HIGH AMPLITUDE SOLUTIONS FOR SMALL DATA IN SYSTEMS OF TWO CONSERVATION LAWS THAT CHANGE TYPE

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Abstract. We study a quadratic system of conservation laws with an elliptic region. The second order terms in the fluxes correspond to type IV in Shearer and Schaeffer classification. The viscosity matrix is the identity so the DRS point lies on the elliptic boundary. We prove that high amplitude Riemann solutions arise from Riemann data with arbitrarily small amplitude in the hyperbolic region near the DRS point. For such Riemann data there is no small amplitude solution. This behavior is related to the bifurcation of one of the codimension-3 nilpotent singularities studied by Dumortier, Roussarie and Sotomaior.

1. Introduction

A famous theorem of Lax [7] states that systems of n conservation laws with small data have Riemann solution consisting of n small waves, rarefactions or shocks, separated by constant states, under certain hypotheses. What happens if the hypotheses are violated? T.P. Liu [8] showed in 1974 that if the hypothesis of genuine nonlinearity is violated, the rarefactions and shocks can join. Still, they form n groups separated by $n-1$ constant states.

In this work, we find an example of a system of two equations for which the Riemannn solution consists of two shocks with $O(1)$ amplitude no matter how small the data is, provided it is close to a special point on the locus where the characteristic speeds coincide, i.e, the data is close to a special point on the boundary of the elliptic region.

Though our example occurs in a system with quadratic flux functions, such a point exists generically for systems that change from hyperbolic to elliptic type. This point is associated to a local bifurcation of the traveling wave ODE for the viscous conservation law, studied by Dumortier, Roussarie and Sotomaior in [4]. In their classification, this is called an elliptic bifurcation. Thus the existence of large Riemann solutions for small data is generic.

Dumortier, Roussarie and Sotomaior proved the existence of three types of codimension three bifurcations for planar vector fields: elliptic, saddle and focus. Azevedo, Marchesin, Plohr and Zumbrum in [1] proved that saddle bifurcation are associated to the existence of local Riemann solutions containing three waves for systems of two conservation laws. This solution has more waves than dimensions, and one of these waves is not a Lax wave. In [1] it was also proved that foci bifurcations do not occur for ODE's originating from systems of two diffusive conservation laws.

Therefore, we conjecture that for diffusive systems of two conservation laws the consequences of violations of Lax theorem's hypotheses are understood.

In Sec. 2 we review some results for systems of two conservation laws in one space dimension. In Sec. 3 we present our results. Proofs are in Sec. 4. In the Sec. 5 we present some remarks about our result.

Key words and phrases. Riemann problems, conservation laws, mixed type, non local solution.

2. Background

In this section we review some results for systems of two conservation laws in one space dimension. These are partial differential equations of the form

$$U_t + F(U)_x = 0, \tag{2.1}$$

where $U(x,t) = (u,v)^T \in \mathbb{R}^2$ for $x \in \mathbb{R}$ and $t \geq 0$, $F \in C^2(\mathbb{R}^2, \mathbb{R}^2)$.

A Riemann problem is an initial value problem with constant states on the left and right hand sides of the origin, called U_L and U_R, that is

$$U(x,0) = \begin{cases} U_L & \text{if} \quad x < 0, \\ U_R & \text{if} \quad x > 0. \end{cases} \tag{2.2}$$

We are concerned with solutions of (2.1) and (2.2) of the form

$$U(x,t) = \begin{cases} U_L & \text{if} \quad x < s_1 t, \\ U_M & \text{if} \quad s_1 t < x < s_2 t, \\ U_R & \text{if} \quad s_2 t < x, \end{cases} \tag{2.3}$$

i.e., they are sequences of two discontinuities (shocks) with speed s_1 and s_2.

Following Gel'fand [5] and Courant-Friedrichs [3] we require that the shocks are traveling waves $U(x,t) = \bar{U}(\eta)$, $\eta = (x - st)/\epsilon$, of the equation

$$U_t + F(U)_x = \epsilon U_{xx} \tag{2.4}$$

with $\lim_{\eta \to \pm\infty} U(\eta) = U_\pm$ in the limit as $\epsilon \searrow 0$, i.e., we impose that the associated ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{2.5}$$

has an orbit connecting the equilibria U_- to U_+. In this case we say that the shock is admissible or that it has a viscous profile. Therefore, each shock must satisfy the following two Rankine-Hugoniot conditions

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0, \tag{2.6}$$

where U_- and U_+ are, respectively, the left and right states of the shock and s is its speed. We denote the shock by the triplet (U_-, U_+, s); we may use $s(U_-, U_+)$ or just s for the shock speed.

