Hermitian star products are completely positive deformations

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Abstract

Let M be a Poisson manifold equipped with a Hermitian star product. We show that any positive linear functional on $C^\infty(M)$ can be deformed into a positive linear functional with respect to the star product.

1 Introduction

A natural question in deformation quantization is whether "classical" positive linear functionals can be deformed into "quantum" positive linear functionals. In [6] we gave an affirmative answer to this question in the case of deformation quantization of symplectic manifolds; in this paper, we extend this result to arbitrary Poisson manifolds.

More precisely, let M be a smooth manifold, and let $C^\infty(M)$ be the algebra of complex-valued smooth functions on M. A *positive linear functional* on $C^\infty(M)$ is a complex linear functional $\omega_0 : C^\infty(M) \to \mathbb{C}$ satisfying $\omega_0(\overline{f} \cdot f) \geq 0$ for all $f \in C^\infty(M)$. (Such functionals are always given by integration with respect to compactly supported positive Borel measures on M, see e.g. [7, App. B]). In the framework of deformation quantization [1], M is quantized by a

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star product $\star$ on $C^\infty(M)[[\lambda]]$, the algebra of formal power series in a real parameter λ with coefficients in $C^\infty(M)$. We assume in addition that $\overline{f \star g} = \overline{g} \star \overline{f}$ so that $(C^\infty(M)[[\lambda]], \star)$ is an associative algebra over $\mathbb{C}[[\lambda]]$ for which the pointwise complex conjugation of functions is an involution. In order to define "quantum" positive linear functionals, we use the natural notion of "asymptotic positivity" in the ring $\mathbb{R}[[\lambda]]$: if $a = \sum_{r=0}^\infty \lambda^r a_r \in \mathbb{R}[[\lambda]]$, then $a > 0$ if and only if $a_{r_0} > 0$, where a_{r_0} is the first non-zero coefficient of a. Then, as before, a $\mathbb{C}[[\lambda]]$-linear functional $\omega_0 : C^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ is called *positive* if $\omega_0(\overline{f} \star f) \geq 0$ for all f.

If ω_0 is a positive linear functional on $C^\infty(M)$, then its λ-linear extension to $(C^\infty(M)[[\lambda]], \star)$ need not be positive. A concrete example is given when $M = \mathbb{R}^{2n}$, ω_0 is the delta functional at 0, and $\star$ is the Weyl-Moyal star product, see e.g. [5, 6]. The natural question is then whether one can find "quantum corrections" $\omega_i : C^\infty(M) \to \mathbb{C}$ so that $\omega = \omega_0 + \sum_{i=1}^\infty \lambda^i \omega_i$ is a positive linear functional on $(C^\infty(M)[[\lambda]], \star)$. A complete answer to this question is provided by Theorem 2.1, which asserts that this is always possible.

The main ingredient in the proof of Theorem 2.1 is the observation that any star product on $\mathbb{R}^n$ can be realized as a subalgebra of the algebra of functions on the "formal cotangent bundle" of $\mathbb{R}^n$ equipped with the Weyl-Moyal star product, a result that relies on [2, 12]. Using this fact, the proof follows the same steps as the one for symplectic star products in [6, Prop. 5.1].

The importance of positive linear functionals in deformation quantization is illustrated by their central role in the representation theory of star products initiated in [5], see [13] for a recent review. In particular, Theorem 2.1 has direct applications to the theory of strong Morita equivalence of star products, see [8].

The paper is organized as follows: Section 2 contains the basic definitions and the statement of the main theorem (Theorem 2.1); Section 3 contains the main construction underlying its proof; Section 4 completes the proof of theorem.

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2 Basic definitions and the main theorem

Let us recall the general algebraic setting in which positive linear functionals and positive deformations can be defined, see e.g. [6].

Let C be a ring of the form $\mathsf{R}(\mathrm{i})$, where R is an ordered ring and $\mathrm{i}^2 = -1$. Let $\mathcal{A}$ be an algebra over C equipped with an involution *. Using the order structure on R, we define a **positive linear functional** on $\mathcal{A}$ to be a C-linear functional $\omega : \mathcal{A} \to \mathsf{C}$ satisfying $\omega(a^* \cdot a) \geq 0$, for all $a \in \mathcal{A}$.

