A new condition characterizing solutions of variational inequality problems.

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Abstract

This paper is devoted to the study of a new alternative necessary condition in variational inequality problems: AGP. A feasible point satisfies such condition if it is the limit of a sequence of approximated solutions of approximations of the variational problem. This condition comes from optimization where the error in the approximated solution is measured by the projected gradient onto the approximated feasible set, which is obtained from a linearization of the restrictions with slacks to make the current point feasible. Thus, coined as Approximated Gradient Projection (AGP) condition.

We state the AGP condition for variational inequality problems and show that it is necessary for a point being a solution even without constraint qualifications (e.g., Abadie's). Moreover, the AGP condition is sufficient in continuous variational inequalities. Sufficiency also holds for variational inequalities involving maximal monotone operators provided boundedness of the vectors in the image of the operator (playing the roll of the gradients). Since AGP is a condition verified by a sequence, it is particularly interesting for iterative methods.

Keywords: optimality conditions, variational inequalities, approximated solutions, polyhedral approximations, approximated gradient projection.

1 Introduction

Constrains qualifications guarantee that optimal solutions of constrained optimization problems satisfy Karush-Kuhn-Tucker conditions. If this happens, these problems can be solved, in principle, via KKT conditions. Constrains qualifications also give sufficient conditions for using the

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linearization of the inequalities that define the feasible set to obtain the normal cone and the tangent cone of any feasible point. Unfortunately, important problems fail to satisfy constraint qualifications. In addition, the solution of non-linear optimization problems, even under constraint qualifications, very often is not generated by direct solution of the KKT system. Instead, an iterative procedure generates a sequence which, under suitable conditions, converges to the solution. Therefore, it seems relevant to investigate conditions on the (generated) sequence which guarantee that the limit points are optimal or at least stationary. These conditions shall also be related with the algorithms in use or shall be easily verified. Moreover, it shall be possible to converge to a solution via sequences which satisfy such conditions.

In the context of inexact restoration methods, the authors of [3, 4] observed that the generated sequences satisfied a condition which, latter on [5], was suited in the setting of optimization and coined "AGP". This condition is more general than KKT, is easily verifiable by sequences generated by inexact restoration methods and, in convex programming, is a necessary and sufficient condition for optimality.

The purpose of this paper is to extend the AGP condition to Variational Inequalities. The basic tools for this extension are the AGP condition for optimization problems and an external penalty principle for variational inequalities involving maximal monotone operators.

This paper is organized as follows. In Section 2 the AGP condition for optimization is presented and its properties are revised. Section 3 is devoted to the case of variational inequalities associated to a continuous function establishing necessity and sufficiency of the AGP condition, without constraints qualifications, for existence of solutions. In Section 4 we recall the external penalty principle for variational inequalities involving maximal monotone operators and state the AGP condition for such problems proving that, even in the absence of constraint qualifications, this is a necessary condition for a point being a solution. Moreover sufficiency is also investigated.

2 AGP for optimization

Consider the optimization problem

$$\min f(x), \quad x \in \Omega \tag{1}$$

where

$$\Omega := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \ i = 1, \dots, p\}. \tag{2}$$

and $f, g_1, \cdots, g_p : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. In principle, the functions g_i, $i = 1, \cdots, p$, may be non-convex.

For $x \in \mathbb{R}^n$ and $\gamma \in [-\infty, 0]$ define (as in [5])

$$\Omega(x, \gamma) := \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{rcl} g_i(x) + \nabla g_i(x)^t (z - x) & \leq & 0 \text{ if } \gamma < g_i(x) < 0, \\ \nabla g_i(x)^t (z - x) & \leq & 0 \text{ if } 0 \leq g_i(x). \end{array} \right\} \tag{3}$$

The set $\Omega(x, \gamma)$ is a "polyhedral approximation" of Ω, thus closed and convex, with some other interesting properties. We use the notation $P_C(y)$ for the orthogonal projection of y onto C, for

$y \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ nonempty, closed and convex. The positive part of a scalar $t \in R$ will be represented by t_+, i.e., $t_+ = \max\{0, t\}$.

