Uniqueness of constant mean curvature surfaces properly immersed in a slab

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Abstract

We study complete properly immersed surfaces contained in a slab of a warped product $\mathbb{R} \times_\varrho \mathbb{P}^2$, where $\mathbb{P}^2$ is complete with nonnegative Gaussian curvature. Under certain restrictions on the mean curvature of the surface we show that such an immersion does not exists or must be a leaf of the trivial totally umbilical foliation $t \in \mathbb{R} \mapsto \{t\} \times \mathbb{P}^2$.

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To prove that a compact hypersurface of constant mean curvature embedded in Euclidean space must be a round sphere Alexandrov [**1**] introduced what nowadays is known as Alexandrov's reflexion method. He observed that the method also works in standard hyperbolic space and that it gives a similar result:

> *Any compact hypersurface embedded with constant mean curvature in hyperbolic space* $\mathbb{H}^{n+1}$ *is a round sphere.*

To see this result in the context of this paper it is convenient to observe that it is completely equivalent to assume *compactness* or *completeness* plus *proper without any point at the asymptotic boundary of* $\mathbb{H}^{n+1}$.

Since the hyperbolic space carries other totally umbilical hypersurfaces, namely, horospheres and hyperspheres, one may want to characterize these too. This was done by do Carmo and Lawson [**4**] making use of Alexandrov's method. In particular, they showed:

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Any complete hypersurface properly embedded with constant mean curvature in hyperbolic space $\mathbb{H}^{n+1}$ with a single point at the asymptotic boundary is a horosphere.

Moreover, they also observed that the statement is no longer true if we replace *embedded* by *immersed* since around that time J. Gomes [**6**] pointed out the existence of counterexamples. In fact, in unit hyperbolic space he proved that any element of the one-parameter family of complete *parabolic rotation hypersurfaces* with constant mean curvature (and parameter) $H \geq 1$ (defined by do Carmo and Dajczer in [**2**]) has a single point at the asymptotic boundary, and auto-intersect along a single $(n-1)-$dimensional horosphere if $H = 1$ and infinite such horospheres if $H > 1$.

Since Lawson [**8**] established what is now known as the *cousin correspondence* between minimal surfaces in Euclidean space $\mathbb{R}^3$ and surfaces with constant mean curvature $H(= \|\vec{H}\|) = 1$ in the unit hyperbolic space $\mathbb{H}^3$, the latter have been extensively studied. The one parameter family of catenoids cousins (see [**11**]) contains the immersed parabolic rotation surface discussed above. For what on the subject directly concerns this paper we also recall the half-space theorem obtained by Rodríguez and Rosenberg [**10**]. They proved that a properly embedded complete surface with $H = 1$ that lies on one side of an horosphere must be an horosphere itself whenever: (i) it is inside the horoball bounded by the horosphere or (ii) lies outside and its mean curvature vector $\vec{H}$ points toward the horoball. In relation to the latter case, there exist catenoids cousins with two points in the asymptotic boundary (see [**12**] or [**13**]) that provide counterexamples if we allow $\vec{H}$ to point in the opposite direction.

By an immersed surface being contained in a *slab* of $\mathbb{H}^3$ we mean that the submanifold lies between two horospheres that share the same point in the asymptotic boundary of $\mathbb{H}^3$. It turns out that each parabolic rotation surface with constant mean curvature $H > 1$ lies inside a slab (see [**6**]) but this is not the case for the cousin catenoid ($H = 1$) in the family. This surface lies on one side of an horosphere but *not* in a slab (because the generating curve is asymptotic to the asymptotic boundary; see [**11**]) and thus shows that the assumption on the mean curvature in Theorem 1 is sharp.

Theorem 1. *If $f\colon \Sigma^2 \to \mathbb{H}^3$ is a properly immersed complete surface with constant mean curvature $\|\vec{H}\| \leq 1$ contained in a slab then $f(\Sigma)$ is a horosphere.*

In fact, the preceding result is a consequence of general theorems on surfaces properly immersed in a large class of ambient spaces, discussed next, that carry a foliation of parallel umbilical surfaces; thus making natural the concept of slab there. On the other hand, there is a nice geometric technique to prove Theorem 1 but that will not work in general cases; see Remark 6.

