ROBUSTLY EXPANSIVE CODIMENSION-ONE HOMOCLINIC CLASSES ARE HYPERBOLIC

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ABSTRACT. We shall prove that C^1-robustly expansive codimension-one homoclinic classes are hyperbolic.

1. INTRODUCTION

Let M be a d-dimensional manifold and $\mathrm{Diff}^1(M)$ be the set of C^1 diffeomorphisms f on M endowed with the C^1 topology. Let p be a hyperbolic periodic point of f and $H(p)$ be its homoclinic class. The diffeomorphism f is α-expansive in $H(p)$ if $\mathrm{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$, with $x, y \in H(p)$, implies $x = y$. It is well known that hyperbolicity implies α-expansiveness for some $\alpha > 0$. But expansiveness alone does not guarantee hyperbolicity, even when one is dealing with a codimension one expansive homoclinic class, as can be seen in [PPV, Section 2]. We note that there are even examples of expansive codimension one homoclinic classes such all of its periodic orbits are hyperbolic that are not hyperbolic.

To see this consider a Smale horse-shoe H in a plane and Λ a non trivial minimal subset of H. In a complementary direction multiply H by a non uniform contraction $\lambda(w)$, depending on the distance from w to Λ, and in such way that $\lambda(w) = 1$ for $w \in \Lambda$. The resulting homoclinic class is expansive, has all periodic points hyperbolic but it is not hyperbolic.

Then we assume that expansiveness holds in a C^1-neighborhood of the homoclinic class, that is, for all diffeomorphism g C^1 near f the homoclinic class $H(p_g)$ of the continuation p_g of p is α-expansive. In [PPV] it was proved that robustly expansive homoclinic classes of a three dimensional manifold are generically hyperbolic. We generalize this result in two ways. First we drop the assumption $\dim(M) = 3$ and get that robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting. Second we prove that robustly expansive codimension-one homoclinic classes with a dominated splitting are hyperbolic.

To announce our results in a precise way let us introduce some notations and definitions.

The homoclinic class $H(p)$ of a hyperbolic periodic point p of $f \in \mathrm{Diff}^1(M)$ is the closure of all transverse intersections of the stable manifold $W^s(p)$ with the unstable manifold $W^u(p)$ of p.

The homoclinic class $H(p)$ has a dominated splitting if $T_{H(p)}M$ splits into a continuous Df-invariant direct sum $E \oplus F$ and there are $c > 0$, $0 < \lambda < 1$ such that for all $x \in H(p)$ it holds

$$\|Df^n/E(x)\| \, \|Df^{-n}/F(f^n(x))\| \leq c\lambda^n$$

for all $n \geq 0$.

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We say that $H(p)$ is a codimension-one homoclinic class if $\dim W^u(p) = 1$ or $\dim W^s(p) = 1$. Our first result is the following

Theorem A. *Robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting $E \oplus F$.*

To obtain the corresponding to Theorem A in [PPV, Section 4] we assumed some kind of generic hypotheses that we remove here.

Our next result is:

Theorem B. *Robustly expansive homoclinic classes with a codimension-one dominated splitting are hyperbolic.*

To prove the corresponding to Theorem B in [PPV, Section 5] we profit from the density of C^2 diffeomorphisms in the C^1 topology proving that the dominated splitting for a C^2 diffeomorphism near the original one is hyperbolic. If in addition the homoclinic class is germ expansive it is proved in [SV] that $H(p)$ is hyperbolic (any codimension).

Next we sketch the proof of the results. To prove Theorem A we follow the same steps as the ones in [PPV, Section 4], dropping the generic assumptions assumed there.

The main step to obtain Theorem B is to prove that center unstable manifolds for all point x in the homoclinic class $H(p)$ are true unstable manifolds, that is, center unstable manifolds are dynamically defined (see definition 3.4 below). To achieve this it is enough to get this property for periodic points homoclinically related to p. For this we use a result, Proposition 3.5, that controls the behavior of periodic points homoclinically related to p. Once this is settled, under the hypothesis that the dominated splitting $E \oplus F$ is not hyperbolic, the fact that center unstable manifolds are dinamically defined allows us to obtain a hyperbolic periodic point q with arbitrarily large period as near $H(p)$ as we wish, and with arbitrarily small rate of contraction along the E-direction, see [PPV]. Then we prove that such hyperbolic periodic points are in fact in $H(p)$, contradicting uniform E-contraction in the period for periodic points in $H(p)$. As in [Ma2] this implies that the sub-bundle F is uniformly expanding, and then $E \oplus F$ is hyperbolic.

2. Proof of Theorem A

As already said the proof of Theorem A follows the same steps as the ones in [PPV, Section 4]. Here we shall only indicate the modifications needed to achieve the proof in the codimension-one case, and leave the details for the reader.

We say that a hyperbolic periodic point $q \in H(p)$ is homoclinically related to p if

$$W^s(q) \cap W^u(p) \neq \emptyset \quad \text{and} \quad W^u(q) \cap W^s(p) \neq \emptyset,$$

where $W^j(q)$, $j = s, u$, stands for the stable (unstable) manifold of q. Denote by $H_r(p)$ the points homoclinically related to p. Then $H_r(p)$ is dense in $H(p)$. So, it suffices to construct the dominated splitting for periodic points in $H_r(p)$.

Step 1. We prove that there is $\delta > 0$ such that if $q \in H_r(p)$ with period $\pi(q)$ and λ is a contracting eigenvalue of $Df^{\pi(q)}(q)$ then $|\lambda| < (1-\delta)^{\pi(q)}$. Similarly, if μ is an expanding eigenvalue of $Df^{\pi(q)}(q)$ then $|\mu| > (1+\delta)^{\pi(q)}$.

The proof of these statements are analogous to the proof of Propositions 4.3, 4.4 and 4.6 of [PPV]. Indeed, as we are dealing with periodic points in $H_r(p)$ we can use transitions matrices to achieve the results in the codimension-one case exactly like in the three dimensional case.

Step 2. Next we prove that there are $\overline{\gamma} > 0$ and $m_0 > 0$ such that for $q \in H_r(p)$ with period $\pi(q) > m_0$ it holds that

$$\angle(E(q), F(q)) > \overline{\gamma},$$

where $\angle(E(q), F(q))$ stands for the angle between $E(q)$ and $F(q)$.

The proof of this statement goes like that of [PPV, Proposition 4.9].

Using that f is expansive in $H(p)$ we obtain that there are only a finite number of periodic points with period bounded by m_0. Therefore, there is $\gamma > 0$ such that for all $q \in H_r(p)$ we have

$$\angle(E(q), F(q)) > \gamma.$$

Step 3 The previous steps allow to prove [PPV, Lemma 4.12] in the codimension-one case, that is, we get

Lemma 2.1. *There are a C^1 neighborhood $\mathcal{V}$ of f, $K \geq 2$, $0 < \lambda < 1$ such that for all $g \in \mathcal{V}$ for all $q \in H_r(p_g)$ with period $\pi(q)$ it holds that*

$$\|Dg^{-\pi(q)}/F(q)\| \leq K \quad \textit{and} \quad \|Dg^{-\pi(q)n}/F(q)\| \leq K\lambda^{\pi(q)n},$$

$$\|Dg^{\pi(q)}/E(q)\| \leq K \quad \textit{and} \quad \|Dg^{\pi(q)n}/E(q)\| \leq K\lambda^{\pi(q)n}.$$

Step 4. With the help of Lemma 2.1 we prove the existence of the dominated splitting as in [PPV, Theorem 4.13]. This result has also been obtained in [SV] for any codimension with a different proof. On the other hand, the proof given in [SV] does not prove Lemma 2.1.

3. Proof of Theorem B

Throughout we assume that $H(p)$ is robustly expansive with α as a constant of expansivity and that $H(p)$ has a codimension-one dominated splitting $T_{H(p)} = E \oplus F$ with $\dim(F) = 1$ and such that for all $x \in H(p)$,

$$\|Df^n/E(x)\|\,\|Df^{-n}/F(f^n(x))\| \leq C\lambda^n. \tag{1}$$

Taking a power of f instead of f itself we may assume (and do) that $C = 1$.

Let us assume that the local manifold $W^{\sigma}_{\varepsilon}(x), \sigma = s, u$ is an embedded topological k-disk, $k \geq 1$. Let Γ be the family of all parameterized continuous arcs $\gamma : [0,1] \to W^{\sigma}_{\varepsilon}(x)$ joining x with the boundary of $W^{\sigma}_{\varepsilon}(x)$.

Definition 3.1. *The size of $W^{\sigma}_{\varepsilon}(x)$ is defined as* $\inf\{\operatorname{diam}(\gamma) \,/\, \gamma \in \Gamma\}$.

It is known that there is a neighborhood $\mathcal{V}$ of $H(p)$ such that if $O(x) \subset \mathcal{V}$ then (1) holds for x. Here $O(x)$ stands for the orbit of x. We call $\mathcal{V}$ an *admissible* neighborhood of $H(p)$, see [Ma1].

The proof of the following lemma can be found, for instance, in [Ma1].

Lemma 3.1. *Assume that $H(p)$ has a dominated splitting $E \oplus F$ and let $x \in M$ such that $O(x) \subset \mathcal{V}$. There is $\varepsilon > 0$ such that the local center unstable manifold $W^{cu}_{\varepsilon}(x)$ is defined and is transverse* *to E, the local center stable manifold $W^{cs}_{\varepsilon}(x)$ is defined and is transverse to F. Such manifolds are of class C^1. Moreover, there is $\delta>0$ such that*

$$\text{if } y\in W^{cs}_{\varepsilon}(x) \text{ and } \operatorname{dist}(f^j(x),f^j(y)\le\delta,\ 0\le j\le n, \text{then } f^j(y)\in W^{cs}_{\varepsilon}(f^j(x)),\ 0\le j\le n.$$

$$\text{If } y\in W^{cu}_{\varepsilon}(x) \text{ and } \operatorname{dist}(f^{-j}(x),f^{-j}(y)\le\delta,\ 0\le j\le n, \text{ then } f^{-j}(y)\in W^{cs}_{\varepsilon}(f^{-j}(x)),0\le j\le n.$$

For $\varepsilon>0$ small, the ε-local unstable manifold of a hyperbolic periodic point q, $W^u_{\varepsilon}(q)$, is an arc tangent to F containing q in its interior. This arc is separated by q into two connected components $W^{+,u}_{\varepsilon}(q)$ and $W^{-,u}_{\varepsilon}(q)$ that, with the point q added, we shall call (local unstable) branches of $W^u_{\varepsilon}(q)$.

