ALMOST ALL COCYCLES OVER ANY HYPERBOLIC SYSTEM HAVE NON-VANISHING LYAPUNOV EXPONENTS

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Abstract. We prove that for any $s > 0$ the majority of C^s linear cocycles over any hyperbolic (uniformly or not) ergodic transformation exhibit some non-zero Lyapunov exponent: this is true for an open dense subset of cocycles and, actually, vanishing Lyapunov exponents correspond to codimension-∞. This open dense subset is described in terms of a rather explicit geometric condition involving the behavior of the cocycle over certain homoclinic orbits of the transformation.

Contents

1. Introduction

In its simplest form, a linear cocycle consists of a dynamical system $f : M \to M$ together with a matrix valued function $A : M \to \mathrm{SL}(d, \mathbb{C})$. More generally, it is a morphism of vector bundles covering the transformation f. Linear cocycles arise in many domains of Mathematics and its applications, from dynamics or foliation theory to spectral theory and mathematical economics. One important special case is when f is differentiable and the cocycle corresponds to its derivative: we call this a dynamical cocycle.

Here the main object of interest is the asymptotic behavior of the products of A along the orbits of the transformation f,

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))\, A(x),$$

especially the exponential growth rate (largest Lyapunov exponent)

$$\lambda^+(A, x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|.$$

The limit exists μ-almost everywhere, relative to any f-invariant probability measure μ on M for which the function $\log \|A\|$ is integrable.

We assume that the ergodic system (f, μ) is hyperbolic, possibly non-uniformly. The main result is that, for any $s > 0$, an open and dense subset of C^s cocycles exhibit $\lambda^+(A, x) > 0$ at almost every point. Exponential growth of the norm is typical also in a measure-theoretical

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sense: full Lebesgue measure in parameter space, for generic parametrized families of cocycles.

This provides a sharp counterpart to recent results of Bochi, Viana [3, 4, 5], where it is shown that for a residual subset of all C^0 cocycles the Lyapunov exponent $\lambda^+(A,x)$ is actually zero, unless the cocycle has a property of uniform hyperbolicity in the projective bundle (dominated splitting). Actually, their conclusions hold also in the, much more delicate, setting of dynamical cocycles.

Precise definitions and statements of our results follow.

1.1. **Linear cocycles.** Let $f : M \to M$ be a continuous transformation on a compact metric space M. A *linear cocycle* over f is a vector bundle automorphism $F : \mathcal{E} \to \mathcal{E}$ covering f, where $\pi : \mathcal{E} \to M$ is a finite-dimensional real or complex vector bundle over M. This means that $\pi \circ F = f \circ \pi$ and F acts as a linear isomorphism on every fiber. The archetypical example is the derivative $F = Df$ of a diffeomorphism on a manifold (dynamical cocycle).

Given $r \in \mathbb{N} \cup \{0\}$ and $0 \le \nu \le 1$, we denote by $\mathcal{G}^{r,\nu}(f,\mathcal{E})$ the space of r times differentiable linear cocycles over f with rth derivative ν-Hölder continuous (for $\nu = 0$ this just means continuity), endowed with the $C^{r,\nu}$ topology. For $r \ge 1$ it is implicit that the space M and the vector bundle $\pi : \mathcal{E} \to M$ have C^r structures. Moreover, we fix a Riemannian metric on $\mathcal{E}$ and denote by $\mathcal{S}^{r,\nu}(f,\mathcal{E})$ the subset of $F \in \mathcal{G}^{r,\nu}(f,\mathcal{E})$ such that $\det F_x = 1$ for every $x \in M$.

Let $F : \mathcal{E} \to \mathcal{E}$ be a measurable linear cocycle over $f : M \to M$, and μ be any invariant probability measure such that $\log \|F_x\|$ and $\log \|F_x^{-1}\|$ are μ-integrable. Suppose first that f is invertible. Oseledets' theorem [14] says that almost every point $x \in M$ admits a splitting of the corresponding fiber

$$\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^k\,, \quad k = k(x), \tag{1}$$

and real numbers $\lambda_1(F,x) > \cdots > \lambda_k(F,x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|F_x^n(v_i)\| = \lambda_i(F,x) \quad \text{for every non-zero } v_i \in E_x^i\,. \tag{2}$$

When f is non-invertible, instead of a splitting one gets a filtration into vector subspaces

$$\mathcal{E}_x = F_x^0 > \cdots > F_x^{k-1} > F_x^k = 0$$

and (2) is true for $v_i \in F_x^{i-1} \setminus F_x^i$ and as $n \to +\infty$. In either case, the *Lyapunov exponents* $\lambda_i(F,x)$ and the *Oseledets subspaces* E_x^i, F_x^i are uniquely defined μ-almost everywhere, and they vary measurably with the point x. Clearly, they do not depend on the choice of the Riemannian structure.

In general, the largest exponent $\lambda^+(F,x) = \lambda_1(F,x)$ describes the exponential growth rate of the norm on forward orbits:

$$\lambda^+(F,x) = \lim_{n\to+\infty} \frac{1}{n} \log \|F_x^n\|\,. \tag{3}$$

Finally, the exponents $\lambda_i(F,x)$ are constant on orbits, and so they are constant μ-almost everywhere if μ is ergodic. We denote by $\lambda_i(F,\mu)$ and $\lambda^+(F,\mu)$ these constants.

1.2. **Hyperbolic systems.** We call *hyperbolic system* any pair (f,μ) where $f : M \to M$ is a C^1 diffeomorphism on a compact manifold M with Hölder continuous derivative Df, and μ is a hyperbolic non-atomic invariant probability measure with local product structure. The notions of hyperbolic measure and local product structure are defined in the sequel:

Definition 1.1. An invariant measure μ is called *hyperbolic* if all Lyapunov exponents $\lambda_i(f,x) = \lambda_i(Df,x)$ are non-zero at μ-almost every $x \in M$.

Given any $x \in M$ such that the Lyapunov exponents $\lambda_i(A,x)$ are well-defined and all different from zero, let E_x^u and E_x^s be the sums of all Oseledets subspaces corresponding to positive, respectively negative, Lyapunov exponents. Pesin's stable manifold theorem (see [11, 16, 17, 20]) states that through μ-almost every point x with non-zero Lyapunov exponents there exist C^1 embedded disks $W^s_{loc}(x)$ and $W^u_{loc}(x)$ such that

(a) $W^u_{loc}(x)$ is tangent to E^u_x and $W^s_{loc}(x)$ is tangent to E^s_x at x.

(b) Given $\tau_x < \min_i |\lambda_i(A,x)|$ there exists $K_x > 0$ such that

$$\begin{aligned} \operatorname{dist}(f^n(y_1), f^n(y_2)) &\le K_x e^{-n\tau_x} \operatorname{dist}(y_1,y_2) \quad \text{for all } y_1,y_2 \in W^s_{loc}(x) \text{ and } n \ge 1, \\ \operatorname{dist}(f^{-n}(z_1), f^{-n}(z_2)) &\le K_x e^{-n\tau_x} \operatorname{dist}(z_1,z_2) \quad \text{for all } z_1,z_2 \in W^u_{loc}(x) \text{ and } n \ge 1. \end{aligned} \tag{4}$$

(c) $f\big(W^u_{loc}(x)\big) \supset W^u_{loc}(f(x))$ and $f\big(W^s_{loc}(x)\big) \subset W^s_{loc}(f(x))$.

(d) $W^u(x) = \bigcup_{n=0}^{\infty} f^n\big(W^u_{loc}(f^{-n}(x)\big)$ and $W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}\big(W^u_{loc}(f^n(x)\big)$.

Moreover, the local stable set $W^s_{loc}(x)$ and local unstable set $W^u_{loc}(x)$ depend measurably on x, as C^1 embedded disks, and the constants K_x and τ_x may also be chosen depending measurably on the point. Thus, one may find compact *hyperbolic blocks* $\mathcal{H}(K,\tau)$, whose μ-measure can be made arbitrarily close to 1 by increasing K and decreasing τ, such that

(i) $\tau_x \ge \tau$ and $K_x \le K$ for every $x \in \mathcal{H}(K,\tau)$ and

(ii) the disks $W^s_{loc}(x)$ and $W^u_{loc}(x)$ vary continuously with x in $\mathcal{H}(K,\tau)$.

In particular, the sizes of $W^s_{loc}(x)$ and $W^u_{loc}(x)$ are uniformly bounded from zero on each $x \in \mathcal{H}(K,\tau)$, and so is the angle between the two disks.

Let $x \in \mathcal{H}(K,\tau)$ and y be any point of $\mathcal{H}(K,\tau)$ in a closed δ-neighborhood $B(x,\delta)$ of x. If δ is small enough, depending only on K and τ, then $W^s_{loc}(y)$ intersects $W^u_{loc}(x)$ at exactly one point, and analogously for $W^u_{loc}(y)$ and $W^s_{loc}(x)$. Let

$$\mathcal{N}^u_x(\delta) = \mathcal{N}^u_x(K,\tau,\delta) \subset W^u_{loc}(x) \quad \text{and} \quad \mathcal{N}^s_x(\delta) = \mathcal{N}^s_x(K,\tau,\delta) \subset W^s_{loc}(x)$$

be the (compact) intersection sets obtained in this way. Reducing $\delta > 0$ if necessary, $W^s_{loc}(\xi) \cap W^u_{loc}(\eta)$ consists of exactly one point $[\xi,\eta]$, for every $\xi \in \mathcal{N}^u_x(\delta)$ and $\eta \in \mathcal{N}^s_x(\delta)$. Let $\mathcal{N}_x(\delta)$ be the set of such points $[\xi,\eta]$. By construction, $\mathcal{N}_x(\delta)$ contains $\mathcal{H}(K,\tau) \cap B(x,\delta)$, and its diameter goes to zero when $\delta \to 0$. Moreover, $\mathcal{N}_x(\delta)$ is homeomorphic to $\mathcal{N}^u_x(\delta) \times \mathcal{N}^s_x(\delta)$ via

$$(\xi,\eta) \mapsto [\xi,\eta]\,. \tag{5}$$

Definition 1.2. A hyperbolic measure μ has *local product structure* if for μ-almost every point x and every small $\delta > 0$ as before, the restriction $\nu = \mu \mid \mathcal{N}_x(\delta)$ is equivalent to the product measure $\nu^u \times \nu^s$, where ν^u and ν^s are the projections of ν to $\mathcal{N}^u_x(\delta)$ and $\mathcal{N}^s_x(\delta)$, respectively.

Lebesgue measure has local product structure if it is hyperbolic; this follows from the absolute continuity of Pesin's stable and unstable foliations [16]. The same is true, more generally, for any hyperbolic probability having absolutely continuous conditional measures along unstable manifolds or along stable manifolds.

1.3. **Uniformly hyperbolic homeomorphisms.** The assumption that f is differentiable will never be used directly: it is needed only to ensure the geometric structure (Pesin stable and unstable manifolds) described in the previous section. Consequently, our arguments remain valid in the special case of uniformly hyperbolic homeomorphisms, where such structure is part of the definition. In fact, the conclusions take a stronger form in this case, as we shall see.

Let us begin by defining precisely what we mean by uniformly hyperbolic homeomorphism. This includes the two-sided shifts of finite type and the restrictions of Axiom A diffeomorphisms to hyperbolic basic sets, among other examples. Let $f : M \to M$ be a continuous transformation on a compact metric space. The stable set of a point $x \in M$ is defined by

$$W^s(x) = \{y \in M : \operatorname{dist}(f^n(x), f^n(y)) \to 0 \text{ when } n \to +\infty\}$$

and the stable set of size $\varepsilon > 0$ of $x \in M$ is defined by

$$W^s_\varepsilon(x) = \{y \in M : \operatorname{dist}(f^n(x), f^n(y)) \le \varepsilon \text{ for all } n \ge 0\}.$$

If f is invertible the unstable set and the unstable set of size ε are defined similarly, with f^{-n} in the place of f^n.

Definition 1.3. We say that a homeomorphism $f : M \to M$ is *uniformly hyperbolic* if there exist $K > 0$, $\tau > 0$, $\varepsilon > 0$, $\delta > 0$, such that for every $x \in M$

(1) $\operatorname{dist}(f^n(y_1), f^n(y_2)) \le Ke^{-\tau n} \operatorname{dist}(y_1, y_2)$ for all $y_1, y_2 \in W^s_\varepsilon(x)$, $n \ge 0$;
(2) $\operatorname{dist}(f^{-n}(z_1), f^{-n}(z_2)) \le Ke^{-\tau n} \operatorname{dist}(z_1, z_2)$ for all $z_1, z_2 \in W^u_\varepsilon(x)$, $n \ge 0$;
(3) if $\operatorname{dist}(x_1, x_2) \le \delta$ then $W^u_\varepsilon(x_1)$ and $W^s_\varepsilon(x_2)$ intersect at exactly one point, denoted $[x_1\,, x_2]$, and this point depends continuously on (x_1, x_2).

The notion of local product structure extends immediately to invariant measures of uniformly hyperbolic homeomorphisms; by convention, every invariant measure is hyperbolic. In this case K, τ, δ may be taken the same for all $x \in M$, and $\mathcal{N}_x(\delta)$ is a neighborhood of x in M. We also note that every equilibrium state of a Hölder continuous potential [9] has local product structure; see for instance [8].

1.4. **Statement of results.** Let $\pi : \mathcal{E} \to M$ be a finite-dimensional real or complex vector bundle over a compact manifold M, and $f : M \to M$ be a C^1 diffeomorphism with Hölder continuous derivative. We say that a subset of $\mathcal{S}^{r,\nu}(f, \mathcal{E})$ has *codimension-∞* if it is locally contained in finite unions of closed submanifolds with arbitrary codimension.

Theorem A. *For every r and ν with $r + \nu > 0$, and any ergodic hyperbolic measure μ with local product structure, the set of cocycles F such that $\lambda^+(F, x) > 0$ for μ-almost every $x \in M$ contains an open and dense subset of $\mathcal{S}^{r,\nu}(f, \mathcal{E})$. Moreover, its complement has codimension-∞.*

The following corollary provides an extension to the non-ergodic case:

Corollary B. *For every r and ν with $r + \nu > 0$, and any invariant hyperbolic measure μ with local product structure, the set of cocycles F such that $\lambda^+(F, x) > 0$ for μ-almost all $x \in M$ contains a residual (dense G_δ) subset $\mathcal{A}$ of $\mathcal{S}^{r,\nu}(f, \mathcal{E})$.*

Now let $\pi : \mathcal{E} \to M$ be a finite-dimensional real or complex vector bundle over a compact metric space M, and $f : M \to M$ be a uniformly hyperbolic homeomorphism. In this case, one recovers the full conclusion of Theorem A even in the non-ergodic case.

Corollary C. *For every r and ν with $r+\nu > 0$, and any invariant measure μ with local product structure, the set of cocycles F such that $\lambda^+(F, x) > 0$ for μ-almost all $x \in M$ contains an open and dense subset $\mathcal{A}$ of $\mathcal{S}^{r,\nu}(f, \mathcal{E})$. Moreover, its complement has codimension-∞.*

The conclusion of Corollary C was obtained before by Bonatti, Gomez-Mont, Viana [7], under the additional assumptions that the cocycle is dominated (a partial hyperbolicity condition, see Section 6.2) and the measure is ergodic. Then the set $\mathcal{A}$ may be chosen independent of μ. Also in that partially hyperbolic setting, Bonatti, Viana [8] get a stronger conclusion: all Lyapunov exponents have multiplicity 1, that is, all Oseledets subspaces E^i are one-dimensional. This should be true in general:

Conjecture. Theorems A and the two corollaries remain true if one replaces $\lambda^+(F, x) > 0$ by all Lyapunov exponents $\lambda_i(F, x)$ having multiplicity 1.

It is important to notice that the regularity hypothesis $r + \nu > 0$ in our statements is necessary: results of Bochi [3] and Bochi, Viana [4, 5] show that generic C^0 cocycles over general transformations often have vanishing Lyapunov exponents. Even more, for generic L^p cocycles, $1 \le p < \infty$, the Lyapunov exponents always vanish, by Arbieto, Bochi [1] and Arnold, Cong [2].

1.5. **Comments on the proofs.** For proving these results it suffices to consider $\nu \in \{0, 1\}$: the Hölder cases $0 < \nu < 1$ are immediately reduced to the Lipschitz one $\nu = 1$ by replacing the metric $\operatorname{dist}(x, y)$ in M by $\operatorname{dist}(x, y)^\nu$. So, we always suppose $r + \nu \geq 1$.

