Blaschke's problem for hypersurfaces

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Abstract

We solve Blaschke's problem for hypersurfaces of dimension $n \geq 3$. Namely, we determine all pairs of Euclidean hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ that induce conformal metrics on M^n and envelop a common sphere congruence in $\mathbb{R}^{n+1}$.

A fundamental problem in surface theory is to investigate which data are sufficient to determine a surface in space. For instance, a generic immersion $f\colon M^2 \to \mathbb{R}^3$ into Euclidean three-space is determined, up to a rigid motion, by its induced metric and mean curvature function. *Bonnet's problem* is to classify all exceptional immersions. Locally, this was accomplished by Bonnet [**Bon**], Cartan [$\mathbf{Ca}_1$] and Chern [**Ch**]. They split into three distinct classes, namely, constant mean curvature surfaces, nonconstant mean curvature *Bonnet surfaces* admitting a one-parameter family of isometric deformations preserving the mean curvature function, and surfaces that admit exactly one such deformation giving rise to a so-called *Bonnet pair*. From a global point of view the problem has been recently taken up by several authors (see [**KPP**] and [**Bob**]).

As another example, a generic $f\colon M^2 \to \mathbb{R}^3$ is also determined up to homothety and translation by its conformal structure and its Gauss map. Classifying the exceptions is known as *Christoffel's problem*. All local solutions were determined by Christoffel himself [**Chr**]. Besides minimal surfaces, the remaining nontrivial solutions are *isothermic surfaces*, which are characterized by the fact that they admit local conformal parameterizations by curvature lines on the open subset of nonumbilic points.

Prescribing the Gauss map of a surface $f\colon M^2 \to \mathbb{R}^3$ can be thought of as giving a plane congruence (i.e., a two-parameter family of two-dimensional affine subspaces of $\mathbb{R}^3$) to be enveloped by f. Christoffel's problem can thus be rephrased as finding which surfaces are not determined by their conformal structure and a prescribed plane congruence which they are to envelop.

A similar problem in the realm of Möbius geometry was studied by Blaschke [**Bl**] and is now known as Blaschke's problem. It consists in finding the surfaces that are not determined, up to Möbius transformations, by their conformal structure and a given sphere congruence (i.e, a two-parameter family of spheres) enveloped by them. Isothermic surfaces show up again as one of the two nontrivial classes of exceptional cases. An apparently unrelated class appears as the other: *Willmore surfaces*, which are best known in connection to the celebrated Willmore conjecture. Willmore surfaces always arise in pairs of *dual* surfaces, as conformal envelopes of their common *central sphere congruence*, whose elements have the same mean curvature as that of the enveloping surfaces at the corresponding points of tangency.

Unlike the case of Willmore dual surfaces, for any isothermic surfaces $f, \tilde{f}\colon M^2 \to \mathbb{R}^3$ that arise as exceptional surfaces for Blaschke's problem the curvature lines of f and $\tilde{f}$ coincide, in which case the sphere congruence is said to be *Ribaucour*. Each element of such a pair is said to be a *Darboux transform* of the other.

Blaschke's problem was recently studied in [**Ma**] for surfaces of arbitrary codimension. On the other hand, the investigation of the analogous to Christoffel's problem for higher dimensional hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$, namely, to determine all hypersurfaces that admit a conformal deformation preserving the Gauss map, was carried out in [**DV**]. The isometric version of the problem had been previously solved in arbitrary codimension in [$\mathbf{DG}_1$].

In this article we solve Blaschke's problem for hypersurfaces: which pairs of hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ envelop a common regular sphere congruence and induce conformal metrics on M^n? (see the beginning of Section 2 for the meaning of the regularity assumption). Since pairs of hypersurfaces that differ by an inversion always satisfy both conditions, they can be regarded as trivial solutions. Thus we look for nontrivial solutions, that is, pairs of hypersurfaces that do not differ by a Möbius transformation of $\mathbb{R}^{n+1}$.

The problem of determining conformal envelopes of *Ribaucour* sphere congruences was recently treated in arbitrary dimension and codimension in [$\mathbf{To}_1$], making use of the extension of the Ribaucour transformation developed in [$\mathbf{DT}_1$] and [$\mathbf{DT}_2$] to that general setting. They were named *Darboux transforms* one of each other, following the standard terminology of the surface case. However, the definition in [$\mathbf{To}_1$] does not exclude the possibility of Darboux pairs that differ by a composition of a rigid motion and an inversion. Thus, here we rule out from the classification in [$\mathbf{To}_1$] the isometric immersions that only admit such trivial Darboux transforms. Unfortunately, no interesting higher dimensional analogues of isothermic surfaces arise: in the hypersurface case, they reduce, up to Möbius transformations of Euclidean space, to cylinders over plane curves, cylinders over surfaces that are cones over spherical curves and rotation hypersurfaces over plane curves (after excluding the ones that only admit conformally congruent Darboux transforms). Our main result is that there are no other solutions of Blaschke's problem for hypersurfaces.

Theorem 1. *Let $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a nontrivial solution of Blaschke's problem. Then $f(M)$ and $\tilde{f}(M)$ are, up to a Möbius transformation of $\mathbb{R}^{n+1}$, open subsets of one of the following:*

(*i*) *A cylinder over a plane curve.*

(ii) A cylinder $C(\gamma) \times \mathbb{R}^{n-2}$, where $C(\gamma)$ denotes the cone over a curve γ in $\mathbb{S}^2 \subset \mathbb{R}^3$.

(iii) A rotation hypersurface over a plane curve.

Conversely, for any hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ that differs by a Möbius transformation of $\mathbb{R}^{n+1}$ from a hypersurface as in either of the preceding cases there exists $\tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ of the same type as f such that $(f, \tilde{f})$ is a nontrivial solution of Blaschke's problem. Moreover, $\tilde{f}$ is a Darboux transform of f.

To prove Theorem 1, we show that for a pair of hypersurfaces $(f, \tilde{f})$, that is a solution of Blaschke's problem, the shape operators are always simultaneously diagonalizable. This reduces the problem to the previously discussed case of Ribaucour sphere congruences. Our approach is as follows. We are first naturally led to study pairs of conformal hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, that satisfy a weaker condition than that of enveloping a common sphere congruence. In order to describe it, we use that a sphere congruence in $\mathbb{R}^{n+1}$ can be regarded as a map $s\colon M^n \to \mathbb{S}_1^{n+2}$ into the Lorentzian hypersphere with constant sectional curvature one of Lorentz space $\mathbb{L}^{n+3}$ (see the beginning of Section 2 for details). We study pairs of conformal hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, that envelop (possibly different) sphere congruences $s, \tilde{s}\colon M^n \to \mathbb{S}_1^{n+2}$ with the same radius function and which induce the same metric on M^n. By the radius function of a sphere congruence $s\colon M^n \to \mathbb{S}_1^{n+2}$ we mean the function that assigns to each point of M^n the (Euclidean) radius of the sphere $s(p)$. We point out that this condition is not invariant under Möbius transformations of Euclidean space. In this way, we are able to restrict the candidates of solutions of Blaschke's problem, in the case in which principal directions are not preserved, to pairs of surface-like hypersurfaces over surfaces that are solutions of Bonnet's problem in three-dimensional space forms. We say that a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is *surface-like* if $f(M)$ is the image by a Möbius transformation of $\mathbb{R}^{n+1}$ of an open subset of one of the following:

(i) a cylinder $M^2 \times \mathbb{R}^{n-1}$ over $M^2 \subset \mathbb{R}^3$;

(ii) a cylinder $CM^2 \times \mathbb{R}^{n-2}$, where $CM^2 \subset \mathbb{R}^4$ denotes the cone over a surface $M^2 \subset \mathbb{S}^3$;

(iii) a rotation hypersurface over $M^2 \subset \mathbb{R}^3_+$.

The proof is then completed by showing that neither of the possible candidates is in fact a solution of Blaschke's problem.

1 Conformally deformable hypersurfaces

Two hypersurfaces $f\colon M^n \to \mathbb{R}^{n+1}$ and $\tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ in Euclidean space are said to be *conformally congruent* if they differ by a conformal transformation of $\mathbb{R}^{n+1}$. By

Liouville's theorem, any such transformation is a composition $T = L \circ \mathcal{I}$ of a similarity L and an inversion $\mathcal{I}$ with respect to a hypersphere of $\mathbb{R}^{n+1}$. Recall that the inversion $\mathcal{I}\colon \mathbb{R}^N \setminus \{p_0\} \to \mathbb{R}^N \setminus \{p_0\}$ with respect to a hypersphere with radius r centered at p_0 is given by

$$\mathcal{I}(p) = p_0 + \frac{r^2}{\|p - p_0\|^2}(p - p_0).$$

If $f\colon M^n \to \mathbb{R}^{n+1}$ is a hypersurface and N is a unit normal vector field to f, then it is easily seen that $\tilde{N} = r^{-2}\|f - p_0\|^2 \mathcal{I}_* N$ defines a unit normal vector field to $\tilde{f} = \mathcal{I} \circ f$. Moreover, the shape operators A_N and $\tilde{A}_{\tilde{N}}$ of f and $\tilde{f}$ with respect to N and $\tilde{N}$, respectively, are related by

$$r^2 \tilde{A}_{\tilde{N}} = \|f - p_0\|^2 A_N + 2\langle f - p_0, N\rangle I, \tag{1}$$

where I stands for the identity endomorphism of TM. Recall that $A_N X = -\bar{\nabla}_X N$ for any $X \in TM$, where $\bar{\nabla}$ denotes the derivative of $\mathbb{R}^{n+1}$. In particular, f and $\tilde{f}$ have common principal directions and the corresponding principal curvatures are related by

$$r^2 \tilde{\lambda}_i = \lambda_i \|f - p_0\|^2 + 2\langle f - p_0, N\rangle. \tag{2}$$

A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$ is said to be *conformally rigid* if any other conformal immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ is conformally congruent to f. The following criterion for conformal rigidity is due to Cartan [**Ca**$_1$].

Theorem 2. *A hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, is conformally rigid if all principal curvatures have multiplicity less than $n-2$ everywhere.*

A conformal immersion $\tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ not conformally congruent to f is said to be a *conformal deformation* of f. It is said to be *nowhere conformally congruent* to f if it is not conformally congruent to f on any open subset of M^n.

It is well-known that an n-dimensional Euclidean hypersurface has a principal curvature of multiplicity at least $n-1$ everywhere if and only if it is conformally flat, hence, highly conformally deformable. By Theorem 2, if an Euclidean hypersurface of dimension $n \geq 5$ has principal curvatures of multiplicity less than $n-1$ everywhere and admits a conformal nowhere conformally congruent deformation, then it must have a principal curvature λ of constant multiplicity $n-2$ everywhere. Such a hypersurface was called in [**DT**$_3$] a *Cartan hypersurface* if, in addition, λ is nowhere zero.

Cartan hypersurfaces of dimension $n \geq 5$ have been classified in [**Ca**$_1$]. We refer to [**DT**$_3$] for a modern account of that classification. They can be separated into four classes, namely, surface-like, conformally ruled, the ones having precisely a continuous 1–parameter family of deformations and those that admit only one deformation.

The approach in [**DT**$_3$] is based on the structure of the *splitting tensor* C of the eigenbundle $\Delta = \ker(A - \lambda I)$ correspondent to the principal curvature λ of multiplicity $n-2$ of a Cartan hypersurface. It is defined by

$$\langle C_T X, Y\rangle = \langle \nabla_X Y, T\rangle \ \text{ for all } \ T \in \Delta \ \text{ and } \ X, Y \in \Delta^\perp.$$

Under a conformal change of metric $\langle\, ,\, \rangle^\sim = e^{2\varphi}\langle\, ,\, \rangle$, the tensor C changes as

$$\tilde{C}_T = C_T - T(\varphi) I \ \text{ for all } \ T \in \Delta. \tag{3}$$

This follows immediately from the formula

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\varphi) Y + Y(\varphi) X - \langle X, Y\rangle \nabla\varphi, \tag{4}$$

that relates the Levi-Civita connections ∇ and $\tilde{\nabla}$ of $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle^\sim$, respectively. Here $\nabla\varphi$ denotes the gradient with respect to $\langle\, ,\, \rangle$.

A key observation on the splitting tensor associated to a Cartan hypersurface is the following result, which we will also need here. It slightly improves Lemma 15 in [**DT**$_3$], so we include its proof.

