Discretization via Homogenization Theory for Elliptic Equations with Rapidly Oscillating Periodic Coefficients

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Abstract

We develop a numerical method to solve

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i} a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega,$$

where the matrix $a(y) = (a_{ij}(y))$ is symmetric positive definite, whose entries are periodic functions of y with periodic cell Y. More specifically we assume $a_{ij} \in C^{1,\beta}(\Re^2)$, $\beta > 0$. It is also assumed that there exists positive constants γ_a and β_a such that $\gamma_a \|\xi\|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \beta_a \|\xi\|^2$ for all $\xi \in \Re^2$ and $y \in \overline{Y}$. The major goal in this paper is to develop a numerical approximation scheme on a mesh size $h > \epsilon$ (or $h >> \epsilon$) with quasi-optimal approximation on L^2 and broken H^1 norms. The new method is based on asymptotic analysis and a careful treatment of the boundary corrector term. This kind of equation has applications in areas such as on the study of flow through porous media and composite materials.

1 INTRODUCTION

On several real world problems the scale ϵ is so smaller than Ω that even with very heavy computer efforts it is impossible to take $h < \epsilon$, h being the scale (mesh-size) of the discrete method used to approximate the solution of

$$L_\epsilon u_\epsilon = -\frac{\partial}{\partial x_i} (a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon = 0 \text{ on } \partial\Omega. \tag{1}$$

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The major goal in this article is to develop a numerical scheme on a mesh size $h > \epsilon$ (or $h >> \epsilon$). We note that when $h > \epsilon$ standard finite element methods do not result in good numerical approximations; see [9] . Recently new numerical methods have been proposed for solving this problem; see for example [7, 8, 13, 15, 1], and [6] for a more general approach for multi-scale problems.

The method proposed here, opposed to the methods [7, 8, 13] is strongly based on asymptotic expansions of u_ϵ. We construct a first order asymptotic expansion for u_ϵ, and then we numerically approximate each term separately. The construction of boundary correctors that are suitable for numerical approximation is a key issue in this paper.

2 NOTATION

We assume that $\Omega = Y = [0,1] \times [0,1]$, and introduce the following notation

$$\begin{aligned} \Gamma_e &= \{x_1 = 1, x_2 \in [0,1]\}, \quad \Gamma_w = \{x_1 = 0, x_2 \in [0,1]\}, \\ \Gamma_n &= \{x_2 = 1, x_1 \in [0,1]\}, \quad \Gamma_s = \{x_2 = 0, x_1 \in [0,1]\}, \end{aligned}$$

where Γ_k, $k \in \{e, w, n, s\}$ denotes a generic side of $\partial\Omega$.

Let $D \subset \Re^2$ be an open set. We use the standard notation $\|\cdot\|_{s,D}$, $\|\cdot\|_{s,p,D}$ for $H^s(D)$ and $W_p^s(D)$ norms, and $|\cdot|_{s,D}$, $|\cdot|_{s,p,D}$ their semi-norms. We define also broken norms by

$$\|v\|_{s,h,D} = \sqrt{\sum_{K_j \in \mathcal{T}_h(D)} \|v\|^2{}_{H^s(K_j)}}.$$

where $\mathcal{T}_h(D) = K_1, K_2, ..., K_m$ is a regular partition of D with size h. Throughout this paper, when we do not make reference to the domain D it is assumed that $D = \Omega$ or Y. It is continually used the Einstein summation convention, i.e. repeated indices indicate summation. In what follows c denotes a generic constant independent of ϵ, h, and functions being evaluated.

3 THEORETICAL APPROXIMATION

3.1 The Asymptotic Expansion

The solution u_ϵ can be approximated by an asymptotic expansion. This approximation can be found using equation (1) and the ansatz

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots,$$

where the functions $u_j(x,y)$ are Y periodic in y. These terms are defined below; for more details see [2, 11, 12] .