In the proofs we focus on the case when the vector bundle is trivial: $\mathcal{E} = M \times \mathbb{K}^d$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then $A(x) = F_x$ may be seen as a $d \times d$ matrix with determinant 1, and we identify $\mathcal{S}^{r,\nu}(f, \mathcal{E})$ with the space $\mathcal{S}^{r,\nu}(M, d)$ of $C^{r,\nu}$ maps from M to $\mathrm{SL}(d, \mathbb{K})$. The $C^{r,\nu}$ topology is defined by the norm

$$\|A\|_{r,\nu} = \max_{0 \leq i \leq r} \sup_{x \in M} \|D^i A(x)\| + \sup_{x \neq y} \frac{\|D^r A(x) - D^r A(y)\|}{\operatorname{dist}(x, y)^\nu}$$

(for $\nu = 0$ omit the last term). The case of a general vector bundle is treated in the same way, using local trivializing charts.

For the time being we do not need μ to be ergodic: ergodicity will intervene only at the very end of the proof in Section 5. Local product structure is used in Sections 3.2, 4.2, and 5.3. In Section 6 we discuss a number of extensions and open problems.

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2. Dominated behavior and invariant foliations

Let μ be a hyperbolic measure and $A \in \mathcal{S}^{r,\nu}(M, d)$ define a cocycle over $f : M \to M$. Let $\mathcal{H}(K, \tau)$ be a hyperbolic block associated to constants $K > 0$ and $\tau > 0$, as in Section 1.2. Given $N \geq 1$ and $\theta > 0$, let $\mathcal{D}_A(N, \theta)$ be the set of points x satisfying

$$\prod_{j=0}^{k-1} \|A^N(f^{jN}(x))\| \, \|A^N(f^{jN}(x))^{-1}\| \leq e^{kN\theta} \quad \text{for all } k \geq 1, \tag{6}$$

together with the dual condition, where f and A are replaced by their inverses.

Definition 2.1. Given $s \geq 1$, we say that x is *s-dominated for A* if it is in the intersection of $\mathcal{H}(K, \tau)$ and $\mathcal{D}_A(N, \theta)$ for some K, τ, N, θ with $s\theta < \tau$.

Notice that if B is an invertible matrix and $B_\#$ denotes the action of B on the projective space, then $\|B\| \, \|B^{-1}\|$ is an upper bound for the norm of the derivatives of $B_\#$ and $B_\#^{-1}$. Hence, this notion of domination means that the contraction and expansion exhibited by the cocycle along projective fibers are weaker, by a definite factor larger than s, than the contraction and expansion of the base dynamics along the corresponding stable and unstable manifolds.

2.1. **Generic dominated points.** Here we prove that almost every point $x \in M$ with $\lambda^+(A, x) = 0$ is s-dominated for A, for every $s \geq 1$.

Lemma 2.2. *For any $\delta > 0$ and almost every $x \in M$ there exists $N \geq 1$ such that*

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{N} \log \|A^N(f^{jN}(x))\| \leq \lambda^+(A, x) + \delta \quad \textit{for all } k \geq 1. \tag{7}$$

Proof. Fix $\varepsilon > 0$ small enough so that $4\varepsilon \sup \log \|A\| < \delta$. Let $\eta \geq 1$ be large enough so that the set Δ_η of points $x \in M$ such that

$$\frac{1}{\eta} \log \|A^\eta(x)\| \leq \lambda^+(A, x) + \frac{\delta}{2}$$

has $\mu(\Delta_\eta) \geq (1 - \varepsilon^2)$. Let $\tau(x)$ be the average sojourn time of the f^η-orbit of x inside Δ_η, and Γ_η be the subset of points for which $\tau(x) \geq 1 - \varepsilon$. By sub-multiplicativity of the norms

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{l\eta} \log \|A^{l\eta}(f^{jl\eta}(x))\| \leq \frac{1}{kl} \sum_{j=0}^{kl-1} \frac{1}{\eta} \log \|A^\eta(f^{j\eta}(x))\| \tag{8}$$

for any $x \in \Gamma_\eta$ and any $k, l \geq 1$. Fix l large enough so that for any $n \geq l$ at most $(1 - \tau(x) + \varepsilon)n$ of the first iterates n of x under f^η fall outside Γ_η. Then the right hand side of the previous inequality is bounded by

$$\frac{\delta}{2} + (1 - \tau(x) + \varepsilon) \sup \log \|A\| \leq \lambda^+(A, x) + \frac{\delta}{2} + 2\varepsilon \sup \log \|A\| < \lambda^+(A, x) + \delta,$$

recall that Lyapunov exponents are constant on orbits. Therefore, x satisfies (7) with $N = l\eta$. On the other hand,

$$\mu(\Gamma_\eta) + (1 - \varepsilon)\mu(M \setminus \Gamma_\eta) \geq \int \tau(x)\, d\mu(x) = \mu(\Delta_\eta) \geq (1 - \varepsilon^2)$$

implies that $\mu(\Gamma_\eta) \geq (1 - \varepsilon)$. Thus, making $\varepsilon \to 0$ we get the conclusion (7) for μ-almost every $x \in M$. □

Remark 2.3. When μ is ergodic the proof of Lemma 2.2 gives some $N \geq 1$ such that

$$\limsup_{l \to \infty} \frac{1}{l} \sum_{j=0}^{l-1} \frac{1}{N} \log \|A^N(f^{jN}(x))\| \leq \lambda^+(A, x) + \delta \quad \text{for } \mu\text{-almost every } x.$$

Indeed, ergodicity implies $\mu(\Gamma_\eta) = 1$. Take $k = 1$. For every $x \in \Gamma_\eta$ the expression in (8) is smaller than $\lambda^+(A, x) + \delta$ if l is large enough.

Corollary 2.4. *Given $\theta > 0$ and $\lambda \geq 0$ such that $d\lambda < \theta$, then μ-almost every $x \in M$ with $\lambda^+(A, x) \leq \lambda$ is in $\mathcal{D}_A(N, \theta)$ for some $N \geq 1$. In particular, μ-almost every $x \in M$ with $\lambda^+(A, x) = 0$ is s-dominated for A, for every $s \geq 1$.*

Proof. Fix δ such that $d\lambda + d\delta < \theta$. Let x and N be as in Lemma 2.2:

$$\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{N} \log \|A^N(f^{jN}(x))\| \leq \lambda^+(A, x) + \delta \quad \text{for all } k \geq 1.$$

Since $\det A^N(z) = 1$ we have $\|A^N(z)^{-1}\| \leq \|A^N(z)\|^{d-1}$ for all $z \in M$. So, the previous inequality implies

$$\frac{1}{kN} \sum_{j=0}^{k-1} \log \left(\|A^N(f^{jN}(x))\| \|A^N(f^{jN}(x))^{-1}\| \right) \leq d\lambda^+(A, x) + d\delta < \theta \quad \text{for all } k \geq 1.$$

This means that x satisfies (6). The dual condition is proved analogously. The second part of the statement is an immediate consequence: given any K, τ, and s, take $s\theta < \tau$ and $\lambda = 0$, and apply the previous conclusion to the points of $\mathcal{H}(K, \tau)$. □

2.2. **Strong-stable and strong-unstable sets.** We are going to show that if $x \in M$ is 2-dominated then the points in the corresponding fiber have strong-stable sets and strong-unstable sets, for the cocycle, which are Lipschitz graphs over the stable set and the unstable set of x. For the first step we only need 1-domination:

Proposition 2.5. *Given K, τ, N, θ with $\theta < \tau$, there exists $L > 0$ such that for any $x \in \mathcal{H}(K, \tau) \cap \mathcal{D}_A(N, \theta)$ and any $y, z \in W^s_{loc}(x)$*

$$H^s_{y,z} = H^s_{A,y,z} = \lim_{n \to +\infty} A^n(z)^{-1} A^n(y)$$

exists and satisfies $\|H^s_{y,z} - \mathrm{id}\| \leq L \operatorname{dist}(y, z)$ and $H^s_{y,z} = H^s_{x,z} \circ H^s_{y,x}$.

We begin with the following observation:

Lemma 2.6. *There exists $C = C(A, K, \tau, N) > 0$ such that $\|A^n(y)\| \, \|A^n(z)^{-1}\| \leq C e^{n\theta}$ for all y, $z \in W^s_{loc}(x)$, $x \in \mathcal{D}_A(N, \theta)$, and $n \geq 0$.*

Proof. By sub-multiplicativity of the norms,

$$\|A^n(y)\| \, \|A^n(z)^{-1}\| \le C_1 \prod_{j=0}^{k-1} \|A^N(f^{jN}(y))\| \, \|A^N(f^{jN}(z))^{-1}\|$$

where $k = [n/N]$ and the constant $C_1 = C_1(A, N)$. Since $A \in \mathcal{S}^{r,\nu}(M, d)$ with $r + \nu \ge 1$, there exists $L_1 = L_1(A, N)$ such that

$$\|A^N(f^{jN}(y))\| / \|A^N(f^{jN}(x))\| \le \exp\left(L_1 \operatorname{dist}(f^{jN}(x), f^{jN}(y))\right) \le \exp\left(L_1 K e^{-jN\tau}\right)$$

and similarly for $\|A^N(f^{jN}(z))^{-1}\| / \|A^N(f^{jN}(x))^{-1}\|$. It follows that

$$\prod_{j=0}^{k-1} \|A^N(f^{jN}(y))\| \, \|A^N(f^{jN}(z))^{-1}\| \le C_2 \prod_{j=0}^{k-1} \|A^N(f^{jN}(x))\| \, \|A^N(f^{jN}(x))^{-1}\|$$

where $C_2 = \exp(L_1 K \sum_{j=0}^{\infty} e^{-jN\tau})$. The last term is bounded by $C_2 e^{kN\theta} \le C_2 e^{n\theta}$, by domination. Therefore, it suffices to take $C = C_1 C_2$. $\square$

Proof of Proposition 2.5. Each difference $\|A^{n+1}(z)^{-1} A^{n+1}(y) - A^n(z)^{-1} A^n(y)\|$ is bounded by

$$\|A^n(z)^{-1}\| \cdot \|A(f^n(z))^{-1} A(f^n(y)) - \operatorname{id}\| \cdot \|A^n(y)\| .$$

Since A is Lipschitz continuous, the middle factor is bounded by

$$L_2 \operatorname{dist}(f^n(y), f^n(z)) \le L_2 K e^{-n\tau} \operatorname{dist}(y, z),$$

for some $L_2 > 0$ that depends only on A. Using Lemma 2.6 to bound the other factors,

$$\|A^{n+1}(z)^{-1} A^{n+1}(y) - A^n(z)^{-1} A^n(y)\| \le C L_2 K e^{n(\theta - \tau)} \operatorname{dist}(y, z). \tag{9}$$

Since $\theta - \tau < 0$, this proves that the sequence is Cauchy and the limit $H^s_{y,z}$ satisfies

$$\|H^s_{y,z} - \operatorname{id}\| \le L \operatorname{dist}(y, z) \qquad \text{with} \quad L = \sum_{n=0}^{\infty} C L_2 K e^{n(\theta - \tau)}.$$

The last claim in the proposition follows directly from the definition of $H^s_{y,z}$. $\square$

Remark 2.7. If x is dominated for A then it is dominated for any other cocycle B in a C^0 neighborhood. More precisely, if $x \in \mathcal{D}_A(N, \theta)$ then, given any $\theta' > \theta$, we have $x \in \mathcal{D}_B(N, \theta')$ if B is uniformly close to A. Using this observation and the fact that the constants L_1, L_2 may be taken uniform in a neighborhood of the cocycle, we conclude that L itself is uniform in a neighborhood of A. The same comments apply to the constant $\hat{L}$ in the next corollary.

Corollary 2.8. *Given K, τ, N, θ with $2\theta < \tau$, there exists $\hat{L} > 0$ such that for any $x \in \mathcal{H}(K, \tau) \cap \mathcal{D}_A(N, \theta)$ and any y, $z \in W^s_{loc}(x)$,*

$$H^s_{f^j(y), f^j(z)} = \lim_{n \to +\infty} A^n(f^j(z))^{-1} A^n(f^j(y)) = A^j(z) \cdot H^s_{y,z} \cdot A^j(y)^{-1}$$

exists for every $j \ge 1$, and satisfies

$$\|H^s_{f^j(y), f^j(z)} - \operatorname{id}\| \le \hat{L} e^{j(2\theta - \tau)} \operatorname{dist}(y, z) \le \hat{L} \operatorname{dist}(y, z).$$

Proof. The first statement follows immediately from

$$A^n(f^j(z))^{-1} A^n(f^j(y)) = A^j(z) \left[A^{n+j}(z)^{-1} A^{n+j}(y)\right] A^j(y)^{-1}.$$

Using Lemma 2.6 and inequality (9), with n replaced by $n + j$, we deduce

$$\|A^{n+1}(f^j(z))^{-1} A^{n+1}(f^j(y)) - A^n(f^j(z))^{-1} A^n(f^j(y))\| \le C e^{j\theta} C L_2 K e^{(n+j)(\theta - \tau)} \operatorname{dist}(y, z).$$

Summing over $n \ge 0$ we get the second statement, with $\hat{L} = CL$. $\square$

2.3. **Dependence of the holonomies on the cocycle.** In the next lemma we study the differentiability of $H^s_{A,x,y}$ as a function of $A \in \mathcal{S}^{r,\nu}(M,d)$. At this point we assume 3-domination. Notice that $\mathcal{S}^{r,\nu}(M,d)$ is a submanifold of the Banach space of $C^{r,\nu}$ maps from M to the space of all $d \times d$ matrices. Thus, each $T_A\mathcal{S}^{r,\nu}(M,d)$ is a subspace of that Banach space.

Lemma 2.9. *Given K, τ, N, θ with $3\theta < \tau$, there is a neighborhood $\mathcal{U} \subset \mathcal{S}^{r,\nu}(M,d)$ of A such that for any $x \in \mathcal{H}(K,\tau) \cap \mathcal{D}_A(N,\theta)$ and y, $z \in W^s_{loc}(x)$, the map $B \mapsto H^s_{B,y,z}$ is of class C^1 on $\mathcal{U}$, with derivative*

$$\partial_B H^s_{B,y,z} : \dot{B} \mapsto \sum_{i=0}^{\infty} B^i(z)^{-1}\big[H^s_{B,f^i(y),f^i(z)}\, B(f^i(y))^{-1}\, \dot{B}(f^i(y)) \\ - B(f^i(z))^{-1}\, \dot{B}(f^i(z))\, H^s_{B,f^i(y),f^i(z)}\big] B^i(y).$$

Proof. By Remark 2.7, for any $\theta' > \theta$ we may find a neighborhood $\mathcal{U}$ of A, such that $x \in \mathcal{H}(K,\tau) \cap \mathcal{D}_B(N,\theta')$ for all $B \in \mathcal{U}$. Choose $3\theta' < \tau$, then $H^s_{B,y,z}$ is well defined on $\mathcal{U}$. Before proving this map is differentiable, let us check that the expression $\partial_B H^s_{B,y,z}$ is also well-defined.

Let $i \geq 0$. By Lemma 2.6, we have $\|B^i(z)^{-1}\|\|B^i(y)\| \leq Ce^{i\theta'}$. Corollary 2.8 gives

$$\|H^s_{B,f^i(y),f^i(z)} - \mathrm{id}\,\| \leq \hat{L}e^{i(2\theta'-\tau)}\,\mathrm{dist}(y,z).$$

It is clear that $\|B(f^i(y))^{-1}\,\dot{B}(f^i(y))\| \leq \|B^{-1}\|_{r,\nu}\|\dot{B}\|_{r,\nu}$. Moreover, since $B \in \mathcal{S}^{r,\nu}(M,d)$ and $\dot{B} \in T_B\mathcal{S}^{r,\nu}(M,d)$ are Lipschitz continuous,

$$\|B(f^i(y))^{-1}\dot{B}(f^i(y)) - B(f^i(z))^{-1}\dot{B}(f^i(z))\| \leq 2L_3\|\dot{B}\|_{r,\nu}\,Ke^{-i\tau}\,\mathrm{dist}(y,z)$$

where $L_3 = \sup\{\|B^{-1}\|_{r,\nu} : B \in \mathcal{U}\}$. This shows that

$$\|\partial_B H^s_{B,y,z} \cdot \dot{B}\| \leq \sum_{i=0}^{\infty} Ce^{i\theta'}\left[2\hat{L}e^{i(2\theta'-\tau)}L_3 + 2L_3Ke^{-i\tau}\right]\mathrm{dist}(y,z)\|\dot{B}\|_{r,\nu}\,.$$

Thus $\|\partial_B H^s_{B,y,z} \cdot \dot{B}\| \leq \sum_{i=0}^{\infty} C_3 e^{i(3\theta'-\tau)}\,\mathrm{dist}(y,z)\|\dot{B}\|_{r,\nu}$ where $C_3 = 2CL_3(\hat{L}+K)$. This proves that the series does converge.