Based on Lax [7] and Conley and Smoller [2], we define:

Definition 2.1. Generic shocks appearing in Riemann solutions are:

- *1-shocks*: U_- is a repeller and U_+ is a saddle ($1S$ in the figures);
- *2-shocks*: U_- is a saddle and U_+ is an attractor ($2S$ in the figures).

Definition 2.2. Other connections are important in our problems, namely:

- *over-compressive shocks:* U_- is a repeller and U_+ is an attractor (C in the figures);
- *left characteristic 1-shocks*: U_- is a repeller-saddle and U_+ is a saddle;
- *left characteristic over-compressive shocks*: U_- is a repeller-saddle and U_+ is an attractor.

For left characteristic shocks U_- is not a hyperbolic equilibrium because one of its eigenvalues on the linearization vanishes.

Definition 2.3. The Rankine-Hugoniot set for a fixed U_- is a one-dimensional set in U-space:

$$\mathcal{H}\left(U_-\right) = \left\{U_+ \in \mathbb{R}^2 : \exists s \in \mathbb{R} \text{ such that equation (2.6) holds}\right\}. \tag{2.7}$$

Each point of the Rankine-Hugoniot set $\mathcal{H}$ is classified according to Definitions 2.1 and 2.2. Typically, there are sectors in $\mathcal{H}$ consisting of 1-shocks, of 2-shocks and of over-compressive shocks, i.e., $\mathcal{H}$ is divided in connected parts such that every U_+ in the sector is a shock of same kind. Similarly, there are (isolated) points in $\mathcal{H}$ representing left characteristic shocks.

Smooth solutions of (2.1) satisfy

$$U_t + DF\left(U\right) U_x = 0. \tag{2.8}$$

Definition 2.4. The set of U in $\mathbb{R}^2$ where $DF\left(U\right)$ has:

- two distinct real eigenvalues is called the *strictly hyperbolic region*;
- two distinct complex conjugate eigenvalues is called the *elliptic region*;
- one double real eigenvalue is called the *coincidence locus*.

In the strictly hyperbolic region the characteristic speeds of $DF\left(U\right)$ are ordered so that the lowest is called 1-speed, $\lambda_1\left(U\right)$, and the highest is called 2-speed, $\lambda_2\left(U\right)$. The eigenvectors of $DF\left(U\right)$ are $\vec{r}_1\left(U\right)$ and $\vec{r}_2\left(U\right)$.

We now state a version of Lax's classical theorem for systems of two equations in a small neighborhood N with $\bar{N}$ in the strictly hyperbolic region, such that $\nabla\lambda_i \cdot \vec{r}_i \neq 0$, $i = 1, 2$.

Theorem 2.5. *Given U_L and U_R in N, there exist two transverse foliations, tangent to $\vec{r}_1$ at U_L and to $\vec{r}_2$ at U_R, and a U_M such that the curve segment from U_L to U_M along the slow speed foliation followed by the curve segment from U_M to U_R along the fast speed foliation parametrize the unique solution of the Riemann problem with data U_L, U_R. These curves represent shocks and rarefactions.*

Corollary 2.6. Let U_M be the middle point of the solution of the Riemann problem with data U_L, U_R. Then $\left|U_M - U_L\right| \searrow 0$ as $\left|U_R - U_L\right| \searrow 0$.

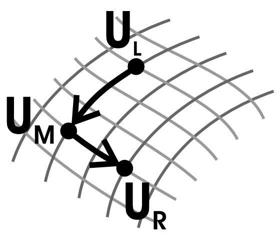

FIGURE 2.1. The transverse set of curves near U_L and the middle point U_M of the Riemann problem solution with data U_L, U_R.

Remark 2.7. Lax's famous shocks inequalities arise from the observation that the eigenvalues of the linearization of the ODE (2.5) at the equilibria U_-, U_+ (in fact, at any equilibrium U) are $\lambda_1\left(U\right) - s$.

3. The Local Riemann Problem with non Local Solution

We study a model of type IV in Shearer and Schaeffer's classification with the flux function

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3u^2+v^2 \\ 2uv \end{pmatrix} + \begin{pmatrix} 2v \\ 0 \end{pmatrix}. \tag{3.1}$$

We set $a = 3$ and $b = 0$ in the classification given in [10]. We expect that other type IV models with nearby parameters lead to similar results.

Since

$$DF\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 3u & v+2 \\ v & u \end{bmatrix} \tag{3.2}$$

the eigenvalues of DF are

$$\lambda_1 = 2u - \sqrt{u^2+(v+1)^2-1} \quad \text{and} \quad \lambda_2 = 2u + \sqrt{u^2+(v+1)^2-1}. \tag{3.3}$$

Notice that $\lambda_1 = \lambda_2$ along the circle $u^2+(v+1)^2 = 1$, the coincidence locus. The interior of this circle is the elliptic region in this model.

