On the periodic Schrödinger-Debye equation

Alexander Arbieto and Carlos Matheus*

November 24, 2005

Abstract

We study local and global well-posedness of the initial value problem for the Schrödinger-Debye equation in the *periodic case*. More precisely, we prove local well-posedness for the periodic Schrödinger-Debye equation with subcritical nonlinearity in arbitrary dimensions. Moreover, we derive a new *a priori* estimate for the H^1 norm of solutions of the periodic Schrödinger-Debye equation. A novel phenomena obtained as a by-product of this *a priori* estimate is the global well-posedness of the periodic Schrödinger-Debye equation in dimensions $1, 2$ and 3 *without* any smallness hypothesis of the H^1 norm of the initial data in the "focusing" case.

1 Introduction

The main theme of this paper is the well-posedness of the initial value problem (IVP) for the *Schrödinger-Debye equation* (SDE):

$$\begin{cases} i\partial_t u + \Delta u = uv, & t \geq 0, \quad x \in \mathbb{T}^n, \\ K\partial_t v + v = \varepsilon |u|^\alpha, & \\ u(0,x) = u_0(x), & v(0,x) = v_0(x), \end{cases} \tag{1}$$

where $\alpha = p-2 > 0$, u is a complex-valued function, v is a real-valued function, $K > 0$, $\varepsilon = \pm 1$ and Δ is the Laplacian operator in the x-variable.

This equation appears naturally in certain *nonlinear optics* phenomena. Indeed, the equation (1) is obtained from the *Maxwell-Debye system*

$$\begin{cases} i\partial_t A + \frac{c}{k\eta_0}\Delta A = \frac{\omega_0}{\eta_0}\nu A, \\ K\partial_t \nu + \nu = \eta_2 |A|^\alpha, \end{cases}$$

via the rescaling

$$\begin{aligned} u(t,x) &= \sqrt{\tfrac{\omega_0|\eta_2|}{\eta_0}}A(t, \sqrt{\tfrac{c}{k\eta_0}}x), \\ v(t,x) &= \tfrac{\omega_0}{\eta_0}\nu(t, \sqrt{\tfrac{c}{k\eta_0}}x). \end{aligned}$$

*The authors were partially supported by Faperj-Brazil.

Physically, the Maxwell-Debye system (with $\alpha = 2$) arises in nonlinear optics describing the non-resonant delayed iteraction of an electromagnetic wave with a certain media. In this system, A denotes the envelope of a light wave that travels through a media. The wave induces a change ν of the refractive index in the material (initially η_0 for an electromagnetic wave of frequency ω_0) with a slight delay K. The parameter η_2 is related with the magnitude and the sign of the coupling of the wave and the matter. Finally, c is the light velocity in the vacuum and k is the wave vector of the incident electromagnetic wave. See [CL] and references therein for discussions of this model.

Mathematically, the well-posedness of the IVP (1) *in the non-periodic case* (i.e., $x \in \mathbb{R}^n$) was recently studied by Bidégaray [Bi1], [Bi2] and Corcho, Linares [CL].

The results proved by Bidégaray, roughly speaking, were local well-posedness in $L^2(\mathbb{R}^n)$ for data $u_0, v_0 \in L^2(\mathbb{R}^n)$ and local well-posedness in $H^1(\mathbb{R}^n)$ for data $u_0, v_0 \in H^1(\mathbb{R}^n)$ $(n = 1, 2, 3)$, although the persistence property was not obtained.

More recently, Corcho and Linares [CL], making an optimal use of *Strichartz's inequalities* for the linear Schrödinger operator, were able to improve Bidégaray's results (and, in fact, obtain new ones).

The strategy used by Bidégaray and Corcho, Linares was a combination of the Strichartz inequality and a fixed point argument.

In this paper we apply the same strategy to the IVP (1), namely, use a fixed point argument and Strichartz inequality in the periodic case. However, there is an extra difficulty in our case because the exact analogue of the Strichartz inequality does not hold in the torus $\mathbb{T}^n$, and Strichartz-like inequalities may only holds locally in time.

The idea to overcome this is to use the work of Bourgain [B], where the correct analogue of Strichartz's inequality was found and applied to the nonlinear periodic Schrödinger equation (again by a fixed point argument).

