) = (\tilde{Z}(\pi, \lambda), \tilde{\Gamma}_{\pi,\lambda}(v)), \qquad \tilde{\Gamma} = \tilde{\Gamma}_{\pi,\lambda} = \Gamma^{-\rho(\pi,\lambda)*}_{\pi,\lambda}.$$

Corollary 5.6. *The functions $\log^+ \|\tilde{\Gamma}^{\pm 1}\|$ are μ_D-integrable, and the smallest Lyapunov exponent of the adjoint Zorich cocycle F_Z^{-1*} for μ is simple if and only if the smallest Lyapunov exponent of $\tilde{F}_Z$ is simple at μ_D-almost every point.*

Proof. Proposition 4.7 implies that $\log^+ \|\Gamma^{\pm N}\|$ are μ-integrable. So, the first statement in the corollary follows immediately from part (2) of Proposition 2.9, applied to $F = F_Z^{-N*}$. Moreover, part (3) of Proposition 2.9 gives that the Lyapunov exponents of $\tilde{F}_Z$ at a generic point x are the products of the Lyapunov exponents of F_Z^{-N*} by some constant $c(x)$, that is, they coincide with the products by $Nc(x)$ of the Lyapunov exponents of F_Z^{-1*}. The last statement in the corollary is a direct consequence. □

Thus, to prove Proposition 5.5 it suffices to show that the smallest exponent of $\tilde{F}_Z$ is simple. Let $\mathcal{C} = \{v \in \mathbb{R}_+^{\mathcal{A}} : v/|v| \in D\}$ be the cone associated to D. The definition of $\tilde{\Gamma}$ implies that

$$\tilde{\Gamma}_{\pi,\lambda}^{-1}(\mathbb{R}_+^{\mathcal{A}}) = \Gamma_{\pi,\lambda}^{\rho *}(\mathbb{R}_+^{\mathcal{A}}) = \Gamma_{\pi,\lambda}^{N*}\big(\Gamma_{\pi^N,\lambda^N}^{(\rho-N)*}(\mathbb{R}_+^{\mathcal{A}})\big) \subset \Gamma_{\pi,\lambda}^{N*}(\mathbb{R}_+^{\mathcal{A}}) \subset \mathcal{C}$$

for every (π, λ). Thus, the cocycle $\tilde{F}_Z$ admits a backward invariant cone which is relatively compact, in the sense that its intersection with the simplex $\Lambda_{\mathcal{A}}$ is relatively compact inside the simplex. So, at this point the proposition is a direct consequence of the following Perron-Fröbenius type result:

Lemma 5.7. *Let $F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d$, $F(x, v) = (f(x), A(x)v)$ be a linear cocycle over a transformation $f : M \to M$, such that $\log^+ \|A^{\pm 1}\|$ are integrable with respect to some f-invariant probability ν. Assume there exists some relatively compact cone $\mathcal{C} \subset \mathbb{R}_+^{\mathcal{A}}$ such that $A(x)^{-1}(\mathbb{R}_+^{\mathcal{A}}) \subset \mathcal{C}$ for all $x \in M$. Then the smallest exponent of F with respect to ν has multiplicity 1 at almost ν-every point.*

Proof. There are two main parts. First, we identify the invariant line bundle associated to the smallest Lyapunov exponent $\lambda(x)$. Then, we prove that any vector outside this invariant subbundle grows, under positive iteration, at exponential rate strictly larger than $\lambda(x)$.

Since $\mathcal{C}$ is relatively compact, it has finite diameter relative to the projective metric on the cone $\mathbb{R}_+^d$ (see Birkhoff [4] for the definition and properties of projective metrics). Consequently, every $A(x)^{-1} : \mathbb{R}_+^d \to \mathcal{C}$ is a contraction with respect to the projective metric, with uniform contraction rate (depending only on $\mathcal{C}$). It follows that the width of $A^n(x)^{-1}(\mathbb{R}_+^{\mathcal{A}})$ is bounded by $C_1 e^{-an}$, for some $C_1 > 0$ and $a > 0$ that depend only on $\mathcal{C}$. In particular, the intersection of all these cones reduces to a half-line at every $x \in M$:

$$\bigcap_{n=1}^{\infty} A^n(x)^{-1}(\mathbb{R}_+^{\mathcal{A}}) = \mathbb{R}_+ \xi(x) \tag{57}$$

for some vector $\xi(x) \in \mathcal{C}$ which we may choose with norm 1. It is clear from (57) that the line bundle $\mathbb{R}\xi(x)$ is invariant under the cocycle F. Let $\lambda(x)$ be the corresponding Lyapunov exponent. We claim that any vector v which is not in $\mathbb{R}\xi(x)$ grows, under positive iteration, at exponential rate larger or equal than $\lambda(x) + a$. This implies that all the other Lyapunov exponents are at least $\lambda(x) + a$, which proves the lemma. Thus, we are left to proving this claim.

Let v be any unit vector outside $\mathbb{R}\xi(x)$. It follows from the definition (57) that some iterate of v is outside the cone $\mathbb{R}_+^{\mathcal{A}}$. Thus, it is no restriction to assume right from the start that $v \notin \mathbb{R}_+^{\mathcal{A}}$. Since $\xi(y) \in \mathcal{C}$ for every $y \in M$, the coefficients of any $\xi(y)$ are uniformly bounded from zero. Hence, there exists $c_1 > 0$, depending only on this bound, such that

$$\frac{A^n(x)v}{\|A^n(x)v\|} + c_1 \frac{A^n(x)\xi(x)}{\|A^n(x)\xi(x)\|} = \frac{A^n(x)v}{\|A^n(x)v\|} + c_1\, \xi(f^n(x)) \in \mathbb{R}_+^{\mathcal{A}} \quad \text{for every } n \geq 1.$$

Then, by the considerations in the previous paragraph, the angle between $\xi(x)$ and

$$\frac{v}{\|A^n(x)v\|} + c_1 \frac{\xi(x)}{\|A^n(x)\xi(x)\|} = A^n(x)^{-1}\Big(\frac{A^n(x)v}{\|A^n(x)v\|} + c_1 \frac{A^n(x)\xi(x)}{\|A^n(x)\xi(x)\|}\Big)$$

is bounded by $C_1 e^{-an}$. Since $v \notin \mathbb{R}_+^{\mathcal{A}}$ and $\xi(x) \in \mathcal{C}$, the angle between $\xi(x)$ and v is bounded below by some constant $c_2 > 0$ that depends only on the cone $\mathcal{C}$. Thus, the previous property implies that

$$\frac{\|A^n(x)\xi(x)\|}{\|A^n(x)v\|} \le C_2 e^{-an},$$

where the constant C_2 depends only on c_1, c_2, and C_1, and so is determined by the cone $\mathcal{C}$. This implies that

$$\|A^n(x)v\| \ge C_2^{-1} e^{an} \|A^n(x)\xi(x)\| \ge c(x) e^{(\lambda(x)+a)n},$$

for every $n \ge 1$ and $v \notin \mathbb{R}_+ \xi(x)$, as claimed. □

At this point the proof of Proposition 5.5 is complete. □

5.3. **Extremal Oseledets subspaces.** Recall that F_Z and the invertible Zorich cocycle $F_{\mathcal{Z}}$ have the same Lyapunov spectrum. Besides, the same is true for the adjoint cocycle $F_{\mathcal{Z}}^{-1*}$, as a consequence of Propositions 2.7 and 5.1. For either of these invertible cocycles, we are going to give an explicit description of the Oseledets subspaces associated to the extremal Lyapunov exponents $\pm\theta_1$.

Let us start with the cocycle $F_{\mathcal{Z}}$. For each $x = (\pi, \lambda, \tau) \in \mathcal{H}$, consider the following subspaces of $\mathbb{R}^{\mathcal{A}}$:

- E^s_x = line spanned by $w = \Omega_\pi(\lambda)$ and E^u_x = line spanned by $h = -\Omega_\pi(\tau)$
- E^c_x = ω_π-symplectic orthogonal to $E^u_x \oplus E^s_x$, that is,

$$E^c_x = \{v \in \mathbb{R}^{\mathcal{A}} : \omega_\pi(v, w) = 0 \text{ for all } w \in E^u_x \oplus E^s_x\}.$$

E^u_x is not symplectic orthogonal to E^s_x: indeed, $\omega_\pi(\Omega_\pi(\lambda), \Omega_\pi(\tau)) = -\lambda \cdot \Omega_\pi(\tau)$ is strictly positive, since both λ and $h = -\Omega_\pi(\tau)$ have only positive coordinates. Thus, E^c_x has codimension 2 and $\mathbb{R}^{\mathcal{A}} = E^u_x \oplus E^c_x \oplus E^s_x$.

Lemma 5.8. *The splitting $E^u \oplus E^c \oplus E^s$ is invariant under the invertible Zorich cocycle $F_{\mathcal{Z}}$. Moreover, E^u corresponds to the largest Lyapunov exponent θ_1, E^s corresponds to the smallest Lyapunov exponent $-\theta_1$, and E^c corresponds to the remaining Lyapunov exponents.*

Proof. Let $(\pi', \lambda', \tau') = \mathcal{R}(\pi, \lambda, \tau)$. The relations (5) and (13) and (24) imply that

$$\Theta(\Omega_\pi(\lambda)) = \Omega_{\pi'}(\Theta^{-1*}(\lambda)) = \Omega_{\pi'}(\lambda') \quad \text{and} \quad \Theta(\Omega_\pi(\tau)) = \Omega_{\pi'}(\tau').$$

This proves that E^u and E^s are invariant under the invertible Rauzy-Veech cocycle $F_{\mathcal{R}}$. Then the symplectic orthogonal E^c is also invariant since, by (13), the cocycle $F_{\mathcal{R}}$ preserves the symplectic form ω_π. It follows that all three subbundles are invariant under the Zorich cocycle $F_{\mathcal{Z}}$ as well.

Since $h = -\Omega_\pi(\tau)$ lies in the positive cone, and the matrices of $F_{\mathcal{Z}}$ have non-negative coefficients, E^u must be contained in the Oseledets subspace corresponding to the largest Lyapunov exponent. As this exponent is simple (Proposition 5.5), it follows that E^u coincides with that Oseledets subspace. This implies that E^s corresponds to the smallest Lyapunov exponent, since it is an invariant direction which is not contained in the symplectic orthogonal to E^u (recall the arguments following Proposition 2.6). Then the complementary invariant subbundle E^c must coincide with the sum of the Oseledets subspaces corresponding to the remaining Lyapunov exponents. □

Now we deal with the cocycle $F_{\mathcal{Z}}^{-1*}$. For each $x = (\pi, \lambda, \tau) \in \mathcal{H}$, define

- E^{s*}_x = line spanned by λ and E^{u*}_x = line spanned by τ
- E^{c*}_x = ω'_π-symplectic orthogonal to $E^{u*}_x \oplus E^{s*}_x$.

E_x^{u*} is not symplectic orthogonal to E_x^{s*} since $\omega_\pi'(\lambda,\tau) = -\lambda \cdot \Omega_\pi(\tau)$ is strictly positive. Thus, E_x^{c*} has codimension 2 and $\mathbb{R}^{\mathcal{A}} = E_x^{u*} \oplus E_x^{c*} \oplus E_x^{s*}$.

Lemma 5.9. *The splitting $E^{u*} \oplus E^{c*} \oplus E^{s*}$ is invariant under the adjoint cocycle $F_{\mathcal{Z}}^{-1*}$. The subspace E^{u*} corresponds to the largest Lyapunov exponent θ_1, the subspace E^{s*} corresponds to the smallest Lyapunov exponent $-\theta_1$, and E^{c*} corresponds to the remaining Lyapunov exponents.*

Proof. The relations (5) and (13) and (24) imply that E^{u*} and E^{s*} are invariant under $F_{\mathcal{Z}}^{-1*}$. From [21, Lemma 10.2] and (53) we get that

$$\Omega_{\pi^n}\Gamma^{-n*}(v) = \Gamma^n\Omega_\pi(v) \quad \text{for every } n \in \mathbb{Z} \text{ and } v \in \mathbb{R}^{\mathcal{A}}. \tag{58}$$

Since the set of combinatorial data π^n is finite, the norms of the Ω_{π^n} are uniformly bounded. Thus, taking $v = \tau$ and $n > 0$ in (58), and using Lemma 5.8,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Gamma^{-n*}(\tau)\| \geq \lim_{n\to+\infty} \frac{1}{n} \log \|\Gamma^n\Omega_\pi(\tau)\| = \theta_1 ,$$

and so the Lyapunov exponent of the cocycle $F_{\mathcal{Z}}^{-1*}$ along the invariant direction $E_x^{u*} = \mathbb{R}\tau$ is equal to θ_1. Analogously, taking $v = \lambda$ and $n < 0$ in (58), and then using Lemma 5.8 once more,

$$\lim_{n\to-\infty} \frac{1}{n} \log \|\Gamma^{-n*}(\lambda)\| \leq \lim_{n\to-\infty} \frac{1}{n} \log \|\Gamma^n\Omega_\pi(\lambda)\| = -\theta_1 .$$

and so the Lyapunov exponent of the cocycle $F_{\mathcal{Z}}^{-1*}$ along the invariant direction $E_x^{s*} = \mathbb{R}\lambda$ is equal to $-\theta_1$. It follows that E^{c*} is also invariant under $F_{\mathcal{Z}}^{-1*}$ and coincides with the sum of the remaining Oseledets subspaces. □

6. Zorich cocycles and Teichmüller flows

In this section we relate the Lyapunov spectrum of the Teichmüller flow, on each connected component of stratum, to the Lyapunov spectrum of the corresponding Zorich cocycle:

Proposition 6.1. *The Lyapunov spectrum of the Teichmüller flow on any connected component $\mathcal{C}$ of a stratum $\mathcal{A}_g(m_1,\ldots,m_\kappa)$ has the form*

$$\{\pm 1 \pm \nu_i : i = 1,\ldots,g\} \cup \{1,\ldots,1\} \cup \{-1,\ldots,-1\}$$

where ± 1 appear with multiplicity $\kappa - 1$, and $\nu_i = \theta_i/\theta_1$ for $i = 1,\ldots,g$.

