# ON THE EXPONENTIAL DECAY OF THE CRITICAL GENERALIZED KORTEWEG-DE VRIES EQUATION WITH LOCALIZED DAMPING

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ABSTRACT. This paper is concerned with the the asymptotic behavior of solutions of the critical generalized Korteweg-de Vries equation in a bounded interval with a localized damping term. Combining multiplier techniques and compactness arguments it is shown that the problem of exponential decay of the energy is reduced to prove the unique continuation property of weak solutions. A locally uniform stabilization result is derived.

## 1. Introduction

We study the exponential decay of the solutions of the critical generalized Korteweg-de Vries equation in the domain $(0, L)$ under the presence of a localized damping:

$$
\begin{cases}
u_t + u_x + u_{xxx} + u^4 u_x + a(x)u = 0 & \text{in } (0, L) \times (0, \infty), \\
u(0, t) = u(L, t) = 0, & t \in (0, \infty) \\
u_x(L, t) = 0, & t \in (0, \infty) \\
u(x, 0) = u_0(x), & x \in (0, L).
\end{cases}
\tag{1.1}
$$

We assume that the real-valued function $a = a(x)$ satisfies the condition

$$
\begin{cases}
a \in L^\infty(0, L) \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } \omega, \\
\text{where } \omega \text{ is a nonempty open subset of } (0, L).
\end{cases}
\tag{1.2}
$$

Multiplying equation (1.1) by $u$ and integrating in $(0, L)$ we get

$$
\frac{dE}{dt} = -\int_0^L a(x)|u(x, t)|^2 dx - \frac{1}{2}|u_x(0, t)|^2
\tag{1.3}
$$

where

$$
E(t) = \frac{1}{2}\int_0^L |u(x, t)|^2 dx.
\tag{1.4}
$$

Note that the term $a(x)u$ plays the role of a feedback damping mechanism and, consequently, one can investigate if the solutions of (1.1) tend to zero as $t \to \infty$ and under what rate they decay.

The above equation is a generalization of the well known Korteweg-de Vries equation, that is,

$$
u_t + u_{xxx} + uu_x = 0.
$$

When it is considered in the real line, it is called critical for various reasons. For instance, if $u$ is a solution of the initial value problem associated to the equation in (1.1) with $a \equiv 0$, then $u_\lambda(x,t) = \lambda^{1/2}u(\lambda x, \lambda^3 t)$ is also a solution and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. That is, the mass remains invariant by scaling. Local results for data in $L^2$ were established by Kenig, Ponce and Vega [7]. On the other hand, Merle in [10] showed that solutions in $H^1$ may blow-up in finite time. Thus this nonlinearity is critical for the long time behavior of solutions. The generalized Korteweg-de Vries equation has been extensively studied for understanding the interaction between the dispersive term and the nonlinearity in the context of the theory of nonlinear dispersive evolution equations. Here, our purpose is to extend, in some sense, the previous results on the subject obtained for the Korteweg-de Vries equation posed on a finite domain, i. e.

$$u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, \quad \text{in } (0,L) \times (0,\infty). \qquad (1.5)$$

In this paper, we are concerned with the exponential decay of $E(t)$. More precisely, our purpose is to prove that, for any $R > 0$, there exist constants $C = C(R)$ and $\alpha = \alpha(R)$ satisfying

$$E(t) \le C(R)E(0)e^{-\alpha(R)t}, \quad \forall \ t > 0,$$

provided $E(0) \le R$, which can be stated in the following equivalent form: Find $T > 0$ and $C > 0$ such that

$$E(0) \le C \int_0^T \Big[ \int_0^L a(x)u^2(x,t)dx + u_x^2(0,t) \Big] dt \qquad (1.6)$$

holds for every finite energy solution of (1.1). Indeed, from (1.6) and (1.3) we have that $E(T) \le \gamma E(0)$ with $0 < \gamma < 1$, which combined with the semigroup property allow us to derive the exponential decay of $E(t)$. Here, due to technical reasons that will become clear in the proofs, we will consider data $u_0$ such that $\|u_0\|_{L^2(0,L)}$ is small.