Lemma 3. *If C is the splitting tensor of Δ, then the dimension of coker $C = (\ker C)^\perp$ is at most two at any point of M^n. Moreover, if it is two everywhere then there exists $S \in$ coker$C$ such that $C_S = aI$ for some nonzero real number a.*

Proof: It was shown in Lemma 14 of [**DT**$_3$] that there exists an operator D on $\Delta^\perp$ such that $\det D = 1$ and $[D, C_T] = 0$ for all $T \in \Delta$. Thus, the image of C lies in the two-dimensional subspace S of linear operators on $\Delta^\perp$ that commute with D. This already implies the first assertion. Assuming that the second assertion does not hold, the subspace spanned by the image of C and the identity operator would have dimension three and be contained in S, a contradiction. ∎

The simplest structure of the splitting tensor C of a Cartan hypersurface occurs when there exists a vector field $\delta \in \Delta^\perp$ such that $C_T = \langle \delta, T\rangle I$ for every $T \in \Delta$. This is equivalent to requiring $\Delta^\perp$ to be an *umbilical distribution* with mean curvature vector field δ. The following classification of the corresponding Cartan hypersurfaces was derived in [**DT**$_3$] as a consequence of the main theorem of [**DFT**] and plays a key role in this paper.

Theorem 4. *Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be a Cartan hypersurface and let Δ be the eigenbundle correspondent to its principal curvature of multiplicity $n-2$. If $\Delta^\perp$ is an umbilical distribution then f is a surface-like hypersurface.*

The preceding result also holds for $n = 3$ if λ is assumed to be constant along Δ, a condition that is always satisfied when the rank of Δ is at least two (see [**DFT**]).

To conclude this section, we point out that conformally deformable Euclidean hypersurfaces of dimensions 3 and 4 have also been studied by Cartan [**Ca**$_2$],[**Ca**$_3$], although in these cases a classification is far from being complete. Even though we do not make use of Cartan's results for these cases, some of our arguments are implicit in his work.

2 A necessary condition for a solution

Let $\mathbb{L}^{n+3}$ be the $(n+3)$–dimensional Minkowski space, that is, $\mathbb{R}^{n+1}$ endowed with a Lorentz scalar product of signature $(+,\ldots,+,-)$, and let

$$\mathbb{V}^{n+2} = \{p \in \mathbb{L}^{n+3}\colon \langle p,p\rangle = 0\}$$

denote the light cone in $\mathbb{L}^{n+3}$. Then

$$\mathbb{E}^{n+1} = \mathbb{E}^{n+1}_w = \{p \in \mathbb{V}^{n+2}\colon \langle p,w\rangle = 1\}$$

is a model of $(n+1)$–dimensional Euclidean space for any $w \in \mathbb{V}^{n+2}$. Namely, choose $p_0 \in \mathbb{E}^{n+1}$ and a linear isometry $D\colon \mathbb{R}^{n+1} \to \operatorname{span}\{p_0,w\}^\perp \subset \mathbb{L}^{n+3}$. Then the triple (p_0,w,D) gives rise to an isometry $\Psi = \Psi_{p_0,w,D}\colon \mathbb{R}^{n+1} \to \mathbb{E}^{n+1} \subset \mathbb{L}^{n+3}$ defined by

$$\Psi(x) = p_0 + Dx - \frac{1}{2}\|x\|^2 w.$$

Hyperspheres can be nicely described in $\mathbb{E}^{n+1}$: given a hypersphere $S \subset \mathbb{R}^{n+1}$ with (constant) mean curvature H with respect to a unit normal vector field N along S, then $v = H\Psi + \Psi_* N \in \mathbb{L}^{n+3}$ is a constant space-like vector. Moreover, the vector v has unit length and $\langle v, \Psi(q)\rangle = 0$ for all $q \in S$; thus

$$\Psi(S) = \mathbb{E}^{n+1} \cap \{v\}^\perp.$$

Therefore, given a hypersurface $f\colon M^n \to \mathbb{R}^{n+1}$, a sphere congruence enveloped by f with radius function $R \in \mathcal{C}^\infty(M)$ can be identified with the map $s\colon M^n \to \mathbb{S}^{n+2}_1$ into the Lorentzian sphere $\mathbb{S}^{n+2}_1 = \{p \in \mathbb{L}^{n+3}\colon \langle p,p\rangle = 1\}$ defined by

$$s(q) = \frac{1}{R(q)}\Psi(f(q)) + \Psi_*(f(q))N(q). \tag{5}$$

The sphere congruence is said to be *regular* if the map s is an immersion.

Proposition 5. *Let $f\colon M^n \to \mathbb{R}^{n+1}$ envelop a sphere congruence $s\colon M^n \to \mathbb{S}^{n+2}_1$ with radius function $R \in \mathcal{C}^\infty(M)$. Then the metrics $\langle\,,\,\rangle$ and $\langle\,,\,\rangle^*$ induced by f and s are related by*

$$\langle X,Y\rangle^* = \langle (A-\alpha I)X, (A-\alpha I)Y\rangle, \tag{6}$$

where $\alpha = 1/R$ and A is the shape operator of f. In particular, the sphere congruence is regular if and only if α is nowhere a principal curvature of f.

Proof: Differentiating (5) we obtain

$$s_* X = X(\alpha)(\Psi\circ f) + \alpha\Psi_* f_* X - \Psi_* AX - \langle N, f_* X\rangle w.$$

The conclusion now follows easily by using that $\langle \Psi,\Psi\rangle = 0$, and hence that $\langle \Psi_* Z, \Psi\rangle = 0$ for any $Z \in \mathbb{R}^{n+1}$. ∎

Corollary 6. *Let $\tilde{f}, f\colon M^n \to \mathbb{R}^{n+1}$ induce conformal metrics $\langle\,,\,\rangle^{\sim} = e^{2\varphi}\langle\,,\,\rangle$ on M^n. Then the following assertions are equivalent:*

(i) *f and $\tilde{f}$ envelop sphere congruences with the same radius function R which induce the same metric on M^n.*

(ii) *There exists $\alpha \in \mathcal{C}^\infty(M)$ such that the tensors $B = A - \alpha I$ and $\tilde{B} = \tilde{A} - \alpha I$ satisfy*

$$\tilde{B}^2 = e^{-2\varphi} B^2. \tag{7}$$

Proof: By Proposition 5, if either (i) or (ii) holds, then so does the other with $\alpha = 1/R$. ∎

Corollary 6 can be extended to pairs of hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{Q}_c^{n+1}$ in any space form with constant sectional curvature c. If, for simplicity, we take $c = \pm 1$ when $c \neq 0$, then the function α in part (ii) is related to the radius function R of the sphere congruence enveloped by f and $\tilde{f}$ by $\alpha = \cot R$ if $c = 1$ and $\alpha = \coth R$ if $c = -1$.

It follows from Corollary 6 that (ii) is a necessary condition for a pair of conformal hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ to be a solution of Blaschke's problem, that is, to envelop a common sphere congruence. This can also be derived directly for hypersurfaces in $\mathbb{Q}_c^{n+1}$ from the fact that f and $\tilde{f}$ enveloping a common sphere congruence with radius function $R \in \mathcal{C}^\infty(M)$ is equivalent to

$$Cf + SN = C\tilde{f} + S\tilde{N}, \tag{8}$$

where we use the standard models $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ of $\mathbb{Q}_c^{n+1}$ when $c = 1$ and $c = -1$, respectively, so that $\langle f, f\rangle = \langle \tilde{f}, \tilde{f}\rangle = c$. Moreover,

$$\begin{cases} C = \cos R, & S = \sin R \quad \text{if } c = 1, \\ C = 1, & S = 1/R \quad \text{if } c = 0, \\ C = \cosh R, & S = \sinh R \quad \text{if } c = -1, \end{cases}$$

whereas N and $\tilde{N}$ are unit vector fields normal to f and $\tilde{f}$, respectively. Differentiating (8) yields

$$X(R)(Sf + CN) + f_*(CI - SA)X = X(R)(S\tilde{f} + C\tilde{N}) + \tilde{f}_*(CI - S\tilde{A})X.$$

Setting

$$\alpha = C/S$$

this gives

$$f_*BX - X(R)(f + \alpha N) = \tilde{f}_*\tilde{B}X - X(R)(\tilde{f} + \alpha\tilde{N}), \tag{9}$$

which implies that

$$\|f_*BX\| = \|\tilde{f}_*\tilde{B}X\|$$

for any $X \in TM$. It follows that

$$\langle f_*BX, f_*BY \rangle = \langle \tilde{f}_*\tilde{B}X, \tilde{f}_*\tilde{B}Y \rangle = e^{2\varphi}\langle f_*\tilde{B}X, f_*\tilde{B}Y \rangle$$

for all $X, Y \in TM$, or equivalently, that $\tilde{B}^2 = e^{-2\varphi}B^2$.

In the remaining of the present section we study pairs of conformal hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ that satisfy condition (ii) of Corollary 6. We point out that the limiting case in which α is identically zero reduces to the problem recently studied by Vlachos [**Vl**] of determining all pairs of conformal hypersurfaces $f, \tilde{f}\colon M^n \to \mathbb{R}^{n+1}$ whose Gauss maps with values in the Grassmannian of n-planes in $\mathbb{R}^{n+1}$ induce the same metric on M^n. In particular, this allows us to adapt to our case some of the arguments used in the proof of the main result of that paper.

Lemma 7. *Let $\tilde{f}, f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, induce conformal metrics $\langle\,,\rangle^\sim = e^{2\varphi}\langle\,,\rangle$ on M^n and satisfy either one of the equivalent conditions in Corollary 6. Assume that the shape operators A and $\tilde{A}$ of f and $\tilde{f}$, respectively, cannot be simultaneously diagonalized at any point of M^n. Then there exist a smooth distribution Δ of rank $n-2$ such that*

(i) *Δ is the common eigenbundle $\ker(A-\lambda I) = \ker(\tilde{A}-\tilde{\lambda}I)$ correspondent to principal curvatures λ and $\tilde{\lambda}$ of f and $\tilde{f}$, respectively,*

(ii) *trace $(A|_{\Delta^\perp}) = 2\alpha =$ trace $(\tilde{A}|_{\Delta^\perp})$,*

(iii) $\ker B|_{\Delta^\perp} = \{0\} = \ker \tilde{B}|_{\Delta^\perp}$.

and an orthogonal tensor T on M^n such that

(iv) *$T|_\Delta = \epsilon I$ for $\epsilon = \pm 1$,*

(v) $\det T|_{\Delta^\perp} = 1$,

(vi) $\tilde{B} = e^{-\varphi}B \circ T$.

Proof: Let $\lambda_1, \ldots, \lambda_n$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ be the principal curvatures of f and $\tilde{f}$, with corresponding principal frames $\{e_1, \ldots, e_n\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$, respectively, which we assume to be orthonormal with respect to the metric induced by f. Set

$$\mu_j = \lambda_j - \alpha \quad \text{and} \quad \tilde{\mu}_j = \tilde{\lambda}_j - \alpha.$$

By condition (ii) of Corollary 6, after re-enumeration of the principal vectors, if necessary, we have

$$\tilde{\mu_j}^2 = e^{-2\varphi}\mu_j^2 \quad \text{for} \quad 1 \leq j \leq n. \tag{10}$$

Write

$$\tilde{e}_i = \sum_{j=1}^{n} a_{ji} e_j \quad \text{for} \quad 1 \le i \le n.$$

Then

$$\tilde{\mu}_i^2 \sum_j a_{ji} e_j = \tilde{B}^2 \tilde{e}_i = e^{-2\varphi} B^2 \tilde{e}_i = e^{-2\varphi} \sum_j a_{ji} \mu_j^2 e_j.$$

Hence,

$$(\tilde{\mu}_i^2 - e^{-2\varphi} \mu_j^2) a_{ji} = 0 \quad \text{for} \quad 1 \le i, j \le n,$$

and it follows from (10) that

$$(\mu_i^2 - \mu_j^2) a_{ji} = 0 = (\tilde{\mu}_i^2 - \tilde{\mu}_j^2) a_{ji} \quad \text{for} \quad 1 \le i, j \le n. \tag{11}$$

Therefore, if $a_{ji} \neq 0$ then $\mu_i^2 = \mu_j^2$ and $\tilde{\mu}_i^2 = \tilde{\mu}_j^2$, or equivalently

$$(\lambda_i + \lambda_j - 2\alpha)(\lambda_i - \lambda_j) = 0 \quad \text{and} \quad (\tilde{\lambda}_i + \tilde{\lambda}_j - 2\alpha)(\tilde{\lambda}_i - \tilde{\lambda}_j) = 0. \tag{12}$$

We now assume the existence of a smooth distribution Δ of rank $n-2$ satisfying (i) and prove the remaining assertions. The principal curvatures can be ordered so that $\Delta = \operatorname{span}\{e_3, \ldots, e_n\} = \operatorname{span}\{\tilde{e}_3, \ldots, \tilde{e}_n\}$, $\lambda_3 = \cdots = \lambda_n := \lambda$ and $\tilde{\lambda}_3 = \cdots = \tilde{\lambda}_n := \tilde{\lambda}$. Since $a_{21} \neq 0$ by the assumption that A and $\tilde{A}$ cannot be simultaneously diagonalized at any point, then (12) is satisfied for $i = 1$ and $j = 2$. Since $\lambda_1 \neq \lambda_2$ and $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ by the same assumption, we obtain that (ii) holds.

