GAUSSIAN ESTIMATES FOR SYMMETRIC SIMPLE EXCLUSION PROCESSES.

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Abstract. We prove Gaussian tail estimates for the transition probability of n particles evolving as symmetric exclusion processes on $\mathbb{Z}^d$, improving results obtained in [9]. We derive from this result a non-equilibrium Boltzmann-Gibbs principle for the symmetric simple exclusion process in dimension 1 starting from a product measure with slowly varying parameter.

1. Introduction

To derive sharp bounds on the rate of convergence to equilibrium is one of the main questions in the theory of Markov processes. In the last decade, this problem has attracted many attention in the context of conservative interacting particle systems in infinite volume. Fine estimates of the spectral gap of reversible generators restricted to finite cubes and logarithmic Sobolev inequalities have been obtained. We refer to [9] for the recent literature on the subject. From these bounds on the ergodic constants, polynomial decay to equilibrium in L^2 has been proved for some processes. For instance, Bertini and Zegarlinski [1], [2] proved that the symmetric simple exclusion process in $\mathbb{Z}^d$ converges to equilibrium in L^2 at rate $t^{-d/2}$. For a class of functions f that includes the cylinder functions, there exists $V(f)$ finite such that

$$\|P_t f - < f >_\alpha \|_2^2 \;\le\; \frac{V(f)}{(1+t)^{d/2}}$$

for all $t \ge 0$. Here P_t stands for the semi–group, $< f >_\alpha$ for the expectation of f with respect to the Bernoulli product measure with density α and $\|f\|_2$ for the L^2 norm of f. Janvresse, Landim, Quastel and Yau [5] and Landim and Yau [10] extended the algebraic decay in L^2 for zero range and Ginzburg-Landau dynamics.

We refine in this article the Gaussian upper bounds obtained in [9] for the transition probabilities of finite symmetric simple exclusion processes evolving on the lattice $\mathbb{Z}^d$. Our approach is based on a logarithmic Sobolev inequality and on Davies [4] method to derive estimates for heat kernels.

Consider $n \ge 2$ indistinguishable particles moving on the d-dimensional lattice $\mathbb{Z}^d$ as symmetric random walks with an exclusion rule which prevents more than one particle per site. The dynamics can be informally described as follows. There are initially n particles on n distinct sites of $\mathbb{Z}^d$. Each particle waits a mean one exponential time at the end of which, being at x, it chooses a site y with probability $p(y-x)$, for some finite range, irreducible, symmetric transition probability $p(\cdot)$.

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If the site is vacant, the particle jumps, otherwise it stays where it is and waits a new mean one exponential time.

The state space of this Markov process, denoted by $\mathcal{E}_n$, is the collection of all subsets A of $\mathbb{Z}^d$ with cardinality n:

$$\mathcal{E}_n \;=\; \{A \subset \mathbb{Z}^d , |A| = n\} \;;$$

while its generator $\mathcal{L}_n$ is given by

$$(\mathcal{L}_n f)(A) \;=\; \sum_{x,y\in\mathbb{Z}^d} p(y-x)[f(A_{x,y}) - f(A)] \;, \tag{1.1}$$

where $A_{x,y}$ stands for the set A with sites x, y exchanged:

$$A_{x,y} \;=\; \begin{cases} (A\setminus\{x\})\cup\{y\} & \text{if } x\in A,\, y\notin A,\\ (A\setminus\{y\})\cup\{x\} & \text{if } y\in A,\, x\notin A,\\ \qquad A & \text{otherwise.}\end{cases}$$

In formula (1.1) summation is carried over all bonds $\{y,z\}$ to avoid counting twice the contribution of the same jump.

It is easy to check that the counting measure on $\mathcal{E}_n$, denoted by ν_n $(\nu_n(A) = 1$ for every A in $\mathcal{E}_n)$, is an invariant, reversible measure for the process.

Fix a set A_0 in $\mathcal{E}_n$ and denote by $f_t = f_t^{A_0}$ the solution of the forward equation with initial data δ_{A_0} :

$$\begin{cases} \partial_t f_t = \mathcal{L}_n f_t \;, \\ f_0(A) = \mathbf{1}\{A = A_0\} \;. \end{cases} \tag{1.2}$$

The main result of the article provides a Gaussian estimate for the transition probability f_t. Denote by $\mathbf{x} = (x_1,\dots,x_n)$ the sites of $(\mathbb{Z}^d)^n$. For a configuration $\mathbf{x}$, let $\mathbf{x}_{i,j}$ be the j-th coordinate of the i-th point of $\mathbf{x}$: $\mathbf{x}_{i,j} = x_i\cdot e_j$, where $\cdot$ stands for the inner product in $\mathbb{R}^d$ and $\{e_1,\dots,e_d\}$ for the canonical basis of $\mathbb{R}^d$. The Euclidean norm of $(\mathbb{R}^d)^n$ is denoted by $\|\mathbf{x}\|$ so that $\|\mathbf{x}\|^2 = \sum_{i,j} x_{i,j}^2$.

Denote by Φ the Legendre transform of the convex function $w^2\cosh w$:

$$\Phi(u) \;=\; \sup_{w\in\mathbb{R}} \left\{uw - w^2\cosh w\right\} \;.$$

An elementary computations shows that $\Phi(w)\sim w^2$ for w small and $\Phi(w)\sim w\log w$ for w large.

Theorem 1.1. *Fix a set $A_0 = \{z_1,\dots,z_n\}$ in $\mathcal{E}_n$. Let f_t be the solution of the forward equation (1.2). There exist finite constants $C_2 = C_2(n,d,p)$, $a_0 = a_0(p)$ such that*

$$f_T(A) \;\le\; \sum_\sigma \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ -\frac{a_0 T}{2(\log T)^2}\, \Phi\Big(\frac{\|\mathbf{x}_\sigma - \mathbf{z}\|\log T}{a_0^2 T}\Big)\Big\}$$

for every $T > C_2$ and every set $A = \{x_1,\dots,x_n\}$. In this formula, summations is performed over all permutations σ of n points and $\mathbf{x}_\sigma$ stands for the vector $(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

The asymptotic behavior of $\Phi(\cdot)$ at the origin shows that for every $\gamma>0$, there exists a constant $a_1 = a_1(p,\gamma)$ such that

$$f_T(A) \;\le\; \sum_\sigma \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ -\frac{\|\mathbf{x}_\sigma - \mathbf{z}\|^2}{a_1 T}\Big\} \tag{1.3}$$

for all $T > C_2$ and all sets A such that $\|\mathbf{x}_\sigma - \mathbf{z}\| \le \gamma T/\log T$ for all permutations σ. Furthermore, since

$$\Phi\Big(\frac{\|\mathbf{x}_\sigma - \mathbf{z}\|\log T}{a_0^2 T}\Big) \;\ge\; \frac{1}{n}\sum_{i=1}^n \Phi\Big(\frac{\|\mathbf{x}_{\sigma(i)} - \mathbf{z}_i\|\log T}{a_0^2 T}\Big)$$

we have that

$$f_T(A) \;\le\; \sum_\sigma \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ - \frac{a_0 T}{2n(\log T)^2} \sum_{i=1}^n \Phi\Big(\frac{\|\mathbf{x}_{\sigma(i)} - \mathbf{z}_i\|\log T}{a_0^2 T}\Big)\Big\}\;.$$

For a fixed $\gamma > 0$, in last formula we may replace $\Phi(w)$ by $C(\gamma)w^2$ if $\|\mathbf{x}_{\sigma(i)} - \mathbf{z}_i\| \le a_0^2 T\gamma/\log T$ and $\Phi(w)$ by $C(\gamma) w\log w$ otherwise.

