Generalizing symmetries in symplectic geometry

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Abstract

We discuss recent results extending the notions of hamiltonian action and reduction in symplectic geometry to the setting of twisted Dirac geometry. We focus on the role of Lie algebroids as infinitesimal symmetries and applications to quasi-Poisson geometry.

1 Introduction

This note discusses several aspects of the hamiltonian theory of twisted Dirac manifolds following [9, 10]. The focus of the exposition is on the interplay between infinitesimal symmetries of Dirac manifolds and reduction, as well as on the close ties between Dirac geometry, quasi-Poisson geometry and the theory of group-valued momentum maps [1, 2, 3].

The classical set-up for hamiltonian theory [22, 23] involves a Poisson manifold (M, π), a Lie algebra $\mathfrak{g}$, and an infinitesimal action $\rho_M : \mathfrak{g} \to \mathfrak{X}(M)$. This action is called **hamiltonian** if there exists a smooth ad^*-equivariant map $J : M \to \mathfrak{g}^*$ relating π and ρ_M by

$$\rho_M(v) = i_{dJ_v}\pi, \quad \forall v \in \mathfrak{g}, \tag{1.1}$$

where $J_v \in C^\infty(M)$ is defined by $J_v(x) = \langle J(x), v\rangle,\ x \in M$. The map J is the **momentum map** of the action. Each level set $J^{-1}(\mu)$ is invariant under $\mathfrak{g}_\mu$, the isotropy Lie algebra at $\mu \in \mathfrak{g}^*$ with respect to the coadjoint action, and the **reduced space** $M_\mu = J^{-1}(\mu)/\mathfrak{g}_\mu$ acquires a Poisson structure induced from the one on M; when M is symplectic, each M_μ is symplectic. Examples of symplectic manifolds obtained by reduction [23] include complex projective spaces and coadjoint orbits; an infinite-dimensional version of this construction produces symplectic structures on moduli spaces in gauge theory [5].

Several interesting generalized notions of hamiltonian action and momentum map have appeared in the last years, see e.g. [28]. One body of generalizations, studied in [24], is based on allowing the target of the momentum map to be an arbitrary Poisson manifold rather than just the dual of a Lie algebra. This theory includes the hamiltonian theory of Poisson-Lie group

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actions [20] and naturally leads to Lie algebroids and symplectic groupoids. We recall it in Section 3.

Another important class of generalizations arises in the context of "quasi"-Poisson geometry [1, 21], the semiclassical limit of the theory of quasi-Hopf algebras [17]. The study of symmetries in this setting does not fit into the usual framework of Poisson geometry since the bivector fields and 2-forms entering the picture are no longer Poisson or symplectic. This will be recalled in Section 4, with emphasis on the theory of group-valued momentum maps [3, 2].

Despite the seemingly different ingredients used in each of these two lines of generalizations of hamiltonian theory, they have several features in common. In particular, both produce (ordinary) symplectic/Poisson spaces via reduction. As we will see, this fact can be explained by looking at all these examples as particular cases of hamiltonian spaces in Dirac geometry. In Section 5, we recall the basic notions of Dirac geometry, its connections with Lie algebroid actions, and its hamiltonian theory with focus on reduction. In Section 6, we revisit various examples of generalized symmetries and explain how they fit into the Dirac-geometric framework.

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2 Poisson geometry and hamiltonian spaces

We recall some basics facts about Poisson geometry to fix our notation. We refer the reader to [13] for details and references.

A **Poisson manifold** is a manifold M equipped with a bivector field $\pi \in \Gamma(\wedge^2 TM)$ satisfying the integrability condition $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket on multivector fields; this condition is equivalent to the requirement that the bracket

$$\{f, g\} := \pi(df, dg), \quad f, g \in C^\infty(M) \tag{2.1}$$

satisfies the Jacobi identity.

Given a function $f \in C^\infty(M)$, its **hamiltonian vector field** is defined by $X_f := \pi^\sharp(df) \in \mathfrak{X}^1(M)$, where $\pi^\sharp$ is the bundle map

$$\pi^\sharp : T^*M \to TM, \quad \beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta), \quad \text{for } \alpha, \beta \in T^*M. \tag{2.2}$$

It follows from the integrability of π that $\mathcal{L}_{X_f}\pi = 0$ for all $f \in C^\infty(M)$.

A Poisson structure π for which $\pi^\sharp$ is invertible is equivalent to a symplectic structure ω by

$$(\pi^\sharp)^{-1} = \omega^\sharp, \tag{2.3}$$

where $\omega^\sharp : TM \to T^*M$ is the bundle map defined by $\omega^\sharp(X) := i_X\omega$. The condition $d\omega = 0$ is equivalent to $[\pi, \pi] = 0$. More generally, if (M, π) is any Poisson manifold, the image of the bundle map $\pi^\sharp : T^*M \to TM$ defines an *integrable* generalized distribution on M whose leaves are locally swept out by flows of hamiltonian vector fields. The restriction of π to each leaf is nondegenerate, so each leaf carries a symplectic structure. Conversely, this singular symplectic foliation completely determines π.