Proof. Let us begin by recalling the construction in [21, Section 20]. The pre-stratum $\hat{\mathcal{S}} = \hat{\mathcal{S}}(C)$ associated to a Rauzy class C is the quotient of the space

$$\hat{\mathcal{H}} = \{(\pi,\lambda,\tau) : \pi \in C, \lambda \in \mathbb{R}_+^{\mathcal{A}}, \tau \in T_\pi^+\}$$

by the equivalence relation generated by

$$(\pi, e^{t_R}\lambda, e^{-t_R}\tau) \sim \mathcal{R}(\pi,\lambda,\tau) = (\pi, \Theta^{-1*}(e^{t_R}\lambda), \Theta^{-1*}(e^{-t_R}\tau)), \tag{59}$$

where the Rauzy renormalization time $t_R = t_R(\pi,\lambda)$ is characterized by

$$|\Theta^{-1*}(e^{t_R}\lambda)| = |\lambda| . \tag{60}$$

By definition, the Teichmüller flow $\mathcal{T}^t$, $t \in \mathbb{R}$ on $\hat{\mathcal{S}}$ is the projection under the quotient map of the flow defined on $\hat{\mathcal{H}}$ by

$$(\pi,\lambda,\tau) \mapsto (\pi, e^t\lambda, e^{-t}\tau). \tag{61}$$

The image $\mathcal{S} \subset \hat{\mathcal{S}}$ of the subset $\mathcal{H} = \{(\pi, \lambda, \tau) \in \hat{\mathcal{H}} : |\lambda| = 1\}$ under the quotient map is a cross section for the Teichmüller flow: the return time coincides with the Rauzy renormalization time and the Poincaré return map is identified with the Rauzy-Veech renormalization $\mathcal{R} : \mathcal{H} \to \mathcal{H}$. From (59) and (61) we see that the derivative of the time-t_R map of the Teichmüller flow has the form

$$D\mathcal{T}^{t_R}(\pi, \lambda, \tau) = \begin{pmatrix} e^{t_R}\Theta^{-1*} & 0 \\ 0 & e^{-t_R}\Theta^{-1*} \end{pmatrix} : \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}. \tag{62}$$

We also consider a smaller cross-section $\mathcal{S}_* \subset \mathcal{S}$ which is the image under the quotient map of $Z_* = \{(\pi, \lambda, \tau) \in Z_0 \cup Z_1 : |\lambda| = 1\}$. Recall $Z_\varepsilon \subset \mathcal{H}$ is the set of all (π, λ, τ) such that (π, λ) has type ε and τ has type ε, for $\varepsilon = 0, 1$. The corresponding Poincaré map coincides with the first return map of $\mathcal{R}$ to the cross-section, which is the Zorich renormalization $\mathcal{Z}$, and the first return time is the *Zorich renormalization time*

$$t_Z = t_Z(\pi, \lambda) = \sum_{j=0}^{n(\pi,\lambda)-1} t_R(R^j(\pi, \lambda)), \tag{63}$$

which is also characterized by (recall (53) and (60))

$$|\Gamma^{-1*}(e^{t_Z}\lambda)| = |\lambda|. \tag{64}$$

Using (53) and (63), one immediately gets an analogue of (62) for the derivative of the time-t_Z map of the Teichmüller flow:

$$D\mathcal{T}^{t_Z}(\pi, \lambda, \tau) = \begin{pmatrix} e^{t_Z}\Gamma^{-1*} & 0 \\ 0 & e^{-t_Z}\Gamma^{-1*} \end{pmatrix} : \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}. \tag{65}$$

These matrices $P(\pi, \lambda, \tau) = D\mathcal{T}^{t_Z}(\pi, \lambda, \tau)$ define a linear cocycle over the Zorich renormalization $\mathcal{Z}$, that we denote by F_P. The nth iterate is described by

$$P^n(\pi, \lambda, \tau) = D\mathcal{T}^{t_Z^n}(\pi, \lambda, \tau) = \begin{pmatrix} e^{t_Z^n}\Gamma^{-n*} & 0 \\ 0 & e^{-t_Z^n}\Gamma^{-n*} \end{pmatrix} \tag{66}$$

where

$$t_Z^n = t_Z^n(\pi, \lambda) = \sum_{j=0}^{n-1} t_Z(Z^j(\pi, \lambda)).$$

We are going to relate the Lyapunov spectra of the Teichmüller flow and the Zorich cocycle through the Lyapunov spectrum of this cocycle F_P. For this we need

Lemma 6.2. *For μ-almost every (π, λ),*

$$\lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi, \lambda) = \theta_1.$$

Proof. From the definition (64),

$$0 \le t_Z(\pi, \lambda) = -\log|\Gamma_{\pi,\lambda}^{-1*}(\lambda)| = \log \frac{|\lambda|}{|\Gamma_{\pi,\lambda}^{-1*}(\lambda)|} \le \log\left(d\,\|\Gamma_{\pi,\lambda}^*\|\right) \tag{67}$$

(take the norm of a matrix to be given by the largest absolute value of the coefficients). Then, since $\|\Gamma_{\pi,\lambda}^*\| = \|\Gamma_{\pi,\lambda}\|$, Proposition 4.7 immediately implies that the function $(\pi,\lambda) \mapsto t_Z(\pi,\lambda)$ is μ-integrable. Thus, we may use the ergodic theorem to conclude that

$$\lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} t_Z(Z^j(\pi,\lambda))$$

exists, at almost every point. Moreover, analogously to (67),

$$t_Z^n(\pi,\lambda) = -\log|\Gamma_{\pi,\lambda}^{-n*}(\lambda)| = \log\frac{|\lambda|}{|\Gamma_{\pi,\lambda}^{-n*}(\lambda)|} \le \log\left(d\,\|\Gamma_{\pi,\lambda}^{n*}\|\right) = \log\left(d\,\|\Gamma_{\pi,\lambda}^{n}\|\right)$$

for every $n \ge 1$. Consequently, applying Theorem 2.1 to the Zorich cocycle F_Z,

$$\lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda) \le \lim_{n\to\infty} \frac{1}{n} \log\|\Gamma_{\pi,\lambda}^n\| = \theta_1\,. \tag{68}$$

Next, fix some compact subset K of the simplex $\Lambda_{\mathcal{A}}$ and some positive constant $c = c(K)$ such that every vector $v \in K$ satisfies $v_\alpha \ge c$ for every $\alpha \in \mathcal{A}$. By ergodicity of the Zorich renormalization ([21, Theorem 8.2]), there exist $n_j \to \infty$ for which $\lambda^{n_j}/|\lambda^{n_j}| \in K$ and so $\lambda_\alpha^{n_j} \ge c\,|\lambda^{n_j}|$ for all $\alpha \in \mathcal{A}$ and all $j \ge 1$. For these iterates,

$$t_Z^{n_j}(\pi,\lambda) = \log\frac{|\lambda|}{|\Gamma_{\pi,\lambda}^{-n_j*}(\lambda)|} \ge \log\left(c\,\|\Gamma_{\pi,\lambda}^{n_j*}\|\right) = \log\left(c\,\|\Gamma_{\pi,\lambda}^{n_j}\|\right).$$

In view of the previous observations, this implies that

$$\lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda) \ge \lim_{n\to\infty} \frac{1}{n} \log\|\Gamma_{\pi,\lambda}^n\| = \theta_1\,. \tag{69}$$

The relations (68) and (69) prove the claim of the lemma. □

From (66) we immediately see that if E_x is an Oseledets subspace for F_Z^{-1*}, corresponding to a Lyapunov exponent θ, then $E_x \times \{0\}$ and $\{0\} \times E_x$ are Oseledets subspaces for F_P, corresponding to exponents

$$\theta + \lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda) \quad \text{and} \quad \theta - \lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda),$$

respectively. Therefore, using Lemma 6.2,

$$\text{Lyap spec}(F_P) = \left(\text{Lyap spec}(F_{\mathcal{Z}}^{-1*}) + \theta_1\right) \cup \left(\text{Lyap spec}(F_{\mathcal{Z}}^{-1*}) - \theta_1\right)$$

(for any ergodic $\mathcal{Z}$-invariant probability). From Propositions 2.7 and 5.1 we get that Lyap spec$(F_{\mathcal{Z}}^{-1*})$ = Lyap spec$(F_{\mathcal{Z}})$. Thus,

$$\begin{aligned} \text{Lyap spec}(F_P) &= (\text{Lyap spec}(F_{\mathcal{Z}}) + \theta_1) \cup (\text{Lyap spec}(F_{\mathcal{Z}}) - \theta_1)\,. \\ &= \{\pm\theta_1 \pm \theta_i : i = 1, \dots, g\} \cup \{\pm\theta_1, \dots, \pm\theta_1\} \end{aligned} \tag{70}$$

where the exponents $\pm\theta_1$ appear with multiplicity $\kappa - 1$. The definition (66) also gives that if E_x is an Oseledets subspace for the derivative $D\mathcal{T}^t$ of the flow, corresponding to Lyapunov exponent θ, then it is also an Oseledets subspace for F_P, corresponding to the exponent

$$\theta \lim_{n\to\infty} \frac{1}{n} t_Z^n(\pi,\lambda).$$

Using Lemma 6.2 once more, we conclude that

$$\text{Lyap spec}(F_P) = \theta_1 \,\text{Lyap spec}(\mathcal{T}). \tag{71}$$

The statement of the proposition follows by combining (70) and (71). □

Notice that $1-\nu_1 = 0 = -1+\nu_1$ and, in view of Proposition 5.5, these are the only vanishing Lyapunov exponents for the Teichmüller flow. The corresponding Oseledets subspace may be described explicitly:

Corollary 6.3. *The vanishing Lyapunov exponents of the Teichmüller flow are associated to the invariant 2-dimensional subbundle*

$$E_x^{00} = (\mathbb{R}\lambda, 0) \oplus (0, \mathbb{R}\tau) \subset \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}, \quad x = (\pi, \lambda, \tau).$$

The dynamics on this subbundle is trivial: up to an appropriate choice of bases, $D\mathcal{T}^t \mid E_x^{00} = \mathrm{id}$ for every $x \in \hat{\mathcal{S}}$ and $t \in \mathbb{R}$. The intersection of E_x^{00} with the tangent space to the hypersurfaces of constant area coincides with the flow direction.

Proof. From the proof of Proposition 6.1 we see that the Lyapunov exponents $1-\nu_1 = 0$ and $-1+\nu_1 = 0$ arise from $(E_x^{s*}, 0)$ and $(0, E_x^{u*})$, where E_x^{s*} and E_x^{u*} are the Oseledets subbundles of $F_{\mathcal{Z}}^{-1*}$ associated, respectively, to the smallest Lyapunov exponent $-\theta_1$ and the largest Lyapunov exponent θ_1. By Lemma 5.8, $E_x^{s*} = \mathbb{R}\lambda$ and $E_x^{u*} = \mathbb{R}\tau$. This proves the first claim in the lemma.

To prove the second one, consider the basis $\{(\lambda, 0), (0, \tau)\}$ of the plane E_x^{00}, defined for each $x = (\pi, \lambda, \tau) \in \mathcal{S}$. From (62) we get that

$$D\mathcal{T}^{t_R}(x)(\lambda, 0) = (\lambda', 0) \quad \text{and} \quad D\mathcal{T}^{t_R}(x)(0, \tau) = (0, \tau'),$$

where $(\pi', \lambda', \tau') = \mathcal{R}(\pi, \lambda, \tau) = \mathcal{T}^{t_R}(\pi, \lambda, \tau)$. This means that $D\mathcal{T}^{t_R}(x) = \mathrm{id}$ for every $x \in \mathcal{S}$, relative to these bases. Then, since $\mathcal{R}$ is the first return map to the cross-section $\mathcal{S}$, there exists a unique extension of the basis of E_x^{00} to every x in the pre-stratum $\hat{\mathcal{S}}$, relative to which $D\mathcal{T}^t(x) = \mathrm{id}$ in all cases.

From the definition (21) of the area we immediately see that the tangent space to the hypersurfaces of constant area at each point $x = (\pi, \lambda, \tau)$ is the hyperplane of all $(\dot{\lambda}, \dot{\tau}) \in \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ such that

$$\dot{\lambda} \cdot \Omega_\pi(\tau) + \lambda \cdot \Omega_\pi(\dot{\tau}) = 0.$$

So, its intersection with E_x^{00} is the space of all $(a\lambda, b\tau)$, $a, b \in \mathbb{R}$ such that

$$a\lambda \cdot \Omega_\pi(\tau) + \lambda \cdot \Omega_\pi(b\tau) = 0, \quad \text{that is} \quad a + b = 0.$$

In other words, the intersection is the line $\mathbb{R}(\lambda, -\tau)$. It is clear from the form of the Teichmüller flow, that the tangent vector field is $(\pi, \lambda, \tau) \mapsto (\lambda, -\tau)$. This completes the proof. □

7. Asymptotic flag theorem: preliminaries

We call *restricted* (respectively, *invertible restricted*) Zorich cocycle the restriction of F_Z (respectively, $F_{\mathcal{Z}}$) to the invariant subbundle $H = \{(\pi, \lambda) \times H_\pi\}$. Recall Lemma 5.2. For simplicity, we also denote these restrictions by F_Z and $F_{\mathcal{Z}}$. According to (29), we may consider them to act on the trivial fiber bundles

$$C \times \Lambda_{\mathcal{A}} \times H^1(M, \mathbb{R}) \quad \text{and} \quad Z_* \times H^1(M, \mathbb{R}),$$

respectively, with their adjoints F_Z^{-1*} and $F_{\mathcal{Z}}^{-1*}$ acting on

$$C \times \Lambda_{\mathcal{A}} \times H_1(M, \mathbb{R}) \quad \text{and} \quad Z_* \times H_1(M, \mathbb{R}),$$

respectively. Consider on H_π and $\mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi$ the Riemann metrics induced by the canonical metric in $\mathbb{R}^{\mathcal{A}}$. Then endow $H_1(M, \mathbb{R})$ and $H^1(M, \mathbb{R})$ with the metrics transported through the identifications in (27) and (28). In view of Proposition 6.1, Theorem C may be restated as

Theorem 7.1. *The Lyapunov spectrum of every restricted Zorich cocycle is simple:* $\theta_1 > \theta_2 > \cdots > \theta_g > -\theta_g > \cdots > -\theta_2 > -\theta_1$.

An outline of the proof of this theorem will be given later in Sections 9 through 12. Here and in the next section we are going to prove Theorem B from Theorem 7.1. For this, we need to recall some terminology.

Let $\sigma = I \times \{0\}$ be the basis horizontal segment in a representation of the surface M as zippered rectangles. We have seen in Section 3 that to each $\gamma(p, l)$ we may associate a vertical segment $\gamma(x, k)$ with endpoints in σ, such that the difference $[\gamma(p, l)] - [\gamma(x, k)]$ is uniformly bounded in the homology. Moreover, k and l are comparable, up to product by the area of the surface. Recall the relations (45) and (46). Up to identifying $H_1(M, \mathbb{R}) \approx \mathbb{R}^{\mathcal{A}} / \ker \Omega_\pi$ through (27), part 2 of Corollary 4.6 may be written as

$$[\gamma(x, z^m(x))] = \sum_{\beta \in \mathcal{A}} [v_\beta] \Gamma^m_{\alpha,\beta} = \sum_{\beta \in \mathcal{A}} [v_\beta] \Gamma^{m*}_{\beta,\alpha} = \Gamma^{m*}_{\pi,\lambda}([v_\alpha]) \tag{72}$$

for every $x \in J^m_\alpha$. The *annihilator* of a subspace $L \subset H_1(M, \mathbb{R})$ is the subspace $L^\perp$ of all $\phi \in H^1(M, \mathbb{R})$ such that $c \cdot \phi = \int_c \phi$ vanishes for every $c \in L$. Recall (30). Then, for any $c \in H_1(M, \mathbb{R})$ and any subspace $L \subset H_1(M, \mathbb{R})$,

$$\operatorname{dist}(c, L) = \max\{|c \cdot \phi| : \phi \in L^\perp, \|\phi\| = 1\}.$$

Let us choose the exponents ν_i and the subspaces L_i in Theorem B as follows:

- $\nu_i = \theta_i / \theta_1$ and
- $L_i \subset H_1(M, \mathbb{R})$ is the sum of the Oseledets subspaces corresponding to the Lyapunov exponents $\theta_1, \ldots, \theta_i$ of the linear cocycle $F_{\mathcal{Z}}^{-1*}$.

