METRIC STABILITY FOR RANDOM WALKS WITH APPLICATIONS IN RENORMALIZATION THEORY

CARLOS G. MOREIRA AND DANIEL SMANIA

Abstract. Consider deterministic random walks $F\colon I \times \mathbb{Z} \to I \times \mathbb{Z}$, defined by $F(x,n) = (f(x), \psi(x) + n)$, where f is an expanding Markov map on the interval I and $\psi\colon I \to \mathbb{Z}$. We study the universality (stability) of ergodic (for instance, recurrence and transience), geometric and multifractal properties in the class of perturbations of the type $\tilde{F}(x,n) = (f_n(x), \tilde{\psi}(x,n) + n)$ which are topologically conjugate with F and f_n are expanding maps exponentially close to f when $|n| \to \infty$. We give applications of these results in the study of the regularity of conjugacies between (generalized) infinitely renormalizable maps of the interval and the existence of wild attractors for one-dimensional maps.

Contents

1. Introduction

1.1. **Metric stability for random walks.** In the study of a dynamical system, some of the most important questions concerns to the stability of their dynamical properties under (most of the) perturbations: how much robust are they?

Here we are most interested in the stability of metric (measure-theoretical) properties of dynamical systems. A well-known example is given by (C^2) Markov expanding maps on the circle: this is a class stable by perturbations and all of them

Date: March 28, 2005.

2000 *Mathematics Subject Classification.* 60G50, 82B41, 37E05, 37C20, 30C65, 37C45, 37D20, 37A50, 37E20.

Key words and phrases. random walk, group extension, skew-product, absolutely continuous, recurrence, transience, stability, rigidity, renormalization, conjugacy, Feigenbaum,period-doubling, universality, Fibonacci, wild attractor, multifractal, strongly quasisymmetric, complete connections, g-measures, Hausdorff, dimension.

The authors would like to thank the hospitality of ICMC-USP and IMPA. The first author was partially supported by CNPq. The second author was partially supported by CNPq 472316/03-6 and FAPESP 03/03107-9.

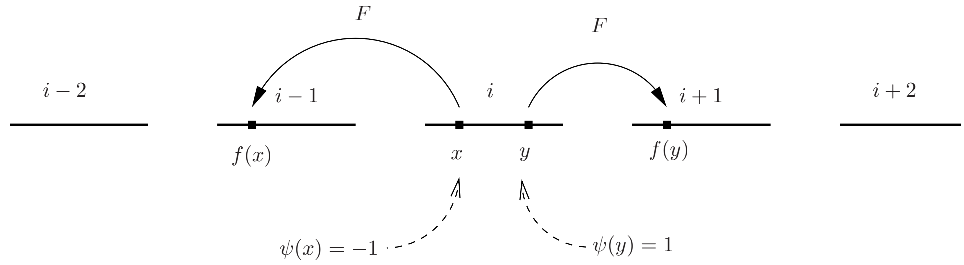

FIGURE 1. A deterministic random walk

have an absolutely continuous and ergodic invariant probability satisfying certain decay of correlations estimative. In particular, in the measure theoretical sense, most of the orbits are dense in the phase space.

Now let's study a slightly more complicated situation: consider a C^2 Markov almost onto expanding map of the interval $f\colon I \to I$ with bounded distortion control and large images (see Section 2 for details) and let $\psi\colon I \to \mathbb{Z}$ be a function which is constant in each interval of the Markov partition of f. We can define $F\colon I \times \mathbb{Z} \to I \times \mathbb{Z}$ as

$$F(x,n) := (f(x), \psi(x) + n).$$

The second entry of (x,n) will be called its **state**. We also assume that

$$\inf \psi > -\infty \tag{1}$$

and that F is topologically mixing.

The map F is refereed in literature in many ways: as a "skew-product between f and the translation on the group $\mathbb{Z}$", a "group extension of f", or even a "deterministic random walk generated by f", and its metric behavior is very well studied: for instance, are most the orbits recurrent? Everything depends on the **mean drift**

$$M = \int \psi d\mu,$$

where μ is the absolutely continuous invariant probability of f (the function ψ will be called **drift function**). Indeed, note that

$$F^n(x,i) = (\ f^n(x)\ ,\ i + \sum_{k=0}^{n-1} \psi(f^k(x))\ ).$$

By the Birkhoff Ergodic Theorem

$$\lim_{n\to\infty} \frac{\pi_2(F^n(x,i)) - \pi_2(x,i)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k(x)) = M.$$

for almost every $x \in I$ (here $\pi_2(x,n) := n$). In particular if $M \neq 0$ then almost every point $(x,i) \in I \times \mathbb{Z}$ is **transient**: in other words we have

$$\lim_{n\to\infty} |\pi_2(F^n(x,i))| = \infty.$$

So most of the points are not recurrent.

On the other hand, if $M = 0$, most of points are going to be recurrent (see Guivarc'h [G]): by the Central Limit Theorem for expanding maps (here we need to assume that ψ in no constant and $f \in aO$: see Section 2) of the interval

$$sup_{\epsilon\in\mathbb{R}}\ |\mu(x \in I\colon \frac{\sum_{k=0}^{n-1}\psi(f^k(x))}{\sigma\sqrt{n}} \leq\ \epsilon) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\epsilon} e^{-\frac{u^2}{2}}\ du| \leq \frac{C}{\sqrt{n}},$$

where δ is a positive constant, so we can easily obtain, taking $\epsilon = n^{-1/4}$ and applying Borel-Cantelli Lemma, that

$$\mu(A_+) := \mu(x \in I\colon\ \ \limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}\psi(f^k(x))}{\sqrt[2+\delta]{n}} = \infty) \geq \frac{1}{2},$$

$$\mu(A_-) := \mu(x \in I\colon\ \ \liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}\psi(f^k(x))}{\sqrt[2+\delta]{n}} = -\infty) \geq \frac{1}{2}.$$

Clearly A_+ and A_- are invariant sets: the ergodicity of f implies that

$$\mu(A_+\ \cap\ A_-)\ =\ 1.$$

Now by the conditions on ψ in Eq. (1), expansion, distortion control usual tricks and the fact that F is transitive, we can conclude that almost every point in $(A_+\ \cap\ A_-) \times \mathbb{Z}$ is a F-recurrent point.

Note that the random walk F is a dynamical system quite similar to expanding circle maps: F is an expanding map, with good bounded distortion properties; but the lack of compactness of the phase space allows the non-existence of an absolutely continuous probability. Moreover, in general the random walk is not even recurrent and the recurrence property lost its stability: given a recurrent random walk (f, ψ), it is possible to obtain a transient random walk just changing a little bit f and ψ.

Since the non compactness of the phase space seems to be the origin of the lack of stability of recurrence and transience properties, a natural question is to ask if such properties are stable by compact perturbations. The answer is yes. Indeed, as we are going to see in Theorems 1-4, the transience and recurrence are preserved even by non-compact perturbations which decreases fast away from state 0. For instance, perturbations like

$$\tilde{F}(x, n) = (f_n(x), \psi(x) + n),$$

where, for some $\lambda \in [0, 1)$,

$$|f_n - f|_{C^3} \leq \lambda^{|n|}. \tag{2}$$

The notations and conventions are more or less obvious: we postponed the rigorous definitions to the next section.

With respect to the stability of transience and recurrence, there is a previous quite elegant result by R. L. Tweedie [T]: if p_{ij} are the transition probabilities of a Markov chain on $\mathbb{Z}$, then any perturbation $\tilde{p}_{ij}$ so that

$$(1 + \epsilon_i)^{-1}p_{ij} \leq \tilde{p}_{ij} \leq p_{ij}(1 + \epsilon_i),\ \ j \neq i,$$

and

$$\prod_{i=0}^{\infty}(1 + \epsilon_i) < \infty$$

preserves the recurrence or transience of the original Markov chain. But Tweedie argument does not seem to work in our setting. Our result coincides with Tweedie result in the very special case where f and f_n are linear Markov maps and $\epsilon_i \sim C\lambda^{|i|}$.

In the transient case we can tell a little more: there will be a conjugacy between the original random walk f and its perturbation which is a martingale strongly quasisymmetric map (for short, mSQS-map) with respect to certain dynamically defined set of partitions. Opposite to the usual class of one-dimensional quasisymmetric functions, which does not share many of most interesting properties of higher dimensional quasisymmetric maps, the one-dimensional mSQS-maps are much closer to their high-dimensional cousins, as quasiconformal maps in dimension 2: for instance, they are absolutely continuous.

We also study the behavior of the Hausdorff dimension of dynamically defined sets: Denote by $\Omega_+(F)$ the set of points which have non-negative states along the positive orbit by F. We prove that $\Omega_+(F)$ has Hausdorff dimension strictly smaller than one if and only if $\Omega_+(\tilde{F})$ has dimension less than one for all perturbation satisfying Eq. (2). Furthermore we give a variational characterization for the Hausdorff dimension $HD(\Omega_+(F))$ as the minimum of $HD(\Omega_+(\tilde{F}))$, where $\tilde{F}$ runs on the set of such perturbations. For these results we study of the stability of the multifractal spectrum of the random walk F under those perturbations.

1.2. **Applications to (generalized) renormalization theory.** An unimodal map is a map with an unique critical point. Under fair conditions (non renormalizable real-analytic maps with negative Schwartzian derivative and non-flat critical point) two unimodal maps with the same topological entropy are indeed topologically conjugated. A key question in one-dimensional dynamics is about the regularity of the conjugacy: is it Holder? Absolutely continuous? Since Dennis Sullivan work in the 80's the quasisymmetry of the conjugacy became a very useful tool to obtain deep results in one-dimensional dynamics. In particular, Lyubich proved that under the fair condition above the conjugacy is quasisymmetric and he used this result to prove the rigidity of the non-renormalizable maps in the real quadratic family. Later on, the density of the hyperbolic maps in the real quadratic family was proved verifying the quasisymmetry of the conjugacies for all combinatorics, including infinitely renormalizable ones.

Note that quasisymmetric maps are not, in general, absolutely continuous. Are the conjugacy between unimodal maps absolutely continuous? The answer is no: M. Martens and W. de Melo [MdM] proved that under the fair conditions above an absolutely continuous conjugacy is actually C^∞, provided the unimodal maps

i. do not have a periodic attractor,

ii. are not infinitely renormalizable,

iii. do not have a wild attractor (the topological and measure-theoretical attractor must coincide).

Since we can change the eigenvalues of the periodic points of maps preserving its topological class, and the eigenvalues are preserved by C^1 conjugacies, we conclude that in general a conjugacy between unimodal maps is not absolutely continuous.

The first condition is clearly necessary. This work shows that the second condition is necessary proving that the conjugacy between two arbitrary Feigenbaum unimodal maps with same critical order is *always* absolutely continuous (Theorem 8). Actually the conjugacy is martingale strongly quasisymmetric with respect to a set of dynamically defined partitions.

Condition iii is never violated when the critical point is quadratic. But for certain topological classes of unimodal maps wild attractor appears when the order of the critical point increases: Fibonacci maps are the simplest kind of such maps. We are going to prove (Theorem 11) that a Fibonacci map with even order has a wild attractor if and only if all Fibonacci maps with same even order are conjugated to each other by an absolutely continuous mapping (in particular all these Fibonacci maps have a wild attractor). So Condition iii is necessary.

In both examples above the previous study about perturbations of transient and recurrent random walks are going to be crucial, as the (generalized) renormalization theory for unimodal maps: for these maps it is possible to construct an induced maps which is essentially a perturbation of a deterministic random walk. In the Fibonacci case the transience of this random walk is equivalent to the existence of a wild attractor. The random walk associated to the Feigenbaum map will be always transient.

For both Feigenbaum and Fibonacci maps there are infinitely many periodic points (indeed in the Fibonacci case the periodic points are also dense in the maximal invariant set). It is well known that the conjugacy between critical circle maps with same irrational rotation number and satisfying certain Diophantine condition is absolutely continuous, but we think that these are the first interesting examples of a similar phenomena for maps with many periodic points.

2. Expanding Markov maps, random walks and its perturbations

In this article we will deal with maps

$$F\colon I\times\mathbb{Z}\to I\times\mathbb{Z}$$

which are piecewise C^2 diffeomorphisms, which means that there is a partition $\mathcal{P}^0$ of $I\times\mathbb{Z}$ so that each element $J\in\mathcal{P}^0$ is an open interval where $F|_{\overline{J}}$ is a C^2 diffeomorphism.

If A_J denotes the unique affine which maps the interval J to $[0,1]$ and preserves orientation, then define, for each $J\in\mathcal{P}^0$,

$$\tau_J^F := A_J\circ F^{-1}\circ A_{F(J)}^{-1}.$$

Along this article we will assume that F satisfies some of the following properties:

- *Markovian (Mk)*: For each $J\in\mathcal{P}^0$, $F(J)$ is a connected union of elements in $\mathcal{P}^0$. In particular we can write $F(x,n)=(f_n(x),n+\psi(x,n))$, where $f_n\colon I\to I$ is a piecewise C^2 diffeomorphism relative to the partition $\mathcal{P}^0_n := \{J\in\mathcal{P}^0\colon\ J\subset I_n\}$ and $\psi\colon I\times\mathbb{Z}\to\mathbb{Z}$, called the **drift function**, is constant on each element of $\mathcal{P}^0$.
- *Lower Bounded Drift (LBD)* F is Markovian and $\min\psi>-\infty$.

- *Large Image (LI)*: F is Markovian and there exists $\delta > 0$ so that for each $J \in \mathcal{P}^0$ we have $|F(J)| \geq \delta$.

- *Onto (On)*: F is Markovian and for each $J \in \mathcal{P}^0$ we have $F(J) = I^n$, for some $n \in \mathbb{Z}$.

- *Bounded Distortion (BD)*: There exists $C > 0$ so that every $J \in \mathcal{P}^0_n$ and map τ_J is a C^2 function satisfying
$$\sup_J \left|\frac{D^2\tau_J}{(D\tau_J)^2}\right| \leq C.$$

- *Strong Bounded Distortion (sBD)*: There exists $C > 0$ so that every $J \in \mathcal{P}^0_n$ and map τ_J is a C^2 function satisfying
$$\sup_J \left|\frac{D^2\tau_J}{(D\tau_J)^2}\right| \leq C|J|.$$

- *Expansivity (Ex)*: If $J \in \mathcal{P}^0_n := \{J \in \mathcal{P}^0 \colon\ J \subset I_n\}$, denote $\phi_J := f_n^{-1}|_{f_n(J)}$. Then either ϕ_J can be extended to a function in a δ-neighborhood of J so that
$$S\phi_J > 0,$$
where $S\phi_J$ denotes the Schwartzian derivative of ϕ_J, or there exists $\theta \in (0,1)$ so that
$$|\phi_J'| < \theta$$
on I.

- *Regularity a (Ra)*: There exists $N \in \mathbb{N}$, $\delta > 0$ and $C > 0$ with the following properties: the intervals in $\mathcal{P}^0_n$ are positioned in I_n in such way that the complement of
$$\bigcup_{J \in \mathcal{P}^0_n} int\ J$$
contains at most N accumulation points
$$c_1^n < c_2^n < \cdots < c_{i_n}^n,$$
with $i_n \leq N$, which is in the interior of I_n. Furthermore $|c^n_{i+1} - c^n_i| \geq \delta$. Moreover, given P and $Q \in \mathcal{P}^0_n$ so that $\overline{P} \cap \overline{Q} \neq \phi$ then
$$\frac{1}{C} \leq \frac{|P|}{|Q|} \leq C.$$

- *Regularity b (Rb)*: Assume *Ra*. There exists $C > 0$, $\lambda \in (0,1)$, $\delta > 0$ so that for each $1 < i < i_n$ we can find a point
$$d_i^n \in (c_i^n, c_{i+1}^n),$$
which does not belong to any $P \in \mathcal{P}^0_n$, and
$$\min\{|c^n_{i+1} - d^n_i|, |d^n_i - c^n_i|\} \geq \delta$$

with the following property: If J is a connected component of

$$I_n \setminus \{d_i^n, c_j^n\}_{i,j}$$

then we can enumerate the set

$$\{P\}_{P\in\mathcal{P}_n^0,\ P\subset J} = \{J_i\}_{i\in\mathbb{N}}$$

in such way that $\partial J_i \cap \partial J_{i+1} \neq \phi$ for each i and

$$\frac{|J_{i+j}|}{|J_i|} \leq C\lambda^j$$

for $i \geq 0$, $j > 0$.

- *Good Drift (GD)*: , if ψ is its drift function then there exists $\gamma \in (0,1)$ and $C > 0$ so that

$$m(\{(x,n)\ s.t.\ \psi(x,n) \geq k\}) \leq C\gamma^k.$$

- *Transitive (T)*: F has a dense orbit.

For convenience of the notation if for instance F is Markovian and it has Bounded Distortion, we will write $F \in Mk + BD$.

A **deterministic random walk** (or simply random walk) is a map

$$F \in Mk + LBD + LI + Ex + BD + GD.$$

It is generated by the pair $(\{f_n\}, \psi)$ if

$$F(x,n) := (f_n(x), \psi(x,n) + n).$$

When $f_n = f \in Mk$ and $\psi(x,n) = \psi(x)$, we say that F is the **spatially homogeneous deterministic random walk** generated by the pair (f,ψ). There is a large literature about such random walks. We will sometimes assume the following property:

- *Almost Onto (aO)*: For every $i,j \in \Lambda$ there exists a finite sequence $i = i_0, i_1, i_2, \ldots, i_{n-1}, i_n = j \in \Lambda$ so that

$$f(I_{i_k}) \cap f(I_{i_{k+1}}) \neq \phi$$

for each $k < j$.

Denote $\pi(x,n) := \pi_2(x,n) := n$. A random walk is called **transient** if for almost every $(x,n) \in I \times \mathbb{Z}$

$$\lim_{k\to\infty} |\pi_2(F^k(x,n))| = \infty,$$

and it is **recurrent** if for almost every $(x,n) \in I \times \mathbb{Z}$

$$\#\{k\colon \pi_2(F^k(x,n)) = n\} = \infty.$$

Making use of usual bounded distortion tricks it is easy to show that every $F \in Mk + LI + Ex + BD + T$ is either recurrent or transient.

A (topological) **perturbation** of a random walk is a random walk $\tilde{F}$, generated by a pair $(\{\tilde{f}_i\}, \tilde{\psi})$, so that $F \circ H = H \circ \tilde{F}$ for some homeomorphism

$$H\colon I \times \mathbb{Z} \to I \times \mathbb{Z}$$

which preserves states: $\pi_2(H(x,i)) = i$.

Define $\mathcal{P}^n(F) := F^{-n}\mathcal{P}^0(F)$. If F and $\tilde{F}$ are random walks and h is a topological conjugacy that preserves states between F and $\tilde{F}$, then for each interval L such that $L \subset J \in \mathcal{P}^n(F)$, define

$$dist_n(L) := \sup_{x\in L} \Big| \ln \frac{DF^n(x)}{DF^n(y)} \Big|,$$

Similarly, if $x \in J \in \mathcal{P}^n(F)$ define

$$dist_n(x) := dist_n(J)$$

and

$$dist_\infty(x) := \sup_n dist_n(x).$$

Another kind of random walk which will have a central role in our results are those which are **asymptotically small** perturbations: these are perturbations $(\{\tilde{f}_i\}, \tilde{\psi})$ of a homogeneous random walk $(\{f_i\}, \psi)$ such that there exists $\lambda \in (0,1)$ and $C > 0$ satisfying either

$$|\log \frac{DF(H(p))}{D\tilde{F}(p)}| \leq C\lambda^{|\pi_2(p)|}, \tag{3}$$

if ψ is bounded, or

$$|\log \frac{DF(H(p))}{D\tilde{F}(p)}| \leq C\lambda^{\pi_2(p)}, \tag{4}$$

for $\pi_2(p) \geq 0$ and $DF(H(p)) = D\tilde{F}(p)$ otherwise, if ψ has only a lower bound.