Let $\mathbb{P}^2$ be a complete Riemannian surface and let $M^3 = \mathbb{R}\times_\varrho \mathbb{P}^2$ denote the product manifold $\mathbb{R}\times\mathbb{P}^2$ endowed with the complete Riemannian warped metric

$$\langle\ ,\ \rangle_{M^3} = dt^2 + \varrho^2(t)\langle\ ,\ \rangle_{\mathbb{P}^2}$$

where $\varrho\colon \mathbb{R}\to(0,+\infty)$ is smooth. The family of surfaces $\mathbb{P}_t = \{t\}\times\mathbb{P}^2$ form a foliation of M^3 by complete totally umbilical leaves of constant mean curvature

$$\mathcal{H}(t) = (\log\varrho)'(t) = (\varrho'/\varrho)(t).$$

Let $s\colon\mathbb{R}\to J$ be given by $s(t) = s(0) - \int_0^t \varrho^{-1}(u)du$, where $J = s(\mathbb{R})$. Then $\mathbb{R}\times_\varrho\mathbb{P}^2$ is isometric to the product manifold $J\times\mathbb{P}^2$ endowed with the conformal metric

$$\langle\ ,\ \rangle = \lambda^2(s)\left(ds^2 + \langle\ ,\ \rangle_{\mathbb{P}^2}\right)\quad\text{with}\ \ \lambda(s) = \varrho(t(s)),$$

by means of the isometry $\tau(t,x) = (s(t),x)$ that is *orientation reversing* since it reverses the orientation in the $\partial/\partial t$ direction. We have that $\mathbb{H}^3 = \mathbb{R}\times_{e^t}\mathbb{R}^2$ since τ is an isometry from $\mathbb{R}\times_{e^t}\mathbb{R}^2$ to $\mathbb{H}^3$ in the half-space model. It is worthwhile to observe that, in general, if $\int_0^{+\infty}\varrho^{-1} < +\infty$ and $\int_{-\infty}^0\varrho^{-1} = +\infty$, then taking $s(0) = \int_0^{+\infty}\varrho^{-1}$ we get $J = (0,+\infty)$, and thus $\mathbb{P}^2$ acts as a boundary at infinite of $\mathbb{R}\times_\varrho\mathbb{P}^2$ as does $\mathbb{R}^2$ in $\mathbb{H}^3$. Hence, the leaves $\mathbb{P}_t$ can be thought as horospheres in a fixed direction of $\mathbb{H}^3$.

In the context of surfaces in $\mathbb{R}\times_\varrho\mathbb{P}^2$ by being contained in a *slab* with boundary $\mathbb{P}_{t_1}\cup\mathbb{P}_{t_2}$ we mean *between* two leaves $\mathbb{P}_{t_1},\mathbb{P}_{t_2}$ with $t_1<t_2$ of the foliation $\mathbb{P}_t$.

Throughout the paper we assume that $\mathbb{P}^2$ is complete, its Gaussian curvature $K_\mathbb{P}$ is nonnegative and the geodesic curvature of the geodesic circles (from a fixed point p_0) of radius $\hat{r}\geq r_0>0$ satisfies $k_g\geq -c/\hat{r}$ for some positive constant c. One of the aforementioned general results is the following.

Theorem 2. *In a slab of $\mathbb{R}\times_\varrho\mathbb{P}^2$ with boundary $\mathbb{P}_{t_1}\cup\mathbb{P}_{t_2}$ there is no complete properly immersed surface with mean curvature satisfying*

$$\sup_\Sigma\|\vec{H}\| < \min_{[t_1,t_2]}\mathcal{H}(t). \tag{1}$$

There are two cases to consider (after normalization) for which $\mathcal{H}(t) = \mathcal{H}_0$ is constant. Either $\varrho = 1$ (thus $\mathcal{H}_0 = 0$) and the ambient space is just a Riemannian product $M^3 = \mathbb{R}\times\mathbb{P}^2$ or $\varrho = e^t$ (thus $\mathcal{H}_0 = 1$) and $M^3 = \mathbb{R}\times_{e^t}\mathbb{P}^2$. In the latter case, M^3 belongs to a class of manifolds called in [**14**] a *pseudo-hyperbolic* space. In particular, we have the following consequence of Theorem 2.

Corollary 3. *There is no properly immersed complete surface* Σ^2 *with mean curvature satisfying* $\sup_\Sigma \|\vec{H}\| < 1$ *contained in a slab of a pseudo-hyperbolic manifold* $\mathbb{R} \times_{e^t} \mathbb{P}^2$.

