Heterodimensional tangencies

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December 6, 2005

Abstract

We consider C^1-diffeomorphisms f defined on three dimensional manifolds having a pair of saddles P_f and Q_f (of unstable indices one and two) whose homoclinic classes coincide persistently. We prove that if the two-dimensional stable manifold of P_f and the two-dimensional unstable manifold of Q_f have some non-transverse intersection (a heterodimensional tangency) the unfolding of such a tangency leads to diffeomorphisms h such that the homoclinic class of Q_h (the continuation of Q_f for h) is robustly non-dominated. This leads to the phenomena of (C^1-locally generic) coexistence of infinitely many sinks or sources and, in some relevant cases, to the coexistence of infinitely many minimal Cantor sets. We give examples where the previous dynamical configuration occurs, providing a natural transition from partially hyperbolic to robustly non-dominated dynamics.

Keywords: dominated splitting, heterodimensional cycle, heterodimensional tangency, homoclinic class, index of a saddle, robustly transitive set.
MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

The main motivation of this paper comes from the so-called *Newhouse C^2-generic coexistence phenomenon*: given any surface M, there are a non-empty open set $\mathcal{U} \subset \mathrm{Diff}^2(M)$ and a residual subset $\mathcal{R}_\mathcal{U}$ of it, such that every diffeomorphism $f \in \mathcal{R}_\mathcal{U}$ has infinitely many sinks, see [22, 23]. On one hand, these results rely on the construction of hyperbolic sets having persistent homoclinic tangencies. On the other hand, the construction of these hyperbolic sets with persistent tangencies is based on the creation of thick horseshoes in the unfolding of homoclinic tangencies. Shortly, [23] claims that *any* homoclinic tangency leads to the C^2-Newhouse coexistence phenomenon. We will discuss briefly the Newhouse's construction, after stating our results, in the end of this introduction.

We point out that the ingredients of the Newhouse's construction are typically C^2 (bounded distortion and fractal dimensions). Moreover, the results in [37] are an indication that those results cannot be adapted to the C^1-topology. In fact, the extension of Newhouse's construction to the

*This paper was partially supported by CNPq (Research Grants and Edital Universal), Faperj (Cientistas do Nosso Estado and Ph.D. Program), and PRONEX (Brazil) and University of Toronto (Canada). L.J.D. and E.R.P. thank the warm hospitality of the Departament of Mathematics of the University of Toronto during their visits while preparing this paper.

C^1-topology is closely related to the Smale's Conjecture on the C^1-density of hyperbolic surface diffeomorphisms (Axiom A and no-cycles condition)[1].

On the other hand, in dimension greater than or equal to three and in the C^1-topology, there are open sets of diffeomorphisms satisfying the Newhouse generic coexistence phenomenon: existence of an open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks, see [6, 7][2]. These examples rely on the construction of persistently *non-dominated homoclinic classes* (see Definitions 1.1 and 1.7). There is the following dichotomy for homoclinic classes of C^1-generic diffeomorphisms: either a homoclinic class has a *dominated splitting* or it is contained in the closude of an infinite set of sinks or sources, see [8]. Therefore, to get C^1-open sets of diffeomorphisms in which the Newhouse phenomenon can be found, it is enough to construct homoclinic classes being persistently non-dominated.

An important step of the construction of non-dominated homoclinic classes in [6, 7] is the generation of persistent *heterodimensional cycles* (see Definition 1.8) associated to saddles having different *indices* (dimension of the unstable bundle) and non-real multipliers (we observe that heterodimensional cycles can only appear in dimension strictly greater than two). We point out that heterodimensional cycles may be persistent in the C^1-topology (this is used in [6, 7]). In contra-position, to get persistent homoclinic tangencies one needs to consider C^2-diffeomorphisms. This is probably the main reason because the C^1-coexistence phenomenon may occur in dimension three (and higher dimensions). The C^1-persistence of heterodimensional cycles involves a blender-like argument, which does not require distortion estimates and roughly plays the role of the thick horseshoes in the Newhouse C^2-construction (we will briefly comment this point in Section 1.2, for a discussion about blenders see [10, Chapter 6]).

In summary, homoclinic tangencies is the natural context for the C^2-Newhouse coexistence phenomenon. On the other hand, the C^1-Newhouse coexistence phenomenon is closely related to the generation of non-dominated homoclinic classes and its natural context are manifolds of dimension three (or greater). Thus a natural question is how to get such a non-dominated dynamical configuration. This paper provides a natural scenario for the C^1-Newhouse coexistence phenomenon (and strong forms of it) in dimension three: the unfolding of *heterodimensional tangencies,* (see Definition 1.2). Roughly, we first consider non-hyperbolic homoclinic classes (persistently) containing a pair of saddles P of index one and Q of index two. Then we study the dynamics at non-transverse intersections (heterodimensional tangencies) between the two-dimensional stable manifold $W^s(P)$ of P and the two-dimensional unstable manifold $W^u(Q)$ of Q. We see that the unfolding of this tangency leads to robustly non-dominated homoclinic classes, therefore (by [8]) to the C^1-Newhouse coexistence phenomenon.

In a somewhat different context, our results also (partially) answer the following question: *how do partially hyperbolic sets/diffeomorphisms bifurcate?* This question is motivated by the series of papers by Newhouse, Palis, and Takens studying bifurcations of hyperbolic sets (of C^2-surface diffeomorphisms) via homoclinic tangencies (see [26, 27, 23, 28, 29, 30]): a hyperbolic set Λ of saddle-type (the homoclinic class of a saddle) losses its hyperbolicity via the creation of a

[1]We recall that the density of hyperbolic dynamics does not hold in the C^2-topology in any dimension greater than or equal to two, (see [22] and the extensions to higher dimensions in [34, 31]) and in the C^1-topology in dimensions higher than or equal to three (see, for instance, the pioneering papers [2], for dimension $n \geq 3$, and [35], for dimension three).

[2]The results in [6, 7] are stated for three dimensional manifolds, but they can be extended to higher dimensions after straightforward modifications.

homoclinic (or heteroclinic) tangency between the invariant manifolds of some saddle(s) of Λ. The unfolding of this tangency leads to a string of bifurcations and interesting dynamical phenomena: period doubling bifurcations ([38]), persistence of tangencies, persistence of coexistence of infinitely many sinks or sources ([23]), and persistence of strange attractors ([20]), among others. See [30] for a comprehensive study of homoclinic bifurcations and [25] for a survey on recent results on this subject.

The dynamical configuration (heterodimensional tangencies) we study here is reminiscent of the one of homoclinic tangencies. Instead bifurcations of hyperbolic homoclinic classes as in the papers above, we consider partially hyperbolic ones. Typically, these non-hyperbolic classes contain saddles of indices one and two. We analyze what happens at heterodimensional tangencies relating these saddles with different indices. In many relevant cases, this kind of bifurcation provides the transition from *partially hyperbolic to non-dominated dynamics.*

1.1 Definitions and statement of results

In this section, we state precisely our results. We begin by recalling the definition of homoclinic class and stating some results about them.

1.1.1 Homoclinic classes and heterodimensional tangencies

Homoclinic classes were introduced by Newhouse in [21] as an abstraction of the basic sets in the Smale theory (see [36]). They are an essential ingredient in the structure of the dynamics of diffeomorphisms and in many relevant cases correspond to the *elementary pieces* of dynamics (see [10, Chapter 10.4]).

Definition 1.1 (Homoclinic class). *Let P be a hyperbolic periodic saddle of a diffeomorphism f. The* homoclinic class *of P, denoted by $H(P,f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit $\mathcal{O}(P)$ of P,*

$$H(P,f) = \overline{W^s(\mathcal{O}(P),f) \pitchfork W^u(\mathcal{O}(P),f)}.$$

The homoclinic class of a hyperbolic periodic saddle P can be also (equivalently) defined as the closure of the set of (hyperbolic) saddles Q *homoclinically related* to P: the stable manifold of the orbit of Q transversely meets the unstable one of the orbit of P and vice-versa. This implies that two saddles P and Q which are homoclinically related have the same index. Moreover, the saddles of $H(P,f)$ having the same index as P form a dense subset of the whole class $H(P,f)$. Recall that the *index* of a saddle is the dimension of its unstable bundle. Finally, any homoclinic class $H(P,f)$ is f-invariant and *transitive* (i.e., there is some $x \in H(P,f)$ whose orbit is dense in $H(P,f)$). In fact, the points of $H(P,f)$ whose forward orbits are dense in the whole class form a residual subset of $H(P,f)$. For these properties about homoclinic classes see, for instance, [24, Section 3].

On the other hand, a homoclinic class $H(P,f)$ may fail to be uniformly hyperbolic: it may contain in a robust way hyperbolic saddles having indices different as the one of P (see, for instance, the constructions in [13, 14, 15] and the examples in [5, 11]). Finally, a homoclinic class may also fail to be locally maximal: for instance, the class can be contained with some persistence in the closure of an infinite set of sinks or sources, (see [6]). In fact, the homoclinic classes we obtain here after unfolding a heterodimensional tangency are not neither hyperbolic nor locally maximal.

For C^1-*generic* diffeomorphisms f (i.e., diffeomorphisms in some residual subset of $\mathrm{Diff}^1(M)$) the homoclinic classes of the periodic points of f form a partition (i.e., a pair of homoclinic classes either coincide or they are disjoint) of a dense subset of the limit set of f (this follows from [32, 3, 12]). Moreover, [1, Lemma 2.1] (which follows from [3, 12] using an argument of genericity) claims the following:

Consider a diffeomorphism f and any pair of (hyperbolic) saddles P_f and Q_f of f of different indices. Then there are a C^1-open neighbourhood $\mathcal{U}_f$ of f and a residual subset $\mathcal{R}_{\mathcal{U}_f}$ of $\mathcal{U}_f$ with the following property: let P_g and Q_g the continuations of the saddles P_f and Q_f for g close to f, then

- *either $H(P_g, g) = H(Q_g, g)$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$;*
- *or $H(P_g, g) \cap H(Q_g, g) = \emptyset$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$.*

In the first case, we say that the saddles P_f and Q_f are persistently linked (in $\mathcal{U}_f$).