Example 2.1 Let $\mathfrak{g}$ be a (real, finite-dimensional) Lie algebra, and let $\mathfrak{g}^*$ be its dual. The **Lie-Poisson structure** $\pi_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is defined by the bracket

$$\{f, g\}(\mu) := \langle \mu, [df(\mu), dg(\mu)] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*),\ \mu \in \mathfrak{g}^*, \tag{2.4}$$

where we used the identification $T^*_\mu\mathfrak{g}^* \cong \mathfrak{g}$. The leaves in this example are the coadjoint orbits, and the leafwise symplectic form is given by

$$\omega(\rho_{\mathfrak{g}^*}(u), \rho_{\mathfrak{g}^*}(v))(\mu) = -\langle \mu, [u, v]\rangle, \quad u, v \in \mathfrak{g}, \ \mu \in \mathfrak{g}^*, \tag{2.5}$$

where $\rho_{\mathfrak{g}^*}(u)(\mu) = \mathrm{ad}^*_u(\mu) = \pi^\sharp_{\mathfrak{g}^*}(u)(\mu)$ is the infinitesimal generator of the coadjoint action.

If (M_1, π_1) and (M_2, π_2) are Poisson manifolds, then a smooth map $\psi : (M_1, \pi_1) \to (M_2, \pi_2)$ is a **Poisson map** if the bivector fields π_2 and π_1 are ψ-related:

$$\pi_1(\psi^*(\alpha), \psi^*(\beta)) = \pi_2(\alpha, \beta) \circ \psi, \quad \text{for } \ \alpha, \beta \in \Omega^1(M_2). \tag{2.6}$$

This is equivalent to requiring that $\psi^* : C^\infty(M_2) \to C^\infty(M_1)$ preserves the brackets (2.1). If π_1 is symplectic, then a Poisson map $\psi : M_1 \to M_2$ is called a **symplectic realization** of M_2. For example, the inclusion of a symplectic leaf into a Poisson manifold is a symplectic realization.

The next result reveals the close relationship between hamiltonian actions and Poisson maps.

Proposition 2.2 *Let (M, π) be a Poisson manifold, and let $J : M \to \mathfrak{g}^*$ be a smooth map. Then the following are equivalent:*

1. *J is a Poisson map;*
2. *The "momentum-map condition"* (1.1) *defines a $\mathfrak{g}$-action on M for which J is equivariant (i.e., J is a momentum map for a hamiltonian $\mathfrak{g}$-action on M).*

Example 2.3 The identity map $\mathfrak{g}^* \to \mathfrak{g}^*$ is the momentum map for the coadjoint action, and each coadjoint orbit is a hamiltonian space with respect to the restricted action and momentum map given by the inclusion map.

3 Generalized symmetries in Poisson geometry

Interesting extensions of the notion of hamiltonian action arise when one allows the momentum map to take values on a general Poisson manifold [24] (see also [16]). Unraveling the infinitesimal symmetries associated with arbitrary Poisson maps $J : M \to P$ naturally leads to Lie algebroids, so we recall the basic definitions, see e.g. [13].

3.1 Infinitesimal actions of Lie algebroids

A **Lie algebroid** over a manifold P is a vector bundle $A \to P$ together with a map $\rho : A \to TP$, called the **anchor**, and a Lie bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$ satisfying the Leibniz rule

$$[a, fb]_A = f[a, b]_A + \mathcal{L}_{\rho(a)}(f)b, \quad \text{for } a, b \in \Gamma(A), \text{ and } f \in C^\infty(M).$$

We often denote a Lie algebroid by the triple $(A, \rho, [\cdot, \cdot]_A)$. Two central features of Lie algebroids are that $\rho(A) \subseteq TP$ defines a generalized integrable distribution (whose leaves are called "orbits" of A) and, at each $y \in P$, the restriction of $[\cdot, \cdot]_A$ to $\ker(\rho)_y$ is a Lie bracket (defining the "isotropy" Lie algebra at y), see e.g. [13].

An **action** of a Lie algebroid $A \to P$ on a manifold M along a map $J : M \to P$ is a Lie algebra homomorphism $\widehat{\rho}_M : \Gamma(A) \to \mathfrak{X}(M)$ satisfying

$$TJ(\widehat{\rho}_M(a)) = \rho(a) \quad \text{and} \quad \widehat{\rho}_M(fa) = J^*f\widehat{\rho}_M(a), \ \ \forall\, a \in \Gamma(A), \ f \in C^\infty(P),$$

see e.g. [18, 25]. One recovers the usual notion of action for Lie algebras when P is a point.

Hamiltonian $\mathfrak{g}$-actions with momentum maps $J : M \to \mathfrak{g}^*$ can be expressed in terms of Lie algebroids as follows. Instead of thinking of $\mathfrak{g}$ as dual to $\mathfrak{g}^*$, we now regard its elements as constant sections of the bundle $T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$. The space $\Omega^1(\mathfrak{g}^*) = C^\infty(\mathfrak{g}^*, \mathfrak{g})$ admits the following Lie bracket extending the one on $\mathfrak{g}$:

$$[u, v](x) = [u(x), v(x)] + \mathcal{L}_{\rho_{\mathfrak{g}^*}(u(x))} v(x) - \mathcal{L}_{\rho_{\mathfrak{g}^*}(v(x))} u(x), \quad u, v \in C^\infty(\mathfrak{g}^*.\mathfrak{g}), \tag{3.1}$$

with $\rho_{\mathfrak{g}^*}$ defined as in Example 2.1. The action $\rho_M : \mathfrak{g} \to \mathfrak{X}(M)$ defined by (1.1) induces a map

$$\widehat{\rho}_M : C^\infty(\mathfrak{g}^*, \mathfrak{g}) \to \mathfrak{X}(M), \quad \widehat{\rho}_M(u)(x) = \rho_M(u(J(x)))(x) \tag{3.2}$$

which is a Lie algebra homomorphism and satisfies the "momentum-map condition"

$$\widehat{\rho}_M(u) = \pi^\sharp(J^*u), \quad u \in \Omega^1(\mathfrak{g}^*). \tag{3.3}$$

The cotangent bundle $T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$ together with the bracket (3.1) and map $\rho_{\mathfrak{g}^*} : T\mathfrak{g} \to T\mathfrak{g}^*$ is a Lie algebroid (in fact, it is a *transformation Lie algebroid* [13]), and the map (3.2) defines an action of $T^*\mathfrak{g}^*$ on M along the momentum map J. The conclusion is that the infinitesimal symmetries encoded in a Poisson map $J : M \to \mathfrak{g}^*$ can be expressed in two alternative ways: either as a $\mathfrak{g}$-action on M defined by (1.1) or as a Lie algebroid action of $T^*\mathfrak{g}^*$ along J defined by (3.3); each action completely determines the other by (3.2).

