THE CONE OF EFFECTIVE DIVISORS OF LOG VARIETIES AFTER BATYREV

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Abstract. The aim of these notes is to discuss Batyrev's theorem on the structure of the cone of nef curves (or its dual cone, the cone of pseudo-effective divisors) on terminal threefolds. We point out a problem in Batyrev's original proof, and explain a way of fixing it. In order to complete Batyrev's proof, we rely on boundedness of terminal Fano threefolds. For this reason, as it stands, this proof does not generalize to the log terminal case, as it has been claimed in previous papers.

In [Bat92] Batyrev studied the cone of pseudo-effective divisors on $\mathbb{Q}$-factorial terminal threefolds and its dual cone, the cone of nef curves. Given a uniruled $\mathbb{Q}$-factorial terminal threefold X, and an ample divisor H on X, he showed that the effective threshold of H (see Definition 1.5 below) is a rational number. Using similar arguments, Fujita generalized this result to log terminal pairs (X, Δ), with $\dim X = 3$ (see [Fuj96]). In [Bat92], using the rationality of the effective threshold and the minimal model program, Batyrev obtained a structure theorem for the cone of nef curves on $\mathbb{Q}$-factorial terminal threefolds. We point out a problem in his proof of the structure theorem. Then we review the argument and use boundedness of terminal Fano threefolds to finish proof. Because of the use of this boundedness result, as it stands, this proof does not generalize to the log terminal case, as it was claimed in [KMMT00] and [Xie05].

In section 1 we recall the main definitions and results of the minimal model program. In section 2 we explain Batyrev's proof of the rationality of the effective threshold. We choose to work in great generality. We consider log terminal pairs (X, Δ), with $\dim X = n$. Then we assume the log minimal model program in dimension n, and obtain the rationality of the log effective threshold. In section 3 we explain the problem in Batyrev's proof of the structure theorem for the cone of nef curves. In section 4 we provide a complete proof of this theorem (generalized to the case of terminal pairs).

We work over some fixed algebraically closed field of characteristic zero.

1. Cone of curves and divisors and the Minimal Model Program

In this section we define some special cones associated to a projective variety X, and recall the main results of the log minimal model program. We refer to [KM98] and [KMM87] for a detailed introduction and proofs.

1.1. **Cones of Curves and Divisors.** Let X be a projective variety of any dimension.

Definition 1.1. Let $N_1(X)$ denote the $\mathbb{R}$-vector space of 1-cycles on X with real coefficients modulo numerical equivalence. Set $N^1(X) = NS(X) \otimes \mathbb{R}$, where $NS(X)$ is the Néron-Severi group of X. Intersection number of divisors and curves defines a perfect pairing between these two vector spaces. The vector spaces $N_1(X)$ and $N^1(X)$ are finitely generated, and their dimension is denoted by $\rho(X)$, the Picard Number of X.

The *cone of curves* of X, $\overline{NE}_1(X)$, is defined to be the closure in $N_1(X)$ of the cone generated by the classes of irreducible curves on X. Its dual cone, $\Lambda_{nef}(X) \subset N^1(X)$, is the closed cone generated by the classes of nef divisors on X. It is called the *nef cone* of X.

Similarly, we define the *cone of pseudo-effective divisors* of X, $\Lambda_{eff}(X)$, to be the closure in $N^1(X)$ of the cone generated by the classes of effective divisors on X. Its dual cone is denoted by $\overline{NM}_1(X) \subset N_1(X)$, and is called the *cone of nef curves* of X.

Definition 1.2. A $\mathbb{Q}$-divisor D on X is said to be *big* if $h^0(X, \mathcal{O}_X(kD)) \geq c.k^{\dim(X)}$ for some $c > 0$ and $k \gg 0$.

Remark 1.3. By Kleiman's ampleness criterion (see [Kle66]), a $\mathbb{Q}$-divisor D on X is ample if and only if D is in the interior of $\Lambda_{nef}(X)$. Therefore $\Lambda_{nef}(X) \subset \Lambda_{eff}(X)$, and $\overline{NM}_1(X) \subset \overline{NE}_1(X)$.

Kodaira's lemma asserts that a $\mathbb{Q}$-divisor D is big if and only if $D = H + E$ for some ample $\mathbb{Q}$-divisor H and some effective $\mathbb{Q}$-divisor E. As a consequence, we have that D is big if and only if D is in the interior of $\Lambda_{eff}(X)$.

Definition 1.4. An *extremal face* F of a cone $N \subset \mathbb{R}^n$ is a subcone of N satisfying:

$$u, v \in N \text{ and } u + v \in F \quad \Rightarrow \quad u, v \in F.$$

A 1-dimensional extremal face R of N is called an *extremal ray*.

Let D be a real function on N. We write

$$N_{D \geq 0} = \{z \in N | D(z) \geq 0\},$$

and similarly for $N_{D=0}$, $N_{D \leq 0}$, etc.

A *D-negative extremal face* is an extremal face $F \subset N$ such that $F \setminus \{0\} \subset N_{D<0}$.