In view of (41), this means that the annihilator of L_i is the sum of the Oseledets subspaces associated to the Lyapunov exponents $\theta_{i+1}, \ldots, \theta_g, -\theta_g, \ldots, -\theta_1$ of the linear cocycle $F_{\mathcal{Z}}$. Then, Theorem B is an immediate consequence of

Theorem 7.2. *For every* $1 \le i < g$ *and any* $\phi \in L_i^\perp \setminus L_{i+1}^\perp$,

$$\limsup_{k \to \infty} \frac{\log \left|[\gamma(x, k)] \cdot \phi\right|}{\log k} = \nu_{i+1} \quad \textit{uniformly in } x \in \sigma. \tag{73}$$

Moreover, $\left|[\gamma(x, k)] \cdot \phi\right|$ *is uniformly bounded, for every* $\phi \in L_g^\perp$.

The proof of Theorem 7.2 occupies both this and the next section. All the arguments in the two sections are for μ-almost every (π, λ). In particular, we assume from the start that the associated interval exchange transformation is uniquely ergodic.

7.1. **Preparatory results.** Recall that we represent by $\{e_\alpha : \alpha \in \mathcal{A}\}$ the canonical basis of $\mathbb{R}^{\mathcal{A}}$. For each $\alpha \in \mathcal{A}$ and $m \ge 1$,

$$\Gamma^{m*}_{\pi,\lambda}(e_\alpha) = \sum_{\beta \in \mathcal{A}} \Gamma^m_{\alpha,\beta} e_\beta$$

is the α-line vector of the matrix of $\Gamma^m_{\pi,\lambda}$.

Proposition 7.3. *For every* $\alpha \in \mathcal{A}$,

$$\lim_{m \to \infty} \frac{1}{m} \log \|\Gamma^{m*}_{\pi,\lambda}(e_\alpha)\| = \theta_1.$$

Proof. The conclusion is independent of the choice of the norm: during the proof we take it to be given by the largest absolute value of the coefficients. From Theorem 2.1 we immediately get that, μ-almost everywhere,

$$\limsup_{m\to\infty}\frac{1}{m}\log\|\Gamma^{m*}_{\pi,\lambda}(e_\alpha)\|\le\lim_{m\to\infty}\frac{1}{m}\log\|\Gamma^{m}_{\pi,\lambda}\|=\theta_1. \tag{74}$$

So, we only have to prove the lower bound: for every $\delta>0$ and $\alpha\in\mathcal{A}$,

$$\liminf_{m\to\infty}\frac{1}{m}\log\|\Gamma^{m*}_{\pi,\lambda}(e_\alpha)\|\ge\theta_1-\delta, \tag{75}$$

μ-almost everywhere. To this end notice that, as a consequence of Corollary 4.4, the entries $\Gamma^{j}_{\alpha,\beta}$ of the matrix of $\Gamma^{j}_{\pi,\lambda}$ are eventually positive, for μ-almost every (π,λ). In particular, $\mu(V_l)\to 1$ when $l\to\infty$, where

$$V_l=\{(\pi,\lambda):\Gamma^{j}_{\alpha,\beta}\ge 1 \text{ for all } \alpha,\beta\in\mathcal{A} \text{ and all } j\ge l\}.$$

Fix ε small enough so that $(\theta_1-\varepsilon)(1-2\varepsilon)\ge\theta_1-\delta$, and then let $l\ge 1$ be fixed such that $\mu(V_l)>1-\varepsilon$. By ergodicity,

$$\lim_{m\to\infty}\frac{1}{m}\#\{0\le j<m: Z^j(\pi,\lambda)\in V_l\}=\mu(V_l),$$

for μ-almost all (π,λ). Thus, on a full μ-measure set we may find $N=N(\pi,\lambda)$ such that for every $m\ge N$ we have

$$\frac{1}{m}\#\{0\le j<m: Z^j(\pi,\lambda)\in V_l\}\ge\mu(V_l)-\varepsilon\ge 1-2\varepsilon \tag{76}$$

and also, recalling (74),

$$\frac{1}{m}\log\|\Gamma^{m}_{\pi,\lambda}\|\in(\theta_1-\varepsilon,\theta_1+\varepsilon). \tag{77}$$

Let $n\ge 2N+l$. Taking $m=n-l$ in (76), we get that there exists $j(n)$ such that $(n-l)(1-2\varepsilon)\le j(n)\le(n-l)$ and

$$Z^{j(n)}(\pi,\lambda)\in V_l. \tag{78}$$

In particular, $j(n)\ge 2N(1-2\varepsilon)\ge N$ (assume $\varepsilon<1/4$ from the start), and so we may take $m=j(n)$ in (77):

$$\log\|\Gamma^{j(n)}_{\pi,\lambda}\|\ge j(n)(\theta_1-\varepsilon). \tag{79}$$

From (78) and $n-j(n)\ge l$ we see that the entries of $\Gamma^{n-j(n)}_{Z^{j(n)}(\pi,\lambda)}$ are all positive. Therefore, for any $\alpha\in\mathcal{A}$,

$$\|\Gamma^{n*}_{\pi,\lambda}(e_\alpha)\|=\|\Gamma^{j(n)*}_{\pi,\lambda}\big(\Gamma^{(n-j(n))*}_{Z^{j(n)}(\pi,\lambda)}(e_\alpha)\big)\|\ge\|\Gamma^{j(n)*}_{\pi,\lambda}\|=\|\Gamma^{j(n)}_{\pi,\lambda}\|\,.$$

Using (79) we conclude that

$$\log\|\Gamma^{n*}_{\pi,\lambda}(e_\alpha)\|\ge j(n)(\theta_1-\varepsilon)\ge(n-l)(1-2\varepsilon)(\theta_1-\varepsilon),$$

and so

$$\liminf_{n\to\infty}\frac{1}{n}\log\|\Gamma^{n*}_{\pi,\lambda}(e_\alpha)\|\ge(1-2\varepsilon)(\theta_1-\varepsilon)\ge\theta_1-\delta$$

for every $\alpha\in\mathcal{A}$. This completes the proof of (75) which, together with (74), implies the proposition. □

Proposition 7.4. *For any $\alpha\in\mathcal{A}$, there exist $0\le l_1<\cdots<l_d$ such that $\{\Gamma^{l_s*}_{\pi,\lambda}(e_\alpha): s=1,\dots,d\}$ is a basis of $\mathbb{R}^{\mathcal{A}}$.*

Proof. By [21, Theorem 28.1] and [21, Remark 19.3], for almost every (π, λ) the intersection of all $\Theta^{l*}(\mathbb{R}_+^{\mathcal{A}})$, $l \geq 0$ coincides with the half-line spanned by λ. Since this intersection is decreasing and, by the definition (53), the Γ^l are a subsequence of the Θ^l, we also have that the intersection of all $\Gamma^{l*}(\mathbb{R}_+^{\mathcal{A}})$, $l \geq 0$ coincides with $\mathbb{R}_+\lambda$. This implies that $\Gamma^{l*}(e_\alpha)$ converges to the direction of λ in the projective space, as $l \to \infty$. Let $E \subset \mathbb{R}^\alpha$ be the subspace generated by the $\Gamma^{l*}(e_\alpha)$, $l \geq 0$. Since E is a closed subset, the previous observation implies that $\lambda \in E$. Suppose the conclusion of the proposition is false, that is, the subspace E has positive codimension. Then the subspace spanned by the (integer) vectors $\Gamma^{l*}(e_\alpha)$, $l \geq 0$ inside $\mathbb{Q}^{\mathcal{A}}$ also has positive codimension. Let $q \in \mathbb{Q}^{\mathcal{A}}$ be a non-zero vector in the orthogonal complement to this subspace. Then every vector in E is rationally dependent:

$$\sum_{\alpha \in \mathcal{A}} q_\alpha v_\alpha = 0 \quad \text{for every } v \in E.$$

This is a contradiction, because λ is rationally independent (for almost every λ). This contradiction proves that the $\Gamma^{l*}(e_\alpha)$, $l \geq 0$ span the whole $\mathbb{R}^{\mathcal{A}}$. Thus, we may choose $l_1 < \cdots < l_d$ as in the statement. □

7.2. **Special subsequence.** The proof of Theorem 7.2 is long and combinatorially subtle. In order to motivate the strategy and help the reader keep track of what is going on, we begin by stating and proving a special case where k runs only over the subsequence of Zorich induction times $z^m(x)$: the arguments are much more direct in this setting, while having the same flavor as the actual proof. This special case is not really used in the sequel, so the reader may also choose to proceed immediately to the next section.

We represent by J^m the domain of the mth Zorich induction $\hat{Z}^m(f)$, for any $m \geq 1$, and by $\{J_\alpha^m\colon \alpha \in \mathcal{A}\}$ the corresponding partition into subintervals. Corresponding to the case $m = 0$, we let $J = I$ and $J_\alpha = I_\alpha$ for any $\alpha \in \mathcal{A}$.

Proposition 7.5. *For every $1 \leq i < g$ and $\phi \in L_i^\perp \setminus L_{i+1}^\perp$,*

$$\limsup_{m\to\infty} \frac{\log \left|[\gamma(x, z^m(x))] \cdot \phi\right|}{\log z^m(x)} = \nu_{i+1} \quad \textit{uniformly in } x \in J^m.$$

This is an immediate consequence of Lemmas 7.6 through 7.8 below.

Lemma 7.6.

$$\lim_{m\to\infty} \frac{1}{m} \log z^m(x) = \theta_1 \quad \textit{uniformly in } x \in J^m.$$

Proof. By Corollary 4.6, $z^m(x) = \sum_{\beta\in\mathcal{A}} \Gamma^m_{\alpha,\beta}$ for every $\alpha \in \mathcal{A}$ and $x \in J_\alpha^m$. Consequently,

$$\min_{\alpha\in\mathcal{A}} \|\Gamma^{m*}(e_\alpha)\| \leq z^m(x) \leq d \max_{\alpha\in\mathcal{A}} \|\Gamma^{m*}(e_\alpha)\|$$

for every $x \in J^m$ (take the norm of a vector to be given by the largest absolute value of its coefficients). Proposition 7.3 asserts that $m^{-1} \log \|\Gamma^{m*}(e_\alpha)\|$ converges to θ_1 for every $\alpha \in \mathcal{A}$. Using this fact on the left hand side and on the right hand side of the previous formula we get that $m^{-1} \log z^m(x)$ converges uniformly to θ_1, as claimed. □

Lemma 7.7. *For every $1 \leq i < g$ and $\phi \in H^1(M, \mathbb{R}) \setminus L_{i+1}^\perp$,*

$$\limsup_{m\to\infty} \frac{1}{m} \log \left|[\gamma(x, z^m(x))] \cdot \phi\right| \geq \theta_{i+1} \quad \textit{uniformly in } x \in J^m.$$

Proof. By (72), we have $[\gamma(x, z^m(x)] = \Gamma^{m*}([v_\alpha])$ for every $x \in J^m_\alpha$. Take $l_1 < \ldots < l_d$ as in Proposition 7.4, such that $\Gamma^{l_s*}(e_\alpha)$, $s = 1, \ldots, d$ generate $\mathbb{R}^{\mathcal{A}}$. Since the Zorich cocycles are locally constant, we may find a simplex $D \subset \Lambda_{\mathcal{A}}$ such that all $(\pi, \lambda') \in \{\pi\} \times D$ share the same Γ^{l_s*} for $s = 1, \ldots, d$. By Poincaré recurrence, there exist infinitely many iterates $0 < k_1 < k_2 < \cdots$ such that $Z^{k_j}(\pi, \lambda) \in \{\pi\} \times D$. Since

$$\Gamma^{l_s*}([v_\alpha]), \quad s = 1, \ldots, d$$

generate $H_1(M, \mathbb{R})$, there exists $c_1 = c_1(\alpha) > 0$ and for each k_j we may find l_s, $s = s(j)$ such that

$$\left|\Gamma^{k_j}(\phi) \cdot \Gamma^{l_s*}([v_\alpha])\right| \geq c_1 \|\Gamma^{k_j}(\phi)\|.$$

This relation may be rewritten as

$$\left|[\gamma(x, z^{m_j}(x))] \cdot \phi\right| = \left|\Gamma^{m_j*}([v_\alpha]) \cdot \phi\right| = \left|\Gamma^{l_s*}([v_\alpha]) \cdot \Gamma^{k_j}(\phi)\right| \geq c_1 \|\Gamma^{k_j}(\phi)\|,$$

where $m_j = k_j + l_s$. The condition $\phi \notin L^\perp_{i+1}$ means that ϕ is outside the sum of the Oseledets subspaces associated to the exponents $\theta_{i+2}, \ldots, \theta_g, -\theta_g, \ldots, -\theta_1$ of the cocycle $F_{\mathcal{Z}}$. So, for any $\varepsilon > 0$ we may find $c_2 = c_2(\varepsilon) > 0$ such that

$$\|\Gamma^k(\phi)\| \geq c_2 e^{(\theta_{i+1} - \varepsilon)k} \|\phi\| \quad \text{for all } k \geq 0.$$

Combining the last inequalities we obtain $\left|[\gamma(x, z^{m_j}(x))] \cdot \phi\right| \geq c e^{(\theta_{i+1} - \varepsilon)k_j}$, where $c = c_1 c_2 \|\phi\|$ depends only on α, ε, and ϕ. It is clear that $m_j/k_j \to 1$ as $j \to \infty$, since l_s takes only finitely many values. So this last inequality implies the conclusion of the lemma. □

Lemma 7.8. *For every $1 \leq i < g$ and $\phi \in L^\perp_i$,*

$$\limsup_{m \to \infty} \frac{1}{m} \log \left|[\gamma(x, z^m(x))] \cdot \phi\right| \leq \theta_{i+1} \quad \textit{uniformly in } x \in J^m.$$

Proof. From the relation (72) we get that, for any $x \in J^m_\alpha$,

$$[\gamma(x, z^m(x))] \cdot \phi = \Gamma^{m*}([v_\alpha]) \cdot \phi = [v_\alpha] \cdot \Gamma^m(\phi).$$

The condition $\phi \in L^\perp_i$ means ϕ belongs to the sum of the Oseledets subspaces associated to the exponents $\theta_{i+1}, \ldots, \theta_g, -\theta_g, \ldots, -\theta_1$ of the cocycle $F_{\mathcal{Z}}$. Hence, given any $\varepsilon > 0$, there exists $c_3 = c_3(\pi, \lambda, \varepsilon) > 0$ such that

$$\|\Gamma^m(\phi)\| \leq c_3 e^{(\theta_{i+1} + \varepsilon)m} \|\phi\| \quad \text{for all } m \geq 1.$$

Combining these observations we find that

$$|[\gamma(x, z^m(x))] \cdot \phi| \leq C e^{(\theta_{i+1} + \varepsilon)m} \|\phi\| \quad \text{for all } m \geq 1, \tag{80}$$

where C is the product of c_3 by an upper bound for the norm of every $[v_\alpha]$. This proves the lemma. □

Remark 7.9. The arguments in Lemma 7.8 remain valid for $i = g$: in the place of (80), one obtains

$$|[\gamma(x, z^m(x))] \cdot \phi| \leq C e^{(-\theta_g + \varepsilon)m} \|\phi\| \quad \text{for all } m \geq 1.$$

Since $-\theta_g < 0$, this implies that $|[\gamma(x, z^m(x))] \cdot \phi|$ converges to zero as $m \to \infty$, uniformly in $x \in J^m$.

8. Asymptotic flag theorem: proof

In this section we prove the full statement of Theorem 7.2. For the reader's convenience, we split the arguments into three main steps, that are presented in Sections 8.1, 8.2, 8.3.