Remark that any homoclinic class containing two different orbits is necessarily infinite. Thus if P_f and Q_f have different indices and are persistently linked then $H(P_f, f) = H(Q_f, f)$ is infinite.

In what follows, we consider saddles (of different indices) which are persistently linked and heterodimensional tangencies associated to them as follows:

Definition 1.2 (Heterodimensional tangency). *Let f be a diffeomorphism and P_f and Q_f two saddles of f with index(P_f) $<$ index(Q_f) and such that $H(P_f, f) = H(Q_f, f)$. We say that f has a* heterodimensional tangency *at y_f associated to P_f and Q_f if $W^s(P_f, f)$ and $W^u(Q_f, f)$ have some non-transverse intersection at some point y_f.*

Note that in the previous definition $\dim(W^s(P_f, f)) + \dim(W^u(Q_f, f)) > \dim(M)$. We observe that (for us) the interesting case in Definition 1.2 occurs when the saddles P_f and Q_f are persistently linked.

Finally, let us recall a definition:

Definition 1.3 (Minimal set). *Let f be a diffeomorphism. A closed f-invariant set Λ_f is* minimal *if the orbit of any point $x \in \Lambda_f$ is dense in the whole Λ_f.*

1.1.2 Statement of results

Before stating our main result let us stand some notation. From now on M is a closed manifold of dimension three and $\mathrm{Diff}^1(M)$ denotes the space of C^1-diffeomorphisms endowed with the usual uniform topology. Given a hyperbolic periodic point P_f of a diffeomorphism f we denote by $\pi(P_f)$ the period of P_f and by P_g the continuation of P_f for g close to f.

Theorem 1. *Let $\mathcal{L}_{P,Q}(M)$ be an open set of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{L}_{P,Q}(M)$, there are saddles P_f and Q_f (depending continuously on f) with different indices which are persistently linked in $\mathcal{L}_{P,Q}(M)$. Assume that there is $f_0 \in \mathcal{L}_{P,Q}(M)$ with a heterodimensional tangency associated to P_{f_0} and Q_{f_0}. Then there are a C^1-open set $\mathcal{V}_0$ and a residual subset $\mathcal{R}_{\mathcal{V}_0}$ of $\mathcal{V}_0$ such that f_0 belongs to the closure of $\mathcal{V}_0$ and for every $h \in \mathcal{R}_{\mathcal{V}_0}$ it holds*

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

In particular, any $h \in \mathcal{R}_{\mathcal{V}_0}$ has either infinitely many sinks or sources.

Moreover, if the jacobians of P_{f_0} and Q_{f_0} verify

$$(\star) \qquad \mathrm{Jac}[f_0^{\pi(P_{f_0})}(P_{f_0})] > 1 > \mathrm{Jac}[f_0^{\pi(Q_{f_0})}(Q_{f_0})] \quad \textit{or} \quad \mathrm{Jac}[f_0^{\pi(P_{f_0})}(P_{f_0})] < 1 < \mathrm{Jac}[f_0^{\pi(Q_{f_0})}(Q_{f_0})],$$

then the sets $\mathcal{V}_0$ and $\mathcal{R}_{\mathcal{V}_0}$ can be taken such that every $h \in \mathcal{R}_{\mathcal{V}_0}$ also satisfies

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\left(\bigcup \text{minimal Cantor set of } h\right).$$

In particular, any $h \in \mathcal{R}_{\mathcal{V}_0}$ has infinitely many non-trivial minimal Cantor sets.

Remark 1.4. *By* [7, Corollary C], *if in Theorem 1 Condition ($\star$) holds one can take the locally residual set $\mathcal{R}_{\mathcal{V}_0}$ such that, for every $h \in \mathcal{R}_{\mathcal{V}_0}$, the homoclinic class $H(P_h, h) = H(Q_h, h)$ is (simultaneously) contained in the closure of the union of infinitely many*

- *disjoint non-trivial homoclinic classes of h,*
- *disjoint non-trivial hyperbolic attractors and repellors of h, and*
- *disjoint non-trivial partially hyperbolic attractors and repellors of h.*

Theorem 1 follows from the following somewhat more general result (Proposition 1.5 below). Assume that the saddles P_f and Q_f have indices one and two, respectively. Consider the subset $\mathcal{L}^c_{P,Q}(M)$ of $\mathcal{L}_{P,Q}(M)$ consisting of diffeomorphisms $f \in \mathcal{L}_{P,Q}(M)$ such that the transverse intersection between $W^s(P_f, f)$ and $W^u(Q_f, f)$ contains a closed curve diffeomorphic to a circle. Note that, by definition, the set $\mathcal{L}^c_{P,Q}(M)$ is open.

Proposition 1.5. *There is a residual subset $\mathcal{R}_{\mathcal{L}}(M)$ of $\mathcal{L}^c_{P,Q}(M)$ of diffeomorphisms g such that*

$$H(P_g, g) = H(Q_g, g) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

Moreover, if P_f and Q_f verify Condition ($\star$) in Theorem 1 then the residual subset $\mathcal{R}_{\mathcal{L}}(M)$ can be taken such that

$$H(P_g, g) = H(Q_g, g) \subset \text{closure}\left(\bigcup \text{minimal Cantor set of } g\right),$$

for all $g \in \mathcal{R}_{\mathcal{L}}(M)$.

To derive Theorem 1 from this proposition it is enough to see that after perturbing a diffeomorphism f with a heterodimensional tangency, associated to saddles P_f and Q_f persistently linked, one can get a diffeomorphism g such that the corresponding invariant manifolds $W^s(P_g, g)$ and $W^u(Q_g, g)$ have a transverse intersection diffeomorphic to a circle.

A context where heterodimensional tangencies arise in a natural way is the one of *robustly transitive sets* we will describe in the next paragraphs. Given $f \in \mathrm{Diff}^1(M)$ and a subset K of M, denote by $\Lambda_f(K) = \cap_{i\in\mathbb{Z}} f^i(K)$ the maximal f-invariant set of f in K.

Definition 1.6 (Robustly transitive set). *Consider $f \in \mathrm{Diff}^1(M)$. An f-invariant compact set Λ_f is* robustly transitive *if there are neighbourhoods U of Λ_f in the ambient manifold M and $\mathcal{V}_f$ of f in $\mathrm{Diff}^1(M)$ such that $\Lambda_f = \Lambda_f(U)$ and, for every $g \in \mathcal{V}_f$, the set $\Lambda_g(U)$ is transitive and contained in the interior of U.*

By [8], every robustly transitive set has a dominated splitting (see Definition 1.7). Moreover, robustly transitive sets are partially hyperbolic in dimension three, [17], and uniformly hyperbolic in dimension two, [19].

Robustly transitive sets of C^1-generic diffeomorphisms are *relative homoclinic classes* (see [6]), meaning the following. Given an open set U of M let $\mathcal{T}(U)$ be the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive. By [6], there is a residual subset $\mathcal{H}(U)$ of $\mathcal{T}(U)$ such that, for every $g \in \mathcal{H}(U)$ and every saddle $P_g \in \Lambda_g(U)$, the set $\Lambda_g(U)$ is the subset of the homoclinic class $H(P_g, g)$ of P_g of points whose whole orbit are contained in U. The fact that robustly transitive sets are (C^1-generically) relative homoclinic classes and the results in [3, 12] will allow us to apply Theorem 1 to robustly transitive sets of non-hyperbolic type. Let us explain briefly this point.

We next consider robustly transitive sets of non-hyperbolic type of three dimensional diffeomorphisms. First, let $\mathcal{N}(U)$ be the subset of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ is not uniformly hyperbolic. Consider now the subset $\mathcal{J}(U)$ of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains (hyperbolic) saddles of indices one and two. This implies that $\mathcal{J}(U) \subset \mathcal{N}(U)$. By definition of robustly transitive set, the set $\mathcal{J}(U)$ is open. Moreover, by [9, Theorem A], the set $\mathcal{J}(U)$ is dense in $\mathcal{N}(U)$.

Take now $f \in \mathcal{J}(U)$, by definition, $\Lambda_f(U)$ contains saddles P_f and Q_f of indices one and two. By the previous results, for C^1-generic f, $\Lambda_f(U)$ is (simultaneously) the relative homoclinic class in U of P_f and Q_f. Moreover, these relative homoclinic classes coincide (this is just a relative version of [12]). Therefore, the saddles P_f and Q_f are persistently linked. In other words, for generic $f \in \mathcal{J}(U)$, there are saddles P_f and Q_f such that $f \in \mathcal{L}_{P,Q}(M)$.

Corollary 2. *Let M be a three dimensional closed manifold, U an open set of M, and $f \in \mathcal{J}(U)$ a diffeomorphism with a heterodimensional tangency associated to saddles P_f and $Q_f \in \Lambda_f(U)$. Then there are an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ whose closure contains f and a residual subset $\mathcal{R}_{\mathcal{U}}$ of $\mathcal{U}$ such that for every $h \in \mathcal{R}_{\mathcal{U}}$*

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

In particular, any $h \in \mathcal{R}_{\mathcal{U}}$ has either infinitely many sinks or sources.