Proposition 2.1. *Let $x \in \mathbb{R}^n$ and $\gamma \in [-\infty, 0]$. Then $x \in \Omega(x, \gamma)$ and, for any $\mu > 0$,*

$$x = P_{\Omega(x,\gamma)}\left(x + \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x)\right).$$

In addition, if $g_1, \cdots, g_p$ are convex, then $\Omega \subseteq \Omega(x, \gamma)$.

Proof. The result concerning the projection is established in the proof of Theorem 2.1 of [5]. $\square$

So, $\Omega(x, \gamma)$ is always nonempty. Now, for $x, v \in \mathbb{R}^n$ and $\gamma \in [-\infty, 0]$ define (also in the spirit of [5]),

$$\rho(x, v, \gamma) := P_{\Omega(x,\gamma)}(x - v) - x. \tag{4}$$

If x^* is a solution of problem (1)-(2) then, *provided* some constraint qualification,

$$\rho(x^*, \nabla f(x^*), \gamma) = 0.$$

An alternative condition is needed to deal with problems without constraint qualification. A point $x^* \in \Omega$ satisfies the *Approximate Gradient Condition* (AGP for short) if for any $\gamma \in [-\infty, 0]$, $\epsilon,\ \delta > 0$ there exist $x \in \mathbb{R}^n$ such that

$$\|x - x^*\| < \delta \qquad \text{and} \qquad \|\rho(x, \nabla f(x), \gamma)\| < \epsilon. \tag{5}$$

Theorem 2.2. *If x^* is a local minimizer of (1)-(2) then it satisfies the AGP condition (5). Consequently, there exists a sequence $\{x^k\}_k \in \mathbb{R}^n$ with*

$$\lim_{k\to\infty} x^k = x^* \qquad \textit{and} \qquad \lim_{k\to\infty} \rho(x^k, \nabla f(x^k), \gamma) = 0. \tag{6}$$

Moreover, the converse is true in the convex case: if f and g_i, $i = 1, \ldots, p$ are convex and for some $\gamma \in [-\infty, 0]$ and $x^* \in \Omega$ it holds that there exists a sequence $\{x^k\}_k \in \mathbb{R}^n$ satisfying (6) then x^* is a minimizer of (1)-(2).

Proof. See [5, Theorem 2.1, Corollary 2.1 and Theorem 3.1]. $\square$

3 AGP for continuous variational inequality problems

In this section we shall extend the AGP condition to variational inequality problems. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator and $\Omega \subseteq \mathbb{R}^n$, a closed set defined as in (2) with the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, continuously differentiable.

$$\Omega := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ \ i = 1, \ldots, p\}.$$

The variational inequality problem $VIP(F,\Omega)$ is to find x^* such that

$$x^* \in \Omega, \quad \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{7}$$

A geometric interpretation of (7) is that $x^* \in \Omega$ is a solution of $VIP(F,\Omega)$ if and only if $F(x^*)$ forms a non-obtuse angle with every vector of the form $y - x^*$ with $y \in \Omega$. That is, $-F(x^*)$ belongs to the *normal cone* to Ω at x^*, $N_\Omega(x^*)$. Recall that the normal cone operator of a closed set $C \subseteq \mathbb{R}^n$ is defined as

$$N_C(x) := \begin{cases} \{v \in \mathbb{R}^n \mid \forall z \in C, v^t(x-z) \geq 0\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

When C is convex, the normal cone at any point $x \in C$ can be obtained throw the polar of the tangent cone of C at x, $T_C(x)$. In fact, in this case, (see [1, 1.3.2])

$$N_C(x) = -T_C(x)^*.$$

The *tangent cone* of C at $x \in C$ is the set of the tangent directions to C at x. A vector $d \in \mathbb{R}^n$ is a *tangent direction* to C at x if there exist a sequence of vectors $\{v^k\} \subset C$, converging to x, and a sequence of positive scalars $\{t_k\}$, converging to zero, such that

$$\lim_{k\to\infty} \frac{v^k - x}{t_k} = d.$$

Recall also that Given a cone K the *dual* or (positive) *polar* cone of K, K^*, is given by

$$K^* = \{d \in \mathbb{R}^n \mid \langle d, y \rangle \geq 0, \quad \forall y \in K\}.$$

