On the Asymptotics of Fast Mean-Reversion Stochastic Volatility Models

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We consider the asymptotic behavior of options under stochastic volatility models for which the volatility process fluctuates on a much faster time scale than that defined by the risk-less interest rate. We identify the distinguished asymptotic limits and, in contrast with previous studies, we deal with small volatility-variance (vol-vol) regimes. We derive the corresponding asymptotic formulae for option prices, and find that the first order correction displays a dependence on the hidden state and a non-diffusive terminal layer. Furthermore, this correction cannot be obtained as the small variance limit of the previous calculations. Our analysis also includes the behavior of the asymptotic expansion, when the hidden state is far from the mean. In this case, under suitable hypothesis, we show that the solution behaves as a constant volatility Black-Scholes model to all orders. In addition, we derive an asymptotic expansion for the implied volatility that is uniform in time. It turns out that the fast scale plays an important role in such uniformity. The theory thus obtained yields a more complete picture of the different asymptotics involved under stochastic volatility. It also clarifies the remarkable independence on the state of the volatility in the correction term obtained by previous authors.

Keywords: Stochastic Volatility Models; Asymptotics; Vol-Vol

1. Introduction and Outline

The standard Black-Scholes (B-S) model [4] of constant volatility, despite its tremendous impact in option trading, suffers from a number of drawbacks: The log-normality of the underlying asset prices is not verified by statistical tests and the corresponding option prices are subject to the so-called smile-effect [7]. Several approaches have been considered to mitigate such difficulties. A popular choice being to allow the volatility in the asset dynamics to depend on a stochastic process. This leads to stochastic volatility models. Such models were proposed and analyzed in [18,23,26]. See also [24,16,2,1,22,14,6,20,21,19,25,27] and [8] for a survey and further references.

Stochastic volatility models have a number of advantages. To cite a few of them:

One can find heavier tails than those in the log-normal model of underlying prices, the distribution could be asymmetric and the smile curve appears naturally [21]. However, stochastic volatility models introduce further difficulties that cannot be easily analyzed. One effective approach for such analysis is to exploit the different scales involved, thus drawing from asymptotic techniques. This program has been implemented in a ground breaking way by [8]. See also [10,9,11,5]. A more complete asymptotic description for the case of no correlation and no risk-premium was derived in [17]. A crucial assumption in [5,8,10,9,11,5,17] is that the variance of the volatility process is of order one. However, statistical studies of available data, already performed in [8], suggest the possibility that such variance could be somewhat smaller than order one.

Instead of arguing on the range of validity for the asymptotic results in [8] *vis a vis* actual market data, we take the viewpoint that all plausible regimes should be investigated. In the present work, using a multiple scale approach, we identify the distinguished asymptotic limits and, in contradistinction with [5,8,10,9,11,5,17], we deal with small volatility-variance regimes. In this case, assuming the volatility to be bounded from above and away from zero, we derive the asymptotic first order correction to the Black-Scholes price using a multiple scales approach. We find that such correction splits in an *inner* and in an *outer* solution. The inner solution is dependent on the hidden state, and it it displays a non-uniform time behavior close to expiration. Such behavior takes the form of a terminal layer where a fast time scale plays an important role. We underscore that the corrections obtained herein are not the small-variance limit of the results in [8,17]. Furthermore, the techniques also differ from the WKBJ approach used on a deterministic volatility problem in [13].

The outline of the paper is the following: We start in Section 2 with some background material on stochastic volatility models. In Section 3 we focus on the dimensional analysis of the pricing equation so as to identify the relevant non-dimensional parameters and the different distinguished regimes. In Section 4 we review the work of Fouque, Papanicolaou, and Sircar [8] in the asymptotic study of stochastic volatility models. Section 5 deals with the limit of moderately small variance. In this case, we show that solution is the B-S price with the mean volatility plus a correction that depends on the volatility state. Also, the remarkable independence on y found by Fouque *et al* becomes more apparent. Moreover, this correction displays a non-uniform behavior in time close to expiration—a terminal layer—where a fast time scale is important. Under the hypothesis that the volatility is bounded away from zero and above, and that it is constant outside a not large compact set, we show that, when its driving process is far from the mean, the price is just the B-S price with the maximum/minimum volatility, and that this change is achieved by a an infinite number of terms jumping order in the expansion. We also derive the asymptotic expansion for the implied volatility. Contrary to previous work, the formula is uniformly bounded even for times close to maturity, and it turns out that the fast time scale plays an important role in such uniformity. The same analysis is applied in the case of small variance, and the results are similar. Finally, in Section 7 we draw some conclusions and present some remarks as well as suggestions for further work.

2. Background on Stochastic Volatility Models

We start by briefly reviewing the classical Black-Scholes (B-S) market model so as to fix the notation. We denote by ζ a risk-less asset (bond or insured bank deposit) and by X a risky asset. In the classical B-S model the assets undergo the following dynamics

$$\mathrm{d}\zeta_t = r\zeta_t \mathrm{d}t \qquad \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \tag{2.1}$$

where W_t is the standard Brownian Motion. Let $P(t, x)$ denote the price of an European option at time t and current stock value x. Standard replication and non-arbitrage arguments lead to the classical Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r\left(x\frac{\partial P}{\partial x} - P\right) = 0 \qquad P(T_E, \cdot) = h \tag{2.2}$$

where h is the payoff at time T_E.

As mentioned in the Introduction, motivated by the need of explaining a number of empirical observations many authors consider stochastic volatility models. More precisely, with [8] and references therein, we consider the dynamics

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \qquad \sigma_t = f(Y_t) \qquad dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$$

where $\widehat{Z}_t$ is a linear combination of two independent Brownian motions (W_t) and (Z_t). As in [12], we assume that f is bounded from above and away from zero. In addition to that we also assume f to be at least two times differentiable. In this model, the risky asset's volatility is controlled by a stochastic process $y = Y_t$, which could be thought of as a hidden process. Such process Y_t, in turn, undergoes an Ornstein-Uhlenbeck dynamics. This choice is motivated by the empirical remark that the volatility tends to return to a historical level after some time. The return rate to such mean is denoted by α.