If $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ is a formal associative deformation of $\mathcal{A}$ in the sense of Gerstenhaber [11], then we call it **Hermitian** if

$$(a_1 \star a_2)^* = a_2^* \star a_1^*, \quad \text{for all } a_1, a_2 \in \mathcal{A}.$$

In this case, the λ-linear extension of the involution * from $\mathcal{A}$ to $\mathcal{A}[[\lambda]]$ makes $\boldsymbol{\mathcal{A}}$ into a *-algebra over $\mathsf{C}[[\lambda]]$. Since $\mathsf{R}[[\lambda]]$ has an order structure induced from that of R (analogous to the one in $\mathbb{R}[[\lambda]]$ discussed in the introduction) and $\mathsf{C}[[\lambda]] = \mathsf{R}[[\lambda]](\mathrm{i})$, the definition of positive linear functionals makes sense for $\boldsymbol{\mathcal{A}}$ as well.

Note that if

$$\omega = \sum_{r=0}^\infty \lambda^r \omega_r : \mathcal{A}[[\lambda]] \longrightarrow \mathsf{C}[[\lambda]]$$

is a positive $\mathsf{C}[[\lambda]]$-linear functional with respect to $\star$, then its classical limit $\omega_0 : \mathcal{A} \longrightarrow \mathsf{C}$ is a positive C-linear functional on $\mathcal{A}$. Conversely, we say that a Hermitian deformation $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ is **positive** [6, Def. 4.1] if for every positive linear functional ω_0 of $\mathcal{A}$ one can find C-linear functionals

$$\omega_r : \mathcal{A} \to \mathsf{C}, \qquad r = 1, 2, \dots,$$

so that $\omega_0 + \sum_{r=1}^{\infty} \lambda^r \omega_r$ is a positive linear functional of $\boldsymbol{\mathcal{A}}$. We say that $\boldsymbol{\mathcal{A}}$ is a **completely positive** deformation if, for each $n \in \mathbb{N}$, the *-algebra $M_n(\boldsymbol{\mathcal{A}})$ is a positive deformation of $M_n(\mathcal{A})$.

A simple example illustrates that not all Hermitian deformations are positive. Let $\mathcal{A}$ be a *-algebra over C, and let $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ denote the multiplication map. If we view $\mathcal{A}$ as an algebra equipped with the *zero* multiplication, then it is a *-algebra for which all linear functionals are positive. Then $\lambda\mu$ provides a Hermitian deformation, which is clearly not positive in general.

In this paper, we are concerned with algebraic deformations arising in the geometric context of deformation quantization: If $(M, \{\cdot, \cdot\})$ is a Poisson manifold, then a **star product** [1] on M is a formal associative deformation $\star$ of the complex algebra $C^\infty(M)$,

$$f \star g = f \cdot g + \sum_{r=1}^{\infty} \lambda^r C_r(f, g),$$

for which each C_r is a bidifferential operator on M and

$$C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}.$$

We call the star product **Hermitian** if $\overline{f \star g} = \overline{g} \star \overline{f}$.

The following is the main result of this paper.

Theorem 2.1 *Any Hermitian star product on a Poisson manifold is a completely positive deformation.*

The proof of Theorem 2.1 will be presented in Section 4.

A key ingredient in the proof is the fact that any star product on $\mathbb{R}^n$ can be realized as a subalgebra of the familiar Weyl-Moyal star product on the “formal cotangent bundle” of $\mathbb{R}^n$. More precisely, let us consider $\mathcal{W} = \mathcal{W}_0[[\lambda]]$, where $\mathcal{W}_0 = C^\infty(\mathbb{R}^n)[[p_1, \dots, p_n]]$, equipped with the formal Weyl-Moyal star product $\star_{\mathrm{Weyl}}$ defined by

$$a \star_{\mathrm{Weyl}} b = \mu \circ \mathrm{e}^{\frac{\mathrm{i}\lambda}{2} \sum_{k=1}^n \left(\partial_{q^k} \otimes \partial_{p_k} - \partial_{p_k} \otimes \partial_{q^k}\right)} a \otimes b, \qquad a, b \in \mathcal{W}. \tag{2.1}$$

Here $\mu(a \otimes b) = ab$ is the undeformed commutative product of $\mathcal{W}$ and $q^1, \dots, q^n$ are the canonical coordinates on $\mathbb{R}^n$. With respect to $\star_{\mathrm{Weyl}}$, $\mathcal{W}$ is an associative *-algebra over $\mathbb{C}[[\lambda]]$ with involution given by complex conjugation. (We treat the formal parameters as real.)