Moreover, if the jacobians of P_f and Q_f verify Condition ($\star$) in Theorem 1 then the sets $\mathcal{U}$ and $\mathcal{R}_{\mathcal{U}}$ can be taken such that every $h \in \mathcal{R}_{\mathcal{U}}$ also satisfies

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\left(\bigcup \text{minimal Cantor set of } h\right).$$

In particular, any $h \in \mathcal{R}_{\mathcal{U}}$ has infinitely many non-trivial minimal Cantor sets.

Let us observe that, in, general, in the corollary the point corresponding to the heterodimensional tangency does not belong to U. Moreover, the orbits of the sinks and the sources, as well as the minimal sets, are not contained in the set U. This situation is similar to the one of homoclinic tangencies associated to hyperbolic sets in the series of papers mentioned above (the tangency does not belong to the hyperbolic set). It is interesting to compare this situation to the one in [33] where the tangency occurs *inside* a horseshoe.

Corollary 2 provides a context where heterodimensional tangencies occur in a natural way, exemplifying the transition from *partially hyperbolic to non-dominated dynamics.* In Section 4, we construct examples of robustly transitive sets $\Lambda_f(U)$ with heterodimensional tangencies. In these examples the robustly transitive sets have partially hyperbolic splittings with three bundles, $T_{\Lambda_f(U)} = E^s \oplus E^c \oplus E^u$, where E^s and E^u are uniformly hyperbolic (note that this is the non-hyperbolic situation closest to the hyperbolic one).

These examples also motivated by the series of papers [26, 27, 28, 29] where the boundary of the set of uniformly hyperbolic diffeomorphisms is crossed via a homoclinic or heteroclinic tangency. Here the situation is similar: the boundary of partially hyperbolic systems is crossed via a heterodimensional tangency. In [26, 27, 28, 29], after crossing the frontier of hyperbolicity one also returns (in fact, infinitely many times) to uniformly hyperbolic regions. The situation here is rather different: after crossing the frontier of partial hyperbolicity one remains in a domain where the dynamics is non-dominated.

1.2 Ingredients of the proofs

We begin by recalling some definitions and results that will play a key role in this paper:

Definition 1.7 (Dominated splitting and partial hyperbolicity). *An f-invariant compact set Λ has a* dominated splitting *if the tangent bundle $T_\Lambda M$ over Λ splits into two Df-invariant bundles E and F, $T_\Lambda M = E \oplus F$, whose fibers $E(x)$ and $F(x)$ have constant dimensions, and there exists an integer $\ell \geq 1$ such that for every point $x \in \Lambda$ and every pair of unit vectors $u \in E(x)$ and $v \in F(x)$ it holds*

$$\|Df^\ell(x)\, u\| \leq 2\, \|Df^\ell(x)\, v\|.$$

If the set Λ has a dominated splitting $T_\Lambda M = E \oplus F$ such that one of the bundles is uniformly hyperbolic we say that Λ is partially hyperbolic.

Definition 1.8 (Heterodimensional cycle). *Let P_f and Q_f be two saddles of different indices of a diffeomorphism f. The diffeomorphism f has a* heterodimensional cycle *associated to P_f and Q_f if*

$$W^s(P_f, f) \cap W^u(Q_f, f) \neq \emptyset \quad \text{and} \quad W^u(P_f, f) \cap W^s(Q_f, f) \neq \emptyset.$$

Remark 1.9. *Assume that the indices of the saddles Q_f and P_f in the definition of heterodimensional cycle are two and one, respectively. Then, after an arbitrarily small perturbation, one can assume that $W^s(P_f, f) \cap W^u(Q_f, f)$ contains transverse intersections and that $W^u(P_f, f)$ and $W^s(Q_f, f)$ meet quasi-transversely along the orbit of some point x_f, i.e., $x_f \in W^u(P_f, f) \cap W^s(Q_f, f)$ and $T_{x_f} W^u(P_f, f) \cap T_{x_f} W^s(Q_f, f) = \{\bar{0}\}$.*

Following [7], we introduce the notions of *wild homoclinic class* and *wild set of diffeomorphisms.* The homoclinic class $H(Q_f, f)$ of a saddle Q_f of $f \in \mathrm{Diff}^1(M)$ is *wild* if there is a neighbourhood $\mathcal{V}_f$ of f in $\mathrm{Diff}^1(M)$ such that, for every $g \in \mathcal{V}_f$, the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting. In this case, we say that the homoclinic class $H(Q_f, f)$ belongs to $W(f)$ (the *set of wild homoclinic classes of f*).

Denote by $\mathcal{W}(M)$ the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms having some wild homoclinic class (i.e., $f \in \mathcal{W}(M)$ if $W(f) \neq \emptyset$). We let $\mathcal{W}_{[1,2,3]}(M)$ the subset of $\mathcal{W}(M)$ of diffeomorphisms f having a wild homoclinic class $H(Q_f, f) \in W(f)$ and a C^1-neighbourhood $\mathcal{V}_f$ such that:

W1) for every $g \in \mathcal{V}_f$, the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting and contains saddles R_g and T_g (R_g and T_g continuously depend on g) having different indices;

W2) either R_g or T_g is homoclinically related to Q_g;

W3) the jacobians of R_g and T_g verify

$$\mathrm{Jac}[g^{\pi(R_g)}(R_g)] > 1 > \mathrm{Jac}[g^{\pi(Q_g)}(Q_g)] \quad \text{or} \quad \mathrm{Jac}[g^{\pi(R_g)}(R_g)] < 1 < \mathrm{Jac}[g^{\pi(Q_g)}(Q_g)].$$

We say that a homoclinic class $H(Q_f, f)$ verifying (W1)–(W3) belongs to $W_{[1,2,3]}(f)$.

Theorem 1.10.

- **([8, Corollary 0.3])** *There is a residual subset $\mathcal{R}(M)$ of $\mathcal{W}(M)$ of diffeomorphisms f such that any homoclinic class $H(P_f, f)$ of f in $W(f)$ is contained in the closure of the (infinite) set of sinks and sources of f;*

- **([7, Theorem B, Heuristic principle, and Corollary C])** *There is a residual subset $\mathcal{R}_{[1,2,3]}(M)$ of $\mathcal{W}_{[1,2,3]}(M)$ of diffeomorphisms f such that any homoclinic class $H(P_f, f) \in W_{[1,2,3]}(f)$ is simultaneously contained in the closures of the (infinite) set of sinks of f, of the (infinite) set of sources of f; and of the (infinite) set of minimal Cantor sets of f*[3].

We are now ready to sketch the proof of our results. In order to simplify the exposition, we prove first our main result in the robustly transitive case (Corollary 2). This allows us to present the main arguments in the proofs while avoiding technical difficulties which are typical in the case of homoclinic classes.

Consider a diffeomorphism f having a robustly transitive set $\Lambda_f(U)$. Assume that f has a heterodimensional tangency associated to saddles P_f and Q_f of $\Lambda_f(U)$, say of indices one and two, respectively. After an arbitrarily small perturbation, we can assume that this tangency is parabolic (see Definition 2.1). Next, after perturbing f, we get a diffeomorphism g arbitrarily close to f such that the transverse intersection between $W^s(P_g, g)$ and $W^u(Q_g, g)$ contains a small contractible circle σ_g (the circle σ_g is contained in a small disk of $W^s(P_g, g)$). Clearly, this property is open. The key step is to see that, after a new perturbation of g, we can assume that the circles σ_g are (robustly) contained in the homoclinic class of Q_g: $\sigma_h \subset H(Q_h, h)$, for every h in an open set whose closure contains g. This is done in Proposition 2.2. Finally, the property of containing small contractible circles prevents the homoclinic class of Q_h of having a dominated splitting, see Proposition 2.9. Corollary 2 now follows from Theorem 1.10.

We now explain the steps of the proof of Proposition 2.2. We first obtain (after a perturbation) a saddle-node S in $\Lambda_f(U)$ (see Lemma 2.3). Unfolding this saddle-node, we get two saddles A_g and B_g of indices one and two, respectively, in $\Lambda_g(U)$. Using the robust transitivity of $\Lambda_g(U)$ and the Connecting Lemma in [18], we get a partially hyperbolic heterodimensional cycle associated to A_g and B_g, (see Lemma 2.4 and Definition 2.5). Finally, by unfolding this heterodimensional cycle, we see that *the (one-dimensional) stable manifold of B_g transversely meets any two-disk Δ transverse to $W^s(A_g, g)$,* see Lemma 2.6 and Figure 1. Following [10, Lemma 6.6], we call this property *distinctive intersection property* of the heterodimensional cycle (although the formulations are not exactly the same). The distinctive intersection property is the key argument in the blender-like construction mentioned above.

[3]These minimal Cantor sets are maximal ones (they contain any invariant set intersecting them) and (clearly) aperiodic (without periodic orbits).

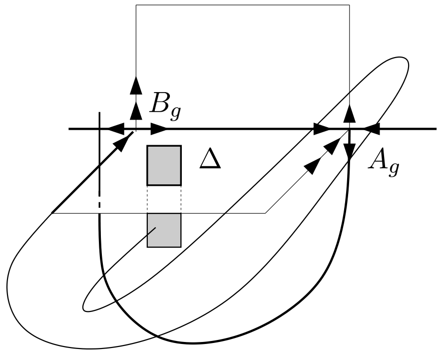

Figure 1: The distinctive intersection property

Using the distinctive property and the fact that B_g is homoclinically related to Q_g and A_g is homoclinically related to P_g, we get that the circles σ_g are (robustly) contained in the homoclinic class of Q_g. Proposition 2.2 now implies that the class $H(Q_g, g)$ does not admit any dominated splitting.

The proof for homoclinic classes follows along the same ideas (see Section 3). The main difference (also the main difficulty) is that we need to check that the saddles A_g and B_g above, obtained after perturbing the dynamics, belong to the homoclinic class of Q_g. We observe that this property follows immediately in the robustly transitive case. To prove this fact for homoclinic classes we use some arguments in [1] and results in [12, 4] about homoclinic classes of C^1-generic diffeomorphisms.

