Local and global well-posedness for the Ostrovsky equation

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Abstract. We consider the initial value problem for

$$\partial_t u - \beta\partial_x^3 u - \gamma\partial_x^{-1} u + uu_x = 0, \quad x, t \in \mathbb{R}, \tag{0.1}$$

where u is a real valued function, β and γ are real numbers such that $\beta \cdot \gamma \neq 0$ and $\partial_x^{-1} f = ((i\xi)^{-1}\widehat{f}(\xi))^{\vee}$.

This equation differs from Korteweg-de Vries equation in a nonlocal term. Nevertheless, we obtained local well posedness in $X_s = \{f \in H^s(\mathbb{R}) : \partial_x^{-1} f \in L^2(\mathbb{R})\}$, $s > 3/4$, using techniques developed in [6]. For the case $\beta \cdot \gamma > 0$, we also obtain a global result in X_1, using appropriate conservation laws.

1. Introduction

In this paper we study local and global well-posedness of the initial value problem (IVP) associated to,

$$(u_t - \beta u_{xxx} + uu_x)_x = \gamma u, \tag{1.2}$$

where $\beta \in \mathbb{R}$, $\beta \neq 0$ and $\gamma > 0$.

The equation above was derived as a model for weakly nonlinear long waves in a rotating frame of reference. It was proposed by Ostrosvky [11] to describe the propagation of surface waves in the ocean (see also [10]). The parameter γ measures the effect of rotation (Coriolis) which is supposed to be small meanwhile β determines the type of dispersion. In the absence of rotation ($\gamma = 0$) the equation (1.2) becomes the well known Korteweg-de Vries (KdV) equation.

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One of the main concerns regarding equation (1.2) was to determine the existence of solitary wave solutions. This question naturally appears due to the similarities between this equation and KdV and Kadomtsev-Petviashvili (KP) equations, which have solitons as solutions. Several works have been devoted to deal with this problem (see [2] and references therein). Using techniques inspired in those applied for the KP equations, it was proved recently in [13], that for $\beta < 0$, there are not solitary wave solutions for (1.2) with phase speed satisfying $c < \sqrt{140\gamma|\beta|}$. It was also proved there that for $\beta > 0$ and $c < 2\sqrt{\gamma\beta}$, solitary waves do exist and that for certain values of c, the set of ground states is stable.

Since closed forms for these kind of solutions are not known, numerical techniques has been used to approximate them, as well as to analyze how an initial perturbation in a KdV soliton profile will be destroyed and in particular, how long it can exist as a structure closed to a KdV soliton when the parameter γ is small.

Here we do not address these problems, but instead we will treat the IVP associated to (1.2). Our main purpose is to establish a local well-posedness theory for data in low regularity anisotropic Sobolev spaces $X_s(\mathbb{R})$ (see the definition below) and determine whether or not it is possible to extend the local solutions globally.

Equation (1.2) can be seen as a perturbation of the KdV by a nonlocal term. So, it is natural to ask whether solutions of the IVP associated to (1.2) enjoy analogous properties to those of the solutions of the IVP associated to the KdV equation. The answer is not obvious and surprisingly affirmative in the case of data in anisotropic Sobolev spaces $X_s(\mathbb{R})$ with indices $s > 3/4$.

To describe our results we first rewrite the equation (1.2) by making use of the anti-derivative,

$$\partial_x^{-1} f(x) = \frac{1}{2}\left(\int_{-\infty}^{x} f(x')dx' - \int_{x}^{\infty} f(x')dx'\right) \tag{1.3}$$

(see [3] for an explanation). With this definition we are led to the following initial value problem

$$\begin{cases} \partial_t u - \beta\partial_x^3 u - \gamma\partial_x^{-1} u + uu_x = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1.4}$$

where u is a real valued function. We will also admit negative values for γ, so we simply assume $\beta \cdot \gamma \neq 0$.

With this choice of the antiderivative we have, $\partial_x^{-1} f = (\frac{\widehat{f}(\xi)}{i\xi})^{\vee}$, so it is natural to define the function space X_s as

$$X_s = \{f \in H^s(\mathbb{R}) : \partial_x^{-1} f \in L^2(\mathbb{R})\}, \; s \in \mathbb{R}. \tag{1.5}$$

Then we consider the integral equivalent formulation of the IVP (1.4), that is,

$$u(t) = \mathcal{U}(t)u_0 + \int_0^t \mathcal{U}(t-t')(u\partial_x u)(t')\,dt' \tag{1.6}$$

where

$$\mathcal{U}(t)u_0 = (e^{-it(\beta\xi^3+\gamma\xi^{-1})+ix\xi}\,\widehat{u}_0(\xi))^{\vee}. \tag{1.7}$$

To prove the local well-posedness we will follow the scheme developed by Kenig, Ponce and Vega [7] to deal with the IVP associated to the KdV equation. More precisely, we will establish a series of estimates exploring the smoothing effects related to the operator $\mathcal{U}(t)$ and then use a contraction mapping principle to obtain the desired result. The analysis of the operator $\mathcal{U}(t)$ however is not trivial and we have to distinguish two cases: when $\beta\cdot\gamma > 0$ and when $\beta\cdot\gamma < 0$.