Definition 1.5. Let X be a projective variety, and let Δ be a boundary divisor on X. Let H be a nef and big $\mathbb{Q}$-Cartier divisor on X, so that $H \in \Lambda_{nef}(X) \cap \mathrm{Int}(\Lambda_{eff}(X))$. The *nef log threshold of H* is defined by

$$\tau(X, \Delta, H) = \sup\{t \in \mathbb{Q} | H + t(K_X + \Delta) \text{ is nef}\}.$$

The *effective log threshold of H* is defined by

$$\sigma(X, \Delta, H) = \sup\{t \in \mathbb{Q} | H + t(K_X + \Delta) \text{ is effective}\}.$$

When there is no ambiguity we write $\tau(H)$ and $\sigma(H)$ for $\tau(X, \Delta, H)$ and $\sigma(X, \Delta, H)$ respectively.

Remark 1.6. In [Fuj96] Fujita defined the log Kodaira Energy $\kappa\epsilon(X, \Delta, H)$. This is related to the effective log threshold $\sigma(X, \Delta, H)$ by the formula

$$\sigma(X, \Delta, H) = -\frac{1}{\kappa\epsilon(X, \Delta, H)}.$$

1.2. **Log Terminal Pairs.** Let (X,Δ) be a log pair, i.e., X is a normal variety, and Δ is a $\mathbb{Q}$-divisor such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier.

Definition 1.7. Let $f : V \to X$ be a proper birational morphism from a nonsingular variety V. Let E_i denote the irreducible components of the exceptional divisor for f, and write $f_*^{-1}\Delta$ for the strict tranform of Δ. Then there are uniquely determined rational numbers $a(E_i, X, \Delta)$ such that

$$K_V + f_*^{-1}\Delta = f^*(K_X+\Delta) + \sum_{E_i} a(E_i, X, \Delta)E_i.$$

We call $a(E_i, X, \Delta)$ the *discrepancy of* (X,Δ) *at* E_i.

We say that (X,Δ) is *log terminal* if

1. $K_X+\Delta$ is $\mathbb{Q}$-Cartier.
2. There exists a projective birational morphism $f : V \to X$ from a nonsingular variety V with exceptional divisors E_i, such that $\sum E_i + f_*^{-1}\Delta$ is simple normal crossing, and $a(E_i, X, \Delta) > -1$ for every E_i.

Remark 1.8. In [Kol92] and [KMM87], the Cone Theorem is established for divisorial log terminal (dlt), and weakly log terminal (wlt) pairs respectively. The dlt and wlt conditons are equivalent, but they are stronger than log terminal in general. However, if X is $\mathbb{Q}$-factorial, i.e., any divisor on X is $\mathbb{Q}$-Cartier, then the three notions coincide. In the next sections we shall assume $\mathbb{Q}$-factoriality of X, and refer to (X,Δ) simply as a log terminal pair.

1.3. **The $(K+\Delta)$-Minimal Model Program.** Let (X,Δ) be a log terminal pair, where X is a $\mathbb{Q}$-factorial projective variety, and Δ is a boundary $\mathbb{Q}$-divisor. The $(K+\Delta)$-Minimal Model Program ($(K+\Delta)$-MMP for short) consists of an inductive sequence of divisorial contractions and log flips $\varphi_i : X_i \dashrightarrow X_{i+1}$, each associated to a $(K_{X_i}+\Delta_i)$-negative extremal ray:

We start with $(X_0,\Delta_0) = (X,\Delta)$. Given (X_i,Δ_i), and assuming $K_{X_i}+\Delta_i$ is not nef, we pick a $(K_{X_i}+\Delta_i)$-negative extremal ray R_i. By the Cone Theorem (Theorem 1.12 below), the contraction of R_i, $f_i : X_i \to Y_i$, exists. (The contraction of R_i is the unique morphism $\varphi : X_i \to Y_i$ such that $\varphi_*\mathcal{O}_{X_i} = \mathcal{O}_{Y_i}$, and, for any curve $C \subset X_i$, $\varphi(C)$ is a point if and only if $[C] \in R_i$.) Assuming that $\dim(Y_i) = \dim(X_i)$, we have that either f_i is a divisorial contraction (i.e. the exceptional locus of f_i is a prime divisor), or the exceptional locus of f_i has codimension ≥ 2 in X_i.

In the first case we put $\varphi_i = f_i$, $X_{i+1} = Y_i$, and $\Delta_{i+1} = \varphi_{i*}\Delta_i$.

In the second case, Y_i is not $\mathbb{Q}$-factorial (in fact $K_{Y_i} + f_{i*}\Delta_i$ is not $\mathbb{Q}$-Cartier). We then assume:

Conjecture 1.9 (Flip Conjecture)**.** *Let X_i, Y_i and f_i be as in the above discussion.*

1. *There exists a unique birational map φ_i, and contraction f_i^+*

$$\begin{array}{ccc} \varphi_i : X_i & \dashrightarrow & X_i^+ \\ f_i \searrow & & \swarrow f_i^+ \\ & Y_i & \end{array}$$

such that X_i^+ is $\mathbb{Q}$-factorial, $K_{X_i^+}+\varphi_{i*}\Delta_i$ is f_i^+-ample, and the exceptional locus of f_i^+ has codimension ≥ 2 in X_i^+. We call $f_i^+ : X_i^+ \to Y_i$ (or the map φ_i itself) a $(K+\Delta)$-flip, or more generaly, a log flip.

