Warped product structure of submanifolds with nonpositive extrinsic curvature in space forms *

Luis A. Florit†

Abstract

By means of a simple warped product construction we obtain examples of submanifolds with nonpositive extrinsic curvature and minimal index of relative nullity in any space form. We then use this to extend to arbitrary space forms four known splitting results for Euclidean submanifolds with nonpositive sectional curvature.

Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an n-dimensional isometrically immersed submanifold of codimension p in a space form $\mathbb{Q}_c^{n+p}$ with nonpositive extrinsic sectional curvature, that is, the sectional curvature K_M of M^n verifies $K_M \leq c$. For such a submanifold, in [**3**] it was shown that the *index of relative nullity* $\nu = \nu^f$ of f, i.e., the dimension of the nullity space Δ of the second fundamental form α of f, $\Delta = \operatorname{Ker}\alpha = \{X \in TM : \alpha(X, \cdot\,) = 0\}$, satisfies that $\nu \geq n - 2p$ everywhere. This result has several deep implications since the positiveness of ν imposes strong restrictions on the manifold M^n and on its isometric immersion f. For example, along connected components of an open dense subset, the relative nullity Δ is an integrable distribution with totally geodesic leaves in both M^n and $\mathbb{Q}_c^{n+p}$. So, the bigger ν is, the more geometrical restrictions we get.

This estimate is in fact sharp since the product immersion $f_1 \times \cdots \times f_p : M^n \to \mathbb{R}^{n+p}$ of p nowhere flat Euclidean hypersurfaces $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+1}$, $1 \leq i \leq p$, with nonpositive sectional curvature satisfies the equality $\nu \equiv n - 2p$, $n = \sum n_i$. It was a surprise to discover that this is the *unique* way to make the estimate sharp: any Euclidean submanifold in codimension p with nonpositive sectional curvature and minimal index of relative nullity must split (locally) as a product of p hypersurfaces; cf.[**6**]. Moreover, for $\nu = n - 2p + 1$ it was shown in [**7**] that the two natural constructions also give all possible examples. Either f splits as the product of $p-1$ Euclidean submanifolds, $p-2$ of which are hypersurfaces and the remaining has codimension two, or f is a composition, $f = h \circ f'$, where $f' : M^n \to \mathbb{R}^{n+p-1}$ splits as the product of $p-1$ hypersurfaces and

* *Mathematics Subject Classification (2000):* Primary 53B25; Secondary 53C40. *Keywords:* nonpositive extrinsic curvature, submanifold splitting, warped products.

†IMPA: Estrada Dona Castorina 110, 22460–320, Rio de Janeiro, Brazil; e-mail: luis@impa.br. Research partially supported by CNPq.

$h : U \subset \mathbb{R}^{n+p-1} \to \mathbb{R}^{n+p}$ is a flat hypersurface. Nothing more is known in this general setting, although for irreducible Euclidean submanifolds with *flat normal bundle* the sharpest estimate is known to be $\nu \geq n - p - 1$; see [**4**]. For closely related results for real Kähler Euclidean submanifolds, see [**8**], [**5**] and [**11**].

The phenomena for a nonpositively extrinsically curved submanifold f in a space form $\mathbb{Q}_c^{n+p}$ of curvature $c \neq 0$ were much less understood, since neither the curvature property nor the structure of the ambient space are preserved by taking products. It was known that such a submanifold does not exist if either $\nu = n-2p = 0$ or $\nu = n-2p+1 = 0$ at some point; cf. Corollary 2 in [**6**] or Theorem 2 in [**7**]. However, we show now a natural way to construct examples using an auxiliary space form $\mathbb{Q}_c^{p-1}$, that we call the *warping factor*, to 'warp' the factors.

For simplicity, we will assume from now on that $c = \pm 1$. Let us consider the usual model of $\mathbb{Q}_c^{r-1}$ as an umbilical submanifold of $\mathbb{E}^r$, where $\mathbb{E}^r$ stands for either the Euclidean space $\mathbb{R}^r$ or the Lorentzian space $\mathbb{L}^r$ with its canonical metric $\langle\ ,\ \rangle$ according with c, i.e., $dx^2 = c\ dx_1^2 + dx_2^2 + \cdots + dx_r^2$. That is,

$$\mathbb{Q}_c^{r-1} = \{x \in \mathbb{E}^r : \langle x, x\rangle = c, \text{ with } x_1 > 0 \text{ if } c = -1\}.$$