Our second general result specifically deals with pseudo-hyperbolic manifolds as ambient spaces and has our Theorem 1 as a corollary.

Theorem 4. *If* $f\colon \Sigma^2 \to M^3 = \mathbb{R} \times_{e^t} \mathbb{P}^2$ *is a properly immersed complete surface with constant mean curvature* $\|\vec{H}\| \leq 1$ *contained in a slab then* $f(\Sigma)$ *is a leaf* $\mathbb{P}_t$.

In the preceding result we are assuming that in (1) equality may hold. The other case in which this may happen, i.e., minimal surfaces in products spaces $\mathbb{R} \times \mathbb{P}^2$, was considered by Rosenberg [**12**] who proved the following half-space theorem.

> *If* $K_{\mathbb{P}} \geq 0$ *and the geodesic curvature of all geodesic circles in* $\mathbb{P}^2$ *of radius at least one from some fixed point is bounded by some constant then any properly immersed minimal surface in a half space* $[0, \infty) \times \mathbb{P}^2$ *is a slice.*

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The proofs

Throughout the paper $M^3 = \mathbb{R} \times_\varrho \mathbb{P}^2$ denotes the product manifold endowed with the complete Riemannian warped metric

$$\langle\ ,\ \rangle = \pi_{\mathbb{R}}^*(dt^2) + \varrho^2(\pi_{\mathbb{R}})\pi_{\mathbb{P}}^*(\langle\ ,\ \rangle_{\mathbb{P}}) \tag{2}$$

where $\varrho : \mathbb{R} \to (0, +\infty)$ is the warping function, $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}}$ are the projections from $\mathbb{R} \times \mathbb{P}^2$ onto each factor, and $\langle\ ,\ \rangle_{\mathbb{P}}$ the Riemannian metric on $\mathbb{P}^2$. Recall from the introduction that $\mathbb{P}^2$ is complete of nonnegative Gaussian curvature and the geodesic curvature of the geodesic circles from a fixed point p_0 of radius $\hat{r} \geq r_0 > 0$ satisfies $k_g \geq -c/\hat{r}$ for a positive constant c.

The *height function* $h \in \mathcal{C}^\infty(\Sigma)$ along an isometric immersion $f\colon \Sigma^2 \to \mathbb{R} \times_\varrho \mathbb{P}^2$ of a Riemannian surface Σ^2 is defined as

$$h = \pi_{\mathbb{R}} \circ f.$$

Hence, that a submanifold lies inside a slab means that its height function is bounded on both sides.

Let $T \in T\mathbb{R}$ denote a smooth unit vector field fixing an orientation for $\mathbb{R}$ and, simultaneously, its lift to a vector field in TM. Thus $T = \partial/\partial t$ coordinate wise. Thus, the gradient of $\pi_{\mathbb{R}} \in \mathcal{C}^\infty(M)$ is $\bar{\nabla}\pi_{\mathbb{R}} = T$, and the gradient of $h \in \mathcal{C}^\infty(\Sigma)$ is

$$\nabla h = (\bar{\nabla}\pi_{\mathbb{R}})^\top = T - \langle T, N\rangle N, \tag{3}$$

where $\langle\ ,\ \rangle$ also stands for the Riemannian metric on Σ^2, $(\ )^\top$ denotes taking the tangential component of a vector field along the immersion and N is a (local) smooth unit normal vector field.

We use next that for (2) we have that $\bar{\nabla}_T T = 0$ and

$$\bar{\nabla}_Z T = \bar{\nabla}_T Z = T(\log\varrho)Z = \frac{\varrho'}{\varrho}Z = \mathcal{H}Z \tag{4}$$

if $Z \in TM$ is the lift of a vector field $Z \in T\mathbb{P}$, where $\bar{\nabla}$ stands for the Levi-Civita connection in M^3 and, as before, $\mathcal{H} = (\log\varrho)' = \varrho'/\varrho$. For simplicity, we are using the same notation for a vector field in $\mathbb{P}^2$ and its lift to M^3, as well as for functions on $\mathbb{R}$ (i.e., ϱ and ϱ') and their lift to M^3 (i.e., $\varrho\circ\pi_{\mathbb{R}}$ and $\varrho'\circ\pi_{\mathbb{R}}$). Later on we also use that

$$\bar{\nabla}_Z W = \hat{\nabla}_Z W - \mathcal{H}\langle Z, W\rangle T, \tag{5}$$

where now $Z, W \in TM$ are both lifts of fields in $T\mathbb{P}$.

