ASYMPTOTIC BEHAVIOR OF THE KORTEWEG-DE VRIES EQUATION POSED IN A QUARTER PLANE

F. LINARES AND A. F. PAZOTO

Abstract. The purpose of this work is to study the exponential stabilization of the Korteweg-de Vries equation in the right half-line under the effect of a localized damping term. We follow the methods in [20] which combine multiplier techniques and compactness arguments and reduce the problem to prove the unique continuation property of weak solutions. Here, the unique continuation is obtained in two steps: we first prove that solutions vanishing on any subinterval are necessarily smooth and then we apply the unique continuation results proved in [27]. In particular, we show that the exponential rate of decay is uniform in bounded sets of initial data.

1. Introduction

In this paper we will address the exponential stabilization problem of solutions of the Korteweg-de Vries equation on the right half-line under the effect of a localized damping term.

We consider the initial-boundary problem (IBVP) for the Korteweg-de Vries equation in the domain $(0,\infty)$ under the presence of a localized damping represented by the function a, that is,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, & x,\, t \in \mathbb{R}^+, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x > 0. \end{cases} \tag{1.1}$$

Bona and Winther in [3] proposed the above boundary-value problem ($a \equiv 0$) to describe the evolution of unidirectional waves generated at one end of a homogeneous stretch of a certain medium and which are allowed to propagate into the initially undisturbed medium beyond a wavemaker. They also gave the first result regarding well-posedness for the IBVP (1.1) which has been the object of great study in the last few years. We should list recent works regarding well-posedness for the IBVP of (1.1), Colin and Gisclon [7], Bona, Sun and Zhang [2], Colliander and Kenig [5], Holmer [14] and Faminskii [9]. The latter works have been motivated by the results obtained for the IVP associated to the KdV equation by Kenig, Ponce and Vega [16] (see also [4] and [6]).

Along this work we assume that the real-valued function $a = a(x)$ satisfies the condition

$$a,\ a' \in L^\infty(0,\infty) \ \text{ and } \ a(x) \geq a_0 > 0 \ \text{ a.e. } \ \text{in } \Omega, \tag{1.2}$$

where Ω is an unbounded open subset of (b,∞), $b > 0$.

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Damped KdV equations have been studied in the past from the point of view of dynamical system. Ghidaglia [12], [13] and Sell and You [28] considered the damped forced KdV equation equation

$$\begin{cases} u_t + u_x + u_{xxx} - \eta u_{xx} + uu_x + \alpha u = f, \quad x \in (0,1),\ t \in \mathbb{R}^+ \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in (0,1) \end{cases} \tag{1.3}$$

posed on the finite interval $(0,1)$ with periodic boundary conditions, where α and η are nonnegative constants and the forcing $f = f(x,t)$ is function of x and t. Assuming that $\eta = 0$, $\alpha > 0$ and that the external excitation f is either time-independent or time-periodic, Ghidaglia [12], [13] proved the existence of a global attractor of finite fractal dimension for the infinite dimensional system described by (1.3). Assuming $\eta > 0$ Sell and You [28] showed that (1.3) possesses an inertial manifold in the case where the external excitation f is time-independent.

In [31], Zhang studied a damped forced KdV equation posed on a finite interval with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t + u_x + u_{xxx} - \eta u_{xx} + uu_x = f, \quad x \in (0,1),\ t \in \mathbb{R}^+ \\ u(0,t) = u(1,t) = u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in (0,1). \end{cases} \tag{1.4}$$

It was shown that if the external excitation f is time periodic with small amplitude, then the system admits a unique time periodic solution which, as a limit cycle, forms an inertial manifold for the infinite dimensional system described by (1.4). Similar results have also been established by Zhang [32] for the damped BBM equation.

More recently, Bona, Sun and Zhang in [1] considered an initial-boundary problem for KdV equation posed in a quarter plane with a damping term appended, namely,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + \alpha u = 0, \quad x,\, t \in \mathbb{R}^+ \\ u(0,t) = h(t), \quad t > 0, \\ u(x,0) = u_0(x), \quad x > 0 \end{cases} \tag{1.5}$$

where $\alpha > 0$ is a constant that is proportional to the strength of the damping effect. The outcome of their development is roughly the following: if the boundary forcing h is a periodic function on the half line $(0,\infty)$ which is small enough in the Sobolev class $H^s(0,\infty)$, then there is a unique time-periodic solution $u^*(x,t)$ of the equation in (1.5) associated to the boundary values $h(t)$, which, for each t, lies in $H^s(0,\infty)$. Moreover, this solution is shown to be either locally or globally exponentially stable depending on whether $s \in (3/4, 1]$ or $s \geq 1$, respectively.

Here we do not get into such questions, but concentrate on the asymptotic behavior of solutions of (1.1) as $t \to \infty$. In order to illustrate our motivation, we observe that if we multiply equation (1.1) by u and integrate in $(0,\infty)$ it is easy to verify that

$$\frac{dE}{dt} = -\int_0^\infty a(x)|u(x,t)|^2 dx - \frac{1}{2}|u_x(0,t)|^2 \tag{1.6}$$

where

$$E(t) = \frac{1}{2}\int_0^\infty |u(x,t)|^2 dx. \tag{1.7}$$

Then, taking conditions (1.2) into account we can see that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, we can investigate if the solutions of (1.1) tend to zero as $t \to \infty$ and under what rate they decay.

To our knowledge this problem has not been addressed in the literature yet and the existing developments do not allow to give an immediate answer to it.