We have seen in Proposition 2.5 that $H^n_{B,y,z} = B^n(z)^{-1}B^n(z)$ converges to $H^s_{B,y,z}$ as $n \to \infty$. By Remark 2.7, this convergence is uniform on $\mathcal{U}$. Elementary differentiation rules give us that each $H^n_{B,x,y}$ is a differentiable function of B, with derivative

$$\partial_B H^n_{B,y,z} \cdot \dot{B} = B^n(z)^{-1}\sum_{i=0}^{n-1} B^{n-i}(f^i(y))B(f^i(y))^{-1}\dot{B}(f^i(y))B^i(y) \\ - \sum_{i=0}^{n-1} B^i(z)^{-1}B(f^i(z))^{-1}\dot{B}(f^i(z))B^{n-i}(f^i(z))^{-1}B^n(y).$$

So, to prove the lemma it suffices to show that $\partial_B H^n_{B,y,z}$ converges uniformly to $\partial_B H^s_{B,y,z}$ when $n \to \infty$. As a first step we rewrite,

$$\partial_B H^n_{B,y,z} \cdot \dot{B} = \sum_{i=0}^{n-1} B^i(z)^{-1}\big[H^{n-i}_{B,f^i(y),f^i(z)}B(f^i(y))^{-1}\dot{B}(f^i(y)) \\ - B(f^i(z))^{-1}\dot{B}(f^i(z))H^{n-i}_{B,f^i(y),f^i(z)}\big]B^i(y).$$

Let $0 \leq i \leq n-1$. From Corollary 2.8 we find that

$$\|H^{n-i}_{B,f^i(y),f^i(z)} - H^s_{B,f^i(y),f^i(z)}\| \leq \hat{L}e^{i\theta}e^{n(\theta-\tau)}\,\mathrm{dist}(y,z).$$

We deduce that the difference between the ith terms in the expressions of $\partial_B H^n_{B,y,z} \cdot \dot{B}$ and $\partial_B H^s_{B,y,z} \cdot \dot{B}$ is bounded by

$$2Ce^{i\theta}\hat{L}e^{i\theta}e^{n(\theta-\tau)}\operatorname{dist}(y,z)\,L_3\|\dot{B}\|_{r,\nu} \le C_4 e^{2i\theta}e^{n(\theta-\tau)}\operatorname{dist}(y,z)\|\dot{B}\|_{r,\nu}$$

with $C_4 = 2C\hat{L}_3 L$. Using the estimates in the previous paragraph to bound the sum of all terms $i \ge n$ in the expression of $\partial_B H^s_{B,y,z} \cdot \dot{B}$, we obtain

$$\|\partial_B H^n_{B,y,z} \cdot \dot{B} - \partial_B H^s_{B,y,z} \cdot \dot{B}\| \le \big(\sum_{i=0}^{n-1} C_4 e^{2i\theta}e^{n(\theta-\tau)} + \sum_{i=n}^{\infty} C_3 e^{i(3\theta-\tau)}\big)\operatorname{dist}(y,z)\|\dot{B}\|_{r,\nu}\,.$$

The right hand side tends to zero uniformly when $n \to \infty$, so the proof is complete. $\square$

2.4. **Holonomy blocks.** The linear cocycle $F_A(x,v) = (f(x), A(x)v)$ induces a *projective cocycle*

$$f_A : M \times \mathrm{P}(\mathbb{K}^d) \to M \times \mathrm{P}(\mathbb{K}^d)$$

in the projective space $\mathrm{P}(\mathbb{K}^d)$ of $\mathbb{K}^d$. For any $y,\ z \in W^s_{loc}(x)$ let $h^s_{y,z} : \mathrm{P}(\mathbb{K}^d) \to \mathrm{P}(\mathbb{K}^d)$ be the projective map induced by $H^s_{y,z}$. We call $h^s_{x,y}$ the *strong-stable holonomy* between the projective fibers of x and y. This terminology is justified by the next lemma, which says that the Lipschitz graph

$$W^s_{loc}(x,\xi) = \{(y, h^s_{x,y}(\xi)) : y \in W^s_{loc}(x)\}$$

is a *strong-stable set* for every point (x,ξ) in the projective fiber of x. Strong-unstable sets $W^u_{loc}(x)$ and strong-unstable holonomies $h^u_{x,y}$ are defined analogously. The next lemma explains this terminology. Since it is not strictly necessary for our arguments, we omit the proof.

Lemma 2.10. *Let $x \in \mathcal{H}(K,\tau) \cap \mathcal{D}_A(N,\theta)$ with $\theta < \tau$. For every $y \in W^s_{loc}(x)$ and ξ in the projective space,*

(1) $\displaystyle\limsup_{n\to+\infty} \frac{1}{n} \log \operatorname{dist}\big(f^n_A(x,\xi), f^n_A(y, h^s_{x,y}(\xi))\big) \le -\tau$ *for all* $\xi \in \mathcal{E}_x$

(2) $\displaystyle\liminf_{n\to+\infty} \frac{1}{n} \log \operatorname{dist}\big(f^n_A(x,\xi), f^n_A(y,\eta)\big) < -\theta$ *if and only if* $\eta = h^s_{x,y}(\xi)$.

We call *holonomy block for A* any compact set $\mathcal{O}$ that is contained in $\mathcal{H}(K,\tau) \cap \mathcal{D}_A(N,\theta)$ for some K,τ,N,θ with $3\theta < \tau$. By Proposition 2.5, points in the local stable set, respectively local unstable set, of a holonomy block have strong-stable, respectively strong-unstable, holonomies Lipschitz continuous with uniform Lipschitz constant $L = L(A,K,\tau,N,\theta)$. More than that, by Remark 2.7,

Corollary 2.11. *Given any K,τ,N,θ with $3\theta < \tau$, there is a neighborhood $\mathcal{U}$ of A in $\mathcal{S}^{r,\nu}(M,d)$ such that any compact subset $\mathcal{O}$ of $\mathcal{H}(K,\tau) \cap \mathcal{D}_A(N,\theta)$ is a holonomy block for every $B \in \mathcal{U}$, and the Lipschitz constant L for the corresponding strong-stable and strong-unstable holonomies may be taken uniform on the whole $\mathcal{U}$.*

3. Invariant measures of projective cocycles

In this section we assume $\lambda^+(A,x) = 0$ for μ-almost every $x \in M$. Let f_A be the projective cocycle associated to A. We are going to analyze the probability measures m on $M \times \mathrm{P}(\mathbb{K}^d)$, invariant under f_A and projecting to μ under $(x,\xi) \mapsto x$. Such measures always exist, by continuity of f_A and compactness of its domain. A *disintegration* of m is a family of probability measures $\{m_z : z \in M\}$ on the fibers $\mathcal{F}_z = \{z\} \times \mathrm{P}(\mathbb{K}^d)$, such that

$$m(E) = \int m_z\big(\mathcal{F}_z \cap E\big)\, d\mu(z)$$

for every measurable subset E. Such a family exists and is essentially unique, meaning that any two coincide on a full measure subset, cf. Rokhlin [18].

3.1. **Invariance along strong foliations.** Let $\mathcal{O} \subset M$ be a holonomy block with positive μ-measure. By definition, $\mathcal{O}$ is contained in some hyperbolic block $\mathcal{H}(K,\tau)$. Let $\delta > 0$ be some small constant, depending only on (K,τ). Fix any point $x \in \operatorname{supp}(\mu \mid \mathcal{O})$ and let $\mathcal{N}_x^s(\delta) = \mathcal{N}_x^s(K,\tau,\delta)$, $\mathcal{N}_x^u(\delta) = \mathcal{N}_x^u(K,\tau,\delta)$, and $\mathcal{N}_x(\delta) = \mathcal{N}_x(K,\tau,\delta)$ be the sets introduced in Section 1.2. Moreover, let $\mathcal{N}_x^s(\mathcal{O},\delta)$, $\mathcal{N}_x^u(\mathcal{O},\delta)$, $\mathcal{N}_x(\mathcal{O},\delta)$ be the subsets of $\mathcal{N}_x^s(\delta)$, $\mathcal{N}_x^u(\delta)$, $\mathcal{N}_x(\delta)$ obtained replacing $\mathcal{H}(K,\tau)$ by $\mathcal{O}$ in the definitions. By construction, $\mathcal{N}_x(\mathcal{O},\delta)$ contains $\mathcal{O} \cap B(x,\delta)$, and so it has positive μ-measure.

Proposition 3.1. *Let m be any f_A-invariant probability measure that projects down to μ. Then the disintegration $\{m_z\}$ of m is invariant under strong-stable holonomy μ-almost everywhere on $\mathcal{N}_x(\mathcal{O},\delta)$: there exists a full μ-measure subset E^s of $\mathcal{N}_x(\mathcal{O},\delta)$ such that*

$$m_{z_2} = \left(h^s_{z_1,z_2}\right)_* m_{z_1}$$

for every z_1, $z_2 \in E^s$ in the same stable leaf $[z,\mathcal{N}_x^s(\delta)]$.

Replacing f by f^{-1} we get that the disintegration is also invariant under strong-unstable holonomy over a full μ-measure subset E^u of $\mathcal{N}_x(\mathcal{O},\delta)$.

The proof of Proposition 3.1 is based on the following slightly specialized version of Theorem 1 of Ledrappier [13]. Let $(M_*,\mathcal{M}_*,\mu_*)$ be a Lebesgue space (complete probability space with the Borel structure of the interval together with a countable number of atoms), $T : M_* \to M_*$ be a one-to-one measurable transformation, and $B : M_* \to \mathrm{GL}(d,\mathbb{C})$ be a measurable map such that $\log\|B\|$ and $\log\|B^{-1}\|$ are integrable. Denote by F_B the linear cocycle and by f_B the projective cocycle defined by B over T. Let $\lambda^-(B,x)$ be the smallest Lyapunov exponent of F_B at a point x. Recall that $\lambda^+(B,x)$ denotes the largest exponent.

Theorem 3.2 (Ledrappier [13]). *Let $\mathcal{B} \subset \mathcal{M}_*$ be a σ-algebra such that*

(1) $T^{-1}(\mathcal{B}) \subset \mathcal{B}$ mod 0 *and* $\{T^n(\mathcal{B}) : n \in \mathbb{Z}\}$ *generates* $\mathcal{M}_*$ mod 0
(2) *the σ-algebra generated by B is contained in $\mathcal{B}$* mod 0.

If $\lambda^-(B,x) = \lambda^+(B,x)$ at μ_*-almost every point then, for any f_B-invariant measure m on $M_* \times \mathrm{P}(\mathbb{C}^d)$, the disintegration $z \mapsto m_z$ of m along projective fibers is $\mathcal{B}$-measurable mod 0.

We also need the following result, whose proof we postpone to Section 3.3:

Proposition 3.3. *There exists $N \geq 1$ and a family of sets $\{S(z) : z \in \mathcal{N}_x^u(\delta)\}$ such that*

(1) *$[z,\mathcal{N}_x^s(\delta)] \subset S(z) \subset W^s_{loc}(z)$ for all $z \in \mathcal{N}_x^u(\delta)$;*
(2) *for all $l \geq 1$ and $z,\zeta \in \mathcal{N}_x^u(\delta)$, if $f^{lN}(S(\zeta)) \cap S(z) \neq \emptyset$ then $f^{lN}(S(\zeta)) \subset S(z)$.*

We are going to deduce Proposition 3.1 from Theorem 3.2 applied to a modified cocycle, constructed with the aid of Proposition 3.3 in the way we now explain. Since Proposition 3.1 is not affected when one replaces f by any iterate, we may suppose $N = 1$ in all that follows. Consider the restriction $\{S(z) : z \in \mathcal{N}_x^u(\mathcal{O},\delta)\}$ of the family in Proposition 3.3. For each $z \in \mathcal{N}_x^u(\mathcal{O},\delta)$ let $r(z) \geq 0$ be largest such that $f^j(S(z))$ does not intersect the union of $S(w)$, $w \in \mathcal{N}_x^u(\mathcal{O},\delta)$ for all $0 \leq j \leq r(z)$ (possibly $r(z) = \infty$). Take $\mathcal{B} \subset \mathcal{M}$ to be the sub-σ-algebra generated by the family $\{f^j(S(z)) : z \in \mathcal{N}_x^u(\mathcal{O},\delta) \text{ and } 0 \leq j \leq r(z)\}$, that is, $\mathcal{B}$ consists of all measurable sets E which, for every z and j, either contain $f^j(S(z))$ or are disjoint from it. Define $B : M \to \mathrm{GL}(d,\mathbb{C})$ by

$$B(x) = A(f^j(z)) = H^s_{f(x),f^{j+1}(z)} \circ A(x) \circ H^s_{f^j(z),x} \tag{10}$$

if $x \in f^j(S(z))$ for some $z \in \mathcal{N}_x^u(\mathcal{O},\delta)$ and $0 \leq j < r(z)$;

$$B(x) = H^s_{f(x),w} \circ A(x) \circ H^s_{f^j(z),x} \tag{11}$$

if $x \in f^j(S(z))$ for some $z \in \mathcal{N}_x^u(\mathcal{O},\delta)$, $j = r(z)$, and $f^{j+1}(S(z)) \subset S(w)$; and

$$B(x) = A(x) \quad \text{in all other cases.} \tag{12}$$

Lemma 3.4. (1) *$f^{-1}(\mathcal{B}) \subset \mathcal{B}$ and $\{f^n(\mathcal{B}) : n \in \mathbb{N}\}$ generates $\mathcal{M}_*$* mod 0.