We finish this introduction by comparing our C^1-construction and the Newhouse's construction of C^2-diffeomorphisms having infinitely many sinks. On one hand, the main step of the Newhouse's construction is to see that the unfolding of a homoclinic tangency generates two one-dimensional Cantor sets presenting persistent intersections. These Cantor sets are obtained as the intersections of the stable and unstable laminations of a hyperbolic set Λ and a curve γ. Denote these sets by K^s and K^u, respectively. The curve γ corresponds to the tangencies between extensions of the stable and unstable laminations. Thus each intersection between these Cantor sets corresponds to a tangency between the leaves of the stable and unstable manifolds of a hyperbolic set. See Figure 2. Finally, each tangency leads to the creation of a sink. The construction of these (so-called *thick*) Cantor sets relies on genuinely C^2-properties (bounded distortion).

On the other hand, our construction (as well as the ones in [8, 7]) contains implicitly the persistence of tangencies between the stable and unstable manifolds of points of the homoclinic class. The existence of such tangencies essentially follows from the distinctive intersection property above and the fact that the homoclinic class contains circles. Roughly, the distinctive property means that the one-dimensional stable manifold of any saddle (of the homoclinic class) of index two topologically behaves as a two-dimensional manifold. Similarly for the one dimensional unstable manifolds of saddles of index one. So one can think of the saddles of the homoclinic class as saddles having simultaneously indices one and two. The distinctive property also allows us to avoid considering persistent intersections of Cantor sets, as in the Newhouse's construction. Moreover, in our context, the distinctive property is also the key for proving that the homoclinic classes contain circles. Finally, the distinctive property also guarantees that our construction can be done in the

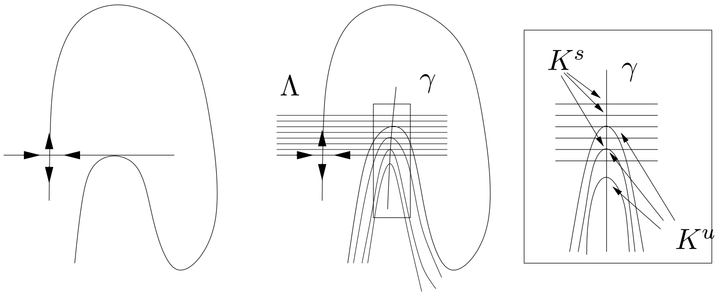

Figure 2: Persistent tangencies

C^1-case.

2 Homoclinic classes without dominated splittings

Recall that $\mathcal{T}(U)$ is the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive and that $\mathcal{J}(U)$ is the (open) subset of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains saddles of indices one and two.

Definition 2.1 (Parabolic heterodimensional tangency). *A diffeomorphism $f \in \mathcal{J}(U)$ has a* parabolic heterodimensional tangency *at the point y_f, associated to the saddles P_f of index 1 and Q_f of index 2 of the set $\Lambda_f(U)$, if there are compact two-disks $K^s_f \subset W^s(P_f, f)$ and $K^u_f \subset W^u(Q_f, f)$ containing y_f in their interiors and local coordinates around $y_f = (0,0,0)$ such that, in these coordinates,*

$$K^s_f = \{z = 0\} \quad \text{and} \quad K^u_f = \{z = x^2 + y^2\}.$$

Thence $W^s(P_f, f)$ and $W^u(Q_f, f)$ have a non-transverse intersection at the heteroclinic point y_f.

For g close to f (g in a small neighbourhood $\mathcal{U}_f \subset \mathcal{J}(U)$ of f), we consider continuations $K^s_g \subset W^s(P_g, g)$ and $K^u_g \subset W^u(Q_g, g)$ of K^s_f and K^u_f. Let $\mathcal{U}^+_f$ be the (open) subset of $\mathcal{U}_f$ of diffeomorphisms g such that $\sigma_g = K^s_g \cap K^u_g$ is a transverse intersection diffeomorphic to a circle. Clearly, f is in the closure of $\mathcal{U}^+_f$.

The main result in this section is the following:

Proposition 2.2. *Let $g \in \mathcal{U}^+_f$. Then there is an open set $\mathcal{V}_g \subset \mathcal{U}^+_f$ whose closure contains g such that, for every $h \in \mathcal{V}_g$, the circle $\sigma_h = K^s_h \cap K^u_h \subset W^s(P_h, h) \cap W^u(Q_h, h)$ is contained in the homoclinic class $H(Q_h, h)$ of Q_h.*

Note that the curve σ_h is contained in a disk of $W^s(P_h, h)$, thus, for some big k, $h^k(\sigma_h)$ is contained in the local stable manifold of P_h.

2.1 Robust transitivity and heterodimensional cycles

In order to prove Proposition 2.2, we begin by collecting some results about the occurrence of saddle-node bifurcations (Lemma 2.3) and (partially hyperbolic) heterodimensional cycles (see Lemma 2.4

and Definition 2.5) inside robustly transitive sets. We also study the dynamics arising from partially hyperbolic heterodimensional cycles (Lemma 2.6).

Consider the subset $\mathcal{S}(U)$ of $\mathcal{J}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains a *periodic saddle-node* (i.e., a periodic point having an unique multiplier in the unitary circle which is equal to one).

Lemma 2.3. ([9, page 203]) *The set $\mathcal{S}(U)$ is dense in $\mathcal{J}(U)$.*

Consider $f \in \mathcal{J}(U)$ and a pair of saddles A_f and B_f of f in $\Lambda_f(U)$. Let $\mathcal{J}_{A,B}(U)$ be an open subset of $\mathcal{J}(U)$ such that for every $g \in \mathcal{J}_{A,B}(U)$ the continuations A_g of A_f and B_g of B_f are defined and belong to $\Lambda_g(U)$.

Next result follows from the Connecting Lemma in [18] and the robust transitivity of $\Lambda_f(U)$:

Lemma 2.4. ([17, Lemma 3.3])

1. *Suppose that the saddles A_f and B_f of f have the same index. Then there is an open and dense subset $\mathcal{H}_{A,B}(U)$ of $\mathcal{J}_{A,B}(U)$ such that for every $g \in \mathcal{H}_{A,B}(U)$ the saddles A_g and B_g are homoclinically related.*

2. *Suppose that the saddles A_f and B_f of f verify* $\operatorname{index}(A_f) < \operatorname{index}(B_f)$. *Then there is a dense subset $\mathcal{C}_{A,B}(U)$ of $\mathcal{J}_{A,B}(U)$ such that every $g \in \mathcal{C}_{A,B}(U)$ has a heterodimensional cycle associated to A_g and B_g as follows:*

   - *$W^s(A_g, g) \cap W^u(B_g, g)$ contains transverse intersections;*
   - *there is $x_g \in W^u(A_g, g) \cap W^s(B_g, g)$ such that $x_g \in \Lambda_g(U)$ and*
   $$T_{x_g}W^u(A_g, g) \oplus T_{x_g}W^s(B_g, g) = T_{x_g}W^u(A_g, g) + T_{x_g}W^s(B_g, g),$$
   that is, $W^u(A_g, g)$ and $W^s(B_g, g)$ meet quasi-transversely at the point y_g.

Definition 2.5 (Partially hyperbolic heterodimensional cycle). *A heterodimensional cycle associated to saddles A_f and B_f, of indices one and two and the same period π, is* partially hyperbolic *if there are a π-periodic connected component Γ_f of the transverse intersection between $W^s(A_f, f)$ and $W^u(B_f, f)$ and a heteroclinic point $x_f \in W^u(A_f, f) \cap W^s(B_f, f)$ such that the set Σ_f obtained as the closure of the orbits of the saddles A_f and B_f and the heteroclinic intersections Γ_f and x_f has a dominated splitting $E^s \oplus E^c \oplus E^u$ having one-dimensional bundles and such that the central bundle E^c is tangent to Γ_f.*

We say that the saddles A_f and B_f and the heteroclinic intersections Γ_f and x_f are the elements of the cycle.

Given a partially hyperbolic heterodimensional cycle (with elements A_f, B_f, Γ_f and x_f), let $\lambda_c(A_f)$ and $\lambda_c(B_f)$ be the eigenvalues of $Df^\pi(A_f)$ and $Df^\pi(B_f)$ corresponding to the central direction E^c. In the partially hyperbolic heterodimensional cycles we consider, the curve Γ_f has extremes A_f and B_f. We call this curve *connection of the cycle* and denote by ℓ_f its length. Finally, we also assume that the map $\log |Df^\pi_{|\Gamma_f}|$ is Lipschitz with Lipschitz constant of L_f (we will explain this point in Remark 2.7). Denote by

$$\Delta_{[B_f,A_f]} = (\mathrm{e}^{-\ell_f L_f}\, \lambda_c(B_f)^k)\, \frac{1-\lambda_c(A_f)}{1-\lambda_c(B_f)^{-1}}, \quad \text{where} \quad k = \max\{i \colon \lambda_c(B_f)^{-i} > (1-\lambda_c(A_f))\}.$$

We say that the cycle is $[B_f, A_f]$-*expanding* if $\Delta_{[B_f,A_f]} > 1$.