2. Boltzmann-Gibbs principle

We prove in this section the Boltzmann-Gibbs principle for the symmetric simple exclusion process out of equilibrium in dimension 1. This result allows the replacement of average of local functions by functions of the empirical density in the fluctuation regime and is the main point in the proof of a central limit theorem around the hydrodynamical limit for interacting particle systems (cf. [6], Chap. 11).

The Boltzmann-Gibbs principle for one-dimensional processes out of equilibrium was proved in [3] through the logarithmic Sobolev inequality. A different version, involving microscopic time integrals, is presented in [8] and uses sharp estimates on the comparison between independent random walks and the symmetric exclusion process.

Fix a profile $\rho_0 : \mathbb{R}^d \to [0,1]$ in $C^1(\mathbb{R}^d)$ with a bounded derivative and consider a sequence of product measures $\{\nu^N_{\rho_0(\cdot)},\, N\ge 1\}$ on $\{0,1\}^{\mathbb{Z}^d}$ associated to this profile so that

$$E_{\nu^N_{\rho_0(\cdot)}}[\eta(x)] \;=\; \rho_0(x/N)\;.$$

Denote by $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ the probability measure on the path space $D(\mathbb{R}_+, \{0,1\}^{\mathbb{Z}^d})$ corresponding to the symmetric simple exclusion process starting from $\nu^N_{\rho_0(\cdot)}$ and speeded up by N^2. Expectation with respect to $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ is denoted by $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}$.

For x in $\mathbb{Z}^d$, let

$$\rho^N(t,x) \;=\; \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\eta_t(x)]\;.$$

Of course, $\rho^N(t,x)$ is the solution of the linear equation

$$\begin{cases} \partial_t \rho^N(t,x) = N^2 \displaystyle\sum_{y\in\mathbb{Z}^d} p(y-x)[\rho^N(t,y) - \rho^N(t,x)]\;, \\ \rho^N(0,x) = \rho_0(x/N)\;. \end{cases}$$

This equation can be written as $\partial_t\rho^N(t,x) = N^2\mathcal{L}_1\rho^N(t,x)$, where $\mathcal{L}_1$ is the generator introduced in (1.1). Next proposition is the main result of this section.

Proposition 2.1. *Let $d=1$ and fix $T>0$, a finite subset A of $\mathbb{Z}$ such that $|A|>2$ and a continuous function H in $L^1(\mathbb{R})$. Then,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\Big(\int_0^T dt\, \frac{1}{N^{1/2}} \sum_{x\in\mathbb{Z}} H(x/N) \prod_{z\in A}[\eta_t(x+z) - \rho^N(t,x+z)]\Big)^2\Big] \;=\; 0\;.$$

The proof of this proposition is presented at the end of this section. The Boltzmann-Gibbs principle is a simple consequence but requires some extra notation.

For a finite subset A of $\mathbb{Z}$ and $0 \le \alpha \le 1$, let

$$\Psi(A, \alpha) \;=\; \prod_{x \in A} [\eta(x) - \alpha] \,.$$

By convention, we set $\Psi(\phi, \alpha) = 1$. Each cylinder function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ can be written as

$$f(\eta) \;=\; \sum_{A \in \mathcal{E}} \mathfrak{f}(A, \alpha) \Psi(A, \alpha) \,.$$

A straightforward computation shows that for each finite set A, $\mathfrak{f}(A, \cdot)$ is a smooth function, in fact a polynomial.

For a cylinder function f, let $\tilde{f} : [0,1] \to \mathbb{R}$ be the real function defined by $\tilde{f}(\alpha) = E_{\nu_\alpha}[f(\eta)]$ and let

$$\Gamma_f(\eta, \alpha) \;=\; f(\eta) \;-\; \tilde{f}(\alpha) \;-\; \tilde{f}'(\alpha)[\eta(0) - \alpha] \,.$$

Note that $\mathfrak{f}(\phi, \alpha) = \tilde{f}(\alpha)$ and that $\sum_{x \in \mathbb{Z}} \mathfrak{f}(\{x\}, \alpha) = \tilde{f}'(\alpha)$. In particular, it follows from a simple computation that

$$\Gamma_f(\eta, \alpha) \;=\; \sum_{z \in \mathbb{Z}} \mathfrak{f}(\{z\}, \alpha)[\eta(z) - \eta(0)] \;+\; \sum_{n \ge 2} \sum_{A \in \mathcal{E}_n} \mathfrak{f}(A, \alpha) \Psi(A, \alpha) \,.$$

Fix a smooth functions H in $L^1(\mathbb{R})$. By the previous formula,

$$\begin{aligned} &\frac{1}{N^{1/2}} \sum_{x \in \mathbb{Z}} H(x/N) \Gamma_f(\tau_x \eta, \rho^N(t,x)) \\ &\quad = \sum_{z \in \mathbb{Z}} \frac{1}{N^{1/2}} \sum_{x \in \mathbb{Z}} H(x/N) \mathfrak{f}(\{z\}, \rho^N(t,x))[\eta(x+z) - \eta(x)] \\ &\qquad + \sum_{n \ge 2} \sum_{A \in \mathcal{E}_n} \frac{1}{N^{1/2}} \sum_{x \in \mathbb{Z}} H(x/N) \mathfrak{f}(A, \rho^N(t,x)) \tau_x \Psi(A, \rho^N(t,x)) \,. \end{aligned}$$

Note that the sums in z, n and A are finite because f is a cylinder function. Since H, $\rho^N(t, \cdot)$ and $\mathfrak{f}(\{z\}, \cdot)$ are smooth functions, a change of variables shows that the first term is of order $N^{-1/2}$ and that the second is equal to

$$\sum_{n \ge 2} \sum_{A \in \mathcal{E}_n} \frac{1}{N^{1/2}} \sum_{x \in \mathbb{Z}} H(x/N) \mathfrak{f}(A, \rho^N(t,x)) \prod_{z \in A} [\eta(x+z) - \rho^N(t, x+z)] \;+\; O(N^{-1/2}) \,,$$

which is exactly the expression appearing in Proposition 2.1. Since $\mathfrak{f}(A, \cdot)$ and $\rho^N(t, \cdot)$ are smooth bounded functions, we have proved the following result, known as the Boltzmann-Gibbs principle.

Corollary 2.2. *Let $d = 1$ and fix $T > 0$, a cylinder function f and a smooth function H in $L^1(\mathbb{R})$. Then,*

$$\lim_{N \to \infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}} \Big[\Big(\int_0^T dt \, \frac{1}{N^{1/2}} \sum_{x \in \mathbb{Z}} H(x/N) \Gamma_f(\tau_x \eta_t, \rho^N(t,x)) \Big)^2 \Big] \;=\; 0 \,.$$

The proof of Proposition 2.1 is based on three lemmas concerning the decay of the space-time correlations of the symmetric exclusion process. We start with a general result which will be used repeatedly.

Fix $n \geq 2$ and denote by $f_t(A, B) = f_t^N(A, B)$ the semi-group associated to the generator $N^2 \mathcal{L}_n$. For a finite subset A of $\mathbb{Z}^d$, let

$$I(A) \;=\; \sum_{x,y \in A} p(y-x) \; .$$

Note that $I(A)$ vanishes unless A contains two sites which are within a distance smaller than the range of the transition probability. Next lemma follows from Theorem 1.1 and a straightforward computation.

