Completely positive inner products and strong Morita equivalence

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Abstract

We develop a general framework for the study of strong Morita equivalence in which C^*-algebras and hermitian star products on Poisson manifolds are treated in equal footing. We compare strong and ring-theoretic Morita equivalences in terms of their Picard groupoids for a certain class of unital *-algebras encompassing both examples. Within this class, we show that both notions of Morita equivalence induce the same equivalence relation but generally define different Picard groups. For star products, this difference is expressed geometrically in cohomological terms.

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1 Introduction

This paper investigates several similarities between two types of algebras with involution: hermitian star products on Poisson manifolds and C^*-algebras. Their connection is suggested by their common role as "quantum algebras" in mathematical physics, despite the fact that the former is a purely algebraic notion, whereas the latter has important analytical features. Building on [11, 12], we develop in this paper a framework for their unified study, focusing on Morita theory; in particular, the properties shared by C^*-algebras and star products allow us to develop a general theory of *strong* Morita equivalence in which they are treated in equal footing.

Our set-up is as follows. We consider *-algebras over rings of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where R is an ordered ring and $\mathrm{i}^2 = -1$. The main examples of R that we will have in mind are $\mathbb{R}$, with its natural ordering, and $\mathbb{R}[[\lambda]]$, with ordering induced by "asymptotic positivity", i.e., $a = \sum_{r=0}^{\infty} a_r \lambda^r > 0$ if and only if $a_{r_0} > 0$, where a_{r_0} is the first nonzero coefficient of a. This general framework encompasses complex *-algebras, such as C^*-algebras, as well as *-algebras over the ring of formal power series $\mathbb{C}[[\lambda]]$, such as hermitian star products. We remark that the case of *-algebras over $\mathbb{C}$ has been extensively studied, see e.g. [33], and [37] for a comparison of notions of positivity.

In our general framework, we define a purely algebraic notion of strong Morita equivalence. The key ingredient in this definition is the notion of *completely positive* inner products, which we use to refine Ara's *-Morita equivalence [1]. One of our main results is that completely positive inner products behave well under the internal and external tensor products, and, as a consequence, strong Morita equivalence defines an equivalence relation within the class of non-degenerate and idempotent *-algebras. This class of algebras includes both star products and C^*-algebras as examples. We prove that important constructions in the theory of C^*-algebras, such as Rieffel's induction of representations [30], carry over to this purely algebraic setting, recovering and improving many of our previous results [11, 12].

In the ordinary setting of unital rings, Morita equivalence coincides with the notion of isomorphism in the category whose objects are unital rings and morphisms are isomorphism classes of bimodules, composed via tensor product. The invertible arrows in this category form the *Picard groupoid* Pic [6], which is a "large" groupoid (in the sense that its collection of objects is not a set) encoding the essential aspects of Morita theory: the orbit of a ring in Pic is its Morita equivalence class, whereas the isotropy groups in Pic are the usual Picard groups of rings. Analogously, we show that our purely algebraic notion of strong Morita equivalence coincides with the notion of isomorphism in a category whose objects are non-degenerate and idempotent *-algebras over a fixed ring C; morphisms and their compositions are given by more elaborate bimodules and tensor products, and the invertible arrows in this category form the *strong Picard groupoid* $\mathsf{Pic}^{\mathrm{str}}$. When restricted to C^*-algebras, we show that $\mathsf{Pic}^{\mathrm{str}}$ defines an equivalence relation which turns out to coincide with Rieffel's (analytical) notion of strong Morita equivalence [31], and its isotropy groups are the Picard groups of C^*-algebras as in [8]; these results are proven along the lines of [2, 11].

In the last part of the paper, we compare strong and ring-theoretic Morita equivalences for unital *-algebras over C by analyzing the canonical groupoid morphism

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}. \tag{1.1}$$

We prove that, for a suitable class of unital *-algebras, including both unital C^*-algebras and hermitian star products, $\mathsf{Pic}^{\mathrm{str}}$ and Pic have the same orbits, i.e., the two notions of Morita equivalence define the *same* equivalence relation. This is a simultaneous extension of Beer's result [5], in the context of C^*-algebras, and [14, Thm. 2], for deformation quantization. Despite the coincidence of orbits, we show that, for both unital C^*-algebras and hermitian star products, the isotropy groups of Pic and $\mathsf{Pic}^{\mathrm{str}}$ are generally different. We note that the obstructions for (1.1) being an equivalence can be described in a unified way for both classes of *-algebras, due to common properties of their automorphism groups. A key ingredient for this discussion in the context of formal deformation quantization is the fact that hermitian star products are always (completely) *positive* deformations, in the sense that positive measures on the manifold can be deformed into positive linear functionals of the star product, see [10, 15].

The paper is organized as follows: In Section 2 we recall the basic definitions and properties of *-algebras over ordered rings and pre-Hilbert spaces. Section 3 is devoted to completely positive inner products, a central notion throughout the paper. In Section 4 we define various categories of representations and prove that internal and external tensor products of completely positive inner products are again completely positive. In Section 5 we define strong Morita equivalence, prove that it is an equivalence relation within the class of non-degenerate and idempotent *-algebras and show that strong Morita equivalence implies the equivalence of the categories of representations introduced in Section 4. In Section 6 we define the strong Picard groupoid and relate our algebraic definition to the C^*-algebraic Picard groupoid, proving their equivalence. In Section 7 we study the map (1.1) for a suitable class of unital *-algebras. Finally, in Section 8, we consider hermitian deformations, and, in particular, hermitian star products.

Acknowledgments: We would like to thank Martin Bordemann for many valuable discussions. We also thank S. Jansen, N. Landsman, I. Moerdijk, R. Nest, and A. Weinstein for useful comments and remarks. H.B. thanks DAAD for financial support and Freiburg University and IPAM-UCLA for their hospitality while part of this work was being done.

Conventions: Throughout this paper C will denote a ring of the form $\mathsf{R}(\mathrm{i})$, where R is an ordered ring and $\mathrm{i}^2 = -1$. Unless otherwise stated, algebras and modules will always be over a fixed ring C. For a manifold M, $C^\infty(M)$ denote its algebra of *complex-valued* smooth functions.

2 *-Algebras, positivity and pre-Hilbert spaces

A ***-algebra** over C is a C-algebra equipped with an anti-linear involutive anti-automorphism. If $\mathcal{A}$ is a *-algebra over C, then there are natural notions of positivity induced by the ordering structure on R: A **positive linear functional** is a C-linear map $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ satisfying $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$, and an algebra element $a \in \mathcal{A}$ is called **positive** if $\omega(a) \geq 0$ for all positive linear functionals ω of $\mathcal{A}$. Elements of the form

$$r_1 a_1^* a_1 + \cdots + r_n a_n^* a_n, \tag{2.1}$$

$r_i \in \mathsf{R}^+$, $a_i \in \mathcal{A}$ are clearly positive. The set of positive algebra elements is denoted by $\mathcal{A}^+$, see [12, Sec. 2] for details. These definitions recover the standard notions of positivity when $\mathcal{A}$ is a C^*-algebra; for $\mathcal{A} = C^\infty(M)$, positive linear functionals coincide with positive Borel measures on M with compact support, and positive elements are positive functions [12, App. B].

A linear map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are *-algebras over C, is called **positive** if $\phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$, and **completely positive** if the canonical extensions $\phi : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ are positive for all $n \in \mathbb{N}$.

Example 2.1 *Let us consider the maps* $\mathrm{tr} : M_n(\mathcal{A}) \longrightarrow \mathcal{A}$ *and* $\tau : M_n(\mathcal{A}) \longrightarrow \mathcal{A}$ *defined by*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}, \quad \textit{and} \quad \tau(A) = \sum_{i,j=1}^{n} A_{ij}, \tag{2.2}$$

where $A = (A_{ij}) \in M_n(\mathcal{A})$*. A direct computation shows that both maps are positive. Replacing* $\mathcal{A}$ *by* $M_N(\mathcal{A})$ *and using the identification* $M_n(M_N(\mathcal{A})) \cong M_{Nn}(\mathcal{A})$*, it immediately follows that* tr *and* τ *are completely positive maps.*

A **pre-Hilbert space** $\mathcal{H}$ over C is a C-module with a C-valued sesquilinear inner product satisfying

$$\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle} \quad \text{and} \quad \langle \phi, \phi \rangle > 0 \text{ for } \phi \neq 0, \tag{2.3}$$

see [12]. We use the convention that $\langle \cdot, \cdot \rangle$ is linear in the second argument. These are direct analogues of complex pre-Hilbert spaces. A ***-representation** of a *-algebra $\mathcal{A}$ on a pre-Hilbert space $\mathcal{H}$ is a *-homomorphism from $\mathcal{A}$ into the adjointable endomorphisms $\mathfrak{B}(\mathcal{H})$ of $\mathcal{H}$, see [11, 12]; the main examples are the usual representations of C^*-algebras on Hilbert spaces and the formal representations of star products, see e.g. [7, 36].

3 Completely positive inner products

3.1 Inner products and complete positivity

Let $\mathcal{A}$ be a *-algebra over C and consider a right $\mathcal{A}$-module $\mathcal{E}$. Thoughout this paper, $\mathcal{A}$-modules are always assumed to have a compatible C-module structure.

Remark 3.1 *When $\mathcal{A}$ is unital, we adopt the convention that $x \cdot \mathbb{1} = x$ for $x \in \mathcal{E}$; morphisms between unital algebras are assumed to be unital.*

An $\mathcal{A}$**-valued inner product** on $\mathcal{E}$ is a C-sesquilinear (linear in the second argument) map $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$ so that, for all $x, y \in \mathcal{E}$ and $a \in \mathcal{A}$,

$$\langle x, y\rangle = \langle y, x\rangle^* \quad \text{and} \quad \langle x, y \cdot a\rangle = \langle x, y\rangle a. \tag{3.1}$$

The definition of an $\mathcal{A}$-valued inner product on a left $\mathcal{A}$-module is analogous, but we require linearity on the first argument. We call an inner product $\langle\cdot,\cdot\rangle$ **non-degenerate** if $\langle x, y\rangle = 0$ for all y implies that $x = 0$, and **strongly non-degenerate** if the map $\mathcal{E} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$, $x \mapsto \langle x, \cdot\rangle$ is a bijection. Two inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ on $\mathcal{E}$ are called **isometric** if there exists a module automorphism U with $\langle Ux, Uy\rangle_1 = \langle x, y\rangle_2$.

An endomorphism $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is **adjointable** with respect to $\langle\cdot,\cdot\rangle$ if there exists $T^* \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ (called an **adjoint** of T) such that

$$\langle x, Ty\rangle = \langle T^* x, y\rangle \tag{3.2}$$

for all $x, y \in \mathcal{E}$. The algebra of adjointable endomorphisms is denoted by $\mathfrak{B}_{\mathcal{A}}(\mathcal{E})$, or simply $\mathfrak{B}(\mathcal{E})$. If $\langle\cdot,\cdot\rangle$ is non-degenerate, then adjoints are unique and $\mathfrak{B}_{\mathcal{A}}(\mathcal{E})$ becomes a *-algebra over C. One defines the C-module $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ of adjointable homomorphisms $\mathcal{E} \longrightarrow \mathcal{F}$ analogously.

We now use the positivity notions in $\mathcal{A}$: An inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{E}$ is **positive** if $\langle x, x\rangle \in \mathcal{A}^+$ for all $x \in \mathcal{E}$, and **positive definite** if $0 \neq \langle x, x\rangle \in \mathcal{A}^+$ for $x \neq 0$.

Definition 3.2 *Consider $\mathcal{E}^n$ as a right $M_n(\mathcal{A})$-module, and let $\langle\cdot,\cdot\rangle^{(n)}$ be the $M_n(\mathcal{A})$-valued inner product on $\mathcal{E}^n$ defined by*

$$\langle x, y\rangle^{(n)}_{ij} = \langle x_i, y_j\rangle, \tag{3.3}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathcal{E}^n$. We say that $\langle\cdot,\cdot\rangle$ is ***completely positive*** *if $\langle\cdot,\cdot\rangle^{(n)}$ is positive for all n.*

Remark 3.3 *Although the direct sum of non-degenerate (resp. positive, completely positive) inner products is non-degenerate (resp. positive, completely positive), this may not hold for positive definiteness: Consider $\mathcal{A} = \mathbb{Z}_2$ as *-algebra over $\mathbb{Z}(\mathrm{i})$, see [11, Sec. 2]; then the canonical inner product on $\mathcal{A}$ is positive definite but on $\mathcal{A}^2$ the vector $(\mathbb{1}, \mathbb{1})$ satisfies $\langle(\mathbb{1}, \mathbb{1}), (\mathbb{1}, \mathbb{1})\rangle = \mathbb{1} + \mathbb{1} = 0$.*

The following observation provides a way to detect algebras $\mathcal{A}$ for which positive $\mathcal{A}$-valued inner products on arbitrary $\mathcal{A}$-modules are automatically completely positive.

Proposition 3.4 *Let $\mathcal{A}$ be a *-algebra satisfying the following property: for any $n \in \mathbb{N}$, if $(A_{ij}) \in M_n(\mathcal{A})$ satisfies $\sum_{ij} a_i^* A_{ij} a_j \in \mathcal{A}^+$ for all $(a_1, \ldots, a_n) \in \mathcal{A}^n$, then $A \in M_n(\mathcal{A})^+$. Then any positive $\mathcal{A}$-valued inner product on an $\mathcal{A}$-module is automatically completely positive.*

Proof: Let $\mathcal{E}$ be an $\mathcal{A}$-module with positive inner product $\langle\cdot,\cdot\rangle$, and let $x_1, \ldots, x_n \in \mathcal{E}$. For $a_1, \ldots, a_n \in \mathcal{A}$, the matrix $A = (\langle x_i, x_j\rangle)$ satisfies

$$\sum\nolimits_{ij} a_i^* \langle x_i, x_j\rangle a_j = \sum\nolimits_{ij} \langle x_i \cdot a_i, x_j \cdot a_j\rangle = \left\langle \sum\nolimits_i x_i \cdot a_i, \sum\nolimits_j x_j \cdot a_j \right\rangle \in \mathcal{A}^+.$$

So the matrix $(\langle x_i, x_j\rangle)$ is positive and $\langle\cdot,\cdot\rangle$ is completely positive. □

The converse also holds, e.g., if $\mathcal{A}$ is unital.