Notice that (4) is tensorial in Z, and thus holds for *any* $Z \in TM$ satisfying $\langle Z, T\rangle = 0$. For every vector field $V \in TM$, we thus have

$$\bar{\nabla}_V T = \bar{\nabla}_{V-\langle V,T\rangle T} T = \mathcal{H}(V - \langle V, T\rangle T). \tag{6}$$

In particular, observe that $Y = \varrho T \in TM$ determines a non-vanishing closed conformal vector field on $\mathbb{R}\times_\varrho \mathbb{P}^2$ (see Remarks 7 below) since

$$\bar{\nabla}_V Y = T(\varrho)V = \varrho' V \quad \text{for any } \ V \in TM.$$

We have from (3) and (6) that

$$\bar{\nabla}_X T = \mathcal{H}(h)(X - \langle X, \nabla h\rangle T) \quad \text{for any } \ X \in T\Sigma.$$

It follows easily that

$$\begin{aligned}\nabla_X(\nabla h) &= (\bar{\nabla}_X(T - \langle T, N\rangle N))^\top\\ &= \mathcal{H}(h)(X - \langle X, \nabla h\rangle\nabla h) + \langle N, T\rangle AX \quad \text{for any } \ X \in T\Sigma,\end{aligned}$$

where ∇ is the Levi-Civita connection in Σ^2 and $A = A_N$ denotes the second fundamental form of f. We conclude that the Laplacian of h is

$$\Delta h = \mathcal{H}(h)(2 - \|\nabla h\|^2) + 2\langle \vec{H}, T\rangle \tag{7}$$

where $\vec{H}$ is the mean curvature vector field of f.

Next observe that any function $\hat{\psi} \in \mathcal{C}^\infty(\mathbb{P})$ defines a function $\bar{\psi} \in \mathcal{C}^\infty(M)$ by

$$\bar{\psi}(t,x) = \hat{\psi}(x).$$

In turn, we may associate to $\hat{\psi} \in \mathcal{C}^\infty(\mathbb{P})$ a function $\psi \in \mathcal{C}^\infty(\Sigma)$ defined by $\psi = \bar{\psi} \circ f$.

Lemma 5. *Along $f \colon \Sigma^2 \to \mathbb{R} \times_\varrho \mathbb{P}^2$ we have that*

$$\bar{\Delta}\bar{\psi} = \Delta\psi - 2(\langle \vec{H}, N\rangle + \mathcal{H}(h)\langle N, T\rangle)\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}} + \hat{\nabla}^2\hat{\psi}(N^*, N^*), \tag{8}$$

where N is a (local) smooth unit normal field and $N^* = \pi_{\mathbb{P}*}(N) = N - \langle N, T\rangle T$.

Proof: Since $\bar{\nabla}\bar{\psi} = \nabla\psi + (\bar{\nabla}\bar{\psi})^\perp$, where $(\ )^\perp$ denotes taking the normal component of a vector field along f, then the Hessians of $\bar{\psi}$ and ψ relate as

$$\bar{\nabla}^2\bar{\psi}(X,X) = \nabla^2\psi(X,X) - \langle A_{(\bar{\nabla}\bar{\psi})^\perp}X, X\rangle$$

where $X \in T\Sigma$. Therefore, along the immersion

$$\bar{\Delta}\bar{\psi} = \Delta\psi - 2\langle \vec{H}, \bar{\nabla}\bar{\psi}\rangle + \bar{\nabla}^2\bar{\psi}(N,N). \tag{9}$$

Observe that $\bar{\nabla}\bar{\psi} = \varrho^{-2}\hat{\nabla}\hat{\psi}$. Moreover, from (5) we get that

$$\bar{\nabla}_{N^*}\hat{\nabla}\hat{\psi} = \hat{\nabla}_{N^*}\hat{\nabla}\hat{\psi} - \mathcal{H}(h)\langle N^*, \hat{\nabla}\hat{\psi}\rangle T$$

and from (4) that $\bar{\nabla}_T\hat{\nabla}\hat{\psi} = \mathcal{H}\hat{\nabla}\hat{\psi}$. We obtain