Lemma 2.3. *For all $T < \infty$, $n \geq 2$, there exists a finite constant C_3, depending only on n, p and T such that*

$$\int_0^t ds \, \frac{1}{(1+sN^2)^{m/2}} \sum_{B \in \mathcal{E}_n} f_{t-s}^N(A,B) \, I(B) \;\leq\; C_3 A_N(m,t)$$

for all A in $\mathcal{E}_n$, $N \geq 1$, $0 \leq t \leq T$; where

$$A_N(m,t) = \begin{cases} N^{-1} & \text{if } m = 0, \\ N^{-2} & \text{if } m = 1, \\ \log N / N^2 \sqrt{1+tN^2} & \text{if } m = 2, \\ 1/N^2\sqrt{1+tN^2} & \text{if } m \geq 3. \end{cases}$$

We now introduce the space-time correlations, also called v-functions in [7]. For a finite subset A of $\mathbb{Z}^d$ and $t \geq 0$, let $\varphi^N(t, \phi) = 1$,

$$\varphi^N(t,A) \;=\; \mathbb{E}_{\nu^N_{\rho_0(\cdot)}} \Big[\prod_{x \in A} \{\eta_t(x) - \rho^N(t,x)\} \Big] \; .$$

Notice that $\varphi^N(t, \{x\})$ vanishes for all x. An elementary computation shows that

$$\begin{cases} \partial_t \varphi^N(t,A) \;=\; N^2 \mathcal{L}_n \varphi^N(t,A) \;+\; G^N(t,A) \; , \\ \varphi^N(0,A) = 0 \; , \end{cases} \tag{2.1}$$

where $n = |A|$ and $G^N(t,A)$ is given by

$$N^2 \sum_{y,z \in A} p(z-y) \Big\{ \varphi^N(t, A \setminus \{z\}) - \varphi^N(t, A \setminus \{y\}) \Big\} \Big\{ \rho^N(t,\{z\}) - \rho^N(t,\{y\}) \Big\}$$

$$- \; (N^2/2) \sum_{y,z \in A} p(z-y) \varphi^N(t, A \setminus \{y,z\}) \Big\{ \rho^N(t,\{z\}) - \rho^N(t,\{y\}) \Big\}^2 \; .$$

Here again summation is carried over all bonds. Notice that the first line vanishes for $n = 2$ and that the second line vanishes for $n = 3$.

The linear differential equation (2.1) has a unique solution which can be represented as

$$\varphi^N(t,A) \;=\; \int_0^t ds \sum_{B \in \mathcal{E}_n} f_{t-s}(A,B) G^N(s,B)$$

so that the space-time correlations $\varphi^N(t,A)$ can be estimated inductively in n.

Next lemma is due to Ferrari, Presutti, Scacciatelli and Vares [7]. In the proof of Proposition 2.1 we don't need such sharp estimates.

Lemma 2.4. *Assume that $d = 1$ and fix $T > 0$. For each $n \geq 1$, there exists a finite constant $C_4 = C_4(n, p, \rho_0, T)$ such that*

$$\sup_{\substack{0 \leq t \leq T \\ A \in \mathcal{E}_{2n}}} \big|\varphi^N(t, A)\big| \;\leq\; \frac{C_4}{N^n}\;, \qquad \sup_{\substack{0 \leq t \leq T \\ A \in \mathcal{E}_{2n+1}}} \big|\varphi^N(t, A)\big| \;\leq\; \frac{C_4 \log N}{N^{n+1}}\;.$$

For $0 \leq s \leq t \leq T$, $1 \leq k \leq n$ and $B \in \mathcal{E}_k$, $A \in \mathcal{E}_n$, let

$$R_N(s, A; t, B) \;=\; \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\prod_{x \in A}[\eta_s(x) - \rho^N(s, x)]\prod_{y \in B}[\eta_t(y) - \rho^N(t, y)]\Big]\;.$$

Since s and A will be fixed, most of the time, we denote $R_N(s, A; t, B)$ by $R_N(t, B)$. Notice that in this definition we do not require A and B to have the same cardinality. An elementary computation shows that $R_N(t, B)$ is the solution of the linear differential equation

$$\begin{cases} \partial_t R_N(t, B) \;=\; N^2 \mathcal{L}_k R_N(t, B) \;+\; H_N(t, B) \quad \text{for } t \geq s\;, \\ R_N(s, B) = J_N(s, A, B)\;, \end{cases} \tag{2.2}$$

where $k = |B|$,

$$J_N(s, A, B) \;=\; \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\prod_{x \in A}[\eta_s(x) - \rho^N(s, x)]\prod_{y \in B}[\eta_s(y) - \rho^N(s, y)]\Big]$$

and $H_N(t, B)$ is given by

$$N^2 \sum_{y, z \in B} p(z - y)\Big\{R_N(t, B \setminus \{z\}) - R_N(t, B \setminus \{y\})\Big\}\Big\{\rho^N(t, \{z\}) - \rho^N(t, \{y\})\Big\}$$
$$-\,(N^2/2) \sum_{y, z \in B} p(z - y) R_N(t, B \setminus \{y, z\})\Big\{\rho^N(t, \{z\}) - \rho^N(t, \{y\})\Big\}^2\;.$$

Notice that $R_N(t, \phi) \;=\; \varphi^N(s, A)$, that $H_N(t, B)$ vanishes for $n \;=\; 1$ and that $J_N(s, A, B)$ is not equal to $\varphi^N(s, A \cup B)$ but given by

$$\sum_C \varphi^N(s, (A \Delta B) \cup C) \prod_{x \in C}[1 - 2\rho^N(s, \{x\})] \prod_{x \in (A \cap B) \setminus C} \rho^N(s, \{x\})[1 - \rho^N(s, \{x\})]\;,$$

where the summation is carried over all subsets C of $A \cap B$ and where $A \Delta B$ stands for the symmetric difference of A and B.

The differential equation (2.2) has a unique solution which can be represented as

$$R_N(t, B) \;=\; \sum_{C \in \mathcal{E}_k} f_r(B, C)\, J_N(s, A, C) \;+\; \int_0^r du \sum_{C \in \mathcal{E}_k} f_{r-u}(B, C) H_N(s + u, C)\;, \tag{2.3}$$

where $r = t - s$. This last notation is systematically used below. Let

$$U_N(t, B) \;=\; \sum_{C \in \mathcal{E}_k} f_r(B, C)\, J_N(s, A, C)\;.$$

Lemma 2.5. *Fix $2 \leq k \leq n$, $0 \leq s \leq t \leq T$ and A in $\mathcal{E}_n$. There exists a finite constant $C_4 = C_4(p, n, T, \rho_0)$ such that*

$$\sup_{B \in \mathcal{E}_k} \big|U_N(t, B)\big| \;\leq\; \frac{C_4 B(n - k)}{(1 + rN^2)^{k/2}}\;,$$

where $B(2j) = N^{-j}$ and $B(2j + 1) = \log N / N^{j+1}$ for $j \geq 0$.