In order to apply the fixed point method to solve the Schrödinger-Debye equation, we start by decoupling the equation (1) to obtain the integral formulation:

$$v(t) = e^{-t/K} v_0(x) + \frac{\varepsilon}{K} \int_0^t e^{-(t-\tau)/K} |u(\tau)|^\alpha d\tau, \tag{2}$$

$$u(t) = U(t)u_0 - i \int_0^t U(t-\tau) w(\tau) d\tau, \tag{3}$$

with $U(t) = e^{it\Delta}$, $w(t) = F_0(u)(t) + F_1(u)(t)$, where

$$F_0(u) = e^{-t/K} u v_0 \quad \text{and} \quad F_1(u) = \frac{\varepsilon}{K} u \int_0^t e^{-(t-\tau)/K} |u(\tau)|^\alpha d\tau.$$

In this setting, we show the following local well-posedness results:

Theorem A $(n = 1)$**.** *The SDE (1) with cubic nonlinearity (i.e., $\alpha = 2$) is locally well-posed for $H^s \times H^s$ initial data for any $s \geq 0$. Also, the SDE (1) is locally well-posed for $H^s \times H^s$ initial data when*

- *either $s > 0$ and $\alpha \leq 4$, or*
- *$s > s_*$ and $\alpha = \frac{4}{1-2s_*} > 6$.*

Theorem B $(n = 2)$**.** *The SDE (1) with cubic nonlinearity (i.e., $\alpha = 2$) is locally well-posed for $H^s \times H^s$ initial data with $s > 0$. Also, the SDE (1) is locally well-posed for $H^s \times H^s$ initial data with $2 \leq \alpha < \frac{2}{1-s}$.*

Theorem C $(n = 3)$**.** *The SDE (1) with cubic nonlinearity (i.e., $\alpha = 2$) is locally well-posed for $H^s \times H^s$ initial data with $s \geq 1$. Also, the SDE (1) is locally well-posed for $H^s \times H^s$ initial data with $2 \leq \alpha < \frac{4}{3-2s}$.*

Theorem D $(n \geq 4)$**.** *The SDE (1) is locally well-posed for $H^s \times H^s$ initial data with $2 \leq \alpha < \frac{4}{n-2s}$.*

From these local well-posedness results, the conservation of the L^2 norm of u and an *a priori* H^1 estimate we obtain the following global well-posedness results:

Theorem E $(n = 1)$**.** *The SDE (1) with cubic nonlinearity $\alpha = 2$ is globally well-posed for initial data in $H^s \times H^s$ with $s \geq 0$. Also, the SDE (1) is globally well-posed for $H^1 \times H^1$ initial data if $\alpha \geq 1$.*

Theorem F $(n = 2)$**.** *The SDE (1) with cubic nonlinearity $\alpha = 2$ is globally well-posed for initial data in $H^s \times H^s$ with $s \geq 1$. Also, the SDE (1) is globally well-posed for $H^1 \times H^1$ initial data if $\alpha \geq 2$.*

Theorem G $(n = 3)$**.** *The SDE (1) with cubic nonlinearity $\alpha = 2$ is globally well-posed for initial data in $H^s \times H^s$ with $s \geq 1$. Also, the SDE (1) is globally well-posed for $H^1 \times H^1$ initial data if $2 \leq \alpha < 3$.*

Remark 1.1. A direct comparision with the global well-posedness theory of the periodic nonlinear Schrödinger equation in the focusing setting [B] reveals a novel phenomena in the global well-posedness in the "focusing case" (i.e., $\varepsilon = -1$) of the SDE (1). Indeed, since the Hamiltonian of the focusing NLS do not control the H^1 norm of the solutions, we need some smallness assumptions of the H^1 norm of the initial data in order to derive global wel-posedness theorems. On the other hand, the structure of the nonlinear term of the SDE (1) allows us to conclude the same global well-posedness results for the SDE *without any smallness hypothesis.* This subtle difference between the NLS and the SDE occurs because the evolution of v in the SDE permits to derive an a priori estimate for the H^1 norm of u, although we do not have conserved Hamiltonians.