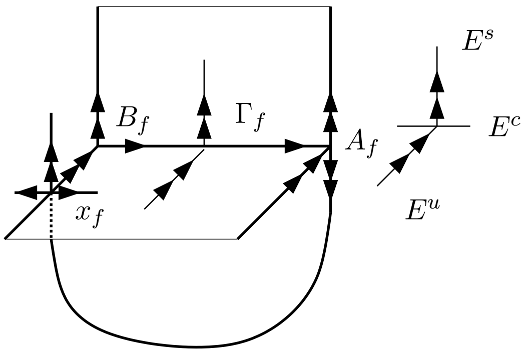

Figure 3: A partially hyperbolic heterodimensional cycle

Lemma 2.6 (Distinctive intersection property, [13]). *Let f be a diffeomorphism having a $[B_f, A_f]$-expanding partially hyperbolic heterodimensional cycle associated to saddles A_f and B_f in a robustly transitive set. Then there is a C^1-open set $\mathcal{U}$ such that $f \in \overline{\mathcal{U}}$ with the following property: for every $g \in \mathcal{U}$ and every two-disk Δ^{cu} intersecting transversely $W^s(A_g, g)$, one has that $W^s(B_g, g)$ transversely meets Δ^{cu}.*

Remark 2.7. Lemma 2.6 follows using the arguments in the proof of [13, Theorem 1.1, pages 302-3]. Let us briefly explain how Lemma 2.6 follows in our setting.

First, [13] considers cycles corresponding to first bifurcations. This assumption, in particular, implies that the cycle is *untwisted,* meaning the following. The heteroclinic point x_f simultaneously belongs to the manifolds with boundary

$$W^u([A_f, B_f]) = W^u(A_f, f) \cup W^u(B_f, f), \qquad W^s([A_f, B_f]) = W^s(A_f, f) \cup W^s(B_f, f).$$

The cycle is *untwisted* if there are neighbourhoods $D^u(x_f)$ and $D^s(x_f)$ of x_f in $W^u([A_f, B_f])$ and $W^s([A_f, B_f])$ such that $D^u(x_f) \cap D^s(x_f) = \{x_f\}$.

Observe that, a priori, the cycle associated to A_f and B_f given by Lemma 2.4 may be twisted. Using the transitivity and the fact that the homoclinic class of A_f is non-trivial, after a perturbation (again using the Connecting Lemma) one gets g arbitrarily close to f with a untwisted (partially hyperbolic) heterodimensional cycle with elements A_g, B_g, Γ_g and y_g, where A_g, B_g and Γ_g are the continuations of the elements A_f, B_f and Γ_f of the initial cycle. The heteroclinic point y_g may involve a longer part of the unstable manifold of P_g. The configuration leading to the untwisted cycle is depicted in Figure 4.

The previous construction implies that if the initial cycle is $[B_f, A_f]$-expanding, one can take the new cycle being $[B_g, A_g]$-expanding (note that this definition only involves the restriction to the connection and the eigenvalues, which continuously depend on the diffeomorphism).

Second, the proof in [13] only involves the restriction of f to the connection Γ_f and the central eigenvalues $\lambda_c(A_f)$ and $\lambda_c(B_f)$. Thus one can repeat the arguments in [13] considering only the dynamics in a neighbourhood of cycle (i.e., in an open set containing the orbits of the elements of the cycle).

Finally, [13] considers C^∞-diffeomorphisms, but using the blender construction in [5] one can carry the same construction to the C^1-case.

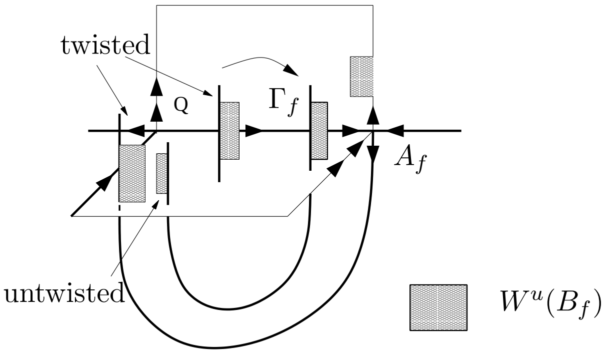

Figure 4: Twisted and untwisted intersections

For an expository explanation of the proof of Lemma 2.6 in the C^1-case, we refer to [10, Chapter 6.1]. We observe that Lemma 2.6 also holds for partially hyperbolic cycles that do not verify the $[B_f, A_f]$-expanding assumption, see [15]. We quote here the result in [13] because we will use it explicitly it in Section 4.

2.2 Proof of Proposition 2.2

We are now ready to prove the proposition. The proof consists of the following steps. Let $g \in \mathcal{U}_f^+$.

Step 1: By Lemma 2.3, there is h_1 arbitrarily close to g such that $\Lambda_{h_1}(U)$ contains a periodic saddle-node S_{h_1} of period $\pi = \pi(S_{h_1})$. Let λ_s, λ_c, and λ_u be the eigenvalues of $Dh_1^\pi(S_{h_1})$, where $|\lambda_s| \le |\lambda_c| = 1 \le |\lambda_u|$. Note that $|\lambda_s| < 1$: otherwise, we can perturb the saddle-node to get a diffeomorphism h_2 (arbitrarily close to h_1) with a source inside U, contradicting the robust transitivity of $\Lambda_{h_1}(U)$. Similarly, $|\lambda_u| > 1$. Therefore, $\lambda_c = 1$ and the multipliers of S_{h_1} are all real.

Step 2: By unfolding the saddle-node S_{h_1}, we get h_2 arbitrarily close to h_1 with two saddles A_{h_2} and B_{h_2}, of period π and indices one and two, contained in $\Lambda_{h_2}(U)$. Moreover, there is a π-periodic segment Υ_{h_2} bounded by A_{h_2} and B_{h_2} and contained in $\Lambda_{h_2}(U)$ and in the transverse intersection between $W^s(A_{h_2})$ and $W^u(B_{h_2})$. Furthermore, the restriction of h_2 to the curve Υ_{h_2} is partially hyperbolic with three bundles (corresponding to the stable, central, and unstable directions of the saddle-node). See Figure 5.

Step 3: Using the second part of Lemma 2.4, we get h_3 arbitrarily close to h_2 with a heterodimensional cycle associated to A_{h_3} and B_{h_3}. This cycle is partially hyperbolic: it is enough to take $\Gamma_{h_3} = \Upsilon_{h_3}$ and the heteroclinic point x_{h_3} given by Lemma 2.4. Moreover, the dynamics of $h_3^{\pi(\Upsilon_{h_3})}$ in Υ_{h_3} is close to the identity (the curve is obtained by perturbing a saddle-node). Thus we can take the heterodimensional cycle being $[B_{h_3}, A_{h_3}]$-expanding. Let us briefly explain this point. First, we can take the diffeomorphism being C^2 around the cycle. Moreover, $h_3^{\pi(\Upsilon_{h_3})}$ is close to the identity and the length ℓ_{h_3} of Υ_{h_3} can be taken arbitrarily small. Thus we can assume that

$$e^{-\ell_{h_3} L_{h_3}} > 1/2.$$

On the other hand, we can take $\lambda_c(A_{h_3})$ and $\lambda_c(B_{h_3})$ close to one in such a way

$$2\,(1 - \lambda_c(A_{h_3})) > \lambda_c(B_{h_3})^{-k} > (1 - \lambda_c(A_{h_3})),$$

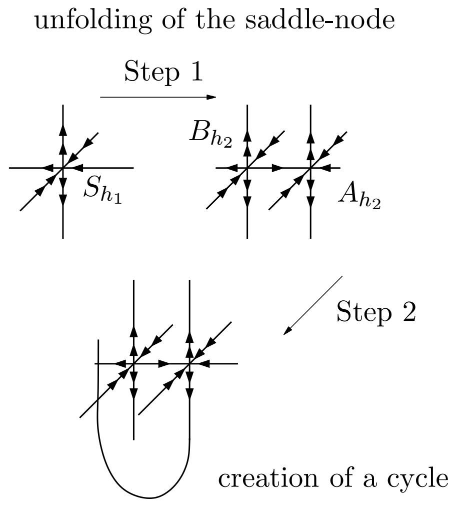

Figure 5: Creation of a partially hyperbolic heterodimensional cycle

for some $k \geq 1$. Hence,

$$\lambda_c(B_{h_3})^k > \frac{1}{2\,(1-\lambda_c(A_{h_3})}.$$

Therefore, putting together all the previous inequalities,

$$\Delta_{[B_{h_3},A_{h_3}]} = (\mathrm{e}^{-\ell_{h_3}\,L_{h_3}}\,\lambda_c(B_{h_3})^k)\,\frac{1-\lambda_c(A_{h_3})}{1-\lambda_c(B_{h_3})^{-1}} > \frac{1}{4\,(1-\lambda_c(B_{h_3})^{-1})}.$$

Hence if $\lambda_c(B_{h_3})$ is close enough to 1, $(1-\lambda_c(B_{h_3})^{-1}) < 1/8$ and thus $\Delta_{[B_{h_3},A_{h_3}]} > 2$.

Step 4: Applying Lemma 2.6 to h_3, we get an open set $\mathcal{U}$, contained in $\mathcal{U}_f^+$ and arbitrarily close to the initial g, such that, for every $h \in \mathcal{U}$, $W^s(B_h, h)$ transversely meets any two disk transverse to $W^s(A_h, h)$.

Last step: Consider now the sets $\mathcal{H}_{A,P}(U)$ and $\mathcal{H}_{B,Q}(U)$ in Lemma 2.4. Note that the saddles A_h and P_h have same index. The same holds for the saddles B_h and Q_h. Thus the first part of Lemma 2.4 implies that the set

$$\mathcal{U}_{[A,P],[B,Q]}(U) = \mathcal{U} \cap \mathcal{H}_{A,P}(U) \cap \mathcal{H}_{B,Q}(U).$$

is open and dense in $\mathcal{U}$.