Let $\pi^* : C^\infty(\mathbb{R}^n) \longrightarrow \mathcal{W}_0$ be the natural inclusion, which is clearly an algebra homomorphism (thought of as dual to the projection π of the “formal cotangent bundle” of $\mathbb{R}^n$ onto $\mathbb{R}^n$). Since $\star_{\mathrm{Weyl}}$ is *homogeneous* in the sense that the degree map

$$\deg = \sum_i p_i \frac{\partial}{\partial p_i} + \lambda \frac{\partial}{\partial \lambda} \tag{2.2}$$

is a derivation of $\star_{\mathrm{Weyl}}$ [3], it follows in particular that the λ-linear extension $\pi^* : C^\infty(\mathbb{R}^n)[[\lambda]] \longrightarrow \mathcal{W}$ satisfies

$$\pi^*(fg) = \pi^*(f) \star_{\mathrm{Weyl}} \pi^*(g).$$

If $\star$ is a star product on $\mathbb{R}^n$ quantizing an arbitrary Poisson structure $\{\cdot,\cdot\}$, then the claim is that π^* can be "deformed" into a $\mathbb{C}[[\lambda]]$-linear algebra homomorphism

$$\tau : (C^\infty(\mathbb{R}^n)[[\lambda]], \star) \longrightarrow (\mathcal{W}, \star_{\mathrm{Weyl}}) \,. \tag{2.3}$$

Remark 2.2 The *classical limit* $\mathrm{cl}(\tau)$ of τ (defined by setting λ to zero) gives an injective homomorphism of Poisson algebras

$$\mathrm{cl}(\tau) : (C^\infty(\mathbb{R}^n), \{\cdot,\cdot\}) \longrightarrow (\mathcal{W}_0, \{\cdot,\cdot\}_{can}) \,, \tag{2.4}$$

where $\{\cdot,\cdot\}_{can}$ is the canonical Poisson bracket on $\mathcal{W}_0$. Since $\mathcal{W}_0$ can be thought of as the algebra of functions on the "formal cotangent bundle" or $\mathbb{R}^n$, we think of (2.4) as a "formal symplectic realization" of $(\mathbb{R}^n, \{\cdot,\cdot\})$, and of τ as its quantization.

3 Constructing the homomorphism τ

The fact that there are no cohomological obstructions for the recursive construction of the map τ as in (2.3) relies on the joint work of Martin Bordemann, Nikolai Neumaier, Claus Nowak and Stefan Waldmann [2], see also the thesis [12]. We will outline the main ideas here.

Regarding $\mathcal{W}_0 = C^\infty(\mathbb{R}^n)[[p_1,\dots,p_n]]$ as a $C^\infty(\mathbb{R}^n)$-bimodule via left and right multiplication with respect to the usual pointwise product, we consider the *differential (resp. continuous)* Hochschild cohomology of $C^\infty(\mathbb{R}^n)$ with values in $\mathcal{W}_0$, denoted by $\mathrm{HH}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0)$, $k \geq 0$. This is the cohomology of the complex

$$\left(\oplus_{k=0}^\infty \mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0), \delta_0\right),$$

where $\mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0)$ is the space of k-multilinear maps from $C^\infty(\mathbb{R}^n) \times \dots \times C^\infty(\mathbb{R}^n)$ (k times) into $\mathcal{W}_0$ which are differential operators on each argument (resp. continuous with respect to the Fréchet structure of $C^\infty(M)$), and δ_0 is the Hochschild coboundary operator, see e.g. [11].

Similarly, we regard $\mathcal{W} = \mathcal{W}_0[[\lambda]]$ as a bimodule over $C^\infty(M)$ via left and right multiplication with respect to $\star_{\mathrm{Weyl}}$. We denote the corresponding (differential, resp. continuous) Hochschild cochains by $\mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W})$, the Hochschild coboundary operator by δ, and the Hochschild cohomology of $C^\infty(M)$ with values in $\mathcal{W}$ by $\mathrm{HH}^k(C^\infty(\mathbb{R}^n), \mathcal{W})$. The continuous cohomology (which turns out to coincide with the differential one) can be explicitly described if one uses the (topological) Koszul resolution of $C^\infty(\mathbb{R}^n)$, see e.g. [10], and observes that the associated complex can be identified with