We also denote the sphere by $\mathbb{S}^{r-1} = \mathbb{Q}_1^{r-1}$ and the hyperbolic space by $\mathbb{H}^{r-1} = \mathbb{Q}_{-1}^{r-1}$. Consider any open subset $\hat{\mathbb{Q}}_c^{r-1} \subset \{x \in \mathbb{Q}_c^{r-1} : x_i \neq 0,\ 1 \leq i \leq r\}$. Define the *c–product* $M_1^{n_1} \times \underset{c}{-} \times M_r^{n_r}$ of r Riemannian manifolds $(M_i^{n_i}, g_i)$, $1 \leq i \leq r$, as the warped product

$$M_1^{n_1} \times \underset{c}{-} \times M_r^{n_r} := \hat{\mathbb{Q}}_c^{r-1} \times_{\lambda_1} M_1^{n_1} \times_{\lambda_2} \cdots \times_{\lambda_r} M_r^{n_r},$$

where λ_i is the height function $\lambda_i : \hat{\mathbb{Q}}_c^{r-1} \to \mathbb{R}_+$, $\lambda_i(x) = x_i$, $1 \leq i \leq r$. In other words, the metric on the product is given by $g = \pi_0^*\langle\ ,\ \rangle + \sum \lambda_i^2 \pi_i^* g_i$. This construction is symmetric in the factors $M_1, \ldots, M_r$ if $c = 1$, while M_1 has a special role for $c = -1$.

Accordingly, given r isometric immersions $f_1 : M_1^{n_1} \to \mathbb{Q}_c^{n_1+p_1}$ and $f_i : M_i^{n_i} \to \mathbb{S}^{n_i+p_i}$, $2 \leq i \leq r$, we call *c–product (isometric) immersion* the map denoted by

$$f^c = f_1 \times \underset{c}{-} \times f_r : M_1^{n_1} \times \underset{c}{-} \times M_r^{n_r} \to \mathbb{Q}_c^{n+p},$$

with $n = r - 1 + \sum n_i$ and $p = \sum p_i$, that is given by

$$f^c(x, y_1, \ldots, y_r) = (x_1 f_1(y_1), \ldots, x_r f_r(y_r)) \quad \in \quad \mathbb{Q}_c^{n+p} \subset \mathbb{E}^{n+p+1}.$$

In particular, if $p_i = 0$, $1 \leq i \leq r$, we get the isometry Ψ, called a *warped product representation* in [**9**], over an open dense subset of $\mathbb{Q}_c^n$ given by $\Psi : \mathbb{Q}_c^{n_1} \times \underset{c}{-} \times \mathbb{S}^{n_r} \to \mathbb{Q}_c^n$, $\Psi(x, w_1, \ldots, w_r) = (x_1 w_1, \ldots, x_r w_r)$. The c–product and the usual product immersions are then related by $f^c = \Psi \circ (id \times f_1 \times \cdots \times f_r)$. If such a decomposition of $\mathbb{Q}_c^{n+p}$ exists for an isometric immersion f, we say that *f splits as a c–product of r submanifolds.* Observe that although the boundary of the positive quadrant is singular for Ψ, it is not singular for the closure of its image. In this sense, a complete submanifold may also split as a c–product.

The key geometric properties of the c–product $f^c = f_1 \times_{\bar{c}} \times f_r : M^n \to \mathbb{Q}_c^{n+p}$ important to us are the following (see Remark 6):

- f^c has nonpositive extrinsic curvature if and only if all the factors f_i have;
- f^c has flat normal bundle if and only if all the factors f_i have;
- The leaves of the warping factor $\hat{\mathbb{Q}}_c^{r-1}$ are contained in the relative nullity leaves of f^c, as well as the relative nullity leaves of each factor;
- In fact, $\nu^{f^c}(x, x_1, \ldots, x_r) = r - 1 + \sum \nu^{f_i}(x_i)$.

In particular, any c–product of p hypersurfaces with (nowhere vanishing) nonpositive extrinsic curvature is an example of a submanifold with nonpositive extrinsic curvature and minimal index of relative nullity $\nu \equiv n - 2p$. This shows that our belief (ii) in the final comments on [**6**], that was based in its Corollary 2, was false. The last assertion in Theorem 1 below will completely clarify this phenomenon.