Concerning the "polyhedral approximation" of Ω, $\Omega(x,\gamma)$, observe that when x is feasible ($x \in \Omega$) and $\gamma = 0$, the set $\Omega(x,0)$ is exactly the "linearization" cone of Ω at x. That is,

$$\Omega(x,0) = \{z \in \mathbb{R}^n \mid \nabla g_i(x)^t(z-x) \leq 0 \text{ if } g_i(x) = 0\}.$$

Note that $T_\Omega(x) \subset \Omega(x,0)$, but the converse inclusion does not hold in general. When it is true we have the Abadie's constraint qualification at x. Thus, under Abadie's constraint qualification (CQ for short) the tangent cone is polyhedral, hence, under convexity, the normal cone is polyhedral too, which "simplify" the problem of solving the VIP. Even without convexity, Abadie's CQ allows for the extension of Karush-Kuhn-Tucker conditions as necessity conditions for variational inequalities. In fact (following [1, 1.3.4]) if x is a solution of $VIP(F,\Omega)$ and Abadie's CQ holds at x, then there exists a vector $\lambda \in \mathbb{R}^p$ such that:

$$0 = F(x) + \sum_{i=1}^p \lambda_i \nabla g_i(x) \tag{8}$$

$$0 \leq \lambda \perp g(x) \leq 0. \tag{9}$$

In this work we avoid assumptions under which $T_\Omega(x)$ becomes a polyhedral cone. Instead, we present an alternative condition (AGP) that is necessary for a point being a solution of the variational inequality problem even without constraint qualifications. In order to do so, we extend the AGP condition to the new setting replacing ∇f, in (5), by F. We say that a point $x^* \in \Omega$ satisfies the AGP condition for $VIP(F, \Omega)$ if for any $\gamma \in [-\infty, 0]$, $\epsilon,\ \delta > 0$ there exist $x \in \mathbb{R}^n$ such that

$$\|x - x^*\| < \delta \qquad \text{and} \qquad \|\rho(x, F(x), \gamma)\| < \epsilon. \tag{10}$$

Note that $-\rho(x, F(x), \gamma)$ is the *natural map* associated to $VIP(F, \Omega(x, \gamma))$. Thus, x is a solution of $VIP(F, \Omega(x, \gamma))$ if, and only if, $\rho(x, F(x), \gamma) = 0$ (see [1, 1.5.8]).

Theorem 3.1. *If x^* is a solution of $VIP(F, \Omega)$, (7), then it satisfies the AGP condition* (10).

Proof. Assume that x^* is a solution of $VIP(F, \Omega)$ and let $\gamma \in [-\infty, 0]$, $\epsilon,\ \delta > 0$. By continuity of F there exists $\tilde{\delta} = \delta(x^*, \epsilon) > 0$, $\tilde{\delta} \le \delta$, such that, for any $x \in \mathbb{R}^n$ with $||x - x^*|| < \tilde{\delta}$, it holds

$$||F(x) - F(x^*)|| < \frac{\epsilon}{2}. \tag{11}$$

Observe that x^* is a solution of $VIP(F, \Omega)$ if, an only if, for any $x \in \Omega$

$$\langle F(x^*), x\rangle \ge \langle F(x^*), x^*\rangle,$$

which is equivalent to

$$x^* \in \arg\min_{x\in\Omega} \langle F(x^*), x\rangle \tag{12}$$

Apply now the first part of Theorem 2.2 to the optimization problem (12) in order to get $x \in \mathbb{R}^n$ satisfying $||x - x^*|| < \tilde{\delta}$ and