Let χ^j be the Y periodic solution with zero average on Y of

$$\nabla_y \cdot a(y)\nabla_y \chi^j = \nabla_y \cdot a(y)\nabla_y y_j = \frac{\partial}{\partial y_i} a_{ij}(y). \tag{2}$$

We have that $\chi^j \in C^{2,\beta}(\Re^2)$ when $a_{ij} \in C^{1,\beta}(\Re^2)$; see Theorem 12.1 from [10]. Define the matrix:

$$A_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi^i)\frac{\partial}{\partial y_m}(y_j - \chi^j)dy. \tag{3}$$

It is easy to see that the matrix A is symmetric positive definite. Define $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of

$$-\nabla . A \nabla u_0 = f \quad \text{in} \quad \Omega, \qquad u_0 = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$

and let

$$u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x).$$

Note that $u_0 + \epsilon u_1$ does not satisfy the zero Dirichlet boundary condition on $\partial\Omega$. In order to correct this, the boundary corrector term $\theta_\epsilon \in H^1(\Omega)$ is introduced as the solution of

$$-\nabla \cdot a(x/\epsilon)\nabla\theta_\epsilon = 0 \quad \text{in} \quad \Omega, \qquad \theta_\epsilon = -u_1(x, \frac{x}{\epsilon}) \quad \text{on} \quad \partial\Omega. \tag{5}$$

Therefore we obtain $u_0 + \epsilon u_1 + \epsilon\theta_\epsilon \in H_0^1(\Omega)$ and it can be shown [11] that the following estimates hold

$$\|u_\epsilon - (u_0 + \epsilon u_1 + \epsilon\theta_\epsilon)\|_0 \le c\epsilon^2 \|u_0\|_3,$$

and

$$\|u_\epsilon - (u_0 + \epsilon u_1 + \epsilon\theta_\epsilon)\|_1 \le c\epsilon \|u_0\|_2.$$

3.2 Boundary Corrector Approximation

Note that the coefficients $a_{ij}(x/\epsilon)$ and the boundary values $-u_1(x, \frac{x}{\epsilon})$ of the Equation (5) are highly oscillatory, hence it is not a trivial problem to obtain a good discretization for θ_ϵ. We propose an analytical approximation for θ_ϵ, denoted by ϕ_ϵ that satisfies the oscillating boundary condition and is more suitable for numerical approximation.

Note that $u_0 = 0$ along $\partial\Omega$ implies $\nabla u_\epsilon|_{\partial\Omega} = \eta\partial_\eta u_0$, where η denotes the unity outward normal vector on $\partial\Omega$ and $\partial_\eta u_0$ denotes the unity outward derivative of u_0 (see Remark 3.1). We then decompose $\theta_\epsilon = \tilde{\theta}_\epsilon + \bar{\theta}_\epsilon$ where

$$-\nabla \cdot a(x/\epsilon)\nabla\tilde{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \tilde{\theta}_\epsilon = -u_1 - \chi^*\partial_\eta u_0 = (\chi^j(\frac{x}{\epsilon})\eta_j - \chi^*)\partial_\eta u_0 \text{ on } \partial\Omega, \tag{6}$$

and

$$-\nabla \cdot a(x/\epsilon)\nabla\bar{\theta}_\epsilon = 0 \text{ in } \Omega, \quad \bar{\theta}_\epsilon = \chi^*\partial_\eta u_0 \text{ on } \partial\Omega, \tag{7}$$

where $\chi^*|_{\Gamma_k} = \chi_k^*$ are properly chosen constants . In Remark 3.1 we show that $\partial_\eta u_0|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$, hence $\chi^*\partial_\eta u_0 \in H^{1/2}(\partial\Omega)$, and therefore problems (6) and (7) are well posed. The approximation ϕ_ϵ for θ_ϵ is defined later as $\tilde{\phi}_\epsilon + \bar{\phi}_\epsilon$, where $\tilde{\phi}_\epsilon \approx \tilde{\theta}_\epsilon$ and $\bar{\phi}_\epsilon \approx \bar{\theta}_\epsilon$.

Next we define constants χ_k^* for which the approximation $\tilde{\phi}_\epsilon$ decays exponentially to zero away from the boundary and is suitable for numerical approximation. Also $\tilde{\phi}_\epsilon$ satisfies the correct Dirichlet condition $-u_1(x, \frac{x}{\epsilon}) - \chi^*\partial_\eta u_0$ on $\partial\Omega$.