Let $P = P(t, x, y)$ be the price of an European option at time t given that the current stock price is x and its driving state is y. Once again, using a non-arbitrage argument it is well-known [14] that $P(t, x, y)$ satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2} + r(x\frac{\partial P}{\partial x} - P) + (\alpha(m - y) - \beta\Lambda(t, x, y))\frac{\partial P}{\partial y} = 0 \tag{2.3}$$

where

$$\Lambda(t, x, y) = \rho\frac{\mu - r}{f(y)} + \gamma(t, x, y)\sqrt{1 - \rho^2}$$

with final condition $P(T_E, \cdot, \cdot\cdot) = h(\cdot)$. Here the function γ can be interpreted as the risk premium factor from the second source of randomness that drives the volatility (Z_t). To avoid technical difficulties, we assume γ to be bounded and continuous. Furthermore, in Sections 4, 5, and 6, as in [8], we shall assume that γ depends only on y.

Equation (2.3) can be interpreted, as done in [8], considering the operator

$$\begin{aligned}&\frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) + \\ &+ \rho\beta x f(y) \frac{\partial^2}{\partial x \partial y} \\ &+ \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y} - \beta\Lambda\frac{\partial}{\partial y}\end{aligned}$$

The first line consists of the standard Black-Scholes operator with (stochastic) volatility $f(y)$. The second one consists of a correlation term. The third one is the generator for the O-U process added to a premium term associated to the market price of volatility risk.

3. Dimensional Analysis

We now consider the dimensional analysis of Equation (2.3). We subsume the classical Black-Scholes model as giving the correct balance between the quantities

$$\frac{\partial P}{\partial t} \text{ and } r(x\frac{\partial P}{\partial x} - P)$$

as well as

$$\frac{\partial P}{\partial t} \text{ and } \frac{1}{2} f^2(y) x^2 \frac{\partial^2 P}{\partial x^2} \ .$$

Since $x\partial P/\partial x$ and $x^2 \partial^2 P/\partial x^2$ do not depend on the x scale, it is natural to take the adimensional time $\widehat{t}$ as $\widehat{t} = rt$ and the adimensional squared volatility $\widehat{f}^2(y)$ as

$$\widehat{f}^2(y) = r f^2(y).$$

Apparently there is no *a priori* scale for $\Lambda(t, x, y)$, since it does not appear in the classical B-S setting of complete markets. The occurrence of the function Λ in Equation (2.3) is a consequence of the market's incompleteness due to stochastic volatility. It has natural properties, namely:

$$\Lambda(t, \lambda x, y) = \Lambda(t, x, y) \ , \ \lambda > 0$$

so as to preserve the scale invariance of the underlying asset price x. We note that Λ has units of time to the power $-1/2$.

In fact, there is a whole continuum of scaling possibilities for $\widehat{\Lambda}$ parametrized by two numbers, c_1 and c_2 say, namely:

$$\widehat{\Lambda}(\widehat{t}, \widehat{x}, \widehat{y}) = \beta^{-c_1} r^{(c_1-1)/2+c_2} \alpha^{-c_2} \Lambda(r^{-1}\widehat{t}, \widehat{x}, \widehat{y}) \ .$$

We can mention at least two plausible adimensional choices for $\widehat{\Lambda}$:

Fig. 1. An asymptotic line: a pictorial description of the different scales under consideration with emphasis on the range of parameters for the scalings considered here and the one employed in [8].

1.
$$\widehat{\Lambda}(\widehat{t},\widehat{x},\widehat{y}) = r^{-1/2}\Lambda(r^{-1}\widehat{t},\widehat{x},\widehat{y})$$

2.
$$\widehat{\Lambda}(\widehat{t},\widehat{x},\widehat{y}) = \beta^{-1}\Lambda(r^{-1}\widehat{t},\widehat{x},\widehat{y})$$

The first one would suggest that the market will charge premium comparable to the risk-less rate, while the second suggests a premium based on the market's perception of the volatility.

The authors of [8] implicitly chose the scale in Item 1 above. Indeed, although the analysis of [8] is dimension dependent we note that Equation (5.9) of [8] implies a balance between the terms

$$xf(y)\frac{\partial^2 P}{\partial x \partial y} \text{ and } \Lambda\frac{\partial P}{\partial y} \ .$$

This implies the scaling of Item 1.

We shall henceforth *work with the adimensional variables* $\widehat{t}$, $\widehat{f}$, $\widehat{\Lambda}$, $\widehat{y}$, $\widehat{x}$, and drop the hats altogether. Equation (2.3) thus becomes

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2\frac{\partial^2 P}{\partial x^2} + \rho\nu x f(y)\frac{\partial^2 P}{\partial x\partial y} + \frac{\nu^2}{2}\frac{\partial^2 P}{\partial y^2} + \\ \left(x\frac{\partial P}{\partial x} - P\right) + \left[\epsilon^{-1}(m-y) - \nu\Lambda(t,x,y)\right]\frac{\partial P}{\partial y} = 0, \\ P(T,x,y) = h(x),\end{aligned} \tag{3.1}$$

with

$$T = rT_E, \quad \epsilon = r\alpha^{-1} \quad \text{and} \quad \nu = r^{-1/2}\beta. \tag{3.2}$$

We thus have the following scale regimes for ν:

1. $\nu^2 \sim \epsilon^{-1}$ corresponds to the scaling considered in [8] and leads to a balance between the terms
$$\frac{\nu^2}{2}\frac{\partial^2 P}{\partial y^2} \text{ and } \epsilon^{-1}(m-y)\frac{\partial P}{\partial y} \ .$$
2. $\nu^2 \ll \epsilon^{-1}$, which will be treated in Sections 5 and 6.
3. $\nu^2 \gg \epsilon^{-1}$ will be treated elsewhere.