$$\left(\oplus_{k=0}^\infty \mathcal{W} \otimes \wedge^k (\mathbb{R}^n)^*, \mathrm{i}\lambda d_p\right),$$

where we view elements in $\mathcal{W} \otimes \bigwedge^k(\mathbb{R}^n)^*$ as k-forms on $\mathbb{R}^{2n}$ of type $\omega^{i_1\cdots i_k}(q,p) dp_{i_1} \wedge \dots \wedge dp_{i_k}$, with $\omega^{i_1\cdots i_k}(q,p) \in \mathcal{W}$, and d_p is the exterior derivative d with respect to the $p_1,\dots,p_n$ variables. An application of the Poincaré's lemma shows the next result.

Proposition 3.1 *We have the following isomorphisms of* $\mathbb{C}[[\lambda]]$*-modules:*

$$\mathrm{HH}^0(C^\infty(\mathbb{R}^n), \mathcal{W}) \cong C^\infty(\mathbb{R}^n)[[\lambda]], \tag{3.1}$$

$$\mathrm{HH}^k(C^\infty(\mathbb{R}^n), \mathcal{W}) \cong (\mathcal{W}_0 \otimes \textstyle\bigwedge^k(\mathbb{R}^n)^*)_{\mathrm{closed}} \quad \text{for} \quad k \geq 1. \tag{3.2}$$

Here $(\mathcal{W}_0 \otimes \bigwedge^k(\mathbb{R}^n)^*)_{\mathrm{closed}}$ denotes the set of elements in $\mathcal{W} \otimes \bigwedge^k(\mathbb{R}^n)^*$ which do not depend on λ and lie in the kernel of d_p. The fact that continuous and differential cohomologies coincide follows from techniques similar to [9], see also [12].

Observe that, due to the additional $\mathrm{i}\lambda$ multiplying d_p, the cohomology is *not* trivial (it would be zero if we used formal Laurent series in λ instead). However, (3.2) has the following consequence.

Corollary 3.2 *If ϕ is a differential Hochschild k-cocycle with $k \geq 1$, then $\lambda\phi$ must be a coboundary.*

Let $\mathrm{cl} : \mathcal{W} \longrightarrow \mathcal{W}_0$ denote the classical limit map (setting λ equal to zero), which is a bimodule homomorphism with respect to the $C^\infty(\mathbb{R}^n)$-bimodule structures. We keep the same notation for the induced maps

$$\mathrm{cl} : \mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}) \longrightarrow \mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0). \tag{3.3}$$

Let Alt denote the antisymmetrization operator on Hochschild cochains.

Lemma 3.3 *Let $\phi \in \mathrm{HC}^{k\geq 1}(C^\infty(\mathbb{R}^n), \mathcal{W})$ be a cocycle with $\mathrm{Alt}(\mathrm{cl}(\phi)) = 0$. Then ϕ is a coboundary.*

PROOF: Since $\mathrm{cl} : \mathcal{W} \longrightarrow \mathcal{W}_0$ is a bimodule homomorphism, it follows that the maps (3.3) satisfy

$$\mathrm{cl} \circ \delta = \delta_0 \circ \mathrm{cl}. \tag{3.4}$$

As a result, if $\phi \in \mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W})$ is a cocycle, i.e., $\delta\phi = 0$, then $\mathrm{cl}(\phi) \in \mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0)$ satisfies $\delta_0 \mathrm{cl}(\phi) = 0$. Since $\mathcal{W}_0 = C^\infty(\mathbb{R}^n) \otimes \mathbb{C}[[p_1, \ldots, p_n]]$, treating $p_1, \ldots, p_n$ as formal parameters we note that the usual Hochschild-Kostant-Rosenberg theorem for $C^\infty(\mathbb{R}^n)$ implies that, for $k \geq 1$, any cocycle in $\mathrm{HC}^k(C^\infty(\mathbb{R}^n), \mathcal{W}_0)$ is cohomologous to its skew symmetric part. Hence there exists a cochain $\psi \in \mathrm{HC}^{k-1}(C^\infty(\mathbb{R}^n), \mathcal{W}_0)$ such that

$$\mathrm{cl}(\phi) = \mathrm{Alt}(\mathrm{cl}(\phi)) + \delta_0\psi. \tag{3.5}$$