We close the introduction with the scheme of this paper. In section 2, we revisit the restriction of the Fourier transform method of Bourgain. In particular, we recall the definition of the Bourgain spaces $X^{s,b}$ and some of its properties. Also, we revisit the Strichartz type estimates which are the basic tools to deal with the nonlinear terms of the SDE. In section 3, we prove the local well-posedness results in the theorems A, B, C and D. In section 4, we derive an a priori estimate for the H^1 norm. By standar arguments, this implies the global well-posedness theorems E, F and G. Finally, we briefly discuss some questions related to the well-posedness results in this paper.

2 Preliminaries

This section is devoted to introduce the reader to the setting of Bourgain [B].

2.1 Restriction of the Fourier transform

The first main ingredient of the proofs of our results is Bourgain's technique of restriction of the Fourier transform below.

As Bourgain [B, p. 136], we are going to find a solution u of the Schrödinger-Debye equation (1) which is local in time, that is, take a function $0 \leq \psi_1 \leq 1$ such that $\mathrm{supp}(\psi_1) \subset [-2\delta, 2\delta]$, $\psi_1 \equiv 1$ on $[0, \delta]$. In the sequel $n := d-1$.

Inspired by equation (3), our goal is to construct a function u satisfying

$$u(t) = \psi_1(t) U(t) u_0 - i \psi_1(t) \int_0^t U(t-\tau) w(\tau) d\tau. \tag{4}$$

If we write u_0, u, w as Fourier series

$$u_0(x) = \sum_{\xi \in \mathbb{Z}^{d-1}} \hat{u_0}(\xi) e^{2\pi i <x,\xi>}$$

$$u(x,t) = \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i <x,\xi>} \int_{-\infty}^{\infty} e^{2\pi i \lambda t} \hat{u}(\xi, \lambda)\ d\lambda$$

$$w(x,t) = \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i <x,\xi>} \int_{-\infty}^{\infty} e^{2\pi i \lambda t} \hat{w}(\xi, \lambda)\ d\lambda.$$

Then the integral equation (4) is

$$u(x,t) = \psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} \hat{u_0}(\xi) e^{2\pi i (<x,\xi> + t|\xi|^2)} + \tag{5}$$

$$+ \frac{1}{2\pi} \psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i (<x,\xi> + t|\xi|^2)} \int_{-\infty}^{\infty} \frac{e^{2\pi i (\lambda - |\xi|^2) t} - 1}{\lambda - |\xi|^2} \hat{w}(\xi, \lambda) d\lambda,$$

We denote by Φ the map defined by (5), i.e.,

$$\Phi(u)(t,x) = \psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} \hat{u_0}(\xi) e^{2\pi i (<x,\xi> + t|\xi|^2)} +$$

$$+\frac{1}{2\pi}\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i(<x,\xi>+t|\xi|^2)} \int_{-\infty}^{\infty} \frac{e^{2\pi i(\lambda-|\xi|^2)t}-1}{\lambda-|\xi|^2} \hat{w}(\xi,\lambda) d\lambda,$$

Consider a function ψ_2 such that $0 \leq \psi_2 \leq 1$, $\psi_2 = 1$ on $[-1,1]$ and supp $\psi_2 \subset [-2,2]$.

If we write,

$$\psi_1(t) \int_{-\infty}^{\infty} \frac{e^{2\pi i(\lambda-|\xi|^2)t}-1}{\lambda-|\xi|^2} \hat{w}(\xi,\lambda) d\lambda =$$

$$\sum_{k\geq 1} \frac{(2\pi i)^k}{k!} \psi_1(t) t^k \int \psi_2(\lambda-|\xi|^2)(\lambda-|\xi|^2)^{k-1} \hat{w}(\xi,\lambda) d\lambda$$

$$+\psi_1(t) \int (1-\psi_2(\lambda-|\xi|^2)) \frac{e^{2\pi i(\lambda-|\xi|^2)t}}{\lambda-|\xi|^2} \hat{w}(\xi,\lambda) d\lambda$$

$$-\psi_1(t) \int (1-\psi_2(\lambda-|\xi|^2)) \frac{\hat{w}(\xi,\lambda)}{\lambda-|\xi|^2} d\lambda.$$

then the right hand side of (5) is controled by the contributions

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} \hat{u_0}(\xi) e^{2\pi i(<x,\xi>+t|\xi|^2)}; \tag{6}$$