4. Large Mean Reversion Asymptotics

In the present section we review the asymptotics developed in [8]. The rationale behind their calculations is that, if the volatility is fast mean-reverting, then the market will see, to leading order, an effective constant volatility plus small corrections. This is modeled by subsuming that the mean reversion time $\tilde{\epsilon} := 1/\alpha$ is small as compared to the other time scales. After introducing such scaling, Equation (2.3) becomes

$$\left(\frac{1}{\tilde{\epsilon}}\mathcal{L}_0 + \frac{1}{\sqrt{\tilde{\epsilon}}}\mathcal{L}_1 + \mathcal{L}_2)\right) P^{\tilde{\epsilon}} = 0 \ , \tag{4.1}$$

where

$$\begin{aligned}
\mathcal{L}_0 =& \tilde{\nu}^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \ , \\
\mathcal{L}_1 =& \tilde{\nu}\rho\sqrt{2}xf(y)\frac{\partial^2}{\partial x \partial y} - \tilde{\nu}s(t,x,y)\frac{\partial}{\partial y} \ , \\
\mathcal{L}_2 =& \frac{\partial}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \ ,
\end{aligned}$$

$\tilde{\nu}^2 := \beta^2/(2\alpha)$, and $s(t,x,y) := (\beta/\alpha)\Lambda(t,x,y)$. Furthermore, ν and $s(t,x,y)$ are assumed to be $\mathcal{O}(1)$.

4.1. *The Perturbation Expansion*

The following formal expansion was considered in [8]:

$$P^{\tilde{\epsilon}} = P_0 + \tilde{\epsilon}^{1/2}P_1 + \tilde{\epsilon}P_2 + \tilde{\epsilon}^{3/2}P_3 + \mathcal{O}(\tilde{\epsilon}^2)$$

After substituting such expansion into Equation (4.1) and grouping terms of same order they get

$$\begin{array}{rcl}
\mathcal{O}(\tilde{\epsilon}^{-1}) & \mathcal{L}_0 P_0 = 0 & \Rightarrow P_0 = P_0(t,x). \\
\mathcal{O}(\tilde{\epsilon}^{-1/2}) & \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0 & \Rightarrow P_1 = P_1(t,x). \\
\mathcal{O}(1) & \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0 & \Rightarrow \mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0.
\end{array}$$

The $\mathcal{O}(1)$ equation implies the solvability condition $\langle \mathcal{L}_2 P_0 \rangle = 0$ upon applying the Fredholm alternative, where $\langle g \rangle := \int g(y)\Phi(y)\mathrm{d}y$ and $\mathcal{L}_0\Phi = 0$. Applying the solvability condition we get

$$\langle \mathcal{L}_2 P_0 \rangle = \frac{\partial P_0}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + r\left(\frac{\partial P_0}{\partial x} - P_0\right) = 0$$

where $\tilde{\sigma}^2 = \langle f \rangle$ is an effective volatility. The next order leads to

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0 \ .$$

Once again, applying the solvability condition we get

$$\langle \mathcal{L}_2 P_1 \rangle = \tilde{V}_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \tilde{V}_3 x^3 \frac{\partial^3 P_0}{\partial x^3}$$

and

$$P_1 = -(T-t)\left[\tilde{V}_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \tilde{V}_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right].$$

Thus, the explicit formula for the corrected price is given by

$$P = P_0 - (T-t)\left[V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right] + \mathcal{O}(\tilde{\epsilon}),$$

where $V_2 = \tilde{\epsilon}\tilde{V}_2$ and $V_3 = \tilde{\epsilon}\tilde{V}_3$.

We notice that, up to first order in ϵ, the price P^ϵ does not depend on the hidden state y. This fact has far reaching implications, because it allows a much easier calibration.

In what follows, we consider what happens if, differently from the previous analysis, we assume that $\tilde{\nu} \ll 1$?

As already observed, $\tilde{\nu}$ represents the volatility of the volatility (vol-vol) and it is conceivable that it can be small in some market scenarios. This is indeed suggested by some results in [8].

Thus in Section 5 we consider the case of $\nu = \mathcal{O}(1)$, while in Section 6 we tackle the case $\nu = \mathcal{O}(\epsilon^{1/2})$. To allow comparison with earlier results, we notice that $\tilde{\nu} = \epsilon^{1/2}\nu/\sqrt{2}$. Hence, we have that $\nu = \mathcal{O}(1)$ corresponds to $\tilde{\nu} = \mathcal{O}(\epsilon^{1/2})$ and that $\nu = \mathcal{O}(\epsilon^{1/2})$ corresponds to $\tilde{\nu} = \mathcal{O}(\epsilon)$.

4.2. *Implied Volatility Asymptotics*

We now review the asymptotic correction to the implied volatility, caused by the correction to the B-S model, that was also derived in [8].

Let $N(d)$ denote the cumulative normal distribution. Then, the B-S price of an European Call is given by

$$P_0 = xN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2) \tag{4.2}$$

with

$$d_{1,2} = \frac{\log(x/K) + \left(1 \pm \frac{1}{2}\tilde{\sigma}\right)(T-t)}{\tilde{\sigma}\sqrt{T-t}}.$$

The implied volatility is defined as the volatility that should go in the B-S model in order to fit the observed prices. More precisely, let us write $P_0 = P_0(\sigma)$, ignoring for the moment the dependence on the other (assumed fixed) variables and parameters. Then, for a given price P, we define $I = P_0^{-1}(P)$.