$$\begin{aligned}
\bar{\nabla}_N\bar{\nabla}\bar{\psi} &= \langle N, T\rangle T(\varrho^{-2})\hat{\nabla}\hat{\psi} + \varrho^{-2}\bar{\nabla}_N\hat{\nabla}\hat{\psi} \\
&= \varrho^{-2}\hat{\nabla}_{N^*}\hat{\nabla}\hat{\psi} - \varrho^{-2}\mathcal{H}\langle N, T\rangle\hat{\nabla}\hat{\psi} - \varrho^{-2}\mathcal{H}\langle N^*, \hat{\nabla}\hat{\psi}\rangle T \\
&= \varrho^{-2}\left(\hat{\nabla}_{N^*}\hat{\nabla}\hat{\psi} - \mathcal{H}\langle N, T\rangle\hat{\nabla}\hat{\psi}\right) - \mathcal{H}\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}}T,
\end{aligned}$$

where $\varrho = \varrho(h)$ and $\mathcal{H} = \mathcal{H}(h)$, and therefore

$$\bar{\nabla}^2\bar{\psi}(N,N) = \hat{\nabla}^2\hat{\psi}(N^*, N^*) - 2\mathcal{H}\langle N, T\rangle\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}}. \tag{10}$$

On the other hand,

$$\langle \vec{H}, \bar{\nabla}\bar{\psi}\rangle = \langle \vec{H}, N\rangle\varrho^{-2}\langle N^*, \hat{\nabla}\hat{\psi}\rangle = \langle \vec{H}, N\rangle\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}}, \tag{11}$$

and (8) follows from (9) using (10) and (11). ∎

Proof of Theorem 2: We claim that Σ^2 is *parabolic* in the sense that it does not admit a non-constant subharmonic function bounded from above. This is clear if Σ^2 is compact. To prove the claim when Σ^2 is noncompact, by a result of Khas'misnkii [**7**] (see also [**5**, Corollary 5.4]) it suffices to show that there exists a function $g \in \mathcal{C}^\infty(\Sigma)$ that is superharmonic outside a compact set and such that $g(q) \to +\infty$ as $q \to \infty$. Here $q \to \infty$ means that q is leaving any compact subset of Σ^2.

Take $\hat\psi = \log \hat r$ where $\hat r(q) = d_{\mathbb{P}}(p_0, q)$. By the Laplacian comparison theorem $\hat\psi$ is superharmonic since

$$\hat\Delta\hat\psi = \hat r^{-1}(\hat\Delta \hat r - \hat r^{-1}) \le 0.$$

From $\bar\Delta\bar\psi = \varrho^{-2}\hat\Delta\hat\psi$ we have that $\bar\psi$ is also superharmonic, and (8) yields

$$\Delta\psi \le 2(\langle \vec H, N\rangle + \mathcal{H}(h)\langle N, T\rangle)\langle N^*, \hat\nabla\hat\psi\rangle_{\mathbb{P}} - \hat\nabla^2\hat\psi(N^*, N^*). \tag{12}$$

Observe that

$$\|N^*\|_{\mathbb{P}} = \varrho^{-1}(h)\|N^*\| = \varrho^{-1}(h)\|\nabla h\|,$$

where $\|\nabla h\|^2 = 1 - \langle T, N\rangle^2 \le 1$. By assumption

$$-\infty < \underline{h} := \inf_\Sigma h \le h \le \bar h := \sup_\Sigma h < +\infty,$$

so that $\inf_\Sigma \varrho(h) = \min_{t\in[\underline{h},\bar h]} \varrho(t) > 0$ and

$$\|N^*\|_{\mathbb{P}} \le \frac{\|\nabla h\|}{\inf_\Sigma \varrho(h)} \le \frac{1}{\inf_\Sigma \varrho(h)}. \tag{13}$$

If $v, w \in T_q\mathbb{P}$ and $\hat r \ge r_0 > 0$, then

$$\begin{aligned}\hat\nabla^2\hat\psi(v,w) &= \langle \hat\nabla_v(\hat r^{-1}\hat\nabla\hat r), w\rangle_{\mathbb{P}}\\ &= \hat r^{-1}\hat\nabla^2\hat r(v,w) - \hat r^{-2}\langle\hat\nabla\hat r, v\rangle_{\mathbb{P}}\langle\hat\nabla\hat r, w\rangle_{\mathbb{P}}.\end{aligned}$$