8.1. **Preparation.** Given $x \in \sigma$ and $k \geq 1$, let $m = m(x,k)$ be the largest integer such that the orbit segment $f^j(x)$, $0 \leq j \leq k$ hits the interval J^m at least twice. That is,

$$m = m(x,k) = \max\left\{l \geq 0 : \#\{0 \leq j \leq k : f^j(x) \in J^l\} \geq 2\right\}. \tag{81}$$

Note that if $x \in J^n$ then $m(x, z^n(x)) = n$, because $z^n(x)$ is the first return time to J^n. Thus, the next result is an extension of Lemma 7.6:

Lemma 8.1.

$$\lim_{k\to\infty} \frac{\log k}{m(x,k)} = \theta_1 \quad \textit{uniformly in } x \in \sigma.$$

Proof. Let $x_j = f^j(x)$, where $j \geq 0$ is the first time x hits J^m. By the definition of m, the orbit of x_j returns to J^m before time $k-j$. So, using part 3 of Corollary 4.6,

$$k \geq k - j \geq z^m(x_j) \geq \min_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{A}} \Gamma^m_{\alpha,\beta} \geq \min_{\alpha\in\mathcal{A}} \|\Gamma^{m*}(e_\alpha)\| \,. \tag{82}$$

By the definition of m, the orbit segment $f^j(x)$, $0 \leq j \leq k$ intersects J^{m+1} at most once. Suppose for a while that, in fact, there is no intersection. Since we take the interval exchange f to be minimal, there are iterates $-r < 0 < k < s$ such that $f^{-r}(x)$ and $f^s(x)$ belong to J^{m+1}. Take r and s smallest and denote $x_{-r} = f^{-r}(x)$. Then, using once more part 3 of Corollary 4.6,

$$k \leq r + s = z^{m+1}(x_{-r}) \leq \max_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{A}} \Gamma^{m+1}_{\alpha,\beta} \leq d \max_{\alpha\in\mathcal{A}} \|\Gamma^{(m+1)*}(e_\alpha)\| \,.$$

In general, if the orbit segment $f^j(x)$, $0 \leq j \leq k$ does intersect J^{m+1}, we may apply the previous argument to the subsegments before and after the intersection. In this way we find that

$$k \leq 2 \max_{\alpha\in\mathcal{A}} \sum_{\beta\in\mathcal{A}} \Gamma^{m+1}_{\alpha,\beta} \leq 2d \max_{\alpha\in\mathcal{A}} \|\Gamma^{(m+1)*}(e_\alpha)\| \,. \tag{83}$$

This relation also ensures that m goes to infinity, uniformly in x, when k goes to infinity. Now let $\varepsilon > 0$. By Proposition 7.3 there is $n_\varepsilon > 0$ such that

$$\frac{1}{n} \log \|\Gamma^{n*}(e_\alpha)\| \in \left(\theta_1 - \varepsilon, \theta_1 + \varepsilon\right) \quad \text{for all } n \geq n_\varepsilon \text{ and } \alpha \in \mathcal{A}.$$

Assume k is large enough to ensure $m \geq n_\varepsilon$. Then (82) and (83) yield

$$m(\theta_1 - \varepsilon) \leq \log k \leq \log(2d) + (m+1)(\theta_1 + \varepsilon).$$

Dividing by m and passing to the limit as $k \to \infty$, we obtain

$$\theta_1 - \varepsilon \leq \liminf_{k\to\infty} \frac{\log k}{m} \leq \limsup_{k\to\infty} \frac{\log k}{m} \leq \theta_1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the claim in the lemma. □

To complete the proof of Theorem 7.2 we only need the following three propositions (compare Lemmas 7.7 and 7.8 and Remark 7.9).

Proposition 8.2. *For every $\phi \in H^1(M,\mathbb{R}) \setminus L_{i+1}^{\perp}$,*

$$\limsup_{k\to\infty} \frac{1}{m(x,k)} \log \left|[\gamma(x,k)]\cdot\phi\right| \geq \theta_{i+1} \quad \textit{uniformly in } x\in\sigma.$$

Proposition 8.3. *For every $\phi \in L_i^{\perp}$,*

$$\limsup_{k\to\infty} \frac{1}{m(x,k)} \log \left|[\gamma(x,k)]\cdot\phi\right| \leq \theta_{i+1} \quad \textit{uniformly in } x\in\sigma.$$

Proposition 8.4. *There exists $C > 0$ such that $\left|[\gamma(x,k)]\cdot\phi\right| \leq C\|\phi\|$ for any $x\in\sigma$, any $k \geq 1$, and any $\phi\in L_g^{\perp}$.*

The proofs are given in the next two subsections. Before that, we need to introduce some terminology. Given any $x\in\sigma$ and $k\geq 1$, define

$$s(x,k;\pi,\lambda) = \sum_{\beta\in\mathcal{A}} s_\beta(x,k;\pi,\lambda)e_\beta$$

where $\{e_\beta : \beta\in\mathcal{A}\}$ is the canonical basis of $\mathbb{R}^{\mathcal{A}}$ and each $s_\beta(x,k;\pi,\lambda)$ is the number of visits of x to the subinterval I_β before time k:

$$s_\beta(x,k;\pi,\lambda) = \#\{0\leq j<k : f^j(x)\in I_\beta\}.$$

Observe that, whenever $x\in J_\alpha^m$, part 1 of Corollary 4.6 gives

$$s_\beta(x,z^m(x);\pi,\lambda) = \#\{0\leq j< z^m(x) : f^j(x)\in J_\beta\} = \Gamma^m_{\alpha,\beta}(e_\alpha),$$

for all $\beta\in\mathcal{A}$. Equivalently,

$$s(x,z^m(x);\pi,\lambda) = \Gamma^{m*}(e_\alpha). \tag{84}$$

Observe, in addition, that $s(x,k;\pi,\lambda)\in\mathbb{R}^{\mathcal{A}}$ corresponds to the homology class $[\gamma(x,k)]$ under the identification (27). In what follows, $v\in H_\pi$ is the vector corresponding to $\phi\in H^1(M,\mathbb{R})$ under the identification (28). Then,

$$[\gamma(x,k)]\cdot\phi = s(x,k;\pi,\lambda)\cdot v. \tag{85}$$

8.2. **Lower bound.**

For the proof of Proposition 8.2 we need the following auxiliary result:

Lemma 8.5. *Let $\alpha\in\mathcal{A}$ be the first symbol on the top line of π. Then there exists $r\geq 1$ and a sequence $(n_j)_j\to\infty$ such that*

$$\liminf_{j\to\infty} \frac{1}{n_j}\log|\Gamma^{n_j*}_{\pi,\lambda}(e_\alpha)\cdot v| \geq \theta_{i+1} \quad \textit{and} \quad J^{n_j+r}\subset J^{n_j}_\alpha \ \textit{for all } j\geq 1.$$

Proof. The condition $\phi\notin L_{i+1}^{\perp}$ means that ϕ (thus, v) is outside the sum of the Oseledets subspaces associated to the Lyapunov exponents θ_{i+1}, …, θ_g, $-\theta_g$, …, $-\theta_1$ of the cocycle $F_{\mathcal{Z}}$. So, for any $\varepsilon>0$, there exists $c_0=c_0(\pi,\lambda,\varepsilon)>0$ such that

$$\|\Gamma^l_{\pi,\lambda}(v)\| \geq c_0 e^{(\theta_{i+1}-\varepsilon)l}\|v\| \quad \text{for every } l\geq 1. \tag{86}$$

By Proposition 7.4, there exist $l_1<\cdots<l_d$ such that

$$\Gamma^{l_s*}_{\pi,\lambda}(e_\alpha), \quad s=1,\ldots,d \quad \text{is a basis of } \mathbb{R}^{\mathcal{A}}. \tag{87}$$

It follows from the definition of the induction operator, recalled in Section 1.1, that the first symbol on the top line of π^n is always α, for all $n\geq 1$. Thus, the left endpoint of J^n_α coincides with $\partial J^n = 0$ for every n. By [21, Corollary 5.2], the diameter of J^n goes to zero as $n \to \infty$. Then, there exists $r \geq 1$ such that

$$J^{l_s+r} \subsetneq J^{l_s}_\alpha \quad \text{for every } s = 1, \ldots, d. \tag{88}$$

By continuity, (88) remains valid for any $(\hat\pi, \hat\lambda)$ in a small neighborhood U of (π, λ). Reducing U if necessary, we may suppose that

$$\Gamma^{l_s *}_{\hat\pi,\hat\lambda}(e_\alpha) = \Gamma^{l_s *}_{\pi,\lambda}(e_\alpha), \quad \text{for every } (\hat\pi, \hat\lambda) \in U \text{ and } s = 1, \ldots, d. \tag{89}$$

By Poincaré recurrence, there exist infinitely many iterates $t_1 < \cdots < t_j < \cdots$ such that

$$Z^{t_j}(\pi, \lambda) \in U.$$

In view of (87), there exists $c_1 > 0$ and for each j there exists some $s = s(j)$ such that

$$|\Gamma^{l_s *}_{\pi,\lambda}(e_\alpha) \cdot \Gamma^{t_j}_{\pi,\lambda}(v)\| \geq c_1 \|\Gamma^{t_j}_{\pi,\lambda}(v)\| .$$

Take $n_j = t_j + l_s$. In view of (89), the previous inequality leads to

$$|\Gamma^{n_j *}_{\pi,\lambda}(e_\alpha) \cdot v| = |\Gamma^{l_s *}_{Z^j(\pi,\lambda)}(e_\alpha) \cdot \Gamma^{t_j}_{\pi,\lambda}(v)| = |\Gamma^{l_s *}_{\pi,\lambda}(e_\alpha) \cdot \Gamma^{t_j}_{\pi,\lambda}(v)| \geq c_1 \|\Gamma^{t_j}_{\pi,\lambda}(v)\| .$$

Combined with (86), this gives that

$$|\Gamma^{n_j *}_{\pi,\lambda}(e_\alpha) \cdot v| \geq c_0 c_1 e^{(\theta_{i+1}-\varepsilon)t_j} \|v\| \quad \text{for every } j \geq 1.$$

Clearly, $|n_t - t_j| \leq \max\{l_s : s = 1, \ldots, d\}$ for all $j \geq 1$, and so t_j/n_j converges to 1 as $j \to \infty$. So, the previous inequality implies that there exists $c_2 = c_2(\varepsilon)$ such that

$$|\Gamma^{n_j *}_{\pi,\lambda}(e_\alpha) \cdot v| \geq c_2 e^{(\theta_{i+1}-\varepsilon)t_j} \|v\| \quad \text{for every } j \geq 1.$$

This proves the first claim in the lemma. To prove the second one, observe that

$$J^n = [0, |\hat\lambda^n|) \quad \text{and} \quad J^n_\alpha = [0, \hat\lambda^n_\alpha) \quad \text{for all } n \geq 1,$$

where $(\pi^n, \hat\lambda^n) = \hat Z^n(\pi, \lambda)$. Keep in mind that $Z^n(\pi, \lambda) = (\pi^n, \lambda^n)$ for all n, where $\lambda^n = \hat\lambda^n / |\hat\lambda^n|$. Denote $(\pi^{n,l}, \hat\lambda^{n,l}) = \hat Z^l(\pi^n, \lambda^n)$, for every $n \geq 1$ and $l \geq 0$. Then

$$\hat\lambda^{n,l} = \frac{\hat\lambda^{n+l}}{|\hat\lambda^n|} \quad \text{for every } n \geq 1 \text{ and } l \geq 0.$$

The relation (88), applied to the points $Z^{t_j}(\pi, \lambda) \in U$, means that

$$|\hat\lambda^{t_j, l_s+r}| < \hat\lambda^{t_j, l_s}_\alpha \quad \text{for all } j \geq 1.$$

Multiplying both sides by $|\hat\lambda^{t_j}|$ we obtain that

$$|\hat\lambda^{t_j+l_s+r}| < \hat\lambda^{t_j+l_s}_\alpha$$

and this implies that $J^{n_j+r} \subset J^{n_j}_\alpha$, for all $j \geq 1$. □

Proof of Proposition 8.2. Given $r \geq 1$ and $(n_j)_j$ as in Lemma 8.5, let us define $p_j = p_j(x)$ to be the first time the orbit of x hits the interval J^{n_j+r}, that is,

$$p_j = \min\{n \geq 0 : f^n(x) \in J^{n_j+r}\}.$$

It is clear from the definition (81) that $m(x, p_j) \leq n_j + r$, and so

$$\limsup_{j\to\infty} \frac{1}{m(x,p_j)} \log |s(x, p_j; \pi, \lambda) \cdot v| \geq \limsup_{j\to\infty} \frac{1}{n_j + r} \log |s(x, p_j; \pi, \lambda) \cdot v|$$
$$= \limsup_{j\to\infty} \frac{1}{n_j} \log |s(x, p_j; \pi, \lambda) \cdot v| .$$

If the limit on the right hand side is greater or equal than θ_{i+1} then the same is true for the limit on the left hand side which, in view of (85), implies that the conclusion of the proposition holds. So, we may assume that the limit is strictly less than θ_{i+1}: there exist $a > 0$ and $c_3 > 0$ such that

$$|s(x, p_j; \pi, \lambda) \cdot v| \le c_3 e^{n_j(\theta_{i+1}-a)} \|v\| \quad \text{for all } j \ge 1. \tag{90}$$

Then let $q_j = q_j(x)$ be the first time the orbit of x returns to J^{n_j} after time p_j:

$$p_j = \min\{n > p_j : f^n(x) \in J^{n_j}\}.$$

In other words, $q_j = p_j + z^{n_j}(x_j)$, where $x_j = f^{p_j}(x)$. Clearly,

$$s(x, q_j; \pi, \lambda) = s(x, p_j; \pi, \lambda) + s(x_j, z^{n_j}(x_j); \pi, \lambda).$$

By construction, $x_j \in J^{n_j+r} \subset J_\alpha^{n_j}$. Thus, using (84), this relation may be rewritten as

$$s(x, q_j; \pi, \lambda) = s(x, p_j; \pi, \lambda) + \Gamma_{\pi,\lambda}^{n_j *}(e_\alpha).$$

It follows, using (86) and (90), that

$$\begin{aligned} |s(x, q_j; \pi, \lambda) \cdot v| &\ge |\Gamma_{\pi,\lambda}^{n_j *}(e_\alpha) \cdot v| - |s(x, p_j; \pi, \lambda) \cdot v| \\ &\ge c_0 e^{n_j(\theta_{i+1}-\varepsilon)} \|v\| - c_3 e^{n_j(\theta_{i+1}-a)} \|v\| . \end{aligned}$$

Taking $\varepsilon < a$, this implies that there exists $c_4 = c_4(\pi, \lambda, \varepsilon) > 0$ such that

$$|s(x, q_j; \pi, \lambda) \cdot v| \ge c_4 e^{n_j(\theta_{i+1}-\varepsilon)} \|v\|$$

for all $j \ge 1$. In view of (85), this implies that

$$\limsup_{k\to\infty} \frac{1}{m} \log \big|[\gamma(x,k) \cdot \phi\big| = \limsup_{k\to\infty} \frac{1}{m} \log \big|s(x, k; \pi, \lambda) \cdot v\big| \ge \theta_{i+1} - \varepsilon,$$

uniformly. Proposition 8.2 follows, since $\varepsilon > 0$ is arbitrary. □

8.3. **Upper bound.** The strategy to prove Proposition 8.3 is to stratify the orbit segment $f^j(x)$, $0 \le j \le k$ according to increasing renormalization depth, relating each stratification level to some subsegment that starts and ends at returns to a domain J^l. Let us explain this in more detail, with the help of Figure 12.