3.2.1 Calculating the Constants χ_k^*

Associated to each side of Ω define the functions v_k, $k \in \{e, w, n, s\}$, as:

1. Let $G_e = \{(-\infty, 0] \times [0, 1]\}$ and v_e the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_e = 0 \text{ in } G_e,\\
&v_e(0, y_2) = \chi^1(1, y_2) \quad \text{for} \quad 0 < y_2 < 1,\\
&v_e(y_1, 0) = v_e(y_1, 1), \quad \text{for} \quad -\infty < y_1 < 0,\\
&\text{and} \quad \tfrac{\partial v_e}{\partial y_i}\exp(-\gamma y_1) \in L^2(G_e), \quad i = 1, 2.
\end{aligned}$$

2. Let $G_w = \{[0, \infty) \times [0, 1]\}$ and v_w the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_w = 0 \quad \text{in} \quad G_w,\\
&v_w(0, y_2) = -\chi^1(1, y_2) \quad \text{for} \quad 0 < y_2 < 1,\\
&v_w(y_1, 0) = v_w(y_1, 1), \quad \text{for} \quad 0 < y_1 < \infty,\\
&\text{and} \quad \tfrac{\partial v_w}{\partial y_i}\exp(\gamma y_1) \in L^2(G_w), \qquad i = 1, 2.
\end{aligned}$$

3. Let $G_n = \{[0,1] \times (-\infty, 0]\}$ and v_n the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_n = 0 \ \text{ in } \ G_n, \\
&v_n(y_1, 0) = \chi^2(y_1, 1) \ \text{ for } \ 0 < y_1 < 1, \\
&v_n(0, y_2) = v_n(1, y_2) \ \text{ for } \ -\infty < y_2 < 0, \\
&\text{and } \ \tfrac{\partial v_n}{\partial y_i}\exp(-\gamma y_2) \in L^2(G_n), \quad , \quad i = 1, 2.
\end{aligned}$$

4. Let $G_s = \{[0,1] \times [0, \infty)\}$ and v_s the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2)\nabla_y v_s = 0 \ \text{ in } \ G_s, \\
&v_s(y_1, 0) = -\chi^2(y_1, 0) \ \text{ for } \ 0 < y_1 < 1, \\
&v_s(0, y_2) = v_s(1, y_2) \ \text{ for } \ 0 < y_2 < \infty, \\
&\text{and } \ \tfrac{\partial v_s}{\partial y_i}\exp(\gamma y_2) \in L^2(G_s), \quad i = 1, 2.
\end{aligned}$$

From [11] Section 6 there exists a unique solution for each of the above equations. Let

$$\begin{aligned}
\chi_k^* \quad = \quad & \frac{1}{(A\eta^k, \eta^k)} \int_{\Gamma_k} \left[\chi^l a_{ij} \left(\delta_{jm} - \frac{\partial \chi^m}{\partial y_j} \right) \eta_i^k \eta_m^k \eta_l^k \right] \Bigg|_{\Gamma_k} ds \\
& + \int_{G_k} (a(y_1, y_2)\nabla_y v_k \cdot \nabla_y v_k) dy,
\end{aligned}$$

where η^k denotes the unity outward normal at Γ_k and η_i^k its ith component. It can be shown [11] that v_e decays exponentially to zero for $y_1 \to -\infty$, i.e.

$$(v_e - \chi_e^*)\exp(-\gamma y_1) \in L^2(G_e).$$

Similar results hold also when $k \in \{w, n, s\}$.

3.2.2 Approximating $\tilde{\theta}_\epsilon$

We note by Remark 3.1 that $(u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0)|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$. Thus we can split $\tilde{\theta}_\epsilon = \sum_{k \in \{e,w,n,s\}} \tilde{\theta}_\epsilon^k$, where

$$L_\epsilon \tilde{\theta}_\epsilon^k = 0 \ \text{ in } \ \Omega, \qquad \tilde{\theta}_\epsilon^k = \begin{cases} -u_1(x, \frac{x}{\epsilon}) - \chi^* \partial_\eta u_0 & \text{on } \ \Gamma_k, \\ 0 & \text{on } \ \partial\Omega \setminus \Gamma_k. \end{cases}$$