When $v = w = \hat\nabla\hat r$, we get

$$\hat\nabla^2\psi(v,v) = -\hat r^{-2}.$$

When $v = \tau \perp \hat\nabla\hat r$ of unit length, we have $\hat\nabla^2\hat\psi(\hat\nabla\hat r, \tau) = 0$ and

$$\hat\nabla^2\hat\psi(\tau,\tau) = \hat r^{-1}\hat\nabla^2\hat r(\tau,\tau) = \hat r^{-1}k_g(q).$$

Thus, for any $v \in T_q\mathbb{P}$ we obtain

$$\begin{aligned}\hat\nabla^2\hat\psi(v,v) &= -\hat r^{-2}\langle v, \hat\nabla\hat r\rangle_{\mathbb{P}}^2 + \hat r^{-1}k_g(q)\langle v,\tau\rangle_{\mathbb{P}}^2\\ &\ge -\hat r^{-2}\langle v, \hat\nabla\hat r\rangle_{\mathbb{P}}^2 - c\hat r^{-2}\langle v,\tau\rangle_{\mathbb{P}}^2\\ &\ge -C\hat r^{-2}\|v\|_{\mathbb{P}}^2\end{aligned}$$

where $C = \max\{1, c\}$. In particular, from (13) we conclude that

$$\hat{\nabla}^2\hat{\psi}(N^*, N^*) \geq \frac{-C\|\nabla h\|^2}{r^2(\inf_\Sigma \varrho(h))^2} \geq \frac{-C}{r^2(\inf_\Sigma \varrho(h))^2}, \tag{14}$$

when $r = \hat{r} \circ f$ is larger than r_0. On the other hand, from (1) we see that

$$\begin{aligned}(\langle \vec{H}, N\rangle + \mathcal{H}(h)\langle N, T\rangle)\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}} &\leq (\|\vec{H}\| + \mathcal{H}(h))\|N^*\|_{\mathbb{P}}\|\hat{\nabla}\hat{\psi}\|_{\mathbb{P}} \\ &\leq \frac{\sup_\Sigma \|\vec{H}\| + \mathcal{H}(h)}{r \inf_\Sigma \varrho(h)} \\ &\leq \frac{\inf_\Sigma \mathcal{H}(h) + \sup_\Sigma \mathcal{H}(h)}{r \inf_\Sigma \varrho(h)}, \end{aligned} \tag{15}$$

where $\inf_\Sigma \mathcal{H}(h) = \min_{t \in [\underline{h}, \bar{h}]} \mathcal{H}(t) > 0$ and $\sup_\Sigma \mathcal{H}(h) = \max_{t \in [\underline{h}, \bar{h}]} \mathcal{H}(t) < +\infty$.

Summing up, we conclude from (12) jointly with (14) and (15) that

$$\Delta\psi \leq a\left(\frac{1}{r} + \frac{1}{r^2}\right) \tag{16}$$

for certain positive constant a when r is larger than r_0.

Let $g \in \mathcal{C}^\infty(\Sigma)$ be given by

$$g = \psi - \sigma(h) = \log r - \sigma(h),$$

where $r = \hat{r} \circ f$ and $\sigma(t) = \int^t \varrho(u)du$ satisfies $\sigma'(t) = \varrho(t)$. We have that the subsets $K_j = f^{-1}([\underline{h}, \bar{h}] \times \bar{B}(p_0, j))$ are compact because f is proper. Therefore, since Σ^2 is noncompact, then r satisfies $r(q) \to +\infty$ as $q \to \infty$, and hence the second condition needed to conclude that g is parabolic is satisfied.

On the other hand, from (7) we have

$$\Delta\sigma(h) = 2\varrho(h)\,(\mathcal{H}(h) + \langle \vec{H}, T\rangle). \tag{17}$$

From (1) we get

$$\mathcal{H}(h) + \langle \vec{H}, T\rangle \geq \inf_\Sigma \mathcal{H}(h) - \sup_\Sigma \|\vec{H}\| > 0.$$

Hence,

$$\Delta\sigma(h) \geq 2\inf_\Sigma \varrho(h)\,(\inf_\Sigma \mathcal{H}(h) - \sup_\Sigma \|\vec{H}\|) > 0.$$

Therefore, we obtain from (16) that

$$\Delta g \leq a\left(\frac{1}{r} + \frac{1}{r^2}\right) - 2\inf_\Sigma \varrho(h)\,(\inf_\Sigma \mathcal{H}(h) - \sup_\Sigma \|\vec{H}\|) \leq 0$$

if $r \geq r_1$ for certain $r_1 \geq r_0$. As a consequence, Σ^2 is parabolic.