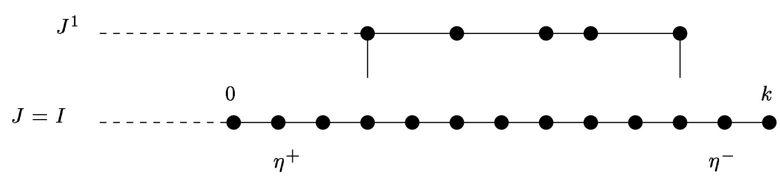

FIGURE 12.

Recall J^1 denotes the domain of the Zorich induction $\hat{Z}(f)$ of the transformation f. Given $x \in \sigma$ and $k \ge 1$, define

$$\begin{aligned} \eta^+ &= \eta^+(x, k; \pi, \lambda) = \min\{j \ge 0 : f^j(x) \in J^1\} \quad \text{and} \\ \eta^- &= \eta^-(x, k; \pi, \lambda) = \min\{j \ge 0 : f^{k-j}(x) \in J^1\}. \end{aligned} \tag{91}$$

In other words, η^+ is the first time and $k-\eta^-$ is the last time the orbit segment hits the interval J^1. Denote $x_1 = f^{\eta^+}(x)$. Then, time $k-\eta^+-\eta^-$ is a return of the point x_1 to the interval J^1 under the map f, and so

$$f^{k-\eta^+-\eta^-}(x_1) = \hat{Z}(f)^{k_1}(x_1) \tag{92}$$

for some $k_1 \geq 1$. It is clear that

$$\begin{aligned} s(x,k;\pi,\lambda) &= s(x_1, k-\eta^+-\eta^-;\pi,\lambda) \\ &\qquad + s(x,\eta^+;\pi,\lambda) + s(f^{k-\eta^-}(x),\eta^-;\pi,\lambda). \end{aligned} \tag{93}$$

Compare Figure 12. The first term on the right hand side will be estimated through the following recurrence relation:

Lemma 8.6. *For every $x \in \sigma$ and $k \geq 1$,*

$$s(x_1, k-\eta^+-\eta^-;\pi,\lambda) = \Gamma^*_{\pi,\lambda}\left(s(x_1,k_1;\pi^1,\lambda^1)\right)$$

where $(\pi^1,\lambda^1) = Z(\pi,\lambda)$ and the number $k_1 \geq 1$ is defined by (92).

Proof. Denote $g = Z(f)$. By (92), we have $f^{k-\eta^+-\eta^-}(x_1) = g^{k_1}(x_1)$. Clearly,

$$s_\beta(x_1, k-\eta^+-\eta^-;\pi,\lambda) = \sum_{i=0}^{k_1-1} s_\beta(g^i(x_1), z^1(g^i(x_1));\pi,\lambda),$$

for every $\beta \in \mathcal{A}$. By part 1 of Corollary 4.6,

$$s_\beta(g^i(x_1), z^1(g^i(x_1));\pi,\lambda) = \#\{0 \leq j < z^1(g^i(x_1)) : f^j(g^i(x_1)) \in I_\beta\} = \Gamma_{\alpha,\beta}$$

whenever $g^i(x_1) \in J^1_\alpha$. Replacing in the previous relation,

$$\begin{aligned} s_\beta(x_1, k-\eta^+-\eta^-;\pi,\lambda) &= \sum_{\alpha\in\mathcal{A}} \#\{0 \leq i < k_1 : g^i(x_1) \in J^1_\alpha\}\,\Gamma_{\alpha,\beta}. \\ &= \sum_{\alpha\in\mathcal{A}} \Gamma_{\alpha,\beta} s_\alpha(x_1,k_1;\pi^1,\lambda^1). \end{aligned}$$

This means that $s(x_1, k-\eta^+-\eta^-;\pi,\lambda) = \Gamma^*\left(s(x_1,k_1;\pi^1,\lambda^1)\right)$, as claimed. □

The sum of the last two terms in (93) will be bounded using the next lemma. Recall we take the norm of a vector to be given by the largest absolute value of its coefficients.

Lemma 8.7. *For every $x \in \sigma$, $k \geq 1$, and $l \geq 1$,*

$$\|s(x,\eta^+;\pi,\lambda) + s(f^{k-\eta^-}(x),\eta^-;\pi,\lambda)\| \leq 2\|\Gamma_{\pi,\lambda}\|\,.$$

Proof. Take $r \geq 0$ minimum such that $\bar{x} = f^{-r}(x) \in J^1$. This is well defined, since the interval exchange f is minimal. Then $r+\eta^+$ is the first return time of $\bar{x}$ to J^1, that is, $r+\eta^+ = z(\bar{x})$. Clearly, $s_\beta(x,\eta^+;\pi,\lambda) \leq s_\beta(\bar{x},z(\bar{x});\pi,\lambda)$ for every $\beta \in \mathcal{A}$. From part 1 of Corollary 4.6 we get that

$$s_\beta(\bar{x},z(\bar{x});\pi,\lambda) = \#\{0 \leq j < z(\bar{x}) : f^j(\bar{x}) \in I_\beta\} \leq \max_{\alpha\in\mathcal{A}} \Gamma_{\alpha,\beta}$$

for every $\beta \in \mathcal{A}$. Therefore,

$$\|s(x,\eta^+;\pi,\lambda)\| \leq \|s(\bar{x},z(\bar{x});\pi,\lambda)\| \leq \max_{\alpha,\beta\in\mathcal{A}} \Gamma_{\alpha,\beta} = \|\Gamma\|.$$

Analogously, $\|s(f^{k-\eta^-}(x),\eta^-;\pi,\lambda)\| \leq \|\Gamma\|$. The lemma follows. □

Replacing Lemmas 8.6 and 8.7 in (93), we obtain that

$$s(x,k;\pi,\lambda) = \Gamma^*_{\pi,\lambda} \cdot s(x_1,k_1;\pi^1,\lambda^1) + r(x,k;\pi,\lambda) \tag{94}$$

with $\|r(x,k;\pi,\lambda)\| \le 2\|\Gamma_{\pi,\lambda}\|$, for every $x \in \sigma$ and $k \ge 1$.

Applying this relation to the orbit segment $Z(f)^i(x_1)$, $0 \le i < k_1$, we obtain

$$s(x_1,k_1;\pi^1,\lambda^1) = \Gamma^*_{\pi^1,\lambda^1} \cdot s(x_2,k_2;\pi^2,\lambda^2) + r(x_1,k_1;\pi^1,\lambda^1),$$

where $(\pi^2,\lambda^2) = Z^2(\pi,\lambda)$ and $\|r(x_1,k_1;\pi^1,\lambda^1)| \le 2\|\Gamma_{\pi^1,\lambda^1}\|$. Thus,

$$\begin{aligned} s(x,k;\pi,\lambda) &= \Gamma^*_{\pi,\lambda} \cdot \left[\Gamma^*_{\pi^1,\lambda^1} \cdot s(x_2,k_2;\pi^2,\lambda^2) + r(x_1,k_1;\pi^1,\lambda^1)\right] + r(x,k;\pi,\lambda) \\ &= \Gamma^{2*}_{\pi,\lambda} \cdot s(x_2,k_2;\pi^2,\lambda^2) + \Gamma^*_{\pi,\lambda} \cdot r(x_1,k_1;\pi^1,\lambda^1) + r(x,k;\pi,\lambda). \end{aligned}$$

Write $(\pi^j,\lambda^j) = Z^j(\pi,\lambda)$ for $j \ge 0$. Repeating this procedure m times, we obtain (compare Figure 13)

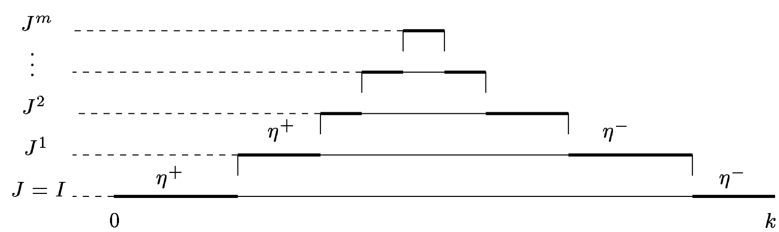

FIGURE 13.

Lemma 8.8. *For every $x \in \sigma$ and $k \ge 1$,*

$$s(x,k;\pi,\lambda) = \Gamma^{m*}_{\pi,\lambda} \cdot s(x_m,k_m;\pi^m,\lambda^m) + \sum_{j=0}^{m-1} \Gamma^{j*}_{\pi,\lambda} \cdot r(x_j,k_j;\pi^j,\lambda^j)$$

with $\|s(x_m,k_m;\pi^m,\lambda^m)\| \le 2\|\Gamma_{\pi^m,\lambda^m}\|$ and $\|r(x_j,k_j;\pi^j,\lambda^j)\| \le 2\|\Gamma_{\pi^j,\lambda^j}\|$ for every $0 \le j \le m-1$.

Proof. All that is left to prove is the bound on the norm of $s(x_m,k_m;\pi^m,\lambda^m)$. Let $\hat\Gamma = \Gamma_{\pi^m,\lambda^m}$ and let $\hat\Gamma_{\alpha,\beta}$, $\alpha,\beta \in \mathcal{A}$ be its coefficients. Denote $g = Z^m(f)$. The definition of m implies that the orbit segment $g^j(x_m)$, $0 \le j \le k_m$ intersects J^{m+1} at most once. Suppose first that there is no intersection. Since g is minimal, there exist $-r < 0 \le k_m < s$ such that both $x_{-r} = g^{-r}(x_m)$ and $x_s = g^s(x_m)$ are in J^{m+1}. Take r and s minimum. Then $r+s$ coincides with the first Zorich inducing time $z_{\pi^m,\lambda^m}(x_{-r})$ of the point x_{-r} for the transformation g. So, using part 1 of Corollary 4.6,

$$s_\beta(x_m,k_m;\pi^m,\lambda^m) \le s_\beta(x_{-r},r+s;\pi^m,\lambda^m) \le \max_{\alpha\in\mathcal{A}} \hat\Gamma_{\alpha,\beta}$$

for every $\beta \in \mathcal{A}$. It follows that

$$\|s(x_m,k_m;\pi^m,\lambda^m)\| = \max_{\beta\in\mathcal{A}} s_\beta(x_m,k_m;\pi^m,\lambda^m) \le \max_{\alpha,\beta\in\mathcal{A}} \hat\Gamma_{\alpha,\beta} = \|\hat\Gamma\|.$$

If $g^j(x)$, $0 \le j \le k_p$ does intersect J^{m+1}, we may apply the same argument as before to the subsegments before and after the intersection. Then, adding the two bounds, we find that $\|s(x_m,k_m;\pi^m,\lambda^m)\| \le 2\|\hat\Gamma\|$, as claimed. □

Proof of Proposition 8.3. The condition $\phi \in L_i^\perp$ means that ϕ (thus, v) belongs to the sum of the Oseledets subspaces associated to the exponents $\theta_{i+1}, \ldots, \theta_g, -\theta_g, \ldots, -\theta_1$ of the cocycle $F_{\mathcal{Z}}$. Hence, given any $\varepsilon > 0$, there exists $c_0 = c_0(\pi, \lambda, \varepsilon)$ such that

$$\|\Gamma^j_{\pi,\lambda}(v)\| \le c_0 e^{(\theta_{i+1}+\varepsilon)j}\|v\| \quad \text{for every } j \ge 0. \tag{95}$$

By Proposition 4.7, the function $\phi(\tilde{\pi}, \tilde{\lambda}) = \log \|\Gamma_{\tilde{\pi},\tilde{\lambda}}\|$ is μ-integrable. So, we may apply Remark 2.5 to conclude that, for any $\varepsilon > 0$ there is $c_1 = c_1(\pi, \lambda, \varepsilon)$ such that

$$\|r(x_j, k_j; \pi^j, \lambda^j)\| \le 2\|\Gamma_{\pi^j,\lambda^j}\| \le c_1 e^{\varepsilon j} \quad \text{for every } j \ge 0. \tag{96}$$

Using Lemma 8.8, we find that

$$s(x, k; \pi, \lambda) \cdot v = s(x_m, k_m; \pi^m, \lambda^m) \cdot \Gamma^m_{\pi,\lambda}(v) + \sum_{j=0}^{m-1} r(x_j, k_j; \pi^j, \lambda^j) \cdot \Gamma^j_{\pi,\lambda}(v).$$

and so, using also (95) and (96),

$$\big|s(x, k; \pi, \lambda) \cdot v\big| \le \sum_{j=0}^{m} c_0 e^{(\theta_{i+1}+\varepsilon)j}\|v\|\, c_1 e^{\varepsilon j} = c_0 c_1 \|v\| \sum_{j=0}^{m} e^{(\theta_{i+1}+2\varepsilon)j}. \tag{97}$$

Assuming $\varepsilon > 0$ is small enough, the exponent $\theta_{i+1} + 2\varepsilon$ is positive, and so the sum is bounded by a multiple of the last term. Thus, there exists $c_2 = c_2(\pi, \lambda, \varepsilon)$ such that

$$\big|s(x, k; \pi, \lambda) \cdot v\big| \le c_2 e^{(\theta_{i+1}+2\varepsilon)m} \tag{98}$$

for every $x \in \sigma$ and $k \ge 1$. In view of (85), this implies that

$$\limsup_{k\to\infty} \frac{1}{m} \log \big|[\gamma(x, k) \cdot \phi\big| = \limsup_{k\to\infty} \frac{1}{m} \log \big|s(x, k; \pi, \lambda) \cdot v\big| \le \theta_{i+1} + 2\varepsilon,$$

uniformly. As $\varepsilon > 0$ is arbitrary, the conclusion of Proposition 8.3 follows. □

Proof of Proposition 8.4. This is similar to the proof of Proposition 8.3. The condition $\phi \in L_g^\perp$ means that ϕ (thus, v) belongs to the sum of the Oseledets subspaces associated to the exponents $-\theta_g, \ldots, -\theta_1$ of the cocycle $F_{\mathcal{Z}}$. Fix $0 < 2a < \theta_g$. Then there exists $c_3 = c_3(\pi, \lambda) > 0$ such that

$$\|\Gamma^j_{\pi,\lambda}(v)\| \le c_3 e^{-2aj}\|v\| \quad \text{for every } j \ge 0. \tag{99}$$

Just as in (96), there is also $c_4 = c_4(\pi, \lambda) > 0$ such that

$$\|r(x_j, k_j; \pi^j, \lambda^j)\| \le 2\|\Gamma_{\pi^j,\lambda^j}\| \le c_4 e^{aj} \quad \text{for every } j \ge 0. \tag{100}$$

Then, analogously to (97),

$$\big|s(x, k; \pi, \lambda) \cdot v\big| \le c_3 c_4 \|v\| \sum_{j=0}^{m} e^{-aj} \tag{101}$$

and this is bounded by $c_5\|v\|$ for some constant $c_5 = c_5(\pi, \lambda) > 0$. This proves Proposition 8.4. □

9. Simplicity criterium

In these last four sections we outline the proof of Theorem 7.1. Here we state an abstract sufficient condition for the Lyapunov spectra of a certain class of linear cocycles to be simple. The main steps in the proof are presented in the next section. Then, we explain how this criterium may be used to obtain the theorem.