One of the key estimates in [7] was the global smoothing effect of Kato's type satisfied by solutions of the linear problem,

$$\begin{cases} v_t + v_{xxx} = 0, & x,t\in\mathbb{R},\\ v(x,0) = v_0(x), \end{cases} \tag{1.8}$$

denoted by $v(x,t) = e^{t\partial_x^3}v_0$, that is,

$$\sup_x \Big(\int_{-\infty}^{\infty} |\partial_x e^{t\partial_x^3}v_0(x)|^2\,dt\Big)^{1/2} \le c\|v_0\|_{L^2}. \tag{1.9}$$

For $\beta\cdot\gamma < 0$ we get an analogous estimate, but because of the singularity present in the symbol $\varphi(\xi) = -\beta\xi^3 - \gamma\xi^{-1}$, we were only able to obtain a local smoothing of this type (see (3.27) below) when $\beta\cdot\gamma > 0$. However, this is enough for our purposes. It is interesting to observe that the Strichartz estimates behave in the opposite way: for $\beta\cdot\gamma > 0$ we obtain a global estimate meanwhile for the case $\beta\cdot\gamma < 0$ we just get a local one. Roughly speaking, the behavior commented above is because the smoothing effect is related to the first derivative of the symbol (this determines the dispersive character of the equation). On the other hand, the Strichartz estimates are connected with the "curvature" of the symbol, i.e. the second derivative of the symbol. We shall also mention that somehow the structure of the KP equation appears when we looked for maximal function estimates. Our main tool to prove these estimates was a careful analysis of the oscillatory integral defining the solutions of the linear problem associated to (1.2) and the theory developed in [5].

**** These estimates and the contraction mapping principle give us local well-posedness for data in $H^s(\mathbb{R})$, $s > 3/4$. To show that we have indeed local well-posedness in X_s, $s > 3/4$, we use the solution previously obtained given in the integral form (1.6) and show that $\partial_x^{-1}u(t) \in L^2(\mathbb{R})$.****

To show the global existence of solutions for the IVP (1.4) we observe that the following quantities are conserved by the flow:

$$I_1(u(t)) = \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = I_1(u_0) \tag{1.10}$$

and

$$I_2(u(t)) = \int_{-\infty}^{\infty} \big(\beta(\partial_x u)^2 + \frac{\gamma}{2}(\partial_x^{-1}u)^2 + \frac{1}{3}u^3\big)(x,t)\, dx = I_2(u_0). \tag{1.11}$$

Therefore if we consider data in X_1 we will obtain an *a priori* estimate that will allow us to extend globally in time the local solutions. Later on we will justify the validity of these two identities.

This paper is organized as follows. We give the statements of our results in Section 2. Section 3 will be dedicated to establish all the linear estimates needed in the proofs of the local results. The local theory will be established in Section 4 and the global result will be proved in Section 5.

2. Main Results

In this section we list the main results in this work. First we have the local well-posedness theory for the IVP (1.4) in the usual Sobolev spaces $X_s(\mathbb{R})$.

Theorem 2.1. *Let $u_0 \in X_s(\mathbb{R})$, $s > 3/4$. If $\beta\cdot\gamma \neq 0$, then there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution u of the IVP* (1.4) *such that*

$$u \in C([0,T] : X_s(\mathbb{R})), \tag{2.12}$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \tag{2.13}$$

$$\|\partial_x u\|_{L_T^4 L_x^\infty} < \infty \tag{2.14}$$

and

$$\|u\|_{L_x^2 L_T^\infty} < \infty. \tag{2.15}$$

In addition, for any $T' \in (0,T)$ there exists a neighborhood U of u_0 such that the map data–solution is Lipschitz from H^s into the class defined by (2.12)–(2.15).

With the local theory at hand and the conserved quantities (1.10) and (1.11) we obtain the next global result.

Theorem 2.2. *If $u_0 \in X_1(\mathbb{R})$ and $\beta\cdot\gamma > 0$, the solutions given in Theorem 2.1 can be extended to any interval of time $[0,T]$.*

Remark 2.3. *For the case $\beta\cdot\gamma < 0$ we only establish the local theory because we cannot use the conserved quantity* (1.11) *directly to obtain an a priori estimate. It is not clear whether we can control $\|\partial_x^{-1}u(t)\|$ by $\|u(t)\|_{H^1}$. In this situation an a priori estimate could be established.*

Remark 2.4. *Our results improve by far the previous ones obtained in* [12] *where the only case treated was* $\beta, \gamma > 0$*.*

Remark 2.5. *Our local results seem to be far of being sharp if we compare them with the ones obtained for KdV equation* [8]*. On the other hand, since the symbol associated to equation* (1.2) *is not homogeneous, a scaling argument cannot be used to predict what the critical Sobolev space should be to prove local well-posedness. We also notice that when* $\beta < 0$ *and* $\gamma > 0$*, the symbol has a good algebraic property similar to the one of the KPII equation. This probably allows one to use the methods in* [1] *and* [8] *introduced to study the KdV equation. So far we have not been able to put these methods forward.*