When $a(x) \geqslant a_0 > 0$ almost everywhere in $\mathbb{R}^+$, it is very simple to prove that the energy $E(t)$ decays exponentially as t tends to infinity. The problem of stabilization when the damping is effective only on a subset of $\mathbb{R}^+$ is much more subtle. In this paper we are concerned with this problem. More precisely, our purpose is to prove that, for any $R > 0$, there exist constants $C = C(R)$ and $\alpha = \alpha(R)$ satisfying

$$E(t) \leq C(R)E(0)e^{-\alpha(R)t}, \ \ \forall \ \ t > 0,$$

provided $E(0) \leq R$. This can be stated in the following equivalent form: Find $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \Big[\int_0^\infty a(x)u^2(x,t)dx + u_x^2(0,t) \Big] dt \tag{1.8}$$

holds for every finite energy solution of (1.1). Indeed, from (1.8) and (1.6) we have that $E(T) \leq \gamma E(0)$ with $0 < \gamma < 1$, which combined with the semigroup property allow us to derive the exponential decay for $E(t)$.

The analysis described above extend, in some sense, the previous results on the subject obtained for the Korteweg-de Vries equation posed on a finite domain. Menzala, Vasconcellos and Zuazua [20] and Pazoto [21] considered the damped KdV equation

$$u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, \quad \text{in } (0,L) \times \mathbb{R}^+, \tag{1.9}$$

with $a = a(x)$ as in (1.2), satisfying the homogeneous boundary conditions

$$u(0,t) = u(L,t) = u_x(L,t) = 0, \quad \text{in } \mathbb{R}^+ \tag{1.10}$$

and initial condition

$$u(x,0) = u_0(x), \quad \text{in } (0,L). \tag{1.11}$$

In [20] and [21] the main tools employed for obtaining (1.8) when u is a solution of (1.9)-(1.11) follow closely the multiplier techniques developed in [23] for the analysis of controllability properties of (1.9) under the boundary conditions given in (1.10). However, when using multipliers, the nonlinearity produces extra terms that were handled by compactness arguments what reduces the problem to showing that the unique solution of (1.1), such that $a(x) = 0$ everywhere and $u_x(0,t) \in L^2(0,L)$ for all time t, has to be the trivial one. This problem may be viewed as a unique continuation one since $au = 0$ implies that $u = 0$ in $\{a > 0\} \times (0,T)$.

The same problem has been intensively investigated in the wave equation context but there are fewer results for the KdV type equation. The case where the damping term is active simultaneously in a neighborhood of both extremes of the interval $(0,L)$ was addressed in [20] where the unique continuation property (UCP) stated above was solved in two steps: first, by extending the solution as being zero outside the interval $(0,L)$, one gets a compactly supported (in space) solution of the Cauchy problem for the KdV equation on the whole line. Then, one applies the classical smoothing properties in [15] showing that the solution is smooth. This allows to apply the unique continuation property results in [30] on smooth solutions to conclude that $u = 0$. Later on, performing as in [20], the general case was solved in [21] showing that solutions vanishing on any subinterval are necessarily smooth which yields enough regularity on

u to apply the unique continuation results obtained in [27]. This result has also been extended to the generalized KdV model in the finite boundary value problem setting by Rosier and Zhang [26] and Linares and Pazoto [18].

The proof of our main result combines compactness arguments and the multiplier methods introduced in [23]. The main difficulty in this context comes from the structure of the nonlinear term and the loss of compactness in the whole half-line. Both problems require more delicate analysis and lead us to estimate the solutions in terms of the energy estimates concentrated on bounded sets of the form $\{|x| \leq r\} \times (0,T)$ to proceed as in the previous works. Indeed, the desired estimate (1.8) will not hold directly since lower order additional terms will appear. So, to absorb them we shall use the so called compactness-uniqueness argument that reduces the question to a unique continuation problem that will be solved by applying the results proved in [27] by Calerman estimates. But due to the lack of regularity of the solutions we are dealing with, i.e., finite energy solutions, the unique continuation result may not directly be applied. To overcome this problem, we proceed as in [21] and we first guarantee that solutions are smooth enough (see Lemmas 2.4 and 2.5). We should mention that to obtain the extra regularity we need to estimate $v = u_t$. Doing so we shall assume some decay in the initial data, more precisely $x^{\frac{3}{2}} u_0 \in L^2(0,\infty)$ and $\sqrt{x} v_0 \in L^2(0,\infty)$, where $v_0(x) = u_t(x,0)$. The latter condition seems to be just of technical nature.

Before stating our main result we observe that the results of this paper show, in particular, how the methods of [10] and [23] may be combined to obtain decay results and Unique Continuation Properties for weak solutions of KdV type equations in unbounded domains.

The main result in this work is next:

Theorem 1.1. *Let u be the solution of problem* (1.1) *given by Theorem 2.1 below and Ω and $a = a(x)$ as in* (1.2)*. Then, for any $R > 0$, there exist positive constants $c = c(R)$ and $\mu = \mu(R)$ such that*

$$E(t) \leq c \, \|u_0\|^2_{L^2(0,\infty)} e^{-\mu t} \tag{1.12}$$

holds for all $t > 0$ and u_0 satisfying $\|u_0\|_{L^2(0,\infty)} \leq R$.

Remark 1.2. *Our result is of local nature in the sense that the exponential decay rate is uniform on bounded sets of initial data, i.e., in balls B_R of $L^2(0,\infty)$. But the results obtained in this paper do not provide any estimate on how the decay rate depends on the radius R of the ball. This has been done for nonlinear models, as far as we know, in very few cases and always using some structural conditions on the nonlinearity.*

To end this section we should mention that Rosier [24] studied the exact boundary controllability for the linear problem associated to the KdV on the half-line, the control being applied at the left endpoint $x = 0$. The proof of the main result obtained in [24] combines Fursikov-Ymanuvilov's approach [11] for the boundary controllability of the Burgers equation on bounded domains and, for the extension to unbounded domain, Rosay's proof of Malgrange-Ehrenpresis's theorem [22]. The nonlinear problem is open.