$$||\rho(x, F(x^*), \gamma)|| < \frac{\epsilon}{2}. \tag{13}$$

Thus, using non expansiveness of the projection, we get

$$\begin{aligned}
||\rho(x, F(x), \gamma)|| &\le ||\rho(x, F(x), \gamma) - \rho(x, F(x^*), \gamma)|| + ||\rho(x, F(x^*), \gamma)|| \\
&= ||P_{\Omega(x,\gamma)}(x - F(x)) - P_{\Omega(x,\gamma)}(x - F(x^*))|| + ||\rho(x, F(x^*), \gamma)|| \\
&\le ||F(x) - F(x^*)|| + ||\rho(x, F(x^*), \gamma)|| < \epsilon,
\end{aligned}$$

where the last inequality follows from (11) and (13). $\square$

The converse of Theorem 3.1 is also true for feasible points under convexity. We shall prove it using the "sequential" formulation of the AGP condition.

Corollary 3.2. *Assume that $x^* \in \Omega$ and $\gamma \in [-\infty, 0]$. The following two statements are valid.*

1. *If x^* is a solution of* (7) *then there exists a sequence $\{x^k\}_k \in \mathbb{R}^n$ satisfying*
$$\lim_{k\to\infty} x^k = x^* \qquad \textit{and} \qquad \lim_{k\to\infty} \rho(x^k, F(x^k), \gamma) = 0. \tag{14}$$

2. *Conversely, if each function g_i is convex, $i = 1, \dots, p$ and there exists $\{x^k\}_k \in \mathbb{R}^n$ satisfying* (14)*, then x^* solves the $VIP(F, \Omega)$.*

Proof. Item (1.) follows directly from Theorem 3.1. For the converse, item (2.), note that by characterization of the orthogonal projection over $\Omega(x^k, \gamma)$

$$\left\langle \left(x^k - F(x^k)\right) - P_{\Omega(x^k,\gamma)}\left(x^k - F(x^k)\right), x - P_{\Omega(x^k,\gamma)}\left(x^k - F(x^k)\right)\right\rangle \le 0, \qquad \forall x \in \Omega(x^k, \gamma).$$

Since $\Omega \subset \Omega(x^k, \gamma)$ (see Proposition 2.1) and $P_{\Omega(x^k,\gamma)}(x^k - F(x^k)) = \rho(x^k, F(x^k), \gamma) + x^k$

$$\langle -F(x^k) - \rho(x^k, F(x^k), \gamma), x - (\rho(x^k, F(x^k), \gamma) + x^k)\rangle \le 0, \qquad \forall x \in \Omega.$$

Taking limits, as k goes to ∞, we get

$$\langle F(x^*), x - x^*)\rangle \ge 0, \qquad \forall x \in \Omega.$$

Hence, x^* is a solution. $\square$

4 AGP for the (set valued) monotone case

In this section we are concerned with variational inequality problems involving maximal monotone operators. A point-to-set operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is *monotone* if

$$v \in T(x), u \in T(y) \Rightarrow \langle v - u, x - y\rangle \ge 0,$$

for any $x, y \in \mathbb{R}^n$. Moreover, if there exists some $\alpha > 0$ such that

$$v \in T(x), u \in T(y) \Rightarrow \langle v - u, x - y\rangle \ge \alpha\|x - y\|^2,$$

for any $x, y \in \mathbb{R}^n$, then T is *strongly monotone.* A monotone operator T is *maximal monotone* if it admits no proper monotone extension, i.e.,

$$\left.\begin{array}{l} M : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n), \text{monotone} \\ \left(\forall x, T(x) \subseteq M(x)\right) \end{array}\right\} \Rightarrow T = M.$$

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be maximal monotone, possibly point-to-set, and $C \subset \mathbb{R}^n$ be a nonempty, closed and convex set. Problem $VIP(T, C)$ is to find x^* and v^* such that

$$x^* \in C,\ v^* \in T(x^*), \quad \langle v^*, x - x^*\rangle \ge 0, \quad \forall x \in C. \tag{15}$$

We considered the case where C is the finitely representable set Ω defined in (2), with the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, p$ convex and (continuously) differentiable.