We show that non local solutions arise from Riemann problems with arbitrarily small data. This result is stated in the following theorems.

Theorem 3.1. *Let $\mathcal{O}$ be $(0,0)$. There exists an open set B with $\mathcal{O} \in \partial B$ in the strictly hyperbolic region having the following property. Given a small $\beta > 0$, for any $U_R \in B$ with $|U_R - \mathcal{O}| < \beta$ the solution of the Riemann problem with data $U_L = \mathcal{O}$, U_R has amplitude close to* 4*.*

This behavior can be extended for U_L, U_R in open sets near $\mathcal{O}$ in the hyperbolic region. Let $T(\beta)$ be the family of open triangles in the hyperbolic region

$$T(\beta) = \left\{(u,v) \in \mathbb{R}^2 : 0 < v < \beta^2/9 \text{ and } -v < u < v\right\}. \tag{3.4}$$

The choice $\beta^2/9$ is explained in the proof of Lemma 4.1.

Theorem 3.2. *Let be $\beta \gtrsim 0$. For every $U_L \in T(\beta)$ there is a non empty open set $A(U_L,\beta)$ closer than β from U_L with the following properties. i) The set $A(U_L,\beta)$ lie in the hyperbolic region; ii) For all points U_R in $A(U_L,\beta)$ the solution of the Riemann problem with data U_L, U_R has amplitude larger than* 4*.*

We remark that both $T(\beta)$ and $A(U_L,\beta)$ approach $\mathcal{O}$ as β vanishes.

4. Proof of the theorems

Substituting (3.1) in the Rankine-Hugoniot relation (2.6) yields

$$-s(u_+ - u_-) + 3(u_+^2 - u_-^2)/2 + (v_+^2 - v_-^2)/2 + 2(v_+ - v_-) = 0 \tag{4.1a}$$

$$-s(v_+ - v_-) + u_+v_+ - u_-v_- = 0. \tag{4.1b}$$

For $U_- = \mathcal{O} = (0,0)$, Eqs. (4.1) reduce to the quadratic curves

$$Q \equiv \frac{3}{2}\left(u_+ - \frac{s}{3}\right)^2 + \frac{1}{2}(v_+ + 2)^2 = 2 + \frac{s^2}{6} \quad \text{and} \quad (u_+ - s)\,v_+ = 0. \tag{4.2}$$

The Rankine-Hugoniot locus $\mathcal{H}(\mathcal{O})$ defined in (2.7) consists of the horizontal axis $v_+ = 0$ and of the circle $u_+^2 + (v_+ + 2)^2 = 4$. On the horizontal axis the shock velocity is given by $s = \frac{3}{2}u_+$. On the circle, $s = u_+$, so that $s < \lambda_1(U_+)$ if and only if $u_+ > 0$ and $-2 < v_+ < 0$; also $s > \lambda_2(U_+)$ if and only if $u_+ < 0$ and $-2 < v_+ < 0$.

Now we can classify the points in $\mathcal{H}(\mathcal{O})$ according to the Definitions 2.1 and 2.2 as shown in Figure 4.1. Notice that the 1-shock $(\mathcal{O}, \mathcal{O}', 0)$ is left characteristic, i.e., $s(\mathcal{O}, \mathcal{O}') = \lambda_1(\mathcal{O}) = 0$. The points D_1, D_2 and D_3 will be used later.

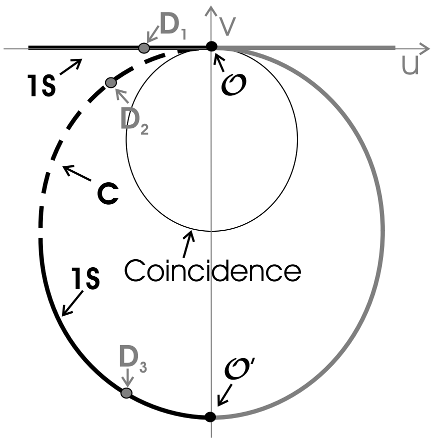

FIGURE 4.1. The curve $\mathcal{H}(\mathcal{O})$. The 1-shocks: solid curve; over-compressive shoks (C): dashed.

The intersections of the two curves in (4.2) are the equilibria of the associated ODE (2.5). If $s = 0$ there are just two equilibria, $\mathcal{O}$ and $\mathcal{O}' = (0, -4)$, see Figure 4.2. The equilibrium $\mathcal{O}'$ plays an important role.