It is easy to see that properties *Ra*, *Rb* and *GD* are invariant by asymptotically small perturbations (if we allow to change the constants described in these properties).

Let $F = (\{f_i\}_i, \psi)$ be a random walk, where ψ is Lebesgue integrable on compact subsets of $I \times \mathbb{Z}$. We say that F is **strongly transient** if there exists $K > 0$ so that

$$\mathbb{E}(\psi \circ F^n | \mathcal{P}_{n-1}) > K$$

for every $n \geq 1$. As the notation suggest, every strongly transient random walk is transient. Moreover we have the following large deviations result:

Proposition 2.1. *Every strongly transient random walk $F \in Ra + Rb$ is transient. Furthermore there exist $\lambda \in [0,1)$ and $C > 0$ so that for each $P \in \mathcal{P}^0_n$ we have*

$$\mu(p \in P\colon\ \pi_2(F^n(p)) - \pi_2(p) < (K-\epsilon)n) \leq C\lambda^n |P|.$$

We will postpone the proof of this result to Section 5.

Remark 2.2. By the Birkhoff Ergodic Theorem it is easy to see that a sufficiently high iteration of a homogeneous random walk with positive mean drift is strongly transient (see the proof of Proposition 5.1 for details).

3. Statements of results

3.1. **Stability of transience.**

Theorem 1 (Stability of Transience I)**.** *Assume that the random walk F defined by the pair $(\{f_i\}_i, \psi)$ is strongly transient. Then every asymptotically small perturbation G of F is also transient. Indeed there is a topological conjugacy between F and G which is an absolutely continuous map and preserves the states.*

We have a similar theorem for all transient homogeneous random walks:

Theorem 2 (Stability of Transience II)**.** *Suppose that the homogeneous random walk F defined by the pair (f, ψ) is transient. Then every asymptotically small perturbation of F is topologically conjugated to F by an absolutely continuous map which preserves the states.*

We can be more precise regarding the regularity of the conjugacy if the drift is non-negative:

Let $\mathcal{A}_0,\ \mathcal{A}_1,\ \cdots, \mathcal{A}_n,\ \mathcal{A}_{n+1}, \cdots$ be succession of finer and finer partitions by intervals of $I \times \mathbb{Z}$ whose union generates the Borelian algebra of $\sqcup_n I_n$. We say that $h\colon \sqcup_n I_n \to \sqcup_n I_n$ is a **martingale strongly quasisymmetric (mSQS)** map with respect to the **stochastic basis** $\cup_n \mathcal{A}_n$ if there exist $C > 0$ and $\alpha \in (0, 1]$ so that

$$\frac{m(h(B))}{|h(J)|} \leq C \left(\frac{m(B)}{|J|} \right)^{\alpha}$$

for all Borelian $B \subset J \in \cup_n \mathcal{A}_n$, and the same inequality holds replacing h by h^{-1} and $\cup_n \mathcal{A}_n$ by $\cup_n h(\mathcal{A}_n)$.

Theorem 3 (Strongly quasisymmetric rigidity)**.** *Let F be either a strongly transient random walk or a transient homogeneous random walk with positive mean drift. Moreover assume in both cases that $\psi \geq 0$. Then every asymptotically small perturbation G of F is topologically conjugated to F by an absolutely continuous map h which preserves the states. Furthermore h on $\cup_{i \geq 0} I_i$ is a martingale strongly quasisymmetric mapping with respect to the stochastic basis $\cup_i \mathcal{P}^i$.*

3.2. **Stability of recurrence.**

In the recurrent case, we are going to restrict ourselves to the stability of the metric properties of homogeneous random walks under asymptotically small perturbations: it is easy to see that the recurrence is not stable by perturbations which are not asymptotically small. Nevertheless

Theorem 4 (Stability of Recurrence)**.** *Suppose that $F \in aO + T$ is a recurrent homogeneous random walk generated by the pair (f, ψ). Then every asymptotically small perturbation of F is also recurrent.*

Note that we can not expect, as in the transient case, an absolutely continuous conjugacy which preserves states between F and G, once asymptotic small perturbations do not preserve (in general) the spectrum of the periodic points and:

Proposition 3.1 (Rigidity)**.** *Suppose that the random walk $F \in On$ generated by a pair $(\{f_i\}_i, \psi)$ is recurrent. If there is an absolutely continuous conjugacy which preserves states H between F and a random walk G, then H is C^1 in each state. In particular the spectrum of the corresponding periodic points of F and G are the same.*

3.3. **Stability of the multifractal spectrum.** Let F be a random walk and denote

$$\Omega_+(F) := \{p\colon\ \pi_2(F^j p) \geq 0,\ for\ j \geq 0\},$$

$$\Omega_+^k(F) := \{(x,k)\colon\ \pi_2(F^j(x,k)) \geq 0,\ for\ j \geq 0\}$$

and

$$\Omega_{+\beta}^k(F) := \{(x,k) \in \Omega_+^k\ s.t\ \underline{\lim}_n \frac{\pi_2(F^n(x,k))}{n} \geq \beta\}$$

Theorem 5. *Let $F \in Ra + Rb + On$ be a random walk. Then, for all $k \in \mathbb{Z}$ and $\beta > 0$ the Hausdorff dimension $HD(\Omega_{+\beta}^k)$ is invariant by asymptotically small perturbations.*

Besides its inner interest, the previous result will be useful by other reason:

Proposition 3.2. *Let $F \in Ra + Rb + On$ be a homogeneous random walk. Then*

$$HD(\Omega_+^k(F)) = \lim_{\beta \to 0^+} HD(\Omega_{+\beta}^k(F)).$$

and as a consequence of Theorem 5 and Proposition 3.2:

Theorem 6. *Let $F \in Ra + Rb + On$ be a homogeneous random walk. If G is an asymptotically small perturbation of F then*

$$HD(\Omega_+^k(G)) \geq HD(\Omega_+^k(F)). \tag{5}$$

We can not replace the inequality in Eq. (5) by an equality. Indeed, even if $HD(\Omega_+^k(F)) < 1$, we have that $sup\ HD(\Omega_+^k(G)) = 1$, where the supremum is taken on all asymptotically small perturbations G of F. Nevertheless:

Theorem 7. *Let $F \in Ra+Rb+On$ be the homogeneous random walk generated by the pair (f, ψ). Consider $M = \int \psi d\mu$, where μ is the unique absolutely continuous invariant measure of f.*

- *If $M > 0$ then for all asymptotically small perturbations G of F we have $m(\Omega_+(G)) > 0$.*
- *If $M = 0$ then for all asymptotically small perturbations G of F we have $HD(\Omega_+(G)) = 1$ but $m(\Omega_+(G)) = 0$.*
- *If $M < 0$ then for all asymptotically small perturbations G of F we have $HD(\Omega_+(G)) < 1$.*

Remark 3.3. Since the authors are more familiar with deterministic rather than stochastic terminology, we stated and proved Theorems 1-7 for determinist random walks. However Theorems 1-7 could be easilly translated to the theory of chains of complete connections (g-measures, chains of infinite order) and one-side shifts on an infinite alphabet.

3.4. Applications to renormalization theory of one-dimensional maps.

Theorem 8. *Let f and g be unimodal maps which are infinitely renormalizable with the same bounded combinatorial type and even critical order. Then the continuous conjugacy h between f and g is a strongly quasisymmetric mapping with respect to a certain nested sequence of partitions $\mathcal{P}$.*

The set of intervals $\mathcal{P}$ is defined using a map induced by f. See the details in Section 8.1.

Let $\mathcal{F}_d$ be the class of analytic maps with Schwartzian negative derivative which are infinitely renormalizable in the Fibonacci sense with even critical order d (see Section 8.2 for definitions). If f is a Fibonacci map, denote by $J_{\mathbb{R}}(f)$ the maximal invariant set of f. Let $\mathcal{F}_d^{uni}$ be the class of Fibonacci unimodal maps with negative Schwartzian derivative.

Theorem 9 (Metric Universality)**.** *For each even critical order d, one of the following statements holds:*

- $HD(J_{\mathbb{R}}(f)) < 1$*, for all* $f \in \mathcal{F}_d$.
- $HD(J_{\mathbb{R}}(f)) = 1$ *and* $m(J_{\mathbb{R}}) = 0$ *for all* $f \in \mathcal{F}_d$.
- $HD(J_{\mathbb{R}}(f)) = 1$ *and f has a wild attractor (in particular,* $m(J_{\mathbb{R}}(f)) > 0$*) for all* $f \in \mathcal{F}_d$

Theorem 10 (Measurable Deep Point)**.** *Let $f \in \mathcal{F}_d$, and assume that 0 is its critical point. If $J_{\mathbb{R}}(f)$ has positive Lebesgue measure then there exists $\alpha > 0$ and $C > 0$ so that*

$$m(x \in (-\delta, \delta)\colon\ x \notin J_{\mathbb{R}}(f)) \leq C\delta^{1+\alpha}.$$

Remark 3.4. Indeed α can be taken depending only on d.

Theorem 11. *For each even critical order d, the following statements are equivalent:*

1. *There exists $f \in \mathcal{F}_d$ such that $m(J_{\mathbb{R}}(F)) > 0$.*
2. *There exists $f \in \mathcal{F}_d$ with a wild attractor.*
3. *There exist maps $f, g \in \mathcal{F}_d^{uni}$ which are conjugated by a continuous absolutely continuous maps h, but f has a periodic point p whose eigenvalue is different from the eigenvalue of the periodic point $h(p)$ of g.*
4. *All maps in $\mathcal{F}_d$ have wild attractors.*
5. *All maps in $\mathcal{F}_d^{uni}$ can be conjugated with each other by an absolutely continuous conjugacy.*

4. Preliminaries

4.1. Probabilistic tools.
We are going to collect here a handful of probabilistic tools which are going to be useful along the article. A good reference for these results is [B].

Most of the probabilistic results in dynamical systems (large deviation, central limit theorem) assumes the the observable ψ is quite regular: usual regularity assumptions are either Holder continuity or bounded variation. Fix $f \in Mk+BD$. We are interested in $\mathcal{P}_0$-measurable observables with integer values which does not have such properties. Fortunally this is almost true: Denote by $\mathcal{O}(\mathbf{f})$ the class of $\mathcal{P}_0$-measurable functions $\psi\colon I \to \mathbb{Z}$ so that

- $\psi \in L^2(\mu)$,
- If P denotes the Perron-Frobenius-Ruelle operator of f, then $P\psi$ have bounded variation.

For instance, if $(f,\psi) \in Mk + sBD + Ra + Rb + GD$ then $\psi \in \mathcal{O}(f)$. Let μ be the absolutely continuous invariant measure of a Markov map f and $\psi\colon I \to \mathbb{R}$ a measurable function.

Proposition 4.1 (Large Deviations Theorem [B])**.** *For every $\psi \in \mathcal{O}(f)$ and $\epsilon > 0$ there exists $\gamma \in (0,1)$ so that*

$$\mu(\{x \in I\colon |\frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i(x)) - \int \psi d\mu| \geq \epsilon\}) \leq \gamma^n$$

Up to simple modifications in the proofs in [B], we have

Proposition 4.2 (Proposition 6.1 of [B])**.** *For every $\psi \in \mathcal{O}(f)$ the limit*

$$\sigma^2 := \lim_{n\to\infty} \int \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi(f^k(x))\right)^2 d\mu$$

exists. Furthermore $\sigma^2 = 0$ if and only if there exists a function $\alpha \in L^2(\mu)$ so that

$$\psi = \alpha \circ f - \alpha.$$

and

Proposition 4.3 (Central Limit Theorem: Theorem 8.1 in [B])**.** *For every $\psi \in \mathcal{O}(f)$ so that $\sigma^2 \neq 0$ we have*

$$sup_{\epsilon\in\mathbb{R}} \ |\mu(x \in I\colon \frac{\sum_{k=0}^{n-1}\psi(f^k(x))}{\sigma\sqrt{n}} \leq \ \epsilon) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\epsilon} e^{-\frac{u^2}{2}} \ du| \leq \frac{C}{\sqrt{n}}, \tag{6}$$

Indeed we are going to see that the assumption $\sigma^2 \neq 0$ is very weak: to this end we need the following result: Let $f\colon \cup_i I_i \to I$ be a map in Mk + BD + Ex + Ra + Rb.

Proposition 4.4 (Theorem 3.1 in [AD])**.** *Let $\psi\colon \cup_i I_i \to \mathbb{S}^1$ be a $\mathcal{P}_0$-measurable function. If*

$$\psi = \frac{\alpha \circ f}{\alpha},$$

where α is measurable, then α is $\mathcal{P}^\star$-measurable, where $\mathcal{P}^\star$ is the finest partition of I so that $f(I_i)$ is included in an atom of $\mathcal{P}^\star$ for each $i \in \Lambda$.

Proposition 4.5. *Let $\psi\colon \cup_i I_i \to \mathbb{Z}$ be a $\mathcal{P}^0$-measurable function. If $\psi = \alpha\circ f - \alpha$, where α is measurable, then α is constant on $f(I_i)$, for each $i \in \Lambda$.*

Proof. Note that we can assume that $\alpha(x) \in \mathbb{Z}$, for every x. Indeed, the relation $\psi = \alpha \circ f - \alpha$ implies that the function $\beta(x) = \alpha(x) \mod 1$ is f-invariant, so we can replace α by $\alpha - \beta$, if necessary. Fix an irrational number γ. Then

$$e^{2\pi\gamma\psi(x)} = \frac{e^{2\pi\gamma\alpha(f(x))}}{e^{2\pi\gamma\alpha(x)}},$$

so by Proposition 4.4 we have that $e^{2\pi\gamma\alpha(x)}$ is a $\mathcal{P}_0^\star$-measurable function. Since $j \in \mathbb{Z} \to e^{2\pi\gamma j} \in \mathbb{S}^1$ is one-to-one, we get that α is $\mathcal{P}_0^\star$-measurable. $\square$

A Markov map f is almost onto if and only if $\mathcal{P}_0^\star = \{I\}$, so

Corollary 4.6. *On the conditions of Proposition 4.5, if f is almost onto then α is constant.*

Corollary 4.7. *For every non constant $\psi \in \mathcal{O}(f)$ we have that $\sigma^2 \neq 0$. In particular the Central Limit Theorem as given in Eq. (6) holds for every non-constant ψ.*

Let $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ be an increasing sequence of σ-subalgebras of a probability space $(\Omega, \mathcal{A}, \mu)$. A **martingale difference sequence** is a sequence of functions $\psi_n \colon \Omega \to \mathbb{R}$, where ψ_n is $\mathcal{A}_n$-measurable for $n \geq 1$, so that

$$\mathbb{E}(\psi_n | \mathcal{A}_{n-1}) = 0$$

for every n. Here $\mathbb{E}(\psi|\mathcal{B})$ denotes de conditional expectation of ψ relative to the sub-algebra $\mathcal{B}$. When $\mathcal{B}$ is generated by atoms $\{J_i\}_i$ then $\mathbb{E}(\psi|\mathcal{B})$ is the function defined as

$$\mathbb{E}(\psi|\mathcal{B})(x) = \frac{1}{\mu(J_i)} \int_{J_i} \psi \, d\mu$$

for every $x \in J_i$.

The following Proposition is the classic Azuma-Hoeffding inequality: see, for instance Exercise E14.2 in [W]:

Proposition 4.8 (Azuma-Hoeffding inequality)**.** *Let ψ_n as above and furthermore assume that*

$$||\psi_n||_\infty = c_i < \infty.$$

Define

$$\psi := \sum_{i=1}^{n} \psi_i.$$

Then

$$\mu(x \in \Omega \colon \ |\psi - \mathbb{E}(\psi)| > t) \leq 2 \exp(-\frac{t^2}{2\sum_{i=1}^n c_i^2}).$$

4.2. **How to construct asymptotically small perturbations.** As we are going to see in the next Proposition, it is easy to construct asymptotically small perturbations of a random walk:

Proposition 4.9. *Let F and G be random walks satisfying the properties sBD, Ra and Rb, where G is a topological perturbation of F. Assume that there exist $C > 0$ and $\lambda \in (0,1)$ with the following properties: if I_j^n is as in properties Ra and Rb, then*

i. *For every I_j^n in $\mathcal{P}_n^0$ we have*

$$|log \frac{|I_{j+1}^n|}{|I_j^n|} \frac{|H(I_j^n)|}{|H(I_{j+1}^n)|}| \leq C\lambda^{|n|+|j|}.$$

ii. *For every* $J \in \mathcal{P}_{0,n}$ *we have*

$$|\tau^F_J - \tau^G_{H(J)}|_{C^2} \leq C\lambda^{|n|}.$$

iii. *If* $I^n_i = [a^n_i, b^n_i]$ *then*

$$\max_i \max\{|a^n_i - H(a^n_i)|, |b^n_i - H(b^n_i)|\} \leq C\lambda^{|n|}.$$

iv. *Either* ψ *is a bounded funtion or* ψ *has a lower bound and* $F = G$ *on* $\cup_{n<0} I_n$.

Then G *is an asymptotically small perturbation of* F. *Furthermore there exist* $\beta \in [0,1)$ *and* $C > 0$ *so that*

$$|H(p) - p| \leq C\beta^{|\psi(p)|}.$$

Proof. We will assume that ψ is bounded: the other case is analogous. Consider $(x,n) \in \mathbb{Z}\times I$ and $(y,n) = H(x,n)$. Denote $(x_i, n_i) := F^i(x,n)$, $(y_i, n_i) := G^i(y,n)$.

Denote $\delta_i = |y_i - x_i|$ and $\tilde{\delta}_i = |A_{G(H(J_i))}(y_i) - A_{F(J_i)}(x_i)|$. Here $(x_i, n_i) \in J_i \in \mathcal{P}^0$. It is easy to conclude, using iii. and property LI, that

$$\tilde{\delta}_i \leq \frac{\delta_i}{|F(J)|} + C\lambda^{|n_i|} \tag{7}$$

and making use of ii. to get

$$|\tau^G_{H(J)}(A_{G(H(J_i))}(y_i)) - \tau^F_J(A_{F(J_i)}(x_i))| \leq D\tau^F_J(z_i)\frac{\delta_i}{|F(J)|} + C\lambda^{|n_i|}.$$

Here $z_i \in [0,1]$. Since $D\tau^F_J(z_i)|F(J)|/|J| \leq \lambda$ (property Ex), we get, using again iii.

$$\delta_{i-1} \leq \lambda\delta_i + C\lambda^{|n_i|}. \tag{8}$$

Because ψ is bounded, $|n_{i+1} - n_i| \leq B = \max|\psi|$. So if $i < n/2B$ then $|n_i| > |n_0|/2$. Since $\delta_{[\frac{n}{2B}]} \leq 1$, Eq. (8) implies

$$|H(x,n) - (x,n)| = |y_0 - x_0| \leq C\lambda^{\frac{|n|}{2}}.$$

In particular, by Eq. (7) and property ii., we have

$$|D\tau^G_{H(J)}(A_{G(H(J_0))}(y_1)) - D\tau^F_J(A_{F(J_0)}(x_1))| \leq C\lambda^{\frac{|n|}{2}}. \tag{9}$$

By $Ra + Rb$ there exists $\theta \in (0,1)$ so that

$$\theta^{|i|} \leq |I^n_i|. \tag{10}$$

Let i so that $J = I^n_i$.