Note that, although a positive definite inner product is always non-degenerate, a positive inner product which is non-degenerate may fail to be positive definite. This is due to the fact that the **degeneracy space** of an $\mathcal{A}$-module $\mathcal{E}$ with inner product $\langle \cdot, \cdot \rangle$, defined by

$$\mathcal{E}^{\perp} = \{x \in \mathcal{E} \mid \langle x, \cdot \rangle = 0\}, \tag{3.4}$$

might be *strictly* contained in the space

$$\{x \in \mathcal{E} \mid \langle x, x \rangle = 0\}. \tag{3.5}$$

Example 3.5 *Let $\mathcal{A} = \bigwedge^{\bullet}(\mathsf{C}^n)$ be the Grassmann algebra over C^n, with *-involution defined by $e_i^* = e_i$, where $e_1, \ldots, e_n$ is the canonical basis for C^n. Regard $\mathcal{A}$ as a right module over itself, equipped with inner product $\langle x, y \rangle = x^* \wedge y$. Then $\langle e_i, e_i \rangle = 0$. However, $\mathcal{A}^{\perp} = \{0\}$, since $\langle 1, x \rangle = x$.*

Any $\mathcal{A}$-valued inner product on $\mathcal{E}$ induces a non-degenerate one on the quotient $\mathcal{E}/\mathcal{E}^{\perp}$. Moreover, (completely) positive inner products induce (completely) positive inner products. In case $\mathcal{E}^{\perp} = \{x \in \mathcal{E} \mid \langle x, x \rangle = 0\}$, the quotient inner product is *positive definite.*

*-Algebras possessing a "large" amount of positive linear functionals, such as C^*-algebras and formal hermitian deformation quantizations [11, 12], are such that (3.4) and (3.5) coincide.

Example 3.6 *Let $\mathcal{A}$ be a *-algebra over C with the property that, for any non-zero hermitian element $a \in \mathcal{A}$, there exists a positive linear functional ω with $\omega(a) \neq 0$. Under the additional assumption that $2 \in \mathsf{R}$ is invertible, any $\mathcal{A}$-module $\mathcal{E}$ with $\mathcal{A}$-valued inner product is such that* (3.4) *and* (3.5) *coincide. The proof follows from the arguments in [12, Sect. 5].*

3.2 Examples of completely positive inner products

Inner products on complex pre-Hilbert spaces are always completely positive. This result extends in two directions: on one hand, one can replace $\mathbb{C}$ by arbitrary rings C; on the other hand, $\mathbb{C}$ can be replaced by more general C^*-algebras.

Example 3.7 (Pre-Hilbert spaces over C)
If $\mathcal{A} = \mathsf{C}$, then [12, Prop. A.4] shows that the condition in Proposition 3.4 is satisfied. So a positive C-valued inner product on any C-module $\mathcal{H}$ is completely positive. This is the case, in particular, for inner products on pre-Hilbert spaces over C (which are non-degenerate).

Example 3.8 (Pre-Hilbert C^*-modules)
*Let $\mathcal{A}$ be a C^*-algebra over $\mathsf{C} = \mathbb{C}$. Then the condition in Proposition 3.4 holds, see e.g. [29, Lem. 2.28]. So a positive $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ on any $\mathcal{A}$-module $\mathcal{E}$ is completely positive, see also [29, Lem. 2.65]. When $\langle \cdot, \cdot \rangle$ is positive definite, $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ is called a **pre-Hilbert C^*-module** over $\mathcal{A}$.*

Example 3.7 uses the quotients fields of R and C, whereas Example 3.8 uses the functional calculus of C^*-algebras, so neither immediately extend to inner products with values in arbitrary *-algebras. Nevertheless, one can still show the complete positivity of particular inner products.

Example 3.9 (Free modules)
Consider $\mathcal{A}^N$ as a right $\mathcal{A}$-module with respect to right multiplication, equipped with the canonical inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i^* y_i, \tag{3.6}$$

where $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathcal{A}^N$. This inner product is completely positive since, for $x^{(1)}, \ldots, x^{(n)} \in \mathcal{A}^N$, the matrix $X = \big(\langle x^{(\alpha)}, x^{(\beta)}\rangle\big) \in M_n(\mathcal{A})$ can be written as

$$X = \sum_i X_i^* X_i, \quad \textit{where} \quad X_i = \begin{pmatrix} x_i^{(1)} & \ldots & x_i^{(n)} \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix}. \tag{3.7}$$

Note, however, that the inner product (3.6) *need not be positive definite as there may exist elements $a \in \mathcal{A}$ with $a^*a = 0$. If $\mathcal{A}$ is unital then* (3.6) *is strongly non-degenerate; in the non-unital case it may be degenerate.*

Remark 3.10 *Let $\mathcal{E}$ be an $\mathcal{A}$-module with inner product $\langle \cdot, \cdot \rangle$ which can be written as*

$$\langle x, y\rangle = \sum_{i=1}^{m} P_i(x)^* P_i(y), \quad \textit{for } x, y \in \mathcal{E}, \tag{3.8}$$

where $P_i : \mathcal{E} \longrightarrow \mathcal{A}$ are $\mathcal{A}$-linear maps. By replacing $x_i^{(\alpha)}$ with $P_i(x^{(\alpha)})$ in Example 3.9, one immediately sees that (3.8) *is completely positive.*

A direct computation shows that completely positive inner products restrict to completely positive inner products on submodules.

Example 3.11 (Hermitian projective modules)
The restriction of the canonical inner product (3.6) *to any submodule of $\mathcal{A}^n$ is completely positive. In particular, hermitian projective modules, i.e., modules of the form $\mathcal{E} = P\mathcal{A}^n$, where $P \in M_n(\mathcal{A})$, $P = P^2 = P^*$, have an induced completely positive inner product (this also follows from Remark 3.10). If $\mathcal{A}$ is* unital, *then this inner product is strongly non-degenerate.*

The following simple observation concerns uniqueness.

Lemma 3.12 *Let $\mathcal{E}$ be an $\mathcal{A}$-module equipped with a strongly non-degenerate $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$. Let $\langle \cdot, \cdot \rangle'$ be another inner product on $\mathcal{E}$. Then there exists a unique hermitian element $H \in \mathfrak{B}(\mathcal{E})$ such that*

$$\langle x, y\rangle' = \langle x, Hy\rangle, \tag{3.9}$$

and $\langle \cdot, \cdot \rangle'$ is isometric to $\langle \cdot, \cdot \rangle$ if there exists an invertible $U \in \mathfrak{B}(\mathcal{E})$ with $H = U^*U$.

Example 3.13 (Hermitian vector bundles)
Let $\mathcal{A} = C^\infty(M)$ be the algebra of smooth complex-valued functions on a manifold M. As a result of Serre-Swan's theorem [35], hermitian projective modules $P\mathcal{A}^N$ correspond to (sections of) vector bundles over M (since in this case idempotents are always equivalent to projections, see Section 7.1), and $\mathcal{A}$-valued inner products correspond to hermitian fiber metrics.

As noticed in Example 3.11, there is a strongly non-degenerate inner product $\langle\cdot,\cdot\rangle$ on $P\mathcal{A}^N$. For any other inner product $\langle\cdot,\cdot\rangle'$, there exists a unique hermitian element $H \in PM_N(\mathcal{A})P$ such that $\langle x, y\rangle' = \langle x, Hy\rangle$. Since any positive invertible element $H \in PM_N(C^\infty(M))P$ can be written as $H = U^*U$ for an invertible $U \in PM_N(C^\infty(M))P$, it follows from Lemma 3.12 that there is only one fiber metric on a vector bundle over M up to isometric isomorphism. We will generalize this example in Section 7.1.

Example 3.14 (Nontrivial inner products)
Even if the algebra $\mathcal{A}$ is a field, one can have nontrivial inner products. For example, consider $\mathsf{R} = \mathbb{Q}$ and $\mathsf{C} = \mathbb{Q}(\mathrm{i})$. Then $3 \in \mathsf{C}$ is a positive invertible element but there is no $z \in \mathsf{C}$ with $\overline{z}z = 3$ (write $z = a + \mathrm{i}b$ with $a = r/n$, $b = s/n$ with $r, s, n \in \mathbb{N}$, then take the equation $3n^2 = r^2 + s^2$ modulo 4). Hence $\langle z, w\rangle' = 3\overline{z}w$ is completely positive and strongly non-degenerate but not isometric to the canonical inner product $\langle z, w\rangle = \overline{z}w$.

4 Representations and tensor products

4.1 Categories of *-representations

We now discuss the algebraic analogues of Hilbert C^*-modules, see e.g. [24]. Let $\mathcal{D}$ be a *-algebra over C.

Definition 4.1 *A (right)* ***inner-product*** $\mathcal{D}$***-module*** *is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, where $\mathcal{H}$ is a (right) $\mathcal{D}$-module and $\langle\cdot,\cdot\rangle$ is a non-degenerate $\mathcal{D}$-valued inner product. If $\langle\cdot,\cdot\rangle$ is completely positive, we call $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ a* ***pre-Hilbert*** $\mathcal{D}$***-module****.*

Whenever there is no risk of confusion, we will denote an inner-product module (or pre-Hilbert module) simply by $\mathcal{H}$.

We now consider *-representations of *-algebras on inner-product modules, extending the discussion in [7, 11, 12]. Let $\mathcal{A}$ be a *-algebra over C, and let $\mathcal{H}$ be an inner-product $\mathcal{D}$-module.

Definition 4.2 *A* ****-representation*** *of $\mathcal{A}$ on $\mathcal{H}$ is a *-homomorphism $\pi : \mathcal{A} \longrightarrow \mathfrak{B}_\mathcal{D}(\mathcal{H})$.*

An **intertwiner** between two *-representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \varrho)$ is an isometry $T \in \mathfrak{B}_\mathcal{D}(\mathcal{H}, \mathcal{K})$ such that, for all $a \in \mathcal{A}$,

$$T\pi(a) = \varrho(a)T. \tag{4.1}$$

We denote by $^*\text{-}\mathsf{mod}_\mathcal{D}(\mathcal{A})$ the category whose objects are *-representations of $\mathcal{A}$ on inner-product modules over $\mathcal{D}$ and morphisms are intertwiners. The subcategory whose objects are *-representations on pre-Hilbert modules is denoted by $^*\text{-}\mathsf{rep}_\mathcal{D}(\mathcal{A})$. Since both categories contain trivial representations of $\mathcal{A}$, we will consider the following further refinement: A *-representation $(\mathcal{H}, \pi)$ is **strongly non-degenerate** if

$$\pi(\mathcal{A})\mathcal{H} = \mathcal{H}, \tag{4.2}$$

(by Remark 3.1, this is always the case if $\mathcal{A}$ is unital). The category of strongly non-degenerate *-representations of $\mathcal{A}$ on inner-product (resp. pre-Hilbert) $\mathcal{D}$-modules is denoted by $^*\text{-}\mathsf{Mod}_\mathcal{D}(\mathcal{A})$ (resp. $^*\text{-}\mathsf{Rep}_\mathcal{D}(\mathcal{A})$).

Definition 4.3 *An inner-product $\mathcal{D}$-module $\mathcal{H}$ together with a strongly non-degenerate *-representation of $\mathcal{A}$ will be called an $(\mathcal{A}, \mathcal{D})$-* ***inner-product bimodule****; it is an $(\mathcal{A}, \mathcal{D})$-* ***pre-Hilbert bimodule*** *if $\mathcal{H}$ is a pre-Hilbert module.*

These are algebraic analogues of Hilbert bimodules as e.g. in [25, Def. 3.2]. This terminology differs from the one in [2].

An **isomorphism** of inner-product (bi)modules (or pre-Hilbert (bi)modules) is just a (bi)module homomorphism preserving inner products.

More generally, suppose ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}$ is a bimodule equipped with an arbitrary $\mathcal{D}$-valued inner product $\langle\cdot,\cdot\rangle$. We say that $\langle\cdot,\cdot\rangle$ is **compatible** with the $\mathcal{A}$-action if

$$\langle a\cdot x, y\rangle = \langle x, a^*\cdot y\rangle, \tag{4.3}$$

for all $a\in\mathcal{A}$ and $x,y\in\mathcal{H}$. Clearly, any *-representation of $\mathcal{A}$ on an inner-product module $\mathcal{H}$ over $\mathcal{D}$ makes $\mathcal{H}$ into a bimodule for which $\langle\cdot,\cdot\rangle$ and the $\mathcal{A}$-action are compatible. Unless otherwise stated, inner products on bimodules are assumed to be compatible with the actions.

4.2 Tensor products and Rieffel induction of representations

Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras over C. Let $\mathcal{F}_{\mathcal{B}}$ be a right $\mathcal{B}$-module equipped with a $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle^{\mathcal{F}}_{\mathcal{B}}$, and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a bimodule equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$ compatible with the $\mathcal{B}$-action. Following Rieffel [30, 31], there is a well-defined $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}$ on the tensor product $\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ completely determined by

$$\langle y_1\otimes_{\mathcal{B}} x_1, y_2\otimes_{\mathcal{B}} x_2\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}} = \left\langle x_1,\, \langle y_1,y_2\rangle^{\mathcal{F}}_{\mathcal{B}}\cdot x_2\right\rangle^{\mathcal{E}}_{\mathcal{A}} \tag{4.4}$$

for $x_1,x_2\in{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and $y_1,y_2\in\mathcal{F}_{\mathcal{B}}$ (we extend it to arbitrary elements using C-sesquilinearity). An analogous construction works for left modules carrying inner products.