We approximate $\tilde{\theta}_\epsilon^k$ by $\tilde{\phi}_\epsilon^k$ given by

$$
\begin{aligned}
\tilde{\phi}_\epsilon^e(x_1,x_2) &= \varphi_e(x_1)\left(v_e(\frac{x_1-1}{\epsilon},\frac{x_2}{\epsilon})-\chi_e^*\right)\frac{\partial u_0}{\partial x_1}(x_1,x_2), \\
\tilde{\phi}_\epsilon^w(x_1,x_2) &= -\varphi_w(x_1)\left(v_w(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon})-\chi_w^*\right)\frac{\partial u_0}{\partial x_1}(x_1,x_2), \\
\tilde{\phi}_\epsilon^n(x_1,x_2) &= \varphi_n(x_2)\left(v_n(\frac{x_1}{\epsilon},\frac{x_2-1}{\epsilon})-\chi_n^*\right)\frac{\partial u_0}{\partial x_2}(x_1,x_2), \\
\tilde{\phi}_\epsilon^s(x_1,x_2) &= -\varphi_s(x_2)\left(v_s(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon})-\chi_s^*\right)\frac{\partial u_0}{\partial x_2}(x_1,x_2),
\end{aligned}
\tag{8}
$$

where φ_k are nonnegative smooth functions satisfying

$$
\varphi_e(s)=\varphi_n(s)=\begin{cases}1 & \text{if } s=1\\ 0 & \text{if } s=0,\end{cases} \quad \varphi_w(s)=\varphi_s(s)=\begin{cases}0 & \text{if } s=1\\ 1 & \text{if } s=0.\end{cases}
$$

Hence

$$
\tilde{\phi}_\epsilon=\sum_{k\in\{e,w,n,s\}}\tilde{\phi}_\epsilon^k
$$

approximate $\tilde{\theta}_\epsilon$, and $\tilde{\phi}_\epsilon=\tilde{\theta}_\epsilon$ on the boundary of Ω.

3.2.3 Approximating $\bar{\theta}_\epsilon$

The boundary condition imposed on Equation (7) does not depend on ϵ. An effective approximation for $\bar{\theta}_\epsilon$ is given by $\bar{\phi}\in H^1(\Omega)$ the solution of

$$
-\nabla\cdot A\nabla\bar{\phi}=0 \text{ in } \Omega, \quad \bar{\phi}=\chi^*\partial_\eta u_0 \text{ on } \partial\Omega.
$$

We define our theoretical approximation for u_ϵ as $u_0+\epsilon u_1+\epsilon\phi_\epsilon$, where

$$
\phi_\epsilon=\tilde{\phi}_\epsilon+\bar{\phi}.
$$

Note that $\phi_\epsilon|_{\partial\Omega}=\theta_\epsilon|_{\partial\Omega}$, therefore $u_0+\epsilon u_1+\epsilon\phi_\epsilon=0$ on $\partial\Omega$. In [14] we prove the following error bounds

Theorem 3.1 *Assume that $a_{ij}\in C^{1,\beta}(\Re^2)$ and $u_0\in H^2(\Omega)$. Then there exists a constant c, such that*

$$
\|u_\epsilon-u_0-\epsilon u_1-\epsilon\phi_\epsilon\|_1\leq c\epsilon\|u_0\|_2.
$$

Theorem 3.2 *Assume that $a_{ij} \in C^{1,\beta}(\Re^2)$ and $u_0 \in H^3(\Omega)$. Then there exists a constant c, such that*

$$\|u_\epsilon - u_0 - \epsilon u_1 - \epsilon\phi_\epsilon\|_0 \leq c\epsilon^{3/2}\|u_0\|_3.$$

Remark 3.1 *In the case $\Omega = [0,1] \times [0,1]$ we have*

$$\partial_\eta u_0 = \begin{cases} \frac{\partial u_0}{\partial x_1} & on \ \Gamma_e, \\ -\frac{\partial u_0}{\partial x_1} & on \ \Gamma_w, \\ \frac{\partial u_0}{\partial x_2} & on \ \Gamma_n, \\ -\frac{\partial u_0}{\partial x_2} & on \ \Gamma_s. \end{cases}$$

Since u_0 satisfies zero Dirichlet boundary condition on $\partial\Omega$ and $u_0 \in H^2(\Omega)$, we have $\frac{\partial u_0}{\partial x_1}|_{\Gamma_n \cup \Gamma_s} = 0$ and $\frac{\partial u_0}{\partial x_2}|_{\Gamma_e \cup \Gamma_w} = 0$. Therefore

$$\partial_\eta u_0 = \left(\varphi_e \frac{\partial u_0}{\partial x_1} - \varphi_w \frac{\partial u_0}{\partial x_1} + \varphi_n \frac{\partial u_0}{\partial x_2} - \varphi_s \frac{\partial u_0}{\partial x_2}\right)\Bigg|_{\partial\Omega},$$