The plan of this paper is as follows. In section 2 we will list a series of results and estimates needed in the proof of Theorem 1.1. Section 3 will be dedicated to prove our main result.

2. Preliminary Estimates

We establish a series of estimates that will be useful in the proof of our main result. We begin this section stating some well-posedness results for model (1.1):

Theorem 2.1 (See [9], Theorem 6.2)**.** *Let $T > 0$. For any $u_0 \in H_0^1(0,\infty)$, problem (1.1) admits a unique mild solution $u \in \mathcal{C}([0,T]; H_0^1(0,\infty))$.*

Now we will obtain some additional properties of the solutions of (1.1).

Theorem 2.2 (See also [17], Theorem 2.1)**.** *Let u be the solution of problem* (1.1) *given by Theorem 2.1. In addition, if $x^\alpha \in L^2(0,\infty)$ for $\alpha = 2, 3$, then $||xu_x||_{L^2(0,T;H^1(0,\infty))} \leq c$, where $c\,(T, ||u_0||_{L^2(0,\infty)}, ||x^\alpha u_0||_{L^2(0,\infty)})$.*

Proof. The proof is obtained following closely the arguments developed in [17]. Therefore, we will only present the main steps.

Let $\psi_0 \in \mathcal{C}^\infty(0,\infty)$ be a nondecreasing function such that $\psi_0(x) = 0$ for $x \leq \frac{1}{2}$ and $\psi_0(x) = 1$ for $x \geq 1$. For $\alpha \geq 0$ we set $\psi_\alpha(x) = x^\alpha \psi_0(x)$ and note that $\psi_\alpha \in \mathcal{C}^\infty(0,\infty)$ and $\psi'_\alpha(x) \geq 0$ for any $x \in (0,\infty)$.

Multiplying the equation in (1.1) by $u(x,t)\psi_\alpha(x-x_0)$ and integrating by parts over $(0,\infty)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)dx + \frac{3}{2}\int_0^\infty u_x^2(x,t)\psi'_\alpha(x-x_0)dx \\
&\qquad \leq \frac{1}{3}\sup_{x\in(0,\infty)} |u(x,t)\sqrt{\psi'_\alpha(x-x_0)}| \int_0^\infty u^2(x,t)\sqrt{\psi'_\alpha(x-x_0)}dx \qquad (2.13)\\
&+ \big(\frac{1}{2} + \|a\|_{L^\infty(0,\infty)}\big)\int_0^\infty u^2(x,t)\{\psi_\alpha(x-x_0) + \psi'_\alpha(x-x_0) + \psi'''_\alpha(x-x_0)\}dx.
\end{aligned}$$

To estimate $\sup\limits_{x\in(0,\infty)} |u(x,t)\sqrt{\psi'_\alpha(x-x_0)}|$ we use the following inequality:

$$\sup_{x\in(0,\infty)} v^2(x,t) \leq \frac{1}{2}\int_0^\infty |v(x)||v'(x)|dx, \ \forall\, v \in H^1(0,\infty).$$

Then, letting $v(x) = u(x,t)\sqrt{\psi'_\alpha(x-x_0)}$

$$\begin{aligned}
\sup_{x\in(0,\infty)} |u\sqrt{\psi'_\alpha}| &\leq \frac{1}{\sqrt{2}}\Big(\int_0^\infty |u\sqrt{\psi'_\alpha}|\,|u_x\sqrt{\psi'_\alpha} + \frac{u\psi''_\alpha}{2\sqrt{\psi'_\alpha}}|\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}}\Big(\int_0^\infty u_x^2\psi'_\alpha dx\Big)^{\frac{1}{4}}\Big(\int_0^\infty u^2\psi'_\alpha dx\Big)^{\frac{1}{4}} + \frac{1}{2}\Big(\int_0^\infty u^2\psi''_\alpha dx\Big)^{\frac{1}{2}}
\end{aligned}$$

and from (2.13) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)dx+\frac{3}{2}\int_0^\infty u_x^2(x,t)\psi'_\alpha(x-x_0)dx\\
&\leq \frac{1}{6}\Big(\int_0^\infty u^2(x,t)|\psi''_\alpha(x-x_0)|dx\Big)^{\frac{1}{2}}\int_0^\infty u^2(x,t)\sqrt{\psi'_\alpha(x-x_0)}dx\\
&+\frac{1}{3\sqrt{2}}\Big(\int_0^\infty u_x^2(x,t)\psi'_\alpha(x-x_0)dx\Big)^{\frac{1}{4}}\Big(\int_0^\infty u^2(x,t)\psi'_\alpha(x-x_0)dx\Big)^{\frac{1}{4}}\times\\
&\qquad\qquad\qquad\qquad\times\int_0^\infty u^2(x,t)\sqrt{\psi'_\alpha(x-x_0)dx}\\
&+\Big(\frac{1}{2}+\|a\|_{L^\infty(0,\infty)}\Big)\int_0^\infty u^2(x,t)\{\psi_\alpha(x-x_0)+\psi'_\alpha(x-x_0)+|\psi'''_\alpha(x-x_0)|\}dx.
\end{aligned}
$$

Taking the above inequality into account, the result is obtained arguing as in [17] (Lemma 2.1 and Theorem 2.2). We observe that in our case it is sufficient to consider $x_0=0$ and $\alpha=1,2$. $\square$

Proposition 2.3. *Let u be the solution of problem (1.1) given by Theorem 2.1. Then, for any $T>0$,*

$$\|u\|_{L^\infty(0,T;H^1(0,\infty))}\leq C,$$

where $C=C(T,\|u_0\|_{H^1(0,\infty)},\|a\|_{L^\infty(0,\infty)},\|a'\|_{L^\infty(0,\infty)})$ is a positive constant.