Remark 2.6. *Notice that our results here imply the positive stability results in* [13] *for data in* X_1 *close to the solitary waves.*

3. Linear Estimates

Next we consider the linear problem

$$\begin{cases} \partial_t u - \beta\partial_x^3 u - \gamma\partial_x^{-1} u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{3.16}$$

whose solution is given by

$$u(x,t) = \mathcal{U}_{\beta,\gamma}(t)u_0(x) = c\int_{-\infty}^{\infty} e^{ix\xi + it\phi_{\beta,\gamma}(\xi)}\hat{u}_0(\xi)\,d\xi = I_{\beta,\gamma}(t) * u_0(x) \tag{3.17}$$

where $\phi_{\beta,\gamma}(\xi) = -\beta\xi^3 - \gamma\xi^{-1}$ and

$$I_{\beta,\gamma}(t,x) = c\int_{\mathbb{R}-\{0\}} e^{i(t\phi_{\beta,\gamma}(\xi)+x\xi)}d\xi.$$

Let us set $\phi_\pm(\xi) = \mp\xi^3 - \frac{1}{\xi}$ and $I^\pm(t,x) = c\int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)}d\xi$. It is not difficult to see that when $\beta\cdot\gamma > 0$,

$$I^{\beta,\gamma}(t,x) = |\gamma/\beta|^{1/4} I^+(t|\beta\gamma^3|^{1/4} sign(\gamma), x|\gamma/\beta|^{1/4}) \tag{3.18}$$

and that an analogous result also holds when $\beta\cdot\gamma < 0$ with I^- instead of I^+. This allows us to simplify the analysis and take $\beta = \pm 1$ and $\gamma = 1$, without loss of generality. Now we can define the corresponding solutions of (3.16) by

$$u(x,t) = \mathcal{U}_\pm(t)u_0(x) = I^\pm(t) * u_0(x). \tag{3.19}$$

Our first result regards the Strichartz estimates. To prove these estimate we rely in the theory of oscillatory integral established by Kenig, Ponce and Vega [5].

Lemma 3.1. *Let $f \in L^2(\mathbb{R})$. Then*

$$\|D_x^{\theta/4}\,\mathcal{U}_+(t)f\|_{L_t^qL_x^p} \le c\,\|f\|_{L^2} \tag{3.20}$$

where $q = \frac{4}{\theta}$ and $p = \frac{2}{1-\theta}$ with $\theta \in [0,1]$.

Let $f \in L^2(\mathbb{R})$, then

$$\Big(\int_0^T \|D_x^{1/4}\mathcal{U}_-(t)f\|_{L^\infty}^4\,dt\Big)^{1/4} \le c\,(1+T^{1/4})\|f\|_{L^2}. \tag{3.21}$$

Remark 3.2. *The estimate* (3.20) *is global in time in contrast with* (3.21)*. For our purposes the case $\theta = 1$ in* (3.20) *is enough.*

Proof. The symbol $\phi_+(\xi)$ is the general class defined in [5]. Thus Theorem 2.1 in [5] implies that

$$\|\int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)}|6\xi + \frac{2}{\xi^3}|^{\theta/4}\,d\xi\|_{L_t^qL_x^p} \le c\,\|f\|_{L^2}. \tag{3.22}$$

On the other hand, noticing that

$$|\phi''(x)| = \frac{2}{|\xi|^3}(3\xi^4+1) \ge 6|\xi|$$

the estimate (3.20) follows.

To prove (3.21) we first notice that the symbol $\phi_-(\xi)$ is also in the general class defined in [5]. Thus

$$\|\int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\,|\phi_-''(\xi)|^{\theta/4}\,\widehat{f}(\xi)\,d\xi\|_{L_t^q(\mathbb{R}:L^p)} \le c\|f\|_{L^2} \tag{3.23}$$

where $\theta \in [0,1]$, $p = \frac{2}{1-\theta}$ and $q = \frac{4}{\theta}$.

Next we take $\psi \in C_0^\infty(\mathbb{R})$ a cut-off function, i.e. $\psi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ if $|x| \le 1$ and $\varphi \equiv 0$ if $|x| \ge 2$, and write

$$\begin{aligned} D_x^{1/4}\mathcal{U}_-(t)f(\xi) &= \int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}|\xi|^{1/4}\,\widehat{f}(\xi)\psi(\xi)\,d\xi \\ &\quad + \int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}|\xi|^{1/4}\,\widehat{f}(\xi)(1-\psi(\xi))\,d\xi. \end{aligned} \tag{3.24}$$

Using the Sobolev lemma and the regularity of ψ we have

$$\begin{aligned}&\Big(\int_0^T \| \int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\, |\xi|^{1/4}\psi(\xi)\widehat{f}(\xi)\, d\xi\|_{L^\infty}^4\, dt\Big)^{1/4}\\ &\leq c\,T^{1/4}\|D_x^{1/4}(f*\overset{\vee}{\psi})\|_{H^1} \leq c\,T^{1/4}\|f\|_{L^2}.\end{aligned} \tag{3.25}$$