where each term on the right hand side satisfies $\varphi_k \frac{\partial u_0}{\partial x_{j_k}} = 0 \quad on \quad \partial\Omega \setminus \Gamma_k$. Using that $\varphi_k \frac{\partial u_0}{\partial x_{j_k}} \in H^1(\Omega)$ we obtain $\varphi_k \frac{\partial u_0}{\partial x_{j_k}}\Big|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$ and

$$\begin{aligned} \|\chi^* \partial_\eta u_0\|_{H^{1/2}(\partial\Omega)} &\leq \left\| \varphi_e \chi_e^* \frac{\partial u_0}{\partial x_1} - \varphi_w \chi_w^* \frac{\partial u_0}{\partial x_1} + \varphi_n \chi_n^* \frac{\partial u_0}{\partial x_2} - \varphi_s \chi_s^* \frac{\partial u_0}{\partial x_2} \right\|_1 \\ &\leq c(\chi^*)\|u_0\|_2. \end{aligned}$$

Note also that $u_1(x, \frac{x}{\epsilon}) = -\chi^j\left(\frac{x}{\epsilon}\right)\frac{\partial u_0}{\partial x_j}(x)$. Since $\chi^j \in C^{2,\beta}(\Re^2)$ we can use the same argument given in this Remark to show that $u_1|_{\Gamma_k} \in H_{00}^{1/2}(\Gamma_k)$.

4 FINITE ELEMENT APPROXIMATION

We now describe how to numerically approximate the terms u_0, u_1, $\tilde{\phi}_\epsilon$ and $\bar{\phi}$.

- Solve the cell problem (2) with a second order accurate conforming finite element in a partition $\mathcal{T}_{\hat{h}}(Y)$. Call these solutions $\chi_{\hat{h}}^j$.
- Obtain $A^{\hat{h}}$ by

$$A_{ij}^{\hat{h}} = \frac{1}{|Y|}\int_Y a_{lm}(y)\frac{\partial}{\partial y_l}(y_i - \chi_{\hat{h}}^i)\frac{\partial}{\partial y_m}(y_j - \chi_{\hat{h}}^j)dy.$$

- Define $V^h(\Omega) = \{v \in C^0(\Omega);\ v|_K \in \mathcal{Q}_1(K),\ K \in \mathcal{T}_h(\Omega),\ K \text{ rectangular}\}$ and $V_0^h(\Omega) = V^h(\Omega) \cap H_0^1(\Omega)$. Let $u_0^{h,\hat{h}} \in V_0^h$ satisfying

$$\int_\Omega (A^{\hat{h}} \nabla u_0^{h,\hat{h}}, \nabla v^h) dx = \int_\Omega f v^h dx, \qquad \forall v^h \in V_0^h.$$

  The justification for using a rectangular mesh is postponed to Remark 4.1.

- Define $u_1^{h,\hat{h}}$ as

$$u_1^{h,\hat{h}}(x) = -\chi_{\hat{h}}^j \left(\frac{x}{\epsilon}\right) \frac{\partial u_0^{h,\hat{h}}}{\partial x_j}(x).$$

  Note that this leads to a nonconforming approximation for u_1 in the partition $\mathcal{T}_h(\Omega)$.

- Let p be a positive integer and $G_e^p = [-p, 0] \times [0,1]$. Define $\tilde{v}_e \in H^1(G_e^p)$ the solution of

$$\begin{aligned}
&-\nabla_y \cdot a(y_1, y_2) \nabla_y \tilde{v}_e = 0 \text{ in } G_e^p, \\
&\tilde{v}_e(0, y_2) = \chi_{\hat{h}}^1(1, y_2), \quad 0 \le y_2 \le 1, \\
&\partial_\eta \tilde{v}_e = 0, \quad \text{on } \{y \in G_e^p;\ y_1 = -p\}, \\
&\text{and } \tilde{v}_e(y_1, 0) = \tilde{v}_e(y_1, 1), \quad -p \le y_1 \le 0.
\end{aligned}$$

  Let $v_e^{\hat{h},p}$ be a numerical approximation of $\tilde{v}_e$ using a second order accurate conforming finite element on a mesh $\mathcal{T}_{\hat{h}}(G_e^p)$.