Proof. $U_N(t,B)$ is absolutely bounded by

$$\sum_{j=0}^{k} J(s,j) \sum_{\substack{D\subset A, D\in\mathcal{E}_j\\ E\cap A=\phi, E\in\mathcal{E}_{k-j}}} f_r(B, D\cup E)\ , \tag{2.4}$$

where

$$J(s,\ell) \;=\; \sup_{\substack{C\in\mathcal{E}_k\\ |C\cap A|=\ell}} \big|J_N(s,A,C)\big| \;\le\; \max_m \sup_{D\in\mathcal{E}_m} \big|\varphi^N(s,D)\big|\ ,$$

where the maximum is carried over $n+k-2\ell \le m \le n+k+\ell$. Last inequality follows from the explicit formula for $J_N(s,A,C)$. By Lemma 2.4, the previous expression is less than or equal to $C_4 B(n+k-2\ell)$. On the other hand, by Theorem 1.1,

$$\sum_{\substack{D\subset A, D\in\mathcal{E}_j\\ E\cap A=\phi, E\in\mathcal{E}_{k-j}}} f_r(B, D\cup E) \;\le\; \frac{C_2(k,p)}{(1+rN^2)^{j/2}}\ .$$

Therefore,

$$\big|U_N(t,B)\big| \;\le\; C_4 \sum_{j=0}^{k} \frac{B(n+k-2j)}{(1+rN^2)^{j/2}} \;\le\; \frac{C_4 B(n-k)}{(1+rN^2)^{k/2}}\ .$$

This concludes the proof of the lemma. □

We are now in a position to prove the main result towards the Boltzmann-Gibbs principle.

Lemma 2.6. *Fix $n\ge 2$, there exists a finite constant $C_4 = C_4(n,p,T,\rho_0)$ such that*

$$\big|R_N(s,A;t,B)\big| \;\le\; C_4\Big\{\frac{\log N}{N^2} + \frac{1}{1+(t-s)N^2}\Big\}$$

for all A, B in $\mathcal{E}_n$, $0\le s<t\le T$.

Proof. Fix $n\ge 2$, $s\ge 0$ and A in $\mathcal{E}_n$. For $1\le k\le n$, denote by $R_{N,k}(t,\cdot)$ the solution of the linear differential equation (2.2). Since the equation for $R_{N,k}$ involves $R_{N,k-1}$, $R_{N,k-2}$, an induction argument on k is required. A simple pattern appears only for $k\ge 7$. Hence, for $1\le k\le 6$, we need to proceed by inspection, making the proof long and tedious.

Consider $k=1$. In this case H_N vanishes and, by Lemma 2.5,

$$\big|R_{N,1}(t,\{x\})\big| \;=\; \big|U_N(t,\{x\})\big| \;\le\; \frac{a_1 B(n-1)}{(1+rN^2)^{1/2}}\ .$$

Here and below $\{a_j, j\ge 1\}$ are finite constants depending on n, p, T and ρ_0 which may change from line to line.

For $k=2$, since $R_N(t,\phi)$ is time independent and absolutely bounded by $B(n)$, the previous estimates and Lemma 2.5 show that

$$\big|H_{N,2}(t,B)\big| \;\le\; a_2\Big\{\frac{NB(n-1)}{(1+rN^2)^{1/2}} \;+\; B(n)\Big\} I(B)\ .$$

Therefore, by the explicit formula (2.3) for $R_{N,2}(t,B)$ and by Lemmas 2.3, 2.5,

$$\big|R_{N,2}(t,B)\big| \;\le\; a_2\Big\{\frac{B(n-2)}{1+rN^2} \;+\; \frac{B(n-1)}{N}\Big\}$$

because $B(n) \le B(n-1)$. Notice that this inequality proves the lemma for $n = 2$ because $R_{N,2}(t,B) = R_N(s,A;t,B)$.

The estimates for $R_{N,1}$ and $R_{N,2}$ give bounds for $H_{N,3}$ which in turn, together with the explicit formula (2.3) for $R_{N,3}(t,B)$ and Lemmas 2.3, 2.5 show that

$$\big|R_{N,3}(t,B)\big| \;\le\; a_3 \Big\{ \frac{B(n-3)}{(1+rN^2)^{3/2}} \;+\; \frac{B(n-2)\log N}{N(1+rN^2)^{1/2}} \Big\} \; .$$

Here we used the fact that $B(n-3) = NB(n-1)$ to eliminate one of the terms appearing in the expression of $R_{N,3}(t,B)$.

We repeat this procedure for $k = 4$, 5 and 6 to obtain that

$$\begin{aligned}
\big|R_{N,4}(t,B)\big| \;&\le\; a_4 \Big\{ \frac{B(n-4)}{(1+rN^2)^{2}} \;+\; \frac{B(n-3)}{N(1+rN^2)^{1/2}} \;+\; \frac{B(n-2)\log N}{N^2} \Big\} \; , \\
\big|R_{N,5}(t,B)\big| \;&\le\; a_5 \Big\{ \frac{B(n-5)}{(1+rN^2)^{5/2}} \;+\; \frac{B(n-4)}{N(1+rN^2)^{1/2}} \;+\; \frac{B(n-3)}{N^2} \Big\} \; , \\
\big|R_{N,6}(t,B)\big| \;&\le\; a_6 \Big\{ \frac{B(n-6)}{(1+rN^2)^{3}} \;+\; \frac{B(n-5)}{N(1+rN^2)^{1/2}} \;+\; \frac{B(n-4)}{N^2} \Big\} \; .
\end{aligned}$$

For $k = 5$, we used the fact that $B(\ell)\log N \le B(\ell-1)$.

A pattern has been found for $k = 5$, 6. It is now a simple matter to prove by induction that this pattern is conserved so that

$$\big|R_{N,k}(t,B)\big| \;\le\; a_6 \Big\{ \frac{B(n-k)}{(1+rN^2)^{k/2}} \;+\; \frac{B(n-k+1)}{N(1+rN^2)^{1/2}} \;+\; \frac{B(n-k+2)}{N^2} \Big\}$$

for $k \ge 7$. It remains to recall the definition of $B(j)$ and to recollect all previous estimates to conclude the proof of the lemma. □

Notice that we could have set $B(1) = N^{-1}$ for the estimates in the previous lemma. Taking $B(1) = \log N/N$ simplifies slightly the notation since we have that $B(n+2) = B(n)N^{-1}$ for all $n \ge 0$ and we miss only a $\log N$ factor, which is irrelevant for our purposes.

We are now in a position to prove Proposition 2.1. With the notation introduced in this section, the expectation appearing in the statement of the proposition becomes

$$\frac{2}{N} \sum_{x,y\in\mathbb{Z}} H(x/N)\, H(y/N) \int_0^T dt \int_0^t ds\, R_N(s, A+x; t, A+y) \; ,$$

where $A + x$ is the set $\{z + x : z \in A\}$. By Lemma 2.6 and a change of variables, this expression is bounded above by

$$\frac{C(n,p,\rho_0,T)\log N}{N} \Big(\frac{1}{N} \sum_{x\in\mathbb{Z}} \big|H(x/N)\big| \Big)^2 \; ,$$

which proves Proposition 2.1.

We conclude this section with an observation. The same arguments presented above in the proof of Proposition 2.1 shows that

$$
\begin{aligned}
&\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\Big(\int_0^T dt\,\frac{1}{N^{1/2}}\sum_{x\in\mathbb{Z}} H(x/N)[\eta_t(x)-\rho^N(t,x)]\Big)^2\Big]\\
&\le\; 2\int_0^T dt\int_0^t ds\,\frac{1}{N}\sum_{x\in\mathbb{Z}} F(\rho^N(s,x))\,H(x/N)\,(f^N_{t-s}H)(x/N)\\
&+\;C(p,\rho_0,T)\Big(\frac{1}{N}\sum_{x\in\mathbb{Z}}\big|H(x/N)\big|\Big)^2
\end{aligned}
$$

for some finite constant $C(p,\rho_0,T)$. Here $F(a)=a(1-a)$.