Case A. $|i| \geq |n/2|(\log\lambda/\log\theta)$: Due i. and iii. and property Ra, there exists $C > 0$ so that

$$|\log\frac{|H(I^n_i)|}{|I^n_i|}| \leq C\lambda^n.$$

Together with $sBD + LI$ and $iii.$, this implies that for every $p \in I^n_i$, with $|i| \geq |n/2|(\log\lambda/\log\theta)$, we have

$$|\log\frac{DG(H(p))}{DF(p)}| \leq C\lambda^{\frac{|n|}{2}\frac{\log\lambda}{\log\theta}}.$$

Case B. $|i| < |n/2|(\log \lambda / \log \theta)$: In this case, by iii. and Eq. (10) we have

$$\log \frac{|H(I_i^n)|}{|I_i^n|} \leq C \frac{|H(b_i^n) - b_i^n| + |H(a_i^n) - a_i^n|}{|b_i^n - a_i^n|} \leq C\lambda^{\frac{|n|}{2}}.$$

Now using Eq. (9) we can easily obtain

$$|\log \frac{DG(H(p))}{DF(p)}| \leq C\lambda^{\frac{|n|}{2}}.$$

□

5. Stability of transience

We will begin this section with the large deviations result to strongly transient random walks:

Proof of Proposition 2.1. We intend to apply the Azuma-Hoeffding inequality, but since ψ is not necessarily bounded, we need made some adjustments first: Fix $P \in \mathcal{P}^0$ and define $\mathcal{F}_0 := \{P\}$ and $\mathcal{F}_n := \{Q\}_{Q \subset P,\ Q \in \mathcal{P}^n}$. Since $F \in GD$, by the usual distortion control tricks for F, we can find $M > 0$ such that $\alpha(x) := \min\{\psi(x), M\}$ satisfies

$$\mathbb{E}(\alpha \circ F^n | \mathcal{F}_{n-1}) \geq K - \epsilon \tag{11}$$

for every $n \geq 1$.

Define the martingale difference sequence

$$\Psi_n := \alpha \circ F^n - \mathbb{E}(\alpha \circ F^n | \mathcal{F}_{n-1}).$$

Of course $||\Psi_n||_\infty \leq M$, if M is large enough. By the Azuma-Hoeffding inequality we have

$$m(p \in P\colon\ |\sum_{i=1}^{n} \Psi_i(p)| > t) \leq 2\exp(-\frac{t^2}{2nM^2})|P|.$$

Taking $t = \delta\ n$ we obtain

$$m(p \in P\colon\ |\sum_{i=1}^{n} \Psi_i(p)| > \delta\ n) \leq 2\exp(-\frac{\delta^2 n}{2M^2})|P|. \tag{12}$$

Since

$$\sum_{i=1}^{n} \psi(F^i(p)) \geq \sum_{i=1}^{n} \alpha(F^i(p)) = \sum_{i=1}^{n} \Psi_i(p) + \sum_{i=1}^{n} \mathbb{E}(\alpha \circ F^i | \mathcal{P}_{i-1})(x)$$
$$\geq \sum_{i=1}^{n} \Psi_i(p) + (K - \epsilon)n.$$

Due Eq. (12), this implies that

$$m(p \in I_j\colon\ \sum_{i=0}^{n-1} \psi(F^i(p)) < (K - \epsilon - \delta)\ n) \leq 2\exp(-\frac{\delta^2 n}{2M^2}).$$

This completes the proof. □

Proposition 5.1. *Let F be either strongly recurrent or a homogeneous random walk with positive mean drift. Then any asymptotically small perturbation G of F has the following property: there exists $\lambda \in [0,1)$ and $K > 0$ so that for every $P \in \mathcal{P}^0$*

$$m(p \in P\colon \ \sum_{i=0}^{n-1} \psi(G^i(p)) < (K-\epsilon)n) \leq C\lambda^n |P|.$$

In particular G is also transient.

Proof. We will carry out the proof assuming the strongly transience: the homogeneous case is analogous: By Proposition 2.1 we have

$$m(p \in P\colon \ \sum_{i=0}^{n-1} \psi(F^i(p)) < (K-\epsilon)n \ for \ some \ n \geq n_0) \leq C_1 \exp(-C_2 n_0)|P|.$$

Since G is an asymptotically small perturbation, Eq. (3) implies that

$$m(p \in P\colon \ \sum_{i=0}^{n-1} \psi(G^i(p)) < (K-\epsilon)n \ for \ some \ n \geq n_0) \leq C_3 \exp(-C_4 n_0)|P| \tag{13}$$

provided that $P \in \mathcal{P}^0_j$, $j \geq 2\,|\min \psi|\; n_0$. In particular, for such P we have that

$$m(p \in P\colon \ \lim_i \pi_2(G^i(p)) = +\infty) > (1-\delta)|P|, \tag{14}$$

if n_0 is large enough.

Let $E(p) \in \mathbb{N} \cup \{\infty\}$ be the first entry of p into

$$\bigcup_{j > -2|\min \psi| n_0} I_j.$$

An argument similar to the proof of Eq. (13) gives

$$\begin{aligned} m(p \in P\colon \ \sum_{i=0}^{n-1} \psi(G^i(p)) < (K-\epsilon)n \ for \ some \ n \in [\tilde{n}, E(p)]) \\ \leq C_5 \exp(-C_6 \tilde{n})|P| \end{aligned} \tag{15}$$

provided that $P \in \mathcal{P}^0_j$, $j \leq -2\,|\min \psi|\; n_0$, with n_0 large enough. In particular, for such P we have that

$$m(p \in P\colon \ E(p) < \infty) = |P|, \tag{16}$$

if n_0 is large enough.

Denote

$$\mathcal{G}_{n_0}(P) := \{p \in P\colon \ \sum_{i=0}^{n-1} \psi(G^i(p)) > (K-\epsilon)n \ for \ all \ n \geq n_0\}$$

then we have, for $P \in \mathcal{P}^0_j$, $j \geq 2\,|\min \psi|\; n_0$ and n_0 large enough

$$\begin{aligned} &\int_P \sum_{i=0}^{n_0-1} \psi(G^i(p)) \ dm \\ &= \int_{\mathcal{G}_{n_0}(P)} \sum_{i=0}^{n_0-1} \psi(G^i(p)) \ dm + \int_{P-\mathcal{G}_{n_0}(P)} \sum_{i=0}^{n_0-1} \psi(G^i(p)) \ dm \\ &\geq ((1 - C_3 \ exp(-C_4 n_0))(K-\epsilon) - C_3 \ exp(-C_4 n_0)|\min \psi|)\, n_0 \ |P| \end{aligned} \tag{17}$$

$$\geq (K-\delta)\ n_0\ |P|.$$

Now fix n_0 large and denote $N = 2\ |\min\psi|\ n_0$. We claim that for almost every point in $P \in P_j^0$, with $i \leq N$, its orbit converges to infinity. Otherwise, consider a Lebesgue density point p of the complement of those transient points. Denote $p \in P_i \in \mathcal{P}^n$, $G^i(P_i) = Q_i$, where $|Q_i| \geq C$, due the Large Image (LI) property. Note that $\pi_2(Q_i)$ can not be larger than N infinitely often, otherwise due the bounded distortion (BD) property a significant portion of the points in Q_i are not transient, that contradicts Eq. (14). But a similar argument using Eq. (16) implies that $\pi_2(Q_i) \in [-N, N]$ infinitely often. Then we can apply once again BD+LI to conclude that there exists an interval $R \subset I_\ell$, for some $\ell \in [-N, N]$ where almost every point is not transient. But since the transient points are dense in R (since F is transient and G is topologically conjugated to F), this is in contradiction with Eq. (14). This finishes the proof of the claim.

As a consequence there exists n_1 satisfying

$$\int_Q \sum_{i=0}^{n_1} \psi(G^i(p))\ dm > \frac{2N}{\delta}|Q| \tag{18}$$

for every interval $Q \subset I_j$, with $|j| \leq N$ and $|Q| \geq \delta$, where δ is as in property LI. Define an induced random walk $\tilde{G}$ in the following way:

$$\tilde{G}(p) := \begin{cases} G^{n_0}(p) & \text{if } \pi_2(p) \notin [-N, N], \\ G^{n_1}(p) & \text{if } \pi_2(p) \in [-N, N]. \end{cases}$$

By Eq. (14) and Eq. (16) the random walk $\tilde{G}$ is strongly transient. Now we can apply Proposition 2.1 to obtain the wished estimative.

□

Let $n > 0$ and j be integers and F be a deterministic random walk. Then any connected component C of $F^{-n}\ int\ I_j$ is called a **cylinder**. The **lenght** $\ell(C)$ of the cylinder C is n. If C is a cylinder of lenght n so that $F^i(C) \subset I_{j_i}$, for $i < n$, we will denote $C = C(j_0, j_1, \ldots, j_n)$.

Proposition 5.2. *Let F be a random walk induced by the pair $(\{f_i\}, \psi)$. Assume that there exists $\epsilon > 0$ so that for $K > 0$, we have*

$$m(\{p \in I_n : \psi(p) < -K\}) \leq \frac{1}{K^{2+\epsilon}},$$

provided $n \geq n_0$. Then

$$\lim_k m(\{p \in I_{n_k} : \text{ there exists } i \leq k^2 \text{ so that } \psi(F^i(p)) < -k\}) = 0,$$

uniformly for all sequence satisfying $n_k > k^3 + n_0$.

Remark 5.3. *For a homogeneous random walk, the condition on ψ is equivalent to $1_{I_0} \cdot \psi \in L^{2+\epsilon}(m)$.*

Let F and G be random walks which are topologically conjugated by a homeomorphism h that preserves states. For any $p \in I \times \mathbb{Z}$ define

$$dist_i(p) := |\log \frac{DG^i(h(p))}{DF^i(p)|}.$$

Define

$$\Omega_{n_0+}(F) := \{p\colon \pi_2(F^n(p)) \geq n_0, \text{ for all } n \geq n_0\}.$$

Proposition 5.4. *Let F and G be random walks which are conjugated by a homeomorphism h which preserves states. Suppose that there exists a F-forward invariant set Λ so that*

-H1: $\sup_i \ dist_i(p) \leq C_p < \infty$, *for each $p \in \Lambda$.*

then h is absolutely continuous on $\cup_i F^{-i}\Lambda$ and h^{-1} is absolutely continuous on $\cup_i G^{-i}h(\Lambda)$. Furthermore, if

-H2: *There exists $C > 0$, $M > 0$ and $n_0 \in \mathbb{Z} \cup \{-\infty\}$ so that for every $n \geq n_0$ and $P \in \mathcal{P}^0_n$,*

$$m(p \in P \cap \Lambda\colon \ C_p \leq C) \geq M|P|.$$

then h is absolutely continuous on $\cup_i F^{-i}(\Omega_{n_0+}(F))$ and h^{-1} is absolutely continuous on $\cup_i G^{-i}(\Omega_{n_0+}(G))$.

Proof. For each $j \in \mathbb{N}$ denote

$$\Lambda_j := \{p \in \Lambda\colon \sup_i \ dist_i(p) \leq j\}.$$

Note that Λ_i is forward invariant.

We claim that h is absolutely continuous on Λ_j and h^{-1} is absolutely continuous on $h(\Lambda_j)$. Indeed, for each $p \in \Lambda_j$ and $k \in \mathbb{N}$, denote $F^i p = (x_i, n_i)$. Denote by $J_k(x) \in \mathcal{P}^k$ the unique interval which contains x so that F^k maps $J_k(x)$ diffeomorphically onto $Q_k \subset I_{n_k}$. There is some ambiguity here if x is in the boundary of $J_k(x)$, but these points are countable, so they are irrelevant for us.

If we use the analogous notation to $h(x)$ and G, we have $h(J_k(x)) = J_k(h(x))$ and, due the bounded distortion property of the random walks F and G, there exist $C_1, C_2 > 0$ such that

$$C_1 e^{-dist_k(p)} \leq \frac{|h(J_k(x))|}{|J_k(x)|} \leq C_2 e^{dist_k(p)}.$$

So, if $p \in \Lambda_j$ then

$$C_1 e^{-j} \leq \frac{|h(J_k(x))|}{|J_k(x)|} \leq C_2 e^j, \quad \text{for all } k \in \mathbb{N}. \tag{19}$$

Let $A \subset \Lambda_j$ be a set with positive Lebesgue measure. We claim that $h(A)$ also has positive Lebesgue measure. Indeed, choose a compact set $K \Subset A$ with positive Lebesgue measure. Denote $U_k := \cup_{x \in K} J_k(x)$. Since $|J_k(x)| \leq \lambda^k$, we have that $\lim_k m(U_k) = m(K)$ and $\lim_k m(h(U_k)) = m(h(K))$. Since U_k is a countable disjoint union of intervals of the type $J_k(x)$, by Eq. (19)

$$C_1 e^{-j} \leq \frac{m(h(U_k))}{m(U_k)} \leq C_2 e^j, \text{ so } C_1 e^{-j} \leq \frac{m(h(K))}{m(K)} \leq C_2 e^j, \tag{20}$$

and we conclude that $h(K)$ also has positive Lebesgue measure. A identical argument shows that, if $A \in \Lambda_j$ has positive Lebesgue measure, then $h^{-1}A$ also has positive Lebesgue measure. The proof of the claim is finished and so h and h^{-1} are absolutely continuous on $\Lambda = \cup_j \Lambda_j$ and $h(\Lambda) = \cup_j h(\Lambda_j)$.

Now it is easy to conclude that h and h^{-1} are absolutely continuous on $\cup_i F^{-i}\Lambda$ and $\cup_i G^{-i}h(\Lambda)$.

Now assume $H2$. We claim that $\cup_i F^{-i}\Lambda$ has full Lebesgue measure on $\Omega_{n_0+}(F)$. Indeed, Assume that $m(\Omega_{n_0+}(F)\setminus \cup_i F^{-i}\Lambda) > 0$ and choose a Lebesgue density point p of this set. Then

$$\lim_k \frac{m(J_k(p)\cap \Omega_{n_0+}(F)\setminus \cup_i F^{-i}\Lambda)}{|J_k(x)|} = 1.$$

Due the bounded distortion of F, if $F^k(p) = (x_k, n_k)$ and $F^k(J_k(x)) = Q_k \subset I_{n_k}$, with $n_k \geq n_0$, then

$$\limsup_k \frac{m(Q_k\cap \Lambda)}{|Q_k|} \leq C(1-\liminf_k \frac{m(J_k(x)\cap \Omega_{n_0+}(F)\setminus \cup_i F^{-i}\Lambda)}{|J_k(x)|}) = 0,$$

which contradicts H2.

Since on the set $\{p\in P\cap\Lambda\colon\ C_p\leq C\}$ the $dist_k(x)$ is uniformly bounded with respect to k and x, we can use an argument identical to the proof of Eq. (20) to conclude that

$$\frac{m(p\in P\cap\Lambda\colon\ C_p\leq C)}{m(h(p)\in h(P)\cap h(\Lambda)\colon\ C_p\leq C)}\leq C_1,$$

so $m(h(P\cap\Lambda\colon C_p\leq C))\geq \tilde{C}M$, for all $n\geq n_o$ and using an argument as above, we conclude that $\cup_i G^{-i}h(\Lambda)$ has full Lebesgue measure on $\Omega_{n_0+}(G)$. Since h (h^{-1}) is absolutely continuous on $\cup_i F^{-i}\Lambda$ ($\cup_i G^{-i}h(\Lambda)$) and

$$m(\Omega_{n_0+}(F)\backslash\cup_i F^{-i}\Lambda) = m(h(\Omega_{n_0+}(F)\backslash\cup_i F^{-i}\Lambda)) = m(\Omega_{n_0+}(G)\backslash\cup_i G^{-i}h(\Lambda)) = 0,$$

we have that h and h^{-1} are absolutely continuous on $\Omega_{n_0+}(F)$ and $\Omega_{n_0+}(G)$. Now it is easy to prove that h is absolutely continuous on $\cup_i F^{-i}\Omega_{n_0+}(F)$ and h^{-1} is absolutely continuous on $\cup_i G^{-i}\Omega_{n_0+}(G)$. □

Proof of Theorem 1. By Proposition 5.1, G is transient. In particular for all $n_0 \in mathbbZ$ the sets

$$\cup_i F^{-i}\Omega_{n_0+}(F) \text{ and } \cup_i G^{-i}\Omega_{n_0+}(G)$$

have full Lebesgue measure. So by Proposition 5.4, to prove that h and h^{-1} are absolutelly continuous, it is enough to find a forward invariant set satisfying the assumptions H1 and H2. Indeed, fix $\delta > 0$ (we will choose δ latter). Consider the F-forward invariant set

$$\Lambda = \Lambda_\delta := \{p:\ \liminf_k \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \geq \frac{\delta}{3}\}.$$

We claim that Λ satisfies H1. Indeed take $x\in\Lambda$. Then, for $k\geq k_0(x)$ we have $n_k := \pi_2(F^k(p)) \geq k\delta/4$. So

$$dist_k(x) \leq \sum_{i=0}^{k-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| \tag{21}$$

$$\leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{k-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}|$$

$$\leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{k-1} \lambda^{n_i}$$
$$\leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{\infty} \lambda^{i\delta/4}$$
$$\leq K_p + C(\delta).$$

To prove that Λ satisfies H2, By Proposition 2.1 for each $P \in \mathcal{P}_i^0$ we have

$$m(p \in P\colon\ \pi_2(F^k(p)) - \pi_2(p) < \delta k) \leq C\lambda^k |P|, \tag{22}$$

provided δ is small enough. From Eq. (22) we obtain

$$\mu(p \in P\colon\ \pi_2(F^n(p)) - \pi_2(p) \geq \delta n \text{ for all } n \geq n_0) \geq (1 - C\lambda^{n_0})|P|. \tag{23}$$

In particular, we have that, for every n,

$$\pi_2(F^n(p)) \geq \delta(n - n_0) + \pi_2(p) + n_0\ min\psi. \tag{24}$$

in the set in Eq. (23). Using the same argument as in Eq. (21) we can easily obtain $H2$ from Eq. (24) and Eq. (23), choosing n_0 large enough. □

Proof of Theorem 2. Observe that using the argument in the proof of Proposition 5.1, an induced map of a homogeneous random walk with positive drift is strongly transient. From this the proof of Theorem 2 goes exactly as the Theorem 1. □

Proof of Theorem 3. By Proposition 5.1, for every i we have

$$m(p \in I_i\ :\ \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \leq \delta) \leq C\theta^k.$$

and furthermore $\theta := \theta(\delta)$ tends to 0 when δ tends to zero. Using an argument as in the proof of Theorem 1 we can conclude that

$$m(p \in I_i\ :\ \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \geq \delta\ for\ k \geq k_0) \geq 1 - C\theta^{k_0} \tag{25}$$

In particular we can use the argument in the proof of Theorem 2 to conclude that the conjugacy h is absolutely continuous. Indeed, Eq. (25) implies

$$m(p \in I_i\ :\ dist_k(x) \geq \delta n + C\ for\ some\ k) \leq C\theta^n. \tag{26}$$

where $\delta = \sup_p dist_1(p)$. Firstly we will prove Theorem 3 when δ is small.