- Define

$$\begin{aligned}
\chi_e^{*,\hat{h},p} \quad &= \quad \frac{1}{A_{11}^{\hat{h}}} \int_0^1 \left(\chi_{\hat{h}}^1(1, y_2) a_{1k}(1, y_2) \left[\delta_{k1} - \frac{\partial \chi_{\hat{h}}^1(1, y_2)}{\partial y_2} \right] \right) dy_2 \\
&\quad + \int_{G_e^p} (a(y_1, y_2) \nabla_y v_e^{\hat{h},p} \cdot \nabla_y v_e^{\hat{h},p}) dy.
\end{aligned}$$

  The other cases $k \in \{w, n, s\}$ are treated similarly.

- Let $\bar{\phi}^{h,\hat{h},p}$ be a second order accurate finite element approximation in a mesh of size h for the following equation

$$-\nabla A^{\hat{h}} \nabla \psi = 0, \qquad \psi = \chi^{*,\hat{h},p} \partial_\eta u_0^{\hat{h},h} \text{ on } \partial\Omega. \tag{9}$$

Remark 4.1 *Since* $u_0^{\hat{h},h} \in H_0^1(\Omega)$*, the domain* Ω *is rectangular, and bilinear rectangular elements are considered to obtain* $u_0^{\hat{h},h}$*, is easy to see that* $\partial_\eta u_0^{\hat{h},h}$ *is continuous on* $\partial\Omega$ *and linear in every edge of* $\mathcal{T}_h(\partial\Omega)$*. Observe also that the zero Dirichlet boundary condition implies* $\partial_\eta u_0^{\hat{h},h} = 0$ *at the corners of* Ω*. Therefore* $\chi^{*,\hat{h},p}\partial_\eta u_0^{\hat{h},h} \in H^{1/2}(\partial\Omega)$ *and Equation (9) is well posed. Taking* $\bar{\phi}^{h,\hat{h},p} \in V^h$ *allows us to use the same stiffness matrix for obtaining* $u_0^{\hat{h},h}$ *and* $\bar{\phi}^{h,\hat{h},p}$*.*

- Observe that in Equation. (8) the term $v_e(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ appears. Since the approximation $v_e^{\hat{h},p}$ is defined in G_e^p, we can calculate $v_e^{\hat{h},p}(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon})$ only if $x_1 \geq 1 - \epsilon p$. Since the functions $v_k - \chi_k^*$ decays exponentially to zero away from the boundary its is natural to consider the following approximation

$$\tilde{\phi}_\epsilon^{e,h,\hat{h},p}(x_1,x_2) = \begin{cases} \varphi_e(x_1)(v_e^{\hat{h},p}(\frac{x_1-1}{\epsilon}, \frac{x_2}{\epsilon}) - \chi_e^{*,\hat{h},p})\frac{\partial u_0^{h,\hat{h}}}{\partial x_1} & \text{if } \ x_1 > 1-\epsilon p, \\ 0 & \text{if } \ x_1 \leq 1-\epsilon p, \end{cases} \tag{10}$$

and

$$\tilde{\phi}_\epsilon^{h,\hat{h},p} = \sum_{k\in\{e,w,n,s\}} \tilde{\phi}_\epsilon^{k,h,\hat{h},p}.$$

- Approximate θ_ϵ by $\phi_\epsilon^{h,\hat{h},p} = \tilde{\phi}_\epsilon^{h,\hat{h},p} + \bar{\phi}^{h,\hat{h},p}$ and finally construct the numerical approximation for u_ϵ as

$$u_\epsilon^{h,\hat{h},p} = u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon\phi_\epsilon^{h,\hat{h},p}.$$

5 ERROR ANALYSIS

When $p \to \infty$ and $\hat{h} \to 0$ we prove in [14] the following estimates.

Theorem 5.1 *Assume that* $a_{ij} \in C^{1,\beta}(\Re^2)$ *and* $u_0 \in W^{2,\infty}(\Omega)$*. Then there exists a constant c, such that*

$$|u_\epsilon - u_h|_{1,h} \leq c(h+\epsilon)\|u_0\|_{2,\infty}$$

Theorem 5.2 *Assume that* $a_{ij} \in C^{1,\beta}(\Re^2)$ *and* $u_0 \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$*. Then there exists a constant c, such that*

$$\|u_\epsilon - u_h\|_0 \leq c(h^2 + \epsilon^{\frac{3}{2}} + \epsilon h ln(h))(|u_0|_{2,\infty} + \|u_0\|_3)$$