3. Gaussian tail estimates for labeled exclusion processes

Fix $n\ge 2$ and a finite range, symmetric and irreducible transition probability $p(\cdot)$ on $\mathbb{Z}^d$. Consider n labeled particles moving on the d-dimensional lattice $\mathbb{Z}^d$ through stirring. This dynamics can be informally described as follows. The n particles start from n distinct sites of $\mathbb{Z}^d$. For each pair (x,y) of $\mathbb{Z}^d$, at rate $p(y-x)$, particles at x, y exchange their positions. This means that if there is a particle at x (resp. y) and no particle at y (resp. x), the particle jumps from x to y (resp. from y to x). If both sites are occupied, the particles change their position and if none of them are occupied, nothing happens.

The state space of this Markov process, denoted by $\mathcal{B}_n$, consists of all vectors $\mathbf{x}=(x_1,\dots,x_n)$ of $(\mathbb{Z}^d)^n$ with distinct coordinates:

$$
\mathcal{B}_n \;=\; \Big\{\mathbf{x}=(x_1,\dots,x_n)\in(\mathbb{Z}^d)^n\;:\;x_i\neq x_j \text{ for } i\neq j\Big\}
$$

while the generator L_n is given by

$$
(L_n f)(\mathbf{x}) \;=\; \sum_{x,y\in\mathbb{Z}^d} p(y-x)[f(\sigma^{x,y}\mathbf{x})-f(\mathbf{x})]\;. \tag{3.1}
$$

In this formula, for a configuration $\mathbf{x}=(x_1,\dots,x_n)$ in $\mathcal{B}_n$, $\sigma^{x,y}\,\mathbf{x}$ is the configuration defined by

$$
(\sigma^{x,y}\,\mathbf{x})_i \;=\; \begin{cases} y & \text{if } x_i=x,\\ x & \text{if } x_i=y,\\ x_i & \text{otherwise.}\end{cases}
$$

It is easy to check that the counting measure on $\mathcal{B}_n$, denoted by μ_n, is an invariant reversible measure for the process. The goal of this section is to obtain sharp estimates on the transition probability of this Markov process. To state the main results of the section, fix a state $\mathbf{z}$ in $\mathcal{B}_n$ and denote by f_t the solution of the forward equation:

$$
\begin{cases} \partial_t f_t = L_n^* f_t\;,\\ f_0(\mathbf{x}) = \mathbf{1}\{\mathbf{x}=\mathbf{z}\}\;. \end{cases} \tag{3.2}
$$

Recall that we denote by Φ the Legendre transform of the convex function $w^2\cosh w$.

Theorem 3.1. *Fix $n\ge 1$ and a point $\mathbf{z}=(z_1,\dots,z_n)$ in $\mathcal{B}_n$. Let f_t be a solution of the forward equation (3.2). There exist finite constants $C_2=C_2(n,d,p)$, $a_0=a_0(p)$*

such that

$$f_T(\mathbf{x}) \le \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ - \frac{a_0 T}{2(\log T)^2} \, \Phi\Big(\frac{\|\mathbf{x}-\mathbf{z}\| \log T}{a_0^2 T} \Big) \Big\}$$

for every $T > C_2$ and every configuration $\mathbf{x}$.

Since $\Phi(w) \sim w^2$ for w small, for $\gamma > 0$, there exists a finite constant $a_1 = a_1(p,\gamma)$ such that

$$f_T(\mathbf{x}) \le \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ \frac{-\|\mathbf{x}-\mathbf{z}\|^2}{a_1 T} \Big\}$$

for every $T > C_2$ and every configuration $\mathbf{x}$ such that $\|\mathbf{x}-\mathbf{z}\| \le \gamma T/\log T$.

On the other hand, since $x^2 \cosh x \le 2e^{2x}$, $\Phi(u) \ge (u/2)\log(u/4e)$. Hence,

$$f_T(\mathbf{x}) \le \frac{C_2}{(1+T)^{nd/2}} \exp\Big\{ - \frac{\|\mathbf{x}-\mathbf{z}\|}{4a_0 \log T} \log \frac{\|\mathbf{x}-\mathbf{z}\| \log T}{4ea_0^2 T} \Big\}$$

for every $T > C_2$. Of course this estimate is only interesting if $\|\mathbf{x}-\mathbf{z}\| \gg T/\log T$,

Since the proof of Theorem 3.1 follows closely the one of Theorem 2.2 in [9], we present only the main differences. Throughout this section, C_0 stands for a universal constant, which may change from line to line.

We first need a logarithmic Sobolev inequality for the process X_t restricted to cubes. Fix an integer ℓ and decompose the lattice $\mathbb{Z}^d$ into disjoint cubes $\{\Lambda_k : k \ge 1\}$ of length ℓ:

$$\begin{aligned} \Lambda_k \;&=\; x_k + \{1,\dots,\ell\}^d \text{ for some } x_k \text{ in } \mathbb{Z}^d; \\ \Lambda_k \cap \Lambda_j \;&=\; \phi \text{ for } k \ne j \quad \text{and} \quad \bigcup_{k\ge 1} \Lambda_k \;=\; \mathbb{Z}^d. \end{aligned}$$

For a vector $\mathbf{k} = (k_1,\dots,k_n)$, let $\Lambda_{\mathbf{k}}$ be the finite cube of $(\mathbb{Z}^d)^n$ defined by $\Lambda_{\mathbf{k}} = \Lambda_{k_1} \times \cdots \times \Lambda_{k_n}$ and let $L_{\Lambda_{\mathbf{k}}}$ be the generator L_n introduced in (3.1) restricted to the cube $\Lambda_{\mathbf{k}}$. This means that jumps from $\Lambda_{\mathbf{k}}$ to its complement are forbidden as well as jumps from the complement to $\Lambda_{\mathbf{k}}$.

Lemma 3.2. *There exists a finite constant C_1 depending only on the transition probability $p(\cdot)$, the dimension d and the total number of particles n such that*

$$\sum_{\mathbf{x}\in\Lambda_{\mathbf{k}}} f(\mathbf{x}) \log f(\mathbf{x}) \;\le\; C_1 \ell^2 \sum_{\mathbf{x},\mathbf{y}\in\Lambda_{\mathbf{k}}} \Big\{ \sqrt{f(\mathbf{y})} - \sqrt{f(\mathbf{x})} \Big\}^2 , \tag{3.3}$$

for all densities f with respect to the uniform probability measure over $\Lambda_{\mathbf{k}}$. In this formula, the sum on the right hand side of the inequality is carried over all pairs $\mathbf{x}$, $\mathbf{y}$ in $\Lambda_{\mathbf{k}}$ such that $\mathbf{y} = \sigma^{x,y}\mathbf{x}$ for some x, y with $p(y-x) > 0$.

Proof. It is well known that a symmetric random walk evolving on a d-dimensional cube satisfies a logarithmic Sobolev inequality of type (3.3) and that the superposition of independent processes satisfying logarithmic Sobolev inequalities also satisfies a logarithmic Sobolev inequality, the constant being the maximum of the individual constants. This proves (3.3) in the case where the cubes Λ_k are all different: $k_i \ne k_j$ for $i \ne j$.

It remains to consider the case where some cubes are equal. In this situation, the diagonal is forbidden because two particles cannot occupy the same site, and two particles may exchange their position. Fix $2 \le m \le n$ and consider the hypercube $\Lambda_{\mathbf{k}} = \Lambda_k \times \cdots \times \Lambda_k$ of $(\mathbb{Z}^d)^m$. If we do not distinguish particles, we retrieve the symmetric simple exclusion process on Λ_k with m particles. This process satisfies a logarithmic Sobolev inequality of type (3.3) with a constant C_0 depending only on the dimension d and the transition probability $p(\cdot)$ [11]. It is not difficult to recover (3.3) for the random walk X_t on $\Lambda_{\mathbf{k}}$ from this estimate.