Denote $\Lambda_1 := \{p \in I_i\ : h'(x) \leq 1\}$ and, for $n \geq 1$

$$\Lambda_n := \{p \in I_i\ :\ e^{\delta(n-1)} < h'(x) \leq e^{\delta n}\}.$$

By Eq. (26) we have $m(\Lambda_n) \leq C\theta^n$.

Let $B \subset I_i$ be an arbitrary Lebesgue measurable set. Let k_1 be so that

$$\theta^{k_1+1} < |B| \leq \theta^{k_1}.$$

Since h is absolutely continuous we have

$$|h(B)| = \int_B h'\ dm$$
$$= \sum_{n=0}^{k_1} \int_{B\cap\Lambda_n} h'\ dm + \sum_{n=k_1+1}^{\infty} \int_{B\cap\Lambda_n} h'\ dm$$

$$\leq \sum_{n=0}^{k_1} C\theta^{k_1} e^{\delta n} + \sum_{n=k_1+1}^{\infty} C(e^\delta \theta)^n$$
$$\leq C(e^\delta\theta)^{k_1} \leq C|B|^{1+\frac{\delta}{\ln\theta}}.$$

Now if $B \subset J \in \mathcal{P}^n$ and $F^n(J) = Q \subset I_i$, with $|Q| \geq C$ (due Property LI), then due the bounded distortion of F

$$\frac{|h(B)|}{|h(J)|} \leq C\frac{|h(F^n(B)|}{|h(Q)|} \leq C\Big(\frac{|F^n(B)|}{|Q|}\Big)^{1+\frac{\delta}{\ln\theta}} \leq C\Big(\frac{|B|}{|J|}\Big)^{1+\frac{\delta}{\ln\theta}}.$$

To prove a similar inequality to h^{-1}, define

$$\tilde{\Lambda}_n := \{p \in I_i \ : \ e^{\delta(n-1)} < (h^{-1})'(x) \leq e^{\delta n}\}.$$

of course

$$h^{-1}\tilde{\Lambda}_n = \{p \in I_i \ : \ e^{-\delta(n)} < h'(x) \leq e^{-\delta(n-1)}\},$$

so by Eq. (26) we obtain

$$m(h^{-1}\tilde{\Lambda}_n) \leq \theta^n.$$

In particular

$$m(\tilde{\Lambda}_n) = \int_{h^{-1}\tilde{\Lambda}_n} h'(x)\ dm \leq (e^{-\delta}\theta)^n$$

Note that this argument gives us an exponential upper bound even if δ is large.

Now we can switch the roles of F and G to obtain the inequality to h^{-1}, which shows that h is a mSQS-homeomorphism relative to the stochastic basis $\cup_n \mathcal{P}^n$.

To complete the proof when δ is not small do the following: find a continuous path of random walks F_t so that $F_0 = F$ and $F_1 = G$, so that for every $t \in [0,1]$ we have that F_t is a asymptotically small perturbation of F. By the argument above for every $t \in [0,1]$ there exists ϵ_t so that $F_{\tilde{t}}$ is mSQS-conjugated to F_t, provided $|\tilde{t} - t| \leq \epsilon_t$. Using the compactness of $[0,1]$ we can find a finite sequence of random walks $F_{t_0} = F, F_{t_1}, F_{t_2}, \dots F_{t_n} = G$ so that F_{t_i} and $F_{t_{i+1}}$ are conjugated by a map h_i which is mSQS with respect some dynamically defined stochastic basis. Composing these conjugacies we find a mSQS-conjugacy between F and G. □

6. Stability of recurrence

To avoid a cumbersome notation, in this section we make the convention that all inequalities holds only for large n. moreover in this section we assume that ψ is unbounded. Recall that in this case we assume that asymptotically small perturbations G coincides with F on negative states. The case where ψ is bounded is similar.

The following is a easy consequence of the Central Limit Theorem for Birkhoff sums (Proposition 4.3)

Corollary 6.1. *Let a_n be a positive increasing sequence. Then*

$$\mu(\frac{|S_n|}{\sqrt{n}} > a_n) \leq Ce^{-\frac{a_n^2}{2}} + C\frac{1}{\sqrt{n}}.$$

Proof. Use Proposition 4.3 and and note that the estimative

$$\int_{-\infty}^{v} e^{-\frac{u^2}{2}}\ du \leq Ce^{-\frac{v^2}{2}}$$

holds for $v << 0$.

□

Given $n \in \mathbb{N}$, split $[0,2n] \cap \mathbb{N}$ in $\sqrt{\log n}$ blocks (called main blocks) , denoted B_j, with length

$$\frac{n}{\log^{8j} n},\ j = 1, \dots, \sqrt{\log n},$$

and between the main blocks we put little blocks H_j, called holes, of length $\log^4 n$. These holes will warranty the independence between the events in distinct main blocks. Put these blocks in the following order:

$$\cdots < B_{j+1} < H_{j+1} < B_j < H_j < \dots,$$

with $min\ B_{\sqrt{\log n}} = 0$. Note that we let most of the second half of the interval $[0,2n] \cap \mathbb{N}$ uncovered.

Define

$$S(j) = \sum_{i \in B_j} \psi \circ f^i$$

$$H(j) = \sum_{i \in H_j} \psi \circ f^i$$

Denote $|B_j| := \max B_j - \min B_j$.

Lemma 6.2. *We have*

$$\mu(\sum_{i=0}^{|B_j|} \psi \circ f^i \geq \frac{\sqrt{n}}{\log^{4j} n} \log^3 n) \leq C \frac{\log^{4j} n}{\sqrt{n}}.$$

Proof. This follows from Corollary 6.1. □

Proposition 6.3. *For every $\epsilon > 0$ we have*

$$\mu(S(j) > \frac{\sqrt{n}}{\log^{4j} n} \log^3 n,\ for\ some\ j \leq \sqrt{\log n}) \leq C \frac{1}{\sqrt[2+\epsilon]{n}},$$

provided n is large enough.

Proof. For $j \leq \sqrt{\log n}$ define

$$\Lambda_j := \{x \in I\colon\ S(j)(x) > \frac{\sqrt{n}}{\log^{4j} n} \log^3 n\}$$

$$= \{x \in I\colon \sum_{i < |B_j|} \psi \circ f^{i + \min B_j}(x) > \frac{\sqrt{n}}{\log^{4j} n} \log^3 n\}$$

and for each $P \in \mathcal{P}^{\min B_j}$ denote $\Lambda_j(P) := \Lambda_j \cap P$.

Due Lemma 6.2 and the bounded distortion of $f^{\min B_j}$ on P we have

$$m(\Lambda_j(P)) \leq C \frac{\log^{4j} n}{\sqrt{n}} |P|.$$

Summing on j and P

$$m(\bigcup_j \bigcup_P \Lambda_j(P)) \leq \sqrt{\log n}\, \frac{\log^{4j} n}{\sqrt{n}} << C \frac{1}{\sqrt[2+\epsilon]{n}}.$$

□

Proposition 6.4. *For every $\epsilon > 0$ and $d > 0$ we have*

$$\mu(|\sum_{i\in H_j}\psi(f^i(x))| > \log^8 n,\ for\ some\ j \leq \sqrt{\log n}) \leq C\frac{1}{n^d}, \tag{27}$$

provided n is large enough.

Proof. For $i \in H_j - 1$, with $j \leq \sqrt{\log n}$, define

$$\Lambda_i := \{x \in I \colon |\psi(f^i(x))| > \log^4 n.\}.$$

By expanding and bounded distortion properties of f and condition GD we have that

$$\mu(\Lambda_i) \leq C\lambda^{\log^4 n}.$$

Since $|H_j| = \log^4 n$, if x belongs to the set in Eq. (27) then $x \in \Lambda_i$, for some $i \in H_j - 1$, with $j \leq \sqrt{\log n}$. So

$$\begin{aligned}
&\mu(|\sum_{i\in H_j}\psi(f^i(x))| > \log^8 n,\ for\ some\ j \leq \sqrt{\log n})\\
&\leq \mu(\bigcup_{j\leq\sqrt{\log n}}\ \bigcup_{i\in H_j-1}\Lambda_j)\\
&\leq \sqrt{\log n}\ \log^4 n\ n^{\log\lambda\log^3 n}\\
&<< \frac{1}{n^d},
\end{aligned}$$

where the least inequality holds for n large enough. □

Proposition 6.5 (Independence between distant events)**.** *There exists $\lambda < 1$ so that the following holds: For all cylinders C_1 and C_2, we have*

$$\mu(C_1 \cap f^{-(n+d)}C_2) = \mu(C_1)\mu(C_2)(1 + O(\lambda^d)).$$

Here $n = |C_1|$.

Proof. Let J be an interval in C_1 so that $f^n(J) = I$. Define the measure $\rho(A) := \mu(f^{-n}A \cap J)/\mu(J)$. Note that by the bounded distortion property of f, we have that $\log d\rho/dm$ is α-Holder, where α does not depend on n. Furthermore it is bounded by above by a constant which does not depend on n. By the well-know theory of Perron-Frobenius-Ruelle operators for Markov expanding maps, if P is the Perron-Frobenius-Ruelle operator of f, then there exists $\lambda < 1$ so that

$$P^d\frac{d\rho}{dm} = (1 + O(\lambda^d))\frac{d\mu}{dm}.$$

So

$$\begin{aligned}
&\frac{\mu(J \cap f^{-(n+d)}C_2)}{\mu(J)}\\
&= \rho(f^{-d}C_2) = \int 1_{C_2}\circ f^d\ \frac{d\rho}{dm}dm\\
&= \int 1_{C_2}\ P^d\frac{d\rho}{dm}dm\\
&= (1 + O(\lambda^d))\int 1_{C_2}\ \frac{d\mu}{dm}dm\\
&= (1 + O(\lambda^d))\mu(C_2).
\end{aligned}$$

Since C_1 is a disjoint union of intervals J so that $f^nJ = I$, we finished the proof. □

Corollary 6.6. *There exists $M > 0$ so that*

$$\mu(S_j < \frac{\sqrt{n}}{\log^{4j} n}\ M \text{ for all } j \leq \sqrt{\log n}) \leq C\big(\frac{2}{3}\big)^{\sqrt{\log n}}$$

Proof. Choose $M > 0$ so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{u^2}{2}}\ du < \frac{2}{3}$$

Consider the disjoint union of cylinders

$$C_j := \{x \ s.t. \sum_{0}^{|B_j|} \psi \circ f^i(x) < \frac{\sqrt{n}}{\log^{4j} n}\ M\}.$$

The Central Limit Theorem tells us that if n is large enough then

$$\mu(C_j) < \frac{2}{3}$$

for every $j \leq \sqrt{\log n}$.

Recall that between B_j and B_{j+1} there is a hole with length $\log^4 n$. Applying $\sqrt{\log n}$ times Proposition 52 , we obtain

$$\mu(S_j < \frac{\sqrt{n}}{\log^{4j} n}\ M \text{ for all } j \leq \sqrt{\log n}) \leq \big(\frac{2}{3}\big)^{\sqrt{\log n}}(1+O(\lambda^{\log^4 n}))^{\sqrt{\log n}} \leq C\big(\frac{2}{3}\big)^{\sqrt{\log n}}$$

□

Proposition 6.7. *There exists $C > 0$ so that for every k,*

$$\mu(x \in I_k\colon \text{ there exists } i < \ell^3 \text{ so that } \sum_{k=0}^{i} \psi \circ f^k(x) > \frac{\ell}{2}) \geq 1 - C\big(\frac{2}{3}\big)^{\sqrt{3 \log \ell}}$$

Proof. Let M be as in Proposition . Denote $n = \ell^3$ and define

$$A_\ell := \{x\colon \text{ there exists } i < \ell^3 \text{ so that } \sum_{k=0}^{i} \psi \circ f^k(x) > \frac{\ell}{2}\},$$

$$B_\ell := \{x\colon\ |S_j(x)| < \frac{\sqrt{n}}{\log^{4j} n} \log^3 n, \text{ for all } j \leq \sqrt{\log n}\},$$

$$C_\ell := \{x\colon S_j(x) \geq \frac{\sqrt{n}}{\log^{4j} n}\ M, \text{ for some } j \leq \sqrt{\log n}\},$$

$$D_\ell := \{x\colon |H_j(x)| \leq \log^8 n, \text{ for all } j \leq \sqrt{\log n}\}.$$

We claim that if ℓ is large then $B_\ell \cap C_\ell \cap D_\ell \subset A_\ell$. Indeed, let $x \in B_\ell \cap C_\ell \cap D_\ell$. Then for some $j_0 \leq \sqrt{\log n}$,

$$S(j_0) \geq \frac{\sqrt{n}}{\log^{4j_0} n}\ M.$$

We claim that, if $m = max\ B_{j_0}$, then

$$\sum_{0}^{m} \psi \circ f^i(x) > \frac{\ell}{2}.$$

Indeed, since $x \in D_\ell$,

$$|\sum_{i\in H_j,\ j>j_0} \psi\circ f^i(x)| \le \sqrt{\log n}\log^8 n = o(\ell).$$

Moreover, since $x \in B_\ell$,

$$|\sum_{i\in B_j,\ j>j_0} \psi\circ f^i(x)| \le \sum_{j>j_0} \frac{\sqrt{n}}{\log^{4j} n}\log^3 n \le C\frac{\sqrt{n}}{\log^{4j_0+4} n}.$$

So

$$\begin{aligned}\sum_0^m \psi\circ f^i(x) &= \sum_{i\in B_{j_0}} \psi\circ f^i(x) + \sum_{i\in B_j,\ j>j_0} \psi\circ f^i(x) + \sum_{i\in H_j,\ j>j_0} \psi\circ f^i(x)\\ &\ge \big(M - \frac{C}{\log^4 n}\big)\frac{\sqrt{n}}{\log^{4j_0} n} + o(\ell) > C\ell - o(\ell) > \frac{\ell}{2},\end{aligned}$$

and we finished the proof of the claim. To finish the proof, note that by Proposition 6.3, Corollary 6.6 and Proposition 6.4

$$\mu(A_\ell) \ge \mu(B_\ell\cap C_\ell\cap D_\ell) \ge 1 - C\frac{1}{\sqrt[2+\epsilon]{n}} - C\big(\frac{2}{3}\big)^{\sqrt{\log n}} - C\frac{1}{n^d} \ge 1 - C\big(\frac{2}{3}\big)^{\sqrt{\log n}}.$$

□

Proposition 6.8. *There exist ϵ and D so that for every $\ell \ge 0$,*

$$\mu(\{x\in I_\ell\colon \text{ there exists } i \text{ so that } F^i(p) \in \bigcup_{t\in[\min\psi,-\min\psi]} I_t \text{ and } dist_i(p)\le D\}) \ge \epsilon$$

Proof. Define, for $p \in C(i_0, i_1, \ldots, i_{n-1})$,

$$Dist_n(p) := sup_{q\in C(i_0,i_1,\ldots,i_{n-1})}\ dist_n(q).$$

We are going to prove by induction on k that there is $C > 0$ so that, if we define

$$B^\ell_k := \{p\in I_\ell\colon \text{ there exists } j \le \sum_{i=0}^{k-1}\frac{\ell^3}{2^{3i}} \text{ such that } \pi_2(F^j(p)) \le \frac{\ell}{2^k} \text{ and } Dist_j(p) \le \sum_{i=0}^{k-1}\frac{\ell^3}{2^{3i}}\theta^{\frac{\ell}{2^i}}\},$$

then

$$\mu(B^\ell_k) \ge \prod_{i=0}^{k-1}\Big(1 - C\big(\frac{2}{3}\big)^{\sqrt{\log\frac{\ell}{2^i}}}\Big). \tag{28}$$

Indeed, take $p \in B^\ell_k$. Let $p \in L = C(i_0, i_1, \ldots, i_{j-1})$, where j is as in the definition of B^ℓ_k. Note that $L \subset B^\ell_k$ and $F^n(L) = I_r$, for some $r < \ell/2^k$. By Proposition 6.7,

$$\begin{aligned}&\mu(x\in I_r\colon \text{ there exists } i < \frac{\ell^3}{2^{3k}} \text{ so that } \sum_{k=0}^{i}\psi\circ f^k(x) > \frac{\ell}{2^{k+1}})\\ &\qquad \ge 1 - C\big(\frac{2}{3}\big)^{\sqrt{\log\frac{\ell}{2^k}}}.\end{aligned} \tag{29}$$

Denote

$$D_L := \{x\in I_\ell\cap L\colon \text{ there exists } i < \frac{\ell^3}{2^{3k}} \text{ so that } \sum_{k=0}^{i}\psi\circ f^k(f^j(x)) > \frac{\ell}{2^{k+1}}\}$$

Due the bounded distortion property for F, the estimative in Eq. (29) implies

$$\frac{\mu(D_L)}{|L|} \geq 1 - C(\frac{2}{3})^{\sqrt{\log \frac{\ell}{2^k}}}. \tag{30}$$

For $x \in D_L$ take the smallest i so that

$$\sum_{k=0}^{i} \psi \circ f^k(f^j(x)) > \frac{\ell}{2^{k+1}}.$$

Then $\pi_2(F^{j+h}(p)) \geq \frac{\ell}{2^{k+1}}$, for every $0 \leq h < i$, so

$$Dist_i(F^j(p)) \leq \sum_{h=0}^{i} \theta^{\pi(F^{j+h}(p))} \leq \frac{\ell^3}{2^{3k}} \theta^{\frac{\ell}{2^{k+1}}}.$$

So $D_L \subset B^{\ell}_{k+1}$. Since B^{ℓ}_k is a disjoint union of cylinders L, the estimative in Eq. (30) implies Eq. (28).

Define

$$D := \sum_{i=0}^{\infty} \frac{\ell^3}{2^{3i}} \theta^{\frac{\ell}{2^i}} < \infty.$$

Let k be so that $2^k \leq \ell \leq 2^{k+1}$. Now it is easy to check that

$$\begin{aligned} &\mu(\{x \in I_\ell \colon \text{ there exists } i \text{ so that } F^i(p) \in I_0 \text{ and } dist_i(p) \leq D\}) \\ &\geq C\mu(B^{\ell}_k) \geq \prod_{i=0}^{k-1} \left(1 - C(\frac{2}{3})^{\sqrt{\log \frac{\ell}{2^i}}}\right) \geq C \prod_{i=0}^{k-1} \left(1 - C(\frac{2}{3})^{\sqrt{\log \frac{2^k}{2^i}}}\right) \\ &\geq \exp(-C \sum_{i=1}^{\infty} (\frac{2}{3})^{\sqrt{i \log 2}}) > \tilde{C} > 0, \end{aligned}$$

which finishes the proof. □

Proof of the Stability of Recurrence (Theorem 4). Because G coincides with F on negative states, and F is recurrent, of course the orbit by G of almost every point p so that $\pi_2(p) < 0$ will entry

$$\cup_{i \geq 0} I^i.$$

So it is enough to prove that the orbit by G of almost every point $p \in \cup_{i\geq 0} I^i$ hit I^0. Let $\ell \geq 0$.

By the previous Proposition, there exist $D > 0$ and $\epsilon > 0$ so that

$$A_\ell := \{p \in I_\ell \colon \text{ there exists } i \text{ so that } F^i(p) \in \bigcup_{t=\min\psi}^{-\min\psi} I_t \text{ and } Dist_i(p) < D\}$$

satisfies $\mu(A_\ell) > \epsilon$, for all ℓ.