Once we know that Σ^2 is parabolic, it suffices to observe that $\Delta\sigma(h) > 0$ and that $\sigma(h) \leq \sup_\Sigma \sigma(h) = \sigma(\bar{h})$. This implies that $\sigma(h)$ must be constant and $\Delta\sigma(h) = 0$, which is not possible and concludes the proof of Theorem 2. ∎

Proof of Theorem 4: In view of Theorem 2 it suffices to argue for the case $\|\vec{H}\| = 1$. As in the preceding proof we first show that Σ^2 is parabolic. This is clear if Σ^2 is compact. Assume then that Σ^2 is noncompact. In the present case $\mathcal{H}(t) = 1$, and (12) reduces to

$$\Delta\psi \leq 2(1 + \langle N, T\rangle)\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}} - \hat{\nabla}^2\hat{\psi}(N^*, N^*) \tag{18}$$

where $N = \vec{H}$ is a global unit normal vector field along the immersion. In this case we also have $1 + \langle N, T\rangle \geq 0$, and by (13) that

$$(1 + \langle N, T\rangle)\langle N^*, \hat{\nabla}\hat{\psi}\rangle_{\mathbb{P}} \leq (1 + \langle N, T\rangle)\frac{\|\nabla h\|}{r \inf_\Sigma \varrho(h)}.$$

Using this in (18) and (14) we conclude that

$$\begin{aligned}
\Delta\psi &\leq 2(1 + \langle N, T\rangle)\frac{\|\nabla h\|}{r \inf_\Sigma \varrho(h)} + \frac{C\|\nabla h\|^2}{r^2(\inf_\Sigma \varrho(h))^2} \\
&= (1 + \langle N, T\rangle)\left(\frac{2\|\nabla h\|}{r \inf_\Sigma \varrho(h)} + \frac{C(1 - \langle N, T\rangle)}{r^2(\inf_\Sigma \varrho(h))^2}\right) \\
&\leq a(1 + \langle N, T\rangle)\left(\frac{1}{r} + \frac{1}{r^2}\right)
\end{aligned} \tag{19}$$

for certain positive constant a when r is larger than r_0. On the other hand, in this case $\sigma(h) = e^h$ and (17) becomes

$$\Delta e^h = 2(1 + \langle N, T\rangle)e^h \geq 2(1 + \langle N, T\rangle)e^{\underline{h}} \geq 0. \tag{20}$$

Therefore, we obtain from (19) that

$$\Delta g \leq (1 + \langle N, T\rangle)\left(\frac{a}{r} + \frac{a}{r^2} - 2e^{\underline{h}}\right) \leq 0$$

if $r > r_1$ for certain $r_1 \geq r_0$. Thus, reasoning as in the proof of Theorem 2 we see that Σ^2 is parabolic.

To conclude the proof, we have from (20) that $\Delta e^h \geq 0$. Since $e^h \leq e^{\bar{h}}$ we obtain from the parabolicity of Σ^2 that e^h and hence h must be constant. ∎

Remark 6. The following geometric proof of Theorem 1 was given by Harold Rosenberg (private communication) and observed by the referee. Assume first that $H < 1$. In the half-space model of $\mathbb{H}^3$ allow to grow up an equidistant surface of the same mean curvature pointing up until it touches the surface for the first time. At that point the mean curvature of the surface must point in the same direction; and that is a contradiction to the maximum principle. If $H = 1$ assume that the surface is not an horosphere. A similar argument as before works but now one has to start with the embedded half of a catenoid cousin whose boundary is a small circle contained in a plane fully inside the slab, cf. [**11**]. The catenoid goes down, so it can be taken disjoint with the surface, and its mean curvature points up. If we shrink the circle to a point the compact piece of the catenoid inside the slab converges to the plane that contains the circle. As before, we will have a first point of contact that gives a contradiction.

Remark 7. (i) The presence of a closed conformal vector field gives rise of a warped structure as the ones considered in this paper. See [**9**] for a precise statement of this correspondence and interesting additional information related to this article.
(ii) Strong results on the structure of the asymptotic boundary of properly embedded hypersurfaces in $\mathbb{H}^{n+1}$ with constant mean curvature $H \in [0, 1)$ have been given in [**3**].

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