(2) *The σ-algebra generated by B is contained in $\mathcal{B}$.*
(3) *The functions $\log\|B\|$ and $\log\|B^{-1}\|$ are bounded.*
(4) *A and B have the same Lyapunov exponents at μ-almost every x.*

Proof. It is clear that $f(\mathcal{B})$ is the sub-σ-algebra generated by $\{f^{j+1}(S(z)) : z \in \mathcal{N}^u_x(\mathcal{O},\delta)$ and $0 \le j \le r(z)\}$. The Markov property in part (2) of Proposition 3.3 implies that this σ-algebra contains $\mathcal{B}$. Equivalently, $f^{-1}(\mathcal{B}) \subset \mathcal{B}$. More generally, $f^n(\mathcal{B})$ is generated by $\{f^{j+n}(S(z)) : z \in \mathcal{N}^u_x(\mathcal{O},\delta)$ and $0 \le j \le r(z)\}$ for each $n \ge 1$. By (4),

$$\operatorname{diam} f^{j+n}(S(z)) \le \operatorname{const} e^{-\tau n} \to 0$$

uniformly as $n \to \infty$. Hence $f^n(\mathcal{B})$, $n \ge 1$ generate $\mathcal{M}$ mod 0. This proves (1). Definitions (10) and (11) imply that $B^{-1}(E)$ is in the σ-algebra $\mathcal{B}$ for every measurable subset E of $\mathrm{SL}(d,\mathbb{C})$. That is the content of statement (2). Claim (3) is clear, except possibly for case (11) of the definition. To handle that case notice that $H^s_{f^{j+1}(\zeta),w}$ and $H^s_{f^j(z),f^j(\zeta)}$ are uniformly close to the identity, by Proposition 2.5 and Corollary 2.8. To prove (4), we consider two cases. If the forward orbit of x never enters the union of $S(z)$, $z \in \mathcal{N}^u_x(\mathcal{O},\delta)$ then $A^n(x) = B^n(x)$ and the statement is obvious. If the forward orbit of x does enter the union at some time $t \ge 0$ then for any $n \ge t$ there are $z, w \in \mathcal{N}^u_x(\mathcal{O},\delta)$ and $0 \le j \le r(w)$ such that

$$B^n(x) = H^s_{f^n(x),f^j(w)} \circ A^{n-t}(f^t(x)) \circ H^s_{z,f^t(x)} \circ A^t(x).$$

The claim follows, observing that time t is fixed and $H^s_{f^n(x),f^j(w)}$ is at bounded distance from the identity, by Proposition 2.5 and Corollary 2.8. $\square$

Proof of Proposition 3.1. The claim will follow from applying Theorem 3.2 with $M_* = M$, $\mathcal{M}_*$ = completion of the Borel σ-algebra of M relative to $\mu_* = \mu$, $T = f$, and B as constructed above. Notice that $(M_*, \mathcal{M}_*, \mu_*)$ is a Lebesgue space (because M is a separable metric space, see[19, Theorem 9]). Since A takes values in $\mathrm{SL}(d,\mathbb{C})$, the sum of all Lyapunov exponents vanishes identically. Therefore,

$$(d-1)\lambda^-(A,x) + \lambda^+(A,x) \le 0 \le \lambda^-(A,x) + (d-1)\lambda^+(A,x).$$

So, $\lambda^+(A,x) = 0$ if and only if $\lambda^-(A,x) = \lambda^+(A,x)$ and, by part (4) of Lemma 3.4, this is equivalent to $\lambda^-(B,x) = \lambda^+(B,x)$. The other hypotheses of the theorem are also granted by Lemma 3.4. Let m be any f_A-invariant measure as in the statement. Invariance means that

$$A(x)_* m_x = m_{f(x)} \quad \mu\text{-almost everywhere.}$$

Define $\tilde m$ to be the probability measure on $M \times \mathrm{P}(\mathbb{K}^d)$ projecting down to μ and with disintegration $\{\tilde m_x\}$ defined by

$$\tilde m_x = \begin{cases} \left(h_{x,f^j(z)}\right)_* m_x & \text{if } x \in f^j(S(z)) \text{ with } \ z \in \mathcal{N}^u_x(\mathcal{O},\delta) \text{ and } 0 \le j \le r(z) \\ m_x & \text{otherwise.} \end{cases}$$

Let us check that $\tilde m$ is f_B-invariant. If $x \in f^j(S(z))$ with $0 \le j < r(z)$ then, by (10),

$$B(x)_* \tilde m_x = \left(h^s_{f(x),f^{j+1}(z)}\right)_* A(x)_* m_x = \left(h^s_{f(x),f^{j+1}(z)}\right)_* m_{f(x)} = \tilde m_{f(x)} \quad \mu\text{-a.s.}$$

Similarly, if $x \in f^j(S(z))$ with $j = r(z)$ and $f^{j+1}(S(z)) \subset S(w)$ then, by (11),

$$B(x)_* \tilde m_x = \left(h^s_{f(x),w}\right)_* A(x)_* m_x = \left(h^s_{f(x),w}\right)_* m_{f(x)} = \tilde m_{f(x)} \quad \mu\text{-a.s.}$$

Case (12) of the definition is obvious. Thus, $\tilde m$ is indeed f_B-invariant. Using Theorem 3.2, we conclude that $x \mapsto \tilde m_x$ is $\mathcal{B}$-measurable mod 0. This implies that there exists a full measure subset E^s of $\mathcal{N}_x(\mathcal{O},\delta)$ such that

$$z_1, z_2 \in E^s \cap S(z) \Rightarrow \tilde m_{z_1} = \tilde m_{z_2} \Leftrightarrow \left(h^s_{z_1,z}\right)_* m_{z_1} = \left(h^s_{z_2,z}\right)_* m_{z_2} \Rightarrow \left(h^s_{z_1,z_2}\right)_* m_{z_1} = m_{z_2}\,.$$

Since $S(z)$ contains $[z, \mathcal{N}^s_x(\delta)]$, this proves the proposition. $\square$

3.2. **Consequences of local product structure.** Here we use, for the first time, that μ has local product structure. The following is a straightforward consequence of the definitions:

$$\operatorname{supp}(\mu \mid \mathcal{N}_x(\mathcal{O},\delta)) = [\operatorname{supp}(\mu^u \mid \mathcal{N}_x^u(\mathcal{O},\delta)), \operatorname{supp}(\mu^s \mid \mathcal{N}_x^s(\mathcal{O},\delta))]. \tag{13}$$

The crucial point in this section is that the conclusion of the next proposition holds for every, not just almost every, point in the support of $\mu \mid \mathcal{N}_x(\mathcal{O},\delta)$.

Proposition 3.5. *Every f_A-invariant measure m projecting down to μ admits a disintegration $\{\tilde{m}_z : z \in M\}$ such that*

(1) $\sup(\mu \mid \mathcal{N}_x(\mathcal{O},\delta)) \ni z \mapsto \tilde{m}_z$ *is continuous relative to the weak topology.*
(2) $\tilde{m}_z$ *is invariant under strong-stable and strong-unstable holonomies everywhere on* $\sup(\mu \mid \mathcal{N}_x(\mathcal{O},\delta))$*:*

$$\tilde{m}_x = \left(h^s_{z,x}\right)_* \tilde{m}_z \quad \textit{and} \quad \tilde{m}_y = \left(h^u_{z,y}\right)_* \tilde{m}_z$$

whenever z, x are in the same local stable manifold, and z, y are in the same local unstable manifold.

Proof. Let $E = E^s \cap E^u$, where E^s and E^u are the full measure subsets of $\mathcal{N}_x(\mathcal{O},\delta)$ given by Proposition 3.1. Since $\mu(\mathcal{N}_x(\mathcal{O},\delta) \setminus E) = 0$ and $\mu \approx \mu^u \times \mu^s$, we have

$$\mu\big([\xi, \mathcal{N}_x^s(\mathcal{O},\delta)] \cap (\mathcal{N}_x(\mathcal{O},\delta) \setminus E)\big) = 0$$

for μ^u-almost every $\xi \in \mathcal{N}_x^u(\mathcal{O},\delta)$. Fix any such ξ. Consider the family $\{\bar{m}_z : z \in M\}$ of probabilities obtained by starting with an arbitrary disintegration $\{m_z : z \in M\}$ of m and forcing strong-unstable invariance from $[\xi, \mathcal{N}_x^s(\mathcal{O},\delta)]$. What we mean by this is that, by definition,

$$\bar{m}_z = (h^u_{\eta,z})_* m_\eta$$

if $z \in [\mathcal{N}_x^u(\mathcal{O},\delta), \eta]$ for some $\eta \in [\xi, \mathcal{N}_x^s(\mathcal{O},\delta)]$, and $\bar{m}_z = m_z$ at all other points. From the definition and the local product structure, we get that $\bar{m}_z = m_z$ at μ-almost every $z \in M$. So, this new family is still a disintegration of m. Moreover, $\bar{m}_z$ varies continuously with z along every unstable leaf $[\mathcal{N}_x^u(\mathcal{O},\delta), \eta]$, as a consequence of the Lipschitz property of holonomies in Proposition 2.5.

Next, fix $\eta \in \mathcal{N}_x^s(\mathcal{O},\delta)$ such that $\mu\big([\mathcal{N}_x^u(\mathcal{O},\delta), \eta] \cap (\mathcal{N}_x(\mathcal{O},\delta) \setminus E)\big) = 0$ and let $\{m_z^s : z \in M\}$ be the family of probabilities obtained starting with the disintegration $\{\bar{m}_z : z \in M\}$ and forcing strong-stable invariance from $[\mathcal{N}_x^u(\mathcal{O},\delta), \eta]$. For the same reasons as before, this third family is again a disintegration of m. By construction, this disintegration is invariant under strong-stable holonomies everywhere on $\mathcal{N}_x(\mathcal{O},\delta)$. Most important, m_z^s varies continuously with z on the whole $\mathcal{N}_x(\mathcal{O},\delta)$.

By a dual procedure, we obtain a disintegration $\{m_z^u : z \in M\}$ varying continuously with z on $\mathcal{N}_x(\mathcal{O},\delta)$ and invariant under strong-stable holonomies everywhere on $\mathcal{N}_x(\mathcal{O},\delta)$. Then m_z^s and m_z^u must coincide almost everywhere. Hence, by continuity, $m_z^s = m_z^u$ at every point $z \in \operatorname{supp}(\mu \mid \mathcal{N}_x(\mathcal{O},\delta))$. Define $\tilde{m}_z = m_z^s = m_z^u$ if $z \in \mathcal{N}_x(\mathcal{O},\delta)$ and $\tilde{m}_z = m_z$ otherwise. The properties in the conclusion of the proposition follow immediately from the construction. □

3.3. **A Markov type construction.** Here we prove Proposition 3.3. Fix $N \geq 1$ such that $Ke^{-N\tau} < 1/4$, then let $g = f^N$. For each $z \in \mathcal{N}_x^u(\delta)$ define $S_0(z) = [z, \mathcal{N}_x^s(\delta)]$ and

$$S_{n+1}(z) = S_0(z) \cup \bigcup_{(j,w) \in Z_n(z)} g^j(S_n(w)) \tag{14}$$

where $Z_n(z)$ is the set of pairs $(j,w) \in \mathbb{N} \times \mathcal{N}_x^u(\delta)$ such that $g^j(S_n(w))$ intersects $S_0(z)$. By induction, $S_{n+1}(z) \supset S_n(z)$ and $Z_{n+1}(z) \supset Z_n(z)$ for all $n \geq 0$. Define

$$S_\infty(z) = \bigcup_{n=0}^{\infty} S_n(z) \quad \text{and} \quad Z_\infty(z) = \bigcup_{n=0}^{\infty} Z_n(z).$$

Then $Z_\infty(z)$ is the set of $(j,w) \in \mathbb{N} \times \mathcal{N}^u_x(\delta)$ such that $g^j(S_\infty(w))$ intersects $S_0(z)$, and

$$S_\infty(z) = S_0(z) \cup \bigcup_{(j,w)\in Z_\infty(z)} g^j(S_\infty(w)). \tag{15}$$

Finally, define

$$S(z) = S_\infty(z) \setminus \bigcup_{(k,\xi)\in V(z)} g^k(S_\infty(\xi)) \tag{16}$$

where $(k,\xi) \in V(z)$ if and only if $g^k(S_\infty(\xi))$ is not contained in $S_\infty(z)$.

Lemma 3.6. *We have $S_0(z) \subset S(z) \subset S_\infty(z) \subset W^s_{loc}(z)$ for all $z \in \mathcal{N}^u_x(\delta)$.*

Proof. Relation (15) and the definition of $V(z)$ imply that $g^k(S_\infty(\xi))$ is disjoint from $S_0(z)$ for all $(k,\xi) \in V(z)$. Since $S_\infty(z)$ contains $S_0(z)$, it follows that $S_0(z) \subset S(z)$. Next, for each $z \in \mathcal{N}^u_x(\delta)$ and $0 \le n \le \infty$, define internal radii

$$\Delta_n = \sup\{\operatorname{dist}(z,\eta) : \eta \in S_n(z) \text{ and } z \in \mathcal{N}^u_x(\delta)\}.$$

It is clear that Δ_0 goes to zero with δ (linearly). Assume δ is small enough so that the local stable manifold of every $z \in \mathcal{N}^u_x(\delta)$ contains the disk of radius $2\Delta_0$ around z. Our choice of N above implies that $\operatorname{diam} g(E) \le Ke^{-N\tau} \operatorname{diam}(E) < (1/4)\operatorname{diam}(E)$ for all $j \ge 1$ and $E \subset W^s_{loc}(z)$. Therefore, the definition (14) gives

$$\Delta_{n+1} \le \Delta_0 + \frac{1}{4} \sup_{w\in\mathcal{N}^u_x(\delta)} \operatorname{diam} S_n(w) \le \Delta_0 + \frac{1}{2}\Delta_n$$

for all $n \ge 0$. By induction, it follows that $\Delta_n \le 2\Delta_0$ for every $n \ge 1$. Then $\Delta_\infty \le 2\Delta_0$, and so $S_\infty(z) \subset W^s_{loc}(z)$ for every $z \in \mathcal{N}^u_x(\delta)$. □

Lemma 3.7. *Suppose $g^l(S(\zeta)) \cap S_\infty(z) \ne \emptyset$. Then, for any $(k,\xi) \in V(z)$,*

(1) *$g^l(S_\infty(\zeta)) \subset S_\infty(z)$ and*
(2) *if $g^l(S(\zeta)) \cap g^k(S_\infty(\xi)) \ne \emptyset$ then $g^l(S(\zeta)) \subset g^k(S_\infty(\xi))$.*

Proof. If $g^l(S(\zeta)) \subset g^l(S_\infty(\zeta))$ intersects $S_0(z)$ then $(l,\zeta) \in Z_\infty(z)$ and the conclusion follows directly from (15). So, to prove the first claim, we only need to consider the case when $g^l(S(\zeta))$ intersects $g^j(S_\infty(w))$ for some $(j,w) \in Z_\infty(z)$. Suppose first that $l \le j$. Then $S(\zeta)$ intersects $g^{j-l}(S_\infty(w))$ and so, by the definition (16) of $S(\zeta)$, we have that $g^{j-l}(S_\infty(w))$ is contained in $S_\infty(\zeta)$. It follows that $g^j(S_\infty(w)) \subset g^l(S_\infty(\zeta))$. This implies that $(l,\zeta) \in Z_\infty(z)$, because $(j,w) \in Z_\infty(z)$, and so $g^l(S_\infty(\zeta)) \subset S_\infty(z)$. Now suppose that $l > j$. Then $g^{l-j}(S(\zeta))$ intersects $S_\infty(w)$. This is analogous to the hypothesis of the lemma, with z replaced by w and l replaced by $l-j < l$. Hence, by induction on l, we may assume that $g^{l-j}(S_\infty(\zeta)) \subset S_\infty(w)$. It follows that $g^l(S_\infty(\zeta)) \subset g^j(S_\infty(w)) \subset S_\infty(z)$, as claimed.

Now we prove the second claim. Suppose $l \le k$. Then $S(\zeta)$ intersects $g^{k-l}(S_\infty(\xi))$. In view of (16), this implies $g^{k-l}(S_\infty(\xi)) \subset S_\infty(\zeta)$. Then, using also claim (1) in this lemma,

$$g^k(S_\infty(\xi)) \subset g^l(S_\infty(\zeta)) \subset S_\infty(z),$$

contradicting the assumption $(k,\xi) \in V(z)$. So, we must have $l > k$. Then $g^{l-k}(S(\zeta))$ intersects $S_\infty(\xi)$. By claim (1) in this lemma, it follows that $g^{l-k}(S(\zeta)) \subset g^{l-k}(S_\infty(\zeta))$ is contained in $S_\infty(\xi)$. That is, $g^l(S(\zeta)) \subset g^k(S_\infty(\xi))$, as we wanted to prove. □

Proof of Proposition 3.3. The first part is contained in Lemma 3.6, since $[z, \mathcal{N}^s_x(\delta)] = S_0(z)$. Let us prove the second part. Recall that $g = f^N$ and we are assuming $g^l(S(\zeta))$ intersects $S(z)$. Then Lemma 3.7(1) gives that $g^l(S(\zeta)) \subset g^l(S_\infty(\zeta)) \subset S_\infty(z)$. So, in view of (16), to prove that $g^l(S(\zeta))$ is contained in $S(z)$ we only have to show that $g^l(S(\zeta))$ is disjoint from $g^k(S_\infty(\xi))$ for all $(k,\xi) \in V(z)$. This is ensured by Lemma 3.7(2): if $g^l(S(\zeta))$ intersected $g^k(S_\infty(\xi))$ then it would be contained in it, in which case it would not intersect $S(z)$. □

4. Periodic points and obstructions to vanishing exponents

The next goal is to exhibit geometric obstructions to the vanishing of Lyapunov exponents, in terms of holonomies over local stable and local unstable sets of periodic points of f. To this end, we construct holonomy blocks $\tilde{\mathcal{O}}$ containing any number of dominated periodic points.

4.1. **Dominated periodic points.** Let p be a periodic point of f, and $\kappa \geq 1$ be its period. Suppose p is hyperbolic, with hyperbolicity constants K and τ. We fix $s = 3$ in what follows, and say that p is *dominated* if it is in $\mathcal{D}_A(N, \theta)$ for some N and θ with $s\theta < \tau$. An equivalent condition is that there be $P \geq 1$ and θ with $s\theta < \tau$ such that

$$\|A^{\kappa P}(p)\| \, \|A^{\kappa P}(p)^{-1}\| \leq e^{\kappa P \theta}. \tag{17}$$

Indeed, (17) implies $p \in \mathcal{D}_A(\kappa P, \theta)$, by periodicity, and $p \in \mathcal{D}_A(N, \theta)$ implies (17) with $P = N$, by sub-multiplicity of norms.