Proof. Multiplying the equation in (1.1) by u we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,\infty)}+\frac{1}{2}|u_x(0,t)|^2+\int_0^\infty a(x)|u(t)|^2dx=0. \tag{2.14}$$

Consequently, we deduce that

$$\|u(t)\|\ \leq\ \|u_0\|_{L^2(0,\infty)},\ \ \forall\, t>0.$$

Now, we multiply the equation in (1.1) by $-2u_{xx}-u^2$ to bound u in $L^2(0,T;H_0^1(0,\infty))$. Indeed, integrating over $(0,\infty)\times(0,T)$, we get

$$
\begin{aligned}
&\int_0^\infty u_x^2dx+\int_0^T|u_x(0,s)|^2ds+\int_0^T|u_{xx}(0,s)|^2ds+2\int_0^T\int_0^\infty a(x)u_x^2dxds\\
&\quad=-2\int_0^T\int_0^\infty a'(x)uu_xdxdt+\int_0^T\int_0^\infty a(x)u^3dxdt+\frac{1}{3}\int_0^\infty u^3dx\\
&\qquad\qquad+\int_0^\infty u_{0,x}^2dxdt-\frac{1}{3}\int_0^\infty u_0^3dx.
\end{aligned}
\tag{2.15}
$$

The terms on the right-hand side of (2.15) may be estimated as follows:

$$
\begin{aligned}
-2\int_0^T\int_0^\infty a'(x)uu_x dxdt \leq & 2\int_0^T \|a'\|_{L^\infty(0,\infty)}\|u\|_{L^2(0,\infty)}\|u_x\|_{L^2(0,\infty)}dt\\
\leq & \int_0^T \|a'\|^2_{L^\infty(0,\infty)}\|u\|^2_{L^2(0,\infty)}dt + \int_0^T\int_0^\infty u_x^2 dxdt\\
\leq & T\,\|a'\|^2_{L^\infty(0,\infty)}\|u_0\|^2_{L^2(0,\infty)} + \int_0^T\int_0^\infty u_x^2 dxdt,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T\int_0^\infty a(x)u^3 dxdt \ \leq \ & \int_0^T \|a\|_{L^\infty(0,\infty)}\|u\|_{L^\infty(0,\infty)}\|u\|^2_{L^2(0,\infty)}dt\\
\leq \ & C\int_0^T \|a\|_{L^\infty(0,\infty)}\|u_0\|^2_{L^2(0,\infty)}\|u\|_{H^1(0,\infty)}dt\\
\leq \ & \frac{T\,C^2}{2}\|a\|_{L^\infty(0,\infty)}\|u_0\|^4_{L^2(0,\infty)} + \frac{1}{2}\int_0^T\int_0^\infty u_x^2 dxdt
\end{aligned}
$$

where $C > 0$ denotes the Sobolev embedding theorem. In a similar vein, one obtains

$$\int_0^\infty u^3 dx \leq \frac{1}{2}\,\|u(t)\|^2_{H^1(0,\infty)} + \frac{C^2}{2}\,\|u_0\|^4_{L^2(0,\infty)}.$$

A combination of the above estimates yields the inequality

$$
\begin{aligned}
&\|u(t)\|^2_{H^1(0,\infty)} + \int_0^t |u_x(0,s)|^2 ds + \int_0^t |u_{xx}(0,s)|^2 ds\\
&\leq \frac{1}{2}\,\|u(t)\|^2_{H^1(0,\infty)} \,+\, c_1 \,+\, c_2\int_0^t \|u\|^2_{H^1(0,\infty)}ds,
\end{aligned}
$$

valid for any $t \in [0,T]$, where $c_1 = c_1(T, \|u_0\|_{H^1(0,\infty)}, \|a\|_{L^\infty(0,\infty)}, \|a'\|_{L^\infty(0,\infty)})$ and c_2 are positive constants. Consequently, it follows that

$$\sup \|u(t)\|_{H^1(0,\infty)} \,\leq\, c, \ \ \forall\, t \in\, [0,T] \tag{2.16}$$

where $c = c(T, \|u_0\|_{H^1(0,\infty)},\, \|a\|_{L^\infty(0,\infty)},\, \|a'\|_{L^\infty(0,\infty)}) > 0$, which completes the proof. □

To prove our main result, we first differentiate the equation in (1.1) with respect to t and analyze the regularity of $v = u_t$, which is a solution of

$$
\begin{cases}
v_t + v_x + v_{xxx} + (u(x,t)v)_x + a(x)v = 0, \quad x,\, t \in \mathbb{R}^+\\
v(0,t) = 0, \quad t > 0,\\
v(x,0) = v_0(x), \quad x > 0,
\end{cases} \tag{2.17}
$$

where $u \in L^\infty(0,\infty; H_0^1(0,\infty))$ is the weak solution of (1.1) and $v_0 = v(x,0) = u_t(x,0)$ in $H^{-2}(0,\infty)$. Observe that model (2.17) is a linearized KdV equation with $u = u(x,t)$ being a variable coefficient and, therefore, the following holds:

Lemma 2.4. *(See* [9]*, Lemma 4.5) Let u be the solution of problem (1.1). Then, problem (2.17) has a unique mild solution $v \in L^\infty(0,T; L^2(0,\infty))$ whenever $v_0 \in L^2(0,\infty)$.*

Lemma 2.5. *Let v be the solution of* (2.17) *given by Lemma 2.4. In addition, if $\sqrt{x}\, v_0 \in L^2(0,\infty)$, then*

$$\|v\|_{L^\infty(0,T;L^2(0,\infty))} + \|v\|_{L^2(0,T;H_0^1(0,\infty))} \leq C$$

where $C = C(T, \|u_0\|_{H^1(0,\infty)}, \|\sqrt{x}\, v_0\|_{L^2(0,\infty)})$, for all $T > 0$.