Since $P^\epsilon = P_0 + \epsilon^{1/2}P_1 + \mathcal{O}(\epsilon)$, it is straightforward to show that I expands as

$$I = \sigma + \epsilon^{1/2}\left(\frac{\partial P_0}{\partial \sigma}\right)^{-1} P_1(t,x) + \mathcal{O}(\epsilon).$$

From (4.2) we can compute the Vega:

$$\frac{\partial P_0}{\partial \sigma} = \frac{x e^{-d_1^2/2}}{\sqrt{2\pi}\sqrt{T-t}}.$$

Also, we need to compute

$$x^2 \frac{\partial^2 P_0}{\partial x^2} = \frac{x e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} \quad \text{and} \quad x^3 \frac{\partial^3 P_0}{\partial x^3} = \frac{x e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right),$$

since they occur in P_1. The first one is known as the Delta. The second one is not usual in option B-S model calculations and it is termed by [8] the Epsilon. From those calculations we have that

$$I = \tilde{\sigma} + \frac{V_3}{\tilde{\sigma}^3}\left(r + \frac{3}{2}\tilde{\sigma}^2\right) - \frac{V_2}{\tilde{\sigma}} - \frac{V_3}{\tilde{\sigma}^3}\left(\frac{\log(K/x)}{T-t}\right) + \mathcal{O}(\epsilon). \tag{4.3}$$

5. Moderately Small Variance

In this section, we consider the case $\nu = \mathcal{O}(1)$. We write $\nu = \nu_0$, with ν_0 being order one.

We shall follow a multiple-scales approach [15]. We introduce the change

$$t \to T - \epsilon s - \tau$$

Under this hypothesis, Equation (4.1) becomes

$$\epsilon^{-1}\mathcal{H}_0 P^\epsilon + \mathcal{H}_1 P^\epsilon + \mathcal{H}_2 P^\epsilon = 0 \tag{5.1}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{\partial}{\partial s} + (m-y)\frac{\partial}{\partial y}, \\
\mathcal{H}_1 &= -\frac{\partial}{\partial \tau} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + \left(x\frac{\partial}{\partial x} - \cdot\right), \\
\mathcal{H}_2 &= \nu_0 \rho x f(y)\frac{\partial^2}{\partial x \partial y} - \nu_0 \Lambda(t,x,y)\frac{\partial}{\partial y} + \frac{\nu_0^2}{2}\frac{\partial^2}{\partial y^2}.
\end{aligned}$$

Note that, although the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are of the same order, it turns out in the subsequent analysis that the above split is quite convenient.

5.1. *The Inner Perturbation Expansion*

Formally, we write

$$P^\epsilon = P_0(s,\tau,x,y) + \epsilon P_1(s,\tau,x,y) + \epsilon^2 P_2(s,\tau,x,y) + \mathcal{O}(\epsilon^3). \tag{5.2}$$

Substituting (5.2) on (5.1) yields at sequential orders:

$\mathcal{O}(\epsilon^{-1})$

$$\mathcal{H}_0 P_0 = 0, \quad P(0,y) = h(x).$$

which implies

$$P_0 = P_0(\tau, x), \quad P_0(0,x) = h(x).$$

where $P_0(\tau, x)$ remains to be determined.

$\mathcal{O}(\epsilon^{0})$

$$\mathcal{H}_0 P_1 = -\mathcal{H}_1 P_0 - \mathcal{H}_2 P_0, \quad P_1(0,y) = 0.$$

On noticing that $\mathcal{H}_2 P_0 = 0$, we have

$$\mathcal{H}_0 P_1 = -\mathcal{H}_1 P_0.$$

Following [8], we write $\mathcal{H}_1 P_0 := g(y)$, ignoring for the moment its dependence on x and τ, and write the solutions as

$$P_1 = -\int_m^y \frac{g(z)}{m-z}\mathrm{d}z + \tilde{P}_1(\tau,x)\int_m^{(y-m)\mathrm{e}^{-s}+m} \frac{g(z)}{m-z}\mathrm{d}z + \hat{P}_1(\tau,x), \tag{5.3}$$

with the initial conditions

$$\tilde{P}_1(0,x) = \frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2}(0,x) \quad \text{and} \quad \hat{P}_1(0,x) = 0.$$

Equation (5.3) will usually define a non-smooth solution at $y = m$, unless $g(z)/(m-z)$ is integrable at $z = m$. If g is at least Hölder continuous, then it is sufficient to have $g(m) = 0$. Let us write $\sigma = f(m)$, then $g(m) = 0$ is equivalent to the adimensional B-S equation, namely:

$$-\frac{\partial P_0}{\partial \tau} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \left(x\frac{\partial P_0}{\partial x} - P_0 \right) = 0, \tag{5.4}$$

which will shorthand to $\mathcal{H}_1^{\sigma} P_0 = 0$, with the initial condition $P_0(0,x) = h(x)$.

It is convenient to observe that $g(z) = g(z) - g(m)$ and, hence that P_1 can be written as

$$P_1(s,\tau,x,y) = -\frac{x^2}{2}\frac{\partial^2 P_0(\tau,x)}{\partial x^2}\int_m^y \frac{f^2(z)-\sigma^2}{m-z}\mathrm{d}z + \tilde{P}_1(\tau,x)\int_m^{(y-m)\mathrm{e}^{-s}+m} \frac{g(z)}{m-z}\mathrm{d}z + \hat{P}_1(\tau,x)$$

with

$$\tilde{P}_1(0,x) = \frac{x^2}{2}\frac{\partial^2 P_0(0,x)}{\partial x^2} \quad \text{and} \quad \hat{P}_1(0,x) = 0.$$

$\mathcal{O}(\epsilon^{1/2})$

$$\mathcal{H}_0 P_2 = -\mathcal{H}_1 P_1 - \mathcal{H}_2 P_1.$$

The above argument can be repeated to give the following solvability conditions for $\hat{P}_1$ and $\tilde{P}_1$. For $\hat{P}_1$, we have

$$\begin{aligned} \mathcal{H}_1^\sigma \hat{P}_1 = -x^2 \frac{\partial^2 P_0}{\partial x^2} \Big[2\rho\nu_0 f^2(m) f'(m) + \frac{\nu_0^2}{4} \left(f'^2(m) + f(m) f''(m) \right) \\ -\nu_0 \Lambda(m) f(m) f'(m) \Big] - 2\rho\nu_0 f^2 f'(m) \frac{x^3}{2} \frac{\partial^3 P_0}{\partial x^3} \ , \end{aligned} \tag{5.5}$$

with the initial condition given above. The solution for (5.5) can be readily written—as done in [8]—as

$$\begin{aligned} \hat{P}_1(\tau, x, y) = -\tau \Big\{ -x^2 \frac{\partial^2 P_0}{\partial x^2} \Big[2\rho\nu_0 f^2(m) f'(m) + \frac{\nu_0^2}{4} \left(f'^2(m) + f(m) f''(m) \right) \\ -\nu_0 \Lambda(m) f(m) f'(m) \Big] - 2\rho\nu_0 f^2 f'(m) \frac{x^3}{2} \frac{\partial^3 P_0}{\partial x^3} \Big\} \ . \end{aligned}$$