Suppose p is dominated, and let z be any point in the local stable set $W^s_{loc}(p)$. Let $H^s_{p,z} = H^s_{A,p,z}$ and $h^s_{p,z} = h^s_{A,p,z}$ be the corresponding strong-stable holonomies. Recall that

$$H^s_{A,p,z} = \lim_{n \to +\infty} A^{\kappa n}(z)^{-1} A^{\kappa n}(p),$$

and $h^s_{A,p,z}$ is the projectivization of $H^s_{A,p,z}$. In particular, these holonomies depend only on the values of A on the local stable manifold of p.

Proposition 4.1. *Let $p \in M$ be a dominated periodic point for $A \in \mathcal{S}^{r,\nu}(M, d)$. Then there is a neighborhood $\mathcal{U}$ of A such that for any $z \in W^s_{loc}(p)$ the map $B \mapsto h^s_{B,p,z}$ is of class C^1 on $\mathcal{U}$. Moreover, given any linearly independent points $\xi_1, \ldots, \xi_d$ in $\mathrm{P}(\mathbb{K}^d)$,*

$$\mathcal{U} \ni B \mapsto \big(h^s_{B,p,z}(\xi_1), \ldots, h^s_{B,p,z}(\xi_d)\big) \in \mathrm{P}(\mathbb{K}^d)^d \tag{18}$$

is a submersion, even restricted to maps with values prescribed outside a neighborhood of z.

In other words, for every $B \in \mathcal{U}$ and any neighborhood U of z, the restriction of (18) to those maps which coincide with B outside U is differentiable at B and the derivative is surjective.

Remark 4.2. The proof uses the following property of $G = \mathrm{SL}(d, \mathbb{K})$, shared by other matrix groups: given any linearly independent $\eta_1, \ldots, \eta_d \in \mathrm{P}(\mathbb{K}^d)$, the map

$$G \to \mathrm{P}(\mathbb{K}^d)^d, \quad \beta \mapsto (\beta(\eta_1), \ldots \beta(\eta_d))$$

is a submersion. Equivalently (think of the η_i as norm 1 vectors),

$$\{(\dot\beta(\eta_1), \ldots, \dot\beta(\eta_d)) : \dot\beta \in T_\beta G\} + \big(\mathbb{K}\eta_1 \times \cdots \times \mathbb{K}\eta_d\big) = (\mathbb{K}^d)^d,$$

for every $\beta \in G$. The class of matrix groups to which our arguments apply is discussed in Section 6.3.

Firstly, we note that the evaluation $\mathrm{ev}_z : \mathcal{S}^{\nu,r}(M, d) \to \mathrm{SL}(d, \mathbb{K})$, $B \mapsto B(z)$ is always a submersion, even restricted to maps with values prescribed outside a neighborhood of z.

Lemma 4.3. *Let $B \in \mathcal{S}^{r,\nu}(M, d)$, $z \in M$, and U be a neighborhood of z. For every $\dot\beta \in T_{B(z)}\,\mathrm{SL}(d, \mathbb{K})$ there exists a C^1 curve $(-\varepsilon, \varepsilon) \ni t \mapsto B_t \in \mathcal{S}^{r,\nu}(M, d)$ such that $B_0 = B$, $(\partial_t B_t)_{t=0}(z) = \dot\beta$, and $B_t = B$ outside U for all t.*

Proof. Let $(-\varepsilon, \varepsilon) \ni t \mapsto \beta_t \in \mathrm{SL}(d, \mathbb{K})$ be a C^1 curve such that $\beta_0 = B(z)$ and $(\partial_t \beta_t)_{t=0} = \dot\beta$. Let $\tau : M \to [0, 1]$ be a $C^{r,\nu}$ function such that $\tau(z) = 1$ and $\tau(w) = 0$ if $w \notin U$. Define

$$(-\varepsilon, \varepsilon) \ni t \mapsto B_t \in \mathcal{S}^{r,\nu}(M, d) \quad \text{by} \quad B_t(w) = \beta_{t\tau(w)} B(z)^{-1} B(w).$$

Then $B_0 = B$ and $B_t(w) = B(w)$ for all $t \in (-\varepsilon, \varepsilon)$ and $w \notin U$. The curve $t \mapsto B_t$ is C^1, with derivative $\tau(w)\partial_t \beta_{t\tau(w)} B(z)^{-1} B(w)$. In particular, $(\partial_t B_t)_{t=0}(z) = (\partial_t \beta_t)_{t=0} = \dot\beta$. $\square$

Proof of Proposition 4.1. The first statement is a direct consequence of Lemma 2.9, since the projectivization $\mathrm{SL}(d,\mathbb{K}) \to \mathrm{PSL}(d,\mathbb{K})$ is a smooth map. To prove the second one, let U be any neighborhood of z. Restricting to cocycles that coincide with B outside U means that we consider tangent vectors $\dot B$ with $\dot B(w) = 0$ for every $w \notin U$. It is no restriction to take U small enough so that it is disjoint from $\{f^j(p), f^j(z) : j \ge 1\}$. Then the expression of the derivative of $B \mapsto H_{B,p,z}$ given in Lemma 2.9 reduces to

$$\partial_B H^s_{B,p,z} \cdot \dot B = -B(z)^{-1} \dot B(z) H^s_{B,p,z}\,.$$

Thus, the derivative of $B \mapsto \big(H^s_{B,p,z}(\xi_i)\big)_{i=1,\dots,d} \in (\mathbb{K}^d)^d$ (think of the ξ_i as norm 1 vectors) is

$$\dot B \mapsto \big(- B(z)^{-1} \dot B(z) H^s_{B,p,z}(\xi_i)\big)_{i=1,\dots,d}\,.$$

Clearly, the $\eta_i = H^s_{B,p,z}(\xi_i)$ are linearly independent. By Lemma 4.3, $\dot B(z)$ takes all the values in $T_{B(z)}\,\mathrm{SL}(d,\mathbb{K})$. Therefore, using the property in Remark 4.2,

$$\big\{\big(\dot B(z)\eta_i\big)_{i=1,\dots,d} : \dot B \in T_B\mathcal{S}^{r,\nu}(M,d)\big\} + (\mathbb{K}B(z)\eta_1 \times \cdots \times \mathbb{K}B(z)\eta_d) = (\mathbb{K}^d)^d.$$

Multiplying by $-B(z)^{-1}$ on the left, we find that

$$\big\{\big(\partial_B H^s_{B,p,z}\dot B(\xi_i)\big)_{i=1,\dots,d} : \dot B \in T_B\mathcal{S}^{r,\nu}(M,d)\big\} \oplus (\mathbb{K}\eta_1 \times \cdots \times \mathbb{K}\eta_d) = (\mathbb{K}^d)^d.$$

Since $h^s_{B,p,z}$ is the projectivization of $H^s_{B,p,z}$, this means that

$$T_B\mathcal{S}^{r,\nu}(M,d) \ni \dot B \mapsto \partial_B h^s_{B,p,z} \cdot \dot B$$

is surjective at every $B \in \mathcal{S}^{r,\nu}(M,d)$ as claimed. □

From Lemma 4.3 we also get the following useful consequence:

Corollary 4.4. *Given periodic points $p_1,\dots,p_k$ of f, with minimum periods $\kappa_1,\dots,\kappa_k$,*

$$A \mapsto \big(A^{\kappa_1}(p_1),\dots,A^{\kappa_k}(p_k)\big) \in \mathrm{SL}(d,\mathbb{K})^k$$

is a submersion at every $A \in \mathcal{S}^{r,\nu}(M,d)$.

Proof. For each $j = 1,\dots,k$, write $\beta_j = A^{\kappa_j}(p_j)$ and let $\dot\beta_j$ be any tangent vector to $\mathrm{SL}(d,\mathbb{K})$ at β_j. Fix a neighborhood U_j of each p_j, small enough so that these neighborhoods be pairwise disjoint and p_j be the unique point in the intersection of U_j with these periodic orbits. Using Lemma 4.3 with $z = p_j$ and $U = U_j$, successively for $j = 1,\dots,k$, we obtain a C^1 curve $(-\varepsilon,\varepsilon) \mapsto A_t$ in $\mathcal{S}^{r,\nu}(M,d)$ such that $A_0 = A$, $A_t = A$ outside $U_1 \cup \cdots \cup U_k$, and

$$(\partial_t A_t)_{t=0}(p_j) = A^{-\kappa_j+1}(p_j)\dot\beta_j \quad \text{for } j = 1,\dots,k.$$

Then $A_t^{\kappa_j}(p_j) = A^{\kappa_j-1}(f(p_j))A_t(p_j)$ and so

$$(\partial_t A_t^{\kappa_j})_{t=0}(p_j) = A^{\kappa_j-1}(f(p_j))(\partial_t A_t)_{t=0}(p_j) = \dot\beta_j\,.$$

This proves that the derivative of $A \mapsto \big(A^{\kappa_j}(p_j)\big)_{j=1,\dots,k}$ is surjective, as claimed. □

4.2. **Holonomy blocks containing periodic points.** Let $M_0 = \{x \in M : \lambda^+(A,x) = 0\}$ and assume $\mu(M_0) > 0$. We are going to prove that there exist holonomy blocks containing any given number of (dominated) periodic points. More precisely,

Proposition 4.5. *Given $\varepsilon > 0$ and $\ell \ge 1$, there exists a holonomy block $\tilde{\mathcal{O}}$ of A such that $\mu(M_0 \setminus \tilde{\mathcal{O}}) < \varepsilon$ and there exist ℓ distinct dominated periodic points $p_1,\dots,p_\ell \in \tilde{\mathcal{O}}$ such that*

1. *every $W^u_{loc}(p_i)$ intersects every $W^s_{loc}(p_j)$ at exactly one point*
2. *and every $p_i \in \operatorname{supp}\big(\mu \mid \tilde{\mathcal{O}} \cap f^{-\kappa_i}(\tilde{\mathcal{O}})\big)$, where $\kappa_i = \operatorname{per}(p_i)$.*

The main tool is the following classical result of Katok [12], that extends the shadowing lemma (see Bowen [9]) to the non-uniformly hyperbolic setting. The Main Lemma in [12] is stated in terms of a family $\Lambda_{\chi,\ell}$ of hyperbolic blocks defined through a number of uniformity conditions, the precise form of which does not concern us here. We take $\mathcal{K}_j = \Lambda_{\chi_j,\ell_j}$ with $\chi_j \to \infty$ and $\ell_j \to \infty$ as $j \to \infty$, and it suffices to know that $\mu(\mathcal{K}_j)$ goes to 1 as $j \to \infty$.

Theorem 4.6 (Katok [12])**.** *Given $j \geq 1$ there are $K > 0$, $\tau > 0$, $\rho > 0$, and given $\gamma > 0$ there is $\varepsilon > 0$ such that, for any $z \in \mathcal{K}_j$ and $\kappa \geq 1$ with $f^\kappa(z) \in \mathcal{K}_j$ and $\operatorname{dist}(f^\kappa(z), z) < \varepsilon$, there exists a periodic point $p \in M$ of period κ such that*

(1) *p is a hyperbolic point for f and the eigenvalues α_s of $Df^\kappa(p)$ satisfy $|\log|\alpha_s|| > \kappa\tau$. Moreover, $\operatorname{dist}(f^n(x), f^n(y)) \leq Ke^{-\tau n}\operatorname{dist}(x,y)$ for all $n \geq 0$ and $x, y \in W^s_{loc}(p)$. Analogously for $W^u_{loc}(p)$ with f^n replaced by f^{-n}.*
(2) *$W^s_{loc}(p)$ has size $> \rho$ and is uniformly transverse (angle $> \rho$) to the unstable sets of all points $w \in \mathcal{K}_j$ in the ρ-neighborhood of z. In particular, $W^s_{loc}(p)$ intersects $W^u_{loc}(w)$ at exactly one point. Analogously, interchanging stable with unstable.*
(3) *$\operatorname{dist}(f^j(p), f^j(z)) < \gamma$ for every $0 \leq j \leq \kappa$.*

The uniform bound on the eigenvalues of p is not explicitly stated in [12], but is easily read out from the proof, for instance from (3.42) and (3.44). The rest of statement (1) is also part of the proof of the Main Lemma of [12]: see Proposition 2.4(ii) and bounds (3.38) and (3.40). The uniform estimates in statement (2) are part of the definition of $(s,1)-$admissible and $(u,1)-$admissible curves, see [12, page 153], and the intersection property is given by Proposition 2.5 in [12]. Let us also comment on the way part (3) is proven in [12], as we shall need a similar argument in a while. Uniform transversality of the local invariant manifolds gives that the local stable manifold of p intersects the local unstable manifold of z at a unique point ζ, and the distances from ζ to p and z are bounded by $C\varepsilon$ for some constant $C > 0$. Then $f^\kappa(\zeta)$ is a heteroclinic point of p and $f^\kappa(z)$, and the distances from it to the latter points are also bounded by $C\varepsilon$. It follows that

$$\begin{aligned} &\operatorname{dist}(f^j(p), f^j(\zeta)) \leq Ke^{-\tau j}\operatorname{dist}(p,\zeta) \leq Ke^{-\tau j}C\varepsilon \quad \text{and} \\ &\operatorname{dist}(f^j(\zeta), f^j(z)) \leq Ke^{-\tau(\kappa-j)}\operatorname{dist}(f^\kappa(\zeta), f^\kappa(z)) \leq Ke^{-\tau(\kappa-j)}C\varepsilon, \end{aligned} \tag{19}$$

and so $\operatorname{dist}(f^j(p), f^j(\zeta)) \leq 2KC\varepsilon$. Choosing ε small with respect to γ, one gets the claim.

Proof of Proposition 4.5. Clearly, it is no restriction to suppose ε is smaller than $\mu(M_0)$. Fix $j \geq 1$ such that $\mu(M_0 \setminus \mathcal{K}_j) < \varepsilon/2$. Take K, τ, ρ as given by Theorem 4.6. Fix $\theta > 0$ such that $s\theta < \tau$. By Corollary 2.4, for μ-almost every $x \in M_0$ there exists $N \geq 1$ such that $x \in \mathcal{D}_A(N,\theta)$. Notice that $\mathcal{D}_A(N,\theta)$ increases when N is replaced by a multiple, by sub-multiplicity of the norm. Thus, we may choose N such that the measure of $M_0 \setminus \mathcal{D}_A(N,\theta)$ is less than $\varepsilon/2$. Then $\mathcal{O} = \mathcal{K}_j \cap \mathcal{D}_A(N,\theta)$ is a holonomy block. Moreover, $\mu(M_0 \setminus \mathcal{O}) < \varepsilon$ and so $\mu(\mathcal{O})$ is positive. Fix any point $x \in \operatorname{supp}(\mu \mid \mathcal{O})$.

Lemma 4.7. *Given $\varepsilon > 0$, there are ℓ distinct points $z_1, \ldots, z_\ell$ and there are $\kappa_1, \ldots, \kappa_\ell \in \mathbb{N}$ such that*

(1) *both z_i and $f^{\kappa_i}(z_i)$ are in $B(x, \rho/2)$, and $\operatorname{dist}(f^{\kappa_i}(z_i), z_i) < \varepsilon$;*
(2) *$z_i \in \operatorname{supp}\big(\mu \mid \mathcal{O} \cap f^{-\kappa_i}(\mathcal{O})\big)$; in particular both z_i and $f^{\kappa_i}(z_i)$ are in $\operatorname{supp}(\mu \mid \mathcal{O})$.*

Moreover, we may choose $\min\{\operatorname{dist}(z_i, z_j) : i \neq j\} \geq r$ with $r > 0$ independent of ε.