It follows that if $\mathrm{Alt}(\mathrm{cl}(\phi)) = 0$, then $\mathrm{cl}(\phi) = \delta_0\psi$. Viewing ψ as cochain with values in $\mathcal{W}$, we have that $\mathrm{cl}(\phi - \delta\psi) = 0$. Thus the cocycle $\phi - \delta\psi$ has the form $\lambda\eta$ for some other cocycle η. It follows from Corollary 3.2 that ϕ is a coboundary. □

Theorem 3.4 *Let $\star$ be a star product on $\mathbb{R}^n$. For each $k \geq 1$, there exists a $\tau_k \in \mathrm{HC}^1(C^\infty(\mathbb{R}^n), \mathcal{W})$, homogeneous of degree k with respect to* deg*, so that*

$$\tau = \pi^* + \sum_{k=1}^{\infty} \tau_k : (C^\infty(\mathbb{R}^n)[[\lambda]], \star) \longrightarrow (\mathcal{W}, \star_{\mathrm{Weyl}})$$

is an injective $\mathbb{C}[[\lambda]]$-linear algebra homomorphism.

Furthermore, if $\star$ is a Hermitian star product, then one can chose τ to be a *-homomorphism, i.e., $\tau(\overline{f}) = \overline{\tau(f)}$.

PROOF: For a sequence $\tau_i \in \mathrm{HC}^1(C^\infty(\mathbb{R}^n), \mathcal{W})$, $i = 0, \ldots, k$, with each τ_i homogeneous of degree i with respect to deg, we define $\tau^{(k)} = \sum_{i=0}^{k} \tau_i$ and consider the error $\epsilon^{(k)} \in \mathrm{HC}^2(C^\infty(\mathbb{R}^n), \mathcal{W})$,

$$\epsilon^{(k)}(f, g) = \tau^{(k)}(f \star g) - \tau^{(k)}(f) \star_{\mathrm{Weyl}} \tau^{(k)}(g) \quad \text{for} \quad f, g \in C^\infty(\mathbb{R}^n). \tag{3.6}$$

We write $\epsilon^{(k)} = \sum_{i=0}^{\infty} \epsilon_i^{(k)}$, with $\epsilon_i^{(k)}$ homogeneous of degree i with respect to deg.

We now construct the desired τ recursively. If $\tau_0 = \pi^*$, then $\epsilon_0^{(0)} = 0$ since π^* is a homomorphism for the undeformed products. Suppose that we have found $\tau_0, \ldots, \tau_{k-1}$ such that $\epsilon_0^{(k-1)} = \cdots = \epsilon_{k-1}^{(k-1)} = 0$. Our goal is to find τ_k, homogeneous of degree k, such that the error $\epsilon^{(k)}$ vanishes up to degree k. Since $\epsilon_i^{(k)} = \epsilon_i^{(k-1)}$ for $i = 0, \ldots, k-1$, we just have to impose the condition $\epsilon_k^{(k)} = 0$. A direct computation shows that

$$\epsilon_k^{(k)}(f,g) = \sum_{i=0}^{k} \lambda^{k-i} \tau_i(C_{k-i}(f,g)) - \sum_{i=1}^{k} \tau_i(f) \star_{\mathrm{Weyl}} \tau_{k-i}(g) = (\delta\tau_k)(f,g) + R_k(f,g), \tag{3.7}$$

where R_k depends only on τ_i, $i = 0, \ldots, k-1$. Explicitly,

$$R_k(f,g) = \sum_{i=0}^{k-1} \lambda^{k-i} \tau_i(C_{k-i}(f,g)) - \sum_{i=1}^{k-1} \tau_i(f) \star_{\mathrm{Weyl}} \tau_{k-i}(g). \tag{3.8}$$

Here C_r is the r-th cochain of the star product $\star$. Hence we are left with showing that there exists τ_k of degree k satisfying the cohomological equation

$$(\delta\tau_k)(f,g) + R_k(f,g) = 0, \tag{3.9}$$

i.e., that R_k is a coboundary. Note that, by (3.6),

$$\delta\epsilon^{(k)}(f,g,h) = f \star_{\mathrm{Weyl}} \epsilon^{(k)}(g,h) - \epsilon^{(k)}(f \star g, h) + \epsilon^{(k)}(f, g \star h) - \epsilon^{(k)}(f,g) \star_{\mathrm{Weyl}} h.$$