If $(\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ is the degeneracy space associated with $\langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}$, then the quotient

$$(\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})\big/(\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$$

acquires an induced inner product, also denoted by $\langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}$, which is non-degenerate, see Section 3.1. Thus the pair

$$\mathcal{F}_{\mathcal{B}}\,\widehat{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} := \Big((\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})\big/(\mathcal{F}_{\mathcal{B}}\otimes_{\mathcal{B}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp},\, \langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}\Big) \tag{4.5}$$

is an inner-product $\mathcal{A}$-module called the **internal tensor product** of $(\mathcal{F}_{\mathcal{B}},\langle\cdot,\cdot\rangle^{\mathcal{F}}_{\mathcal{B}})$ and $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}},\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}})$. As we will see, in many examples the degeneracy space of (4.4) is already trivial.

Lemma 4.4 *If $\mathcal{C}$ is a *-algebra and ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ is a bimodule so that $\langle\cdot,\cdot\rangle^{\mathcal{F}}_{\mathcal{B}}$ is compatible with the $\mathcal{C}$-action, then $\mathcal{F}_{\mathcal{B}}\,\widehat{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ carries a canonical left $\mathcal{C}$-action, compatible with $\langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}$.*

The proof of this lemma is a direct computation. It is also simple to check that internal tensor products have associativity properties similar to those of ordinary (algebraic) tensor products: Let $\mathcal{G}_{\mathcal{C}}$ be a $\mathcal{C}$-module with $\mathcal{C}$-valued inner product, and let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ (resp. ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$) be a bimodule with $\mathcal{B}$-valued (resp. $\mathcal{A}$-valued) inner product compatible with the $\mathcal{C}$-action (resp. $\mathcal{B}$-action).

Lemma 4.5 *There is a natural isomorphism*

$$(\mathcal{G}_{\mathcal{C}}\,\widehat{\otimes}_{\mathcal{C}}\,{}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}})\,\widehat{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong \mathcal{G}_{\mathcal{C}}\,\widehat{\otimes}_{\mathcal{C}}\,({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}\,\widehat{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \tag{4.6}$$

induced from the usual associativity of algebraic tensor products.

Internal tensor products also behave well with respect to maps.

Lemma 4.6 *Let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$, ${}_{\mathcal{C}}\mathcal{F}'_{\mathcal{B}}$ be equipped with compatible $\mathcal{B}$-valued inner products, and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ be equipped with compatible $\mathcal{A}$-valued inner products. Let $S \in \mathfrak{B}({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}, {}_{\mathcal{C}}\mathcal{F}'_{\mathcal{B}})$ and $T \in \mathfrak{B}({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}})$ be adjointable bimodule morphisms. Then their algebraic tensor product $S \otimes_{\mathcal{B}} T$ induces a well-defined adjointable bimodule morphism $S \widehat{\otimes}_{\mathcal{B}} T : {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{C}}\mathcal{F}'_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ with adjoint given by $S^* \widehat{\otimes}_{\mathcal{B}} T^*$. If S and T are isometric then $S \widehat{\otimes}_{\mathcal{B}} T$ is isometric as well.*

Hence, for a fixed triple of *-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we obtain a *functor*

$$\widehat{\otimes}_{\mathcal{B}} : {}^*\text{-}\mathsf{mod}_{\mathcal{B}}(\mathcal{C}) \times {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{C}). \tag{4.7}$$

In the case of unital algebras, one can replace *-mod by *-Mod in (4.7).

A central question is whether one can restrict the functor $\widehat{\otimes}_{\mathcal{B}}$ to representations on *pre-Hilbert modules*, or whether the tensor product (4.4) of two *positive* inner products remains positive. This is the case, for example, in the realm of C^*-algebras, but the proof uses the functional calculus, see e.g. [29, Prop. 2.64]. Fortunately, a purely algebraic result can be obtained if one requires the inner products to be *completely positive*.

Theorem 4.7 *If the inner products $\langle \cdot, \cdot \rangle^{\mathcal{F}}_{\mathcal{B}}$ on $\mathcal{F}_{\mathcal{B}}$ and $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ are completely positive, then the inner product $\langle \cdot, \cdot \rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}}$ on $\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ defined by (4.4) is also completely positive.*

PROOF: Let $\Phi^{(1)}, \ldots \Phi^{(n)} \in \mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We must show that the matrix

$$A = \left(\left\langle \Phi^{(\alpha)}, \Phi^{(\beta)} \right\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}} \right)$$

is a positive element in $M_n(\mathcal{A})$. Without loss of generality, we can write $\Phi^{(\alpha)} = \sum_{i=1}^N y_i^{(\alpha)} \otimes_{\mathcal{B}} x_i^{(\alpha)}$, where N is the same for all α. Let us consider the map

$$f : M_n(M_N(\mathcal{B})) \longrightarrow M_n(M_N(\mathcal{A})), \quad (B^{\alpha\beta}_{ij}) \mapsto \left(\left\langle x_i^{(\alpha)}, B^{\alpha\beta}_{ij} \cdot x_j^{(\beta)} \right\rangle^{\mathcal{E}}_{\mathcal{A}} \right), \tag{4.8}$$

$1 \leq i, j \leq N$, $1 \leq \alpha, \beta \leq n$. We claim that f is a positive map. Indeed, as a consequence of the definition of positive maps in Sect. 2, it suffices to show that $f(B^*B)$ is positive for any $B \in M_n(M_N(\mathcal{B}))$. A direct computation shows that, for $B = (B^{\alpha\beta}_{ij})$,

$$f(B^*B) = \sum_{k=1}^{N} \sum_{\gamma=1}^{n} C^{\gamma}_k \quad \text{with} \quad (C^{\gamma}_k)^{\alpha\beta}_{ij} = \left\langle B^{\gamma\alpha}_{ki} x_i^{(\alpha)}, B^{\gamma\beta}_{kj} x_j^{(\beta)} \right\rangle^{\mathcal{E}}_{\mathcal{A}},$$

which is positive since $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ is completely positive. Since the matrix

$$\left(\left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle^{\mathcal{F}}_{\mathcal{B}} \right) \in M_{nN}(\mathcal{B})$$

is positive, for $\langle \cdot, \cdot \rangle^{\mathcal{F}}_{\mathcal{B}}$ in $\mathcal{F}$ is completely positive, it follows that the matrix

$$\left(\left\langle x_i^{(\alpha)}, \left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle^{\mathcal{F}}_{\mathcal{B}} \cdot x_j^{(\beta)} \right\rangle^{\mathcal{E}}_{\mathcal{A}} \right)$$

is a positive element in $M_{nN}(\mathcal{A})$. Since summation over i, j defines a positive map $\tau : M_{nN}(\mathcal{A}) \longrightarrow M_n(\mathcal{A})$, see Example 2.1, the matrix

$$\sum_{i,j=1}^{N} \left(\left\langle x_i^{(\alpha)}, \left\langle y_i^{(\alpha)}, y_j^{(\beta)} \right\rangle^{\mathcal{F}}_{\mathcal{B}} \cdot x_j^{(\beta)} \right\rangle^{\mathcal{E}}_{\mathcal{A}} \right) = \left(\left\langle \Phi^{(\alpha)}, \Phi^{(\beta)} \right\rangle^{\mathcal{F}\otimes\mathcal{E}}_{\mathcal{A}} \right) = A \tag{4.9}$$

is positive. This concludes the proof. □

As pointed out in Section 3.1, if $\langle\cdot,\cdot\rangle_{\mathcal{A}}^{\mathcal{F}\otimes\mathcal{E}}$ is completely positive, then so is the induced inner product on $\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$.

Corollary 4.8 *If $\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ have completely positive inner products, then $\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a pre-Hilbert module.*

It follows that the functor $\widehat{\otimes}_{\mathcal{B}}$ in (4.7) restricts to a functor

$$\widehat{\otimes}_{\mathcal{B}} : {}^*\text{-}\mathsf{rep}_{\mathcal{B}}(\mathcal{C}) \times {}^*\text{-}\mathsf{rep}_{\mathcal{A}}(\mathcal{B}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathcal{A}}(\mathcal{C}), \tag{4.10}$$

and, from (4.10), we obtain two functors by fixing each one of the two arguments.

Example 4.9 (Rieffel induction)
Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be *-algebras, and fix a $(\mathcal{B},\mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-}\mathsf{rep}_{\mathcal{A}}(\mathcal{B})$. We then have a functor

$$\mathsf{R}_{\mathcal{E}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} \cdot : {}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{B}); \tag{4.11}$$

on objects, $\mathsf{R}_{\mathcal{E}}({}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}) = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}$, and, on morphisms, $\mathsf{R}_{\mathcal{E}}(T) = \mathsf{id} \mathbin{\widehat{\otimes}_{\mathcal{A}}} T$, for $T \in \mathfrak{B}(\mathcal{H},\mathcal{H}')$.
*This functor is called **Rieffel induction** and relates the representation theories of $\mathcal{A}$ and $\mathcal{B}$ on pre-Hilbert modules over a fixed *-algebra $\mathcal{D}$.*

Example 4.10 (Change of base ring)
Similarly, we can change the base algebra $\mathcal{D}$ in ${}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{A})$: Let $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{D}'$ be *-algebras and let ${}_{\mathcal{D}}\mathcal{G}_{\mathcal{D}'} \in {}^*\text{-}\mathsf{rep}_{\mathcal{D}'}(\mathcal{D})$. Then $\widehat{\otimes}_{\mathcal{D}}$ induces a functor

$$\mathsf{S}_{\mathcal{G}} = \cdot \mathbin{\widehat{\otimes}_{\mathcal{D}}} {}_{\mathcal{D}}\mathcal{G}_{\mathcal{D}'} : {}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{rep}_{\mathcal{D}'}(\mathcal{A}) \tag{4.12}$$

defined analogously to (4.11)*.*

A direct consequence of Lemma 4.5 is that the following diagram commutes up to natural transformations:

$$\begin{array}{ccc} {}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{A}) & \xrightarrow{\mathsf{S}_{\mathcal{G}}} & {}^*\text{-}\mathsf{rep}_{\mathcal{D}'}(\mathcal{A}) \\ {\scriptstyle \mathsf{R}_{\mathcal{E}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{R}_{\mathcal{E}}} \\ {}^*\text{-}\mathsf{rep}_{\mathcal{D}}(\mathcal{B}) & \xrightarrow{\mathsf{S}_{\mathcal{G}}} & {}^*\text{-}\mathsf{rep}_{\mathcal{D}'}(\mathcal{B}) \end{array} \tag{4.13}$$

Remark 4.11 *(Rieffel induction for C^*-algebras)*
In the original setting of C^*-algebras [30, 31], Rieffel's construction relates categories of *-representations on Hilbert spaces (in particular, $\mathcal{D} = \mathsf{C} = \mathbb{C}$), so one needs to consider an extra completion of ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}$ with respect to the norm induced by (4.4)*. Since *-representations of C^*-algebras on pre-Hilbert spaces are necessarily bounded, this completion is canonical, so one recovers Rieffel's original construction from this algebraic approach, see [11]. More generally, in this setting, $\mathcal{D}$ could be an arbitrary C^*-algebra.*

Examples of algebraic Rieffel induction of *-representations in the setting of formal deformation quantization can be found, e.g., in [9, 14].

Remark 4.12 (External tensor products)
Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be *-algebras over C, $i = 1, 2$. The tensor products $\mathcal{A} = \mathcal{A}_1 \otimes_{\mathsf{C}} \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_1 \otimes_{\mathsf{C}} \mathcal{B}_2$ are naturally *-algebras. Let $\mathcal{E}_i$ be $(\mathcal{B}_i, \mathcal{A}_i)$-bimodules for $i = 1, 2$ and consider the $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{E} = \mathcal{E}_1 \otimes_{\mathsf{C}} \mathcal{E}_2$. If each $\mathcal{E}_i$ is endowed with an $\mathcal{A}_i$-valued inner product $\langle \cdot, \cdot \rangle_i$, compatible with the $\mathcal{B}_i$-action, then we have an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$, compatible with the $\mathcal{B}$-action, uniquely defined by

$$\langle x_1 \otimes_{\mathsf{C}} x_2, y_1 \otimes_{\mathsf{C}} y_2 \rangle = \langle x_1, y_1 \rangle_1 \otimes_{\mathsf{C}} \langle x_2, y_2 \rangle_2 \tag{4.14}$$

*for $x_i, y_i \in \mathcal{E}_i$. We call the inner product defined by (4.14) the **external tensor product** of $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. Just as for internal tensor products, if $\langle \cdot, \cdot \rangle_i$ are completely positive, then so is $\langle \cdot, \cdot \rangle$. The construction is also functorial in a sense analogous to Lemma 4.6.*

5 Strong Morita equivalence

5.1 Definition

An $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ on an $\mathcal{A}$-module $\mathcal{E}$ is called **full** if

$$\mathsf{C}\text{-span}\{ \langle x, y \rangle^{\mathcal{E}}_{\mathcal{A}} \mid x, y \in \mathcal{E} \} = \mathcal{A}. \tag{5.1}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras over C.

Definition 5.1 *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ and a $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$. We call $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, {}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}, \langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}})$ a ***-equivalence bimodule** if*

- *i.) $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ (resp. ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$) is non-degenerate, full and compatible with the $\mathcal{B}$-action (resp. $\mathcal{A}$-action);*
- *ii.) For all $x, y, z \in \mathcal{E}$ one has $x \cdot \langle y, z \rangle^{\mathcal{E}}_{\mathcal{A}} = {}_{\mathcal{B}}\langle x, y \rangle^{\mathcal{E}} \cdot z$;*
- *iii.) $\mathcal{B} \cdot \mathcal{E} = \mathcal{E}$ and $\mathcal{E} \cdot \mathcal{A} = \mathcal{E}$.*

*If $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ are completely positive, then ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is called a **strong equivalence bimodule**.*

Whenever the context is clear, we will refer to strong or *-equivalence bimodules simply as equivalence bimodules.