Proof. Since $x \in \operatorname{supp}(\mu \mid \mathcal{O})$ and μ is non-atomic, there exist distinct points $\zeta_1, \ldots, \zeta_\ell$ in $B(x, \rho/2) \cap \operatorname{supp}(\mu \mid \mathcal{O})$. Fix any $r > 0$ such that $\operatorname{dist}(\zeta_i, \zeta_j) > r$ for all $i \neq r$. For each $i = 1, \ldots, \ell$ and any $\varepsilon > 0$, we may find a compact set $\Gamma_i \subset B(\zeta_i, \varepsilon/2) \cap \mathcal{O}$ with $\mu(\Gamma_i) > 0$. Moreover, we may choose $\Gamma_i \subset B(x, \rho/2)$ with $\operatorname{dist}(\Gamma_i, \Gamma_j) \geq r$ for all $i \neq j$. By the Poincaré recurrence theorem, there exist $\kappa_i \geq 1$ such that $\Gamma_i \cap f^{-\kappa_i}(\Gamma_i)$ has positive measure. Pick any z_i in the support of $(\mu \mid \Gamma_i \cap f^{-\kappa_i}(\Gamma_i))$. Since Γ_i is contained in $B(x, \rho/2) \cap B(\zeta_i, \varepsilon/2)$, part

(1) of the lemma follows immediately. Since $\Gamma_i \subset \mathcal{O}$, part (2) is also a direct consequence. Finally, it is clear from the construction that $\operatorname{dist}(z_i, z_j) \geq r$. □

We may assume that the κ_i are all multiples of N: it suffices to use the recurrence theorem for f^N instead of f. This observation will be useful in Lemma 4.10. Now from Theorem 4.6 we obtain (see Figure 1),

Corollary 4.8. *For every $\gamma > 0$ there exist ℓ distinct periodic points $p_1, \ldots, p_\ell \in B(x, \rho/2)$, with periods $\kappa_1, \ldots, \kappa_\ell$ satisfying*

1. $\operatorname{dist}(f^n(x), f^n(y)) \leq Ke^{-\tau n} \operatorname{dist}(x, y)$ *for all $n \geq 0$ and $x, y \in W^s_{loc}(p_i)$. Analogously for the unstable manifold, replacing f by its inverse.*
2. $W^s_{loc}(p_i)$ *has size $> \rho$ and intersects $W^u_{loc}(w)$ at exactly one point, for every $w \in \mathcal{O}$ in $B(z_i, \rho) \supset B(x, \rho/2)$, and the same is true if we interchange stable with unstable.*
3. $\operatorname{dist}(f^j(p_i), f^j(z_i)) < \gamma$ *for every $0 \leq j \leq \kappa_i$.*

Proof. It is no restriction to consider $\gamma < r/2$. Let $\varepsilon > 0$ be as in Theorem 4.6 and then take z_i and κ_i as in Lemma 4.7. The theorem gives, for each $i = 1, \ldots, \ell$, a periodic point p_i with period κ_i satisfying (1), (2), (3). For part (2) notice that $B(x, \rho/2) \subset B(z_i, \rho)$, because $z_i \in B(x, \rho/2)$. Finally, the choice of γ ensures that the p_i's are all distinct. □

In particular, from part (2) of the lemma we get that these periodic points are all heteroclinically related, as claimed in part (1) of Proposition 4.5: the local unstable manifold of every p_i intersects the stable manifold of every p_j, transversely, at exactly one point. The main step to get part (2) of Proposition 4.5 is to construct a new holonomy block $\tilde{\mathcal{O}} \supset \mathcal{O}$ such that $p_i \in \tilde{\mathcal{O}}$ and p_i is in the support of $\mu \mid \tilde{\mathcal{O}} \cap f^{-\kappa_i}(\tilde{\mathcal{O}})$ for every $1 \leq i \leq N$. Let us explain how this is done, with the aid of Figure 1.

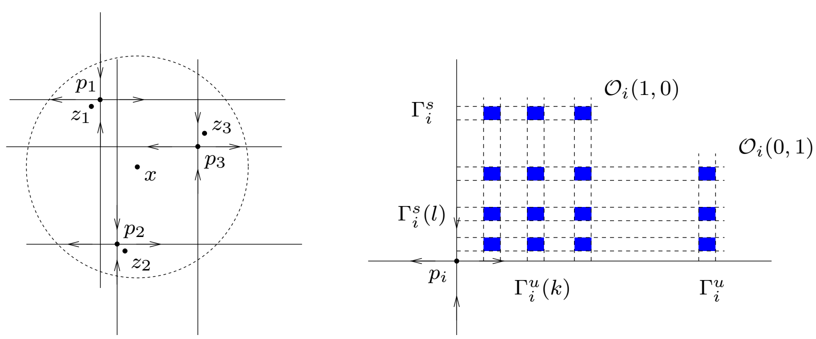

FIGURE 1. Extended holonomy block

Let $\nu > 0$ be small. Since z_i is in the support $(\mu \mid \mathcal{O} \mid f^{-\kappa_i}(\mathcal{O}))$, we may find a compact set with $\mu(\mathcal{O}_i) > 0$ such that $\mathcal{O}_i \subset B(z_i, \nu) \cap \mathcal{O}$ and $f^{\kappa_i}(\mathcal{O}_i) \subset B(f^{\kappa_i}(z_i), \nu) \cap \mathcal{O}$. Reducing ν if necessary, and recalling that $f^{\kappa_i}(z_i)$ and z_i are close to each other, we may suppose that both $\mathcal{O}_i$ and $f^{\kappa_i}(\mathcal{O}_i)$ are contained in $B(z_i, \rho)$. Then we may use part (2) of Corollary 4.8 to conclude that $W^s_{loc}(p_i)$ intersects the local unstable set of every point in $\mathcal{O}_i$ and $W^u_{loc}(p_i)$ intersects the local stable set of every point in $f^{\kappa_i}(\mathcal{O}_i)$. Let $\Gamma^s_i \subset W^s_{loc}(p)$ and $\Gamma^u_i \subset W^u_{loc}(p_i)$ be the corresponding (compact) intersections. Denote

$$\Gamma^u_i(k) = f^{-\kappa_i k}(\Gamma^u_i) \quad \text{and} \quad \Gamma^s_i(l) = f^{\kappa_i l}(\Gamma^s_i), \quad \text{for } k,\, l \geq 0,$$

with the convention $\Gamma^u_i(\infty) = \Gamma^s_i(\infty) = \{p_i\}$. Reducing ν if necessary, the $\Gamma^u_i(k)$ are pairwise disjoint and so are the $\Gamma^s_i(l)$. The λ-lemma (see [15]) implies that, for every $k + l \geq 1$, the local stable manifolds through $\Gamma^u_i(k)$ intersect the local unstable manifolds through $\Gamma^s_i(l)$

transversely, with angles uniformly bounded from zero. Let $\mathcal{O}_i(k,l)$ be the corresponding (compact) intersection set. Notice that $\mathcal{O}_i(1,0) = \mathcal{O}_i$ and $\mathcal{O}_i(0,1) = f^{\kappa_i}(\mathcal{O}_i)$. Finally, define

$$\tilde{\mathcal{O}} = \mathcal{O} \cup \bigcup_{k+l\geq 1} \mathcal{O}_i(k,l).$$

It is clear that $\mu(M_0 \setminus \tilde{\mathcal{O}}) \leq \mu(M_0 \setminus \mathcal{O}) < \varepsilon$. The λ-lemma also implies that local stable manifolds and local unstable manifolds have sizes uniformly bounded from zero, and vary continuously with the point over the whole $\tilde{\mathcal{O}}$. In addition,

Lemma 4.9. *There is $K' > 0$ such that, given any $\xi \in \tilde{\mathcal{O}}$ and $\xi', \xi'' \in W^s_{loc}(\xi)$,*

$$\operatorname{dist}(f^n(\xi'), f^n(\xi'')) \leq K' e^{-\tau n} \operatorname{dist}(\xi', \xi'') \quad \textit{for every } n \geq 0,$$

and analogously for $\xi', \xi'' \in W^u_{loc}(\xi)$ and $n \leq 0$.

Proof. It follows from $\mathcal{O} \subset \mathcal{H}(K,\tau)$ that every local stable manifold through $\mathcal{O}$ is contracted by $\leq K e^{-\tau n}$ under every forward iterate f^n. The same is true for the local stable manifold of any $\xi \in W^s_{loc}(p_i)$, according to part (1) of Corollary 4.8. Then, just continuity, the local stable manifold of any $\xi \in W^s(\Gamma^s_i(k))$ is contracted by $\leq 2Ke^{-\tau n}$ under f^n, if k is large enough and we restrict ourselves to iterates inside some small neighborhood of $W^s_{loc}(p_i)$. Then, choosing a convenient $K_i > 2K$, the local stable manifold of any $\xi \in \mathcal{O}_i(k,l)$ with $k \geq 1$ is contracted by $\leq K_i e^{-\tau n}$ under every f^n with $n \leq \kappa_i k$. By construction, the $f^{\kappa_i k}$-image of any such stable manifold is contained in a local stable manifold through $\mathcal{O}_i(0,1) \subset \mathcal{O}$. So, in view of the first sentence in the proof, we have the conclusion of the lemma as long as $K' > K_i K$. □

This means we may consider $\tilde{\mathcal{O}}$ a subset of a hyperbolic block $\mathcal{H}(K',\tau)$. Hence, the next lemma proves that $\tilde{\mathcal{O}}$ is a holonomy block.

Lemma 4.10. *Fix $\theta' > \theta$ with $s\theta' < \tau$ and assume γ was chosen sufficiently small. Then $\tilde{\mathcal{O}}$ is contained in $\mathcal{D}_A(N,\theta')$.*

Proof. By construction, $\mathcal{O} \subset \mathcal{D}_A(N,\theta) \subset \mathcal{D}_A(N,\theta')$. Therefore, we only have to prove that every $\mathcal{O}_i(k,l)$ is contained in $\mathcal{D}_A(N,\theta')$. Let $\zeta \in \mathcal{O}_i(k,l)$ for some i, k, l. The first step is to observe that, if $k > 0$,

$$\operatorname{dist}(f^j(\zeta), f^j(z_i)) < \gamma \quad \text{for all} \quad 0 \leq j \leq \kappa_i\,. \tag{20}$$

This follows from the same argument as part (3) of Theorem 4.6, recall (19). By a similar calculation, if $k = 0$ and $w \in B(f^{\kappa_i}(z_i), \nu) \cap \mathcal{O}$ is such that $\zeta \in W^s_{loc}(w)$,

$$\operatorname{dist}(f^j(\zeta), w) < \gamma \quad \text{for all} \quad j \geq 0. \tag{21}$$

Notice also that $f_i^{\kappa}(\mathcal{O}_i(k,l)) \subset \mathcal{O}_i(k-1,l+1)$ whenever $k > 0$. Assume γ has been chosen small enough so that

$$\operatorname{dist}(\xi,\eta) \leq \gamma \quad \Rightarrow \quad \|A^N(\xi)\|\,\|A^N(\xi)^{-1}\| \leq e^{N(\theta'-\theta)} \|A^N(\eta)\|\,\|A^N(\eta)^{-1}\|\,.$$

As observed before, we may suppose that the κ_i are multiples of N. Denote $\bar{m} = k\kappa_i/N$. The relation (20) implies that $\operatorname{dist}(f^{jN}(\zeta), z_i) < \gamma$, and so

$$\|A^N(f^{jN}(\zeta))\|\,\|A^N(f^{jN}(\zeta))^{-1}\| \leq e^{N(\theta'-\theta)} \|A^N(f^{jN}(z_i))\|\,\|A^N(f^{jN}(z_i))^{-1}\|$$

for all $j < \bar{m}$. Consequently, recalling that $z_i \in \mathcal{D}_A(N,\theta)$, we obtain

$$\begin{aligned} &\prod_{j=0}^{m-1} \|A^N(f^{jN}(\zeta))\|\,\|A^N(f^{jN}(\zeta))^{-1}\| \\ &\qquad \leq e^{mN(\theta'-\theta)} \prod_{j=0}^{m-1} \|A^N(f^{jN}(z_i))\|\,\|A^N(f^{jN}(z_i))^{-1}\| \leq e^{mN\theta'} \end{aligned} \tag{22}$$

for any $m \le \bar{m}$. Similarly, (21) implies that $\operatorname{dist}(f^{jN}(\zeta), f^{(j-\bar{m})N}(w)) < \gamma$, and so

$$\|A^N(f^{jN}(\zeta))\| \, \|A^N(f^{jN}(\zeta))^{-1}\| \le e^{N(\theta'-\theta)} \|A^N(f^{(j-\bar{m})N}(w))\| \, \|A^N(f^{(j-\bar{m})N}(w))^{-1}\|$$

for every $j \ge \bar{m}$. Therefore, using that $w \in \mathcal{D}_A(N,\theta)$, we obtain

$$\begin{aligned}(23)\qquad &\prod_{j=\bar{m}}^{m-1} \|A^N(f^{jN}(\zeta))\| \, \|A^N(f^{jN}(\zeta))^{-1}\| \\ &\le e^{(m-\bar{m})N(\theta'-\theta)} \prod_{j=0}^{m-\bar{m}-1} \|A^N(f^{jN}(w))\| \, \|A^N(f^{jN}(w))^{-1}\| \le e^{(m-\bar{m})N\theta'}\end{aligned}$$

for every $m > \bar{m}$. Inequalities (22) and (23) show that $\zeta \in \mathcal{D}_A(N,\theta')$, as claimed. □

By construction, $\mathcal{O}$ contains the periodic points p_i for every $i = 1, \dots, N$. Our construction also yields

Lemma 4.11. *We have $p_i \in \operatorname{supp}(\mu \mid \mathcal{O} \cap f^{-\kappa_i}(\mathcal{O}))$ for every $i = 1, \dots, N$.*

Proof. By construction, $\mathcal{O}(1,0) = \mathcal{O}_i$ and $\mathcal{O}(0,1) = f^{\kappa_i}(\mathcal{O}_i)$. Then, more generally,

$$(24)\qquad f^{\kappa_i}(\mathcal{O}_i(k,l-1)) = \mathcal{O}_i(k-1,l) \quad \text{for all } k > 0 \text{ and } l > 0.$$

We claim that $\mu(\mathcal{O}_i(k,l)) > 0$ for all $k + l \ge 1$. In view of (24), and the fact that μ is f-invariant, it suffices to prove this when $l = 0$, say. We do that by induction on k. Notice that the case $k = 1$ corresponds to $\mu(\mathcal{O}_i) > 0$. For the inductive step, suppose it is known that $\mu(\mathcal{O}_i(k,0)) > 0$. By local product structure, it follows that $\mu^u(\Gamma_i^u(k)) > 0$. Moreover, $\mu(\mathcal{O}_i(0,1)) > 0$ implies $\mu^s(\Gamma_i^s(1)) > 0$. Since $\mathcal{O}_i(k,1) = [\Gamma_i^u(k), \Gamma_i^s(1)]$, it follows that $\mu(\mathcal{O}_i(k,1)) > 0$ and, using (24) again, $\mu(\mathcal{O}_i(k+1,0) > 0$. This proves our claim. Now, it is clear that p_i is accumulated by sets $\mathcal{O}_i(k,l)$ with $k > 0$. All these sets are contained in $\tilde{\mathcal{O}}$ and, using (24) once more, also in $f^{-\kappa_i}(\tilde{\mathcal{O}})$. Consequently, p_i is in the support of μ restricted to $\tilde{\mathcal{O}} \cap f^{-\kappa_i}(\tilde{\mathcal{O}})$, as claimed. □

This gives part (2) of Proposition 4.5, and so the proof of the proposition is complete. □

Corollary 4.12. *Let $p_1, \dots, p_\ell$ be dominated periodic points as in Proposition 4.5, and q_i be the point of intersection of $W^s_{loc}(p_i)$ with $W^u_{loc}(p_\ell)$, for $i = 1, \dots, \ell-1$. Consider any points $\xi_a^i \in \mathrm{P}(\mathbb{K}^d)$, $1 \le i \le \ell-1$, $1 \le a \le d$ such that every $\{\xi_1^i, \dots, \xi_d^i\}$ is independent. Then the map*

$$B \mapsto \big(h^s_{p_i,q_i}(\xi_a^i),\ i \in \{1, \dots, \ell-1\},\ a \in \{1, \dots, d\}\big) \in \mathrm{P}(\mathbb{K}^d)^{(\ell-1)d}$$

is a submersion on a neighborhood of A, even restricted to cocycles with values prescribed on a neighborhood of $\{f^{-j}(q_i) : j \ge 1\}$, $1 \le i \le \ell-1$, and $\{f^j(p_i) : 1 \le j \le \kappa_i\}$, $1 \le i \le \ell$.