The purpose of this note is to extend the aforementioned main results in [**6**], [**7**] and [**4**] to arbitrary space forms by showing that, as for Euclidean submanifolds, the above c–product construction exhausts all the possibilities for the lowest indexes of relative nullity. Recalling our notation, from now on $f : M^n \to \mathbb{Q}_c^{n+p}$ will stand for an isometric immersion of a Riemannian manifold with sectional curvature $K_M \leq c = \pm 1$. We then know that $\nu \geq n - 2p$. The extension of Theorem 1 in [**6**] is the following:

Theorem 1. *Assume that $\nu = n - 2p$ everywhere. Then, there exists an open dense subset $\mathcal{U} \subset M^n$ such that, for each connected component $\mathcal{U}'$ of $\mathcal{U}$, $f|_{\mathcal{U}'}$ splits as a c–product of p hypersurfaces. That is, there are p hypersurfaces, $f_1 : M_1^{n_1} \to \mathbb{Q}_c^{n_1+1}$ with $K_{M_1} \leq c$, and $f_i : M_i^{n_i} \to \mathbb{S}^{n_i+1}$ with $K_{M_i} \leq 1$, $2 \leq i \leq p$, such that*

$$\mathcal{U}' = M_1 \times_{\bar{c}} \times M_p \qquad \textit{and} \qquad f|_{\mathcal{U}'} = f_1 \times_{\bar{c}} \times f_p$$

split. In particular, $\nu \geq p - 1$ and hence $n \geq 3p - 1$.

Recall that any hypersurface in a nonflat space form with nonpositive (nonvanishing) extrinsic sectional curvature as above must have relative nullity of codimension two and then can be easily parametrized using the Gauss parametrization: it just coincides with the usual immersion of the unit normal bundle of a surface, its Gauss map (cf. [**2**]).

Let us make a few remarks on the geometry of the submanifolds of Theorem 1. Observe that they all have flat normal bundle. In fact, there is (locally) an orthonormal tangent frame $\{X_1, \ldots, X_n\}$, an orthonormal normal frame $\{\xi_1, \ldots \xi_p\}$, and positive smooth functions $\lambda_1, \ldots, \lambda_{2p}$, such that the second fundamental form decomposes as $\alpha(X_i, X_j) = 0$, $\alpha(X_{2s-1}, X_{2s-1}) = \lambda_{2s-1}\xi_s$, $\alpha(X_{2s}, X_{2s}) = -\lambda_{2s}\xi_s$, and $\Delta =$

span$\{X_{2p+1},\ldots,X_n\}$, where $1\le i\neq j\le n$, $1\le s\le p$. The normal vector fields $\lambda_{2s-1}\xi_s$ and $-\lambda_{2s}\xi_s$ (together with $\xi=0$) are called the *principal normals* of the immersion, whose corresponding principal distributions span$\{X_{2s-1}\}$ and span$\{X_{2s}\}$ have all multiplicity one (cf. [**10**]). This principal directions gives rise to a "warped product net", i.e., $TU'=\bigoplus_{i=0}^p V_i$, with $X_{2s-1},X_{2s}\in V_s$, $1\le s\le p$, and $V_0\subset\Delta$, each V_i being a spherical distribution with integrable complement $V_i^\perp$. The distributions V_s are obtained taking successive derivatives of span$\{X_{2s-1},X_{2s}\}$.

The results in this note are essentially local in nature since there are explicit constructions where the relative nullity actually "jumps" from one factor to another. Indeed, it is easy to construct, for example using the Gauss parametrization, a connected submanifold $f:M^n\to\mathbb{R}^{n+p}$ that has an open dense subset $W\subset M^n$ divided in three irreducible connected components, $W=(N_1^{m+r}\times N_2^s)\,\cup\,(N_3^m\times\mathbb{R}^r\times N_4^s)\,\cup\,(N_5^m\times N_6^{s+r})$, with constant relative nullity $\nu=r$ and with f splitting accordingly along each component. However, if we somehow can control this phenomenon, we get global consequences. For example, this is the case if the estimate in the last assertion of Theorem 1 is also sharp:

Corollary 2. *Suppose that M^n is complete and simply connected, with $n=3p-1$. If $\nu\equiv n-2p$, then f splits globally as a c–product of p complete surfaces of negative extrinsic Gaussian curvature.*

There are two natural ways to construct a submanifold f with nonpositive extrinsic curvature and $\nu=n-2p+1$, and it turns out that these are the only ones. The extension of Theorem 1 in [**7**] then reads:

Theorem 3. *Assume $\nu=n-2p+1$ everywhere. Then, there is an open dense subset $\mathcal{U}\subset M^n$ such that each connected component $\mathcal{U}'$ of $\mathcal{U}$ satisfies $\mathcal{U}'=M_1^{n_1}\times_{\bar{c}}\times M_{p-1}^{n_{p-1}}$ and either:*

i) $f|_{\mathcal{U}'}=f_1\times_{\bar{c}}\times f_{p-1}$ splits as a c–product of $p-1$ submanifolds, all but one of which are hypersurfaces and the remaining having codimension two, or

ii) $f|_{\mathcal{U}'}$ is a composition, $f|_{\mathcal{U}'}=h\circ(f_1\times_{\bar{c}}\times f_{p-1})$, of a c–product of $p-1$ hypersurfaces and an extrinsically flat hypersurface $h:U\subset\mathbb{Q}_c^{n+p-1}\to\mathbb{Q}_c^{n+p}$.

The composition h in part (*ii*) destroys the flat normal bundle of the c–product. For flat normal bundle, we obtain the corresponding generalization of Theorem 1 in [**4**]:

Theorem 4. *Assume that f has flat normal bundle and that $\nu\le n-p-r$ everywhere, for some integer $2\le r\le p$. Then, there exists an open dense subset $\mathcal{U}\subset M^n$ such that, along each connected component $\mathcal{U}'$ of $\mathcal{U}$, $f|_{\mathcal{U}'}$ splits as a c–product of r submanifolds. In particular, $\nu\ge r-1$ and hence $n\ge p+2r-1$.*

1 The proofs

Let $f : M^n \to \mathbb{Q}_c^{n+p} \subset \mathbb{E}^{n+p+1}$, $c = \pm 1$, be an isometric immersion whose second fundamental form is $\alpha = \alpha_f$, and set $\Delta(x) = \Delta_f(x) = \operatorname{Ker}\alpha(x)$ for its relative nullity. Consider the *cone over* f, $F : \tilde{M}^{n+1} = \mathbb{R}_+ \times_{id} M^n \to \mathbb{E}^{n+p+1}$, given by $F(t,x) = tf(x)$. Observe that $\tilde{M}$ is Riemannian for $c = 1$ and Lorentzian for $c = -1$. Denoting by $\imath : \mathbb{Q}_c^{n+p} \to \mathbb{E}^{n+p+1}$ the inclusion, the normal space of $f' = \imath \circ f$ at $x \in M^n$ is given by

$$T^\perp_{f'(x)}M = T^\perp_{f(x)}M \oplus^\perp < f(x) >,$$

where $< v >$ denotes the line spanned by $v \in \mathbb{E}^{n+p+1}$. Thus, we have

$$T^\perp_{F(t,x)}\tilde{M} = T^\perp_{f(x)}M,$$

and then the vector $\partial_t(t,x) := F_{*(t,x)}(1,0) = f(x)$ belongs to the relative nullity of F. In addition, for any $(t,x) \in \tilde{M}$,

$$\alpha_F(t,x)(X,Y) = t\,\alpha_f(x)(X,Y), \quad \forall\; X,Y \in T_xM. \tag{1}$$

We resume now the main geometric properties of the cone that we need:

Lemma 5. *The following relations between f and its cone F hold:*

i) $\Delta_F(t,x) = \Delta_f(x) \oplus^\perp < \partial_t(t,x) >$ and then $\nu^F(t,x) = \nu^f(x) + 1$;

ii) $K_M \le c$ if and only if $K_{\tilde{M}} \le 0$;

iii) f has flat normal bundle if and only if F has flat normal bundle.