3.2 Generalizing the target of momentum maps

Let (P, π_P) be a Poisson manifold. To regard it as the receptacle of a "momentum map", a central fact is that $\mathfrak{p} := (T^*P, \pi_P^\sharp, [\cdot,\cdot])$ is a Lie algebroid, with bracket on $\Omega^1(P)$ given by

$$[\alpha, \beta] := \mathcal{L}_{\pi_P^\sharp(\alpha)}\beta - \mathcal{L}_{\pi_P^\sharp(\beta)}\alpha - d\pi_P(\alpha, \beta).$$

The orbits of this Lie algebroid are the symplectic leaves of P.

If (M, π) is a Poisson manifold and $J : M \to P$ is a smooth map, let us consider, analogously to (3.3), the map

$$\widehat{\rho}_M : \Omega^1(P) \to \mathfrak{X}(M), \quad \alpha \mapsto \pi^\sharp(J^*\alpha). \tag{3.4}$$

We have the following generalization of Proposition 2.2:

Proposition 3.1 *The map $\widehat{\rho}_M$ defines a Lie algebroid $\mathfrak{p}$-action on M along J if and only if J is a Poisson map.*

So if $J : M \to P$ is a Poisson map, it can be seen as a "momentum map" for the "hamiltonian" $\mathfrak{p}$-action defined by the "momentum-map condition" (3.4).

To carry out reduction starting with a Poisson map $J : M \to P$, one notices that the Lie algebroid action of $\mathfrak{p}$ on M defined by (3.4) induces, for each $y \in P$ regular value of J, a Lie algebra action of the isotropy Lie algebra

$$\mathfrak{p}_y = \ker_y(\pi_P^\sharp) \subseteq T_y^*P \tag{3.5}$$

on the level set $J^{-1}(y)$. Assuming that this action is regular (in the sense that its orbits form a simple foliation), then the orbit space $J^{-1}(y)/\mathfrak{p}_y$ acquires a Poisson structure uniquely determined by the fact that the projection $J^{-1}(y) \to J^{-1}(y)/\mathfrak{p}_y$ is a Poisson map. If M is symplectic, then the reduced spaces are also symplectic. This is the infinitesimal version of Mikami-Weinstein reduction [24] for Poisson manifolds, see also [16].

When $P = \mathfrak{g}^*$, the Lie algebras (3.5) are the isotropy Lie algebras for the coadjoint action, and we recover the usual reduction procedure for hamiltonian actions [22, 23]. Another important class of examples is given when P is the a dual Poisson-Lie group G^*; in this case one recovers Lu's hamiltonian theory for Poisson-Lie group actions [20].

Remark 3.2 The description of the *global* symmetries of M associated with a Poisson map $J : M \to P$ involves the theory of symplectic groupoids [27]. If P is an integrable Poisson manifold, and if $J : M \to P$ is a complete Poisson map [13], then the corresponding $\mathfrak{p}$-action on M can be integrated to a symplectic groupoid action of $\mathcal{G}$ on M, where $\mathcal{G}$ is the (source-simply-connected) symplectic groupoid integrating the Lie algebroid $\mathfrak{p} = T^*P$, see e.g. [16, 24].

4 Symmetries beyond Poisson geometry

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose that $\mathfrak{g}$ is equipped with a non-degenerate, invariant, quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. Let $\phi^G \in \Omega^3(G)$ be the Cartan 3-form on G,

$$\phi^G = \frac{1}{12}([\theta,\theta],\theta)_{\mathfrak{g}} = \frac{1}{12}\big([\overline{\theta},\overline{\theta}],\overline{\theta}\big)_{\mathfrak{g}},$$

where $\theta, \overline{\theta} \in \Omega^1(G,\mathfrak{g})$ are the left and right Maurer-Cartan 1-forms, respectively.

A **quasi-hamiltonian $\mathfrak{g}$-space** [3] is a $\mathfrak{g}$-manifold M equipped with an invariant 2-form $\omega \in \Omega^2(M)$, and an Ad-equivariant map $J : M \to G$ such that

$$d\omega = J^*\phi^G, \tag{4.1}$$

$$\ker(\omega)_x = \{\rho_M(v)_x \,|\, v \in \mathfrak{g},\ (\mathrm{Ad}_{J(x)} + 1)v = 0\},\ \ \forall x \in M, \tag{4.2}$$

$$i_{\rho_M(v)}\omega = \frac{1}{2}J^*\big(\theta + \overline{\theta}, v\big)_{\mathfrak{g}} = J^*\sigma(v),\ \ v \in \mathfrak{g}, \tag{4.3}$$

where $\rho_M : \mathfrak{g} \to TM$ is the infinitesimal action and

$$\sigma : \mathfrak{g} \to T^*G,\ \ \sigma(v) = \frac{1}{2}(v_r + v_l, \cdot)_{\mathfrak{g}}, \tag{4.4}$$

where v_r, v_l are the right and left translations of $v \in \mathfrak{g}$. If M is a G-manifold, ω is G-invariant and J is G-equivariant, then it is a **quasi-hamiltonian G-space**. The map J is a **G-valued momentum map**.