Set H_r for the subset of $H(p)$ of hyperbolic periodic points homoclinically related to p. If $q\in H_r$ we write $q\sim p$.

Lemma 3.2. *Given $q\in H_r$ there is $k\in\mathbb{Z}$ and $\delta_0>0$ such that if $W^{+,u}_{\varepsilon}(q)\cap H(p)\neq\emptyset$ then* $\operatorname{diam}(W^{+,u}_{\varepsilon}(f^k((q)))\ge\delta_0$. *A similar result holds for $W^{-,u}_{\varepsilon}(q)$ if $W^{-,u}_{\varepsilon}(q)\cap H(p)\neq\emptyset$.*

Proof. Assume in what follows that $H(p)$ has points in, say, $W^{+,u}_{\varepsilon}(q)$. Then there is a Cantor set of points $y\in W^{+,u}_{\varepsilon}(q)$ belonging to $H(p)$ because, by [PM, λ-Lemma], $W^s(p)$ accumulates on $W^s(q)$ and $W^u(p)$ accumulates on $W^u(q)$. Backward iterations of these points by f approach the orbit of q.

Backward iterates of points in $H(p)\cap W^{+,u}_{\varepsilon}(q)$ rest at a distance less than ε from the orbit of q. Therefore, by expansiveness, forward iterates must separate. Hence there is a first $k>0$ such that $\operatorname{diam}(f^{k+1}(W^{+,u}_{\varepsilon}(q))\ge\varepsilon$. Thus $\operatorname{diam}(f^j(W^{+,u}_{\varepsilon}(q))<\varepsilon$ for $0\le j\le k$ and therefore $f^j(W^{+,u}_{\varepsilon}(q))\subset W^{+,u}_{\varepsilon}(f^j(q))$.

This proves the lemma with $\delta=\delta_0$ defined as $\delta_0=\min\{\delta,\operatorname{diam}(f^{-1}(X))=\delta \text{ and } \operatorname{diam}(X)\ge\varepsilon\}$ where X is any subset of M. □

Let us reduce $\delta_0>0$ if it were necessary to have that if $\operatorname{dist}(y,H(p))\le\delta_0$, $y\in M$, then $y\in\mathcal{V}$, an admissible neighbourhood of $H(p)$.

Lemma 3.3. *Let $q\in H_r$. If $W^{-,u}_{\varepsilon}(q)\cap H(p)=\emptyset$ then either there is $k\in\mathbb{Z}$ such that*

$$\operatorname{diam}(W^{-,u}_{\varepsilon}(f^k((q)))\ge\delta_0$$

or there is a periodic point q' which is an end-point of $W^{-,u}_{\varepsilon}(q)$, of the same period as that of q, such that $\operatorname{dist}(f^j(q),f^j(q'))\le\varepsilon$*, for all $j\in\mathbb{Z}$.*

Proof. If $\operatorname{diam}(W^{-,u}_{\varepsilon}(q))\le\delta_0$ then we will have the end-point q' ($q'\neq q$) of $W^{-,u}_{\varepsilon}(q)$ contained in $\mathcal{V}$ and therefore $\operatorname{Cl}(W^{-,u}(q))$ is tangent to $\mathcal{C}_F$ for all the iterates by f. Here $\operatorname{Cl}(A)$ stands for the closure of A. It follows that q' is periodic of the same period of q or of the double of the period of q, depending on the sign of the expanding eigenvalue of Df^{τ}_q. But if this eigenvalue were negative we would have that $H(p)$ would have points different from q in both branches of $W^u_{\varepsilon}(q)$. Hence the period of q' is the same of that of q, finishing the proof of Lemma 3.3 □

An analogous result holds if $W^{+,u}(q)\cap H(p)=\emptyset$.

Next we announce a result by Pliss [Pl1, Pl2] that we shall use later. A nice proof of this may be found in [Al, Lemma 2.11].

Definition 3.2. *We say that a pair of points $(x, f^n(x))$ contained in $H(p)$, $n > 0$, is a γ-string, $0 < \gamma < 1$, if*

$$\prod_{j=1}^{n} \|Df/E(f^j(x)\| \leq \gamma^n.$$

We say that $(x, f^n(x))$ is a uniform γ-string if $(x, f^k(x))$ is a γ-string for all $0 \leq k \leq n$.

Lemma 3.4. *Let $0 < \gamma_1 < \gamma_2 < 1$ and $(x, f^n(x))$ be a γ_1-string. There exist a positive integer $N = N(\gamma_1, \gamma_2, f)$, $c = c(\gamma_1, \gamma_2, f) > 0$ such that if $n \geq N$ then there exist numbers*

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_l \leq n$$

such that $(f^{n_r}(x), f^n(x))$ are uniform γ_2-strings for all $r = 1, 2, \ldots, l$, with $l \geq cn$. That is,

$$\prod_{i=n_r}^{j} \|Df/E(f^i(x)\| \leq \gamma_2^{j-n_r};\ r = 1, 2, \ldots, l;\ n_r \leq j \leq n.$$

The numbers n_i in Lemma 3.4 are called γ_2-*hyperbolic times* for x, E. Analogously, replacing Df^n/E by Df^{-n}/F in the inequality above we define hyperbolic times for x, F.

Definition 3.3. *Given $0 < \gamma < 1$and $0 < N \leq n$ we say that $(x, f^n(x))$ is an (N, γ)-obstruction if $(f^m(x), f^n(x))$ is not a γ-string for all $n - N \leq m \leq n$.*

3.1. **Dynamically defined center unstable manifolds.**

Definition 3.4. *We say that the local center unstable manifold $W^{cu}_\varepsilon(x)$ is dinamically defined if it is part of the local unstable manifold of x, i.e.: given $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $\varepsilon' \to 0$ when $\varepsilon \to 0$ and for all $y \in W^{cu}_\varepsilon(x)$ we have that* $\operatorname{dist}(f^n(x), f^n(y)) \leq \varepsilon'$ *for all $n \leq 0$.*

A similar definition can be given for local center stable manifolds.

Proposition 3.5. *Let M be a compact manifold, $f : M \to M$ a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant subset with a dominated splitting $E \oplus F$,* $\dim(F) = 1$. *Then there exist $\gamma > 0$, $\varepsilon = \varepsilon(\gamma) > 0$ and $0 < \lambda_1 < 1$ such that for all hyperbolic periodic point $q \in \Lambda$ with* $\dim(W^u(q)) = 1$ *we have that there is $m = m(q) \geq 0$ such that*

(1) *For all $n \geq 0$ we have $\|Df^n/E(f^m(q))\| \leq \lambda_1^n$.*
(2) $W^{cs}_\varepsilon(f^m(q)) \subset W^s_\varepsilon(f^m(q))$.
(3) $f^{-n}(W^{cu}_\varepsilon(f^m(q))) \subset W^{cu}_\gamma(f^{-n+m}(q)), \quad n \geq 0$.

Proof. Let us denote by Λ_1 the set of all $q \in \Lambda$, q hyperbolic periodic point, $\dim(W^u(q)) = 1$. We assume that (1) holds. For the λ given in (1) consider $0 < \lambda < \sqrt{\lambda} < \lambda_1 < 1$. Then, for $q \in \Lambda_1$ we have that

$$\text{either} \quad \|Df^{-\pi(q)}/F(q)\| < \lambda_1^{\pi(q)} \quad \text{or} \quad \|Df^{\pi(q)}/E(q)\| < \lambda_1^{\pi(q)}.$$

Assuming that $\|Df^{\pi(q)}/E(q)\| < \lambda_1^{\pi(q)}$ the proof that given $q \in \Lambda_1$ there is k such that for all $n \geq 0$ $\|Df^n/E(f^k(q))\| \leq \lambda_1^n$ follows from Lemma 3.4 provided that the period $\pi(q)$ of q is greater than the constant $N(\sqrt{\lambda}, \lambda_1)$ given by that Lemma.

The proof that if we have that $\|Df^n/E(q)\| \leq \lambda_1^n$ for all $n \geq 0$ then there is $\gamma > 0$ such that $W_\gamma^{cs}(q) \subset W_\gamma^s(q)$ is contained in [SV, Lemma 5.2] taking into account that, by [Ma3, Lemma II.5] we may replace $\|Df^n/E(q)\| \leq \lambda_1^n$ by $\prod_{j=1}^n \|Df/E(f^j(q)\| \leq C\lambda_2^n$ with $C > 0$, $0 < \lambda_1 \leq \lambda_2 < 1$.

Next we prove (3):
Given $\gamma > 0$ and q a hyperbolic periodic point let us define

$$\varepsilon(q) = \sup\{\varepsilon > 0 \,:\, f^{-n}(W_\varepsilon^{cu}(q)) \subset W_\gamma^{cu}(f^{-n}(q)), n \geq 0\},$$

and $\varepsilon(O(q))$ as

$$\varepsilon(O(q)) = \sup\{\varepsilon(f^j(q)) \,:\, j = 1, \ldots, \pi(q)\}.$$

It is enough to prove that $\inf\{\varepsilon(O(q)) : q \in \Lambda_1\} > 0$.
Since q is hyperbolic we have that $\varepsilon(q) > 0$ and so $\varepsilon(O(q)) > 0$. Arguing by contradiction let us suppose that there is a sequence $\{p_k\}$ of points in Λ_1 such that $\varepsilon(O(p_k)) \to 0$ as $k \to \infty$. Suppose without loss of generality that $\varepsilon(O(p_k)) = \varepsilon(p_k)$. Since

$$f^{-n}(W_{\varepsilon(p_k)}^{cu}(p_k)) \subset W_{\varepsilon(f^{-n}(p_k))}^{cu}(f^{-n}(p_k)) \quad \forall\, n \geq 0$$

and we may assume that $\varepsilon(p_k) < \gamma$, we have that there are $t > 0$ and $m_t > \pi(p_k)$ such that setting $\delta_t = \varepsilon(p_k) + t$ we get $f^{-j}(W_{\delta_t}^{cu}(p_k))) \subset W_\gamma^{cu}(f^{-j}(p_k))$ for $0 \leq j \leq m_t$ and there exists a branch of $W_\gamma^{cu}(f^{-m_t}(p_k)) \setminus \{p_k\}$ that coincides with a branch of $f^{-m_t}(W_{\delta_t}^{cu}(p_k)))$.
Let us set $q_k = f^{-m_t}(p_k)$ and l_k to the branch of $W_\gamma^{cu}(q_k) \setminus \{p_k\}$ coinciding with the corresponding of $f^{-m_t}(W_{\delta_t}^{cu}(p_k)))$. Hence we obtain that $f^{\pi(p_k)}(l_k) \subset l_k$. This implies that $\|Df^{-n}/F(f^n(z))\| \geq \lambda_1^n$ for all $n \geq 0$, for all $z \in l_k$. Otherwise we will have that $f^{-n}(f^n(l_k)) \subsetneq l_k$. Therefore we have that $\|Df^n/E(z)\| \leq \lambda_1^n$ for all $n \geq 0$ for all $z \in l_k$. Then for $z \in l_k$ we have $W_\gamma^{cs}(z) \subset W_\gamma^s(z)$. Thus the stable manifold $W^s(l_k)$ of l_k has volume bounded away from zero. Since for $k \neq k'$ we have that $W^s(l_k) \cap W^s(l_{k'}) = \emptyset$ and M has finite volume we have arrived to a contradiction. This proves (3).