Proof. This is an application of Proposition 4.1. Indeed, the proposition states that every

$$B \mapsto (h^s_{p_i,q_i}(\xi_a^i),\ a \in \{1, \dots, d\}\big) \in \mathrm{P}(\mathbb{K}^d)^d$$

is a submersion on a neighborhood of A, even restricted to cocycles with values prescribed outside any neighborhood V_i of q_i. We may choose these neighborhoods so that their closures be pairwise disjoint. Then the cocycle may be modified independently on each V_i. It follows that the map in the statement of the corollary is a submersion restricted to cocycles with values prescribed on the complement U of $\bar{V}_1 \cup \dots \cup \bar{V}_{\ell-1}$. By further reducing those neighborhoods, we ensure that U is a neighborhood of every $\{f^{-j}(q_i) : j \ge 1\}$, $1 \le i \le \ell-1$, and every $\{f^j(p_i) : 1 \le j \le \kappa_i\}$, $1 \le i \le \ell$. This gives the claim in the corollary. □

Notice that $h^u_{p_\ell,q_i}$ depends only on the values of the cocycle over $\{f^{-j}(q_i) : j \ge 1\} \cup \{p_\ell\}$. Thus, the corollary implies that the stable holonomy map $B \mapsto (h^s_{p_i,q_i}(\xi_a^i), i, a)$ is a submersion, even under perturbations of the cocycle that do not affect the unstable holonomies

$h^u_{p_\ell,q_i}$ nor the value of the cocycle over the periodic orbit p_i. This is the way the corollary will be used in the next section.

5. Proofs of the main results

5.1. **Complex valued cocycles.** Here we prove Theorem A when $\mathbb{K} = \mathbb{C}$. Let (f,μ) be an ergodic hyperbolic system. Suppose $A \in \mathcal{S}^{r,\nu}(M,d)$ is such that $\lambda^+(A,\mu) = 0$. Fix any $\ell \geq 1$. By Proposition 4.5 there is a positive measure holonomy block $\tilde{\mathcal{O}} \subset M$ containing at least 2ℓ periodic points $p_1, \ldots, p_{2\ell}$ such that the local unstable set of every p_i intersects the local stable set of every p_j at exactly one point. By Corollary 2.11 there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{S}^{r,\nu}(M,d)$ such that $\tilde{\mathcal{O}}$ is a holonomy block for every $B \in \mathcal{U}$. Let κ_i be the minimum period of each p_i. By Corollary 4.4 the map

$$\mathcal{U} \ni B \mapsto (B^{\kappa_1}(p_1), \ldots, B^{\kappa_{2\ell}}(p_{2\ell})) \in \mathrm{SL}(d,\mathbb{C})^{2\ell}$$

is a submersion. Let S be the subset of matrices $\alpha \in \mathrm{SL}(d,\mathbb{C})$ such that the norms of the eigenvalues of α are not all distinct. Clearly, S is closed and contained in a finite union of closed submanifolds of $\mathrm{SL}(d,\mathbb{C})$ with codimension ≥ 1. It follows that the subset $\mathcal{Z}_1$ of $B \in \mathcal{U}$ such that $B^{\kappa_i}(p_i) \in S$ for at least ℓ periodic points p_i is closed and contained in a finite union of closed submanifolds with codimension $\geq \ell$.

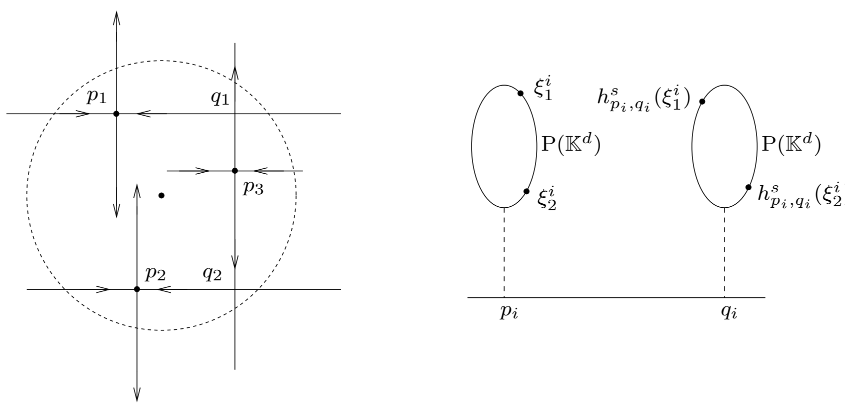

Figure 2. Stable holonomies

For every $B \in \mathcal{U} \setminus \mathcal{Z}_1$ there are at least $\ell+1$ periodic points p_i such that all the eigenvalues of $B^{\kappa_i}(p_i)$ have distinct norms. Restricting to open subsets of $\mathcal{U} \setminus \mathcal{Z}_1$, and renumbering if necessary, may suppose that they are $p_1, \ldots, p_{\ell+1}$. Let $\xi^i_a \in \mathrm{P}(\mathbb{C}^d)$, $a \in \{1, \ldots, d\}$ represent the eigenspaces of $B^{\kappa_i}(p_i)$ and let q_i be the point in $W^s_{loc}(p_i) \cap W^u_{loc}(p_{\ell+1})$, for $i \in \{1, \ldots, \ell+1\}$. By Corollary 4.12 the map

$$\mathcal{U}_1 \ni B \mapsto \left(h^s_{p_i,q_i}(\xi^i_a),\ i \in \{1,\ldots,\ell\},\ a \in \{1,\ldots,d\}\right) \in \mathrm{P}(\mathbb{K}^d)^{\ell d}$$

is a submersion, even restricted to cocycles with values prescribed on the forward orbit of p_i and on $\{f^{-j}(q_i) : j \geq 1\}$, for $1 \leq i \leq \ell$. See Figure 2. It follows that the subset $\mathcal{Z}_2$ of $B \in \mathcal{U} \setminus \mathcal{Z}_1$ such that for every $i \in \{1, \ldots, \ell\}$ there exist $a, b \in \{1, \ldots, d\}$ such that

$$h^s_{p_i,q_i}(\xi^i_a) = h^u_{p_{\ell+1},q_i}(\xi^{\ell+1}_b)$$

is closed and contained in a finite union of closed submanifolds with codimension $\geq \ell$.

For any $B \in \mathcal{U} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$, $i \in \{1, \ldots, \ell\}$, and $a, b \in \{1, \ldots, d\}$,

$$h^s_{p_i,q_i}(\xi^i_a) \neq h^u_{p_{\ell+1},q_i}(\xi^{\ell+1}_b) \tag{25}$$

We claim that $\lambda^+(B,\mu) > 0$ for every $B \in \mathcal{U} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$. Indeed, suppose $\lambda^+(B,\mu)$ vanishes. Let m be any f_B-invariant probability. Proposition 3.5 gives that m admits a disintegration $\{\tilde{m}_z : z \in M\}$ such that

(a) The map $z \mapsto \tilde{m}_z$ is continuous on $\tilde{\mathcal{O}}$, relative to the weak topology.
(b) $\tilde{m}_z$ is invariant under strong-stable and strong-unstable holonomies on the whole $\tilde{\mathcal{O}}$.

Since m is f_B-invariant, $B(z)_*\tilde{m}_z = \tilde{m}_{f(z)}$ for μ-almost every $z \in M$. By Proposition 4.5 each p_i is in the support of $\mu \mid \tilde{\mathcal{O}} \cap f^{-\kappa_i}(\tilde{\mathcal{O}})$. Hence, we may find $z \in \tilde{\mathcal{O}}$ arbitrarily close to p such that $B(z)^{\kappa_i}_* \tilde{m}_z = \tilde{m}_{f^{\kappa_i}(z)}$ and $f^{\kappa_i}(z) \in \tilde{\mathcal{O}}$. Consequently, by continuity (a),

$$B^{\kappa_i}(p_i)_* \tilde{m}_{p_i} = \tilde{m}_{p_i} \quad \text{for all } 1 \le i \le \ell + 1.$$

As $B \notin \mathcal{Z}_1$, this implies that each $\tilde{m}_{p_i}$ is a convex combination of Dirac measures supported on the eigenspaces ξ^i_a, $a \in \{1, \dots, d\}$. Fix a such that $\xi^{\ell+1}_a$ is in the support of $\tilde{m}_{p_{\ell+1}}$. Invariance (b) implies that for every $i \in \{1, \dots, \ell\}$ there is $b \in \{1, \dots, d\}$ such that

$$h^s_{p_i,q_i}(\xi^i_b) = h^u_{p_{\ell+1},q_i}(\xi^i_a).$$

For $B \in \mathcal{U} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ this contradicts (25). This contradiction proves our claim.

Let $\mathcal{Z}_0$ be the subset of $A \in \mathcal{S}^{r,\nu}(M,d)$ such that $\lambda^+(A,\mu) = 0$. We have shown that for every $\ell \ge 1$ and $A \in \mathcal{Z}_0$ there exists a neighborhood $\mathcal{U}$ of A such that $\mathcal{Z}_0 \cap \mathcal{U}$ is contained in a closed nowhere dense subset $\mathcal{Z}_1 \cup \mathcal{Z}_2$ of $\mathcal{U}$, itself contained in a finite union of closed submanifolds with codimension $\ge \ell$. Thus $\mathcal{Z}_0$ has codimension-∞, and its closure $\bar{\mathcal{Z}}_0$ is nowhere dense. Then $\mathcal{A} = \mathcal{S}^{r,\nu}(M,d) \setminus \bar{\mathcal{Z}}_0$ is an open dense subset such that every $A \in \mathcal{A}$ has $\lambda^+(A,\mu) > 0$. The proof of Theorem A is complete, in the complex case.

5.2. **Real valued cocycles.** The previous arguments apply without change in the case $\mathbb{K} = \mathbb{R}$, except for the statement about the set S of matrices whose eigenvalues are not all distinct in norm: this set has non-empty interior in $\mathrm{SL}(d,\mathbb{R})$, corresponding to the existence of pairs of complex conjugate eigenvalues. The way to bypass this is by showing that, up to a perturbation of the cocycle, one may always *choose the periodic points* so that the eigenvalues of the cocycle on the corresponding orbits are all real and distinct in norm. Formally, this means that the first exclusion of a codimension $\ge \ell$ subset $\mathcal{Z}_1$ takes place right after Corollary 4.8, allowing for each p_i to be replaced by a nearby periodic point $\bar{p}_i$ for which the eigenvalues are real and distinct and the corollary remains valid. We are going to outline this step, referring the reader to Section 8 of [8], where the same idea has been used before, for more details.

Start with 2ℓ periodic points p_i as in Corollary 4.8. Fix i for a while. By construction, p_i is dominated and has transverse homoclinic points. Fix some homoclinic point z_i and let H_i be the uniformly hyperbolic set (horseshoe) formed by those points whose orbits remain in a neighborhood of the orbits of p_i and z_i. Taking this neighborhood sufficiently small, the cocycle is dominated restricted to H_i, and so is any perturbation of it. This ensures that the arguments in Section 8 of [8] apply in the present setting. Excluding a codimension 1 subset of cocycles, we may suppose that

(1) all the eigenvalues of $B^{\kappa_i}(p_i)$ are real and have distinct norms, except for $c \ge 0$ pairs of complex conjugate eigenvalues;
(2) $h^s_{p_i,z_i}(E) \cap h^u_{p_i,z_i}(F) = \{0\}$ for any direct sums E and F of eigenspaces of $B^{\kappa_i}(p_i)$ with $\dim E + \dim F \le d$.

Proposition 8.1 of [8] shows how, avoiding another positive codimension subset of cocycles, one can find a new periodic point $\bar{p}_i \in H_i$, with period $\bar{\kappa}_i$ a multiple of κ_i, such that all the eigenvalues of $B^{\bar{\kappa}_i}(\bar{p}_i)$ are real and distinct. Taking the neighborhood of z_i that defines H_i small enough, the conclusion of Corollary 4.8 remains valid for $\bar{p}_i$. In this way, avoiding a codimension ℓ subset of cocycles, we may suppose that the $\bar{p}_i$ are defined for at least ℓ values of i. Up to renumbering, we may suppose they are $i = 1, 2, \dots, \ell$. Now we may replace each

p_i by the corresponding $\bar{p}_i$. From then on the proof of Theorem A proceeds just as in the complex case.

5.3. **Proofs of the corollaries.** We begin with the following simple ergodic decomposition statement:

Lemma 5.1. *If μ is a hyperbolic measure with local product structure, then there exist finite or countably many constants $c_j > 0$ and ergodic hyperbolic probabilities μ_j with local product structure such that $\mu = \sum_j c_j \mu_j$. In the uniformly hyperbolic case the number of ergodic components μ_i is uniformly bounded.*

Proof. Let $M_0 \subset M$ be the full measure subset of points where forward and backward Birkhoff averages exist and coincide, for every continuous function. Consider the equivalence relation defined on M_0 by $x_1 \sim x_2 \Leftrightarrow x_1$ and x_2 have the same Birkhoff averages. The equivalence classes are invariant sets. Let $\mathcal{N}_x(\delta) = \mathcal{N}_x(K, \tau, \delta)$ be as in Section 1.2, for some K, τ, and $x \in \operatorname{supp}(\mu \mid M_0 \cap \mathcal{H}(K, \tau))$. Note that $M_0 \cap \mathcal{N}_x(\delta)$ has positive μ-measure and so, by local product structure, M_0 intersects some unstable set $[\mathcal{N}_x^u(\delta), \eta]$ in a set M_η with positive μ^u-measure. Equivalently, $\mu(M_\eta^s) > 0$ where M_η^s is the union of all stable sets $[\xi, \mathcal{N}_x^s(\delta)]$ through the points of M_η. On the other hand, M_η^s intersects a unique equivalence class, because of the definition of M_0 and the fact that forward (backward) Birkhoff averages of continuous functions are constant on stable (unstable) sets. This proves that there exists an equivalence class $\Gamma_1 \subset M_0$ with $\mu(\Gamma_1) > 0$. Take Γ_1 with largest measure. If $\mu(M_0 \setminus \Gamma_1) > 0$, repeat the argument with M_0 replaced by $M_0 \setminus \Gamma_1$. In this way one constructs a finite or countably many equivalence classes Γ_j with $\mu(\Gamma_j) > 0$ and $\mu(\cup_j \Gamma_j) = 1$. The normalized restrictions $\mu_j = (\mu \mid \Gamma_j)/\mu(\Gamma_j)$ are invariant ergodic probabilities, and $\mu = \sum_j \mu(\Gamma_j) \mu_j$.

It is clear that μ_j is absolutely continuous with respect to μ and so μ_j is a hyperbolic measure. To show that μ_j has local product structure, consider any $\mathcal{N}_z(\delta) = [\mathcal{N}_z^u(\delta), \mathcal{N}_z^s(\delta)]$ with $z \in \operatorname{supp}(\mu \mid \Gamma_j)$. For a measurable set $V \subset \mathcal{N}_z(\delta)$ let V^s be the union of all stable sets $[\xi, \mathcal{N}_z^s(\delta)]$ through points of V, and V^u be the corresponding notion for unstable sets. The hypothesis that μ has local product structure means that $\mu(V) = 0$ if and only if $\mu(V^s) \cdot \mu(V^u) = 0$. Since each stable set and each unstable set intersect at most one equivalence class, $\Gamma_j = \Gamma_j^s = \Gamma_j^u$ up to zero measure sets. So, $(V \cap \Gamma_j)^s = V^s \cap \Gamma_j \bmod 0$ and $(V \cap \Gamma_j)^u = V^u \cap \Gamma_j \bmod 0$. It follows that μ_j has local product structure:

$$\mu_j(V) = 0 \Leftrightarrow \mu(V \cap \Gamma_j) = 0 \Leftrightarrow \mu(V^s \cap \Gamma_j) \cdot \mu(V^u \cap \Gamma_j) = 0 \Leftrightarrow \mu_j(V^s) \cdot \mu_j(V^u) = 0.$$

In the uniformly hyperbolic case K, τ, δ may be taken the same for all $x \in M$. Recall that $\mathcal{N}_x(\delta)$ contains the ball of radius δ around x in M. Since $M_0 \cap \mathcal{N}_x(\delta)$ has full μ-measure in $\mathcal{N}_x(\delta)$, we may choose η such that $M_0 \cap [\mathcal{N}_x^u(\delta), \eta]$ has full μ^u-measure in $[\mathcal{N}_x^u(\delta), \eta]$. Then M_η^s has full measure in $\mathcal{N}_x(\delta)$. Recall that, M_η^s intersects a unique equivalence class. This proves that a full μ-measure subset of M is covered by equivalence classes each of which contains a full measure subset of some δ-ball. Since δ is uniform, there are only finitely many such equivalence classes. The last claim in the lemma follows. $\square$

This immediately leads to the versions of Theorem A for non-ergodic measures stated in the two Corollaries:

Proof of Corollaries B and C. Let μ be any invariant hyperbolic measure with local product structure. By Lemma 5.1, the measure μ has countably many ergodic components μ_j and they have local product structure. Thus, for each j, Theorem A provides an open dense subset $\mathcal{A}_j$ such that for every $A \in \mathcal{A}_j$ we have $\lambda^+(A, \mu_j) > 0$. Then $\mathcal{A} = \cap_j \mathcal{A}_j$ is a residual subset and $\lambda^+(A, x) > 0$ at μ-almost every point, for every $A \in \mathcal{A}$. This completes the proof of Corollary B. In the uniform case the ergodic components are finitely many, and so $\mathcal{A}$ is open and dense. Moreover, the set $\mathcal{Z}_0(\mu)$ of cocycles A such that $\lambda^+(A, x) = 0$ for a positive μ-measure set of points x is contained in the union of the corresponding sets $\mathcal{Z}_0(\mu_j)$ for all ergodic components. Hence, since every $\mathcal{Z}_0(\mu_j)$ has codimension-∞, so does $\mathcal{Z}_0(\mu)$. This proves Corollary C. □

6. Final remarks

To close, we discuss a number of extensions and related open problems.