The main tools we use for obtaining (1.6) follow closely the multiplier techniques developed in [12] for the analysis of controllability properties of (1.5) under boundary conditions given in (1.1). However, when using multipliers, the nonlinearity produces extra terms that we handle by compactness what reduces the problem in showing that the unique solution of (1.1), such that $a(x) = 0$ everywhere and $u_x(0,t)$ for all time $t$, has to be the trivial one. This problem may be viewed as a unique continuation one since $au = 0$ implies that $u = 0$ in $\{a > 0\} \times (0,T)$.

The same problem has been intensively investigated in the context of wave equation but there are fewer results for the KdV type equation. In [9], the case where the damping term is active simultaneously in a neighborhood of both extremes of the interval $(0,L)$ was addressed for (1.5) under boundary conditions as in (1.1). Using multiplier techniques the problem was solved in two steps: first, extending the solution by zero outside the interval $(0,L)$, one gets a compactly supported (in space) solution of the Cauchy problem for the KdV equation on the whole line. Then, one applies the classical smoothing properties in [6] showing that the solution is smooth. This allows one to apply the unique continuation property results in [17] on smooth solutions to conclude that $u = 0$. Later on, performing as in [9], the general case was solved in [11] showing that solutions vanishing on any subinterval are necessarily smooth which yields enough regularity on $u$ to apply the unique continuation results obtained in [15]. More recently, L. Rosier and B.-Y. Zhang considered the generalized KdV model

$$u_t + u_x + u_{xxx} + u^p u_x + a(x)u = 0, \quad \text{in } (0,L) \times (0,\infty) \qquad (1.7)$$

with boundary conditions as in (1.1) and $p = 2, 3$. It was established existence, uniqueness, and persistence properties of solutions corresponding to the given initial data $u_0$, together with continuous dependence on solution upon the initial data $u_0$. Following the methods described above (multiplier techniques, compactness arguments and unique continuation property) the decay of solutions in the energy space was also obtained. At that point we observe that to obtain the decay of solutions they use a new unique continuation property which proof is mainly based on a Carleman type estimate for the Korteweg-de Vries equation established by Rosier in [13] (see also Lemmas 3.1 and 3.2). Both results, decay of solutions and well-posedness, represent a significant advance in the subject since most of the results were obtained for the pure Korteweg-de Vries equation (see for instance [1] and the references therein ).

In recent years, there has been a great interest in the study of the boundary value problems associated to (1.1) (see for instance, [1], [2], [5], [9], [12], [13]). The difficulty to study problem (1.1) is introduced by the nonlinearity and the lack of smoothing effects to deal with it. For our purpose it will be enough to consider a weak solution to (1.1). We will use the results obtained by Faminskii [4] to guarantee the existence of solutions to (1.1).

In what concerns the exponential decay, we argue as in the previous works, using multipliers, what requires the application of a unique continuation result. However, this unique continuation result may not be applied directly because of the lack of regularity of solutions we are dealing with. To overcome this problem, we proceed as in [13] and we first guarantee that the solutions are smooth enough (see Lemmas 3.1 and 3.2). Our result is of local nature in the sense that the exponential decay rate is uniform only for initial data in balls $B_R(0)$ of $L^2(0, L)$ with $0 < R < 1$.

Before leaving this section we would like to comment on the following issue. For the boundary value problem (1.1), according to the dissipation law (1.3), even when $a \equiv 0$, the energy is dissipated through the extreme $x = 0$. In the linear case, as shown in [12] and [9], there are critical lengths $L$ for which the decay is not true. Such lengths $L$ form the set

$$\mathcal{E} = \Big\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2}, \ k \text{ and } l \text{ are positive natural numbers} \Big\}.$$

Whether solutions of the nonlinear problem decay in this case or not is an open problem.

The article is organized as follows: In Section 2 we prepare the needed estimates to prove our main result. Section 3 will be devoted to establish the stabilization result.

We would like to thank A. Faminskii to point out a minor problem in a previous version of this manuscript and the reference [4] which simplified our arguments.