Note that M is *not* symplectic in general, but conditions (4.1) and (4.2) describe the precise way in which ω fails to be closed and nondegenerate according to the geometry of the Lie group G. Condition (4.3) is the analog of the momentum map condition (1.1) (but, unlike (1.1), this condition alone is not enough to determine the infinitesimal action ρ_M).

Example 4.1 Analogously to Example 2.3, each conjugacy class $\mathcal{C}$ in G is a quasi-hamiltonian G-space with respect to the action by conjugation. The momentum map is the inclusion $\iota : \mathcal{C} \hookrightarrow G$ and the 2-form is

$$\omega(\rho_G(u), \rho_G(v)) = \frac{1}{2}\big((\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})(u), v\big)_{\mathfrak{g}}, \tag{4.5}$$

where $\rho_G(u) = u_r - u_l$ is the infinitesimal generator of the action of G on itself by conjugation. This 2-form is analogous to (2.5), though it may be neither nondegenerate nor closed.

In spite of M not being symplectic and J not being a Poisson map, "quasi-hamiltonian" reduction produces honest symplectic spaces (generally singular) [3]: the level set $J^{-1}(e) \hookrightarrow M$ is invariant under the $\mathfrak{g}$-action, and the pull-back of ω to $J^{-1}(e)$ is basic and descends to a *symplectic* form on $J^{-1}(e)/\mathfrak{g}$. Here e is the identity in G, but one can also reduce at different momentum levels.

The main application of quasi-hamiltonian reduction is to give a finite-dimensional construction of the symplectic structure of certain moduli spaces:

Example 4.2 Let G act on G^{2h} by conjugation on each factor, and consider the equivariant map

$$J : G^{2h} \to G, \quad J(a_1, b_1, \dots, a_h, b_h) = \prod_{i=1}^{h} [a_i, b_i]. \tag{4.6}$$

In [3] the authors define a 2-form $\omega \in \Omega^2(G^{2h})$ making G^{2h} into a quasi-hamiltonian space with group-valued momentum map (4.6). The reduced space

$$\mathcal{M} = J^{-1}(e)/G = \{(a_1, b_1, \dots, a_h, b_h) \in G^{2h}, \ \prod_{i=1}^{h}[a_i, b_i] = e\}/G$$

coincides with the representation space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, where Σ is a compact, connected, oriented, 2-manifold of genus h (without boundary), and $\pi_1(\Sigma)$ is its fundamental group. If G is simply connected, the holonomy map identifies $\mathcal{M}$ with the moduli space of gauge equivalence classes of flat connections on $\Sigma \times G$, and the symplectic structure on $\mathcal{M}$ is obtained via quasi-hamiltonian reduction coincides with the one constructed by Atiyah and Bott [5] via infinite-dimensional Marsden-Weinstein reduction.

Remark 4.3 (*Hamiltonian quasi-Poisson actions*)

There is a version of the theory of group-valued momentum maps $J : M \to G$ in which M carries an invariant bivector field rather than a 2-form; these spaces are called *hamiltonian quasi-Poisson manifolds* [2]. Some of their features are analogous to Poisson manifolds: for example, they are associated with Lie algebroids [9, 10] whose orbits define a singular foliation, but unlike Poisson manifolds the bivector field may be degenerate along the leaves. However, one can still find leafwise 2-forms making the leaves into quasi-hamiltonian spaces [2, 9], though the relationship between the bivector field and 2-forms is much more intricate than (2.3). Reduction in this context produces Poisson spaces; an interesting example is the construction of Poisson structures on moduli space of flat connections on surfaces with boundary [2, Sec. 6].

Quasi-Poisson manifolds with group-valued momentum maps fit into the yet more general hamiltonian theory of quasi-Poisson actions developed in [1]. In this setting, momentum maps take values in certain homogeneous spaces associated with Lie quasi-bialgebras [17, 21].

5 Dirac geometry and symmetries

Dirac structures [14, 15] provide a common ground for the study of various "integrable" geometrical structures. We refer the reader to [12] for details, more examples and further references.

5.1 Dirac manifolds and closed 3-forms

Let $\phi \in \Omega^3(M)$ be a fixed closed 3-form on a manifold M. A **ϕ-twisted Dirac structure** on M is a subbundle $L \subset E := TM \oplus T^*M$ such that

1. L is maximal isotropic with respect to the symmetric pairing

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}, \ \ \langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X).$$

   (This means that $\mathrm{rank}(L) = \dim(M)$ and $\langle \cdot, \cdot \rangle|_L = 0$);

2. $\Gamma(L)$ is closed under the bracket $[\![\cdot, \cdot]\!]_\phi : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,

$$[\![(X, \alpha), (Y, \beta)]\!]_\phi = ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + i_{X \wedge Y} \phi). \tag{5.1}$$

We denote ϕ-twisted Dirac manifolds by the triple (M, L, ϕ). The bracket in (5.1) is the **ϕ-twisted Courant bracket** [26], and condition 2. is referred to as the *integrability condition.*

Example 5.1 Bivector fields $\pi \in \Gamma(\wedge^2 TM)$ (resp. 2-forms ω) can be seen as examples of Dirac structures by means of the graphs of the associated bundle maps $\pi^\sharp$ (resp. $\omega^\sharp$) in $TM \oplus T^*M$. The integrability condition 2. amounts to $[\pi, \pi] = 2\pi^\sharp(\phi)$ (resp. $d\omega + \phi = 0$). Hence, for $\phi = 0$, Dirac structures include Poisson structures and closed 2-forms as particular examples.