Observe that $\ker \tilde{B} = \ker B$, in view of (7), thus our assumption on A and $\tilde{A}$ implies that condition (iii) must hold. Therefore, using (7) once more, we obtain that all the remaining conditions in the statement are fulfilled by the tensor T defined by

(i) $T|_{\Delta^\perp} = e^{\varphi}(B|_{\Delta^\perp})^{-1} \tilde{B}|_{\Delta^\perp}$;

(ii) $T|_{\Delta} = \epsilon I$, where $\epsilon = 1$ or $\epsilon = -1$, according as $\tilde{\mu} = \tilde{\lambda} - \alpha$ and $\mu = \lambda - \alpha$ are related by $\tilde{\mu} = e^{-\varphi}\mu$ or $\tilde{\mu} = -e^{-\varphi}\mu$, respectively.

In order to complete the proof, it suffices to show that at each $x \in M^n$ there exists a common eigenspace $\Delta(x)$ of A and $\tilde{A}$ of dimension $n-2$. The assumption on A and $\tilde{A}$ forces $\Delta(x)$ to be maximal with this property, and this implies smoothness of Δ. We consider separately the cases $n \ge 5$, $n = 3$ and $n = 4$.

Case $n \ge 5$. Taking (1) into account, it follows from the assumption on A and $\tilde{A}$ that $f|_U$ and $\tilde{f}|_U$ do not coincide up to a conformal diffeomorphism of Euclidean space on any open subset $U \subset M^n$. Then, existence of a common eigenspace $\Delta(x)$ of A and $\tilde{A}$ of dimension $n-2$ follows from Theorem 2.

Case $n = 3$. All we have to prove in this case is the existence of a common principal direction of A and $\tilde{A}$. First notice that the case in which $\mu_1^2, \mu_2^2, \mu_3^2$ are mutually distinct is ruled out by (11) and the assumption on A and $\tilde{A}$. Therefore, we are left with two possibilities:

(*a*) $\mu_1^2 = \mu_2^2 \neq \mu_3^2$, up to a reordering. Then $a_{13} = a_{23} = 0$ by (11), and hence e_3 is a common principal direction of A and $\tilde{A}$.

(*b*) $\mu_1^2 = \mu_2^2 = \mu_3^2$. We may assume that $-\mu_2 = \mu_3 = \mu_4$ and that $-\tilde{\mu}_2 = \tilde{\mu}_3 = \tilde{\mu}_4$. Then, both A and $\tilde{A}$ have a two dimensional eigenspace and their intersection yields the desired common principal direction.

Case $n = 4$. As before, it follows from (11) and the assumption on A and $\tilde{A}$ that $\mu_1^2, \mu_2^2, \mu_3^2, \mu_4^2$ can not be mutually distinct.

For the remaining cases we need the following facts. The curvature tensors R and $\tilde{R}$ of the metrics induced by f and $\tilde{f}$ are related by

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &- (Q(Y,Z) + \langle Y,Z\rangle|\mathrm{grad}\varphi|^2)X \\ &+ (Q(X,Z) + \langle X,Z\rangle|\mathrm{grad}\varphi|^2)Y - \langle Y,Z\rangle Q_0(X) + \langle X,Z\rangle Q_0(Y),\end{aligned} \tag{13}$$

where $Q_0(X) = \nabla_X \mathrm{grad}\varphi - \langle \mathrm{grad}\varphi, X\rangle \mathrm{grad}\varphi$ and $Q(X,Y) = \langle Q_0(X), Y\rangle$. In particular, if X, Y, Z are orthonormal vectors then

$$\tilde{R}(X,Y)Z = R(X,Y)Z - Q(Y,Z)X + Q(X,Z)Y.$$

From the Gauss equation for f it follows that

$$\langle \tilde{R}(e_r, e_j)e_k, e_r\rangle = -Q(e_j, e_k) \quad \text{if} \quad r \neq j \neq k \neq r \tag{14}$$

and

$$\langle \tilde{R}(e_r, e_j)e_k, e_s\rangle = 0, \;\text{ if all four indices are distinct.} \tag{15}$$

In particular, (14) implies that

$$\langle \tilde{R}(e_r, e_j)e_k, e_r\rangle = \langle \tilde{R}(e_s, e_j)e_k, e_s\rangle, \;\text{ if } \{r,s\} \cap \{j,k\} = \emptyset \text{ and } j \neq k. \tag{16}$$

It will be convenient to single out the following consequence of (16):

FACT: If $e_i = \tilde{e}_i$ and $e_j = \tilde{e}_j$ (up to sign) for some $1 \leq i \neq j \leq 4$ then $\mu_i = \mu_j$ and $\tilde{\mu}_i = \tilde{\mu}_j$.

In order to prove the Fact, for simplicity of notation we assume that $(i,j) = (1,2)$. Set

$$e_3 = \cos\theta\tilde{e}_3 + \sin\theta\tilde{e}_4, \quad e_4 = -\sin\theta\tilde{e}_3 + \cos\theta\tilde{e}_4.$$

It follows from (16) for $(r,j,k,s) = (1,3,4,2)$ and the Gauss equation for $\tilde{f}$ that

$$(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_3 - \tilde{\lambda}_4)\sin\theta\cos\theta = 0$$

or, equivalently, that

$$(\tilde\mu_1 - \tilde\mu_2)(\tilde\mu_3 - \tilde\mu_4)\sin\theta\cos\theta = 0.$$

Since both $\sin\theta\cos\theta = 0$ and $\tilde\mu_3 = \tilde\mu_4$ lead to a contradiction with our assumption on A and $\tilde A$, it follows that $\tilde\mu_1 = \tilde\mu_2$. Reversing the roles of f and $\tilde f$ gives $\mu_1 = \mu_2$, and the proof of the Fact is completed.

We now proceed with the proof of existence of a common two-dimensional eigenspace of A and $\tilde A$.

(a) Assume that $\mu_1^2, \mu_2^2, \mu_3^2$ are mutually distinct and that $\mu_3^2 = \mu_4^2$. Then $a_{i1} = 0$ if $i \neq 1$ and $a_{i2} = 0$ if $i \neq 2$ by (11). Hence $e_1 = \tilde e_1$, $e_2 = \tilde e_2$ (up to sign) and $\operatorname{span}\{e_3, e_4\} = \operatorname{span}\{\tilde e_3, \tilde e_4\}$, in contradiction with the Fact.

(b) Assume that $\mu_1^2 \neq \mu_2^2 = \mu_3^2 = \mu_4^2$. Then $a_{i1} = 0$ if $i \neq 1$, and hence $e_1 = \tilde e_1$ and $\operatorname{span}\{e_2, e_3, e_4\} = \operatorname{span}\{\tilde e_2, \tilde e_3, \tilde e_4\}$. We may assume $-\mu_2 = \mu_3 = \mu_4$ and $-\tilde\mu_2 = \tilde\mu_3 = \tilde\mu_4$. If $\operatorname{span}\{e_3, e_4\} = \operatorname{span}\{\tilde e_3, \tilde e_4\}$ then we are done. Otherwise, we may assume that $e_4 = \tilde e_4$, which gives a contradiction with the Fact.

(c) Assume that $\mu_1^2 = \mu_2^2 \neq \mu_3^2 = \mu_4^2$. Then $a_{ij} = 0$ if $i = 1, 2$ and $j = 2, 3$. Hence $\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{\tilde e_1, \tilde e_2\}$ and $\operatorname{span}\{e_3, e_4\} = \operatorname{span}\{\tilde e_3, \tilde e_4\}$. Since A and $\tilde A$ cannot be simultaneously diagonalized we have to consider only two cases. If $\mu_1 = -\mu_2, \mu_3 = \mu_4$ and $\tilde\mu_1 = -\tilde\mu_2, \tilde\mu_3 = \tilde\mu_4$, then $\operatorname{span}\{e_3, e_4\} = \operatorname{span}\{\tilde e_3, \tilde e_4\}$ is the desired common two dimensional eigenspace of A and $\tilde A$.

Assume that $\mu_1 = -\mu_2, \mu_3 = -\mu_4$ and that $\tilde\mu_1 = -\tilde\mu_2 := \gamma_1 \neq 0, \tilde\mu_3 = -\tilde\mu_4 := \gamma_2 \neq 0$. Setting

$$e_1 = \cos\phi\tilde e_1 + \sin\phi\tilde e_2, \quad e_2 = -\sin\phi\tilde e_1 + \cos\phi\tilde e_2,$$

$$e_3 = \cos\theta\tilde e_3 + \sin\theta\tilde e_4, \quad e_4 = -\sin\theta\tilde e_3 + \cos\theta\tilde e_4,$$

it follows from (15) for $(r, j, k, s) = (1, 3, 4, 2)$ and the Gauss equation for $\tilde f$ that

$$(\tilde\lambda_1 - \tilde\lambda_2)(\tilde\lambda_3 - \tilde\lambda_4)\sin\theta\cos\theta\sin\phi\cos\phi = 0.$$

Thus $\gamma_1\gamma_2\sin\theta\cos\theta\sin\phi\cos\phi = 0$. Hence, we may assume that $e_1 = \tilde e_1$ and $e_2 = \tilde e_2$, and the Fact shows that this case can not occur.

(d) Assume that $\mu_1^2 = \mu_2^2 = \mu_3^2 = \mu_4^2$. If $-\mu_1 = \mu_2 = \mu_3 = \mu_4$ and $-\tilde\mu_1 = \tilde\mu_2 = \tilde\mu_3 = \tilde\mu_4$, then both A and $\tilde A$ have three-dimensional eigenspaces, and their intersection gives a common two-dimensional eigenspace as desired. If $-\mu_1 = \mu_2 = \mu_3 = \mu_4$ and $-\tilde\mu_1 = -\tilde\mu_2 = \tilde\mu_3 = \tilde\mu_4$, then we may assume that $e_2 = \tilde e_2$ and $e_3 = \tilde e_3$, and we get a contradiction with the Fact.