The phase portrait for the ODE (2.5) associated to the shock $(\mathcal{O}, \mathcal{O}', s = 0)$ is shown in Figure 4.4. For this EDO the nilpotent singularity $\mathcal{O}$ is a possibly degenerate elliptic equilibrium in the classification given by Dumortier, Roussarie and Sotomaior, see [4] and [1]. Thus $\mathcal{O}$ is called the DRS point in this phase space. One can verify that the coincidence curve contains an homoclinic orbit of $\mathcal{O}$, thus the orbits that connect the equilibria $\mathcal{O}$ and the saddle $\mathcal{O}'$ lie in the hyperbolic region.

The phase portrait for $U_- = \mathcal{O}$ with shock speed $s_1 \lesssim \lambda_1(\mathcal{O})$ has four equilibria, as it can be easily seen using (4.2), see Figures 4.3 and 4.5. We see that $\mathcal{O}$ splits into three equilibria, $\mathcal{O}$, D_1 and D_2, while $\mathcal{O}'$ moves to D_3; D_1 and D_3 lie on the 1-shock sector $1S$

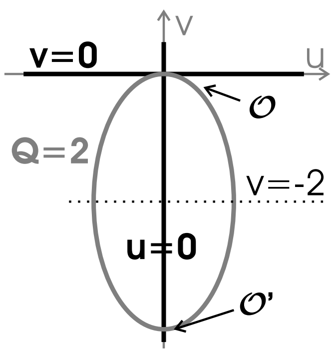

FIGURE 4.2. Quadratic curves for $U_- = \mathcal{O}$ and $s = 0$.

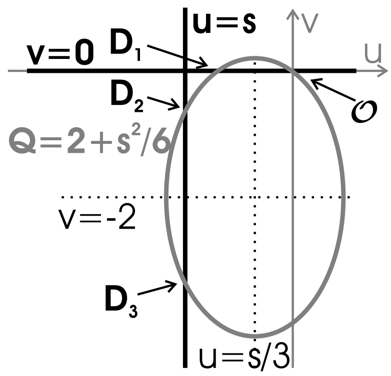

FIGURE 4.3. Quadratic curves for $U_- = \mathcal{O}$ and $s \lesssim 0$.

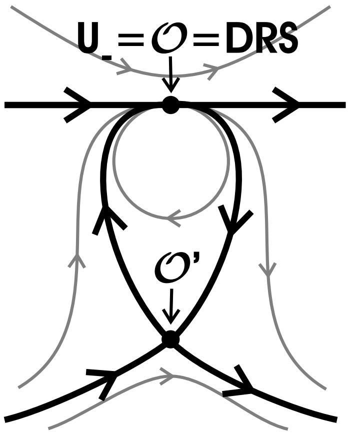

FIGURE 4.4. Phase portrait, $U_- = \mathcal{O}$, $s = 0$. The coincidence curve contains an orbit

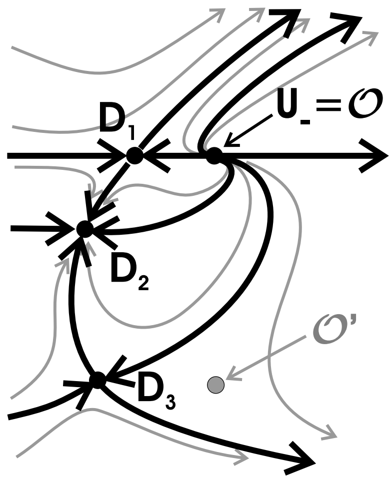

FIGURE 4.5. Phase portrait for $U_- = \mathcal{O}$, $s \lesssim 0$.

of $\mathcal{H}(\mathcal{O})$ while D_2 lies on the over-compressive sector C near $\mathcal{O}'$, see again Figure 4.1. The jacobian DF at the equilibrium $\mathcal{O}$ has only one eigenvector, with eigenvalue $-s_1$, so $\mathcal{O}$ is a non hyperbolic repeller. It is easy to check that D_1 is a saddle and D_2 is an attractor. Since $\mathcal{O}'$ was a saddle, D_3 is also a saddle. Thus, the four equilibria define several shocks (see Figures 4.5 and 4.6): the 1-shocks $(\mathcal{O}, D_1, s_1)$ and $(\mathcal{O}, D_3, s_1)$, the over-compressive shock $(\mathcal{O}, D_2, s_1)$ and the 2-shocks (D_1, D_2, s_1) and (D_3, D_2, s_1). Therefore the Riemann problem with $U_L = \mathcal{O}$ and $U_R = D_2$ has multiple solutions in phase space that coincide in physical space. Now we remove the degeneracy of the Riemann solution.