2. *There is no infinite sequence of $(K+\Delta)$-flips.*

We then put $X_{i+1} = X_i^+$, and $\Delta_{i+1} = \varphi_{i*}\Delta_i$.

The fact that divisorial contractions decrease the Picard number, and the termination assumption for log flips imply that this process must stop. This means that at some point we reach one of the following situations:

(1) $K_{X_n} + \Delta_n$ is nef. In this case we say that X_n is a $(K+\Delta)$-*minimal model*.
(2) The contraction of R_n, $f_n : X_n \to Y_n$, is a $(K+\Delta)$-*Mori fiber space*. This means that Y_n is a normal projective variety with $\dim(Y_n) < \dim(X_n)$, $\rho(Y_n) = \rho(X_n) - 1$, and $-(K_{X_n} + \Delta_n)$ is f_n-ample. In this case $-K_{X_n}$ is also f_n-ample, and thus $f_n : X_n \to Y_n$ is a $\mathbb{Q}$-Fano fibration. This implies that X_n, and hence also X, is uniruled.

We end this section with some established theorems of the log MMP.

Theorem 1.10 (Rationality Theorem)**.** *Let (X, Δ) be a projective dlt pair. Assume $K_X + \Delta$ is not nef, and let H be a nef and big $\mathbb{Q}$-Cartier divisor. Then $\tau(X, \Delta, H)$ is a rational number.* □

The Rationality Theorem gives us a $\mathbb{Q}$-Cartier divisor $H + \tau(H)(K_X + \Delta)$ supporting a $(K_X + \Delta)$-negative extremal face. The next theorem implies that such extremal face can be contracted.

Theorem 1.11 (Basepoint-free Theorem)**.** *Let (X, Δ) be a projective dlt pair. Let D be a nef divisor such that $aD - (K_X + \Delta)$ is nef and big for some $a > 0$. Then $|mD|$ is basepoint-free for $m \gg 0$.* □

The Rationality Theorem and the Basepoint-free Theorem together give the following structure theorem for $\overline{NE}_1(X)$.

Theorem 1.12 (Cone Theorem)**.** *Let (X, Δ) be a projective dlt pair. Then:*

(1) *For any $\varepsilon > 0$, and any ample divisor A on X, there are finitely many rational curves $C_1, \dots, C_r \subset X$ such that $0 < -(K_X + \Delta) \cdot C_i \leq 2\dim(X)$, and*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$
(2) *There are countably many rational curves $C_i \subset X$ such that $0 < -(K_X + \Delta) \cdot C_i \leq 2\dim(X)$, and*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$
(3) *Let $F \subset \overline{NE}(X)$ be a $(K_X + \Delta)$-negative extremal face. Then the contraction of F exists and is unique.* □

In the next section we shall use the relative version of the log MMP. The statements of the results are very similar to the ones above, and we do not include them here in order to keep the notation light. We refer to [KM98] and [KMM87] for the log MMP in the relative setting.

2. The rationality of $\sigma(X, \Delta, H)$

Let (X, Δ) be a log terminal pair, where X is a $\mathbb{Q}$-factorial projective variety. Let H be a nef and big $\mathbb{Q}$-divisor on X, so that $H \in \Lambda_{nef}(X) \cap \mathrm{Int}(\Lambda_{eff}(X))$.

Assume that $(K_X + \Delta) \notin \Lambda_{eff}(X)$. In this case both $\tau(X, \Delta, H)$ and $\sigma(X, \Delta, H)$ are finite. By the Rationality Theorem, $\tau(X, \Delta, H) \in \mathbb{Q}$. In this section we shall rework the argument in [Bat92] and [Fuj96], and, assuming the log minimal model program, prove the following.

Theorem 2.1. *Let X be a $\mathbb{Q}$-factorial projective variety and Δ a boundary divisor such that (X, Δ) is log terminal. Assume the log minimal model program and suppose that $(K_X + \Delta) \notin \Lambda_{eff}(X)$. Then $\sigma(X, \Delta, H) \in \mathbb{Q}$.*

The idea of the proof of Theorem 2.1 is to run the $(K + \Delta)$-minimal model program "oriented" by H: when contracting a $(K_X + \Delta)$-negative extremal ray, we require that it is supported on $H + \tau(H)(K_X + \Delta)$. At each step (X, Δ, H) is replaced with (X', Δ', H'), where $\varphi : (X, \Delta) \dashrightarrow (X', \Delta')$ is either a divisorial contraction or a log flip, $\Delta' = \varphi_*\Delta$, and $H' = \varphi_* H$. In either case $\sigma(X', \Delta', H') = \sigma(X, \Delta, H)$. Eventualy we reach a triple (X'', Δ'', H'') for which $\sigma(X'', \Delta'', H'') = \tau(X'', \Delta'', H'')$. The result then follows from the Rationality Theorem.

Clearly $\tau(X, \Delta, H) \leq \sigma(X, \Delta, H)$, and the next proposition says when equality holds.