Definition 5.2 *Two *-algebras $\mathcal{A}$ and $\mathcal{B}$ are ***-Morita equivalent** (resp. **strongly Morita equivalent**) if there exists a *- (resp. strong) $(\mathcal{B}, \mathcal{A})$-equivalence bimodule.*

The definition of *-Morita equivalence goes back to Ara [1]. Since this notion does not involve positivity, its definition makes sense for ground rings not necessarily of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$.

Remark 5.3 (Formal Morita equivalence of *-algebras)
In our previous work [12] we had a more technical formulation of strong Morita equivalence for *-algebras over C, called formal Morita equivalence. *Definition 5.2, based on completely positive inner products, is conceptually more clear (though, at least for unital algebras, it is equivalent to the one of [12]) and yields refinements of the results in [12].*

Remark 5.4 (Strong Morita equivalence of C^*-algebras)
Rieffel's definition of a strong equivalence bimodule of C^*-algebras [31] (see also [29]) is a refinement of Definition 5.1 involving topological completions which do not make sense in a purely algebraic setting. Nevertheless, one recovers Rieffel's notion as follows [2], [11, Lem. 3.1]: Two C^*-algebras are strongly Morita equivalent in Rieffel's sense if and only if their minimal dense ideals are strongly Morita equivalent (or *-Morita equivalent) in the sense of Def. 5.1. In particular, for minimal dense ideals of C^*-algebras, *- and strong Morita equivalences coincide (see Section 6.2).

As we now discuss, *- and strong Morita equivalences are in fact equivalence relations for a large class of *-algebras. We start with

Lemma 5.5 *The notions of *- and strong Morita equivalences define a symmetric relation.*

For the proof, we just note that if ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a *- (resp. strong) $(\mathcal{B}, \mathcal{A})$-equivalence bimodule, then its *conjugate bimodule* ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ is an *- (resp. strong) $(\mathcal{A}, \mathcal{B})$-equivalence bimodule, see e.g. [12, Sect. 5].

For reflexivity and transitivity, one needs to be more restrictive. Recall that an algebra $\mathcal{A}$ is **non-degenerate** if $a \in \mathcal{A}$, $\mathcal{A} \cdot a = 0$ or $a \cdot \mathcal{A} = 0$ implies that $a = 0$, and it is **idempotent** if elements of the form $a_1 a_2$ span $\mathcal{A}$. The following observation indicates the importance of these classes of algebras.

Let $\mathcal{A}$ be a *-algebra, and let ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ be the natural bimodule induced by left and right multiplications, equipped with the canonical inner products ${}_{\mathcal{A}}\langle a, b\rangle = ab^*$ and $\langle a, b\rangle_{\mathcal{A}} = a^* b$.

Lemma 5.6 *The bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is a *- or strong equivalence bimodule if and only if $\mathcal{A}$ is non-degenerate and idempotent.*

The proof is simple: idempotency is equivalent to the canonical inner products being full and the actions by multiplication being strongly non-degenerate; non-degeneracy is equivalent to the inner products being non-degenerate. The inner products are completely positive by Example 3.9.

We therefore restrict ourselves to the class of non-degenerate and idempotent *-algebras (which contains, in particular, all unital *-algebras). Within this class, *-Morita equivalence is transitive [1], hence it is an equivalence relation. We will show that the same holds for strong Morita equivalence.

The next result follows from arguments analogous to those in [11, Lem. 3.1].

Lemma 5.7 *Let $\mathcal{A}$, $\mathcal{B}$ be non-degenerate and idempotent *-algebras, and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a bimodule with inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ satisfying all the properties of Def. 5.1, except for non-degeneracy. Then their degeneracy spaces coincide, and the quotient bimodule $\mathcal{E}/\mathcal{E}^{\perp}$, with the induced inner products, is a *-equivalence bimodule. If $\langle \cdot, \cdot \rangle_{\mathcal{A}}^{\mathcal{E}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ are completely positive, then the quotient bimodule is a strong equivalence bimodule.*

As a result, within the class of non-degenerate and idempotent *-algebras, one obtains a refinement of the internal tensor product $\widehat{\otimes}$ for equivalence bimodules taking into account both inner products.

Lemma 5.8 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be non-degenerate and idempotent* **-algebras and let* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *and* ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ *be* **- (resp. strong) equivalence bimodules. Then the triple*

$$ {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widetilde{\otimes}_{\mathcal{B}}\, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} := \Big((\, {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \,/\, (\, {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp} \,,\, {}_{\mathcal{B}}\langle\cdot,\cdot\rangle^{\mathcal{F}\otimes\mathcal{E}} \,,\, \langle\cdot,\cdot\rangle_{\mathcal{A}}^{\mathcal{F}\otimes\mathcal{E}} \Big) \tag{5.2} $$

is a **- (resp. strong) equivalence bimodule.*

Clearly $\widetilde{\otimes}$ satisfies functoriality properties analogous to those of $\widehat{\otimes}$. Combining Lemmas 5.5, 5.6 and 5.8, we obtain:

Theorem 5.9 *Strong Morita equivalence is an equivalence relation within the class of non-degenerate and idempotent* **-algebras over* C*.*

5.2 General properties

Let $\mathcal{A}$ and $\mathcal{B}$ be non-degenerate and idempotent *-algebras, and let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a *-isomorphism. A simple check reveals that $\mathcal{B}$, seen as an $(\mathcal{A}, \mathcal{B})$-bimodule via

$$ a \cdot_{\Phi} b \cdot b_1 = \Phi(a) b b_1 \tag{5.3} $$

and equipped with the obvious inner products, is a strong equivalence bimodule. Hence *-isomorphism implies strong Morita equivalence.

On the other hand, [1] shows that *-Morita equivalence (so also strong Morita equivalence) implies *-isomorphism of centers. As a result, for commutative (non-degenerate and idempotent) *-algebras, strong and *-Morita equivalences coincide with the notion of *-isomorphism.

Remark 5.10 (Finite-rank operators)
Let $(\mathcal{E}_{\mathcal{A}}, \langle\cdot,\cdot\rangle_{\mathcal{A}}^{\mathcal{E}})$ *be an inner-product module. The set of "finite-rank" operators on* $\mathcal{E}_{\mathcal{A}}$*, denoted by* $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$*, is the* C*-linear span of operators* $\theta_{x,y}$*,*

$$ \theta_{x,y}(z) := x \cdot \langle y, z\rangle_{\mathcal{A}}^{\mathcal{E}} \,, $$

for $x, y, z \in \mathcal{E}$*. Note that* $\theta_{x,y}^{*} = \theta_{y,x}$ *and* $\mathfrak{F}(\mathcal{E}_{\mathcal{A}}) \subseteq \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$ *is an ideal.*

Within the class of non-degenerate, idempotent **-algebras, an alternative description of* **-Morita equivalence is given as follows [1]: if* $\mathcal{E}_{\mathcal{A}}$ *is a full inner-product module so that* $\mathcal{E}_{\mathcal{A}} \cdot \mathcal{A} = \mathcal{E}_{\mathcal{A}}$*, then* ${}_{\mathfrak{F}(\mathcal{E})}\mathcal{E}_{\mathcal{A}}$ *is a* **-equivalence bimodule, with* $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$*-valued inner product*

$$ (x, y) \mapsto \theta_{x,y}. \tag{5.4} $$

On the other hand, if ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *is a* **-equivalence bimodule, then the* $\mathcal{B}$*-action on* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *provides a natural* **-isomorphism*

$$ \mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}}). \tag{5.5} $$

Under this identification, the **-equivalence bimodule* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *becomes* ${}_{\mathfrak{F}(\mathcal{E})}\mathcal{E}_{\mathcal{A}}$*. As a consequence, if* $\mathcal{E}_{\mathcal{A}}$ *is a pre-Hilbert module with* $\mathcal{E}_{\mathcal{A}} \cdot \mathcal{A} = \mathcal{E}_{\mathcal{A}}$ *and* (5.4) *is completely positive, then* ${}_{\mathfrak{F}(\mathcal{E})}\mathcal{E}_{\mathcal{A}}$ *is a strong equivalence bimodule.*

The following is a standard example in Morita theory, see also [12, Sect. 6].

Example 5.11 (Matrix algebras)
Let $\mathcal{A}$ be a non-degenerate and idempotent *-algebra over C. We claim that $\mathcal{A}$ and $M_n(\mathcal{A})$ are *- and strongly Morita equivalent.

First note that C^n is a strong $(M_n(\mathsf{C}), \mathsf{C})$-equivalence bimodule. In fact, since $\mathfrak{F}(\mathsf{C}^n) = M_n(\mathsf{C})$ and $\mathsf{C}^n \cdot \mathsf{C} = \mathsf{C}^n$, by Remark 5.10 it only remains to check that (5.4) *is completely positive. But if $x, y \in \mathsf{C}^n$, then we can write*

$$\theta_{x,y} = \theta_{x,e_1}\theta^*_{y,e_1},$$

where $e_1 = (1, 0, \ldots, 0) \in \mathsf{C}^n$. So this inner product is of the form (3.8) *(for $m = 1$ and $P_1(x) = \theta_{x,e_1}$), so it is completely positive.*

By tensoring the equivalence bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ with C^n, it follows from Remark 4.12 that the canonical inner products on $\mathcal{A}^n$ are completely positive. It then easily follows that $\mathcal{A}^n$ is a $(M_n(\mathcal{A}), \mathcal{A})$-equivalence bimodule.

For unital *-algebras over C, it follows from the definitions that strong Morita equivalence implies *-Morita equivalence, which in turn implies ring-theoretic Morita equivalence. In particular, (*- or strong) equivalence bimodules are finitely generated and projective with respect to both actions. Using the non-degeneracy of inner products, their compatibility and fullness, one can verify this property directly by checking that any *-equivalence bimodule admits a finite hermitian dual bases. As a consequence, we have

Corollary 5.12 *If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are unital *-algebras and ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ are (*- or strong) equivalence bimodules, then the inner product* (4.4) *on ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is non-degenerate.*

It follows that the quotient in (5.2) is irrelevant. This is always the case for (not necessarily unital) C^*-algebras [24, Prop. 4.5].

5.3 Equivalence of categories of representations

It is shown in [1] that *-Morita equivalence implies equivalence of categories of (strongly non-degenerate) representations on inner-product modules. We now recover this result and show that an analogous statement holds for strong Morita equivalence, generalizing [12, Thm. 5.10].

The next lemma follows from Lemmas 4.5 and 5.7.

Lemma 5.13 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be non-degenerate and idempotent *-algebras. Let ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be *-equivalence bimodules, and let $(\mathcal{H}, \pi) \in {}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{A})$. Then there are natural isomorphisms of inner-product bimodules:*

$$\left({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widetilde{\otimes}_{\mathcal{B}}\, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\right) \widehat{\otimes}_{\mathcal{A}}\, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widehat{\otimes}_{\mathcal{B}} \left({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \,\widehat{\otimes}_{\mathcal{A}}\, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}}\right), \tag{5.6}$$

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \,\widehat{\otimes}_{\mathcal{A}}\, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \cong {}_{\mathcal{A}}\mathcal{H}_{\mathcal{D}} \,\widehat{\otimes}_{\mathcal{D}}\, {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}. \tag{5.7}$$

As a result, when ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ are strong equivalence bimodules, there is a natural equivalence

$$\mathsf{R}_{\mathcal{F}} \circ \mathsf{R}_{\mathcal{E}} \cong \mathsf{R}_{\mathcal{F}\widetilde{\otimes}\mathcal{E}}. \tag{5.8}$$

Using the idempotency and non-degeneracy of $\mathcal{A}$ and $\mathcal{B}$, one shows

Lemma 5.14 *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a (*- or strong) equivalence bimodule. If ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ is its conjugate bimodule, then the following maps are (*- or strong) equivalence bimodule isomorphisms:*

$${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}} \,\widetilde{\otimes}_{\mathcal{B}}\, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \quad \overline{x} \,\widetilde{\otimes}_{\mathcal{B}}\, y \mapsto \langle x, y\rangle^{\mathcal{E}}_{\mathcal{A}}, \tag{5.9}$$

$${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \,\widetilde{\otimes}_{\mathcal{A}}\, {}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}} \longrightarrow {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, \quad x \,\widetilde{\otimes}_{\mathcal{A}}\, \overline{y} \mapsto {}_{\mathcal{B}}\langle x, y\rangle^{\mathcal{E}}. \tag{5.10}$$

Corollary 5.15 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be non-degenerate and idempotent *-algebras, and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a strong equivalence bimodule. Then*

$$\mathsf{R}_{\mathcal{E}} : {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{B}) \tag{5.11}$$

is an equivalence of categories, with inverse given by $\mathsf{R}_{\overline{\mathcal{E}}}$.

Remark 5.16 *Clearly, the functors $\mathsf{S}_{\mathcal{G}}$ satisfy a property analogous to* (5.8)*; similarly to Corollary 5.15, an equivalence bimodule ${}_{\mathcal{D}}\mathcal{G}_{\mathcal{D}'}$ establishes an equivalence of categories $\mathsf{S}_{\mathcal{G}} : {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}'}(\mathcal{A})$. All these properties are direct analogs of the previous constructions by replacing tensor products on the left by those on the right.*

Corollary 5.15 recovers the well-knwon theorem of Rieffel [31] on the equivalence of categories of non-degenerate *-representations of strongly Morita equivalent C^*-algebras on Hilbert spaces, see [2, 11].

6 Picard groupoids

In this section, we introduce the Picard groupoids associated with strong Morita equivalence, in analogy with the groupoid Pic [6] of invertible bimodules in ring-theoretic Morita theory [3, 26]. (See [16, 25] for related constructions.)