Consider a cylinder $C_F = C_F(\ell, k_1, \dots, k_{i-1}, 0) \subset A_\ell$, satisfying $k_j \neq 0$ for $0 < j < i$ and $Dist_i(x) < D$, for every $x \in C_F$. We claim that that corresponding cylinder $C_G = C_G(\ell, k_1, \dots, k_{i-1}, 0)$ for the perturbed random walk G satisfies

$$\frac{1}{C} \leq \frac{|C_G|}{|C_F|} \leq C,$$

where C depends only on D. Since A_ℓ is a disjoint union of cylinders of this type, we obtain that $B_\ell = H(A_\ell)$ satisfies $m(B_\ell) > C\epsilon > 0$, for all ℓ.

To prove that the set of points whose orbits returns infinitely many times to

$$\bigcup_{t=\min\psi}^{-\min\psi} I_t$$

has full Lebesgue measure, it is enough to prove that $\Lambda := \cup_{j>0,\ell} G^{-j} B_\ell$ has full Lebesgue measure.

Indeed, assume by contradiction that Λ is not full. Choose a Lebesgue density point p of the complement of Λ. Then there exist a sequence of cylinders $C_k = C_G(\ell_0, \ell_1, \ldots, \ell_k)$ so that $p \in C_k$ and

$$\frac{m(C_k \setminus \Lambda)}{|C_k|} \to_k 1. \tag{31}$$

But $G^k(C_k) = I_{\ell_k}$, and $m(I_{\ell_k} \cap B_{\ell_k}) \geq C\epsilon |I_{\ell_k}|$. By the bounded distortion property

$$\frac{m(\Lambda \cap C_k)}{|C_k|} > \frac{m(G^{-k} B_{\ell_k} \cap C_k)}{|C_k|} > \tilde{C}\epsilon,$$

which contradicts Eq. (58). Now we can use that G is transitive and has bounded distortion to prove that G is recurrent. □

Proof of Proposition 3.1. Since F is recurrent, almost every point of I^0 returns to I^0 at least once. So the first return map $R_F \colon I^0 \to I^0$ is defined almost everywhere is I^0 and the same can be said about R_G. Of course, the absolutely continuous conjucagy H also cojugates the expanding Markovian maps R_F and R_G. Using the same argument used in Shub and Sullivan [ShSu] and Martens and de Melo [MdM], we can prove that H is actually C^1 on I^0. Using the dynamics, it is easy to prove that H is C^1 everywhere. □

7. Stability of the multifractal spectrum

7.1. **Dynamical defined intervals and root cylinders.** When we are dealing with Markov expanding maps with *finite* Markov partitions, for each arbitrary interval J we can find an element of $\cup_j \mathcal{P}^j$ which covers J and has more or less the same size that J. Note that this is no longer true when the Markov partitions is infinite. Since coverings by intervals are crucial in the study of the Hausdorff dimension of an one-dimensional set, this trick is very useful to estimate the dimension of dynamically defined sets, once we can replace an arbitrary covering by intervals by another one with essentially the same metric properties but whose elements are themselves *dynamically defined* sets (cylinders). The following Lemma is an easy consequence of the regularity properties $Ra + Rb$ and it will be useful to recover that trick for (certain) infinite Markov partitions.

Consider $j \geq 0$ and let $\{C_i\}_i \subset \mathcal{P}^j$ be a finite or countable family of cylinders $\{C_i\}_{i\in\Theta} \subset \mathcal{P}^j$ such that $W := \bigcup_i \overline{C_i}$ is connected and *int* W does not contain any point d_i^n (as defined in property Rb). Then W is called a dynamically defined interval (dd-interval, for short). Define the root cylinder of W as the unique cylinder C_{i_0} with the following property: if $\sharp\Theta = \infty$ then W is a semi-open interval and C_{i_0} will be the cylinder so that $\partial C_{i_0} \cap \partial W \neq \phi$. Otherwise W is closed and let C_{i_0} be the unique cylinder such that $F = \partial C_{i_0} \cap \partial W$ is the boundary of a semi-open dd-interval which contains W. The proof of the following properties of dd-intervals is very simple:

Lemma 7.1. *For every $d \in (0,1)$ there exists $K > 1$ so that for every dd-interval $W := \cup_i \overline{C_i}$ with root cylinder C_{i_0} we have*

$$\frac{1}{K} \leq \frac{|W|^\alpha}{\sum_i |C_i|^\alpha} \leq K$$

$$\frac{1}{K} \leq \frac{|C_{i_0}|^\alpha}{\sum_i |C_i|^\alpha} \leq K$$

for every $\alpha \geq d$. Indeed the constant K depends only on d and constants in the properties $Ra + Rb + Ex + BD$.

Lemma 7.2. *Let N be as in Properties $Ra + Rb$. For every $d \in (0,1)$ there exists $K > 1$ so that the following holds: For every interval $J \subset I \times \mathbb{Z}$ there exists m dd-intervals W_j, all of same level, with $m \leq 2N$, satisfying the following properties:*

- *The interior of these dd-intervals are pairwise disjoint.*
- *The closure of the union of W_j covers J:*

$$J \subset \overline{\bigcup_j W_j}.$$

- *We have*

$$\frac{1}{K} \leq \frac{\sum_{i=1}^m |W_i|^\alpha}{|J|^\alpha} \leq K$$

for every $\alpha > d$.

Indeed the constant K depends only on d and constants in the properties $Ra + Rb + Ex + BD$.

7.2. **Dimension of dynamically defined sets.** Let $f \in Mk + BD + Ex$ and denote by $\mathcal{P}^0$ its Markov partition. Let

$$\mathcal{I} := \{C_i\}_i \subset \cup_i \mathcal{P}^n$$

be a finite or countable family of disjoint cylinders. Define the induced Markov map $f_\mathcal{I} \colon \cup_i C_i \to I$ by

$$f_\mathcal{I}(x) = f^{\ell(C_i)-1}(x), \ if \ x \in C_i.$$

We can also define an induced drift function $\Psi \colon \cup_i C_i \to \mathbb{Z}$ in the following way: Define, for $x \in C \in \mathcal{P}^n_0$,

$$\Psi_\mathcal{I}(x) := \sum_{i=0}^{n-1} \psi(f^i(x)).$$

On the same conditions on x, define $N_\mathcal{I}(x) = n$. The maximal invariant set of $f_\mathcal{I}$ is

$$\Lambda(\mathcal{I}) := \{x \in I \colon \ f^j(x) \in \bigcup_i C_i, \ for \ all \ j \geq 0\}.$$

Denote by $HD(\mathcal{I})$ the Hausdorff dimension of the maximal invariant set of $f_\mathcal{I}$.

We are going to use the following result

Proposition 7.3 (Theorem 1.1 in [MU2])**.** *We have*

$$HD(\mathcal{J}) = \sup\{HD(\mathcal{I}) \colon \mathcal{I} \subset \mathcal{J}, \ \mathcal{I} \ finite\}.$$

The following result was proved to Markov maps with finite Markov partition, however the proof can be adapted to our case.

Before to give the proof of Proposition 3.2 we need to introduce some tools which are useful to estimate the Hausdorff dimension.

Let $\mathcal{J}$ as above. If there exists β such that

$$\sum_{C\in\mathcal{J}} |C|^\beta = 1,$$

we will call β the **virtual Hausdorff dimension** of $f_\mathcal{I}$, denoted $VHD(\mathcal{I})$. The virtual Hausdorff dimension is a nice way to estimate $HD(\mathcal{I})$: indeed if $f_\mathcal{I}$ is linear on each interval of the Markov partition then these values coincide. When the distortion is positive, these values remain related, as expressed in the following result (which is included, for instance, in the proof of Theorem 3, Section 4.2 of [PT]):

Proposition 7.4. *Let $\mathcal{I}$ be a finite family of disjoint cylinders. Then*

$$|HD(\mathcal{I}) - VHD(\mathcal{I})| \leq \frac{d}{\log\lambda - d},$$

where

$$d := \sup_{C\in\mathcal{I}} \sup_{x,y\in C} \log \frac{Df_\mathcal{I}(y)}{Df_\mathcal{I}(x)} \text{ and } \lambda := \inf_{C\in\mathcal{I}} \inf_{x\in C} |Df_\mathcal{I}|.$$

Recall that if $\mathcal{I}$ is finite then $f_\mathcal{I}$ has an invariant probability measure $\mu_\mathcal{I}$ supported on its maximal invariant set $\Lambda(\mathcal{I})$ such that for any subset $S \subset \Lambda(\mathcal{I})$ satisfying $\mu_\mathcal{I}(S) = 1$ we have $HD(S) = HD(\mathcal{I})$.

Note that for a homogeneous random walk F

$$\Omega^k_+(F) = \{k\} \times \{x \in I \ s.t. \sum_{i=0}^{j} \psi(f^j(x)) + k \geq 0, \ for \ j \geq 0\}$$

and

$$\Omega^k_{+\beta}(F) =$$

$$\{k\}\times\{x \in I \ s.t. \sum_{j=0}^{n-1} \psi(f^j(x)) + k \geq 0 \ for \ all \ n \geq 0 \ and \ \underline{\lim}_n \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \geq \beta\}.$$

Define $\pi_1(x, n) := x$. The following is an easy consequence of this observation:

Lemma 7.5. *If F is a homogeneous random walk then $\pi_1(\Omega^0_+(F)) \subset \pi_1(\Omega^k_+(F))$ and $\pi_1(\Omega^0_{+\beta}(F)) \subset \pi_1(\Omega^k_{+\beta}(F))$, for all $k \geq 0$. Furthermore*

$$HD(\Omega^0_+(F)) = HD(\Omega^k_+(F))$$

and

$$HD(\Omega^0_{+\beta}(F)) = HD(\Omega^k_{+\beta}(F)).$$

Proposition 7.6. *Let F be a homogeneous random walk. Then there exists a sequence of finite families of cylinders*

$$\mathcal{F}_s \subset \cup_i \mathcal{P}^i_0$$

so that

- $\Lambda(\mathcal{F}_s) \subset \Omega^0_+(F)$,
- *Denote* $\beta_n := \int \Psi_{\mathcal{F}_s} \, d\mu_{\mathcal{F}_s}$. *Then* $\beta_n > 0$.
- $\lim_{s\to\infty} HD(\mathcal{F}_s) = HD(\Omega^0_+(F))$.

Proof. Denote $d = HD\ \Omega^0_+(F)$. Given any $s \in \mathbb{N}^\star$, $m_{d_s}\ \Omega_+(F) = \infty$, where $d_s := d(1 - 1/s)$. Here m_D denotes the D-dimensional Haussdorf measure. By Theorem 5.4 in [F], for each positive number M we can find a compact subset $\Lambda_s \subset \Omega^0_+(F)$ satisfying $m_{d_s}\ \Lambda_s = M$. We may assume that Λ_s does not have isolated points. We will specify M later.

In particular, for each ϵ small enough the following holds:

i. *For every* family of intervals $\{J_i\}_i$ which covers Λ_s, with $|J_i| < \epsilon$ we have
$$\frac{M}{2} \leq \sum_i |J_i|^{d_s}.$$
ii. *There exists* a family of intervals $\{J_i\}_i$, with $|J_i| \leq \epsilon$, which covers Λ_s and
$$\sum_i |J_i|^{d_s} \leq 2M.$$
Furthermore we can assume that $\partial J_i \subset \Lambda_s$.

Assume that $d_s \geq d/2$. By Lemma 7.2, there exists some K such that we can replace the special covering $\{J_i\}$ in ii. by a new covering by dd-intervals $\{W^\ell_i\}_{i,\ \ell}$, with root cylinders R^ℓ_i, where

$$J_i \cap \Lambda_s \subset \overline{\bigcup_\ell W^\ell_i}, \tag{32}$$

$$W^\ell_i := \bigcup_k \overline{C^{i\ell}_k},\ \ for\ each\ \ell \leq m_{i\ell} \leq 2N, \tag{33}$$

$$\frac{1}{K} \leq \frac{\sum_\ell |R^\ell_i|^{d_s}}{|J_i|^{d_s}} \leq K, \tag{34}$$

$$\frac{1}{K} \leq \frac{\sum_k |C^{i\ell}_k|^{d_s}}{|R^\ell_i|^{d_s}} \leq K, \tag{35}$$

Indeed we can replace W^ℓ_i by a dd-subinterval of it, if necessary, in such way that $R^\ell_i \cap \Lambda_s \neq \phi$ and Eq. (32), Eq. (33), Eq. (34) and Eq. (35) hold, except perhaps the lower bound in Eq. (34). The above estimates, together to the fact that $\{W^\ell_i\}$ covers Λ_s (up to a countable set) gives

$$\frac{M}{2K^2} \leq \sum_{i,\ell,k} |C^{i\ell}_k|^{d_s} \leq 2K^2 M. \tag{36}$$

Since these intervals are cylinders, if necessary we can replace this family of cylinders by a subfamily of disjoint cylinders which covers Λ_s up to a countable number of points and such that each cylinder intersects Λ_s. Indeed we can choose a finite subfamily $\mathcal{F}_s := \{C_r\}_r$ satisfying

$$\frac{M}{3K^2} \leq \sum_r |C_r|^{d_s} \leq 2K^2 M. \tag{37}$$

Let's call this finite subfamily $\mathcal{F}_s$. Note that, since $C_r \cap \Lambda_s \neq \phi$ we have that

$$\sum_{t=0}^{\ell} \psi(f^t(x)) \geq 0$$

for every $x \in C_r$ and $\ell \leq \ell(C_r)$. Choose a very small cylinder $\tilde{C}$ such that

$$\sum_{t=0}^{\ell} \psi(f^t(x)) \geq 0$$

for every $x \in \tilde{C}$ and $\ell < \ell(\tilde{C})$, and moreover satisfying

$$\sum_{t=0}^{\ell(\tilde{C})} \psi(f^t(x)) > 0$$

on $\tilde{C}$, and

$$\frac{M}{3K^2} \leq |\tilde{C}|^{d_s} + \sum_r |C_r|^{d_s} \leq 3K^2 M. \tag{38}$$

Add $\tilde{C}$ to the family $\mathcal{F}_s$. Then, if μ_s is the geometric invariant measure of $f_{\mathcal{F}_s}$, we have

$$\int \Psi_{\mathcal{F}_s}\, d\mu_s > 0.$$

And by Lemma 7.4 and Eq. (38)

$$|HD(\Lambda(f_{\mathcal{F}_s})) - d_s| \leq -\frac{C}{\log \epsilon}.$$

Since ϵ can be taken arbitrary, we can choose $\mathcal{F}_s$ such that

$$HD(\Lambda(f_{\mathcal{F}_s})) \to_s d.$$

$\square$

Corollary 7.7. *If F is a homogeneous random walk we have that*

$$HD(\Omega_+(F)) = \lim_{\beta \to 0^+} HD(\Omega_{+\beta}(F)) = \sup_{\beta > 0} HD(\Omega_{+\beta}(F)).$$

Proof. Due Lemma 7.5, it is enough to prove the Corollary for $k = 0$. Of course $\Omega^0_{+\beta}(F) \subset \Omega^0_+(F)$ and $\beta_0 \leq \beta_1$ implies $\Omega^0_{+\beta_1}(F) \subset \Omega^0_{+\beta_0}(F)$, so

$$\lim_{\beta \to 0^+} HD(\Omega^0_{+\beta}(F)) = \sup_{\beta > 0} HD(\Omega^0_{+\beta}(F)) \leq HD(\Omega^0_+(F)).$$

To obtain the opposite inequality, let $\mathcal{F}_s$ be as in Proposition 7.6. Denote

$$\gamma_s := \int \Psi_{\mathcal{F}_s}\, d\mu_{\mathcal{F}_s}, \text{ and } W_n := \int N_{\mathcal{F}_s}\, d\mu_{\mathcal{I}}$$

and $\beta_s := \gamma_s/W_s$. Then by the Birkhoff Ergodic Theorem there is subset $T_s \subset \Lambda(\mathcal{I}_n)$ such that $\mu_{\mathcal{F}_s}(T_s) = 1$ and

$$\lim_k \frac{1}{k}\sum_{i=0}^{k-1}\psi(f^i(x)) = \lim_k \frac{\sum_{j=0}^{k-1}\Psi_{\mathcal{I}_n}(f^j_{\mathcal{F}_s}(x))}{\sum_{j=0}^{k-1}N_{\mathcal{I}_n}(f^j_{\mathcal{F}_s}(x))} = \frac{\gamma_s}{W_s} = \beta_s > 0.$$

for every $x \in T_s$. Since the Hausdorff dimension of $\mu_{\mathcal{F}_s}$ is equal to $HD(\mathcal{F}_s)$, we have that $HD(T_s) = HD(\mathcal{F}_s)$. Note also that

$$T_s \subset \Omega^0_{+\beta_s},$$

which implies $HD(\mathcal{F}_s) \leq HD(\Omega^0_{+\beta_s})$, so by the choice of $\mathcal{F}_s$, we conclude that

$$HD(\Omega^0_+) = \lim_s\ HD(\mathcal{F}_s) \leq \overline{\lim}_s\ HD(\Omega^0_{+\beta_s}) \leq \sup_{\beta>0}\ HD(\Omega^0_{+\beta}).$$

□

Proof of Theorem 5. Define

$$\Gamma_n(F) := \{x \in \Omega^k_{+\beta}(F)\ s.t.\ \pi_2(F^i(x,k)) \geq \frac{\beta}{2}i,\ for\ all\ i \geq n\}.$$

Of course

$$\Omega^k_{+\beta}(F) = \bigcup_n \Gamma_n(F).$$

To prove the Theorem, it is enough to verify that $HD(\Gamma_n(F)) = HD(\Gamma_n(G))$. Indeed, for every $\epsilon > 0$ and $\alpha \in (HD(\Gamma_n(F)), 1)$ there exists a covering of $\Gamma_n(F)$ by intervals A_i so that

$$\sum_j |A_j|^\alpha \leq \epsilon.$$

Note that we can assume that $\partial A_j \subset \Gamma_n(F)$. Since G is an asymptotically small perturbation of F, it is easy to see that G also satisfies the properties $Ra + Rb$, replacing the points c^n_i and d^n_i by $h(c^n_i)$ and $h(d^n_i)$, and modifying the constant . Indeed can choose constants in the definitions of the properties $Ex+BD+Ra+Rb$ which works for both random walks, so we can take $K > 0$ in the statements of Lemma 7.2 and Lemma 7.1 in such way that it works for both random walks.

In particular (as in the proof of Proposition 7.6) for each A_j we can find at most $2N$ dd-intervals

$$W^\ell_j := \bigcup_k \overline{C^{j\ell}_k},\ with\ \ell \leq m_j \leq 2N$$

which satisfy

$$A_i \cap \Gamma_n(F) \subset \overline{\bigcup_\ell W^\ell_i},$$

and

$$\sum_{k,\ell} |C^{j\ell}_k|^\alpha \leq K|A_j|^\alpha.$$

Furthermore we can assume that the root R^ℓ_j of W^ℓ_j satisfies

$$\frac{1}{K} \leq \frac{|R^\ell_j|^\alpha}{\sum_k |C^{j\ell}_k|^\alpha} \leq K \tag{39}$$

and $R^j_\ell \cap \Gamma_n(F) \neq \phi$.