On the other hand, inequality (3.23) with $\theta = 1$ implies that

$$\begin{aligned}&\Big(\int_0^T \| \int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\, |\xi|^{1/4}(1-\psi(\xi))\widehat{f}(\xi)\, d\xi\|_{L^\infty}^4\, dt\Big)^{1/4}\\ &= \Big(\int_0^T \| \int_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\, |\phi_-''(\xi)|^{1/4}\frac{|\xi|^{1/4}}{|\phi_-''(\xi)|^{1/4}}(1-\psi(\xi))\widehat{f}(\xi)\, d\xi\|_{L^\infty}^4\, dt\Big)^{1/4}\\ &\leq c\,\|\frac{|\xi|^{1/4}(1-\psi(\xi))}{|\phi_-''(\xi)|^{1/4}}\widehat{f}(\xi)\|_{L^2}\\ &\leq c\|f\|_{L^2}\end{aligned} \tag{3.26}$$

where we have used that $\dfrac{|\xi|^{1/4}(1-\psi(\xi))}{|\phi_-''(\xi)|^{1/4}} \in L^\infty$. Combining (3.24), (3.25) and (3.26) the proof is complete. □

The smoothing effects of Kato's type for solutions of (3.16) are given in the following lemma.

Lemma 3.3. *Solutions of the linear problem* (3.16) *satisfy*

$$\|\partial_x \mathcal{U}_-(t)f\|_{L_x^\infty L_T^2} \leq \|f\|_{L^2} \tag{3.27}$$

and

$$\|\partial_x \mathcal{U}_+(t)f\|_{L_x^\infty L_T^2} \leq c\,(1+T^{1/2})\|f\|_{L^2}. \tag{3.28}$$

Proof. The inequality (3.27) follows from Theorem 4.1 in [5].

To prove (3.28), let $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ if $|x| \leq 1$ and $\varphi \equiv 0$ if $|x| \geq 2$. Then we can write

$$\begin{aligned}\partial_x \mathcal{U}_+(t)f &= \int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)} i\xi\hat{f}(\xi)\, d\xi\\ &= \int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)}\, i\xi\hat{f}(\xi)\varphi(\xi)\, d\xi + \int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)}\, i\xi\hat{f}(\xi)(1-\varphi(\xi))\, d\xi.\end{aligned}$$

Thus

$$\sup_x\Big(\int_0^T |\partial_x \mathcal{U}_+(t)f(x)|^2\,dt\Big)^{1/2} \le \sup_x\Big(\int_0^T |\int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)}\,i\xi\hat{f}(\xi)\varphi(\xi)\,d\xi|^2\,dt\Big)^{1/2}$$
$$+\sup_x\Big(\int_0^T |\int_{-\infty}^{\infty} e^{ix\xi+it\phi_+(\xi)}\,i\xi\hat{f}(\xi)(1-\varphi(\xi))\,d\xi|^2\,dt\Big)^{1/2}$$
$$= I_1 + I_2. \tag{3.29}$$

Sobolev's lemma and the regularity of φ gives us

$$I_1 \le c\,T^{1/2}\|\partial_x f * \varphi\|_{H^1} \le c\,T^{1/2}\|f\|_{L^2}. \tag{3.30}$$

On the other hand, since $\phi_+' \neq 0$ for $|\xi| \ge 1$, we can apply Theorem 4.1 in [5] to obtain

$$I_2 \le \int_{-\infty}^{\infty} \frac{|\xi|^2|\hat{f}(\xi)(1-\varphi(\xi))|^2}{|\phi_+'(\xi)|}\,d\xi \le c\|f\|_{L^2}. \tag{3.31}$$

Combining (3.29), (3.30) and (3.31) the result follows. □

Next we establish maximal function like estimates for the solutions of (3.16).

Lemma 3.4. *Let $f \in H^{s_2}(\mathbb{R}) \cap \dot{H}^{-s_1}(\mathbb{R})$, $s_2 > 3/4$ and $s_1 > 1/4$. Then*

$$\|\mathcal{U}_\pm(t)f\|_{L^2_x L^\infty_T} \le c(1+T)^{1/2}(\|f\|_{\dot{H}^{-s_1}} + \|f\|_{H^{s_2}}). \tag{3.32}$$

Proof. We will argue as in [4] and [6].