We consider cocycles $F : \Sigma \times \mathbb{R}^d \to \Sigma \times \mathbb{R}^d$, $F(x,v) = (f(x), A(x)v)$ over a transformation $f : \Sigma \to \Sigma$ together with an invariant ergodic probability measure μ, satisfying the following conditions:

(c1) $f : \Sigma \to \Sigma$ is the shift map on $\Sigma = \mathcal{I}^{\mathbb{Z}}$, where the alphabet $\mathcal{I}$ is either finite or countable

(c2) μ has *bounded distortion*, meaning that it is positive on cylinders and there exists $C = C(\mu) > 0$ such that
$$\frac{1}{C} \le \frac{\mu([i_m, \dots, i_{-1} : i_0, i_1, \dots, i_n])}{\mu([i_m, \dots, i_{-1}])\mu([i_0, i_1, \dots, i_n])} \le C$$
for every $i_m, \dots, i_0, \dots, i_n$ and $m \le n$ with $m \le 0$ and $n \ge -1$.

(c3) $A : \Sigma \to \mathrm{GL}(d, \mathbb{R})$ is locally constant: $A(\dots, i_{-1}, i_0, i_1, \dots) = A(i_0)$.

By *cylinder* we mean any set $[i_m, \dots, i_{-1} : i_0, \dots, i_n]$ of sequences $x \in \Sigma$ such that $x_j = i_j$ for all $j = m, \dots, -1, 0, 1, \dots, n$ (the colon locates the zeroth term; it is omitted when either $m = 0$ or $n = -1$). We also denote
$$\Sigma^+ = \mathcal{I}^{\{n \ge 0\}} \qquad W^s_{loc}(x) = \{y \in \Sigma : y_n = x_n \text{ for all } n \ge 0\}$$
$$\Sigma^- = \mathcal{I}^{\{n < 0\}} \qquad W^u_{loc}(x) = \{y \in \Sigma : y_n = x_n \text{ for all } n < 0\}$$
Condition (c3) above may be relaxed: the theory we are presenting extends to certain continuous cocycles not necessarily locally constant. See [2, 5].

Our simplicity criterium is formulated in terms of the monoid associated to the cocycle. In this context, a *monoid* is just a subset of $\mathrm{GL}(d, \mathbb{R})$ closed under multiplication and containing the identity. The *associated monoid* $\mathcal{B} = \mathcal{B}(F)$ is the smallest monoid that contains all $A(i)$, $i \in \mathcal{I}$. Let $\mathrm{Gr}(\ell, \mathbb{R}^d)$ be the Grassmannian manifold of ℓ-dimensional subspaces of $\mathbb{R}^d$, for any $1 \le \ell < d$.

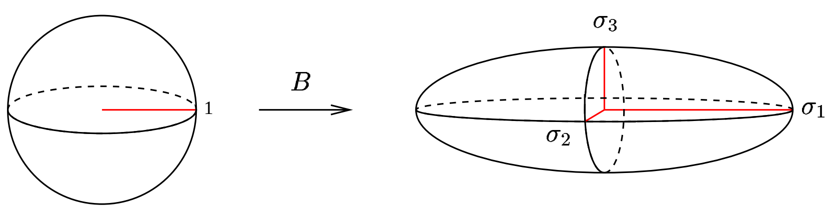

Figure 14.

We need the notion of eccentricity of a linear isomorphism $B : \mathbb{R}^d \to \mathbb{R}^d$, which is defined as follows. Let $\sigma_1^2 \ge \dots \ge \sigma_d^2$ be the eigenvalues of the operator B^*B, in non-increasing order. The eigenvalues are indeed real and positive: if $B^*B(v) = \lambda v$ then $B(v) \cdot B(v) = \lambda(v \cdot v)$. Geometrically, their positive square roots $\sigma_1 \ge \dots \ge \sigma_d > 0$ measure the semi-axes of the ellipsoid $\{B(v) : \|v\| = 1\}$. The *eccentricity* of B is
$$\mathrm{Ecc}(B) = \min_{1 \le \ell < d} \mathrm{Ecc}(\ell, B),$$

where $\mathrm{Ecc}(\ell, B) = \sigma_\ell/\sigma_{\ell+1}$ is called the *ℓ-eccentricity*. See Figure 14. That is, B has large eccentricity if the ratios of any two semi-axes are far from 1.

Definition 9.1. We say that the cocycle F (and the associated monoid $\mathcal{B}$) is

- *pinching* if it contains elements with arbitrarily large eccentricity $\mathrm{Ecc}(B)$
- *twisting* if for any $E \in \mathrm{Gr}(\ell, \mathbb{R}^d)$ and any finite family $G_1, \ldots, G_N$ of elements of $\mathrm{Gr}(d-\ell, \mathbb{R}^d)$ there exists $B \in \mathcal{B}$ such that $B(E) \cap G_i = \{0\}$ for all $j = 1, \ldots, N$.

It is evident from the definition that any monoid that contains a pinching submonoid is also pinching, and analogously for twisting.

Theorem 9.2. *Assume f, μ, F satisfy conditions (c1), (c2), (c3) above. If F is pinching and twisting then its Lyapunov spectrum relative to (f, μ) is simple.*

Remark 9.3. Pinching and twisting are often easy to establish. For instance, suppose a (general) monoid $\mathcal{B}$ contains some element B_1 whose eigenvalues all have distinct norms. Then $\mathcal{B}$ is pinching, since the powers B_1^n have arbitrarily large eccentricity as $n \to \infty$. Suppose, in addition, that the monoid contains some element B_2 satisfying $B_2(V) \cap W = \{0\}$ for any pair of subspaces V and W which are sums of eigenspaces of B_1 and have complementary dimensions. Then $\mathcal{B}$ is twisting. Indeed, given any E, G_1, ..., G_n as in the definition, we have that $B_1^n(E)$ is close to some sum V of ℓ eigenspaces of B_1, and every $B_1^{-n}(G_i)$ is close to some sum W_i of $d-\ell$ eigenspaces of B_1, as long as n is large enough. It follows that

$$B_2(B_1^n(E)) \cap B_1^{-n}(G_i) = \{0\}, \quad \text{that is,} \quad B_1^n B_2 B_1^n(E) \cap G_i = \{0\}.$$

A converse to these observations is given in [3, Lemma A.5].

Example 9.4. Suppose there are symbols t and b in the alphabet $\mathcal{I}$ such that

$$A(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then the associated monoid is pinching and twisting. Indeed,

$$B = A(t)A(b) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is hyperbolic and so its powers have arbitrarily large eccentricity. This proves pinching. To prove twisting, consider E and $G_1, \ldots, G_N \in \mathrm{Gr}(1, \mathbb{R}^2)$. Fix k large enough so that no $A(t)^{-k}(G_i)$ coincides with any of the eigenspaces E^u and E^s of B. Then $B^n(E) \cap A(t)^{-k}(G_i) = \{0\}$, that is, $A(t)^k B^n(E) \cap G_i = \{0\}$ for all i and any sufficiently large n. See Figure 15: the dotted lines express the fact that $A(t)$ and $A(b)$ act by sheer along the horizontal axis and the vertical axis, respectively.

In Section 10 we outline the proof of Theorem 9.2. The strategy is inspired by the following observations. Suppose a cocycle does have $\ell \in \{1, \ldots, d-1\}$ Lyapunov exponents, counted with multiplicity, which are strictly larger than all the other ones. Then the sum $\xi^+(x)$ of the corresponding Oseledets subspaces defines an invariant section of $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ which is an "attractor" for the action of F on the Grassmannian bundle: one may find $\xi^+(x)$ as a limit for $A^n(f^{-n}(x))$, $n \geq 1$ acting on the Grassmannian $\mathrm{Gr}(\ell, \mathbb{R}^d)$, as illustrated in Figure 16. Observe also that $\xi^+(x)$ is constant on local unstable sets $W^u_{loc}(x)$ because, as observed in Remark 2.4, it is determined by the backward iterates of the cocycle alone and, clearly, the sequence of backward iterates is constant on local unstable sets.

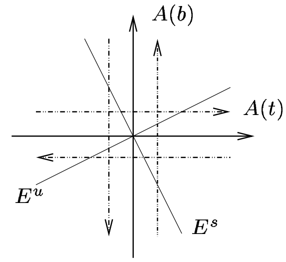

FIGURE 15.

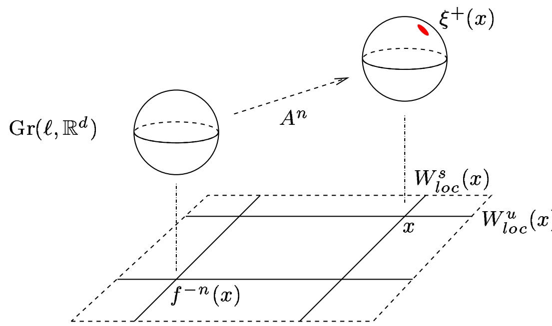

FIGURE 16.

The first main step in the proof of Theorem 9.2 is to show that such an invariant section does exist under the assumptions of the theorem. This is stated in Proposition 10.1 and then we explain how the theorem can be deduced from it. The way we actually construct the invariant section to prove the proposition is as the limit of the iterates under $A^n(f^{-n}(x))$, $n \geq 1$ of certain measures in $\mathrm{Gr}(\ell, \mathbb{R}^d)$. These measures are obtained projecting invariant measures of the cocycle of a special class, that we call *u-states*. The statement is given in Proposition 10.6 and then we explain how Proposition 10.1 may be obtained from it.

The role of u-states is to provide some dynamically meaningful relation between fibers of the Grassmannian bundle $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ at different points, especially points in the same local unstable set. Indeed, these are probability measures on the Grassmannian bundle whose conditional probabilities on the fibers of points in the same local unstable set are all equivalent. For instance, a measure on $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ whose conditional probabilities are Dirac masses, that is, a measure of the form

$$m(X \times Y) = \int_X \delta_{\xi(x)}(Y)\, d\mu(x),$$

is a u-state if and only if the function $\xi : \Sigma \to \mathrm{Gr}(\ell, \mathbb{R}^d)$ is constant on local unstable sets. These observations are important for the proof of Proposition 10.6, that we briefly sketch in the last part of Section 10.

10. Proof of the simplicity criterium

In this section we outline the proof of Theorem 9.2. The presentation is in successive layers, so as to allow the reader to choose an appropriate level of detail. The complete arguments can be found in [2, Appendix] and [3].

10.1. **Invariant section.** First, we explain how Theorem 9.2 can be obtained from the following proposition (see Figure 17):

Proposition 10.1. *Fix $\ell \in \{1, \dots, d-1\}$. Assume F is pinching and twisting. Then there is a measurable section $\xi^+ : \Sigma \to \mathrm{Gr}(\ell, \mathbb{R}^d)$ such that*

1. *ξ^+ is constant on local unstable sets and F-invariant, that is, it satisfies $A(x)\xi^+(x) = \xi^+(f(x))$ at μ-almost every point*
2. *the ℓ-eccentricity $\mathrm{Ecc}(\ell, A^n(f^{-n}(x))) \to \infty$ and the image $E^+(x,n)$ of the ℓ-subspace most expanded under $A^n(f^{-n}(x))$ converges to $\xi^+(x)$ as $n \to \infty$*
3. *for any $V \in \mathrm{Gr}(d-\ell, \mathbb{R}^d)$, the subspace $\xi^+(x)$ is transverse to V at μ-almost every point.*

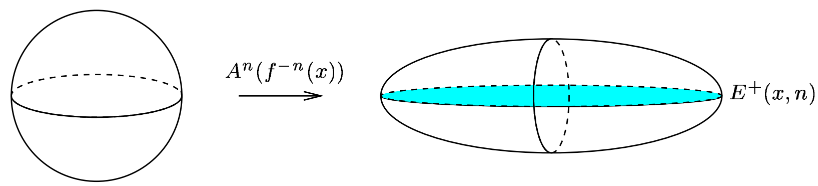

Figure 17.

We want to show that ξ^+ is precisely the sum of the Oseledets subspaces corresponding to the ℓ (strictly) largest Lyapunov exponents. There are three main steps. First, we find a candidate ξ^- to being the sum of the remaining Oseledets subspaces. Next, we check that ξ^+ and ξ^- are transverse to each other at almost every point. Finally, we prove that the Lyapunov exponents of the cocycle along ξ^+ are indeed strictly larger than the exponents along ξ^-. Let us detail each of these steps a bit more.

To begin with, observe that Proposition 10.1 may be applied to the inverse cocycle F^{-1}, since conditions (c1), (c2), (c3) are invariant under time reversal, and we also have

Lemma 10.2. *A monoid $\mathcal{B}$ is pinching and twisting if and only if the inverse $\mathcal{B}^{-1} = \{B^{-1} : B \in \mathcal{B}\}$ is pinching and twisting.*

Considering the action of F^{-1} on the Grassmannian $\mathrm{Gr}(d-\ell, \mathbb{R}^d)$ of complementary dimension, we find an invariant section $\xi^- : \Sigma \to \mathrm{Gr}(d-\ell, \mathbb{R}^d)$ satisfying the analogues of properties (1), (2), (3) in the proposition. In particular, ξ^- is constant on local *stable* sets of f. Next, we need to show that ξ^+ and ξ^- are transverse to each other:

Lemma 10.3. $\xi^+(x) \oplus \xi^-(x) = \mathbb{R}^d$ *for μ-almost every $x \in \Sigma$.*

This is easy to see, with the help of Figure 18. Indeed, suppose the claim fails on a set $Z \subset \Sigma$ with $\mu(Z) > 0$. Using the bounded distortion property (c2), one can see that there exist points $x \in \Sigma$ such that $Z^- \times \Sigma^+$ has positive μ-measure, where

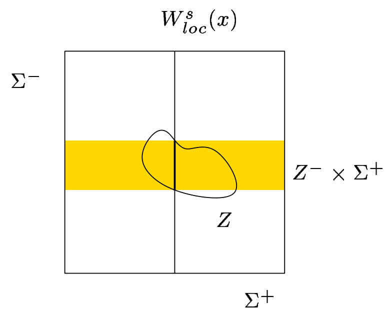

FIGURE 18.

$Z^- = W^s_{loc}(x) \cap Z$. Define $V = \xi^-(x)$. Then $\xi^-(y) = V$ for all $y \in W^s_{loc}(x)$ and ξ^+ is not transverse to V on $Z^- \times \Sigma^+$. This contradicts the last part of Proposition 10.1.

The third and last step in deducing Theorem 9.2 from Proposition 10.1 is

Lemma 10.4. *The Lyapunov exponents of $F \mid \xi^+$ are strictly larger than the Lyapunov exponents of $F \mid \xi^-$.*

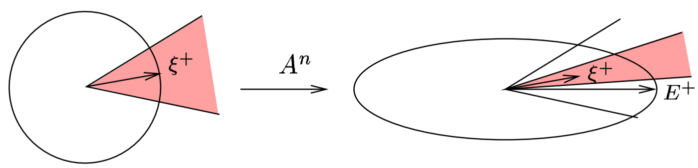

FIGURE 19.