Proof. To do obtain the result we need some a priori estimates which will be obtained in several steps. We first multiply the equation in (2.17) by v and integrate by parts over $(0,\infty)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty v^2 dx + \frac{1}{2}v_x^2(0,t) + \int_0^\infty a(x)v^2 dx = \int_0^\infty uvv_x dx. \tag{2.18}$$

Now, integrating from 0 to T and applying Cauchy-Schwarz and Hölder's inequalities in the right hand side of (2.18), it follows that

$$\begin{aligned}\int_0^\infty v^2(x,T)dx &\leq \int_0^\infty v_0^2 dx + 2\int_0^T\int_0^\infty |uvv_x| dxdt \\ &\leq \int_0^\infty v_0^2 dx + 2\Big(\int_0^T \|uv\|^2_{L^2(0,\infty)} dt\Big)^{\frac{1}{2}}\Big(\int_0^\infty \|v_x\|^2_{L^2(0,\infty)} dt\Big)^{\frac{1}{2}} \\ &\leq \int_0^\infty v_0^2 dx + \int_0^T \|u\|^2_{L^\infty(0,\infty)}\|v\|^2_{L^2(0,\infty)} dt + \int_0^T \|v_x\|^2_{L^2(0,\infty)} dt.\end{aligned} \tag{2.19}$$

In order to estimate the last term in the right hand side of (2.19), we multiply equation in (2.17) by xv and integrate over $(0,\infty)\times(0,T)$. Then, performing integration by parts and using the boundary condition we get

$$\begin{aligned}&\int_0^T\int_0^\infty v_x^2 dxdt + \frac{1}{3}\int_0^\infty xv^2(x,T)dx + \frac{2}{3}\int_0^T\int_0^\infty xa(x)v^2 dxdt \\ &\qquad = \frac{1}{3}\int_0^T\int_0^\infty v^2 dxdt + \frac{1}{3}\int_0^\infty xv_0^2(x)dx \\ &\qquad + \frac{2}{3}\int_0^T\int_0^\infty xuvv_x dxdt + \frac{2}{3}\int_0^T\int_0^\infty uv^2 dxdt\end{aligned} \tag{2.20}$$

or

$$\begin{aligned}\int_0^T\int_0^\infty v_x^2 dxdt &\leq \frac{2}{3}\int_0^T\int_0^\infty xuvv_x dxdt + \frac{2}{3}\int_0^T\int_0^\infty uv^2 dxdt \\ &+ \frac{1}{3}\int_0^T\int_0^\infty v^2 dxdt + \frac{1}{3}\int_0^\infty xv_0^2(x)dx,\end{aligned} \tag{2.21}$$

since the other terms that appears in the first line of (2.20) are positive. Then the Cauchy-Schwarz inequality, properties of the solution u and Young's inequality yield

$$\begin{aligned}\int_0^T\int_0^\infty v_x^2 dxdt \le & \frac{2}{3}\int_0^T \|xu\|_{L^\infty(0,\infty)}\|v\|_{L^2(0,\infty)}\|v_x\|_{L^2(0,\infty)} \\ & + \frac{2}{3}\int_0^T \|u\|_{L^\infty(0,\infty)}\|v\|^2_{L^2(0,\infty)}\, dt + \frac{1}{3}\int_0^T\int_0^\infty v^2 dxdt + \int_0^\infty x v_0^2(x)dx \\ & \le c(\delta)\int_0^T \{1+\|xu\|^2_{H^1(0,\infty)} + \|u\|_{H^1(0,\infty)}\}\|v\|^2_{L^2(0,\infty)} dt \\ & + c\delta \int_0^T\int_0^\infty v_x^2 dxdt + \int_0^\infty x v_0^2(x)dx,\end{aligned} \tag{2.22}$$

where c and δ are positive constants. Consequently, for δ sufficiently small we obtain

$$\begin{aligned}\int_0^T\int_0^\infty v_x^2\, dxdt \le C\Big\{ & \int_0^\infty x v_0^2(x)dx \\ & + \int_0^T (1+\|xu\|^2_{H^1(0,\infty)} + \|u\|_{H^1(0,\infty)})\|v\|^2_{L^2(0,\infty)} dt\Big\},\end{aligned} \tag{2.23}$$

for some positive constant C. So, replacing (2.23) into (2.19) we can apply Gronwall's inequality and Theorem 2.2 to deduce that

$$\|v\|_{L^\infty(0,T;L^2(0,\infty))} \le C, \tag{2.24}$$

where $C = C(T, \|u_0\|_{L^2(0,\infty)}, \|\sqrt{x}v_0\|_{L^2(0,\infty)}) > 0$. On the other hand, combining (2.23), (2.24) and Theorem 2.1, we obtain

$$\|v\|_{L^2(0,T;H_0^1(0,\infty)} \le C, \tag{2.25}$$

where $C > 0$ also depends on $T, \|u_0\|_{L^2(0,\infty)}$ and $\|\sqrt{x}v_0\|_{L^2(0,\infty)}$. This concludes the proof . □

The next result is the key ingredient in the arguments used to show the exponential decay.

Lemma 2.6. *There exists a positive constant $C = C(T, \|u_0\|_{L^2(0,\infty)})$ such that*

$$\|v_0\|^2_{L^2(0,\infty)} \le C\Big\{\int_0^T v_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)v^2 dxdt + \|v_0\|^2_{H^{-2}(0,\infty)}\Big\}, \tag{2.26}$$

holds for every solution v of (2.17).