6.1 The strong Picard groupoid

Let *-Alg (resp. *-Alg^+) be the category whose objects are nondegenerate and idempotent *-algebras over a fixed C, morphisms are isomorphism classes of inner-product (resp. pre-Hilbert) bimodules and composition is internal tensor product (4.5). (The composition is associative by Lemma 4.5.) We call an inner-product (resp. pre-Hilbert) bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ over $\mathcal{A}$ **invertible** if its isomorphism class is invertible in *-Alg (resp. *-Alg^+). Note that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is invertible if and only if there exists an inner-product (resp. pre-Hilbert) bimodule ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ over $\mathcal{B}$ together with isomorphisms

$${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \stackrel{\sim}{\longrightarrow} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \quad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}} \stackrel{\sim}{\longrightarrow} {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}. \tag{6.1}$$

Theorem 6.1 *An inner-product (resp. pre-Hilbert) bimodule $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}})$ is invertible if and only if there exists a $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle^{\mathcal{E}}$ making $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}, {}_{\mathcal{B}}\langle\cdot,\cdot\rangle^{\mathcal{E}})$ into a *-(resp. strong) equivalence bimodule. In particular, *-(resp. strong) Morita equivalence coincides with the notion of isomorphism in *-Alg (resp. *-Alg^+).*

This is an algebraic version of a similar result in the framework of C^*-algebras [25, 34], which we will recover in Section 6.2. We need three main lemmas to prove the theorem.

Lemma 6.2 *Let $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}})$ be an invertible inner-product bimodule. Then $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$ is full and $\mathcal{E}\cdot\mathcal{A} = \mathcal{E}$. (By Remark 5.10, ${}_{\mathfrak{F}(\mathcal{E})}\mathcal{E}_{\mathcal{A}}$ is a *-equivalence bimodule.)*

Proof: Let ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ be an inner-product bimodule such that (6.1) holds. The fullness of $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$ is a simple consequence of the idempotency of $\mathcal{A}$ and the first isomorphism of (6.1).

For the second assertion, note that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\cdot\mathcal{A} \subseteq {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a $(\mathcal{B},\mathcal{A})$ inner-product bimodule. Moreover, using the idempotency of $\mathcal{A}$ and the fact that $\mathcal{A}\cdot {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}} = {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$, it is simple to check that ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ is still an inverse for ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\cdot\mathcal{A}$. By uniqueness of inverses (up to isomorphism), we get

${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cdot \mathcal{A}.$ □

Lemma 6.3 *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an invertible inner-product bimodule and let $\mathcal{F}_{\mathcal{B}}$ be an inner-product $\mathcal{B}$-module. Then the natural map*

$$\mathsf{S}_{\mathcal{E}} : \mathfrak{B}(\mathcal{F}_{\mathcal{B}}) \longrightarrow \mathfrak{B}(\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}), \;\; T \mapsto T \widehat{\otimes}_{\mathcal{B}} \mathsf{id},$$

is an isomorphism.

Proof: Let ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ be as in (6.1). Then we have an induced map

$$\mathsf{S}_{\mathcal{E}'} : \mathfrak{B}(\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \longrightarrow \mathfrak{B}((\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}) \cong \mathfrak{B}(\mathcal{F}_{\mathcal{B}}),$$

since $(\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}} \cong \mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} ({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}) \cong \mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \cong \mathcal{F}_{\mathcal{B}}$. One can check that $\mathsf{S}_{\mathcal{E}}$ and $\mathsf{S}_{\mathcal{E}'}$ are inverses of each other. □
The next result is an algebraic analog of [24, Prop. 4.7].

Lemma 6.4 *Let $\mathcal{F}_{\mathcal{B}}$ be an inner-product $\mathcal{B}$-module so that $\mathcal{F}_{\mathcal{B}} \cdot \mathcal{B} = \mathcal{F}$. Let $\mathcal{E}_{\mathcal{A}}$ be an inner-product $\mathcal{A}$-module and $\pi : \mathcal{B} \longrightarrow \mathfrak{B}(\mathcal{E}_{\mathcal{A}})$ be a *-homomorphism so that $\pi(\mathcal{B}) \subseteq \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$. Then*

$$\mathsf{S}_{\mathcal{E}}(\mathfrak{F}(\mathcal{F}_{\mathcal{B}})) \subseteq \mathfrak{F}(\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}).$$

Proof: Suppose $y_1, y_2 \in \mathcal{F}_{\mathcal{B}}$ and $b \in \mathcal{B}$. Let $y \widehat{\otimes}_{\mathcal{B}} x \in \mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, and let $\theta_{y_1 \cdot b, y_2} \in \mathfrak{F}(\mathcal{F}_{\mathcal{B}})$. Then

$$\mathsf{S}_{\mathcal{E}}(\theta_{y_1\cdot b,y_2})(y \widehat{\otimes}_{\mathcal{B}} x) = \theta_{y_1\cdot b,y_2} y \widehat{\otimes}_{\mathcal{B}} x = y_1 \cdot b \,\langle y_2, y\rangle^{\mathcal{F}}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} x = y_1 \widehat{\otimes}_{\mathcal{B}} \pi(b\,\langle y_2, y\rangle^{\mathcal{F}}_{\mathcal{B}})x. \tag{6.2}$$

For each $y \in \mathcal{F}_{\mathcal{B}}$, consider the map

$$t_y : {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \quad t_y(x) = y \widehat{\otimes}_{\mathcal{B}} x.$$

Then $t_y \in \mathfrak{B}({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})$, with adjoint $t^*_y(y' \widehat{\otimes}_{\mathcal{B}} x') = \pi(\langle y, y'\rangle^{\mathcal{F}}_{\mathcal{B}})x'$. We can rewrite (6.2) as

$$\mathsf{S}_{\mathcal{E}}(\theta_{y_1\cdot b,y_2})(y \widehat{\otimes}_{\mathcal{B}} x) = t_{y_1}\pi(b)t^*_{y_2}(y \widehat{\otimes}_{\mathcal{B}} x).$$

Since $\pi(b) \in \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$, it follows that $\mathsf{S}_{\mathcal{E}}(\theta_{y_1\cdot b,y_2}) \in \mathcal{F}(\mathcal{F}_{\mathcal{B}} \widehat{\otimes}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})$.

For a general θ_{y_1,y_2}, we use the condition that $\mathcal{F}_{\mathcal{B}} \cdot \mathcal{B} = \mathcal{F}_{\mathcal{B}}$ to write $y_1 = \sum_{\alpha=1}^{k} y_1^{\alpha} \cdot b_{\alpha}$ and we repeat the argument above. □

We now prove Theorem 6.1 following [25, 34].
Proof: The fact that an equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is invertible is a direct consequence of (5.9) and (5.10).

To prove the other direction, suppose that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is an invertible inner-product bimodule, with inverse ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$. By Lemma 6.3, we have two isomorphisms

$$\mathfrak{B}(\mathcal{B}) \xrightarrow{\mathsf{S}_{\mathcal{E}}} \mathfrak{B}(\mathcal{E}_{\mathcal{A}}) \xrightarrow{\mathsf{S}_{\mathcal{E}'}} \mathfrak{B}(\mathcal{B}), \tag{6.3}$$

whose composition is the identity. Recall that $\mathcal{B} = \mathfrak{F}(\mathcal{B}) \subseteq \mathfrak{B}(\mathcal{B})$. We claim that

$$\mathsf{S}_{\mathcal{E}'}(\mathfrak{F}(\mathcal{E}_{\mathcal{A}})) \subseteq \mathcal{B}.$$

Indeed, for $x_1, x_2 \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and $b \in \mathcal{B}$, we have $\mathsf{S}_{\mathcal{E}'}(\theta_{b\cdot x_1,x_2}) = b\mathsf{S}_{\mathcal{E}'}(\theta_{x_1,x_2})$, which must be in $\mathcal{B}$ since $\mathcal{B} \subset \mathfrak{B}(\mathcal{B})$ is an ideal. For a general θ_{x_1,x_2}, we use the condition $\mathcal{B} \cdot \mathcal{E} = \mathcal{E}$ to write $x_1 = \sum_{\alpha=1}^{k} b_\alpha \cdot x_1^\alpha$ and apply the same argument. By symmetry, it then follows that

$$\mathsf{S}_{\mathcal{E}}(\mathfrak{F}(\mathcal{E}'_{\mathcal{B}})) \subseteq \mathcal{A}. \tag{6.4}$$

We now claim that

$$\mathsf{S}_{\mathcal{E}}(\mathcal{B}) \subseteq \mathfrak{F}_{\mathcal{A}}(\mathcal{E}). \tag{6.5}$$

By Lemma 6.2, ${}_{\mathfrak{F}(\mathcal{E}')}\mathcal{E}_{\mathcal{B}}$ is a *-equivalence bimodule. Let us consider its conjugate ${}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')}$. Then

$${}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \cong {}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')} \mathbin{\widehat{\otimes}}_{\mathfrak{F}(\mathcal{E}')} {}_{\mathfrak{F}(\mathcal{E}')}\mathcal{E}_{\mathcal{B}},$$

and, as a consequence,

$${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \mathbin{\widehat{\otimes}}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')} \mathbin{\widehat{\otimes}}_{\mathfrak{F}(\mathcal{E}')} ({}_{\mathfrak{F}(\mathcal{E}')}\mathcal{E}_{\mathcal{B}} \mathbin{\widehat{\otimes}}_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \cong {}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')} \mathbin{\widehat{\otimes}}_{\mathfrak{F}(\mathcal{E}')} \mathcal{A}, \tag{6.6}$$

where we regard $\mathcal{A}$ as a left $\mathfrak{F}(\mathcal{E}')$-module via (6.4).

Since ${}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')}$ is a *-equivalence bimodule, it follows that (see Remark 5.10) there is a natural identification

$$\mathcal{B} \cong \mathfrak{F}(\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')}). \tag{6.7}$$

Let us now consider the map

$$\mathsf{S}_{\mathcal{A}} : \mathfrak{B}({}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')}) \longrightarrow \mathfrak{B}({}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')} \mathbin{\widehat{\otimes}}_{\mathfrak{F}(\mathcal{E}')} \mathcal{A}).$$

By (6.4), $\mathfrak{F}(\mathcal{E}'_{\mathcal{B}})$ acts on $\mathcal{A}$ via finite-rank operators; since ${}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')} \cdot \mathfrak{F}({}_{\mathfrak{F}(\mathcal{E}')}\mathcal{E}_{\mathcal{B}}) = {}_{\mathcal{B}}\overline{\mathcal{E}'}_{\mathfrak{F}(\mathcal{E}')}$, we can apply Lemma 6.4 and use (6.6) and (6.7) to conclude that (6.5) holds. We can restrict the isomorphisms in (6.3) to

$$\mathcal{B} \xrightarrow{\mathsf{R}_{\mathcal{E}}} \mathfrak{F}(\mathcal{E}_{\mathcal{A}}) \xrightarrow{\mathsf{R}_{\mathcal{E}'}} \mathcal{B},$$

which implies that $\mathsf{R}_{\mathcal{E}'}(\mathfrak{F}(\mathcal{E}_{\mathcal{A}})) = \mathcal{B}$; since $\mathsf{R}_{\mathcal{E}'}$ is injective, see Lemma 6.3, there is a natural *-isomorphism $\mathcal{B} \cong \mathfrak{F}({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})$, so ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a *-equivalence bimodule, again by Remark 5.10.

If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$ are pre-Hilbert bimodules, by uniqueness of inverses it follows that

$${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}} \cong {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{B}}$$

as pre-Hilbert bimodules, so the $\mathcal{B}$-valued inner product on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ must be completely positive. So ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a strong equivalence bimodule. □

The invertible arrows in *-Alg (resp. *-Alg^+) form a "large" groupoid Pic^* (resp. $\mathsf{Pic}^{\mathrm{str}}$), called the ***-Picard groupoid** (resp. **strong Picard groupoid**). By Theorem 6.1, orbits of Pic^* (resp. $\mathsf{Pic}^{\mathrm{str}}$) are *-Morita equivalence (resp. strong Morita equivalence) classes and isotropy groups are isomorphism classes of self-*-Morita equivalences (resp. self-strong Morita equivalences), called ***-Picard groups** (resp. **strong Picard groups**).

If we restrict Pic, Pic^* and $\mathsf{Pic}^{\mathrm{str}}$ to *unital* *-algebras over C, we obtain natural groupoid homomorphisms

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}^* \longrightarrow \mathsf{Pic}, \tag{6.8}$$

covering the identity on the base. On morphisms, the first arrow "forgets" the complete positivity of inner products, while the second just picks the bimodules and "forgets" all the extra structure.

In Section 7, we will discuss further conditions on unital *-algebras under which the canonical morphism

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic} \tag{6.9}$$

is injective and surjective.

Remark 6.5 *The first arrow in* (6.8) *is generally not surjective since a bimodule may have inner products with different signatures. For the same reason, the second arrow is not injective in general.*

6.2 Strong Picard groupoids of C^*-algebras

Let C^* be the category whose objects are C^*-algebras and morphisms are isomorphism classes of Hilbert bimodules, see e.g. [25]; the composition is given by Rieffel's internal tensor product in the C^*-algebraic sense. The groupoid of invertible morphisms in this context will be denoted by $\mathsf{Pic}^{\mathrm{str}}_{C^*}$. The isotropy groups of $\mathsf{Pic}^{\mathrm{str}}_{C^*}$ are the Picard groups of C^*-algebras as in [8].

It is shown in [19, 25, 34] that Rieffel's notion of strong Morita equivalence of C^*-algebras coincides with the notion of isomorphism in C^*. We will show how this result can be recovered from Theorem 6.1.

For a C^*-algebra $\mathcal{A}$, let $\mathcal{P}(\mathcal{A})$ be its minimal dense ideal, also referred to as its **Pedersen ideal**, see [28]. Just as $\mathcal{A}$ itself, $\mathcal{P}(\mathcal{A})$ is non-degenerate and idempotent. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, let ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ be a Hilbert bimodule (in the C^*-algebraic sense, see e.g. [25, Def. 3.2]) with inner product $\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}}$, and consider the $(\mathcal{P}(\mathcal{B}),\mathcal{P}(\mathcal{A}))$-bimodule

$$\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) := \mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} \cdot \mathcal{P}(\mathcal{A}).$$

Lemma 6.6 *The bimodule $\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$, together with the restriction of $\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}}$, is a pre-Hilbert $(\mathcal{P}(\mathcal{B}),\mathcal{P}(\mathcal{A}))$-bimodule (as in Definition 4.3).*

PROOF: It is clear that $\mathcal{P}(\mathcal{B})\cdot\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) = \mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$, and $\left\langle \mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}), \mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})\right\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}} \subseteq \mathcal{P}(\mathcal{A})$, since $\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}}$ is $\mathcal{A}$-linear and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{A}$ is an ideal.