This lemma is deduced along the following lines. See also Figure 19. Let us consider a cone field C^+ around the invariant section ξ^+. Fix some compact subset K with positive measure and some large number $N \geq 1$, such that

- $E^+(x,n) \subset C^+(x)$ for every $x \in K$ and $n \geq N$. This is possible because part 2 of Proposition 10.1 asserts that $E^+(x,n)$ is close to $\xi^+(x)$ when n is large.
- $A^n(x)C^+(x) \subset C^+(f^n(x))$ for any $x \in K$ and $n \geq N$ such that $f^n(x) \in K$. This is possible because the iterates of the cone $C^+(x)$ approach the image $E^+(f^n(x),n)$ of the most expanded subspace as n goes to infinity.

Reducing K is necessary, we may also assume that no point of K returns to it in less than N iterates. Then the previous property means that the cone field is invariant under the cocycle $\tilde{F}$ induced by F over the first return map. This implies (by a variation of the argument in Lemma 5.7) that there is a gap between the first ℓ Lyapunov exponents of $\tilde{F}$ and the remaining ones. Consequently, by Corollary 5.6, the same is true for the original cocycle F.

This finishes our outline of the proof of Theorem 10.1 from the invariant section Proposition 10.1. In what follows we comment on the proof of the proposition.

10.2. **Invariant u-states.** We are going to explain how Proposition 10.1 can be obtained from a statement about iterations of certain probability measures on the Grassmannian given in Proposition 10.6.

A probability m on $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ is a *u-state* if it projects down to μ and there is $C > 0$ such that

$$\frac{m([i_s, \ldots, i_{-1} : i_0, \ldots, i_p] \times X)}{\mu([i_0, \ldots, i_p])} \leq C \frac{m([i_s, \ldots, i_{-1} : j_0, \ldots, j_q] \times X)}{\mu([j_0, \ldots, j_q])}$$

for every $i_s, \ldots, i_0, \ldots, i_p, j_0, \ldots, j_q$ and $X \subset \mathrm{Gr}(\ell, \mathbb{R}^d)$. Notice that, since μ has bounded distortion, this is the same as saying there is $C' > 0$ such that

$$\frac{m([i_s, \ldots, i_{-1} : i_0, \ldots, i_p] \times X)}{\mu([i_s, \ldots, i_{-1} : i_0, \ldots, i_p])} \leq C' \frac{m([i_s, \ldots, i_{-1} : j_0, \ldots, j_q] \times X)}{\mu([i_s, \ldots, i_{-1} : j_0, \ldots, j_q])}.$$

In other words, up to a uniform factor, the m-measures of any two "parallelepipeds" $[i_s, \ldots, i_{-1} : i_0, \ldots, i_p] \times X$ along the same $[i_s, \ldots, i_{-1}] \subset \Sigma^-$ are comparable to the μ-measures of their "bases" $[i_s, \ldots, i_{-1} : i_0, \ldots, i_p]$. See Figure 20.

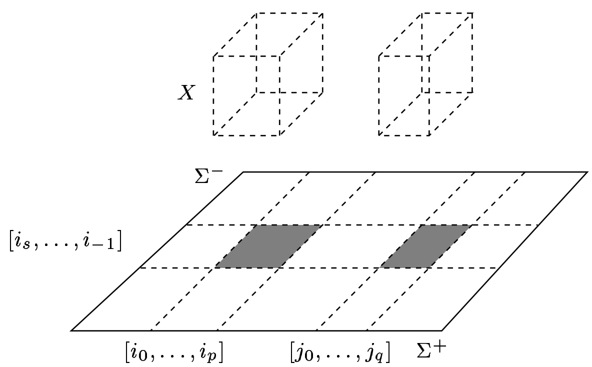

FIGURE 20.

Yet another equivalent formulation is that m is a u-state if it admits a disintegration

$$m = \int_\Sigma m_x \, d\mu(x), \qquad m_x \text{ a probability on } \mathrm{Gr}(\ell, \mathbb{R}^d),$$

where m_x is equivalent to m_y whenever $x \in W^u_{loc}(y)$, with derivative uniformly bounded by C.

It is easy to see that u-states always exist: for instance, $m = \mu \times \nu$ for any probability ν in the Grassmannian. Even more,

Lemma 10.5. *There exist u-states on* $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ *which are invariant under the action of the cocycle on* $\mathrm{Gr}(\ell, \mathbb{R}^d)$.

The arguments are quite standard. The iterates of any u-state under the cocycle are also u-states, with uniform distortion constant C. It follows that the iterates form a relatively compact set, for the weak topology in the space of probability measures in $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$, and every measure in the closure is still a u-state. Hence, any Cesaro weak limit of the iterates is an invariant u-state.

One calls *hyperplane section* of $\mathrm{Gr}(\ell, \mathbb{R}^d)$ associated to any $G \in \mathrm{Gr}(d - \ell, \mathbb{R}^d)$ the subset of all $E \in \mathrm{Gr}(\ell, \mathbb{R}^d)$ such that $E \cap G \neq \{0\}$.

Proposition 10.6. *Let m be an invariant u-state in* $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ *and ν be its projection to* $\mathrm{Gr}(\ell, \mathbb{R}^d)$. *Then*

(1) *the support of ν is not contained in any hyperplane section of the Grassmannian*
(2) *for μ-almost every $x \in M$, the push-forwards $\nu^n(x)$ of ν under $A^n(f^{-n}(x))$ converge to a Dirac measure at some point $\xi^+(x) \in \mathrm{Gr}(\ell, \mathbb{R}^d)$.*

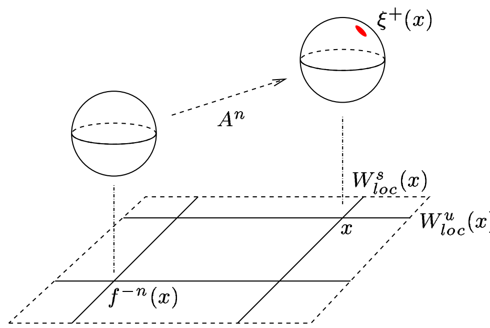

FIGURE 21.

See Figure 21. To deduce Proposition 10.1 from Proposition 10.6 it suffices to use the following linear algebra statement:

Lemma 10.7. *Let $L_n : \mathbb{R}^d \to \mathbb{R}^d$ be a sequence of linear isomorphisms and ρ be a probability measure on $\mathrm{Gr}(\ell, \mathbb{R}^d)$ which is not supported in any hyperplane section of $\mathrm{Gr}(\ell, \mathbb{R}^d)$. If the push-forwards $(L_n)_*\rho$ converge to a Dirac measure δ_ξ then the eccentricity $\mathrm{Ecc}(\ell, L_n) \to \infty$ and the images $E^+(L_n)$ of the most expanded ℓ-subspace converge to ξ.*

10.3. **Convergence to a Dirac mass.** Finally, we comment on the proof of Proposition 10.6. Part 1 of the proposition corresponds to

Lemma 10.8. *If F is twisting then the projection ν of any invariant u-state m is not supported inside any hyperplane section of $\mathrm{Gr}(\ell, \mathbb{R}^d)$.*

Proof. We claim that $B(\operatorname{supp}\nu) \subset \operatorname{supp}\nu$ for every $B \in \mathcal{B}$. The lemma is an easy consequence. Indeed, consider any subspace $F \in \operatorname{supp}\nu$ and suppose the support was contained in an hyperplane section S, associated to some $G \in \mathrm{Gr}(d-\ell, \mathbb{R}^d)$. Then $B(E) \in S$ or, equivalently, $B(E) \cap G \neq \{0\}$ for all $B \in \mathcal{B}$, which would contradict the twisting assumption. Therefore, we only have to prove the claim. Moreover, it suffices to consider the case when $B = A(j_0)$ for some $j_0 \in \mathcal{I}$. Let j_0 be fixed and $\xi \in \mathrm{Gr}(\ell, \mathbb{R}^d)$ be any point in $\operatorname{supp}\nu$. By definition, $m(\Sigma \times V) > 0$ for any neighborhood V of ξ. Equivalently, there exists some $i_0 \in \mathcal{I}$ such that $m([i_0] \times V) > 0$. Since m is a u-state, the measure of any $[i_0] \times V$ is positive if and only if the measure of $[j_0] \times V$ is positive. Hence, $m([j_0] \times V) > 0$ for any neighborhood V of ξ. Since $F([j_0] \times V) \subset \Sigma \times B(V)$ and m is F-invariant, it follows that $m(\Sigma \times B(V))$ is also positive, for any neighborhood V of ξ. This implies that $B(\xi)$ is also in the support of ν, as we wanted to prove. □

Now let us discuss part 2 of Proposition 10.6. There are three main steps. The first, and most delicate, is to show that some subsequence converges to a Dirac measure:

Lemma 10.9. *For almost every x there exist $n_j \to \infty$ such that $\nu^{n_j}(x)$ converges to a Dirac measure.*

Let us give some heuristic explanation of the construction of such a subsequence. See also Figure 22. By hypothesis, there exist elements

$$B_1^p = A(i_{p-1}) \cdots A(i_1)A(i_0)$$

of the associated monoid $\mathcal{B}$ with arbitrarily strong eccentricity. By ergodicity, for μ-almost every $x \in \Sigma$ there exist $m_j \to \infty$ such that $f^{-m_j}(x) \in [i_0, \ldots, i_{p-1}]$, and so

$$A^{m_j}(f^{-m_j}(x)) = C_j B_1^p$$

for some $C_j \in \mathrm{GL}(d, \mathbb{R})$. We want to argue that $C_j B_1^p$ has strong eccentricity, because B_1^p does, and so, using that ν is not supported in a hyperplane section, the measure

$$A^{m_j}(f^{-m_j}(x))_* \nu = \left(C_j B_1^p\right)_* \nu$$

is strongly concentrated near the image of the most expanded ℓ-subspace. In order to justify this kind of assertion, one would need to ensure that, somehow, the strongly pinching behavior of B_1^p is not destroyed by C_j. The following observation, by Furstenberg [8], that the space of projective maps on the Grassmannian has a natural compactification, gives some hope this might be possible.

We call projective map on $\mathrm{Gr}(\ell, \mathbb{R}^d)$ any transformation induced on the Grassmannian by a linear isomorphism of $\mathbb{R}^d$. More generally, we call *quasi-projective map* of $\mathrm{Gr}(\ell, \mathbb{R}^d)$ induced by a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$, the transformation $L_\#$ that assigns to every $E \in \mathrm{Gr}(\ell, \mathbb{R}^d)$ with $E \cap \ker L = 0$ its image $L(E) \in \mathrm{Gr}(\ell, \mathbb{R}^d)$. This is defined on the complement of the *kernel* of the quasi-projective map, defined by

$$\ker L_\# = \left\{E \in \mathrm{Gr}(\ell, \mathbb{R}^d) : E \cap \ker L \neq \{0\}\right\}.$$

We assume L is not identically zero. Then, clearly, $\ker L_\#$ is contained in some hyperplane section of the Grassmannian. We may always consider $\|L\| = 1$, since multiplying L by any constant does not change the definition. Thus, the space of quasi-projective maps inherits a compact topology from the unit ball of linear operators in $\mathbb{R}^d$.

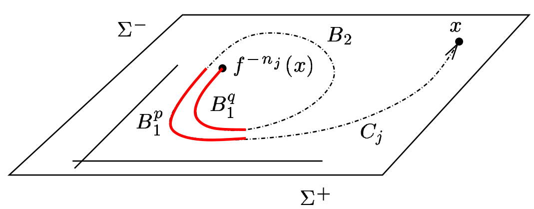

FIGURE 22.

Therefore, the family of all C_j one obtains in the previous construction is contained in some compact set of quasi-projective maps. Of course, this does not yet mean that the effect of the C_j on eccentricity is bounded (which would ensure the strongly pinching behavior of the factor B_1^p prevails). The problem is that the image E_p^+ of the ℓ-dimensional subspace most expanded by B_1^p may be contained in the kernel of any accumulation point $C_\#$ of the sequence C_j in the space of quasi-projective maps: in that case the maps C_j are strongly distorting near $\ker C_\#$ and so they might indeed cancel out the eccentricity of B_1^p. To make the previous arguments work one needs to avoid this situation, that is, one needs to ensure that $C_\#$ may always be chosen so that its kernel does not contain E_p^+. More precisely, one can argue as follows. See Figure 22.

Let B_1^p and $C_\#$ be fixed, as before. Consider another arbitrarily eccentric element

$$B_1^q = A(j_{q-1}) \cdots A(j_1) A(j_0) \in \mathcal{B}$$

and let E_q^+ be the image of its most expanded ℓ-dimensional subspace. By the twisting condition, there exists some

$$B_2 = A(k_s) \cdots A(k_1)$$

that maps E_q^+ outside the kernel of $C_\# B_1^p$. Moreover, by ergodicity, there exist some sequence $n_l = m_{j_l} + q + s \to \infty$ such that

$$f^{-n_l}(x) \in [j_0, \dots, j_{q-1}, k_1, \dots, k_s, i_0, \dots, i_{p-1}]$$

and so $A^{n_l}(f^{-n_l}(x)) = C_j B_1^p B_2 B_1^q$. By construction, E_q^+ is outside the kernel of $C'_\# = C_\# B_1^p B_2$. Thus, the previous arguments now make sense.

Now we move on with the arguments. Let $m^{(n)}(x)$ denote the projection to the Grassmannian of the normalized restriction of m to the cylinder $[i_{-n}, \dots, i_{-1}]$ that contains x. The second step in the proof of part 2 of Proposition 10.6 is

Lemma 10.10. *The sequence $m^{(n)}(x)$ converges almost surely to some probability $m(x)$ on $\mathrm{Gr}(\ell, \mathbb{R}^d)$, and $m(\cdot)$ is almost everywhere constant on local unstable sets.*

The first claim follows from a simple martingale argument. From the construction we easily see that $\{m(x)\}$ is a disintegration of m relative to the partition of $\Sigma \times \mathrm{Gr}(\ell, \mathbb{R}^d)$ into the sets $W^u_{loc}(x) \times \mathrm{Gr}(\ell, \mathbb{R}^d)$, and that gives the second claim.

The final step in the proof of Proposition 10.6 is the following lemma, which is a consequence of the definition of u-state:

Lemma 10.11. *There exists $C = C(m) > 0$ such that*

$$\frac{1}{C} \le \frac{\nu^n(x)}{m^{(n)}(x)} \le C \quad \textit{for all } x.$$

From Lemmas 10.10 and 10.11 we get that, given any $x \in \Sigma$ and any accumulation point $\nu(x)$ of the sequence $\nu^n(x)$,

$$\frac{1}{C} \le \frac{\nu(x)}{m(x)} \le C$$

In particular, any two accumulation points are equivalent. Now, by Lemma 10.9, some accumulation point is a Dirac measure $\delta_{\xi(x)}$, at almost every point. Clearly, this implies the accumulation point is unique, and the sequence $\nu^n(x)$ does converge to a Dirac measure, as we claimed. This finishes our sketch of the proofs of Proposition 10.6 and, thus, Theorem 9.2.

11. Zorich cocycles are pinching and twisting

Now, to deduce Theorem 7.1 we only have to check that Theorem 9.2 may be applied to the restricted Zorich cocycles. Let us begin by verifying that the hypotheses (c1), (c2), (c3).

It was observed in [21, Section 8] that Z has a Markov map: there exists a finite partition $\{\Lambda_{\pi,\varepsilon} : \pi \in C \text{ and } \varepsilon = 0, 1\}$ and a countable refinement

$$\Lambda^*_{\pi,\varepsilon,n} = \{\lambda \in \Lambda_{\pi,\varepsilon} : \varepsilon^1 = \cdots = \varepsilon^{n-1} = \varepsilon \neq \varepsilon^n\}.$$

such that Z maps every $\{\pi\} \times \Lambda^*_{\pi,\varepsilon,n}$ bijectively onto $\{\pi^n\} \times \Lambda_{\pi^n,1-\varepsilon}$. This is not quite a full shift, but it is easy to extend the criterium to this slightly more general version of condition (c1).