Since δ and deg commute and $\epsilon_i^{(k)} = 0$ for $i = 0, \ldots, k-1$, we have

$$\delta\epsilon_k^{(k)}(f,g,h) = f \star_{\mathrm{Weyl}} \epsilon_k^{(k)}(g,h) - \epsilon_k^{(k)}(f \star g, h) + \epsilon_k^{(k)}(f, g \star h) - \epsilon_k^{(k)}(f,g) \star_{\mathrm{Weyl}} h.$$

Using (3.6) and the associativity of $\star$ and $\star_{\mathrm{Weyl}}$, it is simple to check that

$$\epsilon^{(k)}(f \star g, h) - \epsilon^{(k)}(f, g \star h) = \tau^{(k)}(f) \star_{\mathrm{Weyl}} \epsilon^{(k)}(g,h) - \epsilon^{(k)}(f,g) \star_{\mathrm{Weyl}} \tau^{(k)}(h),$$

which in degree k directly implies that $\delta\epsilon_k^{(k)} = 0$. Hence, by (3.7), $\delta R_k = 0$. Now a simple computation shows that $\mathrm{cl}(R_k)$ is symmetric, so Lemma 3.3 implies that R_k is indeed exact. So we can find τ_k solving (3.9), and τ_k is homogeneous of degree k because so is R_k.

For the last part of the theorem, we must check that, if $\star$ is Hermitian, then one can choose τ_k satisfying $\tau_k(\overline{f}) = \overline{\tau_k(f)}$. Recall [6, Sec. 3] that the complex conjugation on $C^\infty(\mathbb{R}^n)$ and $\mathcal{W}$ induce an involution on Hochschild cochains $\phi \in \mathrm{HC}^r(C^\infty(\mathbb{R}^n), \mathcal{W})$ by

$$\phi^*(f_0, \ldots, f_r) := \overline{\phi(\overline{f_r}, \ldots, \overline{f_0})},$$

and $(\delta\phi)^* = (-1)^{r+1}\delta\phi^*$. If $\star$ is Hermitian and $\tau_i^* = \tau_i$ for $i = 1, \ldots, k-1$, then a direct computation shows that R_k defined in (3.8) is Hermitian, i.e., $R_k^* = R_k$. If we pick τ_k with $\delta\tau_k = -R_k$, then $(\delta\tau_k)^* = \delta\tau_k^* = -R_k^* = -R_k$. It follows that

$$\delta(\frac{1}{2}(\tau_k + \tau_k^*)) = -R_k,$$

so we can replace τ_k by its Hermitian part and assume that $\tau_k^* = \tau_k$. $\square$

Remark 3.5 The results in this section hold for any manifold M. This is obtained if one replaces $\mathcal{W}$ by the functions on T^*M depending formally on the 'momentum variables' and $\star_{\mathrm{Weyl}}$ by any homogeneous star product on T^*M, see e.g. [3].

4 Proof of the main theorem

Let us consider complex coordinates $z^k = q^k + \mathrm{i}p_k$ and $\overline{z}^k = q^k - \mathrm{i}p_k$ on $\mathbb{R}^{2n}$, and the derivations $\frac{\partial}{\partial z^k}$ and $\frac{\partial}{\partial \overline{z}^k}$ of $\mathcal{W}_0[[\lambda]]$. For $A, B \in M_N(\mathcal{W}_0)[[\lambda]]$, we consider the *Wick star product*

$$A \star_{\mathrm{Wick}} B = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{i_1,\ldots,i_r} \frac{\partial^r A}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r B}{\partial \overline{z}^{i_1} \cdots \partial \overline{z}^{i_r}}. \tag{4.1}$$

We recall a general observation from [6, Sec. 4].

Lemma 4.1 *If ω_0 is a positive linear functional of $M_N(\mathcal{W}_0)$, then its λ-linear extension to $M_N(\mathcal{W}_0)[[\lambda]]$ is automatically positive with respect to $\star_{\mathrm{Wick}}$.*

We can now prove Theorem 2.1.