6 NUMERICAL EXPERIMENTS

In this section, we present some numerical results for solving our model problem with

$$a(x) = \left(\frac{2 + P\sin(2\pi x_1/\epsilon)}{2 + P\cos(2\pi x_2/\epsilon)} + \frac{2 + \sin(2\pi x_2/\epsilon)}{2 + P\sin(2\pi x_1/\epsilon)} \right) I_{2\times 2}$$

$$f(x) = -1 \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

We compare the solution obtained by our method with the solution obtained by a second order accurate finite element method in a fine mesh of size h_f, which we call u_ϵ^*. Tables I and II provide absolute errors estimates for $u_\epsilon^* - u_\epsilon^{h,\hat{h},p}$, on the $\|\cdot\|_0$ norm and $|\cdot|_{1,h}$ semi norm for different values of h and ϵ. We have used $p = 2$, $\hat{h} = 1/128$, $h_f = 1/2048$, and a triangular mesh with continuous piecewise linear functions to approximate $\chi_{\hat{h}}^j$ and $v_e^{\hat{h},p}$.

Table 1: $\|\cdot\|_0$ error

$\epsilon \downarrow \quad h \rightarrow$	1/8	1/16	1/32	1/64
1/16	2.7085e-04	7.7993e-05		
1/32	2.6300e-04	6.6246e-05	1.7773e-05	
1/64	2.5388e-04	5.9446e-05	1.4414e-05	1.2137e-05

Table 2: $|\cdot|_{1,h}$ error

$\epsilon \downarrow \quad h \rightarrow$	1/8	1/16	1/32	1/64
1/16	0.0097	0.0066		
1/32	0.0089	0.0051	0.0036	
1/64	0.0086	0.0045	0.0026	0.0018

From Tables I and II, we see that for $\epsilon << h$ we have errors of order $O(h^2)$ and $O(h)$ for the L^2 norm and semi norm H^1 respectively. We observe that when we fix h and decrease ϵ the errors almost do not change. This is an evidence that in this case the dominant error term is $O(h)$. Also looking the diagonal values in these tables we see

Table 3:

$\epsilon = 1/64,\ h = 1/32,\ h_f = 1/1024$

	$\|\cdot\|_0$	$\lvert\cdot\rvert_{1,h}$
$u_\epsilon^* - u_0^{h,\hat{h}}$	0.0287	0.0215
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$	0.0213	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon\bar{\phi}^{h,\hat{h},p}$	6.1557e-05	0.0026
$u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}_\epsilon^{h,\hat{h},p})$	6.1557e-05	0.0024

clearly that the numerical error agrees with the theoretical rates from Theorems 5.1 and 5.2.

Table III shows the improvement obtained in the final approximation by considering the numerical approximation for the boundary corrector. We observe a better improvement on the $\|\cdot\|_0$ norm rather then on $|\cdot|_{1,h}$ semi norm. The reason for this is that $\bar{\phi}$ is obtained through the homogenized equation associated to Problem (7), therefore it is a good approximation for $\bar{\theta}_\epsilon$ on $L^2(\Omega)$ norm but not on $|\cdot|_1$ semi norm. The term $\tilde{\phi}_\epsilon$ is defined in a thin boundary layer that mostly force the approximation to satisfies the zero Dirichlet boundary condition.

In our numerical tests we observed a very fast convergence of $v_e^{\hat{h},p}$ to the constant $\chi_e^{*,\hat{h},p}$ as $y_1 \to -p$. Considering $p_1 < p_2 \in \{1, 2...8\}$ we obtained that $\sup_{\{y_2\in[0,1],y_1\in[-p_2,-p_1]\}} |v_e^{\hat{h},p_1}(-p_1, y_2) - v_e^{\hat{h},p_2}(y_1, y_2)| \leq 10^{-14}$. That confirms the numerical approximation for $\tilde{\phi}_\epsilon^e$ given by Formula (10) is reasonable.

The Figures bellow show the error evolution as we include the asymptotic expansion terms in our numerical approximation, for $h_f = 1/100$, $h = 1/10$, $\hat{h} = 1/50$, $p = 2$ and $\epsilon = 1/20$; Figure 1 is the plot of the "exact" solution u_ϵ^*. In Figure 2 from left to right we see that amplitude of error oscillations decreases when we include the approximation for u_1. Its is possible to see an overall improvement in the error from Figure 2 (left) to Figure 3 (right) when the approximation for $\bar{\phi}$ is included, and finally in Figure 3 (left) we see that the zero boundary condition is satisfied when the complete approximation $u_0^{h,\hat{h}} + \epsilon u_1^{h,\hat{h}} + \epsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}^{h,\hat{h},p})$ is considered.