Claim 2.8. *For every $h \in \mathcal{U}_{[A,P],[B,Q]}(U)$ it holds $\sigma_h \subset H(Q_h, h)$.*

This claim immediately implies Proposition 2.2: just note that the open set $\mathcal{U}$ can be taken arbitrarily close to the initial $g \in \mathcal{U}_f^+$ and that $\mathcal{U}_{[A,P],[B,Q]}(U)$ is open and dense in $\mathcal{U}$.

Proof of the claim: Consider $h \in \mathcal{U}_{[A,P],[B,Q]}(U)$ and any point $x \in \sigma_h$. We will see that *$W^s(Q_h, h)$ transversely meets any two-disk $W^u(x)$ contained in $W^u(Q_h, h)$ and containing x in its interior.* Since $W^u(x) \subset W^u(Q_h, h)$, we have that the disk $W^u(x)$ contains a transverse homoclinic point of Q_h. Thus $W^u(x) \cap H(Q_h, h) \neq \emptyset$. As $W^u(x)$ can be taken arbitrarily small and $H(Q_h, h)$ is closed, one has that $x \in H(Q_h, h)$, ending the proof of the claim.

To prove the assertion recall that, by construction, the circle σ_h is contained in the transverse intersection between $W^s(P_h, h)$ and $W^u(Q_h, h)$. Thus $W^s(P_h, h)$ transversely meets $W^u(x)$. As A_h is homoclinically related to P_h, $W^s(A_h, h)$ transversely meets $W^u(x)$. Now, by Step 4, $W^s(B_h, h)$ transversely meets $W^u(x)$. Finally, as Q_h and B_h are homoclinically related, $W^s(Q_h, h)$ transversely meets $W^u(x)$, which implies our assertion. □

The proof of Proposition 2.2 is now complete.

2.3 Non-dominated homoclinic classes

For each $g \in \mathcal{U}_f^+$ consider the set $\mathcal{V}_g$ given by Proposition 2.2 (i.e., g is in the closure of $\mathcal{V}_g$ and, for every $h \in \mathcal{V}_g$, the homoclinic class $H(Q_h, h)$ contains a small contractible circle σ_h). Let

$$\mathcal{N} = \cup_{g \in \mathcal{U}_f^+} \mathcal{V}_g.$$

Note that, by construction, the set $\mathcal{N}$ is open and dense in $\mathcal{U}_f^+$.

Proposition 2.9. *For every $g \in \mathcal{N}$ the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting.*

Proof: For the properties about dominated splittings we use here see, for instance, [10, Appendix B.1]. We need the following lemma (the ideas in its proof are borrowed from [7, Lemma 3.8]).

Lemma 2.10. *Let f be a diffeomorphism defined on a three dimensional manifold. Suppose that there are saddles P and Q of f, of indices 1 and 2, such that $P \in H(Q, f)$ and $H(Q, f)$ has a dominated splitting $E \oplus F$.*

- *If $\dim(E) = 1$ then P has real multipliers λ_{ss}, λ_c, and λ_u, with $|\lambda_{ss}| < |\lambda_c| < 1 < |\lambda_u|$.*
- *If $\dim(F) = 1$ then Q has real multipliers β_s, β_c and β_{uu}, with $|\beta_s| < 1 < |\beta_c| < |\beta_{uu}|$.*

Proof: We prove the lemma when $\dim(E) = 1$. The case $\dim(F) = 1$ follows similarly taking f^{-1}.

First, as $\dim(E^s(P)) = 2$, the domination of $E \oplus F$ implies that $E(P) \subset E^s(P)$. Let $H(P) = F(P) \cap E^s(P)$. This bundle is one-dimensional and Df-invariant. Thus $E^s(P) = E(P) \oplus H(P)$, where the splitting is invariant and dominated.

The proof now is by contradiction. Assume that P has a pair of non-real multipliers, say λ and $\bar{\lambda}$. Since $\dim(E^u(P)) = 1$, these multipliers are contracting. In such a case, $E^s(P)$ does not admit any Df-invariant splitting, contradicting the existence of $E^s(P) = E(P) \oplus H(P)$.

We have that the three multipliers of P are real. Finally, if two multipliers of P have the same modulus (again such multipliers are necessarily contracting) then $E^s(P)$ does not admit any dominated splitting, contradicting the existence of $E^s(P) = E(P) \oplus H(P)$ above. This completes the proof of the lemma. □

The proof of the proposition now follows by contradiction. Consider $g \in \mathcal{N}$ and suppose that $H(Q_g, g)$ has a dominated splitting $T_{H(Q_g,g)}M = E \oplus F$. Suppose, for example, that $\dim(E) = 1$ (the case $\dim(F) = 1$ follows analogously considering g^{-1}). Lemma 2.10 implies that the eigenvalues of $Dg^{\pi(P_g)}(P_g)$ are λ_{ss}, λ_c, and λ_u, where $|\lambda_{ss}| < |\lambda_c| < 1 < |\lambda_u|$. The domination now implies that $E(P) = E^{ss}(P)$.

For each $x \in H(Q_g, g)$ consider the fibers $E(x)$ and $F(x)$ and the angle $\alpha(x)$ between $E(x)$ and $F(x)$. The domination of $E \oplus F$ implies that these fibers depend continuously on x and that there is of some $\alpha_0 > 0$ such that $\alpha(x) > \alpha_0$ for all $x \in H(Q_g, g)$.

Consider now the small contractible curve $\sigma_g \subset W^s(P_g, g) \cap W^u(Q_g, g)$ with $\sigma_g \subset H(Q_g, g)$ given by Proposition 2.2. Then there is a sequence $(i_k)_{k \in \mathbb{Z}}$ such that $g^{i_k}(\sigma_g) \to P_g$ as $i_k \to \infty$ and $g^{i_k}(\sigma_g) \to Q_g$ as $i_k \to -\infty$, where the limits mean convergence in the Hausdorff metric. Since $H(Q_g, g)$ is closed and g-invariant, this implies that $P_g \in H(Q_g, g)$.

Let $\Sigma_g = \cup_{i_k} g^{i_k}(\sigma)$. For each $x \in \Sigma_g$ consider $H(x) = T_x \Sigma_g$. By definition, the family $\mathcal{H} = \{H(x)\}_{x \in \Sigma_g}$ is Dg-invariant. The domination of $E \oplus F$ and the invariance of the family $\mathcal{H}$ imply that either $H(x) \subset E(x)$ for all $x \in \Sigma_g$, or $H(x) \subset F(x)$ for all $x \in \Sigma_g$.

Since the curves $g^{i_k}(\sigma)$ (big i_k) are arbitrarily small contractible circles arbitrarily close to P_g (contained in a small local stable manifold of P), for each big i_k there are points $z, w \in g^{i_k}(\sigma_g)$ such that: (a) $H(z)$ is *parallel* to $E^{ss}(P) = E(P)$ (we write $H(z) \simeq E(P)$) and (b) $H(w)$ is *parallel* to $E^c(P)$, where $E^c(P)$ is the eigenspace associated to the contracting multiplier λ_c of P (we write $H(z) \simeq E^c(P)$). Note that $E^c(P) \subset F(P)$. These possibilities are depicted in Figure 6

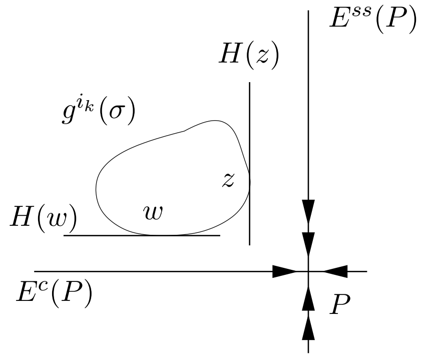

Figure 6: The contradiction

We are not ready to get a contradiction. First, if $H(x) \subset E(x)$ for all $x \in \Sigma_g$ (thus $H(x) = E(x)$) then (b) above gives the contradiction: there is w arbitrarily close to P such that $E(P) = E^{ss}(P)$ and $E(w) = H(w) \simeq E^c(P)$, hence $E(P)$ and $E(w)$ are not close.

Finally, if $H(x) \subset F(x)$ then (a) above gives the contradiction: there is z arbitrarily close to P such that the angle between $F(z)$ and $E(z)$ is arbitrarily close to zero, contradicting the fact that $\alpha(z) > \alpha_0 > 0$ for all $z \in H(P_g, g)$. This completes the proof of the proposition. □

2.4 Consequences: proof of Corollary 2

We are now ready to finish the proof of Corollary 2. Consider $f \in \mathcal{J}(U)$ having a heterodimensional tangency associated to saddles P_f (of index one) and Q_f (of index two) of a robustly transitive set $\Lambda_f(U)$. Proposition 2.9 gives an open set $\mathcal{N}$ whose closure contains f such that, for every $g \in \mathcal{N}$, the homoclinic class of Q_g does not admit any dominated splitting. By the first part of

Theorem 1.10, there is a residual subset $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{N}$ of diffeomorphisms h such that the homoclinic class of $H(Q_h, h)$ is contained in the closure of the (infinite) set of sinks and sources of h.

Moreover, if the homoclinic class $H(Q_f, f)$ contains saddles verifying (W1)–(W3) (i.e., saddles R_f and T_f having different indices and such that $\mathrm{Jac}[f^{\pi(R_f)}(R_f)] > 1 > \mathrm{Jac}[f^{\pi(T_f)}(T_f)]$) the second part of Theorem 1.10 guarantees that the open set $\mathcal{N}$ and the residual subset $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{N}$ above can be chosen such that for every $h \in \mathcal{R}_{\mathcal{N}}$ the homoclinic class of $H(Q_h, h)$ is simultaneously contained in the closure of the (infinite) set of sinks of h, in the closure of the (infinite) set of sources of h, and in the closure of the (infinite) set of minimal Cantor sets of h.