Just as Poisson structures, twisted Dirac structures are always associated with Lie algebroids and singular foliations. Let $L \subset TM \oplus T^*M$ be a ϕ-twisted Dirac structure, and let $\mathrm{pr} : TM \oplus T^*M \to TM$ be the natural projection. Then:

- The triple $(L, \mathrm{pr}|_L, [\![\cdot, \cdot]\!]_\phi|_{\Gamma(L)})$ is a Lie algebroid over M. In particular, the generalized distribution $\mathrm{pr}(L) \in TM$ is integrable, and, at each $x \in M$,

$$L_x \cap T_x^* M = \ker_x(\mathrm{pr}|_L) \tag{5.2}$$

  has a Lie algebra structure induced from $[\![\cdot, \cdot]\!]_\phi$.

- Each leaf $\iota : \mathcal{O} \hookrightarrow M$ of this singular foliation carries a 2-form ω_L defined at $x \in \mathcal{O}$ by

$$\omega_L(X, Y) = \alpha(Y), \tag{5.3}$$

  where $X, Y \in \mathrm{pr}(L)_x$ and $\alpha \in T_x^* M$ is any covector satisfying $(X, \alpha) \in L_x$ (the value of (5.3) turns out to be independent of α), and

$$d\omega_L + \iota^* \phi = 0. \tag{5.4}$$

- The singular foliation and the leafwise 2-forms completely determine L: at each point, we can obtain L from ω_L by

$$L = \{(X, \alpha) \mid X \in \mathrm{pr}(L), \ \alpha|_{\mathrm{pr}(L)} = i_X \omega_L\}. \tag{5.5}$$

All these notions reduce to the ones of Section 3.2 when L is the graph of a Poisson structure. It follows from (5.3) that, at each $x \in M$,

$$\ker(L)_x := T_x M \cap L_x \tag{5.6}$$

is the kernel of the leafwise 2-form ω_L at that point.

We define the opposite of L by $\overline{L} = \{(X, -\alpha) \mid (X, \alpha) \in L\}$.

Remark 5.2 Functions whose differentials are annihilated by $\ker(L)$ are called **admissible**; when $\phi = 0$, they form a *Poisson algebra* with respect to the bracket

$$\{f, g\} := dg(X_f),$$

where X_f is any (local) vector field satisfying $(X_f, df) \in L$. It follows that whenever $\ker(L)$ is the tangent distribution of a simple foliation, the quotient $M/\ker(L)$ has a *Poisson* structure defined via the identification of its functions with admissible functions on M.

Example 5.3 Let G be a Lie group with Lie algebra $\mathfrak{g}$ equipped with a nondegenerate, invariant, quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. We saw in Example 4.1 that the singular foliation of G by conjugacy classes $\iota : \mathcal{C} \hookrightarrow G$ admits a leafwise 2-form ω satisfying $d\omega = \iota^*\phi^G$. This suggests the existence of an underlying $-\phi^G$-twisted Dirac structure L_G on G.

By (5.5) L_G must be given at each point of G by

$$L = \{(X, \alpha) \mid X = \rho_G(v),\ v \in \mathfrak{g},\ \text{ and } \alpha|_{\rho_G(\mathfrak{g})} = i_{\rho_G(v)}\omega\}.$$

By (4.5), if $X = \rho_G(v) = v_r - v_l$, then

$$\alpha|_{\rho_G(\mathfrak{g})} = \frac{1}{2}(v_r + v_l, \cdot)_{\mathfrak{g}} = \sigma(v),$$

where σ is defined in (4.4). It follows that

$$\alpha - \sigma(v) \in \rho_G(\mathfrak{g})^\circ = L \cap T^*M.$$

One can check that σ maps $\ker(\rho_G)$ isomorphically onto $L \cap T^*M$, so $\alpha - \sigma(v)$ is in the image of σ. So there exists $u \in \mathfrak{g}$ satisfying $X = \rho_G(u) = \rho_G(v)$ and $\alpha = \sigma(u)$. It follows that

$$L_G = \{(\rho_G(u), \sigma(u)),\ u \in \mathfrak{g}\} = \{(u_r - u_l, \frac{1}{2}(u_r + u_l, \cdot)_{\mathfrak{g}}),\ u \in \mathfrak{g}\}, \tag{5.7}$$

which is indeed a smooth Dirac structure. Note that

$$\ker(L_G)_g = \{\rho_G(v)_g \mid (1 + \mathrm{Ad}_g)v = 0\}. \tag{5.8}$$

We call L_G the **Cartan-Dirac structure** on G with respect to $(\cdot,\cdot)_{\mathfrak{g}}$.

Remark 5.4 Cartan-Dirac structures fit into a more general class of examples:

Let $\mathfrak{g}$ be a Lie quasi-bialgebra [17, 21]. Let $\mathfrak{d}$ be its Drinfeld double, (G, D) the associated group pair [1], and consider the homogeneous space $S = D/G$. The trivial bundle $\mathfrak{d} \times S \to S$ has the structure of an exact Courant algebroid [4], and $\mathfrak{g} \subseteq \mathfrak{d}$ defines a Dirac structure. Hence a choice of splitting $\mathfrak{d} \times S \cong TS \oplus T^*S$ determines a twisted Dirac structure L_S on S. For a suitable choice of quasi-bialgebra, $S \cong G$, and $L_S = L_G$ (5.7).