For any $n \in \mathbb{N}$, $\mathcal{P}(M_n(\mathcal{A})) = M_n(\mathcal{P}(\mathcal{A}))$. So, by [11, Lem. 3.2], $A \in M_n(\mathcal{P}(\mathcal{A}))^+$ (in the algebraic sense of Section 2) if and only if $A \in M_n(\mathcal{A})^+$. Since $\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}}$ is completely positive, so is its restriction to $\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$ taking values in $\mathcal{P}(\mathcal{A})$. ☐

The next example shows that $\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) \subseteq {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}\cdot\mathcal{P}(\mathcal{A})$ is essencial to guarantee that the restriction of the inner product takes values in $\mathcal{P}(\mathcal{A})$.

Example 6.7 *Let X be a locally compact Hausdorff space, and consider $\mathcal{A} = C_\infty(X)$, the algebra of continuous functions vanishing at ∞, and $\mathcal{B} = \mathbb{C}$. Then $\mathcal{E} = C_\infty(X)$ is naturally a $(\mathcal{B},\mathcal{A})$-Hilbert bimodule. Since $\mathcal{B}$ is unital, $\mathcal{B} = \mathcal{P}(\mathcal{B})$ and $\mathcal{P}(\mathcal{B})\cdot\mathcal{E} = \mathcal{E}$. But $\mathcal{P}(\mathcal{A}) = C_0(X)$ is the algebra of compactly supported functions. If X is not compact, then $\langle\mathcal{E},\mathcal{E}\rangle^{\mathcal{E}}_{\mathcal{A}} \nsubseteq \mathcal{P}(\mathcal{A})$.*

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be C^*-algebras. For Hilbert bimodules ${}_{\mathcal{C}}\widehat{\mathcal{F}}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$, we denote their Rieffel internal tensor product in the C^*-algebraic sense by ${}_{\mathcal{C}}\widehat{\mathcal{F}}_{\mathcal{B}}\overline{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$, see [29–31]. A direct verification gives

Lemma 6.8 *There is a canonical isomorphism $\mathcal{P}({}_{\mathcal{C}}\widehat{\mathcal{F}}_{\mathcal{B}}\overline{\otimes}_{\mathcal{B}}\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) \cong \mathcal{P}({}_{\mathcal{C}}\widehat{\mathcal{F}}_{\mathcal{B}})\,\widehat{\otimes}_{\mathcal{B}}\,\mathcal{P}({}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$.*

Let us write ${}^*\text{-}\mathsf{Alg}^{(+)}$ for either ${}^*\text{-}\mathsf{Alg}$ or ${}^*\text{-}\mathsf{Alg}^+$. With some abuse of notation, it follows from Lemmas 6.6 and 6.8 that we can define a functor

$$\mathcal{P} : \mathsf{C}^* \longrightarrow {}^*\text{-}\mathsf{Alg}^{(+)} \tag{6.10}$$

as follows: on objects, $\mathcal{A} \mapsto \mathcal{P}(\mathcal{A})$; on morphisms, $\mathcal{P}([\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}]) = [\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})]$. Here $[\ \ ]$ denotes the isomorphism class of a (pre-)Hilbert bimodule.

Any pre-Hilbert bimodule of the form $\mathcal{E} = \mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$ must satisfy $\mathcal{E} \cdot \mathcal{P}(\mathcal{A}) = \mathcal{E}$. So the maps on morphisms induced by (6.10), $\mathcal{P} : \mathrm{Mor}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Mor}(\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}})$, are not surjective in general. However, as we will see, the situation changes if we restrict $\mathcal{P}$ to morphisms which are invertible.

Our main analytical tool is the next result, see [2].

Proposition 6.9 *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras.*

i) If ${}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$ is a strong (or *-)equivalence bimodule (as in Def. 5.1), then it can be completed to a C^*-algebraic strong equivalence bimodule ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ in such a way that $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) \cong {}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$.

ii) If ${}_{\mathcal{B}}\widehat{\mathcal{F}}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ are C^*-algebraic strong equivalence bimodules with $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{F}}_{\mathcal{A}}) \cong \mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$, then ${}_{\mathcal{B}}\widehat{\mathcal{F}}_{\mathcal{A}} \cong {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$.

PROOF: The proof of *i*) follows from the results in [2]. Note that $\mathcal{E}_{\mathcal{P}(\mathcal{A})}$ can be completed to a full Hilbert $\mathcal{A}$-module $\widehat{\mathcal{E}}_{\mathcal{A}}$, see e.g. [24, p. 5], so that ${}_{\mathfrak{K}(\widehat{\mathcal{E}}_{\mathcal{A}})}\widehat{\mathcal{E}}_{\mathcal{A}}$ is a strong equivalence bimodule. Here $\mathfrak{K}(\widehat{\mathcal{E}}_{\mathcal{A}})$ denotes the "compact" operators on $\widehat{\mathcal{E}}_{\mathcal{A}}$. Note that $\mathcal{E}$ is naturally an $\mathcal{A}$-module and sits in $\widehat{\mathcal{E}}_{\mathcal{A}}$ as a dense $\mathcal{A}$-submodule. So $\mathcal{P}(\mathcal{B}) = \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ is dense in $\mathfrak{K}(\widehat{\mathcal{E}}_{\mathcal{A}})$, and $\mathcal{B}$ is naturally *-isomorphic to $\mathfrak{K}(\widehat{\mathcal{E}}_{\mathcal{A}})$. So ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ is a C^*-algebraic strong equivalence bimodule. It follows from [2, Thm. 2.4] that any $(\mathcal{P}(\mathcal{B}), \mathcal{P}(\mathcal{A}))$-*-equivalence bimodule is already a strong equivalence bimodule, so the same results hold.

It follows from [2] that

$$\mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} = {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} \cdot \mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} \cdot \mathcal{P}(\mathcal{A}).$$

Since $\mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}} \cdot \mathcal{P}(\mathcal{A}) = {}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$, we have ${}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}} \subseteq \mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} \cdot \mathcal{P}(\mathcal{A})$. On the other hand, since $\mathcal{P}(\mathcal{B}) \subset \mathfrak{B}(\widehat{\mathcal{E}}_{\mathcal{A}})$ is an ideal, it follows that $\mathcal{E} \subset \widehat{\mathcal{E}}$ is $\mathfrak{B}(\widehat{\mathcal{E}}_{\mathcal{A}})$-invariant. By [2, Prop. 1.5], $\mathcal{P}(\mathcal{B}) \cdot {}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}} \cdot \mathcal{P}(\mathcal{A}) \subseteq \mathcal{E}$. This implies that $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}) = {}_{\mathcal{P}_{\mathcal{B}}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$.

Part *ii*) follows from the fact that ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ is a completion of $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$ and any two completions must be isomorphic. $\square$

Corollary 6.10 *A Hilbert bimodule ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ is invertible in C^* if and only if there exists a $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}$ so that $(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}, {}_{\mathcal{B}}\langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}, \langle\cdot,\cdot\rangle^{\widehat{\mathcal{E}}}_{\mathcal{A}})$ is a (C^*-algebraic) strong equivalence bimodule. In particular, two C^*-algebras are strongly Morita equivalent if and only if they are isomorphic in C^*.*

PROOF: If ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ is invertible in C^*, then $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$ is invertible in ${}^*\text{-}\mathsf{Alg}^+$. By Theorem 6.1, there exists a $\mathcal{B}$-valued inner product making $\mathcal{P}(\,{}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}})$ into an equivalence bimodule. By part *i*) of Prop. 6.9, we can complete it to a C^*-algebraic strong equivalence bimodule, isomorphic to ${}_{\mathcal{B}}\widehat{\mathcal{E}}_{\mathcal{A}}$ as a Hilbert bimodule. $\square$

Corollary 6.11 *For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$,*

$$\mathcal{P} : \mathsf{Pic}^{\mathrm{str}}_{C^*}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathsf{Pic}^{\mathrm{str}}(\mathcal{P}(\mathcal{B}), \mathcal{P}(\mathcal{A})) \tag{6.11}$$

is a bijection. As a result, $\mathsf{Pic}^{\mathrm{str}}_{C^*}$ is equivalent to $\mathsf{Pic}^{\mathrm{str}}$ (or Pic^*) restricted to Pedersen ideals.

It follows that the entire strong Morita theory of C^*-algebras is encoded in the algebraic $\mathsf{Pic}^{\mathrm{str}}$. Note that, for unital C^*-algebras, $\mathcal{P}$ is just the identity on objects.

7 Strong versus ring-theoretic Picard groupoids

It is shown in [5] that unital C^*-algebras are strongly Morita equivalent if and only if they are Morita equivalent as rings. In [14], we have shown that the same is true for hermitian star products. In terms of Picard groupoids, these results mean that $\mathsf{Pic}^{\mathrm{str}}$ and Pic, restricted to unital C^*-algebras or to hermitian star products, have the same orbits. In this section, we study the morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ restricted to unital *-algebras satisfying additional properties, recovering and refining these results in a unified way.

7.1 A restricted class of unital *-algebras

We consider algebraic conditions which capture some important features of the functional calculus of C^*-algebras. Let $\mathcal{A}$ be a unital *-algebra over C. The first property is

(I) For all $n \in \mathbb{N}$ and $A \in M_n(\mathcal{A})$, $\mathbb{1} + A^*A$ is invertible.

As a first remark we see that **(I)** also implies that elements of the form $\mathbb{1} + \sum_{r=1}^{k} A_r^* A_r$ are invertible in $M_n(\mathcal{A})$, simply by applying **(I)** to $M_{nk}(\mathcal{A})$. The relevance of this property is illustrated by the following result [23, Thm. 26]:

Lemma 7.1 *Suppose $\mathcal{A}$ satisfies* **(I)***. Then any idempotent $e = e^2 \in M_n(\mathcal{A})$ is equivalent to a projection $P = P^2 = P^* \in M_n(\mathcal{A})$.*

We also need the following property.

(II) For all $n \in \mathbb{N}$, let $P_\alpha \in M_n(\mathcal{A})$ be pairwise orthogonal projections, i.e. $P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha$, with $\mathbb{1} = \sum_\alpha P_\alpha$ and let $H \in M_n(\mathcal{A})^+$ be invertible. If $[H, P_\alpha] = 0$, then there exists an invertible $U \in M_n(\mathcal{A})$ with $H = U^*U$ and $[P_\alpha, U] = 0$.

Most of our results will follow from a condition slightly weaker than **(II)**:

(II$^-$) For all $n \in \mathbb{N}$, invertible $H \in M_n(\mathcal{A})^+$, and projection P with $[P, H] = 0$, there exists a $U \in M_n(\mathcal{A})$ with $H = U^*U$ and $[P, U] = 0$.

On the other hand, our main examples satisfy a stronger version of **(II)**:

(II$^+$) For all $n \in \mathbb{N}$ and $H \in M_n(\mathcal{A})^+$ invertible there exists an invertible $U \in M_n(\mathcal{A})$ such that $H = U^*U$, and if $[H, P] = 0$ for a projection P then $[U, P] = 0$.

Any unital C^*-algebra fulfills **(I)** and **(II$^+$)** by their functional calculus. In Section 8.3 we show that the same holds for hermitian star products. The importance of condition **(II)** and its variants lies in the next result.

Lemma 7.2 *Let $\mathcal{A}$ satisfy* $\mathbf{(II^-)}$ *and let $P = P^2 = P^* \in M_n(\mathcal{A})$. Then any completely positive and strongly non-degenerate $\mathcal{A}$-valued inner product on $P\mathcal{A}^n$ is isometric to the canonical one.*

PROOF: Given such inner product $\langle \cdot, \cdot \rangle'$ on $P\mathcal{A}^n$, we extend it to the free module $\mathcal{A}^n$ by taking $(\mathbb{1} - P)\mathcal{A}^n$ as orthogonal complement with the canonical inner product of $\mathcal{A}^n$ restricted to it. Then the result follows from Lemma 3.12 and the fact that the isometry on $\mathcal{A}^n$ commutes with P. □

Let R be an arbitrary unital ring. An idempotent $e = (e_{ij}) \in M_n(R)$ is called **full** if the ideal in R generated by e_{ij} coincides with R. One of the main results of Morita theory for rings [3] is that two unital rings R and S are Morita equivalent if and only if $S \cong eM_n(R)e$ for some full idempotent e. The next theorem is an analogous result for strong Morita equivalence.

For a projection $P \in M_n(\mathcal{A})$, we consider $P\mathcal{A}^n$ equipped with its canonical completely positive inner product. Note that P is full if and only if this inner product is full in the sense of (5.1).

Theorem 7.3 *Let $\mathcal{A}$, $\mathcal{B}$ be unital *-algebras and let $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, {}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}, \langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}})$ be a *-equivalence bimodule such that $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ is completely positive. If $\mathcal{A}$ satisfies* $\mathbf{(I)}$ *and* $\mathbf{(II^-)}$ *then:*

i.) There exists a full projection $P = P^2 = P^* \in M_n(\mathcal{A})$ such that $\mathcal{E}_{\mathcal{A}}$ is isometrically isomorphic to $P\mathcal{A}^n$ as a right $\mathcal{A}$-module.

ii.) $\mathcal{B}$ is *-isomorphic to $PM_n(\mathcal{A})P$ via the left action on $\mathcal{E}_{\mathcal{A}}$ and the $\mathcal{B}$-valued inner product is, under this isomorphism, given by the canonical $PM_n(\mathcal{A})P$-valued inner product on $P\mathcal{A}^n$.

iii.) ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is completely positive and hence ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a strong equivalence bimodule.

Conversely, if P is a full projection, then $PM_n(\mathcal{A})P$ is strongly Morita equivalent to $\mathcal{A}$ via $P\mathcal{A}^n$.