To conclude the proof we assume that $-\mu_1 = -\mu_2 = \mu_3 = \mu_4$ and $-\tilde\mu_1 = -\tilde\mu_2 = \tilde\mu_3 = \tilde\mu_4$ on an open subset $U \subset M^4$. Then $f|_U$ and $\tilde f|_U$ are Cyclides of Dupin. In particular, U is conformal to an open subset of a Riemannian product $\mathbb{Q}_c^2 \times \mathbb{S}^2$, $c > -1$, with $\Delta_1 = \operatorname{span}\{e_1, e_2\}$ and $\Delta_2 = \operatorname{span}\{e_3, e_4\}$ as the distributions tangent to the factors $\mathbb{Q}_c^2$ and $\mathbb{S}^2$, respectively (cf. [**To**$_1$], Corollary 13). It suffices to argue that for such a Riemannian manifold the pair of orthogonal distributions $\Delta_1 = \mathrm{span}\{e_1, e_2\}$ and $\Delta_2 = \mathrm{span}\{e_3, e_4\}$ is invariant by conformal changes of the metric. For that, let W denote the Weyl curvature tensor of $\mathbb{Q}_c^2 \times \mathbb{S}^2$. Then

$$\langle W(e_i, e_j)e_l, e_k\rangle = (1 + c)/3$$

if $(i, j, k, l) \in \{(1, 2, 2, 1), (2, 1, 1, 2), (3, 4, 4, 3,), (4, 3, 3, 4)\}$,

$$\langle W(e_i, e_j)e_l, e_k\rangle = -(1 + c)/3$$

if $(i, j, k, l) \in \{(1, 2, 1, 2), (2, 1, 2, 1), (3, 4, 3, 4,), (4, 3, 4, 3)\}$, and vanishes otherwise. In particular, if $X = \sum_{i=1}^4 a_i e_i$ and $Y = \sum_{j=1}^4 b_j e_j$ are orthonormal vectors, then

$$\langle W(X, Y)Y, X\rangle = \frac{1 + c}{3}\left((a_1b_2 - a_2b_1)^2 + (a_3b_4 - a_4b_3)^2\right).$$

Hence $\langle W(X, Y)Y, X\rangle = 0$ if and only if there exist $\lambda \in \mathbb{R}$, $Z \in \mathrm{span}\{e_1, e_2\}$ and $U \in \mathrm{span}\{e_3, e_4\}$ such that $X = \lambda Z + U$ and $Y = Z - \lambda U$. In other words,

$$\langle W(X, Y)Y, X\rangle = 0$$

if and only if there exist vectors $S \in \mathrm{span}\{e_1, e_2\}$ and $T \in \mathrm{span}\{e_3, e_4\}$ satisfying that $\mathrm{span}\{X, Y\} = \mathrm{span}\{S, T\}$. By the conformal invariance of W, the pair of orthogonal distributions Δ_1 and Δ_2 is uniquely determined up to conformal changes of the metric by the fact that $W(\sigma) = 0$ for a two-plane σ if and only if σ intersects both Δ_1 and Δ_2. ∎

Remark 8. For $n = 2$, the proof of Lemma 7- (ii) shows that f and $\tilde{f}$ must have a common mean curvature function $H = \alpha$. In particular, this implies the well-known fact that if two surfaces $f \colon M^2 \to \mathbb{R}^3$ and $\tilde{f} \colon M^2 \to \mathbb{R}^3$ induce conformal metrics on M^2 and envelop a common sphere congruence, then the the latter is necessarily their common central sphere congruence. Moreover, the proof can be easily adapted to show that the same conclusion is true for a pair of immersions $\tilde{f}, f \colon M^2 \to \mathbb{Q}_c^3$ into any space form.

Lemma 9. *Under the assumptions of Lemma 7 it holds that*

$$\nabla P(X, Y) + X \wedge Y((P - I)\nabla\varphi) = 0 \tag{17}$$

where $P = B \circ T = e^{-\varphi}\tilde{B}$.

Proof: Since $\tilde{f}, f \colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, induce conformal metrics $\langle\, , \rangle^\sim = e^{2\varphi}\langle\, , \rangle$ on M^n, the corresponding Levi-Civita connections $\tilde{\nabla}$ and ∇ are related by (4). Using this we obtain that

$$\begin{aligned}(\tilde{\nabla}_X\tilde{A})Y &- (\tilde{\nabla}_Y\tilde{A})X \\ &= (\nabla_X\tilde{A})Y - (\nabla_Y\tilde{A})X + (\tilde{A}Y)(\varphi)X - Y(\varphi)\tilde{A}X - (\tilde{A}X)(\varphi)Y + X(\varphi)\tilde{A}Y.\end{aligned}$$

The Codazzi equation for $\tilde{f}$ then gives

$$(\nabla_X \tilde{A})Y - (\nabla_Y \tilde{A})X = (\tilde{A}X)(\varphi)Y - X(\varphi)\tilde{A}Y - (\tilde{A}Y)(\varphi)X + Y(\varphi)\tilde{A}X,$$

which easily implies that

$$\begin{aligned}(\nabla_X \tilde{B})Y - (\nabla_Y \tilde{B})X + X(\alpha)Y - Y(\alpha)X \\ = (\tilde{B}X)(\varphi)Y - X(\varphi)\tilde{B}Y - (\tilde{B}Y)(\varphi)X + Y(\varphi)\tilde{B}X.\end{aligned}$$

Replacing $\tilde{B} = e^{-\varphi}(B \circ T)$ into the preceding equation yields

$$\begin{aligned}&\langle \nabla\varphi, B \circ T(X)\rangle Y - \langle \nabla\varphi, B \circ T(Y)\rangle X \\ &\quad = B((\nabla_X T)Y - (\nabla_Y T)X) + (\nabla_X B)TY - (\nabla_Y B)TX + X(\alpha)Y - Y(\alpha)X,\end{aligned}$$

that is,

$$(\nabla_X P)Y - (\nabla_Y P)X = \langle PX, \nabla\varphi\rangle Y - \langle PY, \nabla\varphi\rangle X + X \wedge Y(\nabla\varphi),$$

which is equivalent to (17). ∎

Lemma 10. *Under the assumptions of Lemma 7, suppose that $\gamma := \lambda - \alpha \neq 0$. Then,*

(i) if the conclusion of Lemma 7 holds with $\epsilon = 1$ then $\Delta^\perp$ is an umbilical distribution;

(ii) if the conclusion of Lemma 7 holds with $\epsilon = -1$ then coker C has dimension 1.

Proof: Case $n \geq 4$. Let $e_1, \ldots, e_n$ be an orthonormal frame field such that $Be_1 = \beta e_1$, $Be_2 = -\beta e_2$ and $Be_i = \gamma e_i$ for $i \geq 3$, where $\beta = \mu_1 - \alpha = \alpha - \mu_2$. Applying (17) for $X = e_i$ and $Y = e_j$, $i \neq j \geq 3$, we obtain that $\nabla\varphi \in \Delta^\perp$. Since $\det T|_{\Delta^\perp} = 1$, we may set

$$Te_1 = \cos\theta e_1 + \sin\theta e_2, \quad Te_2 = -\sin\theta e_1 + \cos\theta e_2$$

for some smooth function θ. Since A and $\tilde{A}$ can not be simultaneously diagonalized at any point of M^n, we have that $\cos\theta \neq 1$ everywhere.

The Codazzi equation for f yields

$$(\nabla_X B)Y + X(\alpha)Y = (\nabla_Y B)X + Y(\alpha)X \tag{18}$$

for all tangent vector fields X, Y. Applying (18) for $X = e_1$ and $Y = e_2$ and taking the e_i-component for $i \geq 3$ yields

$$(\gamma + \beta)\omega_{i2}(e_1) = (\gamma - \beta)\omega_{i1}(e_2), \;\; i \geq 3. \tag{19}$$

Similarly, using (18) for $X = e_1$ and $Y = e_i$, and then for $X = e_2$ and $Y = e_i$, $i \geq 3$, and comparing the e_1-component of the former equation with the e_2-component of the latter, we get

$$(\gamma - \beta)\omega_{i1}(e_1) + (\gamma + \beta)\omega_{i2}(e_2) = 2e_i(\alpha), \;\; i \geq 3. \tag{20}$$

Applying (17) for $X = e_1$ and $Y = e_i$, $i \geq 3$, using (18) and taking the e_1 and e_2-components yields, respectively,

$$\gamma(\epsilon - \cos\theta)\omega_{i1}(e_1) + \beta\sin\theta\omega_{i2}(e_1) + \beta\sin\theta e_i(\theta) + 2\beta\sin\theta\omega_{21}(e_i) = (1-\cos\theta)e_i(\alpha) \tag{21}$$

and

$$\gamma(\epsilon-\cos\theta)\omega_{i2}(e_1) + \beta\sin\theta\omega_{i1}(e_1) + \beta\cos\theta e_i(\theta) - (\gamma+\beta)\sin\theta\omega_{i2}(e_2) = -\sin\theta e_i(\alpha). \tag{22}$$

Similarly, applying (17) for $X = e_2$ and $Y = e_i$, $i \geq 3$, using (18) and taking the e_1 and e_2-components yields, respectively,

$$\gamma(\epsilon - \cos\theta)\omega_{i1}(e_2) + \beta\sin\theta\omega_{i2}(e_2) + \beta\cos\theta e_i(\theta) + (\gamma - \beta)\sin\theta\omega_{i1}(e_1) = \sin\theta e_i(\alpha) \tag{23}$$

and

$$\gamma(\epsilon-\cos\theta)\omega_{i2}(e_2) + \beta\sin\theta\omega_{i1}(e_2) - \beta\sin\theta e_i(\theta) + 2\beta\sin\theta\omega_{12}(e_i) = (1-\cos\theta)e_i(\alpha). \tag{24}$$

Subtracting equation (23) from (22) and adding equations (21) and (24) yields, respectively,

$$\gamma(\epsilon - \cos\theta)(\omega_{i2}(e_1) - \omega_{i1}(e_2)) + \beta\sin\theta(\omega_{i1}(e_1) - \omega_{i2}(e_2)) = 0 \tag{25}$$

and

$$\gamma(\epsilon - \cos\theta)(\omega_{i1}(e_1) + \omega_{i2}(e_2)) + \beta\sin\theta(\omega_{i2}(e_1) + \omega_{i1}(e_2)) = 2(1 - \cos\theta)e_i(\alpha). \tag{26}$$

We now prove (ii). By Lemma 3, if $\operatorname{coker} C$ does not have dimension 1 then there exists $S \in \operatorname{coker} C$ such that $C_S = aI$ for some nonzero real number a. It follows from (20) and (26), respectively, that

$$-a\gamma = S(\alpha)$$

and

$$a\gamma(\epsilon - \cos\theta) = (1 - \cos\theta)S(\alpha),$$

so we get a contradiction since $\gamma \neq 0$.

In order to prove (i), we regard equations (19), (20), (25) and (26) as a system of linear equations in the unknowns $\omega_{i1}(e_1)$, $\omega_{i2}(e_1)$, $\omega_{i1}(e_2)$ and $\omega_{i2}(e_2)$. We obtain that its unique solution is $\omega_{i1}(e_1) = \omega_{i2}(e_2) = e_i(\alpha)/\gamma$ and $\omega_{i2}(e_1) = \omega_{i1}(e_2) = 0$, which implies that $\Delta^\perp$ is an umbilical distribution with mean curvature vector field $\eta = (1/\gamma)(\nabla\alpha)|_\Delta$.

Case $n = 3$. Here we reorder the principal frame e_1, e_2, e_3 so that e_1 spans Δ and

$$Te_1 = e_1, \;\; Te_2 = \cos\theta e_2 + \sin\theta e_3, \;\; \text{and} \;\; Te_3 = -\sin\theta e_2 + \cos\theta e_3.$$

We have

$$Be_1 = \gamma e_1, \ Be_2 = -\beta e_2 \ \text{ and } \ Be_3 = \beta e_3,$$

with $\alpha - \mu_2 = \beta = \mu_3 - \alpha$. From (18) we obtain

$$e_2(\gamma + \alpha) = (\gamma + \beta)\omega_{12}(e_1), \quad e_3(\gamma + \alpha) = (\gamma - \beta)\omega_{13}(e_1), \tag{27}$$

$$\omega_{12}(e_2) = \frac{e_1(\alpha - \beta)}{\gamma + \beta}, \qquad \omega_{13}(e_3) = \frac{e_1(\beta + \alpha)}{\gamma - \beta}, \tag{28}$$

and

$$\omega_{13}(e_2) = \frac{-2\beta}{\gamma - \beta}\omega_{23}(e_1), \qquad \omega_{12}(e_3) = \frac{-2\beta}{\gamma + \beta}\omega_{23}(e_1). \tag{29}$$

Taking into account the first equation in (27), the e_1-component of (17) for $X = e_1$ and $Y = e_2$ gives

$$\cos\theta e_2(\varphi) - \sin\theta e_3(\varphi) = (\cos\theta - 1)\omega_{12}(e_1) - \sin\theta\omega_{13}(e_1). \tag{30}$$

Using the first equations in (28) and (29), the e_2 and e_3-components of (17) for $X = e_1$ and $Y = e_2$ yield, respectively,

$$\gamma e_1(\varphi) = \beta e_1(\theta)\sin\theta - \frac{2\gamma\beta}{\gamma - \beta}\sin\theta\omega_{23}(e_1) - \frac{(\cos\theta - 1)}{\gamma + \beta}(\gamma e_1(\beta) + \beta e_1(\alpha)) \tag{31}$$

and

$$\beta\cos\theta e_1(\theta) - \frac{2\gamma\beta}{\gamma - \beta}(\cos\theta - 1)\omega_{23}(e_1) + \frac{\sin\theta}{\gamma + \beta}(\gamma e_1(\beta) + \beta e_1(\alpha)) = 0. \tag{32}$$