and

$$\sum_{k\geq 1} \frac{(2\pi i)^k}{k!} t^k \cdot \psi_1(t) \cdot \tag{7}$$

$$\left\{ \sum_{\xi} \left[\int \psi_2(\lambda-|\xi|^2)(\lambda-|\xi|^2)^{k-1} \hat{w}(\xi,\lambda)\ d\lambda \right] \cdot e^{2\pi i(<x,\xi>+t|\xi|^2)} \right\};$$

and

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i <x,\xi>} \int \frac{1-\psi_2(\lambda-|\xi|^2)}{\lambda-|\xi|^2} e^{2\pi i \lambda t} \hat{w}(\xi,\lambda)\ d\lambda; \tag{8}$$

and

$$\psi_1(t) \sum_{\xi \in \mathbb{Z}^{d-1}} e^{2\pi i(<x,\xi>+t|\xi|^2)} \int \frac{1-\psi_2(\lambda-|\xi|^2)}{\lambda-|\xi|^2} \hat{w}(\xi,\lambda)\ d\lambda. \tag{9}$$

We apply the Picard's fixed point method in the Bourgain spaces $X^{s,b}$ associated to the norm

$$\|f\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\lambda-\xi^2\rangle^b\widehat{f}(\xi,\lambda)\|_{L^2_{\xi,\lambda}}.$$

Remark 2.1. $X^{s,b}\subset L^\infty_t H^s$ for any $b>1/2$. Thus, we can use the $X^{s,b}$ spaces with $b>1/2$ to prove the well-posedness results.

The idea is to prove that the integral formulation of the SDE (1) is a contraction of a large ball in the $X^{s,b}$. Therefore, the main task is to estimate these four terms. Note that

$$\|(6)\|_{X^{s,b}} \leq \|u_0\|_{H^s}. \tag{10}$$

and

$$\|(7)\|_{X^{s,b}} + \|(8)\|_{X^{s,b}} + \|(9)\|_{X^{s,b}} \leq \|w\|_{X^{s,b'-1}} \tag{11}$$

for $b'>1/2$ and $b'\geq b$. Hence, it suffices to control the expression $\|w\|_{X^{s,b'-1}}$. To accoplish this, we revisit some properties of the Bourgain spaces.

Lemma 2.2. *We have*

$$\|\psi(t)f\|_{X^{s,b}} \lesssim_{\psi,b} \|f\|_{X^{s,b}}$$

for any $s,b\in\mathbb{R}$ and, furthermore, if $-1/2<b'\leq b<1/2$, then for any $0<T<1$ we have

$$\|\psi_T(t)f\|_{X^{s,b'}} \lesssim_{\psi,b',b} T^{b-b'}\|f\|_{X^{s,b}}.$$

Proof. First of all, note that $\langle\lambda-\lambda_0-|\xi|^2\rangle^b \lesssim_b \langle\lambda_0\rangle^{|b|}\langle\lambda-|\xi|^2\rangle^b$, from which we obtain

$$\|e^{it\lambda_0}f\|_{X^{s,b}} \lesssim_b \langle\lambda_0\rangle^{|b|}\|f\|_{X^{s,b}}.$$

Using that $\psi(t)=\int\widehat{\psi}(\lambda_0)e^{it\lambda_0}d\lambda_0$, we conclude

$$\|\psi(t)f\|_{X^{s,b}} \lesssim_b \left(\int|\widehat{\psi}(\lambda_0)|\langle\lambda_0\rangle^{|b|}\right)\|f\|_{X^{s,b}}.$$

Since ψ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s=0$ and, by composition it suffices to treat the cases $0\leq b'\leq b$ or $\leq b'\leq b\leq 0$. By duality, we may take $0\leq b'\leq b$. Finally, by interpolation with the trivial case $b'=b$, we may consider $b'=0$. This reduces matters to show that

$$\|\psi_T(t)f\|_{L^2} \lesssim_{\psi,b} T^b\|f\|_{X^{0,b}}$$

for $0<b<1/2$. Partitioning the frequency spaces into the cases $\langle\lambda-|\xi|^2\rangle\geq 1/T$ and $\langle\lambda-|\xi|^2\rangle\leq 1/T$, we see that in the former case we'll have

$$\|f\|_{X^{0,0}} \leq T^b\|f\|_{X^{0,b}}$$

and the desired estimate follows because the multiplication by ψ is a bounded operation in Bourgain's spaces. In the latter case, by Plancherel and Cauchy-Schwarz