For $\tilde{P}_2(\tau, x)$, we have

$$\mathcal{H}_1^\sigma \tilde{P}_1 = 0, \quad \tilde{P}_1(0, x) = \frac{x^2}{2} \frac{\partial^2 P_0(0, x)}{\partial x^2},$$

with solution given by

$$\tilde{P}_1(\tau, x) = \frac{x^2}{2} \frac{\partial^2 P_0(\tau, x)}{\partial x^2}.$$

Thus, the first two terms of the expansion for P^ϵ are given by

$$\begin{aligned} P^\epsilon(s, \tau, x, y) = P_0 + \epsilon \Big\{ \frac{\partial^2 P_0(\tau, x)}{\partial x^2} \int_y^{(y-m)e^{-s}+m} \frac{g(z)}{m - z} dz - \tau \Big[2\rho\nu_0 f^2(m) f'(m) \frac{x^3}{2} \frac{\partial^3 P_0}{\partial x^3} \\ + \frac{x^2}{2} \frac{\partial^2 P_0}{\partial x^2} \Big(\frac{\nu_0^2}{4} \left(f'^2(m) + f(m) f''(m) \right) + 4\rho\nu_0 f^2(m) f'(m) - \nu_0 \Lambda(m) f(m) f'(m) \Big) \Big] \Big\}. \end{aligned} \tag{5.6}$$

For future reference, we notice that if we write $y = m + \epsilon^{1/2}\upsilon$ we have

$$\begin{aligned} P_1(s, \tau, x, m + \epsilon^{1/2}\upsilon) = -\tau \Big\{ 2\rho\nu_0 f^2(m) f'(m) \frac{x^3}{2} \frac{\partial^3 P_0}{\partial x^3} + \frac{x^2}{2} \frac{\partial^2 P_0}{\partial x^2} \Big[\frac{\nu^2}{4} \left(f'^2(m) + f(m) f''(m) \right) \\ + 4\rho\nu f(m) f'(m) - \nu_0 \Lambda(m) \Big] \Big\} + \epsilon^{1/2} \upsilon (1 - e^{-s}) 2 f(m) 2 f'(m). \end{aligned} \tag{5.7}$$

5.2. *The Outer Perturbation Expansion*

The expansion derived above is not asymptotic for large y, since, for instance, as $y \to +\infty$ we have that

$$P(s,\tau,x,y) \to P_0(\tau,x) - \epsilon s(\sigma^{+2} - \sigma^2)\frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2} + \mathcal{O}(\epsilon^2), \quad \sigma_+ = \lim_{y\to+\infty} f(y).$$

The above expansion is not asymptotic for $s \sim \epsilon^{-1}$. As a matter of fact, it turns out that an infinite number of terms contribute to such non-asymptoticity.

Let $y = \epsilon^{-1}\eta$, $\eta > 0$, and write

$$P = R_0(s,\tau,x,\eta) + \epsilon R_1(s,\tau,x,\eta) + \mathcal{O}(\epsilon^2).$$

For simplicity, we shall assume that f is constant outside a compact set of size asymptotically smaller than ϵ^{-1}. This is not strictly necessary, but simplifies the algebra. Proceeding as in the above analysis, we find that (4.1) becomes

$$\epsilon^{-1}\widehat{\mathcal{H}}_0 + \widehat{\mathcal{H}}_1 + \epsilon\widehat{\mathcal{H}}_2 + \epsilon^2\frac{\nu_0^2}{2}\frac{\partial^2 P}{\partial\eta^2}, \tag{5.8}$$

where

$$\begin{aligned}
\widehat{\mathcal{H}}_0 &= -\frac{\partial}{\partial s} - \eta\frac{\partial}{\partial\eta},\\
\widehat{\mathcal{H}}_1 &= -\frac{\partial}{\partial\tau} + \frac{1}{2}(\sigma_+)^2 x^2\frac{\partial^2}{\partial x^2} + \left(x\frac{\partial}{\partial x} - \cdot\right) + m\frac{\partial}{\partial\eta},\\
\widehat{\mathcal{H}}_2 &= \nu_0\rho x\sigma_+\frac{\partial^2}{\partial x\partial\eta} - \nu_0\Lambda_+\frac{\partial}{\partial\eta}, \quad \Lambda_+ = \lim_{y\to\infty}\Lambda(y).
\end{aligned}$$

So proceeding at sequential orders yields:

$\mathcal{O}(\epsilon^{-1})$:

$$\widehat{\mathcal{H}}_0 R_0 = 0, \quad R_0(0,\eta) = h(x)$$

which leads to

$$R_0 = R_0(\tau,x), \quad R_0(0,x) = h(x).$$

$\mathcal{O}(1)$:

$$\widehat{\mathcal{H}}_0 R_1 = -\widehat{\mathcal{H}}_1 R_0, \quad R_1(0,\eta) = 0.$$

Since R_0 does not depend on η, we have that $\widehat{\mathcal{H}}_1 R_0 = \mathcal{H}_1^{\sigma+}$, hence the solution is

$$R_1(s,\eta) = -s\mathcal{H}_1^{\sigma+}R_0.$$

In order to keep the asymptoticity of the series when $s \sim \epsilon^{-1}$, we have the following secularity condition:

$$\widehat{\mathcal{H}}_1 R_0 = 0, \quad R_0(0,x) = h(x).$$

Hence $P_1 \equiv 0$

$\mathcal{O}(\epsilon)$:

$$\widehat{\mathcal{H}}_0 R_2 = -\widehat{\mathcal{H}}_1 R_1 - \widehat{\mathcal{H}}_2 R_0, \quad R_2(0,\eta) = 0.$$

Since $R_1 \equiv 0$ and R_0 is independent of η, this simplifies to

$$\widehat{\mathcal{H}}_0 R_2 = 0, \quad R_2(0,\eta) = 0,$$

with solution $R_2 \equiv 0$.