The constant K does not depend on α, j or ℓ. In particular the union of all cylinders $C_k^{j\ell}$ covers $\Gamma_n(F)$ up to a countable set and

$$\sum_{j,k,\ell} |C_k^{j\ell}|^\alpha \leq K\epsilon. \tag{40}$$

Note that if $x \in \Gamma_n(F)$ then

$$dist_i(x) \leq r_n := Cn + C\lambda^n$$

for every $i \in \mathbb{N}$. So

$$e^{-r_n} \leq \frac{|\mathcal{P}_F^i(x)|}{|\mathcal{P}_G^i(h(x))|} \leq e^{r_n}.$$

There is a point in the cylinder R_j^ℓ which belongs to $\Gamma_n(F)$, so

$$e^{-\alpha r_n} \leq \frac{|R_j^\ell|^\alpha}{|h(R_j^\ell)|^\alpha} \leq e^{\alpha r_n}. \tag{41}$$

Note that $h(W_j^\ell) = \bigcup_k \overline{h(C_k^{j\ell})}$ is a dd-interval for G and $h(R_j^\ell)$ is its root cylinder. So

$$\frac{1}{K} \leq \frac{|h(R_j^\ell)|^\alpha}{\sum_i |h(C_i^{j\ell})|^\alpha} \leq K \tag{42}$$

But the union of the cylinders $h(C_k^{j\ell})$ covers $\Gamma_n(G)$ up to a countable set and Eq. (39), Eq. (40), Eq. (41) and Eq. (42) gives

$$\sum_{j,k,\ell} |h(C_k^{j\ell})|^\alpha \leq K^3 e^{\alpha r_n}\epsilon.$$

Since $\alpha > HD(\Gamma_n(F))$ and ϵ is arbitrary we obtain that $HD(\Gamma_n(G)) \leq HD(\Gamma_n(F))$. Switching the roles of F and G in the above argument gives the opposite inequality. □

Lemma 7.8. *Let $G \in On + Ra + Rb$ be a random walk. For every $\alpha > 0$ there exist ϵ and C so that*

$$\sum_{P\in\mathcal{P}^n,\ P\subset I_k} |P|^{1-\epsilon} \leq C(1+\alpha)^n, \tag{43}$$

for all n and k.

Proof. For a random walk G, denote by $\mathcal{P}^n := \{P_i^n\}_i$ the Markov partition of G^n. Since $G \in BD + On + Ex$, for each $\delta > 0$, we can choose n_0 large enough so that for every inverse branch ϕ of an iteration of G and an element $P \in \mathcal{P}^{n_0}$, we have

$$1 - \delta \leq \frac{|D\phi(x)|}{|D\phi(y)|} \leq 1 + \delta. \tag{44}$$

for every $x, y \in P$. F

Moreover note that for every $\epsilon < 1$ there exists a constant $K = K_\epsilon > 1$ so that

$$\sum_i |P_i^n|^{1-\epsilon} \leq K^n \tag{45}$$

for every n.

Denote $\mathcal{P}^{n_0} = \{Q^j\}_j$ and $\mathcal{P}^{n_0+1} = \{Q_k^j\}_{j,k}$, in such way that $Q_k^j \subset Q_j$. Indeed, since $G \in BD + Rb$, it is possible to order Q_k^j so that there exists C satisfying

$$\frac{|Q_k^j|}{|Q^j|} \leq C\lambda^k,$$

for every j, k. As a consequence the set of functions

$$h_j(\epsilon) = \sum_k \frac{|Q_k^j|^{1-\epsilon}}{|Q^j|^{1-\epsilon}}$$

is a equicontinuous set of functions in a small neighborhood of 0. In particular, since $h_j(0) = 1$, there exists ϵ so that, for every j,

$$\sum_k \frac{|Q_k^j|^{1-\epsilon}}{|Q^j|^{1-\epsilon}} \leq 1 + \delta. \tag{46}$$

For $n \leq n_0$, it follows from Eq. (45) that there exists C so that, for $n \leq n_0$, we have $\sum_{P \in \mathcal{P}^n} |P|^{1-\epsilon} \leq C$. Assume by induction that we have proved Eq. (43) until some $n \geq n_0$. Denote by $\{\phi_j\}$ the inverse branches of G^{n-n_0}, with $Im\ \phi_i = P_i^{n-n_0}$. Then $\mathcal{P}^{n+1} = \{\phi_i(Q_k^j)\}_{i,j,k}$ and $\mathcal{P}^n = \{\phi_i(Q^j)\}_{i,j}$. By the distortion control in Eq. (44) and the estimative in Eq. (46), for each i, j we have

$$\frac{\sum_k |\phi_i(Q_k^j)|^{1-\epsilon}}{|\phi_i(Q^j)|^{1-\epsilon}} \leq \frac{1+\delta}{1-\delta} \frac{\sum_k |Q_k^j|^{1-\epsilon}}{|Q^j|^{1-\epsilon}} \leq \frac{(1+\delta)^2}{1-\delta}.$$

So

$$\sum_{P \in \mathcal{P}^{n+1}} |P|^{1-\epsilon} = \sum_{i,j,k} |\phi_i(Q_k^j)|^{1-\epsilon} \leq \sum_{i,j} |\phi_i(Q^j)|^{1-\epsilon} \sum_k \frac{|\phi_i(Q_k^j)|^{1-\epsilon}}{|\phi_i(Q^j)|^{1-\epsilon}}$$
$$\leq \frac{(1+\delta)^2}{1-\delta} \sum_{i,j} |\phi_i(Q^j)|^{1-\epsilon} = \frac{(1+\delta)^2}{1-\delta} \sum_{p \in \mathcal{P}^n} |P|^{1-\epsilon}.$$

We finish the proof choosing δ so that $(1+\delta)^2/(1-\delta) \leq (1+\alpha)$. □

From now on we are going to assume that the mean drift is negative: $\int \psi \ d\mu < 0$.

Lemma 7.9. *Let $G \in On + Ra + Rb$ be a random walk with negative mean drift. For every $\alpha > \int \psi \ d\mu$, there exists $\sigma < 1$ so that for any $n_1 \geq n_0$, with n_0 large enough,*

$$m\{p \in I_{n_1}\colon\ \pi_2(G^k(p)) \geq n_0, \text{ for } k \leq n, \text{ and } \pi_2(G^n(p)) - n_1 \geq \alpha n\} \leq \sigma^n. \tag{47}$$

Proof. Denote

$$\Lambda^n_{n_0,n_1}(G) := \{p \in I_{n_1}\colon\ \pi_2(G^k(p)) \geq n_0 \text{ for all } k \leq n \text{ and } \pi_2(G^n(p)) - n_1 \geq \alpha n\}.$$

The statement for F is consequence of the large deviations estimative (see, for instance [B])

$$m\{p \in I\colon |\frac{\sum_{k=0}^{n-1} \psi(f^k(p))}{n} - \int \psi \ d\mu| \geq K\} \leq C_K \sigma^n,$$

which holds for every $K > 0$. In particular choosing $K = \alpha - \int \psi \ d\mu$ we get, for any n_0, and $n_1 \geq n_0$,

$$m\{p \in I_{n_1} : \pi_2(F^n(p)) - n_1 \geq \alpha n\} \leq \sigma^n,$$

which implies (of course)

$$m(\Lambda^n_{n_0,n_1}(F)) \leq \sigma^n. \tag{48}$$

We are going to use this estimative to obtain Eq. (47) for the perturbation of F.

Indeed, for every $\delta > 0$, there is n_0 so that if $\pi_2(x) \geq n_0$ then

$$1 - \delta \leq \frac{|DF(x)|}{|DG(H(x))|} \leq 1 + \delta, \tag{49}$$

Here H is the topological conjugacy between F and G which preserves states. Note that $\Lambda^n_{n_0,n_1}(F)$ is a disjoint union of elements $Q_i \in \mathcal{P}^n(F)$, so $\Lambda^n_{n_0,n_1}(G)$ is a disjoint union of the intervals $H(Q_i)$. Due Eq. (48) and Eq. (49), we have

$$\sum_i |H(Q_i)| \leq \sum_i (1+\delta)^n |Q_i| \leq (1+\delta)^n \sigma^n. \tag{50}$$

Take n_0 large enough so that $(1+\delta)\sigma < 1$. □

We would like to replace n_0 by an arbitrary state in Eq. (47). The following Lemma will be useful for this task:

Lemma 7.10. *Let p_n and q_n sequences of non-negative real numbers such that*

(1) $p_0 + q_0 \leq 1$,
(2) *There exists $\epsilon > 0$ and $\ell \in \mathbb{N}$ such that $s_n := p_n + q_n \leq (1-\epsilon)^\ell p_{n-\ell} + q_{n-\ell}$ and $q_n \leq C\sigma^n + \sum_{k=1}^{n}(1-\epsilon)^k p_{n-k}$, for every $n \geq 1$.*

Then there exists $\delta > 0$ such that $s_n \leq (1-\delta)^n$, for every $n \in \mathbb{N}$.

Proof. If $n \geq \ell$, we have $s_n \leq (1-\epsilon)p_{n-\ell} + q_{n-\ell} = (1-\epsilon)s_{n-\ell} + \epsilon q_{n-\ell}$. It follows by induction that if $n = i\ell + r$, with $r < \ell$, then

$$\begin{aligned} s_n &\leq (1-\epsilon)^i s_r + \sum_{k=0}^{i-1} \epsilon(1-\epsilon)^{k\ell} q_{n-(k+1)\ell} \\ &\leq C(1-\epsilon)^{n/\ell} s_0 + \sum_{k=0}^{n-1} \epsilon(1-\epsilon)^k q_{n-\ell-k} \end{aligned}$$

Since $q_{n-\ell} \leq C(1-\epsilon)^n + \sum_{k=1}^{n-1}(1-\epsilon)^k p_{n-\ell-k}$, we obtain

$$\begin{aligned} s_n &\leq (1-\epsilon)^{n/\ell} s_0 + \epsilon(1-\epsilon)^{n/\ell} + \sum_{k=1}^{n-1} \epsilon(1-\epsilon)^k (p_{n-\ell-k} + q_{n-\ell-k}) \\ &\leq (1-\epsilon)^{n/\ell} C(s_0 + \epsilon) + \sum_{k=1}^{n-1} \epsilon(1-\epsilon)^k s_{n-\ell-k}, \end{aligned}$$

for every $n \geq \ell$.

We claim that there exists $\delta < 1$ and K so that $s_n \leq K(1-\delta)^n$, for every n. Indeed, fix $\delta < 1$, For each n, define $K_n := s_n/(1-\delta)^n$. Note that

$$\begin{aligned} s_n &\leq (1-\epsilon)^{n/\ell} C(s_0 + \epsilon) + \sum_{k=1}^{n-1} \epsilon(1-\epsilon)^k s_{n-\ell-k} \\ &\leq (1-\epsilon)^{n/\ell} C(s_0 + \epsilon) + \sum_{k=1}^{n-1} \epsilon(1-\epsilon)^k K_{n-\ell-k}(1-\delta)^{n-\ell-k} \end{aligned} \tag{51}$$

$$\leq \big[\big(\frac{(1-\epsilon)^{1/\ell}}{1-\delta}\big)^n C(s_0+\epsilon) + \max_{i<\ n-\ell} K_i\ \frac{\epsilon}{(1-\delta)^\ell}\sum_{k=1}^{n-1}\big(\frac{1-\epsilon}{1-\delta}\big)^k\big](1-\delta)^n$$

Choose δ close enough to 1 so that

$$\sigma_1 := \frac{(1-\epsilon)^{1/\ell}}{1-\delta} < 1,\ \textit{and}$$

$$\sigma_2 := \frac{\epsilon}{(1-\delta)^\ell}\sum_{k=1}^{\infty}\big(\frac{1-\epsilon}{1-\delta}\big)^k < 1.$$

Then by Eq. (51) we have $K_n \leq \sigma_2 \max_{i<\ n-\ell} K_i + C\sigma_1^n$, for every $n > \ell$, which easily implies that $\max_i K_i < \infty$.

□

Define

$$\Omega_+^{n_1,n} := \{p \in I_{n_1} : \pi_2(G^k(p)) \geq 0,\ \text{for } 0 \leq k \leq n\}.$$

Lemma 7.11. *There exists $\delta < 1$ so that for every $n_1 \geq 0$ there exists $C = C(n_1)$ satisfying*

$$m(\Omega_+^{n_1,n}(G)) \leq C(1-\delta)^n.$$

Proof. Take n_0 as in Lemma 7.9 and fix $n_1 \geq 0$. Define the sets and sequences

$$s_n := m(\Omega_+^{n_1,n})$$

$$p_n := m(B^n),\ \text{where } B^n := \{p \in \Omega_+^{n_1,n} :\ \pi_2(G^n(p)) \in [0,n_0]\},\ \textit{and}$$

$$q_n := m(C^n),\ \text{where } C^n := \{p \in \Omega_+^{n_1,n} :\ \pi_2(G^n(p)) > n_0\}.$$

To prove Lemma 7.11, it is enough to verify that these sequences satisfy the assumptions of Lemma 7.10. Indeed, of course $p_0 + q_0 \leq 1$. To prove the other assumptions, take $i \in [0, n_0]$. Since G is topologically transitive, there are $\ell_i \in \mathbb{N}$ and intervals $J_i \subset I_i$ so that $\pi_2(G^{\ell_i}(J_i)) < 0$. Denote $\ell = max_{\ 0\leq i\leq n_0}\ell_i$ and $r = min_{\ 0\leq i\leq n_0}|J_i|/|I_i|$.

Clearly $\Omega_+^{n_1,n}(G) = B^n \cup C^n \subset B^{n-\ell}\cup C^{n-\ell}$. Let $J \subset B^{n-\ell}$ be an interval so that $G^{n-\ell}(J) = I_i$, with $0 \leq i \leq n_0$. Note that $B^{n-\ell}$ is a disjoint union of such intervals. By the bounded distortion control for G,

$$\frac{m(J\cap\Omega_+^{n_1,n}(G))}{m(J)} \leq 1 - \frac{m(J\cap G^{-(n-\ell)}J_i)}{m(J)} \leq (1-\frac{r}{c}) \tag{52}$$

Choose ϵ_0 satisfying $(1 - r/c) \leq (1-\epsilon_0)^\ell$. Then Eq. (52) implies

$$m(B^{n-\ell}\cap\Omega_+^{n_1,n}(G)) \leq (1-\epsilon_0)^\ell m(B^{n-\ell})$$

and we obtain

$$s_n = m(B^{n-\ell}\cap\Omega_+^{n_1,n}(G)) + m(C^{n-\ell}\cap\Omega_+^{n_1,n}(G)) \leq (1-\epsilon_0)^\ell p_{n-\ell} + q_{n-\ell}.$$

It remains to prove that $q_n \leq \sum_{k=1}^n (1-\epsilon)^k p_{n-k}$. There are two kind of points p in C^n:

Type 1. For every $j \leq n$ we have $\pi_2(G^j(p)) \geq n_0$ (in particular $n_1 \geq n_0$). We are going to estimate the measure of the set of these points, denoted Θ_1^n. It follows from Lemma 7.9, choosing, for instance, $\alpha = \int \psi \; d\mu/2$, that

$$m(\{p \in I_{n_1}\colon\ \pi_2(G^k(p)) \geq n_0,\ for\ k \leq n \text{ and } \pi_2(G^n(p)) \geq n_1+\alpha n\}) \leq C\sigma^n. \tag{53}$$

But the set in the r.h.s. of Eq. (53) coincides with Θ_1^n provided $n \geq\ (n_0-n_1)/\alpha$. So

$$m(\Theta_1^n) \leq C_{n_1}\sigma^n,$$

for some $\sigma < 1$ which does not depend on n_1.

Type 2. For some $j < n$ we have $\pi_2(G^j(p)) \leq n_0$. Denote the set of these points by Θ_2^n. Denote by $\Theta_{2,k}^n$ the set of points p so that $k \geq 1$ is the smallest natural satisfying $\pi_2(G^{n-k}p) \leq n_0$. Clearly Θ_2^n is a disjoint union of these sets. We are going to estimate their measure. Note that $\Theta_{2,k}^n \subset B^{n-k}$. The set B^{n-k} is a disjoint union of intervals L so that $\pi_2(G^{n-k}L) = I_i$, for some $i \leq n_0$. To estimate

$$\frac{m(\Theta_{2,k}^n \cap L)}{|L|}$$

note that $L \subset B^{n-k}$, and $\Theta_{2,k}^n \cap L$ is the set of points $p \in L$ so that $\pi_2(G^{n-k+j}p) > n_0$, for every $0 < j \leq k$. Define

$$L_y := \{p \in L\colon \psi(G^{n-k}p) = y\}.$$

Firstly note that for $y \leq n_0 - i$ we have

$$|L_y \cap \Theta_{2,k}^n| = 0, \tag{54}$$

since $p \in L_y \cap \Theta_{2,k}^n$ satisfies $\pi_2(G^{n-k+1}p) = i+\psi(G^{n-k}p) = i+y > n_0$. In particular for $y < 0$ we have $|L_y \cap \Theta_{2,k}^n| = 0$, which implies, due the bounded distortion control

$$\frac{m(L \cap \Theta_{2,k}^n)}{|L|} \leq \frac{\sum_{y\geq 0} |L_y|}{|L|} \leq (1-\delta),$$

for some $\delta < 1$ which does not depends on k, L or n_1, which implies

$$m(\Theta_{2,k}^n) \leq (1-\delta)m(B^{n-k}) = (1-\delta)p_{n-k}. \tag{55}$$

Furthermore, using again the distortion control and the regularity condition GD(big jumps are rare) we have

$$\frac{\sum_{y>-\alpha(k-1)} |L_y \cap \Theta_{2,k}^n|}{|L|} \leq \frac{\sum_{y>-\alpha(k-1)} |L_y|}{|L|} \leq C\gamma^k, \tag{56}$$

for some $C \geq 0$ and $\gamma < 1$.

To estimate $|L_y \cap \Theta_{2,k}^n|/|L_y|$, in the case $n_0 - i \leq y \leq -\alpha(k-1)$, recall that $G^{n-k+1}L_y = I_{i+y}$, with $i + y > n_0$. By Lemma 7.9, we have

$$m\{p \in I_{i+y}\colon\ \pi_2(G^m(p)) \geq n_0,\ \text{for } m \leq k-1, \text{ and } \pi_2(G^{k-1}(p)) \geq i+y+\alpha(k-1)\} \leq C\sigma^k.$$

Since $i + y + \alpha(k-1) \leq n_0$, this implies that

$$m\{p \in I_{i+y}\colon\ \pi_2(G^m(p)) \geq n_0,\ \text{for every } m \leq k-1\} \leq C\sigma^k.$$

The points in $L_y \cap \Theta^n_{2,k}$ are exactly the points whose $(n-k+1)$th-iteration belongs to the set in the estimative above. Using the bound distortion control we have

$$\frac{|L_y \cap \Theta^n_{2,k}|}{|L_y|} \leq C\sigma^k,$$

so

$$\frac{|\sum_{n_0-i\leq y\leq -\alpha(k-1)} L_y \cap \Theta^n_{2,k}|}{|L|} \leq C\frac{|\sum_{n_0-i\leq y\leq -\alpha(k-1)} L_y \cap \Theta^n_{2,k}|}{\sum_{n_0-i\leq y\leq -\alpha(k-1)} |L_y|} \leq C\sigma^k. \tag{57}$$

Choose $\epsilon < \epsilon_0$ so that $min\{max\{C\sigma^k, C\gamma^k\}, 1-\delta\} \leq (1-\epsilon)^k$, for every $k \geq 0$, and put together Eq. (54), Eq. (55), Eq. (56) and Eq. (57), to get $m(L \cap \Theta^n_{2,k}) \leq (1-\epsilon)^k|L|$. Since B^{n-k} is a disjoint union of such intervals L, we obtain

$$m(\Theta^n_{2,k}) \leq (1-\epsilon)^k m(B^{n-k}) = (1-\epsilon)^k p_{n-k}$$

and now we can conclude with

$$q_n = m(\Theta^n_1) + \sum_k m(\Theta^n_{2,k}) \leq C_{n_1}\sigma^n + \sum_k (1-\epsilon)^k p_{n-k}.$$

□

Now we are ready to prove Theorem 7:

Proof of Theorem 7. There are three cases:

F **is transient with** $M > 0$**.** If $M > 0$ then the random walk F is transient and it is easy to see that $m(\Omega_+(F)) > 0$. Since the conjugacy with an asymptotically small perturbation G is absolutely continuous (Theorem 2), we conclude that $m(\Omega_+(G)) > 0$.