The map Z admits an invariant probability μ which is ergodic and equivalent to volume $d\lambda$. See [21, Section 30]. This measure μ has bounded distortion, and so condition (c2) is met. Finally, the cocycle F_Z is constant on each atom $\Lambda^*_{\pi,\varepsilon,n}$ of the Markov partition, because

$$\Gamma_{\pi,\lambda} = \Theta^{n(\pi,\lambda)}_{\pi,\lambda}$$

depends only on π and the types of all $R^j(\pi,\lambda)$ with $0 \leq j < n(\pi,\lambda)$. In other words, $\Gamma_{\pi,\lambda}$ depends only on π and ε^j for $1 \leq j < n(\pi,\lambda)$, and so it is constant on every $\Lambda^*_{\pi,\varepsilon,n}$. This gives condition (c3).

Thus, now we only have to check that

Theorem 11.1. *Every restricted Zorich cocycle is twisting and pinching.*

The proof of this theorem will be outlined in the next section. The strategy is to argue by induction on the complexity of the stratum, that is, on the genus g and the number κ of singularities. Indeed, we look for orbits of $\mathcal{T}^t$ that spend a long time close to the boundary of each stratum and, hence, pick up the behavior of the flow on "simpler" strata. Figure 23 illustrates this idea: think of the upper hemisphere as a stratum, whose boundary is a simpler stratum, represented by the equator (the actual geometry of strata near the boundary is much more complex than the figure suggests, and is still poorly understood).

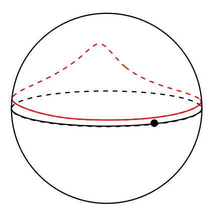

FIGURE 23.

Before we explain in more detail how this strategy is implemented to give the inductive step of the proof of Theorem 11.1, let us note that the initial step of the induction, corresponding to the torus case $g = 1$, $\kappa = 0$, $d = 2$ is easy. Indeed, in this case there is only one permutation pair

$$\pi = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The top case of the renormalization corresponds to $\lambda_A < \lambda_B$, and the bottom case corresponds to $\lambda_B < \lambda_A$. In every case, the cocycle is given by

$$\Theta_{top} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Theta_{bot} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then, arguing as in Example 9.4, we get that F is pinching and twisting.

12. Relating to simpler strata

Here we outline the inductive step in the proof of Theorem 11.1. Fix any permutation pair $\pi \in \mathcal{C}$ and denote by $\mathcal{B}_\pi$ the submonoid of $\mathcal{B}$ corresponding to orbit segments $(\pi^0, \lambda^0), \dots, (\pi^k, \lambda^k)$ such that $\pi^0 = \pi = \pi^k$. It suffices to prove that the action of $\mathcal{B}_\pi$ on the space H_π is pinching and twisting.

The proof is by induction on the complexity of the stratum, that is, the genus and the number of singularities. Recall that Abelian differentials in simpler strata, contained in the boundary of $\mathcal{A}_g(m_1, \dots, m_\kappa)$, may be obtained by collapsing two or more singularities of some Abelian differential in $\mathcal{A}_g(m_1, \dots, m_\kappa)$ together, as illustrated in Figure 24. The multiplicity of the new singularity is the sum of the multiplicities of the original ones.

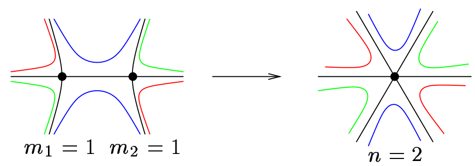

Figure 24.

This strategy is more easily implemented at the level of interval exchange transformations. In that setting, *approaching the boundary* corresponds to *making some coefficient λ_α very small.* Then it remains small for a long time under iteration by the renormalization operator.

12.1. **Simple reductions and simple extensions.** We consider two operations on the combinatorics, that we call simple reduction and simple extension. *Simple reduction* $\pi \mapsto \pi'$ corresponds to removing one letter from both top and bottom lines of the permutation pair. *Simple extension* $\pi' \mapsto \pi$ corresponds to inserting one letter at appropriate positions of both top and bottom lines. See the formula:

$$\pi = \begin{pmatrix} a_1 & \cdots & a_{i-1} & \mathbf{c} & a_{i+1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_d \\ b_1 & \cdots & & \cdots\cdots & \cdots & \cdots & b_{j-1} & \mathbf{c} & b_{j+1} & \cdots & b_d \end{pmatrix}$$

$$\updownarrow$$

$$\pi' = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & \cdots & \cdots & \cdots & a_d \\ b_1 & \cdots & \cdots & \cdots & \cdots & b_{j-1} & b_{j+1} & \cdots & b_d \end{pmatrix}$$

The two operations are not exactly inverse to each other, because there are some restrictions on the insertion locations in the simple extension: the inserted letter can not be last in either line and can not be first in both rows simultaneously.

Lemma 12.1. *Given any π there exists π' such that π is a simple extension of π'. Moreover, either $g(\pi) = g(\pi')$ or $g(\pi) = g(\pi') + 1$.*

We also take advantage of the symplectic structure preserved by the Zorich cocycles. A subspace V of a symplectic space (H, ω) is called *isotropic* if

$$\omega(v_1, v_2) = 0 \quad \text{for any } v_1, v_2 \in V.$$

Let $\mathrm{Iso}(\ell, H) \subset \mathrm{Gr}(\ell, H)$ denote the submanifold of isotropic subspaces with dimension ℓ. The *symplectic reduction* of H by some $v \in \mathbb{P}(H)$ is the quotient H^v by the direction of v of the symplectic orthogonal of v. Note $\dim H^v = \dim H - 2$.

The *stabilizer* of v is the submonoid $\mathcal{B}^v$ of elements of $\mathcal{B}$ that preserve v. The *induced action* of the cocycle on the symplectic reduction is the natural action of the stabilizer $\mathcal{B}^v$ on H^v.

Lemma 12.2. *In the context of Lemma 12.1,*

(1) *If $g(\pi) = g(\pi')$ then there is a symplectic isomorphism $H_{\pi'} \to H_\pi$ that conjugates the action of $\mathcal{B}_{\pi'}$ on $H_{\pi'}$ to the action of some submonoid of $\mathcal{B}_\pi$ on H_π.*
(2) *If $g(\pi) = g(\pi') + 1$, there is some symplectic reduction H_π^v of H_π and some symplectic isomorphism $H_{\pi'} \to H_\pi^v$ that conjugates the action of $\mathcal{B}_{\pi'}$ on $H_{\pi'}$ to the action induced by some submonoid of $\mathcal{B}_\pi$ on H_π^v.*

The proof of Theorem 11.1 may be split into proving two propositions that we state in the sequel. Towards establishing the twisting property, we prove

Proposition 12.3. *The action of $\mathcal{B}_\pi$ on $\mathrm{Iso}(\ell, H_\pi)$ is minimal: any closed invariant set is either empty or the whole ambient space.*

It follows, in particular, that $\mathcal{B}_\pi$ *twists isotropic subspaces* of H_π: given any $E \in \mathrm{Iso}(\ell, H_\pi)$ and any finite family $G_1, \dots, G_N$ of elements of $\mathrm{Gr}(d-\ell, \mathbb{R}^d)$, there exists $B \in \mathcal{B}$ such that $B(E) \cap G_i = \{0\}$ for all $j = 1, \dots, N$. This is a direct consequence of the proposition, and the observation that hyperplane sections

$$\{W : W \cap G_i \neq 0\}$$

have empty interior in $\mathrm{Iso}(\ell, H_\pi)$.

To compensate for this weaker twisting statement, we prove a stronger form of pinching:

Proposition 12.4. *The action of $\mathcal{B}_\pi$ on H_π is strongly pinching: given any $C > 0$ there exist $B \in \mathcal{B}_\pi$ for which*

$$\log \sigma_g > C \quad \textit{and} \quad \frac{\log \sigma_j}{\log \sigma_{j+1}} > C \quad \textit{for all } 1 \leq j < g.$$

Clearly, for symplectic actions in dimension $d = 2$, twisting is equivalent to isotropic twisting and it is also equivalent to minimality. Moreover, pinching is the same as strong pinching. In any dimension,

Lemma 12.5. *Let a monoid $\mathcal{B}$ act symplectically on a symplectic space (H, ω). If $\mathcal{B}$ twists isotropic subspaces and is strongly pinching then it is twisting and pinching.*

This shows that Theorem 11.1 does follow from Propositions 12.3 and 12.4.

12.2. **Proof of minimality.** Here we outline the proof of Proposition 12.3. Given any π, take π' such that π is a simple extension of π'. In the first case of Lemma 12.2 we immediately get, by induction, that the action of $\mathcal{B}_\pi$ on $\mathrm{Iso}(\ell, H_\pi)$ is minimal. In the second case, the starting point of the proof of Proposition 12.3 is the observation that the action $\mathcal{B}_\pi$ on $\mathbb{P}(H)$ is minimal: any closed invariant set is either empty or the whole projective space. Then the proof of the proposition proceeds by induction on the dimension, using the following lemma:

Lemma 12.6. *If the action of $\mathcal{B}$ on $\mathbb{P}(H)$ is minimal and there is $v \in \mathbb{P}(H)$ such that the induced action of $\mathcal{B}^v$ on $\mathrm{Iso}(\ell - 1, H^v)$ is minimal, then the action of $\mathcal{B}$ on $\mathrm{Iso}(\ell, H)$ is minimal.*

The proof of the lemma goes as follows. Consider the fibration

$$\mathcal{I}(H) = \bigcup_{E\in \mathrm{Iso}(\ell,H)} \{E\} \times \mathbb{P}(E) \to \mathbb{P}(H), \quad (E,\lambda) \mapsto \lambda.$$

The fiber over each $\lambda \in \mathbb{P}(H)$ is precisely $\mathrm{Iso}(\ell-1, H^{\lambda})$. There is a natural action of $\mathcal{B}$ on $\mathcal{I}(H)$, and we are going to see that this action is minimal. Indeed, let $C \subset \mathcal{I}(H)$ be a closed invariant set and C_λ denote its intersection with the fiber of each $\lambda \in \mathbb{P}(H)$. The hypothesis implies that C_λ is either empty or the whole $\mathrm{Iso}(\ell-1, H^v)$. In the first case, let Λ be the set of $\lambda \in \mathbb{P}(H)$ for which C_λ is empty. In the second case, let Λ be the set of $\lambda \in \mathbb{P}(H)$ for which C_λ is the whole fiber of λ. In either case, Λ is a closed, non-empty, invariant subset of $\mathbb{P}(H)$, and so it must be the whole projective space. This proves that $C = \emptyset$ in the first case and $C = \mathcal{I}(I)$ in the second case. Thus, the action of $\mathcal{B}$ on $\mathcal{I}(H)$ is minimal, as we claimed. Using the natural projection $\mathcal{I}(H) \to \mathrm{Iso}(\ell, H)$, $(E,\lambda) \mapsto E$ one immediately deduces that the action of $\mathcal{B}$ on the isotropic manifold is minimal.

12.3. **Proof of strong pinching.** Finally, we outline the proof of Proposition 12.4. We denote by $\theta_1(B) \geq \cdots \geq \theta_g(B)$ the non-negative Lyapunov exponents (i.e. logarithms of the norms of the eigenvalues) of a symplectic isomorphism B. We use the following criterium for strong pinching:

Lemma 12.7. *Let $\mathcal{B}$ be a monoid acting symplectically on H, $\dim H = 2g$. Assume for every $C > 0$ there exists some $B \in \mathcal{B}$ such that*

(1) *1 is an eigenvalue of B with 1-dimensional eigenspace*
(2) $\theta_{g-1}(B) > 0$
(3) *$\theta_j(B) > C\theta_{j+1}(B)$ for every $1 \leq j \leq g-2$.*

Then $\mathcal{B}$ is strongly pinching.

Notice that the eigenvalue 1 must have even algebraic multiplicity, because B is symplectic. The second condition ensures the multiplicity is at most two. Thus, B contains an unipotent block

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In terms of the singular values of the powers B^n, this implies that

$$\sigma_g(B^n) \approx n \quad \text{and} \quad \sigma_j(B^n) \approx e^{n\theta_j(B)} \text{ for } j = 1, \ldots, g-1$$

and so $\mathcal{B}$ is indeed strongly pinching.

Another useful observation is that the property of being (or not) strongly pinching is not affected if one replaces the permutation pair π by any other one $\tilde{\pi}$ in the same Rauzy class. That is because one can find monoid elements γ_1 and γ_2 such that

$$\mathcal{B}_\pi = \gamma_1 \mathcal{B}_{\tilde{\pi}} \gamma_2$$

and then it is not difficult to deduce that the action of $\mathcal{B}_\pi$ on H_π is strongly pinching if and only if the action of $\mathcal{B}_{\tilde{\pi}}$ on $H_{\tilde{\pi}}$ is strongly pinching.

The next step is to reduce the general statement to the case when the Rauzy class is *minimal*, meaning that the number of symbols $d = 2g$. In general, $d = 2g + \kappa - 1$ where κ is the number of singularities. Thus, in terms of the Teichmüller flow, this corresponds to reducing the problem to the *minimal stratum* $\mathcal{A}_g(2g-2)$ of Abelian differentials having a unique singularity. It is implemented through the following refinement of Lemma 12.1:

Lemma 12.8. *Let C be a non-minimal Rauzy class, that is, such that $d > 2g$. Then there exists $\pi \in C$ and there exists π' such that π is a simple extension of π' and $g(\pi) = g(\pi')$.*

Then, by Lemma 12.2, the action of $\mathcal{B}_\pi$ on H_π is conjugate to the action of $\mathcal{B}_{\pi'}$ on $H_{\pi'}$, and so the former is strongly pinching if the latter is. Iterating this procedure, one must eventually reach a permutation pair in a minimal component.

The minimal case is more delicate, because we need to relate the minimal stratum of $\mathcal{A}_g$ with some stratum of a *different* moduli space $\mathcal{A}_{g'}$. The crucial ingredient is

Lemma 12.9. *Any minimal Rauzy class contains some permutation pair*

$$\pi = \begin{pmatrix} A & \alpha_2^0 & \cdots & \alpha_{d-1}^0 & Z \\ Z & \alpha_2^1 & \cdots & \alpha_{d-1}^1 & A \end{pmatrix}$$

such that the following reduction is irreducible:

$$\pi' = \begin{pmatrix} \alpha_2^0 & \cdots & \alpha_{d-1}^0 \\ \alpha_2^1 & \cdots & \alpha_{d-1}^1 \end{pmatrix}.$$

Moreover, $g(\pi') = g(\pi) - 1$ and the Rauzy class of π' is also minimal.

This is a consequence of Lemma 20 in Kontsevich, Zorich [14], which expresses at the combinatorial level the surgery procedure they called "bubbling a handle" (or, more precisely, its inverse).

The final step in the proof of the proposition is to use the inductive assumption that the action of $\mathcal{B}_{\tilde{\pi}'}$ on $H_{\pi'}$ is strongly pinching to construct a parabolic element $B \in \mathcal{B}_\pi$ in the way described in Lemma 12.7.

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