Proposition 2.2. *Let (X, Δ) be a projective dlt pair. Assume $(K_X + \Delta) \notin \Lambda_{eff}(X)$. Let H be a $\mathbb{Q}$-Cartier divisor on X such that $H + t(K_X + \Delta)$ is nef and big for some non negative $t \in \mathbb{Q}$. Then $\tau(X, \Delta, H) = \sigma(X, \Delta, H)$ if and only if, for $m \gg 0$, the linear system $|m(H + \tau(H)(K_X + \Delta))|$ induces a contraction $X \to S$, with* $\dim(S) < \dim(X)$.

Proof. Since $(K_X + \Delta) \notin \Lambda_{eff}(X)$, we have that $\tau(X, \Delta, H) \leq \sigma(X, \Delta, H) < \infty$ By the Cone Theorem, $|m(H + \tau(H)(K_X + \Delta))|$ defines a contraction $X \to S$ onto a normal variety for $m \gg 0$. We have that $\tau(X, \Delta, H) = \sigma(X, \Delta, H)$ if and only if $H + \tau(H)(K_X + \Delta) \in \partial\Lambda_{eff}(X)$. By Remark 1.3, this is the case if and only if $H + \tau(H)(K_X + \Delta)$ is not big, which is equivalent to the condition that $\dim(S) < \dim(X)$. $\square$

Remark 2.3. If $(K_X + \Delta) \in \Lambda_{eff}(X)$, then it is possible that $\tau(X, \Delta, H) = \sigma(X, \Delta, H) = +\infty$ (this happens if and only if $K_X + \Delta$ is nef). In this case, for $m \gg 0$, $|m(K_X + \Delta)|$ may or may not define a fibration, depending on whether or not $(K_X + \Delta) \in \partial\Lambda_{eff}(X)$.

Process 2.4 (Running the $(K + \Delta)$-MMP oriented by H)**.**
Let (X, Δ) be a log terminal pair, where X is a $\mathbb{Q}$-factorial projective veriety. Assume that $(K_X + \Delta) \notin \Lambda_{eff}(X)$. Let H be a nef and big $\mathbb{Q}$-divisor on X. From now on we always assume the flip conjecture. If $\tau(X, \Delta, H) < \sigma(X, \Delta, H)$, the Cone Theorem and Proposition 2.2 together imply that, for $m \gg 0$, $|m(H + \tau(H)(K_X + \Delta))|$ induces a birational contraction $f : X \to Y$. There is an ample $\mathbb{Q}$-divisor A_Y on Y such that $f^* A_Y \sim H + \tau(H)(K_X + \Delta)$.

If $f : X \to Y$ is the contraction of a single extremal ray, then we replace (X, Δ, H) either with $(Y, f_*\Delta, f_*H)$, in the case when f is a divisorial contraction, or with $(X^+, \varphi_*\Delta, \varphi_*H)$, in the case when f is a small contraction and $\varphi : X \dashrightarrow X^+$ is the corresponding flip. In the general case, however, f may contract a higher dimensional extremal face, so we run the log MMP relative to $f : X \to Y$ (performing a sequence of log flips and divisorial contractions) until we reach a relative log minimal model $f_1 : X_1 \to Y$,

$$\begin{array}{ccc} \varphi_1 : \quad X & \dashrightarrow & X_1\ . \\ f \downarrow & & \downarrow f_1 \\ Y & & Y \end{array}$$

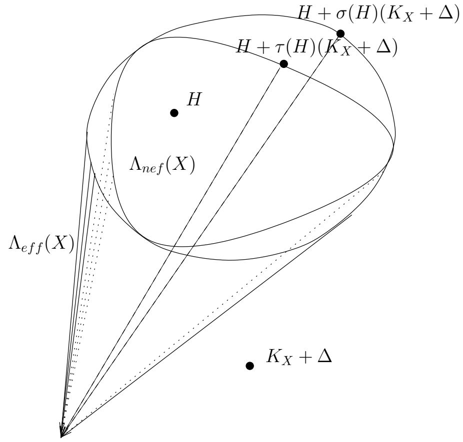

FIGURE 1. Log MMP oriented by H

The divisor $H_1 = \varphi_{1*}H$ is still big. It is not necessarily nef. However, $H_1 + \tau(X, \Delta, H)(K_{X_1} + \Delta_1) = f_1^* A_Y$ is nef and big. Therefore $\tau(X_1, \Delta_1, H_1)$ and $\sigma(X_1, \Delta_1, H_1)$ are well defined. We shall prove that $\sigma(X_1, \Delta_1, H_1) = \sigma(X, \Delta, H)$.

Consider a step in the $(K + \Delta)$-MMP relative to f,

$$\begin{array}{rccl} \rho: & V & \dashrightarrow & W\ . \\ f_V \downarrow & & & \downarrow f_W \\ & Y & & Y \end{array}$$

Let H_V and H_W be the strict transforms of H in V and W respectively, and similarly for Δ_V and Δ_W. Either ρ is a divisorial contraction or a log flip. We consider these two cases separately.