5.2 Dirac maps and infinitesimal symmetries

We now discuss the relationship between Dirac maps and infinitesimal Lie algebroid actions, analogous to Prop. 3.1. Details on Dirac maps can be found e.g. in [12].

Let (M, L, ϕ) and (P, L_P, ϕ^P) be twisted Dirac manifolds. A smooth map $\psi : M \to P$ is a **forward Dirac map** (or simply an **f-Dirac map**) if, for each $x \in M$,

$$(L_P)_{\psi(x)} = \{(T_x\psi(X), \beta) \mid X \in T_xM,\ \beta \in T^*_{\psi(x)}P,\ (X, T\psi^*(\beta)) \in (L_M)_x\}. \tag{5.9}$$

A direct computation shows that if L and L_P are defined by Poisson structures, then (5.9) is equivalent to (2.6). An immediate property of an f-Dirac map is that

$$\ker(L_P)_{\psi(x)} = T_x\psi(\ker(L)_x). \tag{5.10}$$

Let $\mathfrak{p}$ be the Lie algebroid associated with L_P, and let $J : M \to P$ be a smooth map. Following Section 3, it is natural to call a $\mathfrak{p}$-action on M, $\widehat{\rho}_M : \Gamma(L_P) \to \mathfrak{X}(M)$, "hamiltonian" with "momentum map" J if the following extension of the "momentum-map condition" (3.4) holds: if $X = \widehat{\rho}_M(Y,\beta)$, then, at each point of M, X satisfies

$$TJ(X) = Y, \text{ and } (X, (TJ)^*(\beta)) \in L. \tag{5.11}$$

If $\widehat{\rho}_M$ is "hamiltonian" in this sense, then J is an f-Dirac map. However, unlike Prop. 3.1, an f-Dirac map may not completely specify an infinitesial action via (5.11); this is the reason for the quotes in "hamiltonian". If $J : M \to P$ is an f-Dirac map, and given $(Y,\beta) \in \Gamma(L_P)$, one can always find X satisfying (5.11) at each point. We have the following equivalent conditions:

1. For each $(Y,\beta) \in \Gamma(L_P)$, condition (5.11) defines a *unique* X at each point of M.
2. The map $J : M \to P$ satisfies

$$\ker(TJ) \cap \ker(L) = \{0\}. \tag{5.12}$$

3. The restriction of TJ to $\ker(L)$ induces an isomorphism

$$TJ : \ker(L) \xrightarrow{\sim} \ker(L_P). \tag{5.13}$$

We have the following generalization of Prop. 3.1 to Dirac geometry:

Proposition 5.5 *Let (M, L, ϕ) and (P, L_P, ϕ^P) be Dirac manifolds, and let $J : M \to P$ be a smooth map. Suppose that $\phi = J^*\phi^P$ and that J satisfies* (5.12). *Then J is an f-Dirac map if and only if* (5.11) *defines a $\mathfrak{p}$-action on M along J.*

Proposition 5.5 indicates the type of Dirac maps that will play a special role as momentum maps: An f-Dirac map $J : M \to P$ is a **Dirac realization** if

$$\phi = J^*\phi^P, \tag{5.14}$$

and (5.12) holds. If M is presymplectic we call J a **presymplectic realization** of P. We will further discuss the extra conditions (5.12) and (5.14) in the next subsection. A simple example of a presymplectic realization is the inclusion of any presymplectic leaf into a twisted Dirac manifold.

Example 5.6 We now show that quasi-hamiltonian spaces are precisely presymplectic realizations of Cartan-Dirac structures (Example 5.3) [11]; note the analogy with Prop. 2.2 .

Let $J : (M,\omega) \to (G, L_G)$ be a presymplectic realization. The associated Lie-algebroid action of L_G on M (5.11), given by

$$\widehat{\rho}_M(\rho_G(v), \sigma(v)) = X, \text{ where } TJ(X) = \rho_G(v), \text{ and } i_X\omega = J^*\sigma(v),$$

defines a Lie-algebra action $\rho_M : \mathfrak{g} \to \mathfrak{X}(M)$ by $v \mapsto \widehat{\rho}_M(\rho_G(v), \sigma(v))$. A direct consequence of the definition of ρ_M is that J is $\mathfrak{g}$-equivariant and (4.3) holds. By (5.14), ω satisfies (4.1). Using (5.13), we know that $\ker(\omega)_x = \{X \mid TJ(X) \in \ker(L_G)\} \subset \rho_M(\mathfrak{g})_x$. It follows from (5.8) that (4.2) holds. Hence ρ_M is a quasi-hamiltonian action with J as momentum map.

The $\mathfrak{g}$-invariance of ω follows as a consequence: Using the Maurer-Cartan equations $d\theta = (1/2)[\theta,\theta]$ (resp. $d\overline{\theta} = -(1/2)[\overline{\theta},\overline{\theta}]$), we have

$$\begin{aligned}\mathcal{L}_{\rho_M(v)}\omega &= i_{\rho_M(v)}J^*\phi^G + \frac{1}{2}J^*d\big(\theta+\overline{\theta}, v\big)_{\mathfrak{g}} \\ &= i_{\rho_M(v)}J^*\frac{1}{12}([\theta,\theta],\theta)_{\mathfrak{g}} + \frac{1}{4}J^*\big([\theta,\theta]-[\overline{\theta},\overline{\theta}], v\big)_{\mathfrak{g}} \\ &= \frac{1}{4}J^*\big([\theta,\theta], (\mathrm{Ad}_{g^{-1}}-1)v\big)_{\mathfrak{g}} + \frac{1}{4}J^*([\theta,\theta]-\mathrm{Ad}_g([\theta,\theta]), v)_{\mathfrak{g}} \;=\; 0.\end{aligned}$$

Remark 5.7 (*Global symmetries and examples*)
The global objects integrating ϕ-twisted Dirac structures are ϕ-twisted presymplectic groupoids [11, 29]; this generalizes Remark 3.2. Under suitable completeness/integrability assumptions, infinitesimal "hamiltonian" actions in the sense of (5.11) correspond to global actions of presymplectic groupoids [9].