Similarly, taking the e_1-component of (17) for $X = e_1$ and $Y = e_3$ and using the second equation in (27) gives

$$\sin\theta e_2(\varphi) + \cos\theta e_3(\varphi) = -\sin\theta\omega_{12}(e_1) - (\cos\theta - 1)\omega_{13}(e_1). \tag{33}$$

Using the second equations in (28) and (29), the e_3-component of (17) for $X = e_1$ and $Y = e_3$ yields

$$\gamma e_1(\varphi) = -\beta e_1(\theta)\sin\theta + \frac{2\gamma\beta}{\gamma + \beta}\sin\theta\omega_{23}(e_1) + \frac{(\cos\theta - 1)}{\gamma - \beta}(\gamma e_1(\beta) + \beta e_1(\alpha)). \tag{34}$$

Now, using all the equations in (28) and (29) we obtain by taking the e_1-component of (17) for $X = e_2$ and $Y = e_3$ that

$$2\gamma\beta(\cos\theta - 1)\omega_{23}(e_1) + \sin\theta(\gamma e_1(\beta) + \beta e_1(\alpha)) = 0. \tag{35}$$

It follows from (32) and (35) that

$$\beta\cos\theta e_1(\theta) + \frac{2\gamma\sin\theta}{\gamma^2 - \beta^2}(\gamma e_1(\beta) + \beta e_1(\alpha)) = 0. \tag{36}$$

On the other hand, using (35) we get from (31) and (34) that

$$\beta \sin\theta e_1(\theta) - \frac{2\gamma\cos\theta}{\gamma^2-\beta^2}(\gamma e_1(\beta)+\beta e_1(\alpha)) = 0. \tag{37}$$

Since $\beta\gamma \neq 0$, for $\gamma \neq 0$ by assumption and $\beta \neq 0$ by Lemma 7-(iii), we obtain from (36) and (37) that

$$e_1(\theta) = 0 \tag{38}$$

and

$$\gamma e_1(\beta) + \beta e_1(\alpha) = 0. \tag{39}$$

Then, it follows from (28) that

$$\omega_{12}(e_2) = \omega_{13}(e_3). \tag{40}$$

In view of (39), equations (31) and (34) reduce, respectively, to

$$\gamma e_1(\varphi) + \frac{2\gamma\beta}{\gamma-\beta}\sin\theta\omega_{23}(e_1) = 0$$

and

$$\gamma e_1(\varphi) - \frac{2\gamma\beta}{\gamma+\beta}\sin\theta\omega_{23}(e_1) = 0,$$

which imply that

$$e_1(\varphi) = 0 = \omega_{23}(e_1) \tag{41}$$

Then we obtain from (29) that

$$\omega_{13}(e_2) = 0 = \omega_{12}(e_3).$$

Together with (40), this implies that the distribution $\Delta^\perp = \mathrm{span}\{e_2, e_3\}$ is umbilical. ∎

Proposition 11. *Under the assumptions of Lemma 7, suppose further that its conclusion holds with $\epsilon = 1$ and that $\gamma := \lambda - \alpha \neq 0$. Then both f and $\tilde{f}$ are surface-like hypersurfaces.*

Proof: By Lemma 10, both f and $\tilde{f}$ carry principal curvatures λ and $\tilde{\lambda}$, respectively, of constant multiplicity $n-2$ having a common eigenbundle Δ with the property that $\Delta^\perp$ is an umbilical distribution.

Case $n \geq 4$. In this case the conclusion follows from Theorem 4.

Case $n = 3$. We use the notations in the proof of Lemma 10. Using that

$$\tilde{A} = e^{-\varphi}BT + \alpha I$$

and the Gauss equation for $\tilde{f}$, we obtain by taking the e_2-component of (13) for $X = e_1$ and $Y = Z = e_3$ that

$$Q(e_1, e_2) = 0. \tag{42}$$

Similarly, the e_3-component of (13) for $X = e_1$ and $Y = Z = e_2$ gives

$$Q(e_1, e_3) = 0. \tag{43}$$

In view of (41), equations (42) and (43) reduce to

$$e_1 e_2(\varphi) = 0 = e_1 e_3(\varphi),$$

and (30), (33) and (38) imply that

$$e_1(\omega_{12}(e_1)) = 0 = e_1(\omega_{13}(e_1)). \tag{44}$$

Then, using (44) and the second equality in (41), it follows that the derivative of

$$\nabla_{e_1} e_1 = \omega_{12}(e_1)e_2 + \omega_{13}(e_1)e_3$$

with respect to e_1 has no e_2 and e_3-components. This means that the integral curves of e_1 are circles in M^3.

On the other hand, it follows from [**To**$_1$], Lemma 12 that also $\Delta^\perp$ is a spherical distribution, that is, its mean curvature vector field δ satisfies

$$\langle \nabla_Z \delta, e_1 \rangle = 0 \ \text{ for all } \ Z \in \Delta^\perp.$$

We obtain from [**To**$_2$], Theorem 4.3 that M^3 is locally conformal to a Riemannian product $I \times M^2$, the leaves of the product foliation tangent to I and M^2 corresponding, respectively, to the leaves of Δ and $\Delta^\perp$. Then, Theorem 5 in [**To**$_1$] implies that $f(M^3)$ is locally the image by a conformal transformation of Euclidean space of one of the following: a cylinder $N^2 \times \mathbb{R}$ over a surface $N^2 \subset \mathbb{R}^3$, a cone CN^2 over a surface $N^2 \subset \mathbb{S}^3$, a rotation hypersurface over a surface $N^2 \subset \mathbb{R}^3_+$, a cylinder $\gamma \times \mathbb{R}^2$ over a plane curve, a product $C\gamma \times \mathbb{R}$, where $C\gamma$ is the cone over a spherical curve, or a rotation hypersurface over a plane curve $\gamma \subset \mathbb{R}^2_+$. In the three last cases, the hypersurface would have a principal curvature of multiplicity two with $\Delta^\perp$ as eigenbundle, which is not possible by the assumption that A and $\tilde{A}$ can not be simultaneously diagonalized. Therefore $f(M^3)$ and $\tilde{f}(M^3)$ must be (globally) open subsets of hypersurfaces as in one of the first three cases. ∎

3 The main lemma

We now use the results of the previous section to show that a sphere congruence in $\mathbb{R}^{n+1}$, $n \geq 3$, with conformal envelopes is necessarily Ribaucour.

Lemma 12. *Let $f, \tilde{f}$: $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a solution of Blaschke's problem. Then the shape operators A and $\tilde{A}$ of f and $\tilde{f}$, respectively, can be simultaneously diagonalized at any point of M^n.*

Proof: As pointed out after Corollary 6, condition (*ii*) in that result holds for f and $\tilde{f}$, hence so does the conclusion of Lemma 7. Since the set of solutions of Blaschke's problem is invariant under Möbius transformations of $\mathbb{R}^{n+1}$, by composing f and $\tilde{f}$ with such a transformation we may assume, in view of (2), that the function γ defined in Lemma 10 does not vanish. For the same reason and bearing in mind (3), we may suppose that the dimension of $coker(C)$ is not equal to 1, unless $\Delta^\perp$ is an umbilical distribution. It follows from Lemma 10 that $\Delta^\perp$ is indeed umbilical. By Theorem 4, $f(M)$ and $\tilde{f}(M)$ are, up to (possibly different) Möbius transformations $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathbb{R}^{n+1}$, respectively, open subsets of one of the following:

- (i) cylinders $M^2 \times \mathbb{R}^{n-2}$ and $\tilde{M}^2 \times \mathbb{R}^{n-2}$ over surfaces $M^2, \tilde{M}^2 \subset \mathbb{R}^3$, respectively;
- (ii) cylinders $CM^2 \times \mathbb{R}^{n-3}$ and $C\tilde{M}^2 \times \mathbb{R}^{n-3}$, respectively, where $CM^2 \subset \mathbb{R}^4$ denotes the cone over $M^2 \subset \mathbb{S}^3$;
- (iii) rotation hypersurfaces over surfaces M^2 and $\tilde{M}^2$ contained in $\mathbb{R}^3_+$, respectively.

We now make the following key observation.

Lemma 13. *If f is as in (ii), then M^n is conformal to $M^2 \times \mathbb{H}^{n-2}$ with M^2 endowed with the metric induced from $\mathbb{S}^3$. If f is as in (iii)), then M^n is conformal $M^2 \times \mathbb{S}^{n-2}$ with M^2 endowed with the metric induced from the metric of constant sectional curvature -1 on $\mathbb{R}^3_+$ regarded as the half-space model of $\mathbb{H}^3$.*

Proof: An f as in (*ii*) can be parameterized by

$$f = (t_1, \ldots, t_{n-3}, t_{n-2}g_1, \ldots, t_{n-2}g_4),$$

where $g = (g_1, \ldots, g_4)$ parameterizes $M^2 \subset \mathbb{S}^3$. Then the metric induced by f is

$$\langle\,,\,\rangle = \langle\,,\,\rangle_{\mathbb{R}^{n-2}} + t_{n-2}^2\langle\,,\,\rangle_{M^2} = t_{n-2}^2(\langle\,,\,\rangle_{\mathbb{H}^{n-2}} + \langle\,,\,\rangle_{M^2}),$$

where $\langle\,,\,\rangle_{\mathbb{H}^{n-2}}$ is the metric of constant sectional curvature -1 on $\mathbb{R}^{n-2}_+$ regarded as the half-space model of $\mathbb{H}^{n-2}$.

Any f as in (*iii*) can be parameterized by $f = (g_1, g_2, g_3\phi)$, where $g = (g_1, g_2, g_3)$ parameterizes $M^2 \subset \mathbb{R}^3_+$ and ϕ parameterizes $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$. Thus the metric induced by f is

$$\langle\,,\,\rangle = \langle\,,\,\rangle_{M^2} + g_3^2\langle\,,\,\rangle_{\mathbb{S}^{n-2}} = g_3^2(\langle\,,\,\rangle_{\tilde{M}^2} + \langle\,,\,\rangle_{\mathbb{S}^{n-2}}),$$

where $\langle\,,\,\rangle_{M^2}$ denotes the metric on M^2 induced by the Euclidean metric on $\mathbb{R}^3_+$ and $\langle\,,\,\rangle_{\tilde{M}^2}$ the metric on M^2 induced by the hyperbolic metric of constant sectional curvature -1 on $\mathbb{R}^3_+$ regarded as the half-space model of $\mathbb{H}^3$. ∎

Remark 14. If $f: M^n \to \mathbb{R}^{n+1}$ is either a cylinder $C(\gamma) \times \mathbb{R}^{n-2}$, where $C(\gamma)$ is the cone over a curve $\gamma: I \to \mathbb{S}^2 \subset \mathbb{R}^3$, or a rotation hypersurface over a curve $\gamma: I \to \mathbb{R}^2_+$, then the proof of Lemma 13 shows that the metric induced by f is conformal to $ds^2 + d\sigma^2$, where s denotes the arc-length function of γ, regarded as a curve in the half-space model of the hyperbolic plane in the last case, and $d\sigma^2$ is the metric of either the sphere $\mathbb{S}^{n-1}$ or hyperbolic space $\mathbb{H}^{n-1}$, respectively.

Lemma 15. *Let $\langle\,,\,\rangle = \pi_1^*\langle\,,\,\rangle_1 + \pi_2^*\langle\,,\,\rangle_2$ and $\langle\,,\,\rangle^\sim = \pi_1^*\langle\,,\,\rangle_1^\sim + \pi_2^*\langle\,,\,\rangle_2^\sim$ be product metrics on a product manifold $M = M_1 \times M_2$, where π_i denotes the projection of $M_1 \times M_2$ onto M_i for $i = 1, 2$. If $\langle\,,\,\rangle^\sim = \psi^2\langle\,,\,\rangle$ for some $\psi \in \mathcal{C}^\infty(M)$, then ψ is a constant $k \in \mathbb{R}$ and $\langle\,,\,\rangle_i^\sim = k^2\langle\,,\,\rangle_i$ for $i = 1, 2$.*

Proof: Given $i \in \{1, 2\}$, let $X_i \in TM_i$ be a local unit vector field with respect to $\langle\,,\,\rangle_i$. Let $\tilde{X}_i$ be the lift of X_i to M. Then $\langle X_i, X_i\rangle_i^\sim \circ \pi_i = \langle \tilde{X}_i, \tilde{X}_i\rangle^\sim = \psi^2$, and the conclusion follows. ∎

Going back to the proof of Lemma 12, we obtain from Lemmas 13 and 15 that f and $\tilde{f}$ differ by Möbius transformations $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, from surface-like hypersurfaces that are of the same type, with the corresponding surfaces M^2 and $\tilde{M}^2$ being isometric as surfaces as in either $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, respectively.