F **is recurrent** $(M = 0)$**.** if $M = 0$ then F and its asymptotically small perturbations are recurrent by Theorem 4. In particular almost every point visits negative states infinitely many times, so $m(\Omega_+(G)) = 0$. It remains to prove that $HD\ \Omega_+(G) = 1$. By Theorem 6 it is enough to verify that $HD\ \Omega_+(F) = 1$. Indeed, it is easy to show using the Central Limit Theorem that if

$$\int \psi\ d\mu = 0$$

then there exist $C > 0$ and and for each n, subsets $\mathcal{A}_n \subset \mathcal{P}^n$ so that

$$\sum_{i=0}^{n-1} \psi(f^i(x)) > 0$$

for all $x \in J \in \mathcal{A}_n$ and

$$m(\bigcup_{J\in\mathcal{A}_n} J) \geq C > 0.$$

here C does not depend on n. Replacing $\mathcal{A}_n$ by a finite subfamily, if necessary, we can apply Proposition 7.4 to obtain

$$HD\ \Lambda(\mathcal{A}_n) = 1 - O(\frac{1}{n}).$$

If $\mu_{\mathcal{A}_n}$ is the geometric invariant measure of $f_{\mathcal{A}_n}$ then

$$\int \psi_{\mathcal{A}_n}\, d\mu_{\mathcal{A}_n} > 0$$

So by the Birkhoff Ergodic Theorem

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \psi(f^i(x)) = +\infty \tag{58}$$

in a set $S_n \subset \Lambda(\mathcal{A}_n)$ satisfying $\mu_{\mathcal{A}_n}(S_n) = 1$, so $HD\ S_n = 1 - O(1/n)$. In particular the set S of points satisfying Eq.(58) has Hausdorff dimension 1. We can decompose S in subsets B_j defined by

$$B_j := \{x \in S\colon \min_n \sum_{i=0}^{n-1} \psi(f^i(x)) \geq -j\}.$$

Clearly $\sup_j HD\ B_j = 1$.

For each j choose k_j and $J_j \in \mathcal{P}^{k_j}$ so that for all $x \in J_j$ we have

$$\sum_{i=0}^{\ell-1} \psi(f^i(x)) \geq 0$$

for every $\ell \leq k_j$ and

$$\sum_{i=0}^{k_j} \psi(f^i(x)) \geq j.$$

Then

$$(J_j \cap f^{-k_j} B_j) \times \{0\}$$

belongs to $\Omega_+(F)$, for every j. This implies $HD\ \Omega_+(F) \geq HD\ B_j$ so

$$HD\ \Omega_+(F) \geq \sup_j HD\ B_j = 1.$$

F **is transient with** $M < 0$**.** By Lemma 7.11, there is some $\delta \in (0,1)$, which does not depend on n_1, so that

$$m(\Omega^{n_1,n}) \leq C(1-\delta)^n. \tag{59}$$

By Lemma 7.8, there exists ϵ so that

$$\sum_{P\in\mathcal{P}^n,\ P\subset I_k} |P|^{1-\epsilon} \leq C(1-\delta)^{-n/2}. \tag{60}$$

Denote by $\{J_i^n\}_i \subset \mathcal{P}^n$ the family of disjoint intervals so that $\Omega^{n_1,n} = \cup_i J_i^n$. We claim that there exists $C > 0$ satisfying

$$\sum_i |J_i^n|^{1-\epsilon/4} \leq C(1-\delta)^n. \tag{61}$$

Since $sup_i\ |J_i^n| \to_n 0$, this proves that $HD\ \Omega_+^{n_1,\infty} \leq 1 - \epsilon/4$.

Indeed,

$$\begin{aligned}\sum_i |J_i^n|^{1-\epsilon/4} &= \sum_{|J_i|>(1-\delta)^{2n/\epsilon}} |J_i^n|^{1-\epsilon/4} + \sum_{|J_i|\leq(1-\delta)^{2n/\epsilon}} |J_i^n|^{1-\epsilon/4} \\ &\leq (1-\delta)^{-n/2}\sum_i |J_i^n| + (1-\delta)^{3n/2}\sum_i |J_i^n|^{1-\epsilon} \\ &\leq C(1-\delta)^{n/2},\end{aligned}$$

where in the last line we made use of Eq. (59) and Eq. (60). The proof is complete. □

8. Applications to one-dimensional renormalization theory

8.1. **(Classic) infinitely renormalizable maps.** Consider a real analytic unimodal maps $f\colon I \to I$, with negative Schwartzian derivative and even order critical point. The map f is called infinitely renormalizable if there exists an sequence of natural numbers $n_0 < n_1 < n_2 < \dots$ and a nested sequence of intervals

$$I = I_0 \supset I_1 \supset I_2 \supset \cdots$$

so that

- $f^{n_k}\partial I_k \subset \partial I_k$,
- $f^{n_k} I_k \subset I_k$,
- $f^{n_k} \colon I_k \to I_k$ is an unimodal map.

We say that f has bounded combinatorics if there exists $C > 0$ so that $n_{k+1}/n_k \leq C$, for all k. Two infinitely renormalizable maps f and g have the same combinatorics if there exists a homeomorphism $h\colon I \to I$ such that $f \circ h = h \circ g$.

The following result is a deep result in renormalization theory:

Proposition 8.1 ([McM96]). *Let f and g be two infinitely renormalizable unimodal maps with the same bounded combinatorics and same even order. Then for every $r > 0$ there exists $C > 0$ and $\lambda < 1$ so that*

$$||\frac{1}{|I_k^f|}\ f^{n_k}(|I_k^f|\cdot) - \frac{1}{|I_k^g|}\ g^{n_k}(|I_k^g|\cdot)||_{C^r} \leq C\lambda^k.$$

Here $|I_k^f|$ denotes the length of I_k^f.

Proof of Theorem 8. Let f be an infinitely renormalizable map with bounded combinatorics. We are going to define an induced map $F\colon I \to I$, following Y. Jiang (see [J1], [J2]): Let p_k be the periodic point in ∂I_k. Define E as the set

$$\{1, -1, -p_k, p_k, f(p_k), -f(p_k), \dots, f^{n_k-1}(p_k), -f^{n_k-1}(p_k)\} - \{f(p_k), -f(p_k)\}.$$

The set E cuts $I_{k-1} \setminus I_k$ in m_k intervals. Denote these intervals $M_{k-1,i}$, with $i = 1, \dots, m_k$. For each $x \in M_{k-1,i}$, define $n(x) \geq 1$ as the minimal positive integer so that

$$I_k \subset f^{n(x)n_{k-1}} M_{k-1,1}.$$

Note that $f^{n(x)n_{k-1}}$ does not have critical points on $M_{k-1,i}$. Define the induced map F, which is defined everywhere in I, except for a countable set of points:

$$F(x) := f^{n(x)}(x),\ \ for\ x \in I_k \setminus I_{k+1}.$$

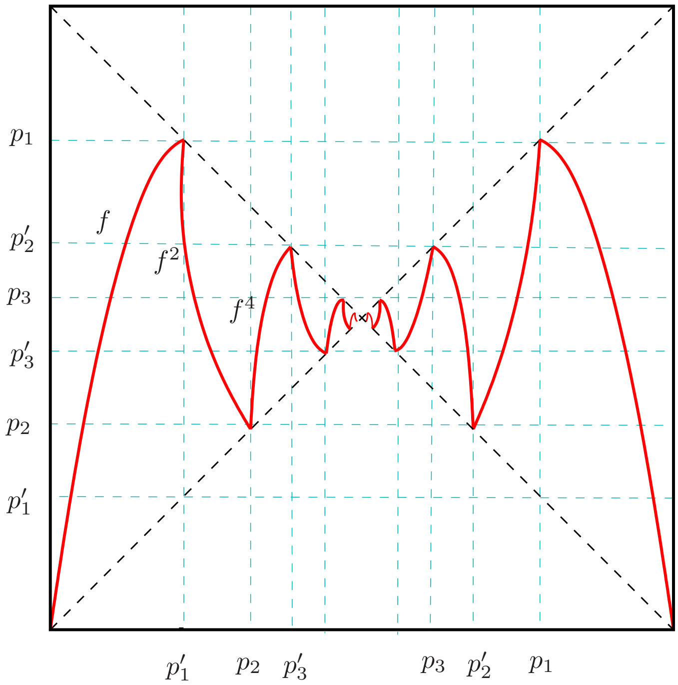

FIGURE 2. The "Bat" map: the induced map F for a Feigenbaum unimodal map

See in Fig. 2 the induced map for an infinitely renormalizable maps satisfying $n_{i+1} = 2n_i$ for all i (the so called Feigenbaum maps). The map F is Markovian with respect to the partition

$$\mathcal{P} := \{M_{k,i}\}_{k\in\mathbb{N}, i\le m_k}.$$

Furthermore, if f and g have the same bounded combinatorics and even order, then by Proposition 8.1, the corresponding induced maps F and G satisfies

$$|| \ \frac{1}{|I_k^f|} \ F(|M_{k,i}^f| \cdot + |I_k^f| - |M_{k,i}^f|) - \frac{1}{|I_k^g|} \ G(|M_{k,i}^g| \cdot + |I_k^g| - |M_{k,i}^g|) \ ||_{C^r([0,1])} \le C\lambda^k.$$

Define L_k as, say, the right component of $I_k \setminus I_{k+1}$ and $\gamma_k \colon I \to L_k$ as the unique bijective order preserving affine map between this two intervals. We are going to define a random walk $\mathcal{F}\colon I \times \mathbb{N} \to I \times \mathbb{N}$ from the map F in the following way:

$$\mathcal{F}(x,k) := \begin{cases} (\gamma_i^{-1} \circ F \circ \gamma_k(x), i) & \text{if } F \circ \gamma_k(x) \in L_i; \\ (\gamma_i^{-1} \circ (-F) \circ \gamma_k(x), i) & \text{if } F \circ \gamma_k(x) \in -L_i. \end{cases}$$

It is easy to see that we can extend $\mathcal{F}\colon I \times \mathbb{Z} \to I \times \mathbb{Z}$ to a strongly transient deterministic random walk with non-negative drift. Furthermore if g is another infinitely renormalizable map with the same bounded combinatorics that f then by Proposition 8.1 and Proposition 4.9 the corresponding random walk $\mathcal{G}$ is an asymptotically small perturbation of $\mathcal{F}$. So we can apply Theorem 3 to conclude that there is a conjugacy between F and G which is strongly quasisymmetric with respect to the nested sequence of partitions defined by the random walk $\mathcal{F}$. We can now easily translate this result in terms of the original unimodal maps f and g saying that the continuous conjugacy h between f and g is a strongly quasisymmetric mapping with respect to $\mathcal{P}$. $\square$

Remark 8.2. An interesting case is when the unimodal map f is a periodic point to the renormalization operator: there exists n_0 and λ, with $|\lambda| < 1$ so that

$$\frac{1}{\lambda} f^{n_0}(\lambda x) = f(x).$$

In this case, if we take $n_k = kn_0$, then the induced map F will satisfy the functional equation

$$F(\lambda x) = \lambda F(x). \tag{62}$$

Define the relation $\sim$ in the following way:

$$x \sim y \text{ iff there exists } i \in \mathbb{Z} \text{ so that } x = \pm\lambda^i y.$$

By Eq. (62), F preserves this relation, so we can take the quotient of F by the relation $\sim$. Note that

$$L_0 = \mathbb{R}^\star / \sim .$$

It is easy to see that if $q = F/\sim\colon L_0 \to L_0$ is a Markov expanding map. Now define $\psi\colon L_0 \to \mathbb{Z}$ as $\psi(x) = k$, if $f(x) \in I_k \setminus I_{k+1}$. Then $\mathcal{F}$ is exactly the homogeneous random walk defined by the pair (q, ψ).

8.2. **Fibonacci maps.** The Fibonacci renormalization is the simplest way to generalize the concept of classical renormalization as described in Section 8.1. Actually we could prove all the results stated for Fibonacci maps to a wider class of maps: maps which are infinitely renormalizable in the generalized sense and with periodic combinatorics and bounded geometry, but we will keep ourselves in the simplest case to avoid more technical definitions and auxiliary results with its long proofs.

Consider the class of real analytic maps f with $Sf < 0$ and defined in a disjoint union of intervals $I_1^0 \sqcup I_1^1$, where $-I_1^0 = I_1^0$, so that

- The map $f\colon I_1^1 \to I_0^0 := f(I_1^1)$ is a diffeomorphism. Furthermore I_1^1 is compactly contained in I_0^0.
- The map $f\colon I_1^0 \to I_0^0$ is an even map which has as 0 as its unique critical point of even order.

We say that f is **Fibonacci renormalizable** if

$$f(0) \in I_1^1,\ f^2(0) \in I_0^1 \textit{ and } f^3(0) \in I_0^1.$$

In this case, the Fibonacci renormalization of f is defined as the first return map to the interval I_1^0 restricted to the connected components of its domain which contain the points $f(0)$ and $f^2(0)$. This new map is denoted $\mathcal{R}f$: it could be Fibonacci renormalizable again and so on, obtaining an infinite sequence of renormalizations $\mathcal{R}f$, $\mathcal{R}^2 f$, $\mathcal{R}^3 f$, ...

We will denote the set of infinitely renormalizable maps in the Fibonacci sense with a critical point of order d by $\mathcal{F}_d$. A map $f \in \mathcal{F}_d$ will be called a **Fibonacci map**.

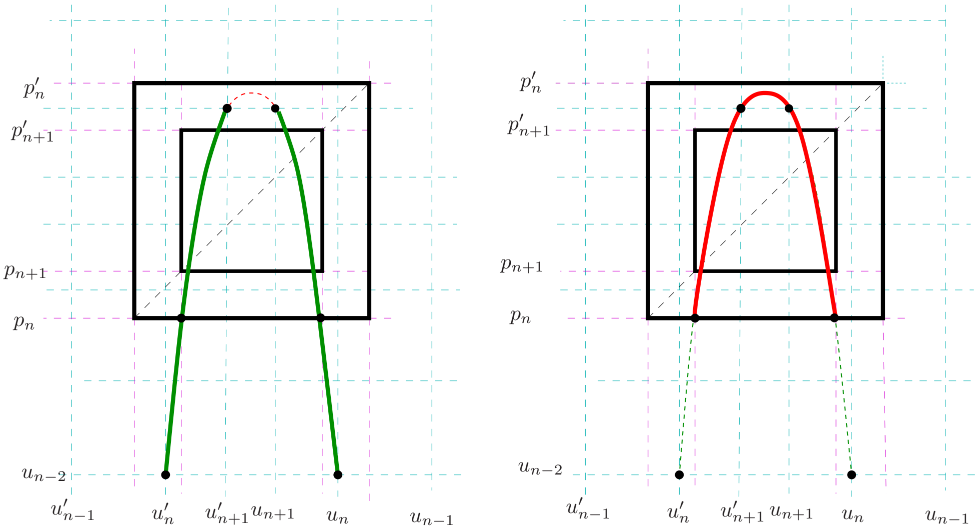

FIGURE 3. On the left figure the (green) solid curves represents the part of the f^{S_n} used in the definition of the induced map. On the right figure the (red) solid curve is the part of f^{S_n} which coincides with the n-th Fibonacci renormalization on its central domain.

As in the original map f, the n-th renormalization $f_n := \mathcal{R}^n f$ of f is a map defined in two disjoint intervals, denoted I_0^n and I_n^1, where $-I_0^n = I_0^n$. Indeed f_n on I_n^0 is an unimodal restriction of the S_n-th iteration of f, where $\{S_n\}$ is the Fibonacci sequence

$$S_0 = 1,\ S_1 = 2,\ S_2 = 3,\ S_3 = 5,\ \dots\ , S_{k+2} = S_{k+1} + S_k, \dots$$

and f_n on I_n^1 is the restriction of the S_{n-1}-th iteration of f.

Denote by p_k the sequence of points $p_k \in \partial I_0^k$ so that

$$f_k(p_{k+1}) = p_k$$

and denote $I_0^k = [p_k, p'_k]$.

It is possible to define a sequence u_k of points satisfying

1. $\cdots <\ p_{k+1} < u_k < p_k < \cdots < p_0$,
2. f^{S_k} is monotone on $[0, u_k]$,
3. $f^{S_k}(u_{k+1}) = u_k$,
4. $f^{S_k}(u_k) = u_{k-2}$.

We are going to define an induced map for an infinitely renormalizable map in the Fibonacci sense in the following way: Firstly, define $f_{-1}\colon I_0^0 \setminus I_0^1$ as an C^3 monotone extension of f_0 on I_1^1 which has negative Schwarzian derivative and

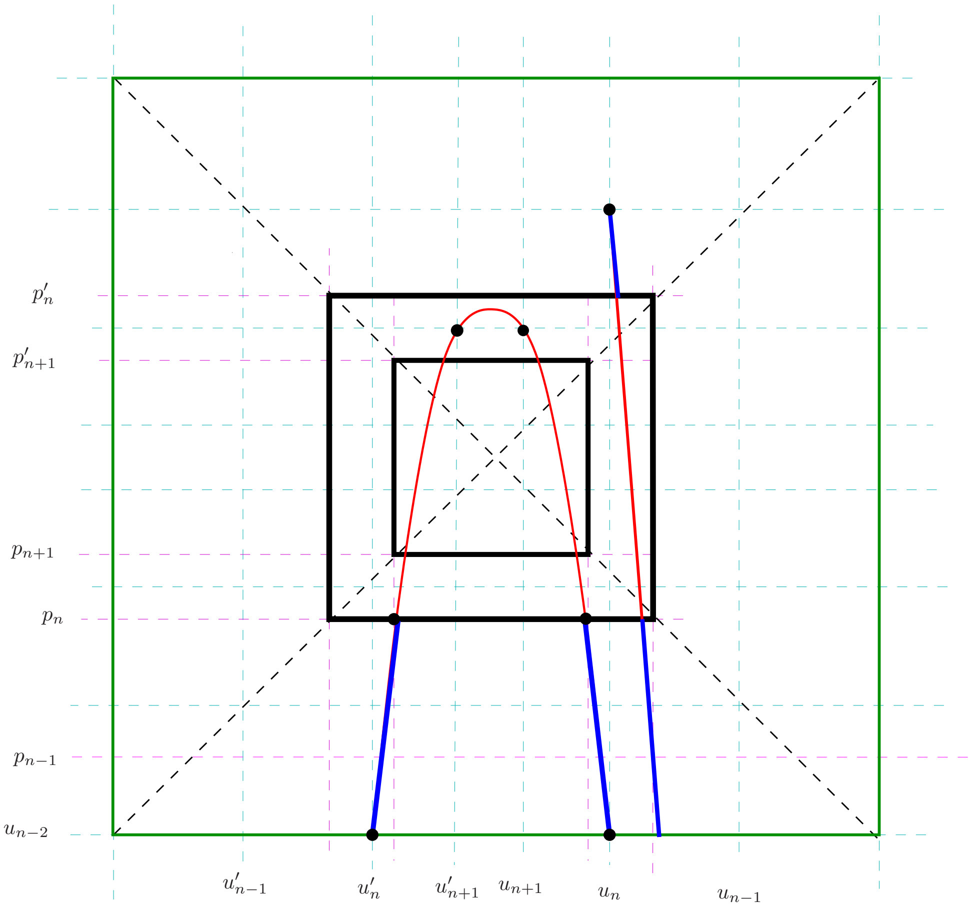

FIGURE 4. The (red) curves inside the medium square is the graph of the n-th Fibonacci renormalization f_n. The (red and blue) curves inside the largest square is the graph of an extension of f_n which has the same maximal invariant set.

bounded distortion. Define $F\colon I_0^0 \to \mathbb{R}$ as

$$F(x) := f^{S_i}(x) \quad if \quad x \in [u_i, -u_i] \setminus [u_{i+1}, -u_{i+1}]$$

for each $i \geq 0$.