## 2. Preliminary Estimates

We establish a series of estimates that will be useful in the proof of Theorem 3.3. We begin by stating the following existence result due to Faminskii [4].

**Theorem 2.1** (See [4], Theorem 1). *Let $u_0 \in L^2(0, L)$ and $T > 0$ be given. Then, there exists a $T^* \in (0, T]$ such that the problem (1.1) admits a unique solution $u \in \mathcal{C}([0, T^*]; L^2(0, L)) \cap L^2((0, T^*); H_0^1(0, L))$.*

In the next proposition we will obtain some additional properties of the solutions to (1.1).

**Proposition 2.2.** *Let $u$ be the solution of problem (1.1) obtained in Theorem (2.1). If $\|u_0\|_{L^2(0,L)} << 1$ then,*

$$\|u\|^2_{L^2(0,T;H^1_0(0,L))} \leq c_1 \frac{\|u_0\|^2_{L^2(0,L)}}{(1 - c_2 \|u_0\|^4_{L^2(0,L)})}$$

*where $c_1 = c_1(T,L)$ and $c_2$ are positive constants. Furthermore, $u_t \in L^{\frac{6}{5}}(0,T;H^{-2}(0,L))$.*

*Proof:* The proof will be done in several steps.

**First estimate:** Multiplying the equation in (1.1) by $u$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,L)} + \frac{1}{2}|u_x(0,t)|^2 + \int_0^L a(x)|u(t)|^2dx = 0. \tag{2.8}$$

Consequently, if $T > 0$, we deduce that

$$\|u\|_{L^\infty(0,T;L^2(0,L))} \leq \|u_0\|_{L^2(0,L)}.$$

**Second estimate:** Now, we multiply the equation in (1.1) by $xu$ to bound $u$ in $L^2(0,T;H^1_0(0,L))$. Indeed, integrating over $(0,L) \times (0,T)$, we get

$$\int_0^T \int_0^L |u_x|^2dxdt + \frac{1}{3}\int_0^L x|u(x,T)|^2dx + \frac{2}{3}\int_0^T \int_0^L xa(x)|u|^2dxdt$$

$$= \frac{1}{3}\int_0^T \int_0^L |u|^2dxdt - \frac{2}{3}\int_0^T \int_0^L x[u]^5u_xdxdt + \frac{1}{3}\int_0^L x|u_0|^2dx. \tag{2.9}$$

Then, integrating by parts and using the boundary it follows that

$$\int_0^T \int_0^L x[u]^5u_x\,dxdt = -\frac{1}{6}\int_0^T \int_0^L |u|^6dxdt. \tag{2.10}$$

Thus, replacing (2.10) into (2.9) and using the first estimate we have the following inequality:

$$\|u\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \frac{(T+L)}{3}\|u_0\|^2_{L^2(0,L)} + \frac{1}{9}\int_0^T \int_0^L |u|^6dxdt. \tag{2.11}$$

On the other hand, from the classical interpolation inequality (Gagliardo-Nirenberg), the Sobolev embedding theorem and (2.8) the following holds

$$\int_0^T \int_0^L |u|^6dxdt \leq C \int_0^T \|u(t)\|^4_{L^\infty(0,L)} \int_0^L |u|^2dxdt$$

$$\leq C \int_0^T \|u(t)\|^2_{L^2(0,L)}\|u_x^m(t)\|^2_{L^2(0,L)}\|u(t)\|^2_{L^2(0,L)}dt$$

$$= C \int_0^T \|u(t)\|^4_{L^2(0,L)}\|u_x(t)\|^2_{L^2(0,L)}dt$$

$$\leq C \|u_0\|^4_{L^2(0,L)} \int_0^T \|u_x^m(t)\|^2_{L^2(0,L)}dt$$

for some constant $C$. The above inequality together with (2.11) and the first estimate, allow us to conclude that

$$(1 - C\|u_0\|_{L^2(0,L)}^4)\|u\|_{L^2(0,T;H_0^1(0,L))}^2 \leq \frac{(T + L)}{3}\|u_0\|_{L^2(0,L)}^2.$$