Indeed, let $\Sigma_{\Lambda_k,m}$ be the subsets of Λ_k with m points: $\Sigma_{\Lambda_k,m} = \{A \subset \Lambda_k : |A| = m\}$, let $\mu_{\Lambda_k,m}$ be the uniform probability measure on $\Sigma_{\Lambda_k,m}$ and, for a density $f : \Lambda_{\mathbf{k}} \to \mathbb{R}_+$ with respect to the uniform measure over $\Lambda_{\mathbf{k}}$, let $\tilde f : \Sigma_{\Lambda_k,m} \to \mathbb{R}_+$ be the density with respect to $\mu_{\Lambda_k,m}$ defined by

$$\tilde f(\{x_1, \dots, x_m\}) \;=\; \frac{1}{m!} \sum_{\sigma} f(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \;,$$

where the summation is performed over all permutations σ of m elements.

With this notation, we may rewrite the left hand side of (3.3) as

$$\sum_{A \in \Sigma_{\Lambda_k,m}} \tilde f(A) \sum_{\mathbf{x} \in A} \frac{f(\mathbf{x})}{\tilde f(A)} \log \frac{f(\mathbf{x})}{\tilde f(A)} \;+\; m! \sum_{A \in \Sigma_{\Lambda_k,m}} \tilde f(A) \log \tilde f(A) \;, \tag{3.4}$$

where the summation over $\mathbf{x}$ is carried over all points $\mathbf{x} = (x_1, \dots, x_m)$ such that $\{x_1, \dots, x_m\} = A$.

It is not difficult to prove a logarithmic Sobolev inequality for the permutation of m points. Let $\mathbb{S}_m$ be the set of all permutations σ of m points. Consider the Dirichlet form $D_{\mathbb{S}_m}$ defined by

$$D_{\mathbb{S}_m}(g) \;=\; \sum_{\sigma, \tilde\sigma \in \mathbb{S}_m} [g(\sigma) - g(\tilde\sigma)]^2 \;.$$

There exists a finite constant C_0 such that

$$\sum_{\sigma \in \mathbb{S}_m} g(\sigma) \log g(\sigma) \;\le\; C_0 D_{\mathbb{S}_m}(\sqrt{g})$$

for all densities g with respect to the uniform probability measure on $\mathbb{S}_m$.

Since $f(\mathbf{x})/\tilde f(A)$ is a density with respect to the uniform probability measure over the set of all permutations, the first term is bounded above by

$$C_0 \sum_{A \in \Sigma_{\Lambda_k,m}} \sum_{\mathbf{x},\mathbf{y} \in A} \Big\{ \sqrt{f(\mathbf{y})} - \sqrt{f(\mathbf{x})} \Big\}^2 \tag{3.5}$$

for some finite universal constant. It remains to connect each point $\mathbf{x}$ in A to each point $\mathbf{y}$ in A by a path $\mathbf{x} = \mathbf{z}_0, \dots, \mathbf{z}_r = \mathbf{y}$ such that $\mathbf{z}_{j+1} = \sigma^{x,y} \mathbf{z}_j$ for some x, y with $p(y-x) > 0$ to estimate the previous term by the right hand side of (3.3). This can be done as follows.

Assume first that $d = 1$. To explain the strategy in a simple way, we allow two particles to occupy the same site in the construction of the path. The modifications needed to respect the exclusion rule are straightforward. Fix $\mathbf{x}$ and $\mathbf{y}$ in a same set A. Since both points belong to the same set, there exists a permutation σ of m points such that $y_i = x_{\sigma(i)}$ for $1 \le i \le m$. The path $\{\mathbf{z}_j\}$ connecting $\mathbf{x}$ to $\mathbf{y}$ is defined as follows. We start changing the first coordinate x_1 of $\mathbf{x}$, keeping all the other constants, moving from $\mathbf{x} = (x_1, \dots, x_m)$ to $\mathbf{w}_1 = (y_1 = x_{\sigma(1)}, x_2, \dots, x_m)$. Note that the last configuration has two particles occupying the same site. At this point, we change the coordinate $x_{\sigma(1)}$, moving from a new configuration $\mathbf{w}_2$, which is obtained from $\mathbf{x}$, by replacing x_1 by $x_{\sigma(1)}$ and $x_{\sigma(1)}$ by $x_{\sigma^2(1)}$, where $\sigma^2 = \sigma \circ \sigma$. We repeat this procedure. If the orbit of 1 for the permutation σ

is the all set $\{1,\dots,m\}$, this algorithm produces a path from $\mathbf{x}$ to $\mathbf{y}$. Otherwise, after completing the orbit of 1 by the map σ, we choose the smallest coordinate not belonging to the orbit of 1 and repeat the procedure.

Denote by $\Gamma_{\mathbf{x},\mathbf{y}}$ the path just constructed. Notice that

1. its length is bounded by $m\ell$ and
2. all coordinates but one of each site $\mathbf{z}$ in $\Gamma_{\mathbf{x},\mathbf{y}}$ belong to the set $\{x_1,\dots,x_m\}$.

Therefore, by Schwarz inequality, (3.5) is bounded above by

$$C_0 \sum_{A\in\Sigma_{\Lambda_k,m}} \sum_{\mathbf{x},\mathbf{y}\in A} |\Gamma_{x,y}| \sum_{b\in\Gamma_{\mathbf{x},\mathbf{y}}} \Big\{\sqrt{f(\mathbf{b}_2)}-\sqrt{f(\mathbf{b}_1)}\Big\}^2$$
$$\le\; C_0\ell m \sum_b \Big\{\sqrt{f(\mathbf{b}_2)}-\sqrt{f(\mathbf{b}_1)}\Big\}^2 \sum_{A\in\Sigma_{\Lambda_k,m}} \sum_{\substack{\mathbf{x},\mathbf{y}\in A\\ b\in\Gamma_{\mathbf{x},\mathbf{y}}}} .$$

The last sum in the first line is performed over all pairs $b=(b_1,b_2)$ of consecutive sites in the path $\Gamma_{\mathbf{x},\mathbf{y}}$, while the first sum in the second line is performed over all pairs $b=(b_1,b_2)$ such that $b_2=\sigma^{x,x+y}$ for some x, y in $\mathbb{Z}^d$ such that $p(y)>0$. Since all but one coordinate of each site in $\Gamma_{\mathbf{x},\mathbf{y}}$ belong to $\{x_1,\dots,x_m\}$, for each fixed bond $b=(b_1,b_2)$ there is at most $m\ell$ possible sets A which might use this bond. For each set A, there is at most $m!$ end points and $m!$ starting points for the path. The last sum is thus bounded by

$$\le\; C_0\ell^2 m^2 (m!)^2 \sum_b \Big\{\sqrt{f(\mathbf{b}_2)}-\sqrt{f(\mathbf{b}_1)}\Big\}^2 .$$

This concludes the proof of the estimate of the first term in (3.4) in dimension 1.

The proof in higher dimension is similar. The idea is to consider a configuration $\mathbf{x}$ as a point in $\mathbb{Z}^{md}$ and repeat the previous algorithm, moving the first coordinate of the first particle, then moving the first coordinate of the $\sigma(1)$-particle, until all first coordinates of all particles are modified. At this point, we change the second coordinate of the first particle and repeat the procedure. This method gives a path of length at most $C_0\ell md$ and whose sites have all but one of the md coordinates equal to the coordinates of $\mathbf{x}$. These two properties permit to derive the estimates obtained in dimension 1, replacing m by md. This proves that the first term in (3.4) is bounded above by the the right hand side of (3.3).