To finish the proof of the proposition we have to find q such that both (1) and (3) hold at the same time for it. Assume than that (1) holds for q. Hence (2) holds too. If it were the case that $\varepsilon(q) = \varepsilon(\gamma)$ then from (3) we are done. Otherwise define

$$m = \min\{0 \leq n \leq \pi(q) \,:\, \varepsilon(f^{-n}(q)) \geq \varepsilon(\gamma);\text{ and } \varepsilon(f^{-n}(q)) \geq \varepsilon(f^{-j}(q)), 0 \leq j \leq n\}.$$

Then it is easy to see that (1), (2) and 3) are verified for $f^{-m}(q)$. □

Corollary 3.6. *Let $\{p_k\}$ be a sequence of hyperbolic periodic points, $p_k \sim p$ converging to x and verifying Proposition 3.5. Then given $0 < \lambda_1 < \lambda_2$ and n_1, a λ_1-hyperbolic time for x, there is $k_0 > 0$ such that for all $k \geq k_0$ we have n_1 is a λ_2-hyperbolic time for p_k.*

Proof. Since n_1 is a λ_1-hyperbolic time for x, E we have

$$\prod_{n_1}^{n-1} \|Df/E(f^j(x))\| \leq \lambda_1^{n-n_1} \quad \forall \quad n \geq n_1.$$

Let $I=\inf\{\|Df/E(y)\|, y\in H(p)\}$. If the corollary does hold, we would have a sub-sequence that for simplicity we still denote by p_k such that for all $k>0$, there is m_k such that

$$\prod_{j=n_1}^{m_k-1}\|Df/E(f^j(p_k))\|>\lambda_2^{m_k-n_1}.$$

Then

$$\lambda^{m_k-n_1}<\prod_{j=n_1}^{m_k-1}\|Df/E(f^j(p_k))\|=\frac{\prod_{j=0}^{m_k-1}\|Df/E(f^j(p_k))\|}{\prod_{j=0}^{n_1-1}\|Df/E(f^j(p_k))\|}<\frac{\lambda_1^{m_k}}{I^{n_1}},$$

implying that

$$(\frac{\lambda_2}{\lambda_1})^{m_k}<(\frac{\lambda_2}{I})^{n_1}\quad\forall k. \tag{2}$$

Since $p_k\to x$ as $k\to\infty$ we obtain $m_k\to\infty$, and so, equation (2) leads to a contradiction.

□

Next we complement the proposition above establishing another properties for the center unstable manifolds of periodic points homoclinically related to p.

Proposition 3.7. *There is $\gamma>0$ such that for all $q\sim p$*

$$W^{cu}_{\varepsilon(\gamma)}(q)=W^{u}_{\varepsilon(\gamma)}(q),$$

where $\varepsilon(\gamma)$ is given by Proposition 3.5.

Proof. Given $q\sim p$, its enough to prove that $\ell(f^{-n}(W^{cu}_{\varepsilon(\gamma)}(q))\to 0$ as $n\to\infty$. The proof goes by contradiction. If it were not true, by Lemma 3.3 there would exist sequences $\gamma_k\to 0$, and periodic points p_k, p'_k with p'_k at the boundary of a branch of $W^{cu}_{\gamma_k}(p_k)$ such that p'_k is either a sink or a saddle-node, and moreover, $\operatorname{dist}(p_k,p'_k)\to 0$ as $k\to\infty$. Let z be a limit point of both p_k and p'_k. Then, by (1) we have

$$\|Df^n/E(z)\|<\lambda^n,\quad\forall n\geq 0.$$

Then, $W^s_\varepsilon(z)$ is well defined and we set $z_k=W^{cu}_{\gamma_k}(p_k)\cap W^s_\varepsilon(z)$ for all k.

Pick p_{k_1}, p_k, p_{k_2} such that p_k, p'_k are both in the region between $W^s_\varepsilon(p_{k_1})$ and $W^s_\varepsilon(p_{k_2})$ and set $z_{k_1}=W^{cu}_{\gamma_k}(p_k)\cap W^s_\varepsilon(p_{k_1})$ and $z_{k_2}=W^{cu}_{\gamma_k}(p_k)\cap W^s_\varepsilon(p_{k_2})$.

Let us suppose first that p'_k is a sink for all k. We set $[p'_k,z_{k_i}]$ for the segment contained in $W^{cu}_{\gamma_k}(p_k)$ connecting z_{k_i} and p'_k.

Assume that $p_k\notin[p'_k,z_{k_1}]$. If $[p'_k,z_{k_1}]$ does not contain any periodic point then since $z_{k_1}\in W^{cu}_{\gamma_k}(p_k)\cap W^s_\varepsilon(p_{k_1})$, we obtain that $f^n(z_{k_1})\to O(p'_k)$, contradicting that $z_{k_1}\in W^s_\varepsilon(p_{k_1})$. Thus there is a periodic point $\tilde{p}_k$ such that $\tilde{p}_k\in(p'_k,z_{k_1}]$. Ordering the segment $[p'_k,z_{k_1}]$ from p'_k to z_{k_1} we claim that if we pick $\tilde{p}_k$ as the supremum of the periodic points at $(p'_k,z_{k_1}]$ then $\tilde{p}_k\in H(p)$. To see this we observe that since p'_k is a sink and $\tilde{p}_k$ is sufficiently near p'_k then $W^{cs}_\varepsilon(\tilde{p}_k)\subset W^s_\varepsilon(\tilde{p}_k)$. Thus $W^s_\varepsilon(\tilde{p}_k)\cap W^u_\varepsilon(p_{k_1})\neq\emptyset$ and $W^u_\varepsilon(\tilde{p}_k)\cap W^s_\varepsilon(p_{k_1})\neq\emptyset$ for some small $\varepsilon>0$. This implies that $\tilde{p}_k\in H(p)$. By construction $\tilde{p}_k\neq p_k$.

Since $\operatorname{dist}(f^n(\tilde{p}_k),f^n(p_k))<\gamma(\varepsilon)$ for all n we arrive to a contradiction with expansiveness of $f/H(p)$.

We arrive to the same result if p'_k is a saddle-node. This finishes the proof.

$\square$

Definition 3.5. *A point $x \in H(p)$ is a border point if there exists a neighborhood N of x, homeomorphic to a ball, such that $N \backslash W^{cs}_{\varepsilon}(x)$ has two connected components and there is only one component of $N \backslash W^{cs}_{\varepsilon}(x)$ with x in its boundary and containing points of $H(p)$ accumulating on x.*

Given $x \in H(p)$, since $\dim(F) = 1$, we may define the branches $W^{+,cu}_{\varepsilon}(x)$ and $W^{-,cu}_{\varepsilon}(x)$ of $W^{cu}_{\varepsilon}(x)$ as before.

Lemma 3.8. *Assume that $x \in H(p)$ is such that $\ell(f^n(W^{cu,+}_{\varepsilon}(x))) \leq \delta_0$ for all $n \geq 0$. Then x is a border point of $H(p)$.*

Proof. Denote $I = W^{+,cu}_{\varepsilon}(x)$ and assume that $\ell(f^n(I)) \leq \delta_0$ for all n. Then, using (1) we obtain that there is $\varepsilon > 0$ such that every $w \in I$ has a local stable manifold $W^s_{\varepsilon}(w)$.

Now we prove that x is a border point of $H(p)$. Indeed, since $\ell(f^n(I)) \leq \delta_0$ for all n we obtain that $\mathrm{dist}(f^n(y), f^n(x)) \leq \delta_0$ for all $n \geq 0$ and $y \in I$. Assume that there is $y \in H(p) \cap I$. Then there is $z \in W^s(p) \cap W^u(p)$ with $\mathrm{dist}(z,y) \leq \varepsilon$, and we conclude, on account of the domination, that

$$\mathrm{dist}(f^n(z), f^n(x)) \leq 2\,\delta_0, \quad \forall\, n \geq 0. \tag{3}$$

As $z \in W^s(p)$ we have $\mathrm{dist}(f^n(z), p) \to 0$ as $n \to \infty$, and this together (3) give $\mathrm{dist}(f^n(x), p) \leq 2\,\delta_0$ for $n \geq n_0$. Thus $x \in W^s(p)$ and since p is hyperbolic, by the λ-lemma we obtain $\ell(f^m(I)) > \delta_0$ for some $m > 0$, a contradiction. This proves the lemma. $\square$

Theorem 3.9. *For all $x \in H(p)$, $W^{cu}_{\varepsilon}(x)$ is dynamically defined.*

Let $\delta_0 > 0$ and $\mathcal{V}$ be an admissible neighborhood of $H(p)$. Theorem 3.9 is a consequence of the following

Lemma 3.10. *There is $\nu > 0$ such that for all periodic point $q \sim p$ it holds that $\ell(W^{\pm,u}_{\varepsilon}(q)) \geq \nu$.*