Define L_i as, say, the right component of $[u_i, -u_i] \setminus [u_{i+1}, -u_{i+1}]$ and $\gamma_i\colon I \to L_i$ as the unique bijective order preserving affine map between these two intervals.

We are ready to define the random walk $\mathcal{F}\colon I \times \mathbb{Z} \to I \times \mathbb{Z}$ as

$$\mathcal{F}(x,k) := \begin{cases} (\gamma_i^{-1} \circ F \circ \gamma_k(x), i) & \text{if } F \circ \gamma_k(x) \in L_i, \\ (\gamma_i^{-1} \circ (-F) \circ \gamma_k(x), i) & \text{if } F \circ \gamma_k(x) \in -L_i. \end{cases}$$

There is a very special Fibonacci map $f^\star$, called the Fibonacci fixed point (see, for instance [Sm]), whose induced map $F^\star$ satisfies (choosing a good u_0)

$$F^\star(\lambda x) = \pm \lambda F^\star(x)$$

for some $\lambda \in (0,1)$. In this case we can use the argument in Remark 8.2 to conclude that $\mathcal{F}^\star$ is a homogeneous random walk. For an arbitrary Fibonacci map f, $\mathcal{F}$ is not homogeneous, however due Proposition 4.9 and the following result $\mathcal{F}$ is an asymptotically small perturbation of $\mathcal{F}^\star$:

Proposition 8.3 (see [Sm]). *For each even integer larger than two the following holds: for every Fibonacci map f, denote*

$$g_i = \alpha_i^{-1} \circ f^{S_i} \circ \alpha_{i+1} \colon I \to I,$$

where $\alpha_i \colon I \to [u_i^f, -u_i^f]$ is an bijective affine map so that $\alpha_i^{-1}(f_{i+1}(0)) > 0$ and consider the correspondent maps $g_i^\star$ for $f^\star$. Then

$$||g_i - g_i^\star||_{C^r} \leq K_r \rho^i$$

for some $\rho < 1$ and every $r \in \mathbb{N}$.

The **real Julia set** of f, denoted $J_{\mathbb{R}}(f)$, is the maximal invariant of the map

$$f \colon I_0^1 \sqcup I_1^1 \to I_0^0,$$

in other words,

$$J_{\mathbb{R}}(f_j) := \cap_i f_j^{-i} I_0^j.$$

Denote

$$\Omega_+^j(F) := \{(x,i) \ s.t.\ \pi_2(F^n(x,i)) \geq j \ \ for\ all\ n \geq 0\}.$$

Proposition 8.4. *There exists some k_0 so that*

$$\Omega_+^{j+1}(F) \subset J_{\mathbb{R}}(f_j) \subset \Omega_+^{j-1}(F).$$

In particular

$$HD\ \Omega_+^{j+1}(F) \leq HD\ J_{\mathbb{R}}(f_j) \leq HD\ \Omega_+^{j-1}(F), \tag{63}$$

and, for the Fibonacci fixed point, since $\Omega_+^{j+1}(F)$ is an affine copy of $\Omega_+^{j-1}(F)$ we have

$$HD\ \Omega_+^j(F) = HD\ J_{\mathbb{R}}(f). \tag{64}$$

for all $j \geq 0$.

Proof. Denote by F_ℓ the restriction of F to $\cup_{i \geq \ell} L_i$. Then the maximal invariant set of F_ℓ

$$\Lambda(F_\ell) := \cap_{i \in \mathbb{N}} F^{-i}\mathbb{R}$$

is $\Omega_+^\ell(F)$. Consider the extension of f_j described in Fig. (4). Let's call this extension $\tilde{f}_j$. An easy analysis of its graph shows that f_j and $\tilde{f}_j$ have the same maximal invariant set. We claim that $\tilde{f}_{j+1}$ is just a map induced by $\tilde{f}_j$. Indeed, the restriction of $\tilde{f}_{j+1}$ to $[u_{j+1}, u'_{j+1}]$ coincides with $\tilde{f}_j^2$ on the same interval. On the rest of $\tilde{f}_{j+1}$-domain $\tilde{f}_{j+1}$ coincides with $\tilde{f}_j$.

By consequence, for $i \geq j$ the map $\tilde{f}_i$ is induced by $\tilde{f}_j$ and, since F_{j+1} restricted to L_i is equal to $\tilde{f}_i$, we obtain that F_{j+1} is a map induced by $\tilde{f}_j$. In particular

$$\Lambda(F_{j+1}) \subset \Lambda(\tilde{f}_j) = J_{\mathbb{R}}(f_j).$$

To prove that $\Lambda(\tilde{f}_j) \subset \Lambda(F_{j-1})$, we are going to prove that

$$x \in \Lambda(\tilde{f}_j) \ implies\ F_{j-1}(x) \in \Lambda(\tilde{f}_j). \tag{65}$$

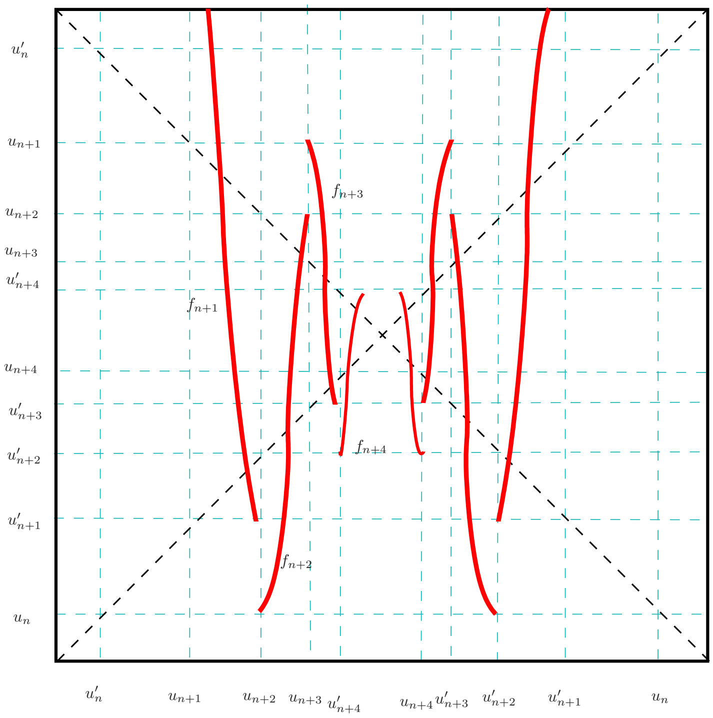

FIGURE 5. Induced map F for a Fibonacci map

If x belongs to the interval $I_1^j \subset L_{j-1}$, where $\tilde{f}_j$ coincides with F_{j-1}, then $F_{j-1}(x) \in \Lambda(\tilde{f}_j)$. Otherwise $x \in I_0^j \subset \cup_{i\geq j} L_i$, so $x \in \Lambda(\tilde{f}_j) \cap \ L_i$, for some $i \geq j$, then F_{j-1} is an iteration of $\tilde{f}_j$ on L_i, so $F_{j-1}(x) \in \Lambda(\tilde{f}_j)$. This finishes the proof of Eq. (65). Since $\Lambda(\tilde{f}_j)$ is invariant by the action of F_{j-1} we have $\Lambda(\tilde{f}_j) \subset \Lambda(F_{j-1})$. □

Proof of Theorem 9. Consider the homogeneous random walk $F^\star = (g, \psi)$ induced by $f^\star$. Denote

$$M = \int \psi \, d\mu,$$

where μ is the absolutely continuous invariant measure of g. Using Thorem 7, there are three cases:

1. M < 0**.** In this case $\mathbf{F}^\star$ is transient and we have that $HD\ \Omega_+(F) < 1$ for every asymptotically small perturbation of $F^\star$, in particular when F is a random walk induced by a Fibonacci map f. By Proposition 8.4, $HD\ J_{\mathbb{R}}(f) < 1$.

2. M $= 0$**.** In this case every asymptotically small perturbation G of $F^\star$ is recurrent and $m(\Omega_+(G)) = 0$ but $HD\ \Omega_+(G) = 1$. By Proposition 8.4 we obtain $m(J_{\mathbb{R}}(f)) = 0$ and $HD\ J_{\mathbb{R}}(f) = 1$.

3. M > 0**.** In this case $\mathbf{F}^\star$ is transient with $m(\Omega_+(F^\star)) > 0$ and the conjugacy between $F^\star$ and any asymptotically small perturbation of it is absolutely continuous on $\Omega^i_+(F^\star)$. In particular $m(\Omega_+(F)) > 0$ for every random walk F induced by a Fibonacci map f so $m(J_{\mathbb{R}}(f)) > 0$ by Proposition 8.4. □

A map $f\colon I \to I$ is called a unimodal map if f has an unique critical point, with even order d, which is a maximum, and $f(\partial I) \subset \partial I$. We will assume that f is real analytic, symmetric with respect the critical point and $Sf < 0$. If the critical value is high enough, then f has a reversing fixed point p. Let $I^0_0 := [-p, p]$. Consider the map of first return R to f: if $x \in I^0_0$ and $f^r(x) \in I^0_0$, but $f^n(x) \notin I^0_0$ for $i < r$, define

$$R(x) := f^r(x).$$

If there exists exactly two connected components I^0_1 and I^1_1 of the domain of R containing points in the orbit of the critical point, and furthermore the map

$$R\colon I^0_1 \cup I^1_1 \to I^0_0$$

is a Fibonacci map, then we will called f an **unimodal Fibonacci map**. The class of all unimodal Fibonacci maps will be denoted $\mathcal{F}^{uni}_d$.

Proof of Theorem 10. We will use the notation in the proof of Theorem 9. Since $m(J_{\mathbb{R}}(f)) > 0$, we conclude that the mean drift M is positive. by Proposition 5.1 any asymptotically small perturbation G of $\mathcal{F}^\star$ has the following property: there exists $\lambda \in [0, 1)$, $C > 0$ and $K > 0$ so that for every $P \in \mathcal{P}^0(G)$

$$m(p \in P\colon \sum_{i=0}^{n-1} \psi(G^i(p)) < Kn) \leq C\lambda^n |P|.$$

This implies that

$$m(p \in I_j\colon \sum_{i=0}^{\ell} \psi(G^i(p)) \geq K\ell \text{ for every } \ell \geq n) \geq (1 - C\lambda^n).$$

so if $j = n|min\psi|$ we obtain

$$m(I_j \cap \Omega^j_+(G)) \geq 1 - C\lambda^{C_1 j}.$$

here $c_1 > 0$. If G is a random walk induced by a Fibonacci map g then this implies that for j large

$$m(L_j \setminus J_{\mathbb{R}}(g)) = m((-L_j) \setminus J_{\mathbb{R}}(g)) \leq C\lambda^{C_1 j}|L_j|.$$

Since

$$[-u_{j+1}, u_{j+1}] = \bigcup_{i \geq j} L_i \cup (-L_i),$$

we conclude that

$$m([u_{j+1}, -u_{j+1}] \setminus J_{\mathbb{R}}(g)) \leq C\lambda^{c_1 j}|u_{j+1}|. \tag{66}$$

For every δ, choose j so that $|u_{j+2}| \leq \delta \leq |u_{j+1}|$. Because $|u_{j+2}| > \theta|u_{j+1}|$, where $\theta \in (0, 1)$ does not depend on j, we have that $|u_j| \geq C\theta^j$. Together with Eq. (66) this implies

$$m([-\delta, \delta] \setminus J_{\mathbb{R}}(g)) \leq C\lambda^{C_1 j}|u_{j+1}| \leq C|u_{j+1}|^{1+\alpha} \leq C|\delta|^{1+\alpha}.$$

□

Proof of Theorem 11. We will prove each one of the following implications:

(1) implies (2): From the proof of Theorem 9, if $m(J_{\mathbb{R}}(f)) > 0$ for some $f \in \mathcal{F}_d$ the mean drift M of the homogeneous random walk $\mathcal{F}^\star$ of $f^\star$ is positive. So $\mathcal{F}^\star$ (and all its asymptotically small perturbations) is transient (to $+\infty$). In terms of the original Fibonacci map f, this means that almost every orbit in $J_{\mathbb{R}}(f)$ accumulates in the post-critical set: So f has a wild attractor.

(2) implies (3): if there exists a wild attractor for f then $m(J_{\mathbb{R}}(f)) > 0$. From the proof of Theorem 9 we obtain that the mean drift M of $\mathcal{F}^\star$ is positive. So there exists a absolutely continuous conjugacy between $\mathcal{F}^\star$ and any asymptotically small perturbation of $\mathcal{F}^\star$. This implies that any two maps $f_1, f_2 \in \mathcal{F}_d$ admits a continuous and absolutely continuous conjugacy

$$h\colon J_{\mathbb{R}}(f_1) \to J_{\mathbb{R}}(f_2).$$

Now consider two arbitrary maps $g_1, g_2 \in \mathcal{F}_d^{uni}$. Then we already know that there exists an absolutely continuous conjugacy

$$h\colon J_{\mathbb{R}}(R_{g_1}) \to J_{\mathbb{R}}(R_{g_2})$$

between the induced Fibonacci maps R_{g_1} and R_{g_2} associated to g_1 and g_2. Of course h is just the restriction of a topological conjugacy between g_1 and g_2. By a Block and Lyubich result (see, for instance, page 332 in [dMvS]), every map of $\mathcal{F}_d^{uni}$ is ergodic with respect the Lebesgue measure. Since g_1 and g_2 have wild attractors, this implies that the orbit of almost every point $x \in I$ hits $J_{\mathbb{R}}(R_{g_1}$ at least once. Let $n(x)$ be a time when this happens.

So consider a arbitrary measurable set $B \subset I$ so that $m(B) > 0$. Then for at least one $n_0 \in \mathbb{N}$ the set

$$B_{n_0} := \{x \in B\colon\ n(x) = n_0\}$$

has positive Lebesgue measure. This implies that $f^{n_0} B_{n_0}$ has positive Lebesgue measure, so $m(h(f^{n_0} B_{n_0})) > 0$. Now it is easy to conclude that $m(h(B_{n_0})$ and $h(B) > 0$. Switching the places of g_1 and g_2 in this argument we can conclude that h is absolutely continuous on I.

Finally note that the eigenvalues of the periodic points are not constant on the class $\mathcal{F}_d^{uni}$.

(3) implies (4): By the argument in Martens and de Melo [MdM], if a Fibonacci map does not have a wild attractor then any continuous absolutely continuous conjugacy with other Fibonacci map is C^1: in particular the conjugacy preserves the eigenvalues of the periodic points. So if (3) holds then we can use the same argument in the proof of the previous implication to conclude that every Fibonacci map has a wild attractor.

(4) implies (5): The proof goes exactly as the proof of (2)$\Rightarrow$ (3).

(5) implies (1): The proof goes exactly as the proof of (3)$\Rightarrow$ (4).

□

References

[AD] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* 1:193–237, 2001.

[B] A. Broise. Transformations dilatantes de l'intervalle et théorèmes limites. in *Études spectrales d'operateurs de transfert et applications*, *Astérisque*, 238, 1996.

[Br] H. Bruin. Topological conditions for the existence of absorbing Cantor sets. *Trans. Amer. Math. Soc.* 350, no. 6, 2229–2263, 1998.

[BKNvS] H. Bruin, G. Keller, T. Nowicki, S. van Strien. Wild Cantor attractors exist. *Ann. of Math.* (2) 143 no. 1, 97–130, 1996.

[G] Y. Guivarc'h. Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés. *Ergodic Theory Dynam. Systems*, 9, no. 3, 433–453, 1989.

[F] K. Falconer. The geometry of fractal sets. Cambridge University Press, Cambridge, 1985.

[J1] Y. Jiang. On the quasisymmetrical classification of infinitely renormalizable maps I. Maps with Feigeinbaum's topology. *Preprint ims91-19a*, IMS-SUNY at Stony Brook, 1991.

[J2] Y. Jiang. On the quasisymmetrical classification of infinitely renormalizable maps II. Remarks om maps with a bounded type topology. *Preprint ims91-19b*, IMS-SUNY at Stony Brook, 1991.

[MdM] M. Martens and W. de Melo. The multipliers of periodic points in one-dimensional dynamics. *Nonlinearity* 12:217–227, 1999.

[MU] R. Mauldin and M. Urbański. Dimensions and measures in infinite iterated function systems. *Proc. Lon. Math. Soc.* (3) 73 , no. 1, 105–154, 1996.

[MU2] R. Mauldin and M. Urbański. Thermodynamic formalism and multifractal analysis of conformal infinite iterated functions systems. *Acta Math. Hungar.*, 96: 27-98, 2002.

[dMvS] W. de Melo and S. van Strien. One-dimensional dynamics Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 25. Springer-Verlag, Berlin, 1993

[McM96] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. Annals of Mathematics Studies, 142, Princeton University Press, Princenton, 1996.

[PT] J. Palis and F. Takens. Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Cambridge University Press, Cambridge, 1993.

[Sm] D. Smania. Puzzle geometry and rigidity: The Fibonacci cycle is hyperbolic. *Eletronic Preprint*, 2002. Avaliable at **www.icmc.usp.br/∼smania/**

[ShSu] M. Shub and D. Sullivan. Expanding endomorphisms of the circle revisited. *Ergodic Theory Dynam. Systems* 5, no. 2, 285–289, 1985.

[T] R. L. Tweedie. Perturbations of countable Markov chains and processes. *Ann. Inst. Statist. Math.* 32: 283–290, 1980.

[V] M. Viana. Stochastic dynamics of deterministic systems. 21º Colóquio Brasileiro de Matemática, IMPA, 1997. Avaliable at **www.impa.br/∼viana/**

[W] D. Williams. Probability with Martingales. Cambridge Mathematical Textbooks, 1991.

Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina, 110, Jardim Botânico, Rio de Janeiro-RJ, CEP 22460-320, Brazil.
E-mail address: gugu@impa.br
URL: www.impa.br/∼gugu/

Departamento de Matemática, ICMC-USP-Campus de São Carlos, Caixa Postal 668, São Carlos-SP, CEP 13560-970, Brazil.
E-mail address: smania@icmc.usp.br
URL: www.icmc.usp.br/∼smania/