Now we say that a point $x^* \in \Omega$ satisfies the AGP condition for $VIP(T, \Omega)$ if for any $\gamma \in [-\infty, 0]$, $\epsilon,\ \delta > 0$ there exist $x,\ v \in \mathbb{R}^n$ such that

$$\|x - x^*\| < \delta, \qquad v \in T(x), \qquad \text{and} \qquad \|\rho(x, v, \gamma)\| < \epsilon. \tag{16}$$

In order to prove necessity of the AGP condition in a set-valued VIP, we shall use an external penalization principle for such problems. The next theorem is a particular case of the results described by Gwinner as "The penalty method for constrained variational inequalities" [2, Section III.].

Theorem 4.1. *Let $M : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be strongly maximal monotone, and let x^* be the unique solution of $VIP(M, \Omega)$. Define, for $\mu > 0$, M_μ,*

$$M_\mu(x) := M(x) + \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x).$$

Then, for each $\mu > 0$, $VIP(M_\mu, \mathbb{R}^n)$ has a unique solution x_μ and

$$\lim_{\mu \to \infty} x_\mu = x^*.$$

Proof. Define $g_+ : \mathbb{R}^n \to \mathbb{R}^p$,

$$g_+(x) := (g_1(x)_+, \cdots, g_p(x)_+),$$

to get the penalty function

$$p(x) = \frac{1}{2}\|g_+(x)\|^2,$$

and apply Theorem 3.1 of [2] with its immediate consequences for finite dimension and a single valued solution set. □

Theorem 4.2. *Assume that x^* is a solution of $VIP(T, \Omega)$, defined in (15). Then x^* satisfies the AGP condition (16). Consequently, there exist sequences $\{x^k\}_k, \{v^k\}_k \subset \mathbb{R}^n$ with*

$$\lim_{k \to \infty} x^k = x^*, \qquad v^k \in T(x^k) \qquad \textit{and} \qquad \lim_{k \to \infty} \rho(x^k, v^k, \gamma) = 0. \tag{17}$$

Proof. Assume that x^* is a solution of $VIP(T, \Omega)$. Take $\gamma \in [-\infty, 0]$, $\epsilon > 0$ and $\delta > 0$ and let $\tilde{T} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be the operator defined, for any $x \in \mathbb{R}^n$, by

$$\tilde{T}(x) = T(x) + \epsilon(x - x^*).$$

Then $\tilde{T}$ is strongly maximal monotone and $VIP(\tilde{T}, \Omega)$ has the unique solution x^*. Thus, in view of Theorem 4.1, there exists $\mu = \mu(\delta) > 0$ such that the unique solution of $VIP(\tilde{T}_\mu, \mathbb{R}^n)$, $x_\mu \in \mathbb{R}^n$, satisfies $\|x_\mu - x^*\| < \min\{\delta, 1\}$. Choose $x = x_\mu$. Since x is the solution of $VIP(\tilde{T}_\mu, \mathbb{R}^n)$, there also exists $v \in T(x)$ with

$$0 = v + \epsilon(x - x^*) + \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x).$$

Consequently,

$$\begin{aligned}
\epsilon \quad &> \quad \epsilon\|x - x^*\| = \left\| -v - \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x) \right\| \\
&= \quad \left\| (x - v) - \left[x + \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x) \right] \right\| \\
&\geq \quad \left\| P_{\Omega(x,\gamma)}(x - v) - P_{\Omega(x,\gamma)} \left(x + \mu \sum_{i=1}^{n} g_i(x)_+ \nabla g_i(x) \right) \right\| \qquad (18) \\
&= \quad \|P_{\Omega(x,\gamma)}(x - v) - x\| = \|\rho(x, v, \gamma)\|. \qquad (19)
\end{aligned}$$

Note that non expansiveness of the orthogonal projection over $\Omega(x, \gamma)$ ensures inequality (18). The last equality (19) follows from definition of ρ, (4), and Proposition 2.1. $\square$

Remark 4.3. Observe that Theorem 4.2 does not ask for constraint qualifications. Indeed, even the usual assumption of maximal monotonicity of $T + N_\Omega$ is not required.