We have a family of equilibria $D_2(s)$ that lies in C. There are two kinds of solutions for the Riemann problem $U_L = \mathcal{O}$ and U_R near D_2 but out of C, see Figure 4.7.

If U_R lies above C (in this case we denote U_R by R_u), the solution is a 1-shock $(\mathcal{O}, D_1, s_1)$ followed by a faster 2-shock $(D_1, R_u, s_u > s_1)$; the equilibria $\mathcal{O}$ and D_1 do not change type and R_u is an attractor like D_2. We remark that the sequence of a 1-shock $(\mathcal{O}, D_3, s_1)$ followed by a 2-shock $(D_3, R_u, s(D_3, R_u))$ has incompatible shock speeds, i.e., $s(D_3, R_u) < s_1$. Therefore, the Riemann problem with data $\mathcal{O}$, R_u has a local solution as established in Lax Theorem.

On other hand if U_R lies below C (in this case we denote U_R by R_d) the solution is a 1-shock $(\mathcal{O}, D_3, s_1)$ followed by a faster 2-shock $(D_3, R_d, s_d > s_1)$; the equilibria $\mathcal{O}$ and D_3 do not change type and R_d is an attractor like D_2. We remark that the sequence of a 1-shock $(\mathcal{O}, D_1, s_1)$ followed by a 2-shock $(D_1, R_d, s(D_1, R_d))$ has incompatible shock speeds, i.e., $s(D_1, R_d) < s_1$. Therefore, the Riemann problem with data $\mathcal{O}$, R_d does not have a local solution, i.e., for such Riemann data there is no small amplitude solution.

Because C touches $\mathcal{O}$ we can choose R_d as close to $\mathcal{O}$ as we wish, so there are Riemann problems with data $U_L = \mathcal{O}$, U_R with non local solutions. The open set B (see Figure 4.8) lies in the gap between C and the part of $\mathcal{H}(\mathcal{O}')$ given by

$$u = \sqrt{-v(v+12)}\,(v+4)\,/\,(v+12)\,. \tag{4.3}$$

The proof of theorem 3.1 is complete.

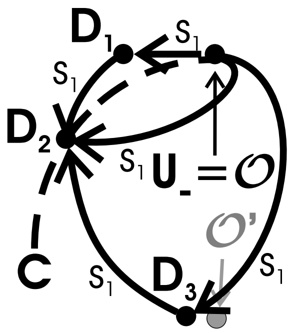

FIGURE 4.6. Phase portrait $U_L = \mathcal{O}$, $s_1 \lesssim 0$.

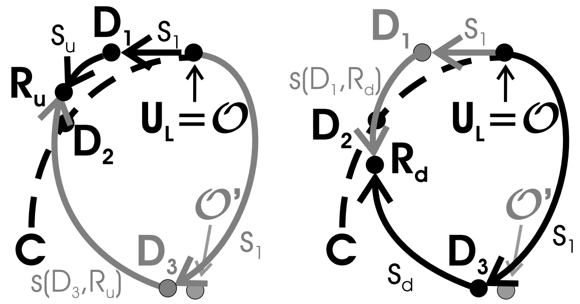

FIGURE 4.7. Solutions of the Riemann problem with $U_L = \mathcal{O}$ and U_R out the compressive sector but near D_2.

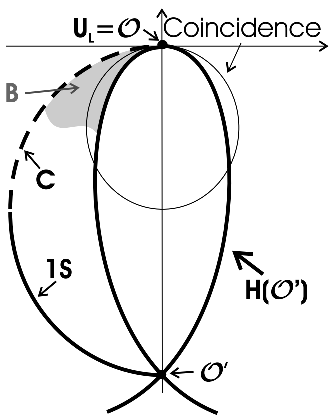

FIGURE 4.8. The open set B in Theorem 3.1.

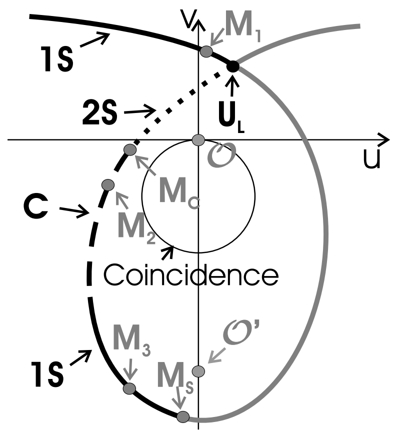

FIGURE 4.9. $\mathcal{H}(U_L)$ for $U_L \in T(\beta)$. The 1-shocks: solid curve; over-compressives shocks: dashed; 2-shocks: dotted curve.