Proof. Assume that the thesis does not hold. Then there is a sequence of periodic points q_k homoclinically related to p such that either $W^{+,u}_{\varepsilon}(q_k)$ or $W^{-,u}_{\varepsilon}(q_k)$ has length less than $1/k$. Assume that this happens to $W^{+,u}_{\varepsilon}(q_k)$. Then, since q_k is periodic, there is $m_k > 0$ such that $\ell(f^{-m_k}(W^{+,u}_{\varepsilon}(q_k))) = \varepsilon$ and so $f^{-m_k}(W^{+,u}_{\varepsilon}(q_k)) = W^{+,u}_{\varepsilon}(f^{-m_k}(q_k))$. Set $f^{-m_k}(q_k) = p_k$ and note that $m_k \to \infty$ as $k \to \infty$. By Propositions 3.5 and 3.7 we can assume

$$W^{cu}_{\varepsilon}(p_k) \subset W^u_{\varepsilon}(p_k), \quad W^{cs}_{\varepsilon}(p_k) \subset W^s_{\varepsilon}(p_k) \quad \text{and} \quad \prod_{j=1}^{j=n} \|Df/E(f^j(p_k))\| \leq \lambda^n,\ \forall\, n \geq 0\,. \tag{4}$$

Taking into account that $H(p)$ is compact, let us assume that $x = \lim_{k\to\infty} p_k$. Moreover, taking subsequences we may also assume that $W^{\pm,u}_{\varepsilon}(p_k)$ converge to arcs $I^{\pm} = I^{\pm}(x)$ with x at their common boundary. Next we prove that I^+ is contained in $W^{cu}_{\varepsilon}(x)$. Indeed,if this were not the case, there would exist $n_0 > 0$ and $y \in I^+$ such that $\mathrm{dist}(f^{-n_0}(x), f^{-n_0}(y)) = \varphi > \varepsilon$. Let $k > 0$ be so great that $y_k \in W^{u,+}_{\varepsilon}(p_k)$ is so close to y that $\mathrm{dist}(f^{-j}(y), f^{-j}(y_k)) < (\varphi - \varepsilon)/3$ and

$\mathrm{dist}(f^{-j}(x), f^{-j}(p_k)) < (\varphi - \varepsilon)/3$ for all $j = 0, 1, \ldots, n_0$. Then $\mathrm{dist}(f^{-n_0}(y_k), f^{-n_0}(p_k)) > \varepsilon$ which is absurd. Thus we may write $W_\varepsilon^{+,u}(x)$ instead of I^+. The same holds for I^- and we write $W_\varepsilon^{-,u}(x) = I^-$.

Since for all $j \in [0, m_k]$ we obtain that $\ell(f^j(W_\varepsilon^{+,u}(p_k))) \le \varepsilon$ and $m_k \to \infty$ when $k \to \infty$ we see that $\ell(f^n(W_\varepsilon^{+,u}(x))) \le \varepsilon$ for all $n \ge 0$.

To conclude the proof we need the following result that we shall prove later. Recall that $\omega(x)$ is the set of points $w \in M$ such that $\exists\, n_j \to +\infty$ such that $f^{n_j}(x) \to w$ as $j \to \infty$.

Theorem 3.11. *If there is $x \in H(p)$ such that $\ell(f^n(W_\varepsilon^{+,cu}(x)) \le \delta_0$ for all $n \ge 0$ then $\omega(x)$ is a periodic orbit.*

Returning to the proof of Lemma 3.10, on account of the above theorem, we have that $x \in W^s(q)$ for some $q \in H(p)$, q periodic. Taking a positive iterate by f of x we may assume that $x \in W_\varepsilon^s(q)$.
Moreover, since $W_\varepsilon^{+,u}(p_k)$ has length bounded away from zero and the angle between the subbundles E and F is bounded away from zero, we claim that $W_\varepsilon^{+,u}(p_k) \cap W_\varepsilon^s(q) \neq \emptyset$. Indeed, to prove this observe that since $\ell(f^n(W_\varepsilon^{+,u}(x))) \le \varepsilon$ for all $n \ge 0$ then, from (1), we get that $W_\varepsilon^{cs}(x)$ is a true stable manifold. Therefore we also have local stable manifolds for all point $y \in I(x)$. Take $r > 0$ small such that $W_\varepsilon^s(x)$ locally separates the ball at x with radius r, $B(x,r)$. Then in the region of $B(x,r) \backslash W_\varepsilon^s(x)$ containing $I(x)$ we cannot have points of the sequence p_k accumulating in x. Otherwise, since for points in that region their local center stable manifolds are local stable ones, we will have points $y \in I(x) \backslash \{x\}$ belonging to $H(p)$. Just take $y = W_\varepsilon^s(p_{k_0}) \cap I(x)$ for a suitable k_0 and then using that $W_\varepsilon^{+,u}(p_k) \to I(x)$, in the Hausdorff sense, we will obtain $y \in H(p)$. But if $y \in I(x)$ we have shown that for all $n \in \mathbb{Z}$ we have that $\mathrm{dist}(f^n(x), f^n(y)) \le \varepsilon$. But this contradicts that $f/H(p)$ is expansive if ε is less than an expansivity constant of $f/H(p)$. Therefore the points p_k accumulate x from the opposite region of that containing $I(x)$ and eventually $W_\varepsilon^{+,u}(p_k)$ will cut $W_\varepsilon^s(x) \subset W_\varepsilon^s(q)$. Since $p_k \sim p$ it follows that $W^u(p)$ cuts $W_\varepsilon^s(q)$, proving the claim.

If q is hyperbolic, by Hayashi's Connecting Lemma we may C^1-perturb f obtaining a diffeomorphism g so that $W^s(p_g) \cap W^u(q_g) \neq \emptyset$ and still having $W^u(p_g) \cap W^s(q_g) \neq \emptyset$.

Since periodic points homoclinically related to p_g cannot have a weak expanding eigenvalue, see [PPV, Section 4], the same is true for q_g, and therefore for q. As $x \in W^s(q) \cap I(x)$, by the λ-lemma, there is n_0 such that $\ell(f^n(I(x))) \ge \varepsilon$ for all $n \ge n_0$. Thus our assumption that there is a sequence $\{q_k\}$ of periodic points homoclinically related to p such that the $\ell(W_\varepsilon^{+,u}(q_k)) \le 1/k$ leads to a contradiction.

Hence there exists $\nu > 0$ such that $\ell(W^{u,\pm}(p')) \ge \nu$ for all periodic point p' homoclinically related to p, finishing the proof of Lemma 3.10 in this case.

If q is not hyperbolic then, by [Fr, Lemma 1.1], we may perturb Df_q/F to obtain hyperbolicity of q without loosing the intersection between $W_\varepsilon^{+,u}(p_k)$ and $W_\varepsilon^s(q)$ and still having p_k accumulating in $W_\varepsilon^s(q)$. So, q is still a point of the homoclinic class (of the perturbed diffeomorphism) and we can reason as above to achieve the same result.

□

Next we extend Lemma 3.10 for all $x \in H(p)$.

Proposition 3.12. *For all $x \in H(p)$ we have that $W^u_\varepsilon(x)$ is a non-trivial set and both branches of $W^u_\varepsilon(x)$ have length greater than $\nu > 0$.*

Proof. Let $x \in H(p)$. Since periodic points homoclinically related with p are dense in $H(p)$ we have p_k homoclinically related with p such that, $p_k \to x$. Taking a converging subsequence, in the Hausdorff sense, of $W^{+,u}_\varepsilon(p_k)$ and $W^{-,u}_\varepsilon(p_k)$, we conclude that $W^{+,cu}_\varepsilon(x)$ and $W^{-,cu}_\varepsilon(x)$ are defined. As both $W^{+,u}_\varepsilon(p_k)$ and $W^{-,u}_\varepsilon(p_k)$ are tangent to F and locally separated by $W^{cs}_\varepsilon(p_k)$ we have that $W^{+,cu}_\varepsilon(x) \neq W^{-,cu}_\varepsilon(x)$. Moreover, both branches are tangent to F and have length greater than ν since this holds for $W^{\pm,u}_\varepsilon(p_k)$. We also have $\ell(f^{-n}(W^{\pm,cu}_\varepsilon(x))) \leq \varepsilon$ because $W^{\pm,cu}(p_k)$ converges to $W^{\pm,cu}(x)$. Hence, $W^{\pm,cu}_\varepsilon(x)$ are in fact part of the local unstable manifold of x. $\square$

Lemma 3.13. *For all $x \in H(p)$ we have that $W^{cu}_\varepsilon(x)$ is dynamically defined, i. e.,*

$$\ell(f^{-n}(W^{\pm,cu}_\varepsilon(x))) \xrightarrow[n\to\infty]{} 0\,.$$

Proof. If it were the case that

$$\ell(f^{-n}(W^{+,cu}_\varepsilon(x))) \not\to 0 \text{ when } n \to +\infty$$

for all $x \in H(p)$ then there would exist $x \in H(p)$, $n_k \to +\infty$, and $\rho > 0$ such that

$$\ell(f^{-n_k}(W^{+,cu}_\varepsilon(x))) \geq \rho\,.$$

By compactness of $H(p)$ we may assume that

$$z = \lim_{k\to+\infty} f^{-n_k}(x) \quad \text{and} \quad I(z) = \lim_{k\to+\infty} f^{-n_k}(W^{+,cu}_\varepsilon(x))\,,$$

the last limit in the Hausdorff metric for compact subspaces of M. Thus, from the construction of $I(z)$, we have that for all $n \geq 0$ it holds

$$\rho \leq \ell(f^n(I(z))) \leq \varepsilon. \tag{5}$$

Moreover $I(z)$ is tangent to F, part of $W^{cu}_\varepsilon(z)$ and therefore an $\varepsilon - E$-interval, following [PS]. Since $I(z) \subset W^{cu}_\varepsilon(z)$ and $W^{cu}_\xi(y)$ is a one-dimensional submanifold for all small $\xi > 0$ we have that $I(z) \supset W^{+,cu}_\xi(z)$ for some small ξ. Therefore, for all $n \geq 0$ we have that $\ell(f^n(I(z)))$ is bounded away from zero.

As in [PS, Section 3.2, Proposition 3.1] we obtain that $\omega(z)$ is a periodic orbit or a periodic circle tangent to F. (Although in [PS] it is assumed that f is C^2 this is not used in that part of the proof of [PS, Proposition 3.1].) As a robustly expansive homoclinic class cannot have a periodic circle nor a Cantor set in $H(p)$ included in a periodic circle tangent to F, we conclude that $\omega(z)$ is a periodic orbit. Using this fact together (5) we arrive to a contradiction reasoning as in Lemma 3.10. $\square$

Let $\nu > 0$ be given in Proposition 3.12.