3 Proof of Theorem 1

Theorem 1 essentially follows from arguments in the robustly transitive case. The main difference concerns Steps 2 and 3 in Section 2.2. More precisely, in the robustly transitive case, after a perturbation, one has a *the robustly transitive set $\Lambda_f(U)$ containing a saddle-node S_f*. This saddle-node is obtained by collapsing two saddles A_g and B_g of different indices of the robustly transitive set $\Lambda_g(U)$. Thus, by definition of robustly transitive set, the resulting saddle-node S_f necessarily belongs to $\Lambda_f(U)$. Using the saddles A_g and B_g, we obtain a partially hyperbolic heterodimensional cycle. The unfolding of such a cycle leads to diffeomorphisms h such that the homoclinic class of the saddle B_h contains the small contractible circle $\sigma_h \subset W^s(P_h, h) \cap W^u(Q_h, h)$. This fact implies that the homoclinic class of B_h (which coincides with the one of Q_h) does not admit any dominated splitting (Proposition 2.9).

In the case of homoclinic classes, one can also obtain a saddle-node S_f as well as the two collapsing saddles A_g and B_g. The main difficulty (and the main difference with the robustly transitive case) is that, a priori, the saddles A_g and B_g may have nothing to do with the homoclinic classes of initial saddles P_g and Q_g being persistently linked. In fact, we will see that one can take the saddles A_g and B_g homoclinically related to P_g and Q_g. This step involves the notion of *chain recurrence class* and the fact that, *C^1-generically, any chain recurrence class containing a periodic saddle P is the homoclinic class of P* (see [4, Remarque 1.10]). After proving this step, the proof follows exactly as in the robustly transitive case. Let us sketch this construction (whose main ingredients can be found in [1]) more precisely.

Given a saddle R_f of a diffeomorphism f, let $\mathrm{Per}_{\mathbb{R}}(H(R_f, f))$ be the subset of saddles T_f of $H(R_f, f)$ such that

- T_f and R_f are homoclinically related (thus the saddles T_f and R_f have the same index);
- the eigenvalues of $Df^{\pi(T_f)}(T_f)$ are real, different in modulus, and of multiplicity one, recall that $\pi(T_f)$ is the period of T_f.

Lemma 3.1. ([1, Proposition 2.6] following [8, Lemma 4.16]) *There is a residual subset $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\mathrm{Per}_{\mathbb{R}}(H(R_f, f))$ is dense in $H(R_f, f)$, for every non-trivial homoclinic class $H(R_f, f)$ of f.*

Let $\mathcal{L}_{P,Q}(M)$, P_f, Q_f, and f_0 as in Theorem 1 ($\mathcal{L}_{P,Q}(M)$ is open and for all $f \in \mathcal{L}_{P,Q}(M)$ the saddles P_f and Q_f are persistently linked, and f_0 has a heterodimensional tangency associated to P_{f_0} and Q_{f_0}). Let $\mathcal{G}$ be a residual subset of $\mathcal{L}_{P,Q}(M)$ such that $H(P_f, f) = H(Q_f, f)$ for all $f \in \mathcal{G}$.

Assume that the indices of P_f and Q_f are one and two, respectively. By Lemma 3.1, there is an open neighbourhood $\mathcal{U}_0$ of f_0 and a residual subset $\mathcal{R}_{\mathcal{U}_0}$ of $\mathcal{U}_0$ such that, for all $h \in \mathcal{R}_{\mathcal{U}_0}$, there are saddles $P'_h \in \mathrm{Per}_{\mathbb{R}}(H(P_h, h))$ and $Q'_h \in \mathrm{Per}_{\mathbb{R}}(H(Q_h, h))$. Let us suppose that the saddles P'_h and Q'_h are defined in the whole $\mathcal{U}_0$. This implies that

$$H(P'_h, h) = H(P_h, h) = H(Q_h, h) = H(Q'_h, h), \quad \text{for all } h \in \mathcal{G}_{\mathcal{U}_0} = \mathcal{R}_{\mathcal{U}_0} \cap \mathcal{G}. \tag{1}$$

Using the Connecting Lemma in [18], the transitivity of the homoclinic classes, and the fact that the unstable manifold of Q'_h and the stable manifold of P'_h are both two-dimensional, one has (exactly as in Lemma 2.4) the following:

Lemma 3.2. *There are an open and dense subset $\mathcal{T}_0$ of $\mathcal{U}_0$ and a dense subset $\mathcal{D}_0$ of $\mathcal{U}_0$ such that*

1. *$W^s(P'_h, h)$ and $W^u(Q'_h, h)$ have some transverse intersection for all $h \in \mathcal{T}_0$;*
2. *$W^u(P'_h, h) \cap W^s(Q'_h, h) \neq \emptyset$ for all $h \in \mathcal{D}_0$.*

Therefore, there is a dense subset of $\mathcal{U}_0$ of diffeomorphisms h having a heterodimensional cycle associated to P'_h and Q'_h.

Exactly as in the robustly transitive case, unfolding this heterodimensional cycle (associated to saddles with real multipliers) one obtains a saddle-node S_g. This saddle-node is originated by the collapse of two saddles A_f and B_f of indices one and two (exactly as in Step 2). Arguing exactly as in the proof of [1, Theorem 3.2][4], one gets that there is an open set $\mathcal{U}_1$, whose closure contains g, such that for every $f \in \mathcal{U}_1$ the saddles A_f and B_f are defined and verify

$$W^u(Q'_f, f) \cap W^s(A_f, f) \neq \emptyset \quad \text{and} \quad W^u(A_f, f) \cap W^s(P'_f, f) \neq \emptyset, \tag{2}$$

and

$$W^s(P'_f, f) \cap W^u(B_f, f) \neq \emptyset \quad \text{and} \quad W^s(B_f, f) \cap W^u(Q'_f, f) \neq \emptyset. \tag{3}$$

We now need the notion of chain recurrence class. A point x is *chain recurrent* if for any $\varepsilon > 0$, there are closed ε-pseudo orbits containing x. The set of chain recurrent points of f, denoted by $\mathcal{R}(f)$, splits into f-invariant compact sets called *chain recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in the same class if, for any $\varepsilon > 0$, there are ε-pseudo orbits going from x to y and from y to x. [4, Remarque 1.10] states that, for C^1-generic f, any chain recurrence class containing a periodic point P is the homoclinic class of P.

Consider now the residual subset $\mathcal{G}_{\mathcal{U}_1} = \mathcal{G}_{\mathcal{U}_0} \cap \mathcal{U}_1$ of $\mathcal{U}_1$.

Lemma 3.3. *The saddles Q'_f, P'_f, A_f, and B_f are in the same chain recurrence class for every $f \in \mathcal{G}_{\mathcal{U}_1}$.*

Proof: We first prove that the saddles Q'_f, A_f, and P'_f are in the same chain recurrence class. This assertion will follow from equation (2) and the fact that $H(P'_f, f) = H(Q'_f, f)$ for every $f \in \mathcal{G}_{\mathcal{U}_1}$ (see (1)). Let us prove this claim.

[4] In fact the situation in [1] is much more subtle: the saddle P'_f has index k and the saddle Q'_f has index $k+m$, $m \geq 2$, and one gets a saddle A_f of intermediate index ℓ, $m < \ell < m+k$ such that $W^u(Q'_f, f) \cap W^s(A_f, f) \neq \emptyset$ and $W^u(A_f, f) \cap W^s(P'_f, f) \neq \emptyset$.

Taking any point $x \in W^u(Q'_f, f) \cap W^s(A_f, f)$, we get a finite segment of orbit $f^{-m_1}(x)$, $f^{-m_1+1}(x), \ldots, f^{n_1}(x)$, $m_1, n_1 \in \mathbb{N}$, such that $f^{-m_1}(x)$ is arbitrarily close to Q'_f and $f^{n_1}(x)$ is arbitrarily close to A_f. Analogously, taking $y \in W^u(A_f, f) \cap W^s(P'_f, f)$, we get a segment of orbit $f^{-m_2}(y)$, $f^{-m_2+1}(y), \ldots, f^{n_2}(y)$, $m_2, n_2 \in \mathbb{N}$, such that $f^{-m_2}(y)$ is arbitrarily close to A_f and $f^{n_2}(y)$ is arbitrarily close to P'_f. Finally, as $H(P'_f, f) = H(Q'_f, f)$ and this set is transitive, there is a segment of orbit $z, f(z), \ldots, f^{n_3}(z)$, $n_3 \in \mathbb{N}$, starting arbitrarily close to P'_f and ending arbitrarily close to Q'_f. The announced pseudo-orbit containing the points Q'_f, A_f, and P'_f is obtained concatenating the three segments of orbits above. This shows that the saddles Q'_f, A_f, and P'_f are in the same chain recurrence class.

Similarly, from equation (3) and $H(P'_f, f) = H(Q'_f, f)$, one has that the saddles Q'_f, B_f, and P'_f are in the same chain recurrence class. Thus, for all $f \in \mathcal{G}_{\mathcal{U}_1}$, the saddles Q'_f, A_f, B_f, and P'_f are in the same chain recurrence class. This completes the proof of the lemma. $\square$

Since (C^1-generically) any chain recurrence class containing a saddle is the homoclinic class of such a saddle, [4, Remarque 1.10], we get a residual subset $\mathcal{R}_{\mathcal{U}_1}$ of $\mathcal{U}_1$ of diffeomorphisms f such that

$$H(P'_f, f) = H(A_f, f) = H(B_f, f) = H(Q'_f, f).$$

Applying now Lemma 3.2 to the saddles A_f and B_f (which are arbitrarily close to a saddle-node) we get a heterodimensional cycle associated to A_f and B_f. Exactly as in Step 3, this cycle can be taken partially hyperbolic and $[B_f, A_f]$-expanding. The proof of the result now follows exactly as in the robustly transitive case noting that

$$H(P_f, f) = H(P'_f, f) = H(A_f, f) = H(B_f, f) = H(Q'_f, f) = H(Q_f, f).$$

This completes the sketch of the proof of Theorem 1.