If $\mathcal{G}$ is a twisted presymplectic groupoid over (P, L_P) with source and target maps s and t, then t is an f-Dirac map (though it may not satisfy (5.12)) and is a "momentum map" for the "hamiltonian" action of $\mathcal{G}$ on itself by left multiplication; the map $(\mathsf{t},\mathsf{s}) : \mathcal{G} \to P \times \overline{P}$ is a Dirac realization, corresponding to the $\mathcal{G}\times\mathcal{G}$-action on $\mathcal{G}$ by $(g,h)\cdot x = gxh^{-1}$.

5.3 Reduction

We now turn to reduced spaces in Dirac geometry, unifying the reduction procedure of Section 3.2 and the quasi-hamiltonian reduction of Section 4. The key point is to understand when these more general Dirac-reduced spaces carry ordinary Poisson structures.

Let (M, L, ϕ) and (P, L_P, ϕ^P) be twisted Dirac manifolds and suppose that $\widehat{\rho}_M$ is a $\mathfrak{p}$-action on M along $J : M \to P$ satisfying the "momentum-map condition" (5.11); in particular, J is f-Dirac but may not satisfy (5.12). Following Section 3.2, let $y \in P$ be a regular value of J, and consider the submanifold $J^{-1}(y) \hookrightarrow M$ and the isotropy Lie algebra $\mathfrak{p}_y = L_P \cap T^*_yP$. If $(0,\beta) \in \mathfrak{p}_y$ and $X = \widehat{\rho}_M((0,\beta))$, then (5.11) implies that $TJ(X) = 0$. Hence $\widehat{\rho}_M$ restricts to a Lie-algebra action ρ_M of $\mathfrak{p}_y$ on $J^{-1}(y)$, which we assume to be regular.

The level set $\iota : J^{-1}(y) \hookrightarrow M$ inherits a "pull-back" Dirac structure ι^*L from the ambient manifold M with leaves given by $\mathcal{O}\cap J^{-1}(y)$, where $\mathcal{O}$ is a leaf of L in M, and leafwise 2-form $\iota^*\omega_L$. It is simple to check that

$$\rho_M(\mathfrak{p}_y) \subseteq \ker(\iota^*L). \tag{5.15}$$

If $\iota^*\phi$ is basic with respect to the $\mathfrak{p}_y$-orbits on $J^{-1}(y)$, then the orbit space $M_{red} = J^{-1}(y)/\mathfrak{p}_y$ inherits a Dirac structure which is generally *degenerate and twisted*, and uniquely characterized by $J^{-1}(y) \to M_{red}$ being an f-Dirac map.

Example 5.8 Let $\mathcal{G}$ be a (source-connected) twisted presymplectic groupoid over (P, L_P), and consider the infinitesimal $\mathfrak{p}$-action on $\mathcal{G}$ by left multiplication (with momentum $J = \mathsf{t}$ the target map, see Remark 5.7). The action of $\mathfrak{p}_y$ on $t^{-1}(y)$ is regular, and the reduced space can be naturally identified with the leaf of L_P through y. The reduced Dirac structure is the twisted presymplectic structure on the leaf.

Let us focus on the case where $M_{red} = J^{-1}(y)/\mathfrak{p}_y$ inherits an honest Poisson structure. To avoid the 3-form twist on M_{red}, it suffices to assume that $\iota^*\phi = 0$ on $J^{-1}(y)$ (or that it is zero along each leaf of ι^*L). On the other hand, following Remark 5.2, the reduced Dirac structure on M_{red} is nondegenerate if and only if (5.15) is an equality:

$$\rho_M(\mathfrak{p}_y) = \ker(\iota^*L), \tag{5.16}$$

since in this case we can identify functions on M_{red} with admissible functions on $(J^{-1}(y), \iota^*L)$.

Lemma 5.9 *The $\mathfrak{p}_y$-action on $J^{-1}(y)$ satisfies* (5.16) *if and only if J satisfies* (5.12) *at each $x \in J^{-1}(y)$.*

PROOF: Note that $(X, 0) \in \ker(\iota^*L)_x$ if and only if $TJ(X) = 0$ and there is a $\beta \in T_y^*P$ such that $(X, J^*(\beta)) \in L_x$. Since J is an f-Dirac map, it follows that $(TJ(X), \beta) = (0, \beta) \in (L_P)_y$. If (5.12) holds, then this implies that $(X, J^*(\beta)) = \widehat{\rho}_M(0, \beta)$, showing that (5.16) holds.