Lemma 3.14. *For all $0 < \xi \leq \nu$ we have that there is $m_0 = m_0(\xi) > 0$ such that for all $x \in H(p)$ $\ell(f^n(W^{u,\pm}_\xi(x))) \geq \nu$ for all $n \geq m_0$.*

Proof. Given $0 < \xi \le \nu$, arguing as in the proof of Lemma 3.10 we may see that given $x \in H(p)$ there exists $m_0(x,\xi) > 0$ such that

$$\ell(f^{m_0(x,\xi)-j}(W^{+,u}_{\xi}(x))) \le \varepsilon,\ j = 1,\dots,m_0(x,\xi) \quad \text{but} \quad \ell(f^{m_0}(W^{+,u}_{\xi}(x))) > \varepsilon.$$

Let $\xi \ge c' > 0$ be so that $f^{m_0(x,\xi)}(W^{+,u}_{c'}(x)) = W^{+,u}_{\varepsilon}(f^{m_0(x,\xi)}(x))$. Hence $\ell(W^{+,u}_{\varepsilon}(f^{m_0(x,\xi)}(x))) > \nu$. For $n = m_0(x,\xi)+1$ either $f^n(W^{+,u}_{c'}(x)) = W^{+,u}_{\varepsilon}(f^n(x))$ or $\ell(f^n(W^{+,u}_{c'}(x))) > \varepsilon$. In the first case $\ell(f^n(W^{+,u}_{c'}(x))) > \nu$ by Proposition 3.12. In the second case we certainly have $\ell(f^n(W^{+,u}_{c'}(x))) > \nu$ too. In this later case we choose $0 < c'' < c'$ such that $f^n(W^{+,u}_{c''}(x)) = W^{+,u}_{\varepsilon}(f^n(x))$. By induction in $n \ge m_0(x,\xi)$ we conclude that $\ell(f^n(W^{+,u}_{\xi}(x))) > \nu$ for all $n \ge m_0(x,\xi)$.

By compactness of $H(p)$ there exists a common m_0 for all $x \in H(p)$. This ends the proof. □

3.2. **Proof of Theorem 3.11.** The proof of Theorem 3.11 will be done in several steps. We shall assume in this sub-section that $x \in H(p)$ is such that $\ell(f^n(W^{+,cu}_{\varepsilon}(x)) \le \delta_0$ for all $n \ge 0$, where δ_0 is such that the δ_0-neighborhood of $H(p)$ is admissible.

Lemma 3.15. *There is $\varepsilon > 0$ small such that $W^{cs}_{\varepsilon}(x) \subset W^{s}_{\varepsilon}(x)$.*

Proof. Let p_k be as in Lemma 3.10. By (4)

$$\prod_{j=1}^{j=n} \|Df/E(f^j(p_k)\| \le \lambda^n, \quad \forall n \ge 0.$$

Since $p_k \to x$ the same holds for x. Thus $W^{cs}_{\varepsilon}(x) \subset W^{s}_{\varepsilon}(x)$. □

Pick some $\gamma \in I\!R$ such that $0 < \sqrt{\lambda} < \gamma < 1$ and let $0 \le n_1 < \cdots < n_l < \cdots$ be the maximal sequence of hyperbolic times for Df/E along $O^+(x)$ associated to that choice of γ. Thus we have that

$$\prod_{j=n_i}^{n} \|Df/E(f^j(x))\| < \gamma^{n-n_i} \quad \text{for} \quad n \ge n_i\,.$$

Lemma 3.16. *Assume that there exists $L > 0$ such that for all i we have $n_{i+1} - n_i \le L$. Then $E/\omega(x)$ is a uniformly contracting sub-bundle.*

Proof. Let $z \in \omega(x)$. Then we have that there exists $n_r \to +\infty$ such that $f^{n_r}(x) \to z$. By the definition of δ_0 we know that if $\mathrm{dist}(y,z) < \delta_0$ then

$$\frac{\|Df/E(z)\|}{\|Df/E(y)\|} \le 1 + c.$$

For $r > 0$ large enough let $J_r > 0$ be such that $\mathrm{dist}(f^j(z), f^j(f^{n_r}(x))) < \delta_0$ for all $j \in [-J_r, J_r]$. Then $J_r \to +\infty$ when $r \to +\infty$. Let $K = \sup\{\|Df_y\| : y \in M\}$. For a given r let $i = i(r)$ be such that $n_i < n_r \le n_{i+1}$. Then $n_{i+1} - n_r < L$ and therefore

$$\prod_{j=0}^{n-1} \|Df/E(f^j(z))\| \le (1+c)^n \prod_{j=0}^{n-1} \|Df/E(f^{n_r+j}(x))\| \le$$

$$\le (1+c)^n K^{n_{i+1}-n_r} \prod_{j=0}^{(n-1-n_{i+1}+n_r)} \|Df/E(f^{n_{i+1}+j}(x))\| \le$$
$$\le [(1+c)K]^{n_{i+1}-n_r}[(1+c)]\lambda_2]^{n-1-n_{i+1}+n_r} \le$$
$$[(1+c)K]^L \lambda_3^{n-1-L} \quad \forall n \le J_r.$$

Since $J_r \to +\infty$ we conclude that $E/\omega(x)$ is uniformly contracting. □

Lemma 3.17. *Let $E/\omega(x)$ be uniformly contracting and assume that $\ell(f^n(W_\varepsilon^{+,cu}(x)) \not\to 0$ when $n \to +\infty$. Then $\omega(x)$ is a periodic orbit.*

Proof. If we have that $\ell(f^n(W_\varepsilon^{+,cu}(x)) \not\to 0$, as in the proof of [PS, Proposition 3.1] we may conclude, on account of (1), that there is a subsequence $f^{n_j}(W_\varepsilon^{+,cu}(x))$ converging to an arc L tangent to F. We remark that the hypothesis of f being of class C^2 is not necessary for this part of the proof given in [PS, Proposition 3.1]. Moreover, again by the domination property (equation (1)) and the boundedness of the lengths of the forward iterates of $W_\varepsilon^{+,cu}(x)$, if $y \in L$ then $W_\varepsilon^s(y)$ has uniform size. Thus, as $W_\varepsilon^s(y)$ is tangent to E, $W_\varepsilon^s(L) = \cup_{y\in L} W_\varepsilon^s(y)$ is a neighborhood of L. As in [PS, page 989] we may conclude that $\omega(x)$ is either a periodic orbit or $\omega(x)$ is contained in a C^1 simple closed curve $\mathcal{C}$ invariant by f^m, some m, which attracts a neighborhood of itself.

If f were C^2 we would conclude, as in [PS] that $\omega(x)$ would be a periodic orbit or $\omega(x)$ is the whole curve $\mathcal{C}$. As we do not assume that f is C^2 we proceed in a different way to get directly that $\omega(x)$ is a periodic orbit. The proof goes by contradiction. Assume that $\omega(x)$ is not a periodic orbit. We already know that $\omega(x) \subset \mathcal{C}$. By the attracting properties of $\mathcal{C}$, derived from (1), it follows that any point in a neighbourhood of $\mathcal{C}$ is asymptotic to $\mathcal{C}$. But $x \in H(p)$ and therefore its omega-limit set is in $H(p)$. Take $z \in H(p)$ a point such that its forward and backward orbit is dense in $H(p)$. There is a residual set of such points in $H(p)$. Then z has to visit $\mathcal{C}$ because $\omega(x) \subset \mathcal{C}$. But as $\mathcal{C}$ attracts a neighborhood of itself we have that $z \in \mathcal{C}$ and therefore $H(p) \subset \mathcal{C}$. In particular $p \in \mathcal{C}$ and the C^1-curve $\mathcal{C}$ which is tangent to F and transverse to E, has to self-accumulate (for instance) in p in the disk $W_\varepsilon^s(p)$ tangent to E. But this is not possible and we finish the proof of Lemma 3.17. □

From now on we assume that $E/\omega(x)$ a uniformly contracting sub-bundle and

$$\ell(f^n(W_\varepsilon^{+,cu}(x))) \to 0, \quad n \to +\infty.$$

As a consequence of Propositions 3.5 and 3.7 together with the fact that the angle between E and F is uniformly bounded away from 0 we obtain that for all k big enough

(6) $$W^u(p_k) \cap W_\varepsilon^s(x) = \{z_k\} \quad \text{and} \quad W^s(p_k) \cap W_\varepsilon^u(x) = \{y_k\}.$$

Claim 3.1. $z_k, y_k \in H(p)$.

Proof. Fix k_1 and consider the intersections $z_{k_1,k} = W_\varepsilon^u(p_{k_1}) \cap W_\varepsilon^s(p_k)$. Then $z_{k_1,k} \to z_{k_1}$ as $k \to \infty$. As $z_{k_1,k} \in H(p)$ we conclude that $z_{k_1} \in H(p)$. In the same way we prove that $y_k \in H(p)$. □

If there are subsequences y_{k_i} and y_{k_j} of y_k with $y_{k_i} \in W^{+,cu}(x)$ for all i and $y_{k_j} \in W^{-,cu}(x)$ for all j we have by Lemma 3.2 that

(7) $$\ell(f^n(W_\varepsilon^{\pm,cu}(x)) \not\to 0 \quad \text{as} \quad n \to \infty.$$

By construction, either $\ell(f^n(W_\varepsilon^{+,cu}(x)) \leq \varepsilon$ for all $n \geq 0$ or $\ell(f^n(W_\varepsilon^{-,cu}(x)) \leq \varepsilon$ for all $n \geq 0$. Thus (7) leads to a contradiction. Assume then that $y_k \in W_\varepsilon^{-,cu}(x)$ for all k.

Let $I_0 = [x,y]$ be the segment tangent to F containing $W_\varepsilon^{+,cu}(x)$ maximal with respect to

- for all $I_0' \subset I_0$ we have $\ell(f^n(I_0')) \leq \delta_0$ for all $n \geq 0$.

Then $\ell(f^n(I_0)) \leq \delta_0$ for all $n \geq 0$ and $\ell(f^n(I_0)) \to 0$. Indeed, if $\ell(f^n(I_0)) \not\to 0$, Lemma 3.17 implies that $\omega(x)$ is a periodic orbit and we are done. Moreover, for all $J \supset I_0$ we have $\ell(f^n(J)) \not\to 0$, for otherwise we would also have that $\omega(x)$ is a periodic orbit.