Consider the following open covering of $\mathbb{R} - \{0\}$,

$$\Omega_k = (-2^{k+1}, -2^{k-1}) \cup (2^{k-1}, 2^{k+1}), k \in \mathbb{Z},$$

and a subordinated partition of unity $\{\varphi_k\}_{k=-\infty}^{\infty}$ and let

$$I_k^\pm(t,x) = c\int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)}\varphi_k(\xi)d\xi. \tag{3.33}$$

We will prove that for any $k \in \mathbb{Z}$, there exists a function $H_k^\pm \in L^1(\mathbb{R})$ satisfying

$$|I_k^\pm(t,x)| \le H_k^\pm(x), \tag{3.34}$$

for any $x \in \mathbb{R}$ and $|t| \le T$ and such that

$$\|H_k^\pm\|_{L^1(\mathbb{R})} \le c\begin{cases} (1+T)^{1/2}2^{3k/2} & , k \ge 1 \\ (1+T) & , -1 \le k \le 0 \\ (1+T)^{1/2}2^{-k/2} & , k \le -2. \end{cases}$$

To do that, let us take $t \in [-T, T]$. We shall consider three different cases.

3.1. **Case 1: $k \geq 1$.** If $\xi \in \Omega_k$ then $|\phi_\pm'(\xi)| \leq 6 \cdot 2^{2k}$. Then for $|x| > 12 \cdot 2^{2k}T$, we have $|t\phi_\pm'(\xi) + x| > \frac{1}{2}|x| > \frac{|x|}{3}$.

Assume that $12 \cdot 2^{2k}T > 1$ and let us consider a function $h \in C^\infty(\mathbb{R})$ such that supp$h \subset \left\{\xi : |t\phi_\pm'(\xi) + x| \leq \frac{|x|}{2}\right\}$ and that equals one in $\left\{\xi : |t\phi_\pm'(\xi) + x| \leq \frac{|x|}{3}\right\}$. Performing two integrations by parts and using the remarks above we obtain that when $|x| > 12 \cdot 2^{2k}T$,

$$\left| \int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)} \varphi_k(\xi)(1 - h(\xi)) d\xi \right| \leq c \frac{2^k}{|x|^2}. \tag{3.35}$$

If $\xi \in \Omega_k \cap \left\{\xi : |t\phi_\pm'(\xi) + x| \leq \frac{|x|}{2}\right\}$, we have that

$$\begin{aligned} |t\phi_\pm''(\xi)| &= \frac{2|t|}{|\xi|} |3\xi^2 \pm \frac{1}{\xi^2}| \\ &\geq \frac{1}{|\xi|} |t\phi_\pm'(\xi)| \\ &\geq |x| \, 2^{-k}, \end{aligned} \tag{3.36}$$

in the second line we have used that if $|\xi| \geq 1$ then

$$3\xi^4 - 1 \geq \frac{1}{2}(3\xi^4 + 1). \tag{3.37}$$

Now we can use Van der Corput's lemma to get,

$$\left| \int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)} \varphi_k(\xi) h(\xi) d\xi \right| \leq c \frac{2^{k/2}}{|x|^{1/2}}. \tag{3.38}$$

Thus, the lemma follows by choosing

$$H_k^\pm(x) = \begin{cases} 2^k & , |x| \leq 1 \\ \frac{2^k}{|x|^2} + \frac{2^{k/2}}{|x|^{1/2}} & , 1 < |x| \leq 12 \cdot 2^{2k}T \\ \frac{2^k}{|x|^2} & , |x| > 12 \cdot 2^{2k}T, \end{cases}$$

when $12 \cdot 2^{2k}T > 1$ and

$$H_k^\pm(x) = \begin{cases} 2^k & , |x| \leq 1 \\ \frac{2^k}{|x|^2} & , |x| > 1, \end{cases}$$

otherwise. In the first case we have $\|H_k^\pm\|_{L^1} \leq c(1+T)^{1/2} 2^{3k/2}$ and in the second $\|H_k^\pm\|_{L^1} \leq c2^k$.

3.2. **Case 2:** $k \le -2$**.** Now $|\phi'_{\pm}(\xi)| \le 5 \cdot 2^{-2k}$ if $\xi \in \Omega_k$.

To estimate I_k^-, we do not have inequalities such as in (3.36), but since for $\xi \in \Omega_k$ it holds $|\xi| \le \frac{1}{\sqrt{3}} \le \frac{1}{3^{1/4}}$ we can use the following calculations,

$$\begin{aligned} |\phi''_{-}(\xi)| &= \frac{2}{|\xi|^3} - 6|\xi| \\ &\ge \frac{1}{|\xi|^3} + 3|\xi| \\ &= |\phi'_{-}(\xi)|. \end{aligned}$$

Similarly as in the previous case we obtain

$$H_k^{\pm}(x) = \begin{cases} 2^k & , |x| \le 1 \\ \frac{2^k}{|x|^2} + \frac{2^{k/2}}{|x|^{1/2}} & , 1 < |x| \le 10 \cdot 2^{-2k}T \\ \frac{2^k}{|x|^2} & , |x| > 10 \cdot 2^{-2k}T, \end{cases}$$

and $\|H_k^{\pm}\|_{L^1(\mathbb{R})} \le c(1+T)^{1/2}2^{-k/2}$ when $10 \cdot 2^{-2k}T > 1$, and

$$H_k^{\pm}(x) = \begin{cases} 2^k & , |x| \le 1 \\ \frac{2^k}{|x|^2} & , |x| > 1, \end{cases}$$

with $\|H_k^{\pm}\|_{L^1(\mathbb{R})} \le c2^k$ otherwise.