PROOF: We know that $\mathcal{E}_{\mathcal{A}}$ is finitely generated and projective. By $\mathbf{(I)}$ we can find a projection P with $\mathcal{E}_{\mathcal{A}} \cong P\mathcal{A}^n$ and by $\mathbf{(II^-)}$ we can choose the isomorphism to be isometric to the canonical inner product, according to Lemma 7.2, proving the first statement.

Since ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ is full, the left action map is an injective *-homomorphism of $\mathcal{B}$ into $\mathfrak{B}(\mathcal{E}_{\mathcal{A}})$. By compatibility, ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ has to be the canonical one and again by fullness we see that $\mathcal{B}$ is *-isomorphic to $\mathfrak{B}(\mathcal{E}_{\mathcal{A}}) \cong PM_n(\mathcal{A})P$, proving the second statement.

Since $\langle \cdot, \cdot \rangle^{\mathcal{E}}_{\mathcal{A}}$ is full we find $Px_r, Py_r \in P\mathcal{A}^n$ with $\mathbb{1}_{\mathcal{A}} = \sum_{r=1}^k \langle Px_r, Py_r \rangle$. Since $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\mathcal{A}}^*$ we have

$$\sum_r \langle Px_r + Py_r, Px_r + Py_r \rangle = \mathbb{1}_{\mathcal{A}} + \mathbb{1}_{\mathcal{A}} + \sum_r \langle Px_r, Px_r \rangle + \sum_r \langle Py_r, Py_r \rangle.$$

By $\mathbf{(I)}$ and $\mathbf{(II^-)}$ we find an invertible $U \in \mathcal{A}$ such that for $Pz_r = P(x_r + y_r)U^{-1} \in P\mathcal{A}^n$ we have $\mathbb{1}_{\mathcal{A}} = \sum_r \langle Pz_r, Pz_r \rangle$. By compatibility, we get

$$ {}_{\mathcal{B}}\langle Px, Py \rangle^{\mathcal{E}} = \sum_r {}_{\mathcal{B}}\langle Px, Pz_r \rangle^{\mathcal{E}} \; {}_{\mathcal{B}}\langle Pz_r, Py \rangle^{\mathcal{E}} ; \tag{7.1}$$

the complete positivity of ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle^{\mathcal{E}}$ now follows from Remark 3.10. This also shows the last statement. □

We use Theorem 7.3 to show that condition $\mathbf{(I)}$ and $\mathbf{(II)}$ are natural from a Morita-theoretic point of view.

Proposition 7.4 *Conditions* **(I)** *and* **(II$^+$)** *(resp.* **(II)***), together, are strongly Morita invariant.*

PROOF: Assume that $\mathcal{A}$ satisfies **(I)** and **(II$^-$)** and $\mathcal{B}$ is strongly Morita equivalent to $\mathcal{A}$. Then $\mathcal{B} \cong PM_n(\mathcal{A})P$ for some full projection P. If $B \in \mathcal{B}$, then $\mathbb{1}_{\mathcal{B}} + B^*B$, viewed as element in $PM_n(\mathcal{A})P$, can be extended 'block-diagonally' to an element of the form

$$\mathbb{1}_{M_n(\mathcal{A})} + A^*A \tag{7.2}$$

by addition of $\mathbb{1}_{M_n(\mathcal{A})} - P$. By **(I)**, (7.2) has an inverse in $M_n(\mathcal{A})$. By **(II$^-$)**, the inverse is again block-diagonal and hence gives an inverse of $\mathbb{1}_{\mathcal{B}} + B^*B$. Passing from $M_n(\mathcal{A})$ to $M_{nm}(\mathcal{A})$, one obtains the invertibility of $\mathbb{1}_{M_m(\mathcal{B})} + B^*B$ for $B \in M_m(\mathcal{B})$. Hence $\mathcal{B}$ satisfies **(I)**.

Assume that $\mathcal{A}$ satisfies **(II$^+$)**, and let $H \in \mathcal{B}^+$ be invertible. Then

$$H + (\mathbb{1}_{M_n(\mathcal{A})} - P) \in M_n(\mathcal{A})^+$$

is still positive and invertible. So there is an invertible $V \in M_n(\mathcal{A})$ with $H + (\mathbb{1}_{M_n(\mathcal{A})} - P) = V^*V$, commuting with P since $H + (\mathbb{1}_{M_n(\mathcal{A})} - P)$ commutes with P. Thus $U = PVP$ satisfies $U^*U = H$. Moreover, if $Q \in \mathcal{B}$ is a projection with $[H, Q] = 0$, then $PQ = Q = QP$ and hence Q commutes with $H + (\mathbb{1}_{M_n(\mathcal{A})} - P)$. Thus V commutes with Q, and hence U commutes with Q as well. For $M_m(\mathcal{B})$, the reasoning is analogous. So $\mathcal{B}$ satisfies **(II$^+$)**.

An analogous but simpler argument shows the same result for **(II)**. $\square$

7.2 From $\mathsf{Pic}^{\mathrm{str}}$ to Pic

Let us consider the groupoid morphism

$$\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic} \tag{7.3}$$

from the strong Picard groupoid to the (ring-theoretic) Picard groupoid. The next result follows from Theorem 7.3.

Theorem 7.5 *Within the class of unital *-algebras satisfying* **(I)** *and* **(II$^-$)***, the groupoid morphism* (7.3) *is injective.*

For surjectivity, first note that if we define the **Hermitian K_0-group** of a *-algebra $\mathcal{A}$ as the Grothendieck group $K_0^H(\mathcal{A})$ of the semi-group of isomorphism classes of finitely generated projective pre-Hilbert modules over $\mathcal{A}$ equipped with strongly non-degenerate inner products, then Lemmas 7.1 and 7.2 imply that if $\mathcal{A}$ satisfies **(I)** and **(II$^-$)**, then the natural group homomorphism $K_0^H(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ (forgetting inner products) is an isomorphism. For Picard groupoids, however, we will see that (7.3) is not generally surjective, even if **(I)** and **(II)** hold. In order to discuss this surjectivity problem, we consider pairs of *-algebras satisfying the following rigidity property:

(III) Let $\mathcal{A}$ and $\mathcal{B}$ be unital *-algebras, let $P \in M_n(\mathcal{A})$ be a projection, and consider the *-algebra $PM_n(\mathcal{A})P$. If $\mathcal{B}$ and $PM_n(\mathcal{A})P$ are isomorphic as unital algebras, then they are *-isomorphic.

The following are the motivating examples.

Example 7.6 • *For unital C^*-algebras, condition **(III)** is always satisfied: If $\mathcal{A}$ is a C^*-algebra, then so is $PM_n(\mathcal{A})P$, and **(III)** follows from the fact that two C^*-algebras which are isomorphic as algebras must be *-isomorphic [32, Thm. 4.1.20].*

- *Another class of unital *-algebras satisfying **(III)** is that of hermitian star products on a Poisson manifold M, see Section 8. In this case, condition **(III)** follows from the more general fact that two equivalent star products which are compatible with involutions of the form $f \mapsto \overline{f} + o(\lambda)$ must be *-equivalent, see [14, Lem. 5].*

For unital algebras $\mathcal{A}$ and $\mathcal{B}$, let us consider the action of the automorphism group $\mathrm{Aut}(\mathcal{B})$ on the set of morphisms $\mathsf{Pic}(\mathcal{B}, \mathcal{A})$ by

$$(\Phi, [\mathcal{E}]) \mapsto [\,{}_\Phi\mathcal{E}]; \tag{7.4}$$

here $\mathcal{E}$ is a $(\mathcal{B}, \mathcal{A})$-equivalence bimodule (in the ring-theoretic sense), $\Phi \in \mathrm{Aut}(\mathcal{B})$ and ${}_\Phi\mathcal{E}$ coincides with $\mathcal{E}$ as a C-module, but its $(\mathcal{B}, \mathcal{A})$-bimodule structure is given by

$$b \cdot_\Phi x \cdot a := \Phi(b) \cdot x \cdot a,$$

see e.g. [3, 13].

Proposition 7.7 *If $\mathcal{A}$ and $\mathcal{B}$ are unital *-algebras satifying* **(III)***, and if $\mathcal{A}$ satisfies* **(I)** *and* **(II$^-$)***, then the composed map*

$$\mathsf{Pic}^{\mathrm{str}}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{B}, \mathcal{A})\big/\mathrm{Aut}(\mathcal{B}) \tag{7.5}$$

is onto.

Proof: Let ${}_\mathcal{B}\mathcal{E}_\mathcal{A}$ be an equivalence bimodule for ordinary Morita equivalence. We know that $\mathcal{E}_\mathcal{A} \cong e\mathcal{A}^n$, as right $\mathcal{A}$-modules, for some full idempotent $e = e^2 \in M_n(\mathcal{A})$, and $\mathcal{B} \cong eM_n(\mathcal{A})e$ as associative algebras via the left action.

By properties **(I)** and **(II$^-$)**, we can replace e by a projection $P = P^2 = P^*$ and consider the canonical $\mathcal{A}$-valued inner product on $P\mathcal{A}^n$. Then $PM_n(\mathcal{A})P$ and $\mathcal{A}$ are strongly Morita equivalent via $P\mathcal{A}^n$, see Theorem 7.3. The identification $\mathcal{B} \cong PM_n(\mathcal{A})P$ induces a *-involution † on $\mathcal{B}$, possibly different from the original one. By assumption, there exists a *-isomorphism

$$\Phi : (\mathcal{B}, {}^*) \longrightarrow (\mathcal{B}, {}^\dagger)$$

in such a way that ${}_\Phi\mathcal{E}$ becomes a strong equivalence bimodule. □

Corollary 7.8 *Within a class of unital *-algebras satisfying* **(I)***,* **(II)** *and* **(III)***, ring-theoretic Morita equivalence implies strong Morita equivalence (so the two notions coincide).*

Proposition 7.7 is an algebraic refinement of Beer's result for unital C^*-algebras [5], which is recovered by Corollary 7.8.

The question to be addressed is when (7.3) is surjective, and not only surjective modulo automorphisms. The obstruction for surjectivity is expressed in the next condition.

(IV) For any $\Phi \in \mathrm{Aut}(\mathcal{A})$ there is an invertible $U \in \mathcal{A}$ such that $\Phi^*\Phi^{-1} = \mathrm{Ad}(U^*U)$, where $\Phi^*(a) = \Phi(a^*)^*$.

Lemma 7.9 *Assume that a unital* **-algebra* $\mathcal{A}$ *satisfies* **(I)** *and* **(II$^-$)**, *and let* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *be a ring-theoretic equivalence bimodule whose class* $[{}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}] \in \mathsf{Pic}(\mathcal{B}, \mathcal{A})$ *is in the image of* (7.3)*. Then its entire* $\mathrm{Aut}(\mathcal{B})$*-orbit is in the image of* (7.3) *if and only if* $\mathcal{B}$ *satisfies* **(IV)**.

PROOF: If the isomorphism class of ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is in the image of (7.3), then there is a full completely positive $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$ which is uniquely determined up to isometry by the right $\mathcal{A}$-module structure.

If $\Phi \in \mathrm{Aut}(\mathcal{B})$, then $[{}_{\Phi}\mathcal{E}]$ is in the image of (7.3) if and only if there is an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle'$, necessarily isometric to $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$ by Lemma 7.2, which is compatible with the $\mathcal{B}$-action modified by Φ. In this case, the $\mathcal{B}$-valued inner product is determined by compatibility, and its complete positivity follows from Theorem 7.3. Since there exists an invertible $U \in \mathfrak{B}_{\mathcal{A}}(\mathcal{E}) = \mathcal{B}$ such that

$$\langle x, y\rangle' = \langle U \cdot x, U \cdot y\rangle^{\mathcal{E}}_{\mathcal{A}},$$

condition **(IV)** easily follows from the non-degeneracy of $\langle\cdot,\cdot\rangle^{\mathcal{E}}_{\mathcal{A}}$. □

Corollary 7.10 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital* **-algebras satisfying* **(III)**, *and suppose that* $\mathcal{A}$ *satisfies properties* **(I)** *and* **(II$^-$)**. *Then the first map in* (7.5) *is surjective if and only if* $\mathcal{B}$ *satisfies* **(IV)**.

Corollary 7.11 *Within a class of unital* **-algebras satisfying* **(I)**, **(II$^-$)** *and* **(III)**, *property* **(IV)** *is strongly Morita invariant.*

Example 7.12 (The case of C^*-algebras)
For a unital C^**-algebra* $\mathcal{A}$, *any automorphism* $\Phi \in \mathrm{Aut}(\mathcal{A})$ *can be uniquely decomposed as*

$$\Phi = \mathrm{e}^{\mathrm{i}D} \circ \Psi, \tag{7.6}$$

where Ψ *is a* **-automorphism and* D *is a* **-derivation, i.e., a derivation with* $D(a^*) = D(a)^*$, *see [27, Thm. 7.1] and [32, Cor. 4.1.21]. In this case,* ***(IV)*** *is satisfied if and only if, for any* **-derivation* D, *the automorphism* $\mathrm{e}^{\mathrm{i}D}$ *is inner.*

Let us discuss some concrete examples. If $\mathcal{A}$ *is a simple* C^**-algebra or a* W^**-algebra, then any automorphism is inner, see [32, Thm. 4.1.19]. So* ***(IV)*** *is automatically satisfied, and* (7.3) *is surjective.*

In general, however, there may be automorphisms Φ *with* $\Phi^* = \Phi^{-1}$ *such that* Φ^2 is not *inner, in which case* (7.3) *is not surjective. For example, consider the compact operators* $\mathfrak{K}(\mathcal{H})$ *on a Hilbert space* $\mathcal{H}$ *with countable Hilbert basis* e_n. *Define* $A = A^* \in \mathfrak{B}(\mathcal{H})$ *by*

$$Ae_{2n} = 2e_{2n} \quad \textit{and} \quad Ae_{2n+1} = e_{2n+1}.$$

Then $\mathrm{Ad}(A)$ *induces an automorphism* Φ *of* $\mathfrak{K}(\mathcal{H}) \oplus \mathbb{C}\mathbb{1}$ *which satisfies* $\Phi^* = \Phi^{-1}$ *but whose square is not inner: clearly* $\mathrm{Ad}(A)^2 = \mathrm{Ad}(A^2)$ *and there is no* $B \in \mathfrak{K}(\mathcal{H}) \oplus \mathbb{C}\mathbb{1}$ *with* $\mathrm{Ad}(A^2) = \mathrm{Ad}(B^*B)$.