For each $j \geq 0$ consider the maximal interval I_j tangent to F such that $I_j \supset f^j(W_\varepsilon^{+,cu}(x))$ and for all $I_j' \subset I_j$ we have $\ell(f^n(I_j')) \leq \delta_0$ for all $n \geq 0$. Note that $f(I_j) \subset I_{j+1}$ for all j.

Lemma 3.18. *Fixed $\delta \leq \delta_0/2$ there is j_0 such that for all $j \geq j_0$ it holds $\ell(f^n(I_j)) \leq \delta$ for all $n \geq 0$ or $\omega(x)$ is a periodic orbit.*

Proof. The proof goes by contradiction. If the conclusion does not hold as for all $j' > j$ we have that $f^{j'-j}(I_j) \subset I_j{}'$ we would have the volume $\mathrm{vol}(\mathrm{W}_\varepsilon^{cs}(\mathrm{I_j})) > \nu_0$ for some $\nu_0 > 0$. This implies that $W_\varepsilon^{cs}(I_j)) \cap W_\varepsilon^{cs}(I_j{}')) \neq \emptyset$ for some $j' > j$. And this implies that $\omega(x)$ is a periodic orbit. $\square$

Then we can assume that $\ell(f^n(I_j)) \leq \delta$ for all $n \geq 0$, for all j.

Lemma 3.19. *Let $W_\varepsilon^{+,u}(p_k) \cap W_\varepsilon^s(I_0) = \tilde\beta_k$. Then $\tilde\beta_k$ is an arc tangent to F. Let us pick an arc β_k, $\beta_k \supset \tilde\beta_k$ tangent to F joining z_k to a point $y_k \in W_\varepsilon^{cs}(y)$, where y is the end point of I_0 different from x. Then $\beta_k \subset W_{\varepsilon+\delta}^{+,u}(p_k)$ or $\omega(x)$ is a periodic orbit.*

Proof. If it were not true, there would exist a subsequence of $\{p_k\}$, that we still denote $\{p_k\}$ such that for some m_k we would have $\ell(f^{-m_k}(\beta_k)) > \delta$ but $\ell(f^{-n}(\beta_k)) \leq \delta$ for all $0 \leq n \leq m_k$. Note that $m_k \to \infty$ as $k \to \infty$ because $\ell(f^{-n}(I_0))$ goes to zero as $n \to \infty$. By equation (1) we have that

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| \, \|Df^{-1}/F(f^{j+1}(y_k))\| < \lambda^{m_k+1}. \tag{8}$$

Pick λ_1 such that $0 < \sqrt{\lambda} < \lambda_1 < 1$ and let $N(\sqrt{\lambda}, \lambda_1)$, $c(\sqrt{\lambda}, \lambda_1)$ be the numbers given in Lemma 3.4. Then, for $m_k > N$ there exists a hyperbolic time from $-n_k$ to 0 for E. For from (8) we have that either

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| < \sqrt{\lambda}^{m_k+1} \text{ or } \prod_{j=-m_k}^{0} \|Df^{-1}/F(f^{j+1}(y_k))\| < \sqrt{\lambda}^{m_k+1}.$$

But the last possibility cannot hold. Otherwise we will have a hyperbolic time for F which will contradict that $\ell(f^{-m_k}(\beta_k)) > \delta$. Thus we have that

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| < \sqrt{\lambda}^{m_k+1} \tag{9}$$

which implies the existence of a hyperbolic time $-n_k$ for E. We have that $-m_k \leq -n_k < -m_k + N$, otherwise we will have a hyperbolic time for F between $-m_k$ and $-n_k$, and this again contradicts that $\ell(f^{-m_k}(\beta_k)) > \delta$. On the other hand, since $y_k \in W^s_\varepsilon(I_0)$, we have that

$$\prod_{j=0}^{n} \|Df/E(f^j(y_k))\| < \lambda^{n+1}, \quad \forall\, n \geq 0\,. \tag{10}$$

Therefore, equations (9) and (10) imply that the ε-stable manifold of $f^{-m_k}(\beta_k)$ has volume bounded away from zero. As this holds for all k sufficiently large we eventually have that there exist k, k', $k \neq k'$ such that

$$W^s_\varepsilon(f^{-m_k}(\beta_k)) \cap W^s_\varepsilon(f^{-m_{k'}}(\beta_{k'})) \neq \emptyset\,.$$

This implies that $\omega(x)$ is a periodic orbit. So, if $\omega(x)$ is not a periodic orbit then $\beta_k \subset W^{+,u}_{\varepsilon+\delta}(p_k)$. □

Remark 3.20. (1) *Lemma 3.19 implies that if* $\omega(x)$ *is not a periodic orbit then*

$$W^{+,u}_{\varepsilon+\delta}(p_k) \cap W^{cs}_\varepsilon(y) \neq \emptyset.$$

(2) *Since* $\ell(f^n(\beta_k)) \to 0$ *as* $n \to \infty$ *we obtain from Lemma 3.2 that the arc* β_k *cannot have points in* $H(p)$ *different from its end points.*
(3) *Since* $z_k \in H(p)$ *and we cannot have points in* $\beta_k \cap H(p)$ *different from its end points we obtain that* $W^u(p_k)$ *crosses* $W^s_\varepsilon(I_0)$. *To see that it suffices to pick* $f^{\pi(p_k)}(z_k)$ *where* $\pi(p_k)$ *is the period of* p_k.

Let y_k, z_k be, as above, the end points of β_k. Our next target is to prove that $y_k \in H(p)$. For this we orient $W^{+,u}(p_k)$ from p_k and, as we have noted in remark 3.20 we have that there are no points of $H(p) \cap \beta_k$ different from its end points and there are points $w \in H(p) \cap W^{+,u}(p_k)$ with $w > y_k$ in the ordering established. Denote by $w_k = \inf\{w \in H(p) \cap W^{+,u}(p_k), w > y_k\}$. Since $H(p)$ is closed we have that $w_k \in H(p)$.

Lemma 3.21. $w_k = y_k$.

Proof. By definition we have $w_k \geq y_k$. Assume, by contradiction, that $w_k > y_k$. Then $[z_k, w_k] \supset [z_k, y_k] = \beta_k$.

Assume that $\ell(f^n([z_k, w_k])) \leq \delta_0$ for all $n \geq 0$. Since $\ell(f^n([y_k, w_k])) \not\to 0$ we can choose a sub-sequence $n_j \to \infty$ such that $\ell(f^{n_j}([y_k, w_k])) > \delta$ for all j, for some $0 < \delta < \delta_0$. Let $z = \lim_{j\to\infty} f^{n_j}(y_k)$. As $\ell(f^n([z_k, y_k])) \to 0$ we also have that $z = \lim_{j\to\infty} f^{n_j}(z_k)$ which implies that $z \in H(p)$. Since $\delta < \ell(f^{n_j}([y_k, w_k])) < \delta_0$ we have that $f^{n_j}([y_k, w_k]) \in \mathcal{V}(H(p))$, for all j, where $\mathcal{V}(H(p))$ is an admissible neighborhood of $H(p)$. This implies that $W^s_\varepsilon(f^{n_j}(z_k))$ is well defined for all j. Then, for $j > j'$ sufficiently big $W^s_\varepsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, w_k]) \neq \emptyset$. On the other hand, by Corollary 3.6, for infinitely many points $p_{k'}$ we have $W^s_\varepsilon(f^{n_j}(p_{k'})) \to W^s_\varepsilon(f^{n_j}(x))$ and this implies that $W^s_\varepsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, w_k]) \in H(p)$. Hence, the definition of w_k implies that $W^s_\varepsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, y_k]) = \emptyset$ and so $W^s_\varepsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([y_k, w_k]) \neq \emptyset$, implying that there is $u \in H(p) \cap f^{n_{j'}}([y_k, w_k])$, leading to a contradiction with the definition of w_k.

If $\ell(f^n([z_k, w_k])) > \delta_0$ for some $n \geq 0$ then we pick $w_{k(n)} \in [y_k, w_k]$ so that $\ell(f^n([z_k, w_{k(n)}])) \leq \delta_0$ and repeat the same argument replacing w_k by $w_{k(n)}$. □

Since $I_0 = [x,y]$ and $y_k \to y$ we get

Corollary 3.22. $y \in H(p)$.

Returning to the proof of Theorem 3.11, and keeping the previous notation, take $2(\varepsilon+\delta) < \alpha$, where α is the expansiveness constant for $f/H(p)$. Then, since $z_k, y_k \in W^u_{\varepsilon+\delta}(p_k)$ we obtain

$$\operatorname{dist}(f^n(z_k), f^n(y_k)) \leq \ell(f^n([z_k,y_k])) \leq 2(\varepsilon+\delta) < \alpha \quad \forall n \leq 0\,. \tag{11}$$

On the other hand, since $z_k \in W^s_\varepsilon(x)$ and $y_k \in W^s_\varepsilon(y)$ we have that

$$\operatorname{dist}(f^n(z_k), f^n(y_k)) \leq \tag{12}$$
$$\operatorname{dist}(f^n(z_k), f^n(x)) + \operatorname{dist}(f^n(x), f^n(y)) + \operatorname{dist}(f^n(y), f^n(y_k)) < \varepsilon+\delta+\varepsilon < \alpha \quad \forall n \geq 0\,.$$

Equations (11) and (12) give

$$\operatorname{dist}(f^n(z_k), f^n(y_k)) \leq \alpha \quad \forall\, n \in \mathbb{Z}\,,$$

contradicting the expansiveness of $f/(H(p)$.

All together finishes the proof of Theorem 3.11. □

4. Proof of hyperbolicity of $H(p)$.

In this section we prove that $(E \oplus F)/H(p)$ is hyperbolic. For this, we shall prove first that E is uniformly contracting, and this is done following Mañé's proof of [Ma2, Theorem I.4].