Proof: i). This is a direct consequence of (1).

ii). Consider a plane $\sigma \subset T_xM \subset T_{(t,x)}\tilde{M}$ and an orthonormal basis X, Y of σ. Then, by the Gauss equation, $K_M(\sigma) - c = \langle \alpha_f(X,X), \alpha_f(Y,Y)\rangle - \|\alpha_f(X,Y)\|^2 = t^{-2}K_{\tilde{M}}(\sigma)$, where the last equality is a consequence of $\langle F_{*(t,x)}X, F_{*(t,x)}Y\rangle = t^2\langle f_{*x}X, f_{*x}Y\rangle$. The assertion follows since ∂_t belongs to the relative nullity of F and the relative nullity is always contained in the nullity of the curvature tensor due to the Gauss equation.

iii). Since both $\mathbb{Q}_c^{n+p}$ and $\mathbb{E}^{n+p+1}$ have constant sectional curvature, the Ricci equation for f and F gives for the normal curvature tensors $R_F^\perp(\partial_t, X) = 0$, $R_F^\perp(X,Y) = R_f^\perp(X,Y)$ because the shape operators of f and F in any direction $\xi \in T_x^\perp M = T^\perp_{(t,x)}\tilde{M}$ are related by $A_\xi^F|_{\Delta_F^\perp} = t^{-1}A_\xi^f$. ∎

Remark 6. The above proves all the properties of the c–product f stated in the introduction, since the cone of a c–product of r submanifolds is clearly a product of r cones. For example, $f = f_1 \times_{\bar{c}} \times f_r$ has nonpositive extrinsic sectional curvature if and only if its cone F has, which in turn is equivalent to each factor F_i of F being nonpositively curved, or that each f_i having nonpositive extrinsic sectional curvature.

The idea now is quite simple: to reduce the problems in space forms to Euclidean (or Lorentzian) space using the cone of f. Observe that, although the results in [**6**], [**7**] and [**4**] are stated for the Euclidean space as their ambient space, the same proofs hold for any semiriemannian flat space form $\mathbb{R}^{m,n}$ and, in particular, for $\mathbb{L}^{n+1} = \mathbb{R}^{1,n}$.

Proof of Theorem 4: By Lemma 5 *ii*) and *iii*), the cone F of f has nonpositive sectional curvature and flat normal bundle. Moreover, by part *i*), its relative nullity satisfies $\nu_F = \nu_f + 1 \leq (n+1) - p - r$, where now p is the codimension of F and $n+1$ its dimension. Therefore, by Theorem 1 in [**4**], we have that, locally, F splits as the product of r Euclidean submanifolds $F_i : \tilde{M}_i^{n_i+1} \to V_i^{n_i+p_i+1}$, $1 \leq i \leq r$, of nonpositive sectional curvature: $\mathbb{E}^{n+p+1} = V_1 \oplus^\perp \cdots \oplus^\perp V_r$,

$$\tilde{M} = \tilde{M}_1 \times \cdots \times \tilde{M}_r \quad \text{ and } \quad F = F_1 \times \cdots \times F_r$$

split. In addition, all the $V_i's$ are Euclidean, except, say, V_1, that is Lorentzian for $c = -1$. In fact, no V_i can be degenerate since, by construction, the decomposition of $\mathbb{E}^{n+p+1}$ is as a direct sum of the $V_i's$, and thus $V_i \cap V_i^\perp = 0$.

Claim: Each factor F_i is also a cone with its vertex at the origin.

To prove the claim, observe first that an Euclidean submanifold in $\mathbb{E}^{N+1}$ is a cone with its vertex at the origin if and only if it is invariant by the homotheties $v \mapsto tv$, $t > 0$, $v \in \mathbb{E}^N$. In fact, the submanifold will be a cone over the intersection of the submanifold with $\mathbb{Q}_c^N \subset \mathbb{E}^{N+1}$. So, $(tF_1(y_1), \ldots, tF_r(y_r)) = tF(y) = F(z) = (F_1(z_1), \ldots, F_r(z_r))$, since F is a cone with its vertex at the origin. So, each F_i is invariant under the homotheties, and hence a cone.

Since the result is local in nature, we can assume that f is an embedding. Now, consider the intersections, which are transversal since the factors are cones, $M_1^{n_1} = F_1^{-1}(F_1(\tilde{M}_1) \cap \mathbb{Q}_c^{n_1+p_1})$, and $M_j^{n_j} = F_j^{-1}(F_j(\tilde{M}_j) \cap \mathbb{S}^{n_j+p_j})$, $2 \leq j \leq r$, and set $f_i = F_i|_{M_i}$. Therefore, $\tilde{M}_i = \mathbb{R}_+ \times_{id} M_i$ and F_i is the cone over f_i. Observe that $\|f_1\|^2 = c$, $\|f_j\|^2 = 1$, $2 \leq j \leq r$. We can write for $T = (t_1, \ldots, t_r) \in \mathbb{E}^r$, $F(T, y_1, \ldots, y_r) = (t_1 f_1(y_1), \ldots, t_r f_r(y_r))$ and $\|F(T, y_1, \ldots, y_r)\|^2 = ct_1^2 + t_2^2 + \cdots + t_r^2$. We conclude that, locally, $M^n = F^{-1}(F(\tilde{M}^{n+1}) \cap \mathbb{Q}_c^{n+p}) = M^{n_1} \times \underset{c}{-} \times M^{n_r}$ and $f = F|_M = f_1 \times \underset{c}{-} \times f_r$, as desired. ∎