Theorem 4.4. *Assume that $\gamma \in [-\infty, 0]$ and $x^* \in \Omega$. If there exist bounded sequences $\{x^k\}_k, \{v^k\}_k \subset \mathbb{R}^n$ satisfying*

$$\lim_{k\to\infty} x^k = x^*, \qquad v^k \in T(x^k), \qquad \text{and} \qquad \lim_{k\to\infty} \rho(x^k, v^k, \gamma) = 0,$$

then x^* is a solution of (15).

Proof. The proof follows as in Theorem 3.2. Let $\gamma \in [-\infty, 0]$ and $x^* \in \Omega$. Assume that such sequences $\{x^k\}_k, \{v^k\}_k \subset \mathbb{R}^n$ are given. Let

$$\rho_k = \rho(x^k, v^k, \gamma) = P_{\Omega(x^k,\gamma)}(x^k - v^k) - x^k.$$

Then,

$$\lim_{k\to\infty} \rho_k = 0 \qquad \text{and} \qquad P_{\Omega(x^k,\gamma)}(x^k - v^k) = \rho_k + x^k \quad \stackrel{k\to\infty}{\longrightarrow} x^*.$$

Since $\{v^k\}$ is bounded, we can assume passing to a subsequence if necessary the existence of $v \in \mathbb{R}^n$ such that $v^k \stackrel{k\to\infty}{\longrightarrow} v$. Consequently, $v \in T(x^*)$. In fact, $v^k \in T(x^k)$ and $x^k \stackrel{k\to\infty}{\longrightarrow} x^*$ with T maximal monotone, hence demiclosed.

Now, by characterization of the orthogonal projection over $\Omega(x^k, \gamma)$, we can write

$$\langle x^k - v^k - P_{\Omega(x^k,\gamma)}(x^k - v^k), x - P_{\Omega(x^k,\gamma)}(x^k - v^k)\rangle \leq 0, \qquad \forall x \in \Omega(x^k, \gamma).$$

Thus,

$$\langle -v^k - \rho_k, x - (\rho_k + x^k)\rangle \leq 0, \qquad \forall x \in \Omega,$$

because definition of ρ_k and $\Omega \subset \Omega(x^k, \gamma)$ (see Proposition 2.1). Taking limits, as k goes to ∞, it follows

$$\langle v, x - x^*\rangle \geq 0, \qquad \forall x \in \Omega,$$

with $x^* \in \Omega$ and $v \in T(x^*)$. Hence, x^* is a solution of $VIP(T, \Omega)$. □

Concerning the hypothesis of boundedness of $\{v^k\}$, in the Theorem of sufficiency above 4.4, we should observe that defining the *domain* of a maximal monotone operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ as

$$D(T) := \{x \in \mathbb{R}^n | T(x) \neq \emptyset\},$$

then it is locally bounded at any point of the interior of $D(T)$, $[D(T)]^\circ$; but it is not, at any point of the boundary of $D(T)$ (see [6]). Local boundedness of T at x means (as defined in [6]) the existence of a neighborhood (open set) U of x such that

$$\cup_{u \in U} T(u) \qquad \text{is bounded.}$$

Corollary 4.5. *Assume that $\Omega \subseteq [D(T)]^\circ$. Then $x^* \in \Omega$ is a solution of $VIP(T, \Omega)$ if and only if there exist sequences $\{x^k\}_k, \{v^k\}_k \subset \mathbb{R}^n$ such that*

$$\lim_{k\to\infty} x^k = x^*, \qquad v^k \in T(x^k), \qquad \text{and} \qquad \lim_{k\to\infty} \rho(x^k, v^k, \gamma) = 0.$$

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