3.3. **Case 3:** $-1 \le k \le 0$**.** In this case we have that $|\phi'_{\pm}(\xi)| \le 20$ and we can take

$$H_k^{\pm}(x) = \begin{cases} 1 & , |x| \le 40T \\ \frac{1}{|x|^2} & , |x| > 40T, \end{cases}$$

and $\|H_k^{\pm}\|_{L^1(\mathbb{R})} \le c(1+T)$ when $40T > 1$, and

$$H_k^{\pm}(x) = \begin{cases} 1 & , |x| \le 1 \\ \frac{1}{|x|^2} & , |x| > 1, \end{cases}$$

with $\|H_k^{\pm}\|_{L^1(\mathbb{R})} \le c$ otherwise.

Now that (3.34) has been established we can proceed as in [4] and [6] to obtain the desired result. □

Lemma 3.5. *For any* $f \in H^s(\mathbb{R})$*,* $s > 1/2$*,*

$$\partial_x^{-1}(f\partial_x f) = \frac{1}{2}f^2. \tag{3.39}$$

Proof. Suppose that $f, g \in \mathcal{S}(\mathbb{R})$. Then $\partial_x(fg) = \partial_x fg + f\partial_x g$ and consequently

$$\partial_x^{-1}(\partial_x fg + f\partial_x g) = fg$$

and

$$\|\partial_x^{-1}(\partial_x fg + f\partial_x g)\|_{L^2} = \|fg\|_{L^2} \le \|f\|_s\|g\|_s$$

for any $s > 1/2$. Therefore $\partial_x^{-1}(\partial_x f g + f\partial_x g)$ defines a continuous bilinear form from H^s into L^2. Hence, it can be uniquely continuously extended over H^s and then we get that $\partial_x^{-1}(f\partial_x f) = \frac{1}{2}f^2$ for any $f \in H^s$, $s > 1/2$. □

Remark 3.6. *Observe that for $f \in X_s$ we have*

$$\|f\|_{\dot{H}^{-s_1}} \le c\|f\|_{X_s}, \quad s_1 \le 1. \tag{3.40}$$

To complement the set of linear estimates we will use the Leibniz's rule for fractional derivatives established by Kenig, Ponce and Vega [7].

Lemma 3.7 (Leibniz' rule)**.** *Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Let $p, q, p_1, p_2, q_2 \in (1,\infty)$, $q_1 \in (1,\infty)$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \le c\,\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{3.41}$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

4. Local Theory

We first give the proof of Theorem 2.1 when $\beta \cdot \gamma > 0$.

Proof of Theorem 2.1. We define the space

$$\mathcal{B}_T^a = \{v \in C([0,T] : X_s(\mathbb{R})) : |||v||| \le a\} \tag{4.42}$$

where

$$|||v||| := \|v\|_{L_T^\infty H^s(\mathbb{R})} + \|\partial_x^{-1} v\|_{L_T^\infty L_x^2} + \|v\|_{L_x^2 L_T^\infty} + \|\partial_x v\|_{L_T^4 L_x^\infty} + \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} \tag{4.43}$$

and the operator

$$\Psi(v) = \Psi_{u_0}(v) = \mathcal{U}_+(t)u_0 + \int_0^t \mathcal{U}_+(t-t')(vv_x)(t')\,dt. \tag{4.44}$$

We will show that for a and T suitable positive numbers the map Ψ defines a contraction in $\mathcal{B}_T^a$.

We first estimate the H^s-norm of $\Psi(u)(t)$. So, the Minkowski inequality group properties and the Hölder inequality give

$$\begin{aligned} \|\Psi(u)(t)\|_{L^2} &\le c\,\|u_0\|_{L^2} + \int_0^T \|uu_x(t)\|_{L^2}\,dt \\ &\le c\,\|u_0\|_{L^2} + c\,T^{3/4} \sup_{[0,T]} \|u\|_{L^2} \|u_x\|_{L_T^4 L_x^\infty}. \end{aligned} \tag{4.45}$$

On the other hand, using Cauchy-Schwarz's inequality and Leibniz's rule (3.41) we have that

$$
\begin{aligned}
\|D_x^s\Psi(u)(t)\|_{L^2} &\le \|D_x^s u_0\|_{L^2} + \|D_x^s \int_0^t \mathcal{U}_+(t-t')(uu_x)(t')\,dt'\|_{L^2}\\
&\le \|D_x^s u_0\|_{L^2} + cT^{1/2}\|D_x^s(uu_x)\|_{L^2_xL^2_T}\\
&\le \|D_x^s u_0\|_{L^2} + cT^{1/2}(\|D_x^s u\|_{L^4_TL^2_x}\|u_x\|_{L^4_TL^\infty_x} + \|D_x^s u_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T})\\
&\le \|D_x^s u_0\|_{L^2} + cT^{3/4}\|D_x^s u\|_{L^\infty_TL^2_x}\|u_x\|_{L^4_TL^\infty_x} + cT^{1/2}\|D_x^s u_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T}.
\end{aligned}
\tag{4.46}
$$