Lemma 4.1. *There exist $c > 0$, a positive integer m, and a dense subset $\mathcal{D}$ of $H(p)$ such that for $x \in \mathcal{D}$ we have*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^m/E(f^{-jm}(x))\| \leq -c\,.$$

Proof. We have that $Per(f) \cap H(p)$ is dense in $H(p)$. By Lemma 2.1 we have for any periodic point $q \in H(p) \cap Per(f)$:

$$\|Df^{\tau n}|_{E^s(q)}\| \leq K\lambda^{\tau n}, \forall n \geq 0. \tag{13}$$

Here τ is the period of q and $K > 0$, $0 < \lambda < 1$ are independent of the particular periodic point in $H(p)$. Moreover by [Ma3, Lemma II.5] there exists $m > 0$, $C > 0$, and μ, $0 < \lambda \leq \mu < 1$ such that

$$\prod_{j=1}^{[\tau/m]} \|Df^m_{f^{jm}(q)}|_{E^s}\| \leq C\mu^{[\tau/m]}\,, \tag{14}$$

whenever (13) holds. Here $[\tau/m]$ is the greatest integer less or equal than τ/m.
Combining (13) and (14) and taking logarithms we get

$$\frac{1}{[\tau/m]} \sum_{j=1}^{[\tau/m]} \log \|Df^m/E(f^{jm}(q))\| \leq \frac{\log C}{[\tau/m]} + \log \mu\,. \tag{15}$$

Since $f/H(p)$ is expansive, the number of periodic points of bounded period is finite. Then, letting $\mathcal{D}$ be the set of periodic points of period sufficiently greater than m we have that $\mathcal{D}$ is dense in $H(p)$. Equation (15) together with the fact that $0 < \mu < 1$ imply the result. □

Lemma 4.1 ensures that for $x \in \mathcal{D}$, $\prod_{j=1}^{n} \|Df^m/E(f^{-jm}(x))\|$ converges to 0 exponentially fast. Since if $f : H(p) \to H(p)$ is expansive then the same is true for f^m, we may assume (and do) that $m = 1$ in Lemma 4.1.

Let γ_0 be such that $0 < e^{-c} < \gamma_0 < 1$ where $c > 0$ is given by Lemma 4.1. It follows that for all $x \in \mathcal{D}$ there are infinitely many values of n satisfying

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| < \gamma_0^n .$$

Take $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1 .$$

Let $N_0 = N(\gamma_3, \gamma_4)$ be given by Lemma 3.4.

Lemma 4.2. *If $E_{|H(p)}$ is not a contracting bundle then for all $\varepsilon > 0$ there exist a compact invariant set $\Lambda(\varepsilon) \subset H(p)$ and $N = N(\varepsilon)$ such that every $x \in \Lambda(\varepsilon)$ has the following property: there exist x_0 arbitrarily near x, $n_0 \geq 0$ and $y \in \Lambda(\varepsilon)$ such that* $\mathrm{dist}(f^{n_0}(x_0), y) < \varepsilon$, *$(y, f^n(y))$ is an (N_0, γ_2)-obstruction for all $n \geq N = N(\varepsilon)$ and if $n_0 > 0$ then $(x_0, f^{n_0}(x_0))$ is a uniform γ_4-string. Moreover, $\Lambda(\varepsilon)$ is the closure of its interior.*

Proof. See [Ma2, Lemmas II.6 and II.7]. □

Fix $0 < \gamma < \gamma_0$ and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be as in Lemma 4.2. Choose $k_0 \in (0, 1)$ such that $\gamma < k_0^2 \gamma_1$ and $k_0^{-1} \gamma_4 < 1$, i.e., $1 > k_0 > \max\{\gamma_4, \sqrt{\gamma/\gamma_1}\}$.

Proposition 4.3. *If E is not contracting, for all $\varepsilon > 0$ there exist sequences $\{x_i\} \subset \Lambda(\varepsilon/4)$, ($\Lambda(\varepsilon)$ as in Lemma 4.2) and $n_i > 0$ such that*

(a) $\mathrm{dist}(f^{n_i}(x_i), x_{i+1}) < \varepsilon$
(b) *$(x_i, f^{n_i}(x_i))$ is a uniform γ_4-string but if i is even $(x_i, f^{n_i}(x_i))$ is not a γ_1-string.*
(c) *If $K = min\{\|Df/E(x)\| ; x \in H(p)\}$ then $\gamma_1^{n_i} K^{n_{i-1}} \geq (k_0\gamma_1)^{n_i + n_{i+1}}$ for all even i.*
(d) *There exist odd numbers k, l; $k > l$ such that* $\mathrm{dist}(x_l, x_k) < \varepsilon/2$.

Proof. See [Ma2, page 179] for the proof of (a), (b), (c). Item (d) follows from compactness. □

Corollary 4.4. *Given $\varepsilon > 0$ there are a sequence $x_1, x_2, \ldots, x_k$ and uniform γ_4-strings $(x_i, f^{n_i}(x_i))$ that are not γ_1-strings, so that* $\mathrm{dist}(x_i, x_{i+1}) < \varepsilon$ *and* $\mathrm{dist}(f^{n_k}(x_k), x_1) < \varepsilon$ *for all $1 \leq i \leq, k-1$.*

Let as find $\varepsilon_0 > 0$ such that the cones defining the dominated splitting in $H(p)$ can be extended to $\mathcal{V}$ the ε_0-neighborhood of $H(p)$.

Lemma 4.5. *Let us assume that $(x, f^n(x))$ is a uniform γ string, $0 < \gamma < 1$, $n > 0$, $x \in H(p)$. Then there are $\delta > 0$, $\varepsilon > 0$ and $\eta > 0$ such that if for some constant $c > 0$,* $\mathrm{diam}(f^n(W_c^{\sigma,u}(x))) \geq 2\eta$ *for $n > n_0$ but* $\mathrm{diam}(f^j(W_c^{u,\sigma}(x))) < \varepsilon$; *$j = 0, \ldots, n-1$ then* $\mathrm{diam}(f^n(W_c^{\sigma,cu}(y))) \geq \nu$, *for all $y \in W_\delta^{cs}(x)$, $\sigma = +, -$.*

(Roughly speaking, if Df contracts in $E(f^j(x)); j = 1, \ldots, n$ and in addition $f^j(W_c^{u,\sigma}(x))$ growths in diameter in n iterates then the local center-unstable manifold of points in $W_\delta^{cs}(x)$ have to have the same property.)

Proof. Let $1 > \gamma_1$ and let $\varepsilon_1 > 0$, $\varepsilon_1 < \varepsilon_0$ be so small that if $(z, f^n(z))$ is a uniform γ string then $\mathrm{dist}(f^j(z), f^j(y)) \leq \gamma_1^j \mathrm{dist}(z,y)$ for some $1 > \gamma_2 > \gamma_1$ and for all $y \in W^{cs}_{\varepsilon_1}(z)$. Let ε, η be such that $\eta < \varepsilon/2 < \varepsilon < \varepsilon_1/8$ and if $\mathrm{dist}(z,w) \leq \varepsilon_1 + 2\eta$ then

$$(1-d) < \frac{\|Df|_{E(w)}\|}{\|Df|_{E(z)}\|} < (1+d) \quad \text{and} \quad (1-d) < \frac{\|Df|_{F(w)}\|}{\|Df|_{F(z)}\|} < (1+d)\,. \tag{16}$$

We choose $d > 0$ such that $\gamma_2 = (1+d)\gamma_1 < 1$ and also choose $\delta < \eta/2$. Arguing by contradiction assume that for some $y \in W^{cs}_\delta(x)$ and moreover that for all $j = 0,1,\dots,n-1$ we have that $\mathrm{diam}(f^j(W^{cu,\sigma}_c(y))) < 2\varepsilon$ and $\mathrm{diam}(f^n(W^{cu,\sigma}_c(y))) < \eta$. Hence $\mathrm{dist}(f^j(x), f^j(y')) < 2\varepsilon + \delta$ for all $j = 0,\dots,n$ and all $y' \in W^{cu,\sigma}_c(y)$. Thus there exists $d > 0$ such that if (16) holds then $(y', f^n(y'))$ is a uniform γ_1 string and therefore $\mathrm{dist}(f^j(z'), f^j(y') \leq \gamma_2^j\delta$ for all $z' \in W^{cs}_\delta(y')$. Moreover by (1), for all $x' \in W^{cu,\sigma}_c(x)$ there is $y' \in W^{cu,\sigma}_c(y)$ such that $x' \in W^{cs}_\delta(y')$. Then,

$$\mathrm{dist}(f^n(x), f^n(x')) \leq$$
$$\mathrm{dist}(f^n(x), f^n(y)) + \mathrm{dist}(f^n(y), f^n(y')) + \mathrm{dist}(f^n(y'), f^n(x')) < \delta\gamma_2^n + \eta + \delta\gamma_2^n < 2\eta$$

contradicting the hypothesis.
On the other hand, if for some $0 < j < n$ we have that $\mathrm{diam}(f^j(W^{cu,\sigma}_c(y))) > 2\varepsilon$, choosing J the minimum positive integer with such property, we have, taking a point $y' \in W^{cu,\sigma}_c(y)$ such that $\mathrm{dist}(f^h(y), f^h(y')) < 2\varepsilon$, $0 \leq h < J$ and $\mathrm{dist}(f^J(y), f^J(y')) = 2\varepsilon$, and taking $x' \in W^{u,\sigma}_c(x)$ as above, that

$$\mathrm{dist}(f^J(x), f^J(x') \geq \mathrm{dist}(f^J(y), f^J(y')) -$$
$$-[\mathrm{dist}(f^J(y), f^J(x)) + \mathrm{dist}(f^J(y'), f^J(x'))] > 2(\varepsilon - \delta] > \varepsilon\,.$$

Thus contradicting our assumption about $W^{u,\sigma}_c(x)$ □

The following proposition is similar to a lemma proved in [Li] (see also [Ma2]). The proof there uses that $Df|F$ is uniformly expanding. We, instead, use that the diameter of the unstable manifolds are uniformly bounded away from zero (Proposition 3.12) which is the main result obtained assuming Theorem 3.11.