On the other hand, let $\beta \in \mathfrak{p}_y$, and suppose that $X = \rho_M(\beta)$. By (5.11), $TJ(X) = 0$ and $(X, J^*(\beta)) \in L$. If $X' \in \ker(L) \cap \ker(TJ)$, then $X + X'$ still satisfies these conditions and does not lie in the image of ρ_M unless $X' = 0$. So (5.16) implies that (5.12) holds. □

As a result, we have [9, Thm. 4.11]:

Theorem 5.10 *Let $J : M \to P$ be a Dirac (resp. presymplectic) realization, and suppose that the $\mathfrak{p}_y$-action on $J^{-1}(y)$ is regular. Then there is a unique Poisson (resp. symplectic) structure on the reduced space $M_{red} = J^{-1}(y)/\mathfrak{p}_y$ for which $J^{-1}(y) \to M_{red}$ is an f-Dirac map.*

Thm. 5.10 recovers the reduction of Section 3.2 when P is a Poisson manifold, and quasi-hamiltonian reduction [3] when J is as in Example 5.3. For presymplectic realizations, it coincides with the infinitesimal version of Xu's reduction in [29].

Remark 5.11 There is a more general version of reduction in the spirit of the intertwiner spaces of Xu [29]. If $J_i : (M_i, L_i) \to (P, L_P)$ are Dirac realizations, $i = 1, 2$, we consider the fibred product

$$M = M_1 \times_P M_2 = \{(x_1, x_2) \in M_1 \times M_2 \mid J_1(x_1) = J_2(x_2)\},$$

which we assume to be a submanifold $\iota_M : M \hookrightarrow M_1 \times M_2$. The Lie algebroid $\mathfrak{p}$ acts on M along the map $J : M \to P$, $J(x_1, x_2) = J_1(x_1) = J_2(x_2)$. Let us assume that this action is regular. Consider $M_1 \times M_2$ with the product Dirac structure $\overline{L_1} \times L_2$, and let M be equipped with $L_M = \iota_M^*(\overline{L_1} \times L_2)$. Just as in Theorem 5.10, the twisting of L_M vanishes and

$$\rho_M(\mathfrak{p}) = \ker(L_M),$$

so the orbit space $M_{red} = M/\mathfrak{p}$ inherits a Poisson structure (which is symplectic if M_1 and M_2 are presymplectic [29]). Theorem 5.10 follows from this result if one takes J_1 to be the inclusion of the presymplectic leaf $\mathcal{O}_y$ through y in P. Then $M \cong J^{-1}(\mathcal{O}_y)$, and $J^{-1}(\mathcal{O}_y)/\mathfrak{p} \cong J^{-1}(y)/\mathfrak{p}_y$ are naturally isomorphic.

6 Revisiting symmetries and momentum maps

Let (P, L_P, ϕ^P) be a twisted Dirac manifold. Let us summarize the ingredients of the hamiltonian theory of $\mathfrak{p}$-actions with P-valued momentum maps:

Hamiltonian spaces: Hamiltonian $\mathfrak{p}$-spaces are Dirac realizations[1] $J : M \to P$. They form a category $\mathrm{Mom}(P)$, and presymplectic realizations form a subcategory $\mathrm{Mom}_{ps}(P)$.

Reduction: Reduced spaces are the Poisson/symplectic spaces $J^{-1}(y)/\mathfrak{p}_y$, $y \in P$, of Theorem 5.10 (or, more generally, Remark 5.11)

Global symmetries: Assuming $\mathfrak{p}$ to be integrable, global symmetries are described by actions of ϕ^P-twisted presymplectic groupoids $\mathcal{G}$ over (P, L_P).

This framework unifies the notions of symmetry discussed in Sections 3 and 4; as we now see, each example is recovered by a suitable choice of target P:

- If P is a Poisson manifold and $J : M \to P$ is a Dirac realization, then (5.13) implies that M is necessarily Poisson and J is a Poisson map. So $\mathrm{Mom}(P)$ coincides with the category of Poisson maps into P. The reduced spaces are those of Section 3.2 [24] and global symmetries are given by actions of symplectic groupoids. More specifically:
  - If $P = G^*$, a dual Poisson-Lie group, one recovers the hamiltonian Poisson actions and reduction of Lu [20].
  - If $G^* = \mathfrak{g}^*$, the dual of a Lie algebra, one recovers classical hamiltonian theory.
- Let $P = (S, L_S)$ be a twisted Dirac manifold associated with a Lie quasi-bialgebra as in Remark 5.4. The main result of [10] asserts that there is a correspondence between quasi-Poisson bivector fields and twisted Dirac structures so that $\mathrm{Mom}(S)$ is isomorphic to the category of hamiltonian quasi-Poisson $\mathfrak{g}$-spaces with S-valued momentum maps of [1]. Particular cases are:
  - If $S = G$ equipped with the Cartan-Dirac structure, then $\mathrm{Mom}(G)$ is isomorphic to the category of hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds with G-valued momentum maps [2, 9]. Reduction coincides with the quasi-Poisson reduction of [2].
  - If $S = G$ equipped with the Cartan-Dirac structure, then $\mathrm{Mom}_{ps}(G)$ is exactly the category of quasi-hamiltonian $\mathfrak{g}$-spaces, as shown in Example 5.6; reduction coincides with quasi-hamiltonian reduction [3].

  The presymplectic groupoid associated with (G, L_G) is the AMM-groupoid [6], see [11].

The framework of Dirac geometry sheds light on various aspects of quasi-Poisson geometry, such as the existence of quasi-hamiltonian foliations [9, 10]. Besides reduction, this framework encompasses other key features that different hamiltonian theories share, such as convexity [30] and prequantization [19], and the relationship between momentum map theories corresponding to different target Dirac manifolds can be investigated through Xu's Morita theory for twisted presymplectic groupoids [29].

[1] Hamiltonian spaces for which J does not satisfy (5.12) seem relevant in applications such as [7, 8].

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