From these calculations, it is easy to see that $R_i \equiv 0$, $i = 1, \ldots$. Hence the expansion is just R_0. In order to compare with the solution previously obtained, we need to write the solution R_0 in terms of P_0. This has been accomplished by the following device:

Write

$$\mathcal{H}_1^{\sigma_+} = \mathcal{H}_1^{\sigma} - ({\sigma_+}^2 - \sigma^2)L, \quad L = \frac{x^2}{2}\frac{\partial^2}{\partial x^2}.$$

Then, if we write $R_0 = P_0 + Q_0$, we find that Q_0 can be written in terms of an infinity series, yielding

$$R_0 = P_0 + \sum_{n=1}^{\infty}(-1)^n \frac{\tau^n({\sigma_+}^2 - \sigma^2)^n}{n!} L^n P_0. \tag{5.9}$$

By writing $\tau = \epsilon s$, we find that

$$R_0 = P_0 - \tau(\sigma_+^2 - \sigma^2)LP_0 + \mathcal{O}(\epsilon^2).$$

Actually, it is possible to show that the other terms in the expansion given by (5.9) will match terms coming from higher order terms in the *inner* expansion. Thus, an infinite number of terms jump order when $y \to \infty$.

Remark 5.1. An analogous calculation holds for large negative y, with $\eta < 0$ and σ_+ replaced by $\lim_{y\to-\infty} f(y) = \sigma_-$.

5.3. *Implied Volatility Asymptotics*

A similar analysis to the one reviewed in Section 4, yields

$$I = \sigma + \epsilon P_1(s,\tau,x,y)\frac{1}{\partial P_0/\partial\sigma} + \mathcal{O}(\epsilon^2).$$

Here, however, we have that

$$T - t = \tau + \epsilon s.$$

Hence, the implied volatility expand as

$$\begin{aligned} I =& \sigma + \frac{\epsilon}{2\sigma(\tau+\epsilon s)}\left\{\tau\left[A - B\frac{(1+3\sigma^2/2)}{\sigma^2} + A\frac{\log(x/K)}{\tau+\epsilon s}\right] + \right. \\ & \left. + \epsilon^{1/2} C\upsilon(1-\mathrm{e}^{-s})\right\} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{5.10}$$

with, from (5.7),

$$
\begin{aligned}
A &= 2\rho\nu_0 f(m)f'(m),\\
B &= \frac{\nu_0^2}{4}\left(f'^2(m) + f(m)f''(m)\right) + 4\rho\nu_0 f^2(m)f'(m) - \nu_0\Lambda(m)f(m)f''(m),\\
C &= 2f(m)f'(m).
\end{aligned}
$$

Equation (5.10) should be compared with (4.3). Notice that (5.10) is asymptotic both when $T-t$ is order one, and when $T-t = \mathcal{O}(\epsilon)$. In the former, we have $\tau \neq 0$. Thus (5.10) becomes

$$
I = \sigma + \frac{\epsilon}{2\sigma}\left(B - \frac{A(1+3\sigma^2/2)}{\sigma^2} + A\frac{\log(x/K)}{\tau}\right) + \mathcal{O}(\epsilon^{3/2}).
$$

In the latter, we have $\tau = 0$. In this case, (5.10) becomes

$$
I = \sigma + \epsilon^{1/2}\frac{\upsilon}{2\sigma}C\frac{(1-\mathrm{e}^{-s})}{s} + \mathcal{O}(\epsilon).
$$

6. Small Variance

In this section, we derive an asymptotic expansion for the price, when $\nu = \mathcal{O}(\epsilon^{1/2})$. Hence, we set $\nu = \nu_1\epsilon^{1/2}$ with $\nu_1 = \mathcal{O}(1)$.

Under this hypothesis Equation (4.1) becomes

$$
\epsilon^{-1}\mathcal{H}_0 P^\epsilon + \mathcal{H}_1 P^\epsilon + \epsilon^{1/2}\tilde{\mathcal{H}}_2 P^\epsilon + \epsilon\frac{\nu_1}{2}\frac{\partial^2 P^\epsilon}{\partial y^2} = 0 \tag{6.1}
$$

where now

$$
\begin{aligned}
\mathcal{H}_0 &= -\frac{\partial}{\partial s}(m-y)\frac{\partial}{\partial y},\\
\mathcal{H}_1 &= -\frac{\partial}{\partial \tau} + \frac{1}{2}\left(f(y)\right)^2 x^2\frac{\partial^2}{\partial x^2} + \left(x\frac{\partial}{\partial x} - \cdot\right),\\
\tilde{\mathcal{H}}_2 &= \nu_1\rho x f(y)\frac{\partial^2}{\partial x\partial y} - \nu_1\Lambda(t,x,y)\frac{\partial}{\partial y}.
\end{aligned}
$$

6.1. *The Inner Perturbation Expansion*

We, formally, write

$$
\begin{aligned}
P^\epsilon = P_0(s,\tau,x,y) &+ \epsilon^{1/2}P_1(s,\tau,x,y) + \epsilon P_2(s,\tau,x,y)\\
&+ \epsilon^{3/2}P_3(s,\tau,x,y) + \epsilon^2 P_4(s,\tau,x,y) + \mathcal{O}(\epsilon^{5/2}).
\end{aligned} \tag{6.2}
$$

Substituting (6.2) on (6.1) yields at sequential orders:

$\mathcal{O}(\epsilon^{-1})$

$$\mathcal{H}_0 P_0 = 0, \quad P_0(0, y) = h(x).$$

which implies

$$P_0 = P(\tau, x), \quad P_0(0, x) = h(x),$$

where $P_0(\tau, x)$ remains to be determined.