We now show that this behavior actually occurs also for U_L in triangles above $\mathcal{O}$. However, in this case the over-compressive sector does not touch U_L any more. Let $T(\beta)$ be the family of open triangles defined in (3.4). For U_L in $T(\beta)$ the Rankine-Hugoniot curve is shown in Figure 4.9; the points M_i will be defined later. Because U_L now lies in the hyperbolic region it has two characteristic speeds, and we set $s_0 = \lambda_1(U_L)$. The Lax theorem guarantees that the over-compressive sector does not touch U_L.

The phase portrait for $U_L \in T(\beta)$ with $s \lesssim s_0 = \lambda_1(U_L)$ has four equilibria, U_L is a repeller, see Figure 4.10.a. The equilibria define the following shocks: the 1-shocks (U_L, M_1, s) and (U_L, M_3, s), the over-compressive shock (U_L, M_2, s), and the 2-shocks (M_1, M_2, s) and (M_3, M_2, s).

By increasing the speed back to s_0 the equilibria M_1 and U_L collapse into each other (U_L is a repeller-saddle) but M_2 stays away, see Figures 4.9 and 4.10.b. In this case we rename

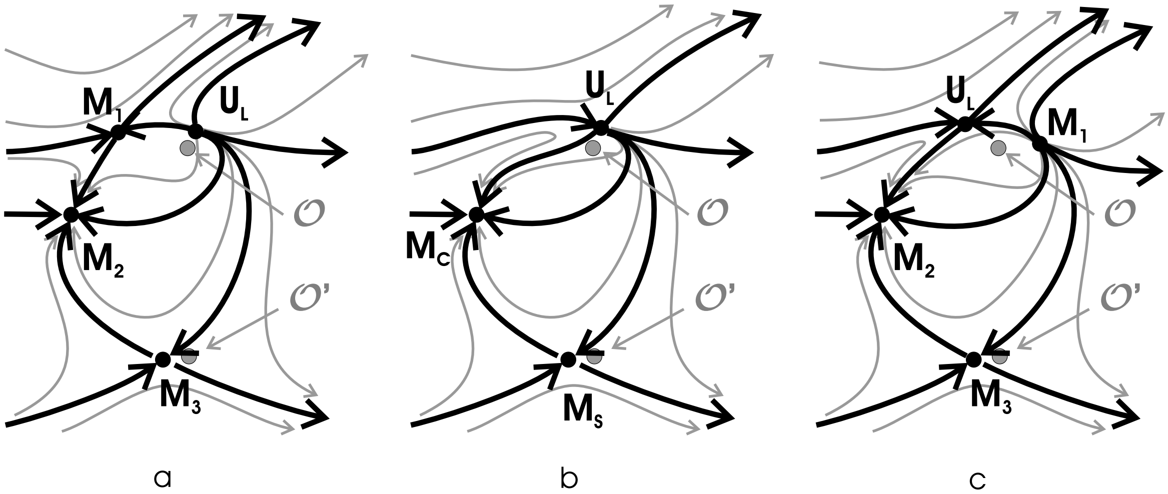

FIGURE 4.10. Phase portraits for $U_L \in T(\beta)$ with different speeds: a) $s \lesssim s_0$; b) $s = s_0$ (the equilibrium $M_1 = U_L$ is a repeller-saddle); c) $s \gtrsim s_0$.

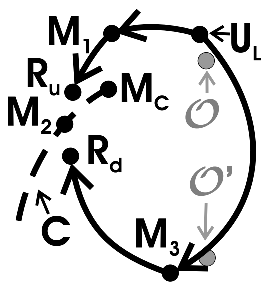

FIGURE 4.11. Solutions of the Riemann problem with $U_L \in T(\beta)$ and U_R out of compressive sector but near M_2.

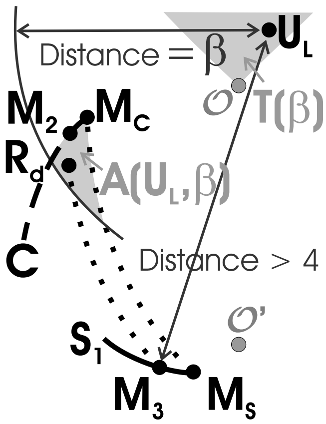

FIGURE 4.12. Solution for $U_L \in T(\beta)$, $U_R \in A(U_L, \beta)$.

M_2 and M_3 as, respectively, M_C and M_S. The equilibria define the following shocks: the left characteristic 1-shock (U_L, M_S, s_0) and the left characteristic over-compressive shock (U_L, M_C, s_0).