Proof. To prove (2.26) we combine multiplier techniques and the so called "compactness-uniqueness" argument. First we multiply the equation in (2.17) by $(T-t)v$ and integrate over $(0,\infty)\times(0,T)$ to obtain

$$\begin{aligned} T\|v_0\|^2_{L^2(0,\infty)} = & \int_0^T\int_0^\infty v^2 dxdt + \int_0^T (T-t)v_x^2(0,t)dt + 2\int_0^T\int_0^\infty (T-t)a(x)v^2 dxdt \\ & + \int_0^T\int_0^\infty (T-t)u_x v^2 dxdt.\end{aligned} \tag{2.27}$$

¿From (2.27) we deduce that

$$\|v_0\|^2_{L^2(0,\infty)} \leq \frac{1}{T}\int_0^T\int_0^\infty v^2dxdt + \int_0^T v_x^2(0,t)dt + 2\int_0^T\int_0^\infty a(x)v^2dxdt + \int_0^T\int_0^\infty |u_x|v^2dxdt, \tag{2.28}$$

and since

$$\begin{aligned}\int_0^T\int_0^\infty |u_x|v^2dxdt &\leq \int_0^T \|u_x\|_{L^2(0,\infty)}\|v\|^2_{L^4(0,\infty)}dt \\ &\leq \|u\|_{L^2(0,T;H_0^1(0,\infty))}\,\|v\|^2_{L^4(0,T;L^4(0,\infty))},\end{aligned} \tag{2.29}$$

it follows from (2.28) and Theorem 2.1 that

$$\|v_0\|^2_{L^2(0,\infty)} \leq c\,\|v\|^2_{L^4(0,T;L^4(0,\infty))} + \int_0^T v_x^2(0,t)dt + 2\int_0^T\int_0^\infty a(x)v^2dxdt, \tag{2.30}$$

where $c > 0$ only depends on T and $\|u_0\|_{L^2(0,\infty)}$. Thus, in order to prove (2.26) it is sufficient to show that for any $T > 0$ there exists a positive constant $C = C(T)$ such that

$$\|v\|^2_{L^4(0,T;L^4(0,\infty))} \leq C\left\{\int_0^T v_x^2(0,t)dt + \int_0^T\int_0^\infty a(x)v^2dxdt + \|v_0\|^2_{H^{-2}(0,\infty)}\right\} \tag{2.31}$$

for any solution of (1.1).

We argue by contradiction. Suppose that (2.31) does not hold. Then, there exists a sequence of functions $v_n \in L^\infty(0,T;L^2(0,\infty)) \cap L^2(0,T;H_0^1(0,\infty))$ that solves (1.1), satisfying

$$\lim_{n\to\infty} \frac{\|v_n\|^2_{L^4(0,T;L^4(0,\infty))}}{\displaystyle\int_0^T |v_{n,x}(0,t)|^2dt + \int_0^T\int_0^\infty a(x)v_n^2dxdt + \|v_{0,n}\|^2_{H^{-2}(0,\infty)}} = \infty. \tag{2.32}$$

Let $\lambda_n = \|v_n\|_{L^4(0,T;L^4(0,\infty))}$ and define $w_n(x,t) = \dfrac{v_n(x,t)}{\lambda_n}$. For each $n \in \mathbb{N}$ the function w_n satisfies

$$\begin{cases} w_{n,t} + w_{n,x} + w_{n,xxx} + (u(x,t)w_n)_x + a(x)w_n = 0 \quad \text{in } (0,\infty)\times(0,T), \\ w_n(0,t) = 0, \quad t \in (0,T), \\ w_n(x,0) = \dfrac{v_n(x,0)}{\lambda_n}, \quad x \in (0,\infty). \end{cases} \tag{2.33}$$

Moreover,

$$\|w_n\|_{L^4(0,T;L^4(0,\infty))} = 1 \tag{2.34}$$

and

$$\int_0^T |w_{n,x}(0,t)|^2dt + \int_0^T\int_0^\infty a(x)w_n^2dxdt + \|w_n(\,.\,,0)\|^2_{H^{-2}(0,\infty)} \to 0 \tag{2.35}$$

as $n \to \infty$.

Using (2.30), (2.34) and (2.35) it follows that $w_n(x,0)$ is bounded in $L^2(0,\infty)$, and therefore, according to Lemma 2.5,

$$\|w_n\|_{L^2(0,T;H_0^1(0,\infty))} \leq C, \tag{2.36}$$

for some constant $C > 0$. On the other hand,

$$\begin{aligned}\|(uw_n)_x\|_{L^2(0,T;L^1(0,\infty))} \leq& \|w_n\|_{L^\infty(0,T;L^2(0,\infty))}\|u\|_{L^2(0,T;H_0^1(0,\infty))}\\ &+ \|u\|_{L^\infty(0,T;L^2(0,\infty))}\|w_n\|_{L^2(0,T;H_0^1(0,\infty))}.\end{aligned} \tag{2.37}$$

So, by (2.37) we obtain that there exists $C > 0$ such that

$$\|(uw_n)_x\|_{L^2(0,T;L^1(0,\infty))} \leq C, \tag{2.38}$$

and therefore,

$$(w_n)_t \text{ is bounded in } L^2(0,T;H^{-2}(0,\infty)). \tag{2.39}$$

Indeed, according to (2.33), $w_{n,t}$ satisfies

$$w_{n,t} = -w_{n,x} - w_{n,xxx} - (u(x,t)w_n)_x - a(x)w_n \quad \text{ in } \mathcal{D}'(0,T;H^{-2}(0,\infty)),$$

and (2.36)-(2.38) guarantee the boundedness (in $L^2(0,T;H^{-2}(0,\infty))$) of the terms appearing in the right hand side of the above equation.