**Third estimate:** To obtain a bound for $u_t$ we have to pay some attention to the nonlinear term $[u]^4 u_x = \frac{1}{5}\partial_x[u]^5$. First, observe that for $\alpha > 0$ (to be chosen later),

$$\int_0^T \int_0^L |[u]^5|^\alpha dxdt = \int_0^T \int_0^L |u|^{5\alpha-2}|u|^2 dxdt \leq \int_0^T \|u\|_{L^\infty(0,L)}^{5\alpha-2} \int_0^L |u|^2 dxdt$$

$$\leq c \int_0^T \|u_x\|_{L^2(0,L)}^2 \|u\|_{L^2(0,L)}^{2-\frac{5\alpha-2}{2}+\frac{5\alpha-2}{2}} dt$$

$$\leq c \int_0^T \|u_x\|_{L^2(0,L)}^{5\alpha-2} \|u_x\|_{L^2(0,L)}^{-\frac{5\alpha-2}{2}} \|u\|_{L^2(0,L)}^{2+\frac{5\alpha-2}{2}} dt$$

$$\leq c \|u_0\|^{1+\frac{5\alpha}{2}} \int_0^T \|u_x\|_{L^2(0,L)}^{\frac{5\alpha-2}{2}} dt.$$

Choosing $\alpha = \frac{6}{5}$, from the first and the second estimates we get

$$\{[u]^5\} \text{ is bounded in } L^{\frac{6}{5}}((0,T) \times (0,L)).$$

On the other hand, since $L^{\frac{6}{5}}(0,L) \hookrightarrow H^{-1}(0,L)$ (because $H_0^1(0,L) \hookrightarrow L^\infty(0,L)$) we conclude that

$$\{[u]^4 u_x\} = \{\frac{1}{5}\partial_x[u]^5\} \text{ is bounded in } L^{\frac{6}{5}}(0,T;H^{-2}(0,L)).$$

**Fourth estimate:** Now, we can obtain a bound for $\{u_t\}$. Indeed, since

$$u_t = -u_{xxx} - u_x - [u]^4 u_x - a(x)u$$

the previous estimates allows to conclude that

$$\{u_t\} \text{ is bounded in } L^{\frac{6}{5}}(0,T;H^{-2}(0,L)).$$

This completes the proof of Proposition 2.2.

## 3. EXPONENTIAL DECAY

Now we concentrate on the stabilization. The following results will be needed:

**Lemma 3.1** (See [14], Lemma 3.6)**.** *Let $0 < t_1 < t_2 < T$ and $u$ the solution of* (1.1) *obtained in Theorem 2.1. Then, there exists a subinterval $(t_1', t_2') \subset (t_1, t_2)$ such that $u \in L^\infty(t_1', t_2'; H^1(0,L))$.*

**Lemma 3.2** (See [14], Lemma 3.5)**.** *Let $\omega$ and $a = a(x)$ be as in (1.2). If $u \in L^\infty(0,T;H^1(0,L))$ solves (1.1) and $u \equiv 0$ in $\omega$, then $u \equiv 0$ in $(0,L) \times (0,T)$.*

Our main result reads as follows:

**Theorem 3.3.** *Let $u$ be the solution of problem* (1.1) *given by Theorem 2.1 and $\omega$ and $a = a(x)$ as in (1.2). Then, for any $0 < R < 1$ and $T > 0$, there exist positive constants $c = c(R,T)$ and $\mu = \mu(R)$ such that*

$$E(t) \leq c \|u_0\|_{L^2(0,L)}^2 e^{-\mu t}$$

*holds for all $t > 0$ and $u_0$ satisfying $\|u_0\|_{L^2(0,L)} \leq R$.*

*Proof:* Multiply equation in (1.1) by $u$ and integrate in $(0, L)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,L)} + \frac{1}{2}|u(0,t)|^2 + \int_0^L a(x)|u(t)|^2 dx = 0. \tag{3.12}$$