Case 1: Suppose $\rho : V \to W$ is a divisorial contraction, and let $E \subset V$ be the exceptional divisor. We have that

$$H_V + \tau(H)(K_V + \Delta_V) \sim f_V^* A_Y \sim \rho^* f_W^* A_Y \sim \rho^*(H_W + \tau(H)(K_W + \Delta_W)).$$

Let $R \subset \overline{NE}_1(V)$ be the $(K_V + \Delta_V)$-negative extremal ray contracted by ρ. Notice that R lies on the hyperplane defined by $H_V + \tau(X, \Delta, H)(K_V + \Delta_V)$ (and hence $\tau(V, \Delta_V, H_V) = \tau(X, \Delta, H)$). Since $(K_V + \Delta_V) \cdot R < 0$, we get that $H_V \cdot R > 0$. This implies that $\rho^* H_W = H_V + aE$ for some positive rational number a.

Let $t \geq \tau(H) = \tau(X, \Delta, H) \geq 0$. Then

$$\begin{aligned} H_V + t(K_V + \Delta_V) - \rho^*(H_W + t(K_W + \Delta_W)) &= \Big(1 - \frac{t}{\tau(H)}\Big)(H_V - \rho^* H_W) \\ &= \Big(\frac{t}{\tau(H)} - 1\Big)aE. \end{aligned}$$

Since $(\frac{t}{\tau(H)} - 1)a \geq 0$, and E is ρ-exceptional, we get that

$$\begin{aligned} H^0(V, m(H_V+&t(K_V + \Delta_V))) \cong \\ &\cong H^0(V, m\rho^*(H_W + t(K_W + \Delta_W)) + m\Big(\frac{t}{\tau(H)} - 1\Big)aE) \\ &\cong H^0(W, m(H_W + t(K_W + \Delta_W))). \end{aligned}$$

Hence $H_V+t(K_V+\Delta_V)$ is big if and only if $H_W+t(K_W+\Delta_W)$ is big. Remark 1.3 then implies that $\sigma(V,K_V,H_V)=\sigma(W,K_W,H_W)$.

Case 2: Suppose $\rho : V \dashrightarrow W$ is a log flip. Then ρ is an isomorphism in codimension 1, and $H_V+t(K_V+\Delta_V)=\rho^*(H_W+t(K_W+\Delta_W))$. So $H_V+t(K_V+\Delta_V)$ is pseudo-effective if and only if so is $H_W+t(K_W+\Delta_W)$.

We have proved that $\sigma(X_1,\Delta_1,H_1)=\sigma(X,\Delta,H)$.

If $\tau(X_1,\Delta_1,H_1)<\sigma(X_1,\Delta_1,H_1)$, then we repeat the process with (X,Δ,H) replaced by (X_1,Δ_1,H_1). By the termination assumption for log flips, and the fact that divisorial contractions decrease the Picard number, this process must stop. That means that eventually we reach a triple (X_n,Δ_n,H_n) for which

$$\tau(X_n,\Delta_n,H_n)=\sigma(X_n,\Delta_n,H_n)<\infty.$$

By Proposition 2.2, for $m\gg 0$, the linear system $|m(H_n+\tau(H_n)(K_{X_n}+\Delta_n))|$ induces a contraction $g : X_n\to S$, with $\dim(S)<\dim(X_n)$.

Now we run the log-MMP relative to $g : X_n\to S$ (performing a sequence of flips and divisorial contractions). At the end we get either a Mori fiber space $X_{n+1}\to Z$, $Z\to S$, with $\dim(Z)<\dim(X_{n+1})$, or a relative minimal model $X'_{n+1}\to S$. We claim that the latter does not occur. Indeed if $f' : X'_{n+1}\to S$ is a relative $(K+\Delta)$-minimal model, then $(K_{X'_{n+1}}+\Delta'_{n+1})$ is f'-nef. Moreover $H'_{n+1}+\tau(H_n)(K_{X'_{n+1}}+\Delta'_{n+1})$ is numerically trivial on the general fiber F of f'. Hence H'_{n+1} is semi-negative on F, contradicting the fact that H'_{n+1} is big (here we are using the assumption that $\dim(S)<\dim(X_{n+1})$, and hence $\dim(F)>0$).

To summarize, let (X,Δ) be a log terminal pair, where X is a $\mathbb{Q}$-factorial projective variety, and $(K_X+\Delta)\notin\Lambda_{eff}(X)$. Then there is a sequence of birational maps $\varphi_i : (X_{i-1},\Delta_{i-1})\dashrightarrow(X_i,\Delta_i)$, each of which is a log flip or a divisorial contraction in the $(K+\Delta)$-MMP, and a Mori fibration $f : X_n\to Z$ such that $-(K_{X_n}+\Delta_n)$ is f-ample:

$$\begin{array}{ccccccccccc} X & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \dots & \dashrightarrow & X_{n-1} & \dashrightarrow & X_n. \\ &&&&&&&&&& \downarrow f \\ &&&&&&&&&& Z \end{array}$$

Moreover, denoting by H_i the strict transform of H in X_i, we have $\sigma(X,\Delta,H)=\sigma(X_i,\Delta_i,H_i)$, $1\le i\le n$, and $\sigma(X_n,\Delta_n,H_n)=\tau(X_n,\Delta_n,H_n)\in\mathbb{Q}$. This proves Theorem 2.1.