At that point we claim that the following holds:

There exists $s > 0$ such that $\{w_n\}$ is bounded in $L^4(0,T;H^s(0,\infty))$, the embedding $H^s_{loc}(0,\infty) \hookrightarrow L^4_{loc}(0,\infty)$ being compact.

In fact, since $\{w_n\}$ is bounded in $L^2(0,T;H_0^1(0,\infty)) \cap L^\infty(0,T;L^2(0,L))$ by interpolation we can deduce that $\{w_n\}$ is bounded in

$$[L^q(0,T;L^2(0,\infty)), L^2(0,T;H_0^1(0,\infty))]_\theta = L^p(0,T;[L^2(0,\infty),H_0^1(0,\infty)]_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$ and $0 < \theta < 1$. Thus, choosing $q = \infty$, $\theta = 1/2$, so that $p = 4$, the claim holds with $s = 1/2$, i.e.,

$$[L^2(0,\infty), H_0^1(0,\infty)]_{\frac{1}{2}} = H^{\frac{1}{2}}(0,\infty).$$

Furthermore, the embedding $H^{\frac{1}{2}}_{loc}(0,\infty) \hookrightarrow L^4_{loc}(0,\infty)$ is compact.

Then, using the statement above, (2.39) and classical compactness results [[29], Corollary 4] we can extract a subsequence of $\{w_n\}$, that we also denote by $\{w_n\}$, such that

$$w_n \to w \text{ strongly in } L^4(0,T;L^4_{loc}(0,\infty)). \tag{2.40}$$

Consequently, (2.34) and the structure of Ω allow us to conclude that

$$\|w\|^4_{L^4(0,T;L^4(0,\infty))} = \int_0^T \int_{\overline{\Omega}} |w|^4 dxdt + \int_0^T \int_{\overline{\Omega}^c} |w|^4 dxdt = 1. \tag{2.41}$$

On the other hand, by weak lower semicontinuity we have

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \Big\{ \int_0^T |w_{n,x}|^2 dt + \int_0^\infty \int_K a(x) w_n^2 dxdt + \|w_n(\cdot,0)\|^2_{H^{-2}(K)} \Big\} \\ &\geq \int_0^T |w_x(0,t)|^2 dt + \int_0^T \int_K a(x) w^2 dxdt + \|w(\cdot,0)\|^2_{H^{-2}(0,K)}. \end{aligned} \tag{2.42}$$

for all $K \subset (0,\infty)$. This implies, in particular, that $w(x,0)=0$. Consequently, the limit w, which solves the system

$$\begin{cases} w_t + w_x + w_{xxx} + (u(x,t)w)_x + a(x)w = 0 \quad \text{in } (0,\infty)\times(0,T), \\ w(0,t) = w_x(0,t) = 0, \quad t\in(0,T), \\ w(x,0) = 0, \quad x \in (0,\infty). \end{cases} \tag{2.43}$$

is identically zero, i.e., $w \equiv 0$. This contradicts (2.41) and, necessarily, (2.31) has to be valid. This completes the proof of Lemma 2.6. □

3. Exponential Decay

In this section we prove our main result.

Proof of Theorem 1.1. Multiply the equation in (1.1) by u and integrate in $(0,L)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,\infty)} + \frac{1}{2}|u(0,t)|^2 + \int_0^\infty a(x)|u(t)|^2dx = 0. \tag{3.44}$$

We claim that for any $T>0$, there exists $c=c(T)>0$ such that

$$\|u_0\|^2_{L^2(0,\infty)} \le c\left[\int_0^T |u_x(0,t)|^2dt + \int_0^T\int_0^\infty a(x)u^2dxdt\right] \tag{3.45}$$

for every solution of (1.1). This fact, together with the energy dissipation law (1.6) and the semigroup property, suffices to obtain the uniform exponential decay. In fact, let us prove (3.45).

Multiplying the equation by $(T-t)u$ and integrating on $(0,\infty)\times(0,T)$ we obtain the identity

$$T\int_0^\infty u_0^2dx = \int_0^T\int_0^\infty |u|^2dxdt + \int_0^T (T-t)|u_x(0,t)|^2dt + 2\int_0^T\int_0^\infty (T-t)a(x)|u|^2dxdt. \tag{3.46}$$

Consequently,

$$\int_0^\infty u_0^2dx \le \frac{1}{T}\int_0^T\int_0^\infty |u|^2dxdt + \int_0^T |u_x(0,t)|^2dt + 2\int_0^T\int_0^\infty a(x)|u|^2dxdt. \tag{3.47}$$

Thus, in order to show (3.45) it suffices to prove that for any $T>0$, there exists a positive constant $C_1(T)$ such that

$$\int_0^T\int_0^\infty |u|^2dxdt \le C_1\Big\{\int_0^T |u_x(0,t)|^2dt + 2\int_0^T\int_0^\infty a(x)|u|^2dxdt\Big\}. \tag{3.48}$$

Let us argue by contradiction following the so-called "compactness-uniqueness" argument (see for instance [33]). Suppose that (3.48) is not valid. Then, we can find a sequence of functions $\{u_n\} \in L^\infty(0,T;L^2(0,\infty)) \cap L^2(0,T;H_0^1(0,\infty))$ that solve (1.1) and such that

$$\lim_{n\to\infty} \frac{\|u_n\|^2_{L^2(0,T;L^2(0,\infty))}}{\displaystyle\int_0^T |u_{n,x}(0,t)|^2dt + \int_0^T\int_0^\infty a(x)u_n^2dxdt} = +\infty.$$