Proposition 4.6. *Given $\delta > 0$, $0 < \gamma < 1$ and $(x_i, f^{n_i}(x_i))$ a sequence of uniform γ-strings in $H(p)$, $i = 1,\dots k$, then there exist $\mu = \mu(\gamma,\delta) > 0$ and $N_0 > 0$ such that if* $\mathrm{dist}(x_i, x_{i+1}) < \mu$ *and* $\mathrm{dist}(f^{n_k}(x_k), x_1) < \mu$ *for all $i = 1,2,\dots,k-1$ and $n_1 + n_2 + \cdots + n_k \geq N_0$ then there exists a periodic point q of f with period $N = n_1 + n_2 + \cdots + n_k$ such that* $\mathrm{dist}(f^n(q), f^n(x_1)) < \delta$ *for all $0 \leq n \leq n_1$ and setting $N_i = n_1 + \cdots + n_i$* $\mathrm{dist}(f^{N_i+n}(q), f^n(x_{i+1})) < \delta$ *for all $0 \leq n \leq n_{i+1}$, $1 \leq i \leq k-1$.*

Proof. Let us first assume $k = 1$. Let $x \in H(p)$ and $W^{cs}_\varepsilon(x)$ be the local invariant manifold given by the dominated splitting which is an embedded disk transverse to F. It is proved in [PS2] that for any $\varepsilon > 0$ there is $r = r(\varepsilon) > 0$ such that the size of $W^{cs}_\varepsilon(x)$ (see Definition 3.1), $\mathrm{size}(W^{cs}_\varepsilon(x)) \geq r$. Moreover, if $x = x_1$ then the fact that $Df_{|E}$ contracts n_1 iterates implies that $W^{cs}_\varepsilon(x_1)$ behaves as a stable manifold for that iterates. That is, there is γ_1, $\gamma < \gamma_1 < 1$ such that for $j = 0,1,\dots,n_1$, if $y \in W^{cs}_\varepsilon(x_1)$ then $\mathrm{dist}(f^j(x_1), f^j(y) < \gamma_1^j\varepsilon$. As usual we choose $\varepsilon > 0$ such that $\varepsilon < \alpha/2$, where $\alpha > 0$ is a constant of expansiveness. On the other hand, by Proposition 3.12, it holds that there is $\eta > 0$ such that $\mathrm{diam}(W_\varepsilon^{u,\sigma}(x_1)) \geq 2\eta$ where $\sigma = +,-$ indicates anyone of the separatrices of $W_\varepsilon^u(x_1)$. Let us consider local center-unstable manifolds of the points $x \in D_1 \subset W_\varepsilon^{cs}(x_1)$ where D_1 is a 2-disk centered in x_1. These center-unstable manifolds are coherent with the local unstable manifolds of points of $H(p) \cap W_\varepsilon^{cs}(x_1)$. By Lemma 3.14 for all $c_0 > 0$ there exists $N_0 > 0$ such that $\mathrm{diam}(f^n(W_c^u(y))) > 2\eta$ for all $y \in H(p)$, $c \geq c_0$ and $n \geq N_0$. Hence, by Lemma 4.5 we have that $\mathrm{diam}(f^{n_1}(W_c^{cu,\sigma}(x))) \geq \eta$ because $\mathrm{diam}(f^{n_1}(W_c^{u,\sigma}(x_1))) \geq 2\eta$.

Let $\mathcal{B}_1$ be a cylinder centered in x_1 given by $\mathcal{B}_1 = \cup_{x\in D_1} U_x$ where $U_x \subset W_\varepsilon^{cu}(x)$ has diameter η and is centered in x. Let $\mathcal{C}_1 \subset \mathcal{B}_1$ be defined by

$$\mathcal{C}_1 = \cup_{x\in D_1}(W_c^{cu}(x)).$$

Take $\mu > 0$ small enough such that when $\mathrm{dist}(f^{n_1}(x_1), x_1) < \mu$ then $f^{n_1}(W_\varepsilon^{cs}(x_1))$ is contained in the interior of $\mathcal{B}_1$ and moreover $W_\varepsilon^{cu}(x)$ cuts $W_\varepsilon^{cs}(y)$ whenever $\mathrm{dist}(x,y) < \mu$ with $x, y \in \mathcal{V}$. Hence $f^{n_1}(\mathcal{C}_1)$ intersects $\mathcal{C}_1$ and any point in the boundary of $\mathcal{C}_1$ is not fixed by f^{n_1}. Therefore, by a standard argument of index theory, see [Do], there exist a fixed point q of f^{n_1} in $\mathcal{C}_1 \cap f^{n_1}(\mathcal{C}_1)$. That is, q is a periodic point of f. Observe that since $q \in \mathcal{V}$ there is z in $W_\varepsilon^{cu}(q) \cap f^{n_1}(D_1)$. Moreover, the distance between $f^{-j}(z)$ and $f^{-j}(q)$ is bounded by ε for all $0 \leq j \leq$ the period of q. On the other hand $\mathrm{dist}(f^{-j}(z), f^{n_1-j}(x_1)) < \mathrm{diam}(\gamma_1^{n_1-j} r)$ where $\gamma_1^{n_1-j} r = \mathrm{diam}(f^{n_1-j}(D_1))$. Therefore $\mathrm{dist}(f^j(q), f^j(x_1)) \leq \varepsilon + r$ for all $j = -1, -2, \ldots - n_1$. Choosing $r < \varepsilon < \delta/2$ we conclude that the orbit of q δ-shadows the orbit of x_1 for all $j = 1, 2, \ldots n_1$, proving Proposition 4.6 when $k = 1$.

For $k = 2$ we proceed as follows. Take a small disk $D_2 \subset W_\varepsilon^{cs}(x_2)$ and $\mathcal{B}_1$ as in the previous case and set $\mathcal{B}_2 = \cup_{x\in D_2}(W^{cu}(x))$ and $\mathcal{C}_2 = \cup_{x\in D_2} W_\beta^{cu}(x)$, where β is such that $\mathrm{diam}(f^{n_2}(W_\beta^{cu}(x))) > \eta$. Find $\mu > 0$ such that $\mathrm{dist}(f^{n_1}(x_1), x_2) < \mu$ and $\mathrm{dist}(f^{n_2}(x_2), x_1) < \mu$ imply $f^{n_1}(D_1) \subset \mathrm{int}(\mathcal{B}_2)$ and $f^{n_2}(D_2) \subset \mathrm{int}(\mathcal{B}_1)$, where $\mathrm{int}(A)$ stands for the interior of A.

Then $f^{n_2}(f^{n_1}(\mathcal{C}_1) \cap \mathcal{C}_2)$ is a small cylinder that cuts $\mathcal{C}_1$ and no point of the boundary of $\mathcal{C}_1 \cap f^{-n_1}(\mathcal{C}_2)$ is fixed by $f^{n_1+n_2}|_{(\mathcal{C}_1 \cap F^{-n_1}(\mathcal{C}_2))}$. Hence, arguing as before, there is a fixed point q of $f^{n_1+n_2}$ and reasoning as in the case $k = 1$ we conclude the proof for $k = 2$.

The general case follows by induction. □

Lemma 4.7. *Let $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1$ be as in Lemma 4.2. Moreover let $0 < \gamma < \gamma_0$ and take $0 < k_0 < 1$ such that $\gamma < k_0^2\gamma_1$ and $k_0^{-1}\gamma_4 < 1$. Then the periodic point q given by Lemma 4.6 satisfies*

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^N.$$

Proof. The proof is given in detail in [Ma2, Section II], pages 179-181 (the reader should be aware that in Mañé's article our γ_3 is denoted $\overline{\gamma_2}$ and γ_4 is denoted by γ_3). □

Proposition 4.8. *We may construct q given in Proposition 4.6 such that $q \in H(p)$.*

Proof. By Proposition 4.7 if $\delta > 0$ is small enough we ensure that $Df_q/E(q)$ contracts in a rate similar to that of $Df_{x_j}/E(x_j)$, $j = 1, 2, \ldots, k$ if $\mathrm{dist}(f^j(q), f^j(x_j)) < \delta$ for all $j = 0, \ldots, N$, in the local center-stable manifold of q. More precisely we have

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\|.$$

And for all $j \in [1,N]$ we have

$$\prod_{n=1}^{j} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^j.$$

As q is periodic of period N this implies that its center-stable manifold is a stable manifold, of size about the same of that of the center-stable manifold of x_j. Hence the local unstable manifold of a periodic point $q' \in H(p)$ close to x_1 intersects the local stable manifold of q. Therefore $W^u(p)$ accumulates in $W^u(q)$ or there would exist, by [PS, Lemma 3.3.1] another periodic point $\hat{q}$ in the unstable separatrix of $W^{cu}_\varepsilon(q)$ cutting $W^{cs}_\varepsilon(q')$ and between q and $w \in W^{cu}_\varepsilon(q) \cap W^{cs}_\varepsilon(q')$. Let Q be the periodic point closest to $H(p)$, in the linear ordering from q to w. Then $W^u_\varepsilon(Q)$ is accumulated by $W^u(q')$ and therefore also by $W^u(p)$. If we have that for some $j = 1,\dots,k$, $i = 0,\dots,n_j$, $W^{cs}_\varepsilon(f^i(x_j))$ is a true stable manifold then we are done. For in that case $W^s(p)$ would accumulate in $W^s(Q)$ and $W^u(p)$ in $W^u(Q)$ respectively and we obtain that $Q \in H(p)$, concluding the proof of Proposition 4.8.

As we cannot assume that, we have to proceed in a different way. Choose $\mathcal{V}_l(H(p))$, $l \geq 1$, a sequence of admissible neighborhoods of $H(p)$ such that

$$\mathrm{Cl}(\mathcal{V}_{l+1}(H(p))) \subset \mathcal{V}_l(H(p)) \text{ and } \cap_{l\geq 1} \mathcal{V}_l(H(p)) = H(p)\,.$$

For any $\mathcal{V}_l(H(p))$ we may find q_l, like the point Q as above, such that q_l shadows the pseudo-orbit given by a sequence $(x_j, f^{n_j}(x_j))$, $j = 1,\dots,k_l$ of uniform γ_4-strings that are not γ-strings as in Proposition 4.3. Take an accumulation point x of $\{q_l\}$. If there is a subsequence $\{q_{l_h}\}$ of $\{q_l\}$ such that the periods of q_{l_h} are uniformly bounded then $x \in H(p) \cap \mathrm{Per}(f)$ and x is the desired periodic point. Otherwise the periods of q_l are unbounded and $x \in H(p)$ is a uniform γ_4-string for all $n \geq 0$. Therefore by arguments similar to those used in [PS, Corollary 3.3] we obtain a stable manifold for x that the unstable manifold of q_l will cut for $l \geq l_0$. Thus q_l is homoclinically related with p completing the proof. $\square$

Proof of Theorem B

By Propositions 4.6, 4.8, and Lemma 4.7, if E is not a contracting sub-bundle there exists $q \in H(p) \cap \mathrm{Per}(f)$ such that Df_q contracts in E rather weakly in the period contradicting Lemma 2.1. Therefore E is a uniform contracting bundle. This in turn implies, using the arguments of [Ma3, section II], that F is a uniform expanding bundle. Thus $E \oplus F$ is a hyperbolic splitting. This implies that the same is true for g C^1- close to f. $\square$

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