3. Numerical Pullback of Curves

Let X be an n-dimensional projective $\mathbb{Q}$-factorial terminal variety, and Δ a boundary divisor such that (X,Δ) is log terminal. For any nef and big $\mathbb{Q}$-divisor H on X, Process 2.4 yields a birational map $\varphi : X\dashrightarrow X'$ and a Mori fibration $f : X'\to S$ such that $\tau(X',\Delta',H')=\sigma(X',\Delta',H')=\sigma(X,\Delta,H)$, where $\Delta'=\varphi_*\Delta$, and $H'=\varphi_*H$. By Miyaoka and Mori's numerical criterion for uniruledness (see [MM86]), we can choose a covering family of rational curves $\mathcal{C}'$ lying on fibers of f so that the $(-K_{X'})$-degree of its members is bounded by $2n$. Then we would like to pull back to X a general curve from the family $\mathcal{C}'$. There is a problem when X' is singular and all curves from $\mathcal{C}'$ meet the singular locus of X'. In this case, taking the strict transform of the curves is not enough, and we have to consider their numerical pullback, which we define next.

Definition 3.1. Let $\varphi : X \dashrightarrow Z$ be a birational map between $\mathbb{Q}$-factorial projective varieties, and assume it is surjective in codimension 1.

Taking pullback of divisors on Z defines an injective linear map $\varphi^{1*} : N^1(Z) \to N^1(X)$. Taking pushforward of divisors on X defines a surjective linear map $\varphi_* : N^1(X) \to N^1(Z)$. The composition $\varphi_* \circ \varphi^{1*}$ is the identity on $N^1(Z)$.

We shall define an injective linear map $\varphi_1^* : N_1(Z) \to N_1(X)$. Choose dual basis $\{z_i\} \subset N^1(Z)$ and $\{l_i\} \subset N_1(Z)$ (so that $z_i \cdot l_j = \delta_{ij}$). Consider the subset $\{\alpha_i = \varphi^{1*} z_i\} \subset N^1(X)$. Extend it to a basis of $N^1(X)$ by adding the classes of the exceptional divisors for φ, $\beta_j = [E_j]$. Let $\{m_i, n_j\}$ be a basis for $N_1(X)$ dual to $\{\alpha_i, \beta_j\}$, i.e.,

$$\alpha_i \cdot m_j = \delta_{ij},\ \beta_i \cdot m_j = 0 = \alpha_i \cdot n_j,\ \beta_i \cdot n_j = \delta_{ij}.$$

Define $\varphi_1^* : N_1(Z) \to N_1(X)$ by putting $\varphi_1^*(l_i) = m_i$.

Remark 3.2. It is easy to check that the map φ_1^* is in fact injective, and it is the unique linear map satisfying the following conditions.

(1) If $z \in N^1(Z)$ and $l \in N_1(Z)$, then $\varphi^{1*}(z) \cdot \varphi_1^*(l) = z \cdot l$.
(2) If $\beta \in \ker \varphi_*$ and $m \in \operatorname{im} \varphi_1^*$, then $\beta \cdot m = 0$.

Now let C' be a general curve in the family $\mathcal{C}'$, and let $C = \varphi_1^*[C']$. There are classes $\beta, \beta' \in \ker \varphi_*$ such that $[H + \sigma(K_X + \Delta)] = \varphi^{1*}[H' + \sigma(K_{X'} + \Delta')] + \beta$ and $[K_X] = \varphi^{1*}[K_{X'}] + \beta'$ in $N^1(X)$, where $\sigma = \sigma(X, \Delta, H) = \sigma(X', \Delta', H') = \tau(X', \Delta', H')$. Properties (1) and (2) above imply the following.

(1) $(H + \sigma(K_X + \Delta)) \cdot C = (H' + \sigma(K_{X'} + \Delta')) \cdot C' = 0$, and
(2) $-K_X \cdot C = -K_{X'} \cdot C' \leq 2n$.

Loosely speaking Batyrev's theorem states that, when $\dim(X) = 3$ and (X, Δ) is terminal, such pullback classes generate the half cone of $\overline{NM}_1(X)$ where $K_X + \Delta$ is negative. More precisely:

Theorem 3.3. *Let X be a $\mathbb{Q}$-factorial threefold and Δ a boundary divisor such that (X, Δ) is terminal. Then*

(a) *For any $\varepsilon > 0$, and any ample divisor A on X, there are finitely many classes of curves $C_1, \ldots, C_r \in N_1(X)$ such that*

(1) $0 < -K_X \cdot C_i \leq 6$,
(2) *There is a Mori fiber space $f_i : X_i \to S_i$, which can be obtained from X by running the $(K + \Delta)$-MMP, such that C_i is the pullback class of a rational curve lying on a general fiber of f_i, and*
(3) $\overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X) = \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \sum \mathbb{R}_{\geq 0} C_i$.