$\mathcal{O}(\epsilon^{-1/2})$

$$\mathcal{H}_0 P_1 = 0, \quad P_1(0, y) = 0.$$

which also implies in $P_1 = P_1(\tau, x)$, yet undetermined at this order.

$\mathcal{O}(\epsilon^{0})$

$$\mathcal{H}_0 P_2 = -\mathcal{H}_1 P_0, \quad P_2(0, y) = 0.$$

As in the previous analysis, we the solution can be written as

$$P_2 = -\int_m^y \frac{\mathcal{H}_1^\sigma P_0}{m - z} \mathrm{d}z + \tilde{P}_2(\tau, x) \int_m^{(y-m)\mathrm{e}^s + m} \frac{\mathcal{H}_1^\sigma P_0}{m - z} \mathrm{d}z + \hat{P}_2(\tau, x).$$

together with the solvability condition

$$\mathcal{H}_1^\sigma P_0 = 0, \quad P_0(0, x) = h(x).$$

$\mathcal{O}(\epsilon^{1/2})$

$$\mathcal{H}_0 P_3 = -\mathcal{H}_1 P_1 - \tilde{\mathcal{H}}_2 P_0,$$

which, upon noticing that P_0 depends only on τ and x, simplifies to

$$\mathcal{H}_0 P_3 = -\mathcal{H}_1 P_1, \tag{6.3}$$

and leads to the following solvability conditions:

$$\mathcal{H}_1^\sigma P_1 = 0, \quad P_1(0, x) = 0.$$

The solution to this problem is $P_1(\tau, x) \equiv 0$.

$\mathcal{O}(\epsilon)$

$$\mathcal{H}_0 P_4 = -\mathcal{H}_1 P_2 - \tilde{\mathcal{H}}_2 P_1 - \frac{\nu_1^2}{2}\frac{\partial^2 P_0}{\partial y^2}, \quad P_4(0,y) = 0.$$

This simplifies to

$$\mathcal{H}_0 P_4 = -\mathcal{H}_1 P_2,$$

that will lead to the following solvability condition:

$$\mathcal{H}_1^\sigma \tilde{P}_2 = 0, \quad \tilde{P}_2(0,x) = -\frac{x^2}{2}\frac{\partial^2 P_0(0,x)}{\partial x^2}.$$

i.e

$$\tilde{P}_2(\tau,x) = -\frac{x^2}{2}\frac{\partial^2 P_0(\tau,x)}{\partial x^2}.$$

Also we have

$$\mathcal{H}_1^\sigma \hat{P}_2 = 0, \quad \hat{P}_2(0,x) = 0,$$

with solution $\hat{P}_2 \equiv 0$. The solution is then given by

$$P^\epsilon = P_0(t,x) + \epsilon \frac{\partial^2 P_0(\tau,x)}{\partial x^2}\int_y^{(y-m)\mathrm{e}^{-s}+m} \frac{g(z)}{m-z}\mathrm{d}z + \mathcal{O}(\epsilon^2). \tag{6.4}$$

6.2. *Outer Perturbation expansion*

The outer perturbation expansion of this problem is very similar to the previous case. Hence, we give only an outline. If we let $y = \epsilon^{-1}\eta$, $\eta > 0$, Equation (4.1) now becomes

$$\epsilon^{-1}\widehat{\mathcal{H}}_0 + \widehat{\mathcal{H}}_1 + \epsilon^{3/2}\widehat{\mathcal{H}}_2 + \epsilon^3 \frac{\nu_1^2}{2}\frac{\partial^2 P}{\partial \eta^2}, \tag{6.5}$$

where the operators $\widehat{\mathcal{H}}_i$, $i = 0,1,2$, were defined in Section 5.2. Then again, we have at sequential orders:

$\mathcal{O}(\epsilon^{-1})$:

$$\widehat{\mathcal{H}}_0 R_0 = 0, \quad R_0(0,\eta) = h(x)$$

Implying

$$R_0 = R_0(\tau,x), \quad R_0(0,x) = h(x).$$

$\mathcal{O}(\epsilon^{-1/2})$:

$$\widehat{\mathcal{H}}_0 R_1 = -\widehat{\mathcal{H}}_1 R_0, \quad R_1(0,\eta) = 0.$$

The solution is

$$R_1(s,\eta) = -s\widehat{\mathcal{H}}_1 R_0.$$

Which again lead to the secularity condition

$$\mathcal{H}_1^{\sigma+} R_0 = 0, \quad R_0(0,x) = h(x).$$

Hence $P_1 \equiv 0$. From now on, it is an easy matter to see that, to all orders, we have

$$\widehat{\mathcal{H}}_0 R_i = 0, \quad R_i(0,\eta) = 0,$$

which implies $R_i(s,\eta) \equiv 0$, $i = 1, \cdots$. As in the previous case, a similar analysis holds for large negative y.

6.3. *Implied Volatility Asymptotics*

In this case, the calculation is more involved, and one needs to consider the following expansion for I:

$$I = \sigma + \epsilon P_2 \frac{1}{\partial P_0/\partial\sigma} + \epsilon^{3/2} P_3 \frac{1}{\partial P_0/\partial\sigma} \epsilon^2 \left[P_4 \frac{1}{\partial P_0/\partial\sigma} - P_2^2 \left(\frac{1}{\partial P_0/\partial\sigma} \right)^3 \right] + \mathcal{O}(\epsilon^{5/2}). \tag{6.6}$$

After a long calculation, we find a formula that is nearly identical to (5.10), namely

$$\begin{aligned} I =& \sigma + \frac{\epsilon^{3/2}}{2\sigma(\tau+\epsilon s)} \left\{ \tau \left[A - B\frac{(1+3\sigma^2/2)}{\sigma^2} + A\frac{\log(x/K)}{\tau+\epsilon s} \right] + \right. \\ & \left. + \epsilon^{1/2} C\upsilon (1 - \mathrm{e}^{-s}) \right\} + \mathcal{O}(\epsilon^{5/2}), \end{aligned} \tag{6.7}$$

where $y = m + \epsilon\upsilon$, and A, B and C are the same as before with ν_0 replaced by ν_1.