We claim that for any $T > 0$, there exists $c = c(T) > 0$ such that

$$\|u_0\|^2_{L^2(0,L)} \leq c\left[\int_0^T |u_x(0,t)|^2 dt + \int_0^T \int_0^L a(x)u^2 dx dt\right] \tag{3.13}$$

for every solution of (1.1). This fact, together with the energy dissipation law (1.3) and the semigroup property, suffices to obtain the uniform exponential decay. In fact, let us prove (3.13). We multiply equation in (1.1) by $xu$, integrate over $(0, L) \times (0, T)$ and proceed as in steps (2.9)-(2.11) to obtain

$$\|u\|^2_{L^2(0,T;H_0^1(0,L))} \leq C(T), \tag{3.14}$$

for some $C(T) > 0$.

On the other hand, multiplying the equation by $(T - t)u$ and integrating on $(0, L) \times (0, T)$ we obtain the identity

$$T\int_0^L u_0^2 dx = \int_0^T \int_0^L |u|^2 dx dt + \int_0^T (T-t)|u_x(0,t)|^2 dt + 2\int_0^T \int_0^L (T-t)a(x)|u|^2 dx dt. \tag{3.15}$$

Consequently,

$$\int_0^L u_0^2 dx \leq \frac{1}{T}\int_0^T \int_0^L |u|^2 dx dt + \int_0^T |u_x(0,t)|^2 dt + 2\int_0^T \int_0^L a(x)|u|^2 dx dt. \tag{3.16}$$

Therefore, in order to show (3.13) it suffices to prove that for any $T > 0$, there exists a positive constant $C_1(T)$ such that

$$\int_0^T \int_0^L |u|^2 dx dt \leq C_1\left\{\int_0^T |u_x(0,t)|^2 dt + 2\int_0^T \int_0^L a(x)|u|^2 dx dt\right\}. \tag{3.17}$$

Let us argue by contradiction following the so-called "compactness-uniqueness" argument (see for instance [18]). Suppose that (3.17) is not valid. Then, we can find a sequence of functions $\{u_n\} \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$ that solve (1.1) and such that

$$\lim_{n\to\infty} \frac{\|u_n\|^2_{L^2(0,T;L^2(0,L))}}{\displaystyle\int_0^T |u_{n,x}(0,t)|^2 dt + \int_0^T \int_0^L a(x)u_n^2 dx dt} = +\infty.$$

Let $\lambda_n = \|u_n\|_{L^2(0,T;L^2(0,L))}$ and define $w_n(x,t) = u_n(x,t)/\lambda_n$. For each $n \in \mathbb{N}$ the function $w_n$ solves

$$\begin{cases} w_{n,t} + w_{n,x} + w_{n,xxx} + \lambda_n^4 w_n^4 w_{n,x} + a(x)w_n = 0 & \text{in } (0,L)\times(0,T), \\ w_n(0,t) = w_n(L,t) = 0, \quad t \in (0,T), \\ w_{n,x}(L,t) = 0, \quad t \in (0,T), \\ w_n(x,0) = w_{0,n} = u_n(x,0)/\lambda_n, \quad x \in (0,L). \end{cases} \tag{3.18}$$

Moreover,

$$\|w_n\|_{L^2(0,T;L^2(0,L))} = 1 \tag{3.19}$$

and

$$\int_0^T \int_0^L |w_{n,x}(0,t)|^2 dxdt + \int_0^T \int_0^L a(x)w_n^2 dxdt \longrightarrow 0 \tag{3.20}$$

as $n \to \infty$.

Using (3.16) it follows that $w_n(\cdot, 0)$ is bounded in $L^2(0, L)$. Then, by (3.14) it follows that

$$\|w_n\|_{L^2(0,T;H_0^1(0,L))} \leq C, \quad \forall n \in \mathbb{N} \tag{3.21}$$

for some constant $C > 0$. Also, since

$$w_{n,t} = -w_{n,x} - w_{n,xxx} - \lambda_n^4 w_n^4 w_{n,x} - a(x)w_n \text{ in } \mathcal{D}'(0,T;H^{-2}(0,L))$$

performing as in the previous section, the above estimates guarantee that

$$\{w_{n,t}\} \text{ is bounded in } L^{\frac{6}{5}}(0,T;H^{-2}(0,L)). \tag{3.22}$$