(b) *There are countably many classes of curves $C_i \in N_1(X)$ such that*

(1) $0 < -K_X \cdot C_i \leq 6$,
(2) *There is a Mori fiber space $f_i : X_i \to S_i$, which can be obtained from X by running the $(K + \Delta)$-MMP, such that C_i is the pullback class of a rational curve lying on a general fiber of f_i, and*
(3) $\overline{NE}_1(X)_{(K_X+\Delta)\geq 0} + \overline{NM}_1(X) = \overline{NE}_1(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0} C_i$.

Remark 3.4. The rays $\mathbb{R}_{\geq 0} C_i$ above are called *coextremal rays*.

Part (b) follows from part (a). In order to prove part (a), we need to show that, for any compact set $B \subset N_1(X)_{(K_X+\Delta)<0}$, there are only finitely many $C_i \in B$ satisfying (2) above. Batyrev achieves this by claiming that the pullback classes of rational curves lying on general fibers of Mori fiber spaces are integral (he explicitly assumes this throughout [Bat92]). This is not always true though, as the next example shows.

Example 3.5. Let Y be the cone over the Veronese surface. Then Y is a $\mathbb{Q}$-factorial terminal Fano threefold of Picard number 1. Let $\pi : X \to Y$ be the blowup of the vertex of the cone, and let $E \cong \mathbb{P}^2$ be the exceptional divisor. Let $l \subset Y$ be a ruling of the cone. A simple computation shows that $\pi_1^* l = \tilde{l} + \frac{1}{2}e$, where $\tilde{l}$ is the strict transform of l and e is a curve on E corresponding to a line on $\mathbb{P}^2$ under the isomorphism $E \cong \mathbb{P}^2$.

One way to fix this problem is the following. We fix a basis m_i for $N_1(X)_{\mathbb{Q}}$. For each Mori fiber space $f : X' \to S$ that can be obtained from X by running the log MMP, we fix a covering family of rational curves $\mathcal{C}'$ lying on fibers of f. We also require that $-K_{X'} \cdot C' \leq 6$ for C' a general member in this family. We can write the numerical pullback of C' as $C = \sum a_i m_i$, where the a_i's are suitable rational numbers. Then all we need to do is to find some universal bound on the denominators of the a_i's.

Another possibility is to find some universal constant N satisfying the following condition. For any $\mathbb{Q}$-factorial terminal Fano threefold X' of Picard number 1, there exists a curve $C' \subset X'$ obtained as the intersection of 2 very ample divisors on X' and such that $-K_{X'} \cdot C' \leq N$. Then, for every Mori fiber space $f : X' \to S$ that can be obtained from X by running the log MMP, we can take the strict transform of a curve C' on a general fiber of $X' \to S$ avoiding the indeterminancy locus of $X' \dashrightarrow X$ and such that $K_{X'} \cdot C' \leq N$. This strict transform is of course integral and coincides with the numerical pullback of C'.

These two possible strategies are morally the same, and in the next section we work out the latter.

4. Proof of Theorem 3.3

Let X be a $\mathbb{Q}$-factorial threefold and Δ a boundary divisor such that (X, Δ) is terminal. First notice that if there exists a nef curve $C \subset X$ such that $(K_X + \Delta) \cdot C < 0$, then $(K_X + \Delta) \notin \Lambda_{eff}(X)$, and we are back to the setting of section 2.

We will need the following result.

Lemma 4.1. *There exists a constant N such that the following holds. If X is a projective $\mathbb{Q}$-factorial terminal Fano threefold of Picard number 1, and $B \subset X$ is a subset of codimension at least 2, then there exists a proper curve $C \subset X \setminus B$ such that $-K_X \cdot C \leq N$.*

Proof. By [Kaw92], $\mathbb{Q}$-factorial terminal Fano threefolds of Picard number 1 form a bounded family. In particular, there exist universal constants M and D such that the following holds. For any $\mathbb{Q}$-factorial terminal Fano threefold X of Picard number 1, $-MK_X$ is very ample and $(-K_X)^3 \leq D$. (An explicit bound for M is given in [Kol93].) Let $B \subset X$ be a subset of codimension at least 2. By intersecting 2 general members of the linear system $|-MK_X|$, we obtain a proper curve $C \subset X \setminus B$ such that $-K_X \cdot C = -K_X \cdot M^2(-K_X)^2 \leq M^2 D =: N$. $\square$

Let N be as in Lemma 4.1. Given ε and A as in Theorem 3.3(a), there are finitely many classes of integral curves $C \in N_1(X)$ such that $0 < -(K_X + \Delta) \cdot C \leq N$, and $[C] \notin \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0}$. Indeed, all such curves satisfy $A \cdot C < N/\varepsilon$.

Pick such classes C_i for which there exists a Mori fiber space $f_i : X_i \to S_i$ obtained from X by running the $(K+\Delta)$-MMP, so that C_i is the pull back class of a curve lying on a general fiber of f_i and avoiding the indeterminancy locus of $X_i \dashrightarrow X$. Now set

$$W = \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \sum \mathbb{R}_{\geq 0} C_i.$$

We prove that $W = \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X)$. (A standard argument about cones in $\mathbb{R}^n$ shows that both these cones are closed.) Clearly $W \subset \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X)$. Suppose they are different. Then, since these are both convex cones, there exists an element $D \in N^1(X)$ such that:

(1) $D \cdot z_0 = 0$ for some $z_0 \in (\overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X)) \setminus \{0\}$.
(2) $D \cdot z \geq 0$ for every $z \in \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X)$.
(3) $D \cdot z > 0$ for every $z \in W \setminus 0$.