Remark 6.1. Formula (6.7) is also uniformly valid in time and, when $\tau \neq 0$ is very similar to (4.3).

Remark 6.2. The solutions given by (5.6) and (6.4) turn out to be formally asymptotic throughout the whole (τ, x, y) domain. However, since the the volatility process is mean-reverting, heuristically we expect that the large y-region is not important in applications.

The calculations in this and in the previous section shows that the solutions display a very special asymptotic behavior. Such behavior is illustrated in Figure 2.

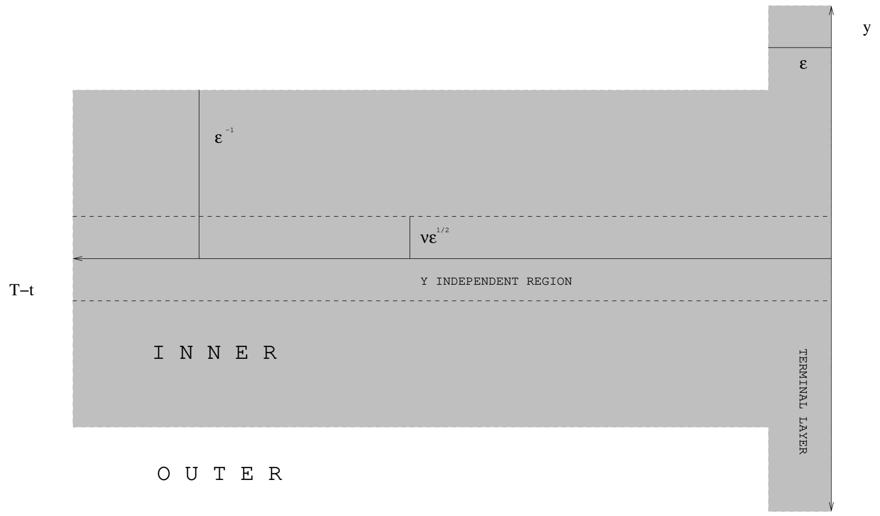

Fig. 2. Asymptotic structure of the solution

7. Conclusions

In this work, we derived the asymptotic expansion for stochastic volatility models under fast mean reversion using a multiple scales formalism [3,15].

A preliminary dimensional analysis suggests that relevant adimensional parameters are $\epsilon = r\alpha^{-1}$ and $\nu = r^{-1/2}\beta$, where α, β, and r define the underlying asset dynamics as in Equation (2.1). Using such adimensionalization we identified the distinguished asymptotic limits. One of such limits, namely $\nu = \mathcal{O}(\epsilon^{-1/2})$ has been the subject of extensive analysis [8]. This corresponds to an $\mathcal{O}(1)$ variance (vol-vol) of the volatility driving process.

In contradistinction, in the present work we focused on the case $\nu = \mathcal{O}(1)$ and $\nu = \mathcal{O}(\epsilon^{1/2})$. The former corresponds to a vol-vol of order $\epsilon^{1/2}$, whereas the latter corresponds to a vol-vol of order ϵ. In either case, we found the first order correction to the option prices to be dependent on the state of the volatility process (y) and also on a fast time scale, in addition to the usual dependence on the underlying price (x) and the time to maturity ($T-t$). We notice that these corrections are not the small variance limits of those found in [8]. Moreover, the correction we obtained for either case shows that the fast scale part of the solution is governed by a transport equation. This can be interpreted as a terminal layer, which differs in spirit from the one found in [17] for the specific case of $\nu = \mathcal{O}(\epsilon^{-1/2})$, $\rho = 0$, and $\Lambda = 0$. The structure of the solution we have found displays an outer and an inner layer, with respect to y. In particular, assuming f to be constant outside a compact set of length asymptotically less than ϵ^{-1}, we found the outer layer solution to be y independent. Furthermore, all but the leading order term of such expansion vanish. We remark, in passing, that the transition from the inner solution to the outer one leads to an infinite number of terms jumping order.

Following [8], we used our asymptotic results to derive an asymptotic expansion for the implied volatility. In contrast with the formula found, albeit in a different regime, by [8] which blows up at expiration, we found a formally uniform asymptotic expression valid for all times.

As pointed out in Section 4, the expansion obtained by [8] is y independent up to the first correction, while this is not so for (5.6) and (6.4). This raises the quandary of the nature of such y independence.

Notice that the structure of the calculations depend on the leading order operator, which in [8] is a second-order one, while here we deal with a first order operator. Indeed, in the first case, the Fredholm condition will be equivalent to centering with respect to the invariant distribution of the y process. Thus, if we write $y = \delta\eta$, in order to have a second-order leading order operator, we need $\delta^{-2}\nu^2 \sim \epsilon^{-1}$. This leads to $\delta \sim \epsilon^{1/2}\nu$. Thus, if $\nu = \mathcal{O}(\epsilon^{-1/2})$, the solution will be y-independent at least for $y = \mathcal{O}(1)$.

For $\nu = \mathcal{O}(1)$ and $\nu = \mathcal{O}(\epsilon^{1/2})$ the y region where we can guarantee y independence is of order $\epsilon^{1/2}$ and ϵ respectively. In addition, the calculations in Sections 5 and 6, show that this cannot be improved without further assumption on f.

Moreover, writing $y = m + \delta y$, we can derive from (5.6) and (6.4) expansions that depend solely on x and t. This y independent region can be interpreted as the region where the volatility process has diffused from the mean.

Yet another important point, is the fact that there is multitude of choices for the adimensionalization of the market price of risk Λ. Some of those choices may substantially affect the asymptotic behavior of the model. Hence, it is important to have better estimates for Λ from the market data. A natural follow up to the present work would be to develop ways of associating the distinguished asymptotic regimes to the corresponding parameter space regions for ν and α.

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