Proof of Theorem 1: As above, the cone of f has $K_{\tilde{M}} \leq 0$ and minimal index of relative nullity. Thus it has flat normal bundle by Theorem 1 in [**4**]. Then, so do f and we can apply Theorem 4 for $r = p$. ∎

Proof of Corollary 2: Since the relative nullity is all absorbed by the warping factor, the decomposition of F is globally well defined. The only singular point of F is the origin, so each factor f_i is complete. ∎

Proof of Theorem 3: Similarly, we take the cone F of f for which it applies Theorem 1 in [**7**]. We have that either F splits as a product of $p-1$ cones, or it is a composition of a product of $p-1$ nowhere hypersurfaces F_i of nonpositive sectional curvature and a flat hypersurface $H : U \subset \mathbb{E}^{n+p} \to \mathbb{E}^{n+p+1}$, $F = H \circ (F_1 \times \cdots \times F_{p-1})$. The first case occurs when the nullity of F (or, equivalently, of f) coincides with its relative nullity; see the remark below Theorem 1 in [**7**]. The same argument as for Theorem 4 takes care of the first case, so we can restrict ourselves to the second one.

We have that the second fundamental form of F decomposes orthogonally as scalar bilinear forms, $\alpha_F = \beta_1 \oplus^\perp \cdots \oplus^\perp \beta_p$, where $\beta_p = \alpha_H|_{T\tilde{M}\times T\tilde{M}}$ has relative nullity Δ_p of codimension one, $\beta_j = \alpha_{F_j}$ has relative nullity Δ_j of codimension two, $1 \le j \le p-1$, and $\Delta_F^\perp = \Delta_1^\perp \oplus \cdots \oplus \Delta_p^\perp$. Therefore, by (1) the same property holds for f. That is, $T_f^\perp M$ decomposes orthogonally as line bundles, $T_f^\perp M = L_1 \oplus \cdots \oplus L_p$, each L_i spanned by a unitary vector field ξ_i, $\operatorname{rank} A^f_{\xi_p} = 1$, $\operatorname{rank} A^f_{\xi_j} = 2$, and

$$\Delta_f^\perp = \operatorname{Im} A^f_{\xi_1} \oplus \cdots \oplus \operatorname{Im} A^f_{\xi_p}, \tag{2}$$

for the shape operators of f. Let α' the orthogonal projection of α onto $L = L_p^\perp$. Since $A^f_{\xi_p}$ has rank one, α' satisfies the Gauss equation for M^n in space of constant curvature c. That α' also satisfies the Codazzi and Ricci equations with the connection on L induced by the normal connection $\nabla^\perp$ of f was proved in Proposition 5 of [**7**] using just (2), since these equations are independent of the curvature of the ambient space form. So, there is an isometric immersion $f' : M^n \to \mathbb{Q}_c^{n+p-1}$ whose second fundamental form is $\tau \circ \alpha'$, for some parallel bundle isometry $\tau : L \to T_{f'}^\perp M$. In particular, $\nu_{f'} = n - 2(p-1)$. Therefore, Theorem 1 applies to f' which is then a c–product of $p-1$ hypersurfaces. It only remains to show that f is a composition $f = h \circ f'$.

The Codazzi equation for $A^f_{\xi_p}$ gives

$$A^f_{\xi_p}[X,Y] = A^f_{\nabla^\perp_X \xi_p} Y - A^f_{\nabla^\perp_Y \xi_p} X \in \Delta^\perp_{\alpha'}, \qquad \forall\, X, Y \in \operatorname{Ker} A^f_{\xi_p}.$$