Using again the invariance of the set of solutions of Blaschke's problem under Möbius transformations of $\mathbb{R}^{n+1}$, we may assume that $\mathcal{I}_1$ is the identity map of $\mathbb{R}^{n+1}$. Since the shape operators of f and $\tilde{f}$ satisfy trace $(A|_{\Delta^\perp})$ = trace $(\tilde{A}|_{\Delta^\perp})$, as follows from Lemma 7, we obtain that trace $(\tilde{A}|_{\Delta^\perp})$ is constant along Δ. Taking (2) into account once more, this easily implies that $\mathcal{I}_2$ must be a similarity in cases (i) and (ii), and a composition of a similarity with an inversion with respect to a hypersphere centered at a point of the rotation axis in case (iii). In either case $\mathcal{I}_2 \circ \tilde{f}$ is still a hypersurface as in (i), (ii) or (iii), respectively.

In summary, we can assume that f and $\tilde{f}$ are both as in (i), (ii) or (iii). We now use that

$$f + RN = \tilde{f} + R\tilde{N}, \tag{45}$$

and that

$$2/R = \text{trace } (A|_{\Delta^\perp}) = \text{trace } (\tilde{A}|_{\Delta^\perp}),$$

by Lemma 7. Suppose first that f and $\tilde{f}$ are (open subsets of) cylinders $M^2 \times \mathbb{R}^{n-2}$ and $\tilde{M}^2 \times \mathbb{R}^{n-2}$, respectively. Then R is constant along Δ and we can write (45) as

$$g(q) + t + R(q)N(q) = \tilde{g}(q) + \tilde{t} + R(q)\tilde{N}(q), \tag{46}$$

where g and $\tilde{g}$ denote the position vectors of M^2 and $\tilde{M}^2$, respectively, and $t, \tilde{t}$ parameterize the rulings of f and $\tilde{f}$, respectively. Differentiating (46) with respect to a vector $T \in \Delta$ implies that f and $\tilde{f}$ are cylinders with respect to the same orthogonal decomposition $\mathbb{R}^{n+1} = \mathbb{R}^3 \times \mathbb{R}^{n-2}$. The same argument shows that also in case (ii) the hypersurfaces f and $\tilde{f}$ are cylinders with respect to the same orthogonal decomposition $\mathbb{R}^{n+1} = \mathbb{R}^4 \times \mathbb{R}^{n-3}$. We now show that CM^2 and $C\tilde{M}^2$ are cones over surfaces M^2, $\tilde{M}^2$ in the same hypersphere of $\mathbb{R}^4$. In fact, if g and $\tilde{g}$ denote the position vectors of M^2 and $\tilde{M}^2$, respectively, then, disregarding the common components in $\mathbb{R}^{n-3}$, we can now write (45) as

$$P_0 + t(g(q) - P_0) + t(1/H(q))N(q) = \tilde{P}_0 + t(\tilde{g}(q) - \tilde{P}_0) + t(1/H(q))\tilde{N}(q),$$

where P_0 and $\tilde{P}_0$ are the vertices of CM^2 and $C\tilde{M}^2$, respectively, H is the common mean curvature function of M^2 and $\tilde{M}^2$, and N and $\tilde{N}$ are unit normal vector fields to M^2 and $\tilde{M}^2$, respectively. Letting t go to 0 yields $P_0 = \tilde{P}_0$, as asserted.

In case (iii), we claim that the rotation hypersurfaces f and $\tilde{f}$ must have the same "axis". In fact, in this case (45) can be written as

$$\begin{aligned} P_0 &+ (g_1(q), g_2(q), g_3(q)\phi(t)) + R(q)(N_1(q), N_2(q), N_3(q)\phi(t)) \\ &= \tilde{P}_0 + (\tilde{g}_1(q), \tilde{g}_2(q), \tilde{g}_3(q)\phi(t)) + R(q)(\tilde{N}_1(q), \tilde{N}_2(q), \tilde{N}_3(q)\phi(t)), \end{aligned} \tag{47}$$

where (g_1, g_2, g_3) and $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ parameterize M^2 and $\tilde{M}^2$, N and $\tilde{N}$ are unit normal vector fields to M^2 and $\tilde{M}^2$, respectively, and ϕ parameterizes the unit sphere in $\mathbb{R}^{n-2}$. Coordinates in different sides of (47) are possibly with respect to different orthonormal bases of $\mathbb{R}^{n+1}$. Differentiating (47) with respect to t_i yields

$$(f_3(q) + R(q)N_3(q))\frac{\partial \phi}{\partial t_i} = (\tilde{f}_3(q) + R(q)\tilde{N}_3(q))\frac{\partial \phi}{\partial t_i},$$

which implies that the rotation axes coincide up to translation. In particular, we may choose a common orthonormal basis of $\mathbb{R}^{n+1}$ to parameterize f and $\tilde{f}$ as in (47). Then we obtain

$$(P_0)_i - (\tilde{P}_0)_i = \phi_{i+2}(t)(-g_3(q) - R(q)N_3(q) + \tilde{g}_3(q) + R(q)\tilde{N}_3(q)),$$

which implies that $(P_0)_i = (\tilde{P}_0)_i$ for $i = 3, \ldots, n+1$, and proves our claim.

Now, intersecting the spheres of the congruence enveloped by f and $\tilde{f}$ with either the three-dimensional subspace $\mathbb{R}^3$ orthogonal to their common Euclidean factor $\mathbb{R}^{n-2}$ in case (i), the hypersphere $\mathbb{S}^3 \subset \mathbb{R}^4$ containing M^2 and $\tilde{M}^2$ in case (ii), or the affine subspace $\mathbb{R}^3$ of $\mathbb{R}^{n+1}$ containing the profiles M^2 and $\tilde{M}^2$ in case (iii), gives a sphere congruence in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively, which is enveloped by M^2 and $\tilde{M}^2$. Moreover, since the shape operators of f and $\tilde{f}$ can not be simultaneously diagonalized at any point, the same holds for the shape operators of M^2 and $\tilde{M}^2$.

The proof of Lemma 12 is now completed by the following result.

Proposition 16. *If $f_1, f_2 \colon M^2 \to \mathbb{Q}_c^3$ are isometric immersions whose shape operators can not be simultaneously diagonalized at any point of M^2 then they can not envelop a common sphere congruence.*

Proof: Assuming the contrary, the common sphere congruence enveloped by f and $\tilde{f}$ is necessarily their common central sphere congruence (see Remark 8). In particular, f_1 and f_2 have the same mean curvature function, and hence (f_1, f_2) is a solution of Bonnet's problem in $\mathbb{Q}_c^3$. We argue separately for each of the three types of solutions of that problem, as discussed (for $\mathbb{R}^3$) at the beginning of the introduction.

Assume first that $f_1, f_2 \colon M^2 \to \mathbb{Q}_c^3$ are isometric immersions with the same *constant* mean curvature function. Using that the radius function of the sphere congruence enveloped by f and $\tilde{f}$ is constant we obtain from (9) that

$$f_* BX = \tilde{f}_* \tilde{B} X$$

for every $X \in TM^2$, where $B = A - HI$ and $\tilde{B} = \tilde{A} - HI$. Since $\tilde{B}$ is invertible by Lemma 7-(iii), the preceding equation can be rewritten as

$$\tilde{f}_* X = f_* \Phi X$$

for every $X \in TM^2$, where $\Phi = B \circ \tilde{B}^{-1}$. Regarding $\omega = \tilde{f}_*$ as a one-form in M^2 with values in either $\mathbb{R}^3$, $\mathbb{R}^4$ or $\mathbb{L}^4$, according as $c = 0, 1$ or -1, respectively, we obtain by taking the normal component to M^2 in the equation $d\omega = 0$ that

$$\Phi^t A_\xi = A_\xi \Phi \tag{48}$$

for every normal vector field ξ to M^2, regarded as a surface in $\mathbb{R}^3$, $\mathbb{R}^4$ or $\mathbb{L}^4$, respectively (see [**DT**$_2$], Proposition 1). Moreover, the fact that f and $\tilde{f}$ are isometric implies that Φ is an orthogonal tensor. Then we obtain from (48) that $\mathrm{trace} A_\xi = 0$ for every normal vector ξ, a contradiction.

If f and $\tilde{f}$ are Bonnet surfaces admitting a one-parameter family of isometric deformations preserving the Gauss map then they are isothermic surfaces (cf. [**Bob**]). But isothermic surfaces are characterized as the only surfaces whose central sphere congruence is Ribaucour (see [**H-J**], Lemma 3.6.1). Thus, this case is also ruled out.

Finally, suppose that $f_1, f_2 \colon M^2 \to \mathbb{Q}_c^3$ form a Bonnet pair of surfaces with mean curvature function H with respect to unit normal vector fields η_j, $1 \le j \le 2$. For simplicity we take $c = 0$ or $c = \pm 1$. We have from [**Bob**] and [**Te**] that in conformal coordinates

$$ds^2 = e^u (dx^2 + dy^2)$$

their second fundamental forms are given by

$$\begin{cases} A_{\eta_j} X = (H + he^{-u}) X + \epsilon\, ke^{-u} Y \\ A_{\eta_j} Y = \epsilon\, ke^{-u} X + (H - he^{-u}) Y, \end{cases} \tag{49}$$

where $X = \partial/\partial x$, $Y = \partial/\partial y$ are the coordinate tangent vector fields, $\epsilon = 1$ if $j = 1$ and $\epsilon = -1$ if $j = 2$. Moreover, $k \neq 0$ is a constant and $h \in \mathcal{C}^\infty(M)$ satisfies the Codazzi equations

$$\begin{cases} H_x e^u - h_x = 0 \\ H_y e^u + h_y = 0. \end{cases} \tag{50}$$

The integrability condition of (50) is

$$2H_{xy} + H_x u_y + H_y + H_x = 0. \tag{51}$$

We first consider the case $c = 0$. We show that the surfaces

$$F^j = f_j + \frac{1}{H}\eta_j, \quad 1 \leq j \leq 2,$$

are isometric but never isometrically congruent. The coordinate vector fields of F^j are given by

$$\begin{cases} F^j_* X = \dfrac{-e^{-u}}{H}(\epsilon\, kY + hX) + \left(\dfrac{1}{H}\right)_x \eta_j \\ F^j_* Y = \dfrac{-e^{-u}}{H}(\epsilon\, kX - hY) + \left(\dfrac{1}{H}\right)_y \eta_j. \end{cases}$$

Then,

$$N_j = (\epsilon\, kH_y + hH_x)X + (\epsilon\, kH_x - hH_y)Y - H(k^2 + h^2)\eta_j$$

is a normal vector field to F_j. Thus, we have

$$\begin{cases} \|F^1_* X\| = \dfrac{e^{-u}}{H^2}(h^2 + k^2) + \dfrac{H_x^2}{H^4} = \|F^2_* X\| \\ \|F^1_* Y\| = \dfrac{e^{-u}}{H^2}(h^2 + k^2) + \dfrac{H_y^2}{H^4} = \|F^2_* Y\| \\ \langle F^1_* X, F^1_* Y\rangle = \dfrac{H_x H_y}{H^4} = \langle F^2_* X, F^2_* Y\rangle, \end{cases}$$

and

$$\|N_j\|^2 = (h^2 + k^2)(H^2 + e^u(H_x^2 + H_y^2)).$$

Therefore, the surfaces F^1, F^2 are isometric and $\|N_1\| = \|N_2\|$.

Now, a long but straightforward computation using (49), (50) and (51) shows that the second fundamental forms B^j_N of F_j, $j = 1, 2$, satisfy that

$$\langle B^1_N X, F^1_* X\rangle = \langle B^2_N X, F^2_* X\rangle, \quad \langle B^1_N X, F^1_* Y\rangle = \langle B^2_N X, F^2_* Y\rangle$$

and

$$\langle B^1_N X, F^1_* Y\rangle = \langle B^2_N X, F^2_* Y\rangle$$

are independent of ϵ. On the other hand,

$$\langle B_N^1 X, F_*^1 Y\rangle = \epsilon\,\frac{k}{H}\left(e^u(H_x^2+H_y^2)+H^2(k^2+h^2)\right) = \langle B_N^2 X, F_*^2 Y\rangle$$

never vanishes. Thus the surfaces cannot be isometrically congruent.