Proof: We first deal with the local case. Consider $\mathbb{R}^n$ (or any contractible open subset of it) equipped with an arbitrary Poisson structure, and let $\star$ be a Hermitian star product. Let us consider the formal Weyl algebra $(\mathcal{W} = \mathcal{W}_0[[\lambda]], \star_{\mathrm{Weyl}})$. By Theorem 3.4, we have an injective map $\tau : C^\infty(\mathbb{R}^n)[[\lambda]] \longrightarrow \mathcal{W}$, where $\tau = \sum_{k=0}^{\infty} \tau_k$, $\tau_0 = \pi^* : C^\infty(\mathbb{R}^n) \longrightarrow \mathcal{W}_0$ is the canonical inclusion, each τ_k is homogeneous of degree k, and

$$\tau(f \star g) = \tau(f) \star_{\mathrm{Weyl}} \tau(g) \quad \text{and} \quad \tau(\overline{f}) = \overline{\tau(f)}. \tag{4.2}$$

It is clear that τ extends to a *-homomorphism for the matrix algebras,

$$\tau : M_N(C^\infty(\mathbb{R}^n)[[\lambda]]) \longrightarrow M_N(\mathcal{W}).$$

Using the complex coordinates $z^k = q^k + \mathrm{i}p_k$ and $\overline{z}^k = q^k - \mathrm{i}p_k$, we recall that the operator

$$S = \mathrm{e}^{\lambda\Delta} \quad \text{with} \quad \Delta = \sum\nolimits_k \frac{\partial^2}{\partial z^k \partial \overline{z}^k} \tag{4.3}$$

is an invertible $\mathbb{C}[[\lambda]]$-linear endomorphism of $M_N(\mathcal{W}_0)[[\lambda]]$ which is a *-equivalence between $\star_{\mathrm{Wick}}$ and $\star_{\mathrm{Weyl}}$ [4], i.e.,

$$S(A^*) = S(A)^* \quad \text{and} \quad S(A \star_{\mathrm{Wick}} B) = SA \star_{\mathrm{Weyl}} SB, \quad \text{for } A, B \in M_N(\mathcal{W}_0)[[\lambda]]. \tag{4.4}$$

Let $\Omega_0 : M_N(C^\infty(\mathbb{R}^n)) \longrightarrow \mathbb{C}$ be a positive linear functional. The canonical inclusion $\iota : \mathbb{R}^n \longrightarrow T^*\mathbb{R}^n$ leads, at the algebra level, to a map $\iota^* : \mathcal{W}_0 \longrightarrow C^\infty(\mathbb{R}^n)$ (just setting the 'momentum variables' $p_1, \ldots, p_n$ to zero) which is a *-homomorphism for the undeformed products and satisfies $\iota^*\pi^* = \mathsf{id}$. It follows that $\Omega_0 \circ \iota^* : M_N(\mathcal{W}_0) \longrightarrow \mathbb{C}$ is also a positive linear functional. Hence, by Lemma 4.1,

$$\Omega_0 \circ \iota^* : M_N(\mathcal{W}_0)[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]$$

is a positive linear functional with respect to $\star_{\mathrm{Wick}}$, and by (4.3) we see that $\Omega_0 \circ \iota^* \circ S^{-1}$ is positive with respect to $\star_{\mathrm{Weyl}}$. Finally, by (4.2), the functional

$$\Omega = \Omega_0 \circ \iota^* \circ S^{-1} \circ \tau : C^\infty(\mathbb{R}^n)[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]] \tag{4.5}$$

is positive with respect to $\star$. Since $\iota^*\pi^* = \mathsf{id}$, it is easy to see that Ω is actually a deformation of Ω_0. Therefore $\star$ is a completely positive deformation. Note that these results also hold for any contractible open subeset of $\mathbb{R}^n$.

For the global result, we proceed just as in the symplectic case [6, Prop. 5.1]. For a given Poisson manifold M equipped with a Hermitian star product $\star$, we consider a locally finite open cover of M by contractible open sets $\{\mathcal{O}_\alpha\}$ subordinate to a quadratic partition of unity $\{\chi_\alpha\}$ with $\sum_\alpha \overline{\chi_\alpha}\chi_\alpha = 1$. If Ω_0 is a positive linear functional on $M_N(C^\infty(M))$, then, on each $\mathcal{O}_\alpha$, we have a deformation Ω_α of the restriciton of Ω_0 to $\mathcal{O}_\alpha$. Then

$$\Omega(f) = \sum_\alpha \Omega_\alpha(\overline{\chi_\alpha} \star f \star \chi_\alpha)$$

defines a positive linear functional on $(C^\infty(M)[[\lambda]], \star)$ which is a global deformation of Ω_0. $\square$

Remark 4.2 One can directly prove Theorem 2.1 globally for M by using Remark 3.5 together with a Wick star product defined via an almost complex structure on the "formal cotangent bundle" T^*M [4].

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