This and (2) imply that $\langle \nabla^\perp_X \xi_p, \xi_j \rangle A^f_{\xi_j} Y = \langle \nabla^\perp_Y \xi_p, \xi_j \rangle A^f_{\xi_j} X$, for all $X, Y \in \operatorname{Ker} A^f_{\xi_p}$, $1 \le j \le p-1$. Again by (2) and $\operatorname{rank} A^f_{\xi_j} = 2$ we easily obtain $\nabla^\perp_X \xi_p = 0$, for all $X \in \operatorname{Ker} A^f_{\xi_p}$. Thus the bundle $W := \{\tilde{\nabla}_X \xi_p : X \in TM\} \subset \operatorname{Im} A^f_{\xi_p} \oplus L$ is a line bundle with $W \cap L = 0$. We conclude that

$$\Gamma := W^\perp \cap (\operatorname{Im} A^f_{\xi_p} \oplus L)$$

is a vector bundle of rank $p-1$, with $\Gamma \cap TM = 0$. Moreover, if $\Phi := Id_{TM} \oplus \tau$, by definition of Γ and α', and the parallelism of τ we get that $\tilde{\nabla}_X \eta \in TM \oplus L$, and

$$\tilde{\nabla}'_X \Phi \eta = \Phi \tilde{\nabla}_X \eta, \qquad \forall\, X \in TM, \tag{3}$$

where $\tilde{\nabla}$ and $\tilde{\nabla}'$ stand for the connections of $\mathbb{Q}_c^{n+p}$ and $\mathbb{Q}_c^{n+p-1}$, respectively.

Let $G : \Gamma \to \mathbb{Q}_c^{n+p-1}$, $G(\eta_x) = exp'_{f'(x)}\Phi\eta_x$, where $\eta_x \in \Gamma(x)$, and exp' (resp. exp) stands for the exponential map of $\mathbb{Q}_c^{n+p-1}$ (resp. $\mathbb{Q}_c^{n+p}$). By the transversallity between Γ and TM, there is an open neighborhood $N \subset \Gamma$ of the 0-section such that $G|_N$ is a local diffeomorphism onto the open subset $U = G(N) \subset \mathbb{Q}_x^{n+p-1}$. Let h': $N \to \mathbb{Q}_c^{n+p}$ be the immersion $h'(\eta_x) = exp_{f(x)}\eta_x$. Using (3) we easily check that G and h' define the same metric on N^{n+p-1}. Hence, $h = h' \circ G^{-1}|_U : U \subset \mathbb{Q}_x^{n+p-1} \to \mathbb{Q}_c^{n+p}$ is an isometric immersion and $(h \circ f')(x) = h(G(0_x)) = h'(0_x) = f(x)$. ∎

Remark 7. The construction in the last part of the above proof shows that Proposition 8 in [**1**] also holds for any space form as ambient space.

References

[**1**] Dajczer, M., Florit, L.: *Compositions of isometric immersions in higher codimension.* Manuscripta Math. **105** (2001), 507–517.

[**2**] Dajczer, M., Gromoll, D.: *Gauss parametrizations and rigidity aspects of submanifolds.* J. Differential Geom. **22** (1985), 1–12.

[**3**] Florit, L.: *On submanifolds with nonpositive extrinsic curvature.* Math. Ann. **298** (1994), 187–192.

[**4**] Florit, L.: *A splitting theorem for euclidean submanifolds of nonpositive sectional curvature.* Math. Z. **225** (1997), 685–690.

[**5**] Florit, L., Hui, W., Zheng, F.: *On real Kähler Euclidean submanifolds with nonnegative Ricci curvature.* J. Eur. Math. Soc. (JEMS) **7** (2005), 1–11.

[**6**] Florit, L., Zheng, F.: *On nonpositively curved euclidean submanifolds: splitting results.* Comm. Math. Helv. **74** (1999), 53–62.

[**7**] Florit, L., Zheng, F.: *On nonpositively curved Euclidean submanifolds: splitting results, II.* J. Reine Angew. Math. **508** (1999), 1–15.

[**8**] Florit, L., Zheng, F.: *A local and global splitting result for real Kähler Euclidean submanifolds.* Arch. Math. (Basel) **84** (2005), 88–95.

[**9**] Nolker, S.: *On isometric immersions of warped products.* Differential Geom. Appl. **6** (1996), 1–30.

[**10**] *H. Reckziegel*, Krümmungsflächen von isometrischen Immersionen in Räume Konstanter Krümmung, Math. Ann. **223** (1976), 169–181.

[**11**] Zheng, F.: *Isometric embedding of Kähler manifolds with nonpositive sectional curvature.* Math. Ann. **304** (1996), 769–784.