At that point, we claim that the following holds:

*There exists $s > 0$ such that $\{w_n\}$ is bounded in $L^4(0,T;H^s(0,L))$, the embedding $H^s(0,L) \hookrightarrow L^4(0,L)$ being compact.*

In fact, since $\{w_n\}$ is bounded in $L^2(0,T;H_0^1(0,L)) \cap L^\infty(0,T;L^2(0,L))$ by interpolation we can deduce that $\{w_n\}$ is bounded in

$$[L^q(0,T;L^2(0,L)), L^2(0,T;H_0^1(0,L))]_\theta = L^p(0,T;[L^2(0,L), H_0^1(0,L)]_\theta),$$

where $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$ and $0 < \theta < 1$. Thus, choosing $q = \infty$, $\theta = 1/2$, so that $p = 4$, the claim holds with $s = 1/2$, i.e.,

$$[L^2(0,L), H_0^1(0,L)]_{\frac{1}{2}} = H^{\frac{1}{2}}(0,L).$$

Furthermore, the embedding $H^{\frac{1}{2}}(0,L) \hookrightarrow L^4(0,L)$ is compact.

Due to the statement above, (3.22) and classical compactness results ([16], Corollary 4) we can extract a subsequence of $\{w_n\}$, that we also denote by $\{w_n\}$, such that

$$w_n \to w \text{ strongly in } L^4(0,T;L^4(0,L)), \tag{3.23}$$

and by (3.19),

$$\|w\|_{L^2(0,T;L^2(0,L))} = 1. \tag{3.24}$$

Also,

$$\begin{aligned}
0 &= \liminf_{n\to\infty} \left\{ \int_0^T |w_{n,x}|^2 dt + \int_0^T \int_0^L a(x)w_n^2 dxdt \right\} \\
&\geq \int_0^T |w_x(0,t)|^2 dt + \int_0^T \int_0^L a(x)w^2 dxdt.
\end{aligned} \tag{3.25}$$

We now distinguish two situations:

(a) There exists a subsequence of $\{\lambda_n\}$ also denoted by $\{\lambda_n\}$ such that

$$\lambda_n \longrightarrow 0.$$

In this case, the limit $w$ satisfies the linear problem

$$\begin{cases} w_t + w_x + w_{xxx} + a(x)w = 0 & \text{in } (0,L) \times (0,T), \\ w(0,t) = w(L,t) = 0, & t \in (0,T), \\ w_x(L,t) = 0, & t \in (0,T), \\ w \equiv 0, & \text{in } \omega \times (0,T). \end{cases}$$

Then, by Holmgren's Uniqueness Theorem, $w \equiv 0$ in $(0,L) \times (0,T)$ and this contradicts (3.24).

(b) There exists a subsequence of $\{\lambda_n\}$ also denoted by $\{\lambda_n\}$ and $\lambda > 0$ such that

$$\lambda_n \longrightarrow \lambda.$$

In this case, the limit function $w$ solves system (3.18) and so, we apply the Unique Continuation Property (UCP) proved in [14] for the subset $\omega$ obtaining that $w \equiv 0$ in $(0,L) \times (0,T)$ and again, this is a contradiction. Indeed, let $t_1 \in (0,T)$ and let $t_2 \in (t_1,T)$. According to Lemma 3.1, $w \in L^\infty(t'_1, t'_2; H^1(0,L))$ for some interval $(t'_1, t'_2) \subset (t_1, t_2)$. Then, it follows from Lemma 3.2 that $w \equiv 0$ on $(t'_1, t'_2) \times (0,L)$. As $t_2$ is arbitrary close to $t_1$, we obtain by continuity of $w$ in $H^{-1}$ that $w(.,t) = 0$. Thus, $w \equiv 0$.

In summary, we see that, in each of the possible situations (a) and (b) we are in a contradiction. Then, necessarily, (3.17) holds. This complete the proof of Theorem 3.3.

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