8 Hermitian deformation quantization

We now show that, just as C^*-algebras, hermitian star products can be treated in the framework of Section 7. The key observation is that the properties considered in Section 7 are rigid under deformation quantization.

8.1 Hermitian and positive deformations of *-algebras

Let $\mathcal{A}$ be a *-algebra over C. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be an associative deformation of $\mathcal{A}$, in the sense of [20]. We call this deformation **hermitian** if

$$(a_1 \star a_2)^* = a_2^* \star a_1^*,$$

for all $a_1, a_2 \in \mathcal{A}$. In this case, * can be extended to a *-involution making $\boldsymbol{\mathcal{A}}$ into a *-algebra over $\mathsf{C}[[\lambda]]$. Note that $\mathsf{C}[[\lambda]] = \mathsf{R}[[\lambda]](\mathrm{i})$, and $\mathsf{R}[[\lambda]]$ has a natural ordering induced from R, see Section 1, so all the notions of positivity of Section 2 make sense for $\boldsymbol{\mathcal{A}}$. We assume λ to be real, so $\overline{\lambda} = \lambda > 0$.

If $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r : \mathcal{A}[[\lambda]] \longrightarrow \mathsf{C}[[\lambda]]$ is a positive $\mathsf{C}[[\lambda]]$-linear functional with respect to $\star$, then its classical limit $\omega_0 : \mathcal{A} \longrightarrow \mathsf{C}$ is a positive C-linear functional on $\mathcal{A}$. We say that a hermitian deformation $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ is **positive** [10, Def. 4.1] if every positive linear functional on $\mathcal{A}$ can be *deformed* into a positive linear functional of $\boldsymbol{\mathcal{A}}$. In the spirit of complete positivity, we call a deformation $\boldsymbol{\mathcal{A}}$ **completely positive** if, for all $n \in \mathbb{N}$, the *-algebras $M_n(\boldsymbol{\mathcal{A}})$ are positive deformations of $M_n(\mathcal{A})$. We remark that not all hermitian deformations are positive [15].

In the following, we shall consider unital *-algebras and assume that hermitian deformations preserve the units.

8.2 Rigidity of properties (I) and (II)

The next observation is a direct consequence of the definitions.

Lemma 8.1 *Let $\boldsymbol{\mathcal{A}}$ be a positive deformation of $\mathcal{A}$. If $\boldsymbol{a} = a + o(\lambda) \in \boldsymbol{\mathcal{A}}$ is positive, then its classical limit $a \in \mathcal{A}$ is also positive.*

A property of a *-algebra $\mathcal{A}$ is said to be **rigid** under a certain type of deformation if any such deformation satisfies the same property. Clearly, property **(I)** is rigid under hermitian deformations. More interestingly,

Proposition 8.2 *Property* **(II)** *is rigid under completely positive deformations.*

PROOF: Let $\boldsymbol{H} = H + o(\lambda) \in M_n(\boldsymbol{\mathcal{A}})^+$ be positive and invertible, and let $\boldsymbol{P}_\alpha = P_\alpha + o(\lambda) \in M_n(\boldsymbol{\mathcal{A}})$ be pairwise orthogonal projections satisfying $\sum_\alpha \boldsymbol{P}_\alpha = \mathbb{1}$ and $[\boldsymbol{H}, \boldsymbol{P}_\alpha]_\star = 0$. By Lemma 8.1, $H \in M_n(\mathcal{A})$ is positive and invertible. Since $[P_\alpha, H] = 0$, by **(II)** there exists an invertible $U \in M_n(\mathcal{A})$ with $H = U^* U$ and $[P_\alpha, U] = 0$. In particular, $P_\alpha U P_\alpha \in P_\alpha M_n(\mathcal{A}) P_\alpha$ is invertible, with inverse $P_\alpha U^{-1} P_\alpha$; here we consider $P_\alpha M_n(\mathcal{A}) P_\alpha$ as a unital *-algebra with unit P_α as before. Hence

$$P_\alpha H P_\alpha = P_\alpha U^* P_\alpha P_\alpha U P_\alpha. \tag{8.1}$$

But $\boldsymbol{P}_\alpha \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}_\alpha$ induces a hermitian deformation $\star_\alpha$ of $P_\alpha M_n(\mathcal{A}) P_\alpha$, so we can apply [9, Lem. 2.1] to deform (8.1), i.e., there exists an invertible $\boldsymbol{U}_\alpha \in \boldsymbol{P}_\alpha \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}_\alpha$ such that

$$\boldsymbol{P}_\alpha \star \boldsymbol{H} \star \boldsymbol{P}_\alpha = \boldsymbol{P}_\alpha \star \boldsymbol{U}_\alpha^* \star \boldsymbol{P}_\alpha \star \boldsymbol{P}_\alpha \star \boldsymbol{U}_\alpha \star \boldsymbol{P}_\alpha. \tag{8.2}$$

If we set $\boldsymbol{U} = \sum_\alpha \boldsymbol{U}_\alpha$, then it is easy to check that $\boldsymbol{U}$ commutes with the $\boldsymbol{P}_\alpha$, is invertible and $\boldsymbol{H} = \boldsymbol{U}^* \star \boldsymbol{U}$. □

By only considering the projections P and $(\mathbb{1} - P)$, one can show that property **(II$^-$)** is rigid under completely positive deformations as well.

As a consequence, any completely positive deformation of a *-algebra $\mathcal{A}$ satisfying **(I)** and **(II)** (or **(II$^-$)**) also satisfies these properties and those resulting from them, as discussed in Section 7.

8.3 Hermitian star products

A **star product** [4] on a Poisson manifold $(M, \{\cdot,\cdot\})$ is a formal deformation $\star$ of $C^\infty(M)$,

$$f \star g = fg + \sum_{r=1}^{\infty} \lambda^r C_r(f,g),$$

for which each C_r is a biddiferential operator and

$$C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}.$$

Following Section 8.1, a star product is **hermitian** if $\overline{(f \star g)} = \overline{g} \star \overline{f}$.

In [10, Prop. 5.1], we proved that any hermitian star product on a symplectic manifold is a positive deformation. This turns out to hold much more generally.

Theorem 8.3 *Any hermitian star product on a Poisson manifold is a completely positive deformation.*

The proof consists of showing that any hermitian star product can be realized as a subalgebra of a formal Weyl algebra, and then use the results in the symplectic case [10], see [15].

Since $C^\infty(M)$ satisfies **(I)** and **(II)**, we have

Corollary 8.4 *Hermitian star products on Poisson manifolds satisfy properties* **(I)** *and* **(II)**.

Corollary 8.5 *Let $E \longrightarrow M$ be a hermitian vector bundle. Then any hermitian deformation of $\Gamma^\infty(\mathrm{End}(E))$ satisfies* **(I)** *and* **(II)**.

PROOF: Any such deformation is strongly Morita equivalent to some hermitian star product on M, see [9, 14], so the result follows from Prop. 7.4. □

Knowing that hermitian star products are completely positive deformations, we can use the star exponential to show that they satisfy a property which is much stronger than **(II)**.

Proposition 8.6 *Let $\star$ be a hermitian star product on M. Then any positive invertible $H \in M_n(C^\infty(M)[[\lambda]])^+$ has a unique positive invertible $\star$-square root $\sqrt[\star]{H}$ such that $[\sqrt[\star]{H}, A]_\star = 0$ if and only if $[H, A]_\star = 0$, $A \in M_n(C^\infty(M)[[\lambda]])$. In particular, $\star$ satisfies* **(II$^+$)**.

PROOF: If $H = H_0 + o(\lambda)$ then, by Lemma 8.1, H_0 is positive in $M_n(C^\infty(M))$ and invertible. This implies that H_0 has a unique real logarithm $\ln(H_0) \in M_n(C^\infty(M))$. Using the star exponential as in [13, 14], extended to matrix-valued functions, we conclude that there exists a unique real star logarithm $\mathrm{Ln}(H) = \ln(H_0) + o(\lambda)$ of H, whence $\mathrm{Exp}(\mathrm{Ln}(H)) = H$. It follows that $\sqrt[\star]{H} = \mathrm{Exp}(\frac{1}{2}\mathrm{Ln}(H))$ has the desired property. □

This shows that many important features of the functional calculus of C^*-algebras are present in formal deformation quantization.

8.4 The strong Picard groupoid for star products

Since hermitian star products satisfy **(I)**, **(II$^+$)** and **(III)**, it follows that Thm. 7.5 and Prop. 7.7 hold.

Corollary 8.7 *For hermitian star products,* $\mathsf{Pic}^{\mathrm{str}}$ *and* Pic *have the same orbits and the canonical morphism* $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ *is injective.*

Corollary 8.7 recovers [14, Thm. 2]. The orbits and isotropy groups of the Picard groupoid in deformation quantization were studied in [13, 14, 22].

The next result reveals an interesting similarity between the structure of the automorphism group of C^*-algebras and hermitian star products, see Example 7.12.

Proposition 8.8 *Let $\star$ be a hermitian star product on a Poisson manifold M, and let $\Phi \in \mathrm{Aut}(\star)$ be an automorphism of $\star$. Then there exists a unique *-derivation D and a unique *-automorphism Ψ such that*

$$\Phi = \mathrm{e}^{\mathrm{i}\lambda D} \circ \Psi. \tag{8.3}$$

PROOF: Writing $\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r$, we know that $\Phi_0 = \varphi^*$ is the pull-back by some Poisson diffeomorphism $\varphi : M \longrightarrow M$. In particular, $\Phi_0(\overline{f}) = \overline{\Phi_0(f)}$.

Let us define a new star product $\star'$ by $f \star' g = \varphi^*(\varphi_* f \star \varphi_* g)$. Then $\star'$ is hermitian and *-isomorphic to $\star$ via φ^*. If we write $\Phi = T \circ \varphi^*$, then $T = \mathsf{id} + o(\lambda)$. Hence $\star$ and $\star'$ are equivalent via T.

By [14, Cor. 4], there exists a *-equivalence $\tilde{T}$ between $\star$ and $\star'$, so $\Psi^{(1)} = \varphi^* \circ \tilde{T}$ is a *-automorphism of $\star$ deforming φ^*. By [14, Lem. 5], there is a unique derivation $D^{(1)}$ so that $\Phi = \mathrm{e}^{\mathrm{i}\lambda D^{(1)}} \circ \Psi^{(1)}$, and we can write $D^{(1)} = D_1^{(1)} + \mathrm{i} D_2^{(1)}$, where each $D_i^{(1)}$ is a *-derivation. Now the Baker-Campbell-Hausdorff formula defines a derivation $D^{(2)}$ by $\mathrm{e}^{\mathrm{i}\lambda D} \circ \mathrm{e}^{\lambda D_2^{(1)}} = \mathrm{e}^{\mathrm{i}\lambda D^{(2)}}$, in such a way that the imaginary part of $D^{(2)}$ is of order λ. By induction, we can split off the *-automorphisms $\mathrm{e}^{\lambda D_2^{(k)}}$ to obtain (8.3). A simple computation shows the uniqueness of this decomposition. □

Using this result, we proceed in total analogy with the case of C^*-algebras. A derivation of a star product $\star$ is **quasi-inner** if it is of the form $D = \frac{\mathrm{i}}{\lambda}\mathrm{ad}(H)$ for some $H \in C^\infty(M)[[\lambda]]$.

Theorem 8.9 *Let $\star$, $\star'$ be Morita equivalent hermitian star products. Then*

$$\mathsf{Pic}^{\mathrm{str}}(\star, \star') \longrightarrow \mathsf{Pic}(\star, \star') \tag{8.4}$$

is a bijection if and only if all derivations of $\star$ are quasi-inner.

PROOF: We know that (8.4) is injective, and it is surjective if and only if any automorphism Φ of $\star$ satisfies

$$\overline{\Phi} \circ \Phi^{-1} = \mathrm{Ad}(\overline{U} \star U)$$

for some invertible function U, see Corollary 7.10. Using (8.3), this is equivalent to the condition that, for any *-derivation D, $\mathrm{e}^{-2\mathrm{i}\lambda D} = \mathrm{Ad}(\overline{U} \star U)$.

Since $\overline{U} \star U = \overline{U_0} U_0 + o(\lambda)$ for some invertible U_0, we can use the unique real star logarithm $\mathrm{Ln}(H)$ of $H = \overline{U} \star U$ to write

$$\mathrm{e}^{-2\mathrm{i}\lambda D} = \mathrm{Ad}(\mathrm{Exp}(\mathrm{Ln}(H))) = \mathrm{e}^{\mathrm{ad}(\mathrm{Ln}(H))}.$$

Hence we have the equivalent condition $D = \frac{\mathrm{i}}{\lambda}\mathrm{ad}(\frac{1}{2}\mathrm{Ln}(H))$ for any *-derivation. Since any derivation can be decomposed into real and imaginary parts, each being a *-derivation, the statement follows. □

If $\star$ is a hermitian star product for which Poisson derivations can be deformed into $\star$-derivations in such a way that hamiltonian vector fields correspond to quasi-inner derivations, then $\star$-derivations modulo quasi-inner derivations are in bijection with formal power series with coefficients in the first Poisson cohomology, see e.g. [13, 21]. In this case, (8.4) is an isomorphism if and only if the first Poisson cohomology group vanishes. We recall that any Poisson manifold admits star products with this property [17], and any symplectic star product is of this type.

Corollary 8.10 *If $\star$ is a hermitian star product on a symplectic manifold M, then* (8.4) *is an isomorphism if and only if* $\mathrm{H}^1_{\mathrm{dR}}(M, \mathbb{C}) = \{0\}$.

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