We focus now on the second term of (3.4). By the logarithmic Sobolev inequality for m exclusion particles in a cube Λ_k, this expression is less than or equal to

$$Cm! \sum_{A\in\Sigma_{\Lambda_k,m}} \sum_{x,y\in\mathbb{Z}^d} p(y)\Big\{\sqrt{\tilde f(A_{x,x+y})}-\sqrt{\tilde f(A)}\Big\}^2$$

for some finite constant C depending only on $p(\cdot)$ and d. By Schwarz inequality, this expression is bounded by the right hand side of (3.3). □

The second main ingredient in the proof of Theorem 3.1 is an estimate of the action of the generator L_n on certain exponential functions.

For a vector $\theta=(\theta_1,\dots,\theta_n)$, θ_i in $\mathbb{R}^d$, denote by ψ_θ the function $\psi_\theta\colon \mathcal{B}_n\to\mathbb{R}$ defined by $\psi_\theta(\mathbf{x})=\exp\{\theta\cdot\mathbf{x}\}$. Here, $\mathbf{x}\cdot\mathbf{y}$ represents the inner product in $(\mathbb{Z}^d)^n$. An elementary computation shows that there exists a finite constant a_0, depending only on the transition probability $p(\cdot)$, such that

$$\frac{1}{\psi_\theta(\mathbf{x})}(L_n\psi_\theta)(\mathbf{x}) \;\le\; R(\theta)\;, \qquad \sum_{x,y\in\mathbb{Z}^d} p(y-x)\Big\{\frac{\psi_\theta(\sigma^{x,y}\mathbf{x})}{\psi_\theta(\mathbf{x})}-1\Big\}^2 \;\le\; R(\theta) \tag{3.6}$$

for all $\mathbf{x}$ in $\mathcal{B}_n$, where

$$R(\theta) \;=\; a_0\sum_{j=1}^{d}\sum_{i=1}^{n}\Big(\cosh\{a_0\theta_{i,j}\}-1\Big)\;. \tag{3.7}$$

Next result relies mainly on Lemma 3.2 and on the bounds (3.6). Its proof follows closely the one of Lemma 4.3 in [9] and is therefore omitted. For a positive function $\psi:\mathcal{B}_n\to\mathbb{R}$, denote by $\mathfrak{D}_\psi$ the Dirichlet formula defined by

$$\mathfrak{D}_\psi(u) = (1/2)\sum_{\mathbf{x}\in\mathcal{B}_n}\sum_{x,y\in\mathbb{Z}^d} p(y-x)\big\{u(\sigma^{x,y}\mathbf{x})-u(\mathbf{x})\big\}^2\,\psi(\mathbf{x})\;.$$

Lemma 3.3. *Fix a vector θ in $(\mathbb{R}^d)^n$, $\ell\ge 2$, denote by C_1 the constant introduced in Lemma 3.2 and let $\psi=\psi_\theta$. There exists a finite constant a_0, depending only on the transition probability $p(\cdot)$, such that*

$$\int f\,\log f\,\psi\,d\mu_n \;\le\; -\int f\,\psi\,\log\psi\,d\mu_n \;-\; \log|\Lambda_\ell|^n \;+\; 4C_1\ell^2\mathfrak{D}_\psi(\sqrt{f}) \;+\; R(\theta)\ell^2$$

for every density $f:\mathcal{B}_n\to\mathbb{R}$ with respect to $\psi\,d\mu_n$.

The estimates (3.6) permit also to prove the following bound. Recall that f_t is the solution of the forward equation (3.2) and that μ_n is the counting measure on $\mathcal{B}_n$.

Lemma 3.4. *Fix a smooth increasing function $p:\mathbb{R}_+\to(1,\infty)$ and a smooth function $\lambda=(\lambda_1,\dots,\lambda_n):\mathbb{R}_+\to(\mathbb{R}^d)^n$. Let $\psi_t(\mathbf{x})=\exp\{\lambda(t)\cdot\mathbf{x}\}$ and let $h_t=f_t/\psi_t$, $u_t=h_t^{p(t)/2}$. There exists a finite constant a_0, depending only on the transition probability $p(\cdot)$, such that*

$$\begin{aligned}\frac{d}{dt}\int h_t^{p(t)}\,\psi_t\,d\mu_n \;\;&\le\;\; \frac{\dot p(t)}{p(t)}\int u_t^2\,\log u_t^2\,\psi_t\,d\mu_n-(p(t)-1)\int u_t^2\,\dot\psi_t\,d\mu_n\\ &-\;\; \frac{(p(t)-1)}{p(t)}\mathfrak{D}_{\psi_t}(u_t)\;+\;R(\lambda(t))\,p(t)\int u_t^2\,\psi_t\,d\mu_n\;.\end{aligned}$$

The proof of Lemma 3.4 relies on the estimates (3.6) and follows closely the proof of Lemma 5.1 in [9].

We are now in a position to prove Theorem 3.1. Recall that f_t is the solution of the forward equation (3.2). Fix $T>0$ large, set $q=1+(\log T)^{-1}$, $q'=\log T$ and consider a smooth increasing function $p\colon[0,T]\to[q,q']$ such that $p(0)=q$, $p(T)=q'$. At the end of the proof, $p(t)$ will be taken as a rescaling of the function $[1-(s/T)^\alpha]^{-1}$ for some $0<\alpha<1/2$.

Following Davies [4], fix $\theta=(\theta_1,\dots,\theta_n)$ in $(\mathbb{R}^d)^n$, define $\psi_t\colon\mathcal{B}_n\to\mathbb{R}_+$ by

$$\psi_t(\mathbf{x}) \;=\; \exp\Big\{\frac{p(t)}{p(t)-1}\,\theta\cdot\mathbf{x}\Big\}\;,$$

denote $\theta_i\,p(t)/[p(t)-1]$ by $\lambda_i(t)$ and let $h_t=f_t/\psi_t$.

For a function $g\colon \mathcal{B}_n \to \mathbb{R}$ and $1 \le p < \infty$, denote by $\|g\|_{\psi,p}$ the L^p norm of g with respect to the measure $\psi\, d\mu_n$:

$$\|g\|_{\psi,p}^p \;=\; \sum_{\mathbf{x}\in\mathcal{B}_n} |g(\mathbf{x})|^p \psi(\mathbf{x})\ .$$

A straightforward computation gives that

$$\frac{d}{dt}\log\|h_t\|_{\psi_t,p(t)} \;=\; -\frac{\dot p(t)}{p(t)}\log\|h_t\|_{\psi_t,p(t)} \;+\; \frac{1}{p(t)}\frac{1}{\|h_t\|_{\psi_t,p(t)}^{p(t)}}\frac{d}{dt}\|h_t\|_{\psi_t,p(t)}^{p(t)}\ . \tag{3.8}$$

Denote $h_t^{p(t)}$ by u_t^2 and $u_t^2/\|u_t\|_2^2$ by v_t^2. By Lemma 3.4, the second term on the right hand side of last formula is bounded above by

$$\begin{aligned}&\frac{\dot p(t)}{p(t)^2}\int v_t^2\,\log v_t^2\,\psi_t\,d\mu_n \;+\; \frac{\dot p(t)}{p(t)}\log\|h_t\|_{\psi_t,p(t)} \;+\; R(t)\\ &\quad -\frac{p(t)-1}{p(t)^2}\mathfrak{D}_{\psi_t}(v_t) \;-\; \frac{p(t)-1}{p(t)}\int v_t^2\,\dot\psi_t\,d\mu_n\ ,\end{aligned} \tag{3.9}$$

where $R(t)=R(\lambda(t))$. Notice that the second term in this expression cancels with the first term in the previous formula and that v_t^2 is a density with respect to the measure $\psi_t\,d\mu_n$. By Lemma 3.3, the first term of this formula is bounded by

$$\frac{\dot p(t)}{p(t)^2}\Big\{-\int v_t^2\,\psi_t\,\log\psi_t\,d\mu_n \;-\; \log|\Lambda_\ell|^n \;+\; 4C_1\ell^2\mathfrak{D}_{\psi_t}(v_t) \;+\; \ell^2 R(t)\Big\} \tag{3.10}$$

for all $\ell \ge 2$.