We now consider the case $c \neq 0$. We take $\mathbb{S}^3 \subset \mathbb{R}^4$ and $\mathbb{H}^3 \subset \mathbb{L}^4$, and thus $\langle f_j, f_j\rangle = c$. As before, we show that the surfaces

$$F^j = Cf_j + S\eta_j, \quad 1 \le j \le 2,$$

are isometric but never isometrically congruent. Here

$$\begin{cases} C = \cos R, \quad S = \sin R \quad \text{where} \quad \cot R = H \qquad \text{if } \ c = 1, \\ C = \cosh R, \ S = \sinh R \quad \text{where} \quad \coth R = H \quad \text{if } \ c = -1. \end{cases}$$

Using (49) we easily obtain that the coordinate vector fields of F^j are

$$\begin{cases} F_*^j X = R_x(C\eta_j - cSf_j) - She^{-u}X - \epsilon kSe^{-u}Y, \\ F_*^j Y = R_y(C\eta_j - cSf_j) + She^{-u}Y - \epsilon kSe^{-u}X, \end{cases}$$

where $R_x = -S^2 H_x$ and $R_y = -S^2 H_y$. Then,

$$N_j = S(h^2+k^2)(C\eta_j - cSf_j) + (hR_x + \epsilon kR_y)X - (hR_y - \epsilon kR_x)Y.$$

is a normal vector field to F_j. Then, we have

$$\begin{cases} \|F_*^1 X\| = S^2 e^{-u}(h^2+k^2) + R_x^2 = \|F_*^2 X\| \\ \|F_*^1 Y\| = S^2 e^{-u}(h^2+k^2) + R_y^2 = \|F_*^2 Y\| \\ \langle F_*^1 X, F_*^1 Y\rangle = R_x R_y = \langle F_*^2 X, F_*^2 Y\rangle, \end{cases}$$

and

$$\|N_j\|^2 = (h^2+k^2)(S^2(h^2+k^2) + e^u(R_x^2+R_y^2)).$$

Therefore, the surfaces F^1, F^2 are isometric and $\|N_1\| = \|N_2\|$.

As before, the second fundamental forms B_N^j of F_j, $j = 1, 2$, satisfy that

$$\langle B_N^1 X, F_*^1 X\rangle = \langle B_N^2 X, F_*^2 X\rangle, \ \ \langle B_N^1 Y, F_*^1 Y\rangle = \langle B_N^2 Y, F_*^2 Y\rangle$$

and

$$\langle B_N^1 Y, F_*^1 Y\rangle = \langle B_N^2 Y, F_*^2 Y\rangle$$

are independent of ϵ On the other hand,

$$\langle B_N^1 X, F_*^1 Y\rangle = -\epsilon\,\frac{k}{S}\left(e^u(H_x^2+H_y^2) + S^2C^2(k^2+h^2)\right) = \langle B_N^2 X, F_*^2 Y\rangle$$

never vanishes, hence the surfaces cannot be isometrically congruent. ∎

Remark 17. In Lemma 12 one does not need to assume regularity of the enveloped sphere congruence.

4 Proof of Theorem 1

By Lemma 12, the shape operators of f and $\tilde{f}$ can be simultaneously diagonalized at any point of M^n. Moreover, by assumption the sphere congruence enveloped by f and $\tilde{f}$ is regular, thus the inverse of its radius function is nowhere a principal curvature of either hypersurface by Proposition 5. Therefore f and $\tilde{f}$ are Ribaucour transforms one of each other, as defined in [**DT**$_2$]. Furthermore, since they induce conformal metrics on M^n, they are in fact *Darboux* transforms one of each other in the sense of [**To**$_1$]. Isometric immersions $f\colon M^n \to \mathbb{R}^N$, $n \geq 3$, that admit Darboux transforms have been classified in [**To**$_1$], Theorem 20. In order to state that result we first recall some preliminary facts.

Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion of a simply-connected Riemannian manifold with second fundamental form denoted by $\alpha\colon TM \times TM \to T_f^\perp M$. We have from [**DT**$_2$], Theorem 17 that if $\tilde{f}\colon M^n \to \mathbb{R}^N$ is a Ribaucour transform of f then there exist $\varphi \in \mathcal{C}^\infty(M)$ and $\beta \in T_f^\perp M$ satisfying

$$\alpha(X, \nabla\varphi) + \nabla_X^\perp \beta = 0 \ \text{ for all } \ X \in TM \tag{52}$$

such that

$$\tilde{f} = f - 2\nu\varphi\mathcal{F}, \tag{53}$$

where $\mathcal{F} = df(\operatorname{grad}\varphi) + \beta$ and $\nu^{-1} = \langle \mathcal{F}, \mathcal{F} \rangle$. Therefore $\tilde{f}$ is completely determined by (φ, β), or equivalently, by φ and $\mathcal{F}$. We denote $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)$. Moreover, we have that

$$\mathcal{S}_{\varphi,\beta} := \operatorname{Hess}\varphi - A_\beta$$

is a Codazzi tensor on M^n such that

$$\alpha(\mathcal{S}_{\varphi,\beta}X, Y) = \alpha(X, \mathcal{S}_{\varphi,\beta}Y) \ \text{ for all } \ X, Y \in TM$$

and

$$d\mathcal{F} = df \circ \mathcal{S}_{\varphi,\beta}.$$

Conversely, given (φ, β) satisfying (52) on an open subset $U \subset M^n$ where

$$D := I - 2\nu\varphi\mathcal{S}_{\varphi,\beta}$$

is invertible, then $\tilde{f}$ given by (53) defines a Ribaucour transform of $f|_U$, and the induced metrics of f and $\tilde{f}$ are related by

$$\langle X, Y \rangle^\sim = \langle DX, DY \rangle. \tag{54}$$

It follows from (54) and the symmetry of D that if f and $\tilde{f}$ induce conformal metrics on M^n then $D^2 = r^2 I$ for some $r \in \mathcal{C}^\infty(M)$. Therefore, either $D = \pm rI$ or TM splits orthogonally as $TM = E_+ \oplus E_-$, where E_+ and E_- are the eigenbundles of D correspondent to the eigenvalues r and $-r$, respectively. In the first case, it follows from the results in [$\mathbf{DT}_2$] that there exists an inversion I in $\mathbb{R}^N$ such that $L'(\tilde{f}) = I(L(f))$, where L and L' are compositions of a homothety and a translation. The immersion $\tilde{f}$ is said to be a Darboux transform of f if the second possibility holds, in which case E_+ and E_- are also the eigenbundles of $\mathcal{S}$ correspondent to its distinct eigenvalues $\lambda = h(1-r)$ and $\mu = h(1+r)$, respectively, where $h^{-1} = 2\nu\varphi$. Thus, $\tilde{f}$ is a Darboux transform of f if and only if the associated Codazzi tensor $\mathcal{S}$ has exactly two distinct eigenvalues λ, μ everywhere satisfying

$$(\lambda + \mu)\varphi = \nu^{-1} = \langle \mathcal{F}, \mathcal{F} \rangle. \tag{55}$$

The classification of isometric immersions $f\colon M^n \to \mathbb{R}^N$, $n \geq 3$, that admit Darboux transforms is as follows.

Theorem 18. *Let $f\colon M^n \to \mathbb{R}^N$, $n \geq 3$, be an isometric immersion that admits a Darboux transform $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)\colon M^n \to \mathbb{R}^N$. Then there exist locally a product representation $\psi\colon M_1 \times M_2 \to M^n$ of (E_+, E_-), a homothety H and an inversion I in $\mathbb{R}^N$ such that one of the following holds:*

(i) *ψ is a conformal diffeomorphism with respect to a Riemannian product metric on $M_1 \times M_2$ and*

$$f \circ \psi = H \circ I \circ g,$$

where $g = g_1 \times g_2\colon M_1 \times M_2 \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ is an extrinsic product of isometric immersions. Moreover, there exists $i \in \{1, 2\}$ such that either M_i is one-dimensional or $g_i(M_i)$ is contained in some sphere $\mathbb{S}^{N_i-1}(P_i; r_i) \subset \mathbb{R}^{N_i}$.

(ii) *ψ is a conformal diffeomorphism with respect to a warped product metric on $M_1 \times M_2$ and*

$$f \circ \psi = H \circ I \circ \Phi \circ (g_1 \times g_2),$$

where $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m}(1) \to \mathbb{R}^N$ denotes an isometry and $g_1\colon M_1 \to \mathbb{R}^m_+$ and $g_2\colon M_2 \to \mathbb{S}^{N-m}(1)$ are isometric immersions.

Conversely, any such isometric immersion admits a Darboux transform.

By a product representation $\psi\colon M_1 \times M_2 \to M^n$ of the orthogonal net (E_+, E_-) we mean a diffeomorphism that maps the leaves of the product foliation of $M_1 \times M_2$ induced by M_1 (respectively, M_2) onto the leaves of E_+ (respectively, E_-). In part (*ii*) the isometry Φ and the isometric immersions g_1 and g_2 may be taken so that $g_2(M_2)$ is not contained in any hypersphere of $\mathbb{S}^{N-m}(1)$.

The preceding definition of a Darboux transform $\tilde{f}\colon M^n \to \mathbb{R}^N$ of an isometric immersion $f\colon M^n \to \mathbb{R}^N$ does not rule out the possibility that $\tilde{f}$ be conformally congruent to f. In fact, we now prove that in either case of Theorem 18 this always occurs whenever both factors M_1 and M_2 have dimension greater than one.

Proposition 19. *Let $f\colon M^n \to \mathbb{R}^N$ be as in (i) or (ii) of Theorem 18. If both M_1 and M_2 have dimension greater than one then any Darboux transform $\tilde{f}\colon M^n \to \mathbb{R}^N$ of f is conformally congruent to it.*

For the proof of Proposition 19 we will need the following fact on Codazzi tensors on warped products.

Lemma 20. *Let $M^n = M_1 \times_\rho M_2$ be a warped product, let (E_1, E_2) be its product net and let $\mathcal{S} = \lambda\Pi_1 + \mu\Pi_2$ be a Codazzi tensor on M^n with $\lambda \neq \mu$ everywhere, where Π_i denotes orthogonal projection of TM onto E_i for $i = 1, 2$. If both factors have dimension greater than one then $\lambda = A \in \mathbb{R}$ and $\mu = B(\rho \circ \pi_1)^{-1} + A$ for some $B \neq 0$.*

Proof: Since E_1 and E_2 are the eigenbundles of the Codazzi tensor $\mathcal{S}$, both E_1 and E_2 are umbilical with mean curvature normals given, respectively, (see [**Re**] or [$\mathbf{To}_2$], Proposition 5.1) by

$$\eta = (\lambda - \mu)^{-1}(\nabla\lambda)_{E_2} \quad \text{and} \quad \zeta = (\mu - \lambda)^{-1}(\nabla\mu)_{E_1}. \tag{56}$$

Here, writing a vector subbundle as a subscript of a vector field indicates taking the orthogonal projection of the vector field onto that subbundle.

On the other hand, since (E_1, E_2) is the product net of a warped product with warping function ρ we have (see [**MRS**], Proposition 2)

$$\eta = 0 \quad \text{and} \quad \zeta = -\nabla \log \circ \rho \circ \pi_1. \tag{57}$$

It follows from (56) and (57) that there exists $\tilde{\lambda} \in \mathcal{C}^\infty(M_1)$ such that $\lambda = \tilde{\lambda} \circ \pi_1$. Since any eigenvalue of a Codazzi tensor is constant along its eigenbundle whenever the latter has rank greater than one (see [$\mathbf{To}_2$], Proposition 5.1), we obtain that $\tilde{\lambda} = A$ for some $A \in \mathbb{R}$ and that μ is constant along M_2. Hence $\nabla\mu = (A - \mu)\nabla \log \circ \rho \circ \pi_1$ by the second equation in (56). This implies that $\nabla(\mu(\rho \circ \pi_1)) = A\nabla\rho \circ \pi_1$, and the conclusion follows. ∎

Corollary 21. *Let $M^n = M_1 \times M_2$ be a Riemannian product, let (E_1, E_2) be its product net and let $\mathcal{S} = \lambda\Pi_1 + \mu\Pi_2$ be a Codazzi tensor on M^n with $\lambda \neq \mu$ everywhere. If both factors have dimension greater than one then both λ and μ are constants.*

Proof of Proposition 19: We first consider f as in case (i) of Theorem 18. It suffices to prove that under the assumption on the dimensions of M_1 and M_2 any Darboux transform

$$\tilde{g} = g - 2\varphi\nu\mathcal{F}\colon M_1 \times M_2 \to \mathbb{R}^N$$

of an extrinsic product $g = g_1 \times g_2\colon M_1 \times M_2 \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ of isometric immersions, such that the eigenbundle net of the associated Codazzi tensor $\mathcal{S}_{\varphi,\beta} = \operatorname{Hess}\varphi - A_\beta$ is the product net of $M_1 \times M_2$, is conformally congruent to g.