Combining (4.45) and (4.46) we have

$$
\begin{aligned}
\sup_{[0,T]}\|\Psi(u)(t)\|_{H^s} \le c\|u_0\|_{H^s} &+ c\,T^{3/4}\|u\|_{L^\infty_TH^s_x}\|u_x\|_{L^4_TL^\infty_x}\\
&+ c\,T^{1/2}\|D_x^s u_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T}.
\end{aligned}
\tag{4.47}
$$

Next by using the definition of X_s, Minkowski's inequality, Plancherel's indentity and Lemma 3.5 we deduce

$$
\begin{aligned}
\|\partial_x^{-1}\Psi(u)(t)\|_{L^2_x} &\le \|u_0\|_{X_s} + \int_0^T \|\partial_x^{-1}(uu_x)\|_{L^2}\,dt\\
&\le \|u_0\|_{X_s} + cT\|u\|^2_{L^\infty_TH^s}.
\end{aligned}
\tag{4.48}
$$

By group properties, Lemma 3.4, remark 3.6, Lemma 3.5 and the argument used in (4.47) we have that

$$
\begin{aligned}
\|\Psi(u)\|_{L^2_xL^\infty_T} &\le c(1+T)^{1/4}\|u_0\|_{X_s} + \|\mathcal{U}_+(t)\big(\int_0^t \mathcal{U}_+(-t)(uu_x)(t')\,dt'\big)\|_{L^2_xL^\infty_T}\\
&\le c(1+T)^{1/2}\|u_0\|_{X_s} + c(1+T)^{1/2}\|\int_0^T \mathcal{U}_+(-t)(uu_x)(t')\,dt'\|_{X_s}\\
&\le c(1+T)^{1/2}\|u_0\|_{X_s} + c(1+T)^{1/2}\Big(\int_0^T \|\partial_x^{-1}(uu_x)\|_{L^2} + T^{1/2}\,\|uu_x\|_{H^s_xL^2_T}\Big)\\
&\le c(1+T)^{1/2}\|u_0\|_{X_s} + cT(1+T)^{1/2}\|u\|^2_{L^\infty_TH^s_x}\\
&\quad + c(1+T)^{1/2}\Big(T^{3/4}\|u\|_{L^\infty_TH^s_x}\|u_x\|_{L^4_TL^\infty_x} + T^{1/2}\|D_x^s u_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T}\Big).
\end{aligned}
\tag{4.49}
$$

Inequality (3.20) in Lemma 3.1, Cauchy-Schwarz's inequality and the argument in (4.47) give

$$\begin{aligned}\|\partial_x\Psi(u)\|_{L^4_TL^\infty_x} &\le \|D_x^{1/4}\mathcal{U}_+(t)D_x^{3/4}u_0\|_{L^2}\\&\quad + \|D_x^{1/4}\mathcal{U}_+(t)\Big(\int_0^T \mathcal{U}_+(-t')D_x^{3/4}(uu_x)(t')\,dt'\Big)\|_{L^4_TL^\infty_x}\\&\le c\,\|u_0\|_{H^s} + c\,T^{1/2}\|uu_x\|_{H^s_xL^2_T}\\&\le c\,\|u_0\|_{H^s} + cT^{3/4}\,\|u\|_{L^\infty_TH^s_x}\|u_x\|_{L^4_TL^\infty_x} + c\,T^{1/2}\|D^s_xu_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T}.\end{aligned} \tag{4.50}$$

Finally, Minkowski's inequality, Lemma 3.3, Cauchy-Schwarz's inequality and (4.46) yield

$$\begin{aligned}\|D^s_x\partial_x\Psi(u)\|_{L^\infty_xL^2_T} &\le c(1+T^{1/2})\|D^s_xu_0\|_{L^2} + c\,(1+T^{1/2})T^{1/2}\|D^s_x(uu_x)\|_{L^2_TL^2_x}\\&\le c\,(1+T^{1/2})\|u_0\|_{H^s} + c\,T^{3/4}(1+T^{1/2})\|u\|_{L^\infty_TH^s_x}\|u_x\|_{L^4_TL^\infty_x}\\&\quad + c\,T^{1/2}(1+T^{1/2})\|D^s_xu_x\|_{L^\infty_xL^2_T}\|u\|_{L^2_xL^\infty_T}.\end{aligned} \tag{4.51}$$

From (4.47)–(4.51) we obtain

$$\begin{aligned}\|\Psi(u)\|_{L^\infty_TH^s_x} &\le c\,\|u_0\|_{X_s} + c\,T^{1/2}(1+T^{1/4})|\!|\!|u|\!|\!|^2,\\ \|\partial_x^{-1}\Psi(u)\|_{L^\infty_TL^2_x} &\le c\,\|u_0\|_{X_s} + c\,T|\!|\!|u|\!|\!|^2,\\ \|\Psi(u)\|_{L^2_xL^\infty_T} &\le c(1+T)^{1/2}\|u_0\|_{X_s} + c\,T^{1/2}(1+T)^{1/2}(1+T^{1/4}+T^{1/2})|\!|\!|u|\!|\!|^2,\\ \|\partial_x\Psi(u)\|_{L^4_TL^\infty_x} &\le c\,\|u_0\|_{X_s} + c\,T^{1/2}(1+T^{1/4})|\!|\!|u|\!|\!|^2,\end{aligned}$$

and

$$\|D^s_x\partial_x\Psi(u)\|_{L^\infty_xL^2_T} \le c(1+T^{1/2})\|u_0\|_{X_s} + c\,T^{1/2}(1+T^{1/4})(1+T^{1/2})|\!|\!|u|\!|\!|^2.$$