4 Examples of diffeomorphisms with heterodimensional tangencies

In this section, we construct diffeomorphisms with heterodimensional tangencies obtained as *bifurcations* of partially hyperbolic sets. These examples illustrate a natural transition from partially hyperbolic to non-dominated dynamics. The examples are motivated by the construction of robustly transitive sets (which are non-hyperbolic homoclinic classes) arising from Morse-Smale systems via bifurcations through heterodimensional cycles in [13]. Our construction is similar to the one in [24, page 89-90] for diffeomorphisms with homoclinic tangencies. Let us recall the main steps of the construction in [24], .

Consider a diffeomorphism f having a two-dimensional Plykin attractor Σ in a disk D of $\mathbb{R}^2$. Take a fixed saddle p of f in Σ and note that Σ is the homoclinic class of p. Multiply now f by a linear expansion $g(x) = \lambda\, x$ on the line $\mathbb{R}$. We take λ big enough in such a way $D \times \{0\}$ is normally hyperbolic (any expansion of the derivative of g in the disk D is strictly upper bounded by λ). Note that the point $S = (p, 0)$ is a saddle of index two of the product diffeomorphism $\phi = (f, g)$ and that $\Lambda = \Sigma \times \{0\} \subset D \times \{0\}$ is a hyperbolic set of ϕ which is the homoclinic class of the saddle $S = (p, 0)$. Finally, the unstable bundle of Λ is two-dimensional.

We can now select points z_1 in the strong unstable manifold of S (tangent to $\{0, 0\} \times \mathbb{R}$) and z_2 in the stable manifold of S and perform a semi-local perturbation of ϕ preserving the hyperbolic set Λ as follows: there is an arc $(\phi_t)_{t\in[0,1]}$, with $\phi_0 = \phi$, such that for every $t \in [0,1)$ the set Λ is a hyperbolic set of ϕ_t equal to the homoclinic class of S, and, for $t = 1$, ϕ_1 has a homoclinic tangency associated to S at the point z_2. The perturbation is depicted in Figure 7. A key condition here (that we will use later) is that Λ does not contain points in $z > 0$.

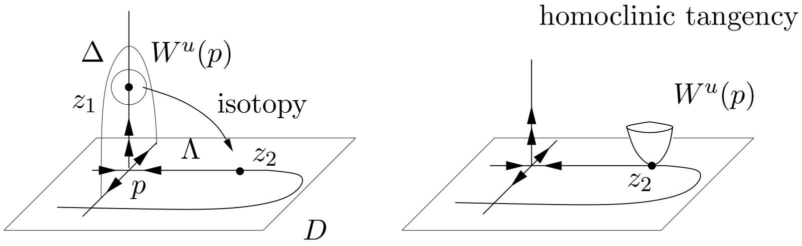

Figure 7: Creation of a homoclinic tangency

We now sketch our construction and explain how the arguments above can be used in our setting. For the sake of simplicity we use local coordinates in $\mathbb{R}^3$, but our construction clearly work in the broader setting of diffeomorphisms bifurcating from Morse-Smale diffeomorphisms via expanding partially hyperbolic heterodimensional cycles.

Next construction is analogous to the one in [16, Section 2]. We begin with a diffeomorphism f having a heterodimensional cycle related to saddles $Q = (0,0,0)$ and $P = (0,4,0)$, of indices 2 and 1, such that dynamics around the cycle is the following. In the cube $R = [-1,1] \times [-1,5] \times [-1,1]$,

$$f(x,y,z) = (\lambda_s\, x, F(y), \lambda_u\, z), \qquad 0 < \lambda_s < 1 < \lambda_u,$$

where F is a strictly increasing function such that $F'(0) = \beta > 1$ and $F'(4) = \lambda < 1$, $0 < \lambda < 1$, and $\lambda_s < F'(x) < \lambda_u$ for every $x \in [0,4]$. By construction, the restriction of f to the cube R has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with three non-trivial bundles given by the coordinate directions. Moreover,

$$\begin{array}{lcl} \{0\} \times [-1,4) \times [-1,1] \subset W^u(Q) & \text{and} & [-1,1] \times \{(0,0)\} \subset W^s(Q), \\ \{(0,4)\} \times [-1,1] \subset W^u(P) & \text{and} & [-1,1] \times (0,5] \times \{0\} \subset W^s(P). \end{array}$$

Thus $W^s(P)$ and $W^u(Q)$ meet transversely along the curve $\{0\} \times (0,4) \times \{0\}$.

Take the points $A = (0,4,-\frac{1}{2}) \in W^u(P)$ and $B = (-\frac{1}{2},0,0) \in W^s(Q)$. We assume that there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(A) = B$ and $f^j(A) \notin R$ for all $0 < j < k_0$. Thus $W^u(P) \cap W^s(Q) \neq \emptyset$ and f has a heterodimensional cycle associated to P and Q.

Take a small compact neighbourhood U_A of A such that $U_A, \dots, f^{k_0}(U_A)$ are pairwise disjoint, $U_A, f^{k_0}(U_A) \subset R$, and $f^j(U_A) \cap R = \emptyset$, for all $0 < j < k_0$. We assume that transition f^{k_0} from U_A to $f^{k_0}(U_A)$ is of the form

$$f^{k_0}(x,y,z) = (x - \frac{1}{2}, y - 4, z + \frac{1}{2}), \quad \text{for every } (x,y,z) \in U_A.$$

Thus $W^u(P)$ and $W^s(Q)$ meet *quasi-transversely* throughout the orbit of the heteroclinic point B:

$$T_B W^u(P) + T_B W^s(Q) = T_B W^u(P) \oplus T_B W^s(Q) = \mathbb{XZ}.$$

We can now extend f to a diffeomorphism defined in a three dimensional closed manifold M in the boundary of the Morse-Smale systems. Moreover, we can choose the restriction F of f to $\{0\} \times [0,4] \times \{0\}$ in such a way f has a $[Q,P]$-expanding partially hyperbolic heterodimensional cycle whose elements are P, Q, $\Gamma_f = \{0\} \times [0,4] \times \{0\}$, and $x_f = B$.

We select a small neighbourhood of the cycle

$$W = V \cup U_A \cup f(U_A) \cup f^2(U_A) \cup \cdots \cup f^{k_0-1}(U_A) \cup f^{k_0}(U_A),$$

where V is a small neighbourhood of the curve $\{0\} \times [0,4] \times \{0\}$ contained in the cube R such that $f^{-1}(U_A) \subset V$ and $f^{k_0+1}(U_A) \subset V$. This configuration is depicted in Figure 8

Given g close to f, let Λ_g be the maximal invariant set of g in W. By construction, the set Λ_g is partially hyperbolic (with a splitting with three one-dimensional bundles). Moreover, if we take g such that A is a transverse homoclinic point of Q (i.e., after unfolding the cycle), this maximal invariant set Λ_g is robustly transitive and equal to the non-hyperbolic homoclinic class of Q (note that by definition $P \in \Lambda_g$). Thus the set $\Lambda_g = H(Q,g)$ verifies the hypotheses of Theorem 1. For details see the proof of [13, Theorem 3] (see also [16, Theorem 2.1]).

We now take h close to g such that $\Lambda_h = H(Q,h)$ is robustly transitive. We can now use the arguments above to get a heterodimensional tangency associated to P and Q. A key point here is that the set Λ_h does not intersect the regions $z > 0$ and $x < 0$. We now exactly repeat the procedure in [24, page 89-90] taking, for instance, $z_1 = (0,0,1/2) \in W^u(Q,h)$ and $z_2 = (-1/2,1/2,0) \in W^s(P,h)$. As above, there is an arc $(h_t)_{t\in[0,1]}$, with $h_0 = h$, such that for every $t \in [0,1)$ the set $\Lambda_{h_t} = H(Q,h_t)$ is robustly transitive (we do not modify the dynamics around the cycle) and for $t = 1$, h_1 has a heterodimensional tangency associated to Q and P at the point z_2. See Figure 8. This completes the sketch of our construction.

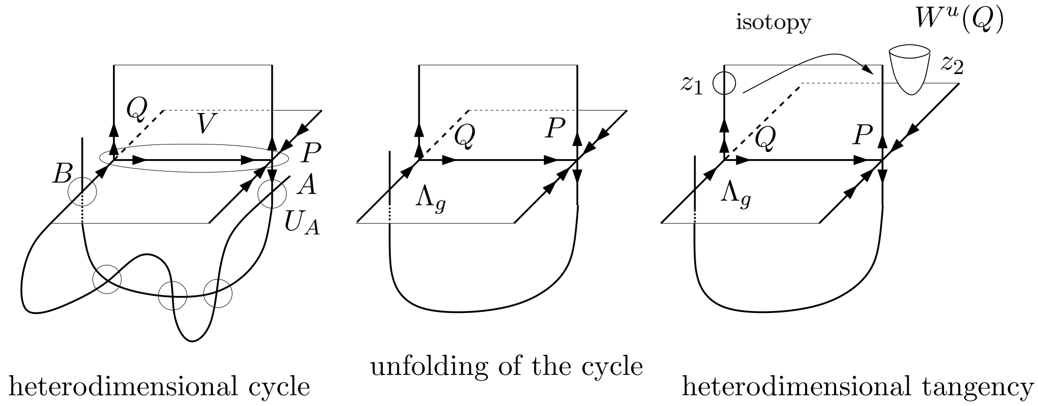

Figure 8: Creation of a heterodimensional tangency

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