For $s \gtrsim s_0$ there is just one shock starting at U_L, namely the 2-shock from U_L to M_2, see Figures 4.9 and 4.10.c.

For right states near the over-compressive sector C of $\mathcal{H}(U_L)$ there are two kinds of solutions, see Figure 4.11. If U_R lies above C (in this case we denote U_R by R_u), the solution is a 1-shock from U_L to M_1 followed by a faster 2-shock from M_1 to R_u (the equilibria U_L and M_1 do not change type and R_u is an attractor as M_2). We remark that the sequence of a 1-shock from U_L to M_3 followed by a 2-shock from M_3 to R_u has incompatible shock speeds. Therefore, the Riemann problem with data U_L, R_u has a unique local solution with middle state M_1 as established in Lax Theorem.

On other hand if U_R lies below C (in this case we denote U_R by R_d), see again Figure 4.11, the solution is a 1-shock from U_L to M_3 followed by a faster 2-shock from M_3 to R_d (the equilibria U_L and M_3 do not change type and R_d is an attractor as M_2). We will show that M_3 stays away from U_L, therefore, the Riemann problem with data U_L, R_d has a large amplitude solution, i.e., for such Riemann data there is no small amplitude solution. We remark that the sequence of a 1-shock from U_L to M_1 followed by a 2-shock from M_1 to R_d has incompatible shock speeds.

We need to determine where the over-compressive sector ends, i.e., we must locate the point M_C separating the 2-shock sector from the over-compressive sector.

Lemma 4.1. *For $U_L \in T(\beta)$ with small β we have $|U_L - M_C| < \beta$ and $|U_L - M_S| > 4$.*

Proof. Let us find the location of $M_C \equiv (u_C, v_C)$ and $M_S \equiv (u_S, v_S)$. If $U_L = (\alpha v_L, v_L) \in T(\beta)$, with $-1 < \alpha < 1$ and $0 < v_L < \beta^2/9$, straightforward calculations using (4.1) with $s = \lambda_1(U_L)$ lead to

$$v_C = -v_L - 2 + b, \quad v_S = -v_L - 2 - b \quad \text{and} \quad u_i = 2\alpha v_L - a - (\alpha v_L^2 + a v_L)/v_i \tag{4.4}$$

for $i = C, S$, with

$$a = \sqrt{2v_L + (1+\alpha^2)v_L^2} \qquad \text{and} \qquad b = \sqrt{4 + (6\alpha a - 2)v_L - 6(\alpha^2 - 2)v_L^2}. \tag{4.5}$$

The quantity a is real in the hyperbolic region; b is real in part of the hyperbolic region, e.g. where $v_L < \sqrt{3}u_L + 1$ and $v_L > -\frac{1}{2}$, or where $v_L > -\sqrt{3}u_L + 1$ and $v_L < -\frac{1}{2}$. For small positive β both M_C and M_S lie in the strictly hyperbolic region.

Expanding the distances from U_L to M_S and M_C in power series in v_L we have:

$$|U_L, M_C| \simeq \left(5\sqrt{2v_L} - \alpha v_L\right)/3 \quad \text{and} \quad |U_L, M_S| \simeq 4 + 7v_L/4, \tag{4.6}$$

with error $O\left(v_L^{3/2}\right)$, so for small positive v_L we have

$$|U_L, M_C| < 3\sqrt{v_L} < \beta \quad \text{and} \quad |U_L, M_S| > 4. \tag{4.7}$$

□

Lets us examine the Riemann solution for U_R lying in the region below the part of C to the left of M_C (see Figure 4.12). The 1-shock from U_L to M_3 near M_S has speed s_1 slightly lower than $\lambda_1(U_L)$; the 2-shock from M_3 to R_d near M_2 and M_C has speed higher than s_1. By continuity we have $|U_L, M_3| > 4$ and $|U_L, R_d| < \beta$.

The tangent dU to $\mathcal{H}(U_L)$ at M_C is given by $\big(DF(M_C) - sI\big)dU - (M_C - U_L)\,ds = 0$ and the tangent of $\mathcal{H}(M_S)$ at M_C is given by $\big(DF(M_C) - sI\big)dU - (M_C - M_S)\,ds = 0$. Therefore $\mathcal{H}(U_L)$ and $\mathcal{H}(M_S)$ are transverse at M_C because either U_L, M_C and M_S are not collinear neither $\lambda_1(U_L)$ equals any characteristic speed of M_S. So we can define an angular open set $A(U_L, \beta)$, see Figure 4.12, with vertices on M_C and angle given by the tangents of $\mathcal{H}(U_L)$ and $\mathcal{H}(M_S)$ at M_C and distance to U_L that is smaller than β.