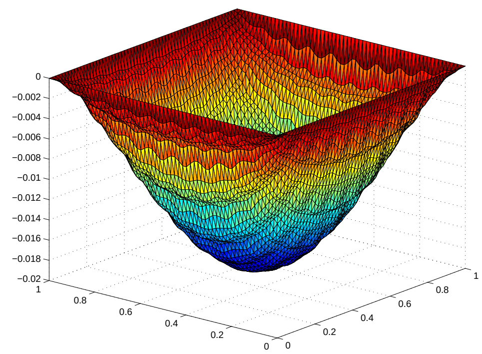

Figure 1: u_ϵ^*

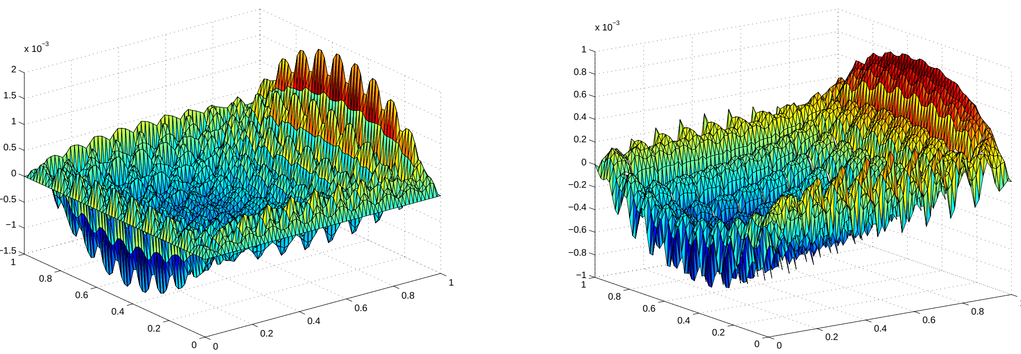

Figure 2: $u_\epsilon^* - u_0^{h,\hat{h}}$ (left), and $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}}$ (right)

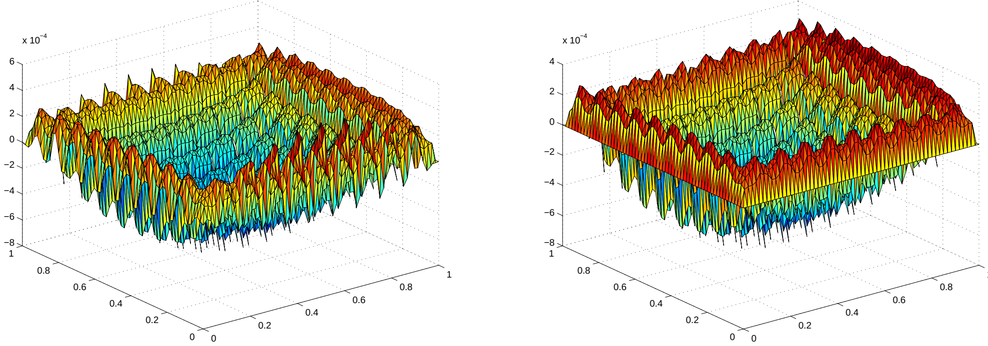

Figure 3: $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon \bar{\phi}^{h,\hat{h},p}$ (left), and $u_\epsilon^* - u_0^{h,\hat{h}} - \epsilon u_1^{h,\hat{h}} - \epsilon(\bar{\phi}^{h,\hat{h},p} + \tilde{\phi}^{h,\hat{h},p})$ (right)

7 CONCLUSIONS

We propose a new method for approximating numerically the solution of Equation (1). This method is strongly based on periodicity of the coefficients a_{ij}, and for this reason it has relative low computational cost with quasi optimal error convergence rate. Although the convergence analysis presented in [14] does not work for the quasi periodic case $a_{ij}(x, x/\epsilon)$, we believe that the numerical approximation presented here can be generalized for this case. This would be done by approximating matrix $a(x, x/\epsilon)$ by $\sum_j a^j(x/\epsilon) I_{K_j}(x)$, where I_{K_J} is the characteristic function for $K_j \in \mathcal{T}_k(\Omega)$, and then solving cell problem in each sub-domain K_j.

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