By definition of ψ_t,

$$-\frac{p(t)-1}{p(t)}\,\dot\psi_t \;=\; \frac{\dot p(t)}{p(t)^2}\,\psi_t\log\psi_t\ ,$$

so that the first term of formula (3.10) cancels with the fifth term of formula (3.9). Denote by $[a]$ the integer part of a real a. If we set $\ell=\ell(t)$ as

$$\ell(t) = \Big[\sqrt{\frac{p(t)-1}{4C_1\dot p(t)}}\,\Big]\ ,$$

a straightforward computation shows that the Dirichlet form in formula (3.9) cancels with the Dirichlet form appearing in (3.10). The inequality $\ell(t)\ge 2$ imposes conditions on $p(t)$ that will need to be checked when defining $p(t)$.

Up to this point we proved that

$$\frac{d}{dt}\log\|h_t\|_{\psi_t,p(t)} \;\le\; -\frac{\dot p(t)}{p(t)^2}\log|\Lambda_\ell|^n \;+\; 2R(t)$$

because $\ell(t)^2\dot p(t)\le p(t)^2$ by definition of $\ell(t)$. Integrating in time, we obtain that

$$\|h_T\|_{\psi_T,p_T} \;\le\; \|h_0\|_{\psi_0,p_0}\exp\Big\{-(nd/2)\int_0^T dt\,\frac{\dot p(t)}{p(t)^2}\log\frac{p(t)-1}{8C_1\dot p(t)} \;+\; 2\int_0^T dt\,R(t)\Big\}\ .$$

because $\ell(t)^2 \ge [p(t)-1]/8C_1\dot p(t)$. By definition of the density f, $\|h_0\|_{\psi_0,p_0} = f(\mathbf{z})\psi_0(\mathbf{z})^{1-p_0/p_0} = \exp\{\theta\cdot\mathbf{z}\}$. On the other hand, $\|h_T\|_{\psi_T,p_T}$ is bounded below by $f_T(\mathbf{x})\psi_T(\mathbf{x})^{1-p_T/p_T} = \exp\{\theta\cdot\mathbf{x}\}$ for every $\mathbf{x}$ in $\mathcal{B}_n$. Moreover, since $-\dot p(t)/p(t)^2 = (1/p(t))'$,

$$-\log(8C_1)\int_0^T dt\,\frac{\dot p(t)}{p(t)^2} \;\le\; \log(8C_1)$$

because $p(0) = 1 + (\log T)^{-1}$, $p(1) = \log T$. Finally, since $p(t)$ is an increasing function, $p(t)/[p(t)-1] \le 1+\log T \le 2\log T$ for $T \ge e$. Therefore, if we assume that $|\theta_{i,j}| \le B/2\log T$ for some finite constant B, $R(t) \le C(a_0, B)\|\theta\|^2 p(t)^2/(p(t)-1)^2$, where $C(a_0, B) = 2a_0^3 M(a_0 B)$ and

$$M(r) \;:=\; \sup_{|w|\le r} \frac{\cosh w - 1}{w^2} \;\le\; \cosh r\ .$$

Putting together all previous estimates, we obtain that

$$\begin{aligned} f_T(\mathbf{x}) \;\le\; C(n,d)\exp\{-\theta\cdot(\mathbf{x}-\mathbf{z})\}\, \exp\Big\{ -(nd/2)\int_0^T dt\, \frac{\dot p(t)}{p(t)^2}\log\frac{p(t)-1}{\dot p(t)}\Big\} \;\times \\ \times\; \exp\Big\{ C(a_0,B)\|\theta\|^2 \int_0^T dt\, \frac{p(t)^2}{[p(t)-1]^2}\Big\} \end{aligned}$$

provided $|\theta_{i,j}| \le B/2\log T$.

It remains to choose an appropriate increasing smooth function $p : [0,T] \to [q, q']$ which connects $1 + (\log T)^{-1}$ to $\log T$ to conclude the proof of the theorem. Let $q(s) = p(sT)/p(sT) - 1$ and notice that $q(0) = \log T + 1$, $q(1) = \log T/\log T - 1$. With this notation, a change of variables and an elementary computation shows that the two previous integrals become

$$\begin{aligned} &(nd/2)\int_0^1 ds\, \frac{q'(s)}{q(s)^2}\log\frac{q(s)-1}{-q'(s)} \;-\; (nd/2)\log T\Big(\frac{\log T}{\log T+1} - \frac{1}{\log T}\Big) \\ &\text{and}\quad C(a_0,B)\|\theta\|^2 T\int_0^1 ds\, q(s)^2\ . \end{aligned}$$

The second term of the first line is bounded by $\log T^{-nd/2} + C(n,d)$, which is responsible for the diagonal estimate of the density.

Let $g(s) = s^{-\alpha}$ for some $0 < \alpha < 1/2$. It is easy to show

$$\int_0^1 ds\, \frac{-g'(s)}{g(s)^2}\log\frac{g(s)-1}{-g'(s)} \;<\; \infty\ , \qquad \int_0^1 ds\, g(s)^2 \;<\; \infty\ .$$

Defining $q(s) = g(a + (b-a)s)$ for appropriate constants a, b, we deduce that

$$f_T(\mathbf{x}) \;\le\; \frac{C(n,d)}{T^{nd/2}}\exp\{-\theta\cdot(\mathbf{x}-\mathbf{z}) + C(a_0,B)\|\theta\|^2 T\}$$

provided $|\theta_{i,j}| \le B/2\log T$. An elementary computation shows that with this choice $\ell(t) \ge 2$ for all $0 \le t \le T$ provided T is chosen large enough: $T \ge C_2(n,d)$.

Fix $\mathbf{x}$, let $\mathbf{y} = \mathbf{x} - \mathbf{z}$ and choose $\theta = B(2\log T)^{-1}\mathbf{y}/\|\mathbf{y}\|$ so that $|\theta_{i,j}| \le B/2\log T$. With this choice, the expression inside braces in the previous formula becomes bounded by

$$-\frac{\|\mathbf{x}-\mathbf{z}\|\, B}{2\log T} \;+\; \frac{C(a_0,B)B^2 T}{(2\log T)^2}$$

for every $B > 0$. Recall the definition of $C(a_0, B)$, change variables as $B' = a_0 B$ and minimize over B' to obtain that the previous expression is bounded above by

$$-\frac{a_0 T}{2(\log T)^2}\,\Phi\Big(\frac{\|\mathbf{x}-\mathbf{z}\|\log T}{a_0^2 T}\Big)\ ,$$

where Φ is the convex conjugate of $w^2\cosh w$. This concludes the proof of Theorem 3.1.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 3.1 since the evolution of n random walks evolving with exclusion can be obtained from the evolution of n labeled random walks by just ignoring the labels.

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