Claim 4.2. *There exists an ample $\mathbb{R}$-divisor H of the form $D - a(K_X + \Delta)$ with $a \in \mathbb{R}, a > 0$.*

Proof. We need to prove that $\mathrm{Int}(\Lambda_{nef}(X)) \cap \mathrm{Int}(\mathbb{R}_{\geq 0}[D] + \mathbb{R}_{\geq 0}[-(K_X+\Delta)]) \neq \emptyset$. Assume otherwise. Then there exists an element $l \in \overline{NE}_1(X) \setminus \{0\}$ such that $D \cdot l \leq 0$ and $-(K_X+\Delta) \cdot l \leq 0$. The last inequality implies that $l \in W \setminus \{0\}$, but this contradicts the choice of D above. □

Let $H = D - a(K_X + \Delta)$ be as in the claim. Then $\sigma(X, \Delta, H) = a$, and $H + \sigma(H)(K_X + \Delta) = D$. (See Figure 2).

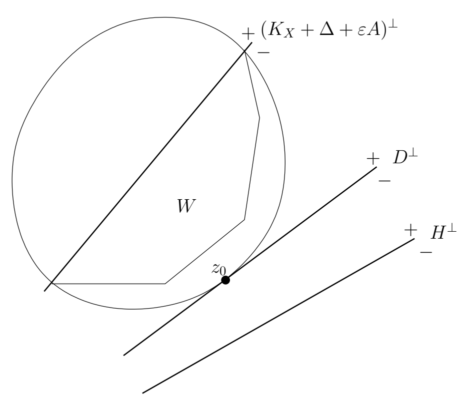

FIGURE 2. Finding the ample $\mathbb{R}$-divisor H

Choose a sequence of ample $\mathbb{Q}$-divisors H_i on X such that $\lim_{i\to\infty} H_i = H$. Then $\lim_{i\to\infty} \sigma(H_i) = \sigma(H) = a$. For each i, Process 2.4 yields a birational map $\varphi_i : X \dashrightarrow X_i$ and a Mori fibration $f_i : X_i \to S_i$ such that $(H_i + \sigma(H_i)(K_X + \Delta)) \cdot (\varphi_i)^*_1(l_i) = 0$, where l_i is the class of a curve on a general fiber of f_i. We note that the general fiber of f_i is a terminal Fano variety of dimension ≤ 3. So, by Lemma 4.1, we can take l_i to be the class of an integral curve avoiding the indeterminancy locus of $X_i \dashrightarrow X$ and such that

$$\begin{aligned} 0 < -(K_{X_i} + \Delta_i) \cdot l_i &= -(K_X + \Delta) \cdot (\varphi_i)_1^*(l_i) \\ &\leq -K_X \cdot (\varphi_i)_1^*(l_i) \\ &= -K_{X_i} \cdot l_i \leq N. \end{aligned}$$

Hence $(\varphi_i)_1^*(l_i) \in W$, and the intersection numbers $H \cdot (\varphi_i)_1^*(l_i)$ are all bounded by some positive constant c. Since there are finitely many integral classes of curves of H-degree bounded by c, some class $(\varphi_{i_0})_1^*(l_{i_0})$ must appear infinitely many times in the sequence $\{(\varphi_i)_1^*(l_i)\}$. Thus

$$D \cdot (\varphi_{i_0})_1^*(l_{i_0}) = \lim_{i\to\infty} [(H_i + \sigma(H_i)(K_X + \Delta)) \cdot (\varphi_i)_1^*(l_i)] = 0,$$

contradicting the choice of D above. Hence $W = \overline{NE}_1(X)_{(K_X+\Delta+\varepsilon A)\geq 0} + \overline{NM}_1(X)$.

To conclude the proof we just need to observe the following. Let $\varphi_i : X \dashrightarrow X_i$, $f_i : X_i \to S_i$ and $l_i \in N_1(X_i)$ be as above. By Miyaoka and Mori's numerical criterion for uniruledness, we can find a rational curves m_i lying on a general fiber of f_i such that $0 \leq -K_X \cdot (\varphi_i)_1^*(m_i) \leq 6$. Moreover $\mathbb{R}_{\geq 0}(\varphi_i)_1^*(m_i) = \mathbb{R}_{\geq 0}(\varphi_i)_1^*(l_i)$.

Remark 4.3. If (X, Δ) is log terminal, then the argument above shows that

$$\overline{NE}_1(X)_{(K_X+\Delta)\geq 0} + \overline{NM}_1(X) = \overline{NE}_1(X)_{(K_X+\Delta)\geq 0} + \overline{\sum \mathbb{R}_{\geq 0} C_i},$$

where the C_i are pullback classes of curves on fibers of Mori fiber spaces obtained from X by running the $(K + \Delta)$-MMP. However, this proof does NOT show that the part of the cone where $K_X + \Delta$ is negative is a locally finite polyhedral cone, as it was claimed in [KMMT00] and [Xie05].

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