Since both M_1 and M_2 have dimension greater than one, it follows from Corollary 21 that the eigenvalues of $\mathcal{S}_{\varphi,\beta}$ are constant, say, $a_1, a_2 \in \mathbb{R}$. Integrating $d\mathcal{F} = dg \circ \mathcal{S}_{\varphi,\beta}$ gives

$$\mathcal{F} = (a_1(g_1 - P_1), a_2(g_2 - P_2))$$

for some $P_1 \in \mathbb{R}^{N_1}$ and $P_2 \in \mathbb{R}^{N_2}$. Using that $\varphi(a_1 + a_2) = \nu^{-1} = \langle \mathcal{F}, \mathcal{F} \rangle$, as follows from (55), we obtain that $a_1 + a_2 \neq 0$ and

$$\begin{aligned}
\tilde{g} &= g - 2\varphi\nu\mathcal{F} \\
&= (g_1, g_2) - \frac{2}{a_1 + a_2}\,(a_1(g_1 - P_1), a_2(g_2 - P_2)) \\
&= \frac{a_1 - a_2}{a_1 + a_2}\left(-g_1 + \frac{2a_1}{a_1 + a_2}P_1, g_2 + \frac{2a_2}{a_1 + a_2}P_2\right).
\end{aligned}$$

Now we consider f as in case (ii) of Theorem 18. Again, it suffices to prove that if M_1 and M_2 both have dimension greater than one then any Darboux transform

$$\tilde{g} = \mathcal{R}_{\varphi,\beta}(g)\colon M_1 \times M_2 \to \mathbb{R}^N$$

of a warped product $g = \Phi \circ (g_1 \times g_2)\colon M_1 \times M_2 \to \mathbb{R}^N$ of isometric immersions

$$g_1\colon M_1 \to \mathbb{R}^m_+ \quad \text{and} \quad g_2\colon M_2 \to \mathbb{S}^{N-m}(1),$$

where $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m}(1) \to \mathbb{R}^N$ is an isometry and $g_2(M_2)$ is not contained in any hypersphere of $\mathbb{S}^{N-m}(1)$, is conformally congruent to g, whenever the eigenbundle net of the Codazzi tensor $\mathcal{S}_{\varphi,\beta} = \operatorname{Hess}\varphi - A_\beta$ associated to $\tilde{g}$ is the product net of $M_1 \times M_2$. Set $g_1 = (h_1, \ldots, h_m)$, so that $g = (h_1, ..., h_{m_1}, h_m g_2)$ and h_m is the warping function of the warped product metric induced by g. By Lemma 20, the Codazzi tensor $\mathcal{S}_{\varphi,\beta}$ has eigenvalues

$$\lambda = A \in \mathbb{R} \;\text{ and }\; \mu = A + Bh_m^{-1}, \;\; B \neq 0.$$

Integrating $d\mathcal{F} = dg \circ \mathcal{S}_{\varphi,\beta}$ and $d\varphi = \langle \mathcal{F}, dg \rangle$ gives

$$\mathcal{F} = Ag + B(0, g_2) + V$$

and

$$\varphi = \frac{A}{2}\|g_1\|^2 + Bh_m + \langle g, V \rangle + c,$$

where $V = (V_1, \ldots, V_N) \in \mathbb{R}^N$ and $c \in \mathbb{R}$.

The condition (55) that g and $\tilde{g}$ induce conformal metrics on $M_1 \times M_2$ gives

$$\begin{aligned}
\tilde{g} &= g - 2\varphi\nu\mathcal{F} \\
&= g - \frac{2h_m}{B + 2Ah_m}(Ag + B(0, g_2) + V) \\
&= \frac{1}{B + 2Ah_m}(B(h_1, \ldots, h_{m-1}, -h_m g_2) - 2h_m V)
\end{aligned} \tag{58}$$

and

$$\|V\|^2 + 2B\langle g_2, V\rangle = 2cA + \frac{BA}{2h_m}\|g_1\|^2 + \frac{B}{h_m}\langle g, V\rangle + \frac{cB}{h_m}. \tag{59}$$

In particular, the preceding equation implies that $\langle g_2, V\rangle$ is a constant, hence $V_j = 0$ for $m \leq j \leq N$ by the condition that $g_2(M_2)$ is not contained in any hypersphere of $\mathbb{S}^{N-m}(1)$. Assume first that $A = 0$. Then (58) reduces to

$$\tilde{g} = \bar{g} - 2\frac{h_m}{B}V,$$

where $\bar{g} = (h_1, \ldots, h_{m-1}, -h_m g_2)$. On the other hand, (59) gives

$$\frac{h_m}{B} = \frac{1}{\|V\|^2}(\langle g, V\rangle + c) = \frac{1}{\|V\|^2}(\langle \bar{g}, V\rangle + c),$$

and we obtain that $\tilde{g}$ is the composition of g with the reflection that sends $(0, g_2)$ to $(0, -g_2)$ followed by the reflection with respect to the hyperplane orthogonal to V and the translation by the vector $-2cV/\|V\|^2$.

Now suppose that $A \neq 0$, and set

$$K = \frac{2}{BA}(\|V\|^2 - 2cA).$$

Then (59) reads as

$$\|g_1\|^2 + \frac{2}{A}\langle g, V\rangle + \frac{2c}{A} = Kh_m. \tag{60}$$

Composing $\tilde{g}$ as in (58) with the reflection that sends $(0, g_2)$ to $(0, -g_2)$ and the translation by V/A, we obtain F that is isometric to $\tilde{g}$ and is given by

$$F = \frac{B}{B + 2Ah_m}\left(g + \frac{V}{A}\right).$$

On the other hand, composing g with the inversion with respect to a hypersphere with radius r given by

$$r^2 = \frac{BK}{2A} = \frac{1}{A^2}(\|V\|^2 - 2cA)$$

and center $-V/A$ yields

$$G = -\frac{V}{A} + \frac{BK}{2A\|g + V/A\|^2}\left(g + \frac{V}{A}\right).$$

Using (60) we obtain

$$\begin{aligned}\|g + V/A\|^2 &= \|g_1\|^2 + \frac{2}{A}\langle g, V\rangle + \frac{\|V\|^2}{A^2}\\ &= Kh_m + \frac{\|V\|^2}{A^2} - \frac{2c}{A}\\ &= Kh_m + \frac{BK}{2A}\end{aligned}$$

Hence,

$$G + \frac{V}{A} = \frac{B}{B + 2Ah_m}\left(g + \frac{V}{A}\right) = F. \quad \blacksquare$$

In order to complete the proof of Theorem 1 we need the following fact.

Lemma 22. *Let ϕ: $I \subset \mathbb{R} \to \mathbb{Q}_c^2 \subset \mathbb{O}$ be a unit-speed curve with nowhere vanishing curvature k, where $\mathbb{O}$ denotes either $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{L}^3$ according as $c = 0, 1$ or -1, respectively. Then,*

(i) *The linear system of ODE's*

$$\begin{cases} h_1' = kh_2 + (A - c)h_3, \quad A \in \mathbb{R}, \\ h_2' = -kh_1, \\ h_3' = h_1, \end{cases} \tag{61}$$

has the first integral

$$h_1^2 + h_2^2 + (c - A)h_3^3 = K \in \mathbb{R}. \tag{62}$$

(ii) *If (h_1, h_2, h_3) is a solution of (61) with initial conditions chosen so that the constant K in the right-hand-side of (62) vanishes, and n denotes a unit normal vector to ϕ in $\mathbb{Q}_c^2$ so that $\{\phi', n\}$ is positively oriented, then $\tilde{\phi}$: $I \to \mathbb{O}$ given by*

$$\tilde{\phi} = \phi - 2\frac{h_3\gamma}{\langle\gamma, \gamma\rangle}, \quad \textit{where } \gamma = h_1\phi' + h_2 n + ch_3\phi$$

is a unit-speed Ribaucour transform of ϕ in $\mathbb{Q}_c^2$.

Proof: Using (61) we obtain that the derivative of $h_1^2 + h_2^2 + (c - A)h_3^3$ vanishes identically, which gives (i). Using that $\phi'' = kn - c\phi$ and $n' = -k\phi'$ we obtain

$$\gamma' = (h_1' - kh_2 + ch_3)\phi' + (h_2' + kh_1)n + c(h_3' - h_1)\phi,$$

and hence $\gamma' = Ah_3\phi'$ by (61). Moreover, we have $h_3' = h_1 = \langle\gamma, \phi'\rangle$. Thus $\tilde{\phi}$ is a Ribaucour transform of ϕ. For $c \neq 0$, we have $\langle\gamma, \phi'\rangle = h_3$, which implies that $\langle\tilde{\phi}, \tilde{\phi}\rangle = \langle\phi, \phi\rangle$, that is, $\tilde{\phi}(I) \subset \mathbb{Q}_c^2$. Finally,

$$\langle\tilde{\phi}'(s), \tilde{\phi}'(s)\rangle = (1 - 2h_3\langle\gamma, \gamma\rangle^{-1}(Ah_3))^2 = 1$$

by (62). $\blacksquare$

Now let f: $M^n \to \mathbb{R}^{n+1}$ be either a cylinder $\phi \times \mathbb{R}^{n-1}$ over a plane curve ϕ: $I \to \mathbb{R}^2$, a cylinder $C(\phi) \times \mathbb{R}^{n-2}$, where $C(\phi)$ is the cone over a curve ϕ: $I \to \mathbb{S}^2 \subset \mathbb{R}^3$, or a rotation hypersurface over a curve ϕ: $I \to \mathbb{R}^2_+$. We prove that f admits a Darboux transform not conformally congruent to it. Let $\tilde{\phi}$: $I \to \mathbb{Q}^2_c$ be any Ribaucour transform of ϕ given by Lemma 22, where in the case $c = -1$ we use the half-plane model of $\mathbb{H}^2$ on $\mathbb{R}^2_+$. Let $\tilde{f}$ be either the cylinder $\tilde{\phi} \times \mathbb{R}^{n-1}$ over $\tilde{\phi}$, the cylinder $C(\tilde{\phi}) \times \mathbb{R}^{n-2}$, where $C(\tilde{\phi})$ is the cone over $\tilde{\phi}$: $I \to \mathbb{S}^2 \subset \mathbb{R}^3$, or the rotation hypersurface over $\tilde{\phi}$: $I \to \mathbb{R}^2_+ = \mathbb{H}^2$. By Remark 14 the metrics induced by f and $\tilde{f}$ are conformal, for ϕ and $\tilde{\phi}$ have the same arc-length function. Now, since ϕ and $\tilde{\phi}$ are Ribaucour transforms one of each other, there exists a congruence of circles in $\mathbb{Q}^2_c$ having ϕ and $\tilde{\phi}$ as envelopes. For each such circle, consider the hypersphere of $\mathbb{R}^{n+1}$ that intersects either $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{R}^2_+ = \mathbb{H}^2$ orthogonally along it. This gives a sphere congruence in $\mathbb{R}^{n+1}$ that is enveloped by f and $\tilde{f}$. We conclude that f and $\tilde{f}$ are Darboux transforms one of each other. This completes the proof of Theorem 1.

Remark 23. If we drop the assumption that f and $\tilde{f}$ are not conformally congruent in Theorem 1 then we have the following further possibilities.

(i) There exists an inversion I in $\mathbb{R}^{n+1}$ such that $L'(\tilde{f}) = I(L(f))$, where L and L' are compositions of a homothety and a translation.

(ii) The immersions f and $\tilde{f}$ are as in either case of Theorem 18 with both factors of dimension greater than one.

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