Let $\lambda_n = \|u_n\|_{L^2(0,T;L^2(0,\infty))}$ and define $w_n(x,t) = u_n(x,t)/\lambda_n$. For each $n \in \mathbb{N}$ the function w_n solves

$$\begin{cases} w_{n,t} + w_{n,x} + w_{n,xxx} + \lambda_n w_n w_{n,x} + a(x)w_n = 0 \quad \text{in } (0,\infty)\times(0,T), \\ w_n(0,t) = 0, \quad t \in (0,T), \\ w_n(x,0) = w_{0,n} = u_n(x,0)/\lambda_n, \quad x \in (0,\infty). \end{cases} \tag{3.49}$$

Moreover,

$$\|w_n\|_{L^2(0,T;L^2(0,\infty))} = 1 \tag{3.50}$$

and

$$\int_0^T \int_0^\infty |w_{n,x}(0,t)|^2 dxdt + \int_0^T \int_0^\infty a(x)w_n^2 dxdt \longrightarrow 0 \tag{3.51}$$

as $n \to \infty$.

Using (3.47) it follows that $w_n(\cdot,0)$ is bounded in $L^2(0,L)$. Then, by Proposition 2.3 it follows that

$$\|w_n\|_{L^2(0,T;H_0^1(0,\infty))} \le C, \ \ \forall n \in \mathbb{N} \tag{3.52}$$

for some constant $C > 0$. Also, since

$$w_{n,t} = -w_{n,x} - w_{n,xxx} - \lambda_n w_n w_{n,x} - a(x)w_n \ \text{ in } \mathcal{D}'(0,T;H^{-2}(0,\infty))$$

performing as in the previous section, the above estimates guarantee that

$$\{w_{n,t}\} \text{ is bounded in } L^2(0,T;H^{-2}(0,\infty)). \tag{3.53}$$

Furthermore, we can extract a subsequence of $\{w_n\}$, that we also denote by $\{w_n\}$, such that

$$w_n \to w \text{ strongly in } L^2(0,T;L^2_{loc}(0,\infty)). \tag{3.54}$$

Thus, the structure of Ω and (3.50) give us that

$$\|w\|^2_{L^2(0,T;L^2(0,\infty))} = \int_0^T \int_{\overline{\Omega}} |w|^2 dxdt + \int_0^T \int_{\overline{\Omega}^c} |w|^2 dxdt = 1. \tag{3.55}$$

Also,

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \left\{ \int_0^T |w_{n,x}|^2 dt + \int_0^T \int_K a(x) w_n^2 dxdt \right\} \\ &\ge \int_0^T |w_x(0,t)|^2 dt + \int_0^T \int_K a(x) w^2 dxdt. \end{aligned} \tag{3.56}$$

for all $K \subset (0,\infty)$. We now distinguish two situations:

(a) There exists a subsequence of $\{\lambda_n\}$ also denoted by $\{\lambda_n\}$ such that

$$\lambda_n \longrightarrow 0.$$

In this case, the limit w satisfies the linear problem

$$\begin{cases} w_t + w_x + w_{xxx} + a(x)w = 0 \quad \text{in } (0,\infty)\times(0,T), \\ w(0,t) = 0, \quad t \in (0,T), \\ w \equiv 0, \quad \text{in } \Omega \times (0,T). \end{cases}$$

Then, by Holmgren's Uniqueness Theorem, $w \equiv 0$ in $(0,\infty)\times(0,T)$ and this contradicts (3.55).

(b) There exists a subsequence of $\{\lambda_n\}$ also denoted by $\{\lambda_n\}$ and $\lambda > 0$ such that

$$\lambda_n \longrightarrow \lambda.$$

In this case, the limit function w solves the system (3.49) and, according to (3.56), $a(x)w = 0$ in $K \times (0,T)$ and $w_x(0,t) = 0$ a. e.. So, we apply the Unique Continuation Property (UCP) proved in [27] for the subset Ω obtaining that $w \equiv 0$ in $(0,\infty) \times (0,T)$ and again, this is a contradiction.

At that point, we observe that to apply the UCP mentioned above we need to prove that $u \in L^2(0,T;H^2(0,\infty))$. Indeed, differentiating (3.49) with respect to t, we obtain system (2.17) with

$$v_0(x) = w_t(x,0) = -w_{0,x} - w_{0,xxx} - \lambda w_0 w_{0,x} - a(x)w_0 \in H^{-2}(0,\infty).$$

On the other hand, if $w_x(0,t) = 0$ and $a(x)w = 0$ vanishes then, $v_x(0,t) = 0$ and $a(x)v = 0$ as well. Consequently, by the assumptions on the damping potential $a = a(x)$, $v \equiv 0$ in $\Omega \times (0,T)$ and according to Lemma 2.6 we obtain $v_0 \in L^2(0,\infty)$. Then, combining Lemma 2.5 and system (1.1) we get $w_t = v \in L^2(0,T;H_0^1(0,\infty))$ which allows to conclude that $w \in L^2(0,T;H^3(0,\infty)) \cap H^1(0,T,L^2(0,\infty))$.

In summary, we see that, in each of the possible situations (a) and (b) we are in a contradiction. Then, necessarily, (3.48) holds. This complete the proof of Theorem 1.1. □

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IMPA, Estrada Dona Castorina 110, Rio de Janeiro, 22460–320, Brazil
E-mail address: linares@impa.br

Institute of Mathematics, Federal University of Rio de Janeiro, UFRJ, P.O. Box 68530, CEP 21945-970, Rio de Janeiro, RJ, Brasil
E-mail address: ademir@acd.ufrj.br