6.1. **Non-invertible transformations.** Theorem A remains true if one replaces *hyperbolic* by *expanding* throughout the statements. We define expanding systems (uniformly or not) and explain how these extensions may be deduced, via natural extensions.

The *natural extension* of a (non-invertible) transformation $f : M \to M$ is the map $\hat{f} : \hat{M} \to \hat{M}$ defined as follows: $\hat{M}$ is the space of all sequences $(x_n)_{n\le 0}$ in M such that $f(x_n) = x_{n+1}$ for all $n < 0$, endowed with the metric

$$\hat{d}\big((x_n)_n\,,(y_n)_n\big) = \sum_{n=-\infty}^{0} 2^n \min\{d(x_n\,,y_n),1\},$$

and $\hat{f} : \hat{M} \to \hat{M}$ is the shift map

$$\hat{f}(\ldots, x_n\,, \ldots, x_{-1}\,, x_0) = (\ldots, x_{n+1}\,, \ldots, x_0\,, f(x_0)).$$

Notice that f is bijective and $\pi_0 \circ \hat{f} = f \circ \pi_0$, where $\pi_0 : \hat{M} \to M$ is the canonical projection onto the zeroth coordinate. We always assume M is compact and f is continuous. Then $(\hat{M}, \hat{d})$ is a compact space and $\hat{f}$ is a homeomorphism.

The functions $\psi \circ \pi_0 \circ \hat{f}^{-n} : \hat{M} \to \mathbb{R}$ with $n \ge 0$ and $\psi : M \to \mathbb{R}$ measurable, generate the space of measurable functions on $\hat{M}$. Let μ be an f-invariant probability measure on M. The *lift* of μ is the $\hat{f}$-invariant probability $\hat{\mu}$ defined by

$$\int (\psi \circ \pi_0 \circ \hat{f}^{-n})\, d\hat{\mu} = \int \psi\, d\mu$$

for every $n \ge 1$ and any measurable function $\psi : M \to \mathbb{R}$.

Let $f : M \to M$ be a C^1 local diffeomorphism on a compact manifold M, such that the derivative is Hölder continuous. An f-invariant probability μ is (non-uniformly) *expanding* if all Lyapunov exponents $\lambda_i(f,x) = \lambda_i(Df,x)$ are positive μ-almost everywhere. We call (f,μ) an expanding system.

We say that a continuous transformation $f : M \to M$ on a compact metric space is *uniformly expanding* if it is locally injective and there exist constants $K > 0$, $\tau > 0$, $\varepsilon > 0$, such that for every $x \in M$ and $n \ge 1$ there exists an inverse branch $f_x^{-n} : B(f^n(x),\varepsilon) \to M$ satisfying

(1) $f_x^{-n}(f^n(x)) = x$ and $f^n \circ f_x^{-n} = \mathrm{id}$ on the ball $B(f^n(x),\varepsilon)$;
(2) $\operatorname{dist}(f_x^{-n}(y), f_x^{-n}(z)) \le Ke^{-\tau n} \operatorname{dist}(y,z)$ for every $y,z \in B(f^n(x),\varepsilon)$.

Uniformly expanding maps include, among other examples, one-sided shifts of finite type, as well as local diffeomorphisms $f : M \to M$ on manifolds whose derivative expands uniformly every tangent vector.

If $f : M \to M$ is uniformly expanding then $\hat{f} : \hat{M} \to \hat{M}$ is a uniformly hyperbolic homeomorphism. Indeed, denoting $x = (x_n)_n$, $y = (y_n)_n$, $z = (z_n)_n$, if $\varepsilon > 0$ is small then

(a) $W^s_\varepsilon(x)$ consists of the points $z \in \hat{M}$ such that $z_0 = x_0$ and $\hat{d}(x,z) \le \varepsilon$;
(b) $W^u_\varepsilon(y)$ contains points z with z_0 close to y_0 and $z_n = f^n_{y_n}(z_0)$ for all $n < 0$.

Property (1) in Definition 1.3 follows from the definition of $\hat{d}$, and property (2) follows from condition (b) above. Moreover, $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ may intersect only at the point z defined by $z_0 = x_0$ and $z_n = f^n_{y_n}(z_0)$ for $n < 0$. If x and y are close enough, this point z is well-defined, and it is in the intersection of the two ε-manifolds. This gives property (3) in Definition 1.3.

Similarly, if μ is an expanding measure then $(\hat{f}, \hat{\mu})$ has well defined local stable and local unstable sets at almost every point: local unstable sets project injectively to M under π_0, and the set of all z with $z_0 = x_0$ is a local stable set for each $x \in \hat{M}$. We say that an invariant measure μ has *local product structure* for f if the lift $\hat{\mu}$ has local product structure for the natural extension $\hat{f} : \hat{M} \to \hat{M}$. This is the case, for instance, if (f, μ) has bounded distortion, in the sense that the inverse branches of f^n, $n \geq 1$ admit Jacobians with respect to the measure μ that form an equicontinuous family.

Cocycles defined on M lift canonically to $\hat{M}$, preserving the regularity class $C^{r,\nu}$, in the sense that the extension is $C^{r,\nu}$ along the horizontal and constant along the vertical. The arguments used before for hyperbolic systems apply, without change, to the natural extensions of expanding systems (although $\hat{M}$ is usually not a manifold). So, to get these extensions it suffices to observe that all the perturbations are carried out within the space of cocycles on $\hat{M}$ lifted from M.

6.2. **Uniformity and continuity.** The conclusion of Corollary C was first obtained by Bonatti, Gomez-Mont, Viana [7] under the assumption that the cocycle is partially hyperbolic (or dominated). One says that the cocycle defined by a ν-Hölder function A over a transformation $f : M \to M$ is *dominated* if f is uniformly hyperbolic (or uniformly expanding), with hyperbolicity constants K, τ, and there exists $N \geq 1$ such that

$$\|A^N(x)\| \, \|A^N(x)^{-1}\| < \left(e^{\tau N}\right)^{\nu} \quad \text{for every } x \in M \tag{26}$$

Note that the definition does not depend on the choice of the metric $\|\cdot\|$ on the vector bundle $\hat{\Sigma}_T \times \mathbb{C}^d$, as long as the metric varies ν-Hölder continuously with the base point x. Also, as explained before, one may get rid of the Hölder constant in the definition by replacing the metric $d(\cdot,\cdot)$ in M by a new metric $d(\cdot,\cdot)^\nu$, relative to which A is Lipschitz. In this situation one may choose the subset $\mathcal{A}$ in the statement independent of the measure μ. Our methods fall short of extending this conclusion to the general (non-dominated) case:

Problem 1.

(a) Can the residual subset in Corollary B be chosen the same for every hyperbolic invariant measure μ with local product structure ?

(b) Can the open dense subset in Corollary C be chosen the same for every invariant measure μ with local product structure ?

In either setting, in view of the arguments in Section 5.3 it suffices to consider the ergodic case. A possible approach to Problem 1 goes as follows. For each ergodic measure μ with local product structure each and cocycle A such that $\lambda^+(A, \mu) = 0$, we have found a neighborhood $\mathcal{U}$ of A and a closed nowhere dense subset $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$ containing all the $B \in \mathcal{U}$ such that $\lambda^+(B, \mu) = 0$. This closed set is defined in terms of certain dominated periodic points of f contained in the support of μ. As μ varies, so do the periodic points, and the set $\mathcal{Z}$ with them. However, since there are only countably many of them, the union of all these $\mathcal{Z}$ would be a meager set containing all cocycles having vanishing exponents for some ergodic measure with local product structure. The difficulty with this approach, currently, is that the neighborhood $\mathcal{U}$ itself, where those periodic points remain dominated, also depends on the measure μ.

Problem 2.

(a) Does the closure of the set $\mathcal{Z}_0 \subset \mathcal{S}^{r,\nu}(M, d)$ of cocycles with $\lambda^+(A, \mu) = 0$ have codimension-∞ ?

(b) Is the set $\mathcal{Z}_0$ closed in $\mathcal{S}^{r,\nu}(M, d)$ relative to the $C^{r,\nu}$ topology ?

Of course, the second question is stronger than the first one. Both would follow immediately if we knew that Lyapunov exponents $F \mapsto \lambda_i(F, \mu) = \int \lambda_i(F, x)\, d\mu$ vary continuously on $\mathcal{S}^{r,\nu}(f, \mathcal{E})$ relative to the $C^{r,\nu}$ topology, when $r + \nu > 0$. However, the latter is not true in general. Indeed, Bochi, Viana [3, 4, 5] have given a necessary and sufficient condition for a cocycle F over an arbitrary transformation to be a point of continuity of $F \mapsto \lambda_i(F,\mu)$ relative to the $C^0 = C^{0,0}$ topology: the Oseledets splitting must be either dominated or else trivial over almost every orbit. More precisely, for $d = 2$ say, [3] showed that, unless the Oseledets splitting is uniformly hyperbolic, the cocycle may be C^0 approximated by another whose Lyapunov exponents vanish almost everywhere. A closer look at the arguments shows that they provide examples of discontinuity of the Lyapunov exponents in the $C^{0,\nu}$ topology for small $\nu > 0$. I am grateful to Jairo Bochi for a conversation in the course of which we realized this fact.

Problem 3.

(a) When do Lyapunov exponents $F \mapsto \lambda_i(F,\mu)$ vary continuously on $\mathcal{S}^{r,\nu}(f,\mathcal{E})$ relative to the $C^{r,\nu}$ topology, with $r + \nu > 0$?
(b) In particular, when the base dynamics is uniformly hyperbolic, do Lyapunov exponents vary continuously in the subset of dominated cocycles in $\mathcal{S}^{r,\nu}(f,\mathcal{E})$?

6.3. **Other matrix groups.** We have focussed on normalized cocycles, with values in the group $\mathrm{SL}(d,\mathbb{K})$, but the same arguments apply to general cocycles in $\mathrm{GL}(d,\mathbb{K})$. The statement is: for an open dense subset of $\mathcal{G}^{r,\nu}$ the spectrum is not reduced to a point and, indeed, one-point spectrum has codimension-∞.

More generally, one may consider cocycles with values in a given subgroup $G \subset \mathrm{GL}(d,\mathbb{K})$. For our previous arguments to apply, the group should be sufficiently rich:

(a) Every $\alpha \in G$ can be "approximated" by matrices in G whose eigenvalues are all distinct in norm.
(b) The map $G \ni B \mapsto (B(\xi_1),\ldots,B(\xi_d)) \in \mathrm{P}(\mathbb{K}^d)^d$ is a submersion, for any choice $\{\xi_1,\ldots,\xi_d\}$ of a basis of $\mathbb{K}^d$.

That property (a) holds for $G = \mathrm{SL}(d,\mathbb{C})$ was used in Section 5.1 to prove density. In the case $G = \mathrm{SL}(d,\mathbb{R})$ this assumption took a much more subtle format (which is the reason we write "approximated" in quotation marks), as discussed in Section 5.2. Concerning property (b), recall Remark 4.2. This condition requires $\dim G \geq d(d-1)$ and is not satisfied by the symplectic group $\mathrm{Symp}(d,\mathbb{K})$, for instance.

On the other hand, these two sufficient conditions are probably not optimal for getting the conclusion of the theorem:

Problem 4.

(a) Characterize the class of groups $G \subset \mathrm{GL}(d,\mathbb{K})$ for which the theorem is valid.
(b) Does it include the symplectic group $\mathrm{Symp}(d,\mathbb{K})$?

6.4. **Cocycles over partially hyperbolic maps and flows.** While it is clear that our arguments rely on the base dynamics being fairly "chaotic", it is probably not necessary to assume the full strength of (non-uniform) hyperbolicity. One possible extension would be to cocycles over partially hyperbolic maps with some indecomposability property such as accessibility. We recall the main notions, referring the reader to [6, 10] and references therein for motivations and more information.

A diffeomorphism $f : M \to M$ is *partially hyperbolic* if there exists a Df-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ and there exists $\lambda < 1$ and $N \geq 1$ such that

$$\|Df^N \mid E^s_x\| < \lambda \quad \text{and} \quad \|(Df^N \mid E^s_x)(Df^N \mid E^c_x)^{-1}\| < \lambda$$

and

$$\|(Df^N \mid E^u_x)^{-1}\| < \lambda \quad \text{and} \quad \|(Df^N \mid E^u_x)^{-1}(Df^N \mid E^c_x)\| < \lambda$$

for all x. Assume all three subbundles have positive dimension. One calls f *accessible* if any two points may be joined by a smooth curve $t \mapsto \gamma(t)$ such that $\dot\gamma(t) \in E^u_{\gamma(t)} \cup E^s_{\gamma(t)}$ at every point; the velocity $\dot\gamma$ is allowed to vanish at a finite number of points.

Problem 5.

(a) Almost all $C^{r,\nu}$ cocycles, $r + \nu > 0$ over an accessible partially hyperbolic volume preserving diffeomorphism have some non-zero Lyapunov exponents (additional technical assumptions may be needed).

(b) The problem is equally interesting when the base system is a dissipative diffeomorphism endowed with some physical (Sinai-Ruelle-Bowen) measure.

A cocycle over a flow $f^t : M \to M$, $t \in \mathbb{R}$ is a flow $F^t : M \times \mathbb{K}^d \to M \times \mathbb{K}^d$, $t \in \mathbb{R}$ of the form $F^t(x, v) = (f^t(x), A^t(x)v)$. The cocycle is $C^{r,\nu}$ if $x \mapsto A^t(x)$ is $C^{r,\nu}$ for every $t \in \mathbb{R}$.

Problem 6.

(a) Almost all $C^{r,\nu}$ cocycles, $r + \nu > 0$ over a hyperbolic flow (f^t, μ) with local product structure have some non-zero Lyapunov exponents.

(b) Even more interesting: replace uniformly hyperbolic by Lorenz-like flow.

6.5. **Non-linear cocycles.** Our approach is based on analyzing the projective cocycle $f_A : M \times \mathrm{P}(\mathbb{K}^d) \to M \times \mathrm{P}(\mathbb{K}^d)$ associated to A. The fact that this skew-product comes from a linear map is not really crucial for the arguments, and this suggests that these ideas might be useful in more general, non-linear, settings. One situation we have in mind, are skew-products

$$F : M \times N \to M \times N, (x, \xi) \mapsto (f(x), F_x(\xi))$$

where M and N are compact manifolds, $f : M \to M$ is a homeomorphism, and $x \mapsto F_x$ takes values in the group of diffeomorphisms of N. Assume the derivative $DF_x(\xi)$ is uniformly continuous on (x, ξ), and its norm, as well as the norm of its inverse, are uniformly bounded.

Let m be an F-invariant probability measure and let $\mu = \pi_* m$, where $\pi : M \times N \to M$ is the canonical projection. The (largest) *Lyapunov exponent* of F at (x, ξ) is

$$\lambda(F, x, \xi) = \lim_{n\to\infty} \frac{1}{n} \log \|DF_x^n(\xi)\|.$$

This limit exists and is constant m-almost everywhere if the measure m is ergodic. It would be interesting to understand the behavior of these Lyapunov exponents and, in particular, when they vanish.

A first step in this direction is to extend Ledrappier's Theorem 3.2 to this non-linear setting, and that is indeed possible. A proof will appear elsewhere. For instance, when f is a two-sided subshift of finite type, if F_x depends only on the positive coordinates of x, and the Lyapunov exponent vanishes at almost every point, when deduces that the measure m admits a disintegration into conditional measures along the fibers $\{x\} \times N$ that also depends only on the positive coordinates.

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