The proof of theorem 3.2 is complete.

5. Remarks

Dumortier, Roussarie and Sotomaior studied (see [4]) the versal bifurcation for a nilpotent singularity for a planar vector field with three parameters. They classify the codimension-3 bifurcations as saddle, focus and elliptic type. In [1] it is proved that saddle and elliptic bifurcations occur in quadratic models; moreover for a type IV flux with identity viscosity

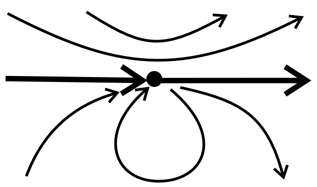

FIGURE 5.1. Phase portrait for the elliptic singularity DRS. From [4]. (Reproduced by permission.)

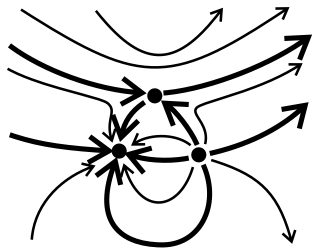

FIGURE 5.2. One of the possible perturbations of the phase portrait for the elliptic singularity DRS.

matrix the singularity is elliptic. The phase portrait for this kind of nilpotent singularity is sketched in Figure 5.1. One of the sixteen stable deformation is shown on Figure 5.2. No high amplitude solutions arise directly from the local bifurcation. In fact, looking only for local solutions would lead to nonexistence of Riemann solution. However the phase portraits of the solution for $U_L \in T(\beta)$ contain an extra equilibrium M_3 near $\mathcal{O}'$ which is fundamental for defining the non local solution, see again Figure 4.10.a.

So, in this work, we show that the elliptic bifurcation is associated to nonlocal solutions of local Riemann problems, which do not lie in Lax Theorem scope. In [1], it was shown that the saddle bifurcation is associated to Riemann solutions which require three waves separated by constant states (see Figure 5.4). One of the waves is a saddle-to-saddle connection called transitional or undercompressive wave, separating the 1-wave and the 2-wave. Again the necessity of three waves for solving a planar Riemann problem with small data lies outside Lax Theorem scope.

For Riemann problem with a type IV umbilic point, it is shown in [6] that in the high amplitude solutions do not appear. The singularity is not nilpotent any more, since taking for U_L the umbilic point it contains all the four equilibria points. In other words, the phase portrait for U_L equal to the umbilic point with speed lower than characteristic is topologically equivalent to the phase portrait for $\mathcal{O}$ with $s \lesssim 0$, see again Figure 4.5. However for left characteristic speed they are not topologically equivalent any more: there is just one equilibrium in the umbilic case and two equilibria ($\mathcal{O}$ and $\mathcal{O}'$) in the our case.

In our work we restrict ourself to identity matrix. A natural question is to ask what happens if we allow real viscosity matrices. This is a motivation for future work.

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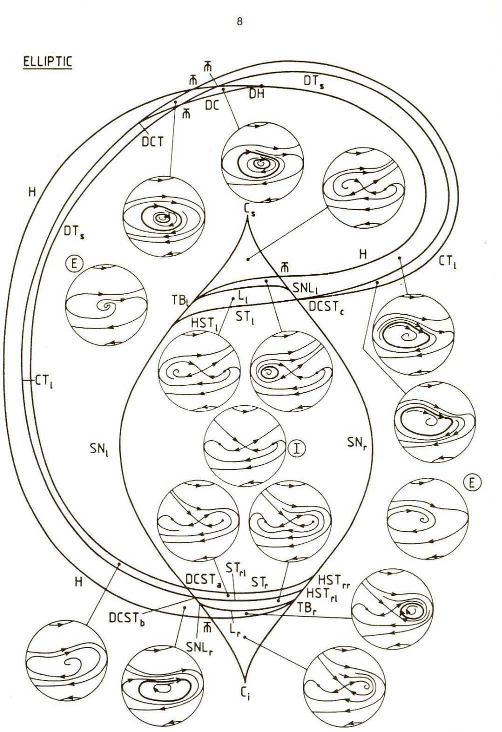

FIGURE 5.3. Versal unfolding for a codimension-3 elliptic nilpotent singularity. (Reproduced by permission.)

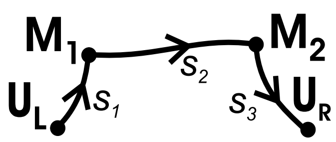

FIGURE 5.4. Local solution for saddle equilibrium. Waves: 1-shock (U_L, M_1, s_1); transitional (M_1, M_2, s_2); 2-shock (M_2, U_R, s_3).

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