Hence choosing $a = 2c\,(1+T)^{1/2}\,\|u_0\|_{H^s}$ and $T>0$ such that

$$c\,T^{1/2}(1+T)^{1/2}(1+T^{1/4}+T^{1/2})a \ll \frac{1}{2} \tag{4.52}$$

we have that $\Psi : \mathcal{B}^a_T \to \mathcal{B}^a_T$. The same argument shows that Ψ is a contraction in $\mathcal{B}^a_T$. Thus the contraction mapping principle guarantees the existence of a unique u in $\mathcal{B}^a_T$ solving the integral equation (4.44). To show the continuous dependence we follow a similar argument as the one described above. The uniqueness into the space

$$\mathcal{B}_T = \{v \in C([0,T] : X_s(\mathbb{R})) : |\!|\!|v|\!|\!| < \infty\} \tag{4.53}$$

follows using a standard argument so we will omit it.

To complete the proof of Theorem 2.1 we need to consider the case $\beta\cdot\gamma < 0$. This situation uses a similar argument as the one given in the proof of the case $\beta\cdot\gamma > 0$. The only difference resides in the estimate (4.50) where we use the inequality (3.21) instead of (3.20). Thus we will omit the details and so the proof is finished. □

5. Global Theory

In this section we will extend the local solutions obtained in Theorem 2.1, globally in time. To achieve our goal we will use the conserved quantities (1.10) and (1.11).

Formally, the identities (1.10) and (1.11) can be obtained multiplying by u, integrating by parts and using the antisymmetry of the operator ∂_x^{-1}. To justify this procedure we need to use the results in Theorem 2.1 when s is sufficiently large and the continuous dependence. In fact, let $u_0 \in X_1(\mathbb{R})$ and $\{u_{0j}\} \in X_s(\mathbb{R})$, s sufficiently large, such that

$$\|u_{0j} - u_0\|_{H^s} + \|\partial_x^{-1}(u_{0j} - u_0)\|_{L^2} \to 0 \quad \text{as} \quad j \to \infty.$$

Now let u_j be the solutions with initial data $u_j(\cdot, 0) = u_{0j}(\cdot)$. By Theorem 2.1 u_j exists in $[0, T]$ for sufficiently large j and $u_j \to u$ in $C([0, T] : X_1(\mathbb{R}))$.

The identity (1.10) can be easily justified for $u_j(t) \in X_s$ if s is sufficiently large. To obtain (1.11) for regular initial data, we can proceed as was made in [9] to justify the conservation of the energy for KP I. They used an exterior regularization of the equation by a sequence of smooth functions that cut the low frequencies and a limiting argument.

Once we have (1.10) and (1.11) for regular initial data, we let $j \to \infty$ and noting that $u_j(t) \to u(t)$ in $X_1(\mathbb{R})$, we obtain these identities also for u.

Now we can use (1.10) and (1.11) to obtain an *a priori* estimate in $X_1(\mathbb{R})$.

Observe that (1.10) and a Young's inequality type inequality imply that

$$\begin{aligned} \int_{-\infty}^{\infty} u^3(x,t)\,dx &\le \|u(t)\|_{L^\infty}\|u_0\|_{L^2}^2 \le \sqrt{2}\,\|u_0\|_{L^2}^{5/2}\|\partial_x u(t)\|_{L^2}^{1/2} \\ &\le \frac{\beta}{2}\,\|\partial_x u(t)\|_{L^2}^2 + C. \end{aligned} \tag{5.54}$$

On the other hand, the identity (1.11) and (5.54) yield

$$\begin{aligned} \beta\,\|\partial_x u(t)\|_{L^2}^2 + \frac{\gamma}{2}\,\|\partial_x^{-1}u(t)\|_{L^2}^2 &= I_2(u_0) - \frac{1}{3}\,\|u(t)\|_{L^3}^3 \\ &\le |I_2(u_0)| + \frac{\beta}{2}\,\|\partial_x u(t)\|_{L^2}^2 + C. \end{aligned} \tag{5.55}$$

Therefore, for any β and γ positive

$$\beta\,\|\partial_x u(t)\|_{L^2}^2 + \gamma\,\|\partial_x^{-1}u(t)\|_{L^2}^2 \le C. \tag{5.56}$$

Thus we obtain an *a priori* estimate in X_1 and we can reapply Theorem 2.1 to extend the solutions. This shows Theorem 2.2.

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