$$\|f(t)\|_{L^2_x} \leq \|\widehat{f(t)}(\xi)\|_{L^2_\xi} \leq \left\| \int_{\langle \lambda - |\xi|^2 \rangle \leq 1/T} |\widehat{f}(\lambda, \xi)| d\lambda) \right\|_{L^2_\xi}$$

$$\leq_b T^{b-1/2} \left\| \int \langle \lambda - |\xi|^2 \rangle^{2b} |\widehat{f}(\lambda, \xi)|^2 d\lambda)^{1/2} \right\|_{L^2_\xi} = T^{b-1/2} \|f\|_{X^{s,b}}.$$

Integrating this against ψ_T concludes the proof of the lemma. $\square$

In order to keep a precise control of the nonlinear term w, we recall the Strichartz-type inequalities in the periodic setting derived in [B].

2.2 Some one-dimensional estimates

In the 1-dimensional case, specially for the cubic nonlinearity, the following Strichartz estimate will be useful:

Lemma 2.3. *It holds $X^{0,3/8} \subset L^4_{x,t}(\mathbb{T} \times [0,1])$. More precisely,*

$$\|f\|_{L^4(\mathbb{T} \times [0,1])} \leq c \|f\|_{X^{0,3/8}}.$$

Next we introduce the definition:

Definition 2.4. Let $d \geq 1$, $S \subset \mathbb{Z}^d$ and $p > 2$. We define $K_p(S)$ to be the smallest number such that

$$\left\| \sum_{\gamma \in S} a_\gamma e^{2\pi i < x, \gamma >} \right\|_{L^p(\mathbb{T}^d)} \leq K_p(S) \left(\sum |a_n|^2 \right)^{1/2}.$$

Also, when the nonlinearity is not cubic, we will use the following L^6-estimate:

Proposition 2.5. *If $S_N = \{(n, n^2) : |n| \leq N\}$ then*

$$K_6(S_N) < \exp c \frac{\log N}{\log \log N}. \tag{12}$$

In particular,

$$\| \sum_{n \in \mathbb{Z}} a_n e^{i(nx + n^2 t)} \|_{L^6(\mathbb{T}^2)} \ll N^\varepsilon \left(\sum |a_n|^2 \right)^{1/2}, \forall \ \ \varepsilon > 0. \tag{13}$$

Since this proposition is not difficult to show, we include a proof of it here.

Proof. Let $f = \sum_{n=1}^{N} a_n e^{i(nx+n^2t)}$. Then:

$$\|f\|_6 = \|f^3\|_2^2 = \sum_{n,j} \left| \sum_{n_1^2+n_2^2+(n-n_1-n_2)^2=j} a_{n_1} a_{n_2} a_{n-n_1-n_2} \right|^2.$$

Define $r_{n,j} = \#\{(n_1, n_2) : |n_i| \leq N,\ n_1^2 + n_2^2 + (n - n_1 - n_2)^2 = j\}$. We have:

$$\sum_{n,j} \left| \sum_{n_1^2+n_2^2+(n-n_1-n_2)^2=j} a_{n_1} a_{n_2} a_{n-n_1-n_2} \right|^2 \leq \max_{|n|\leq 3N, |j|\leq 3N^2} r_{n,j} \cdot \left(\sum_{|n|\leq N} |a_n|^2 \right)^3.$$

Hence, it remains to prove that $r_{n,j} < \exp c \frac{\log N}{\log\log N}$.

The condition $n_1^2+n_2^2+(n-n_1-n_2)^2 = j$ is $n_1^2+n_2^2-nn_1-nn_2+n_1n_2 = \frac{j-n^2}{2}$, i.e.,

$$\frac{3}{4}(n_1 + n_2)^2 + \frac{1}{4}(n_1 - n_2)^2 - n(n_1 + n_2) = \frac{j - n^2}{2}.$$

If we put $m_1 = n_1 + n_2, m_2 = n_1 - n_2$, then:

$$(3m_1 - 2n)^2 + 3m_2^2 = 6j - 2n^2,$$

which has the form $X^2 + 3Y^2 = A$, where $X, Y, A \in \mathbb{Z}$.

Put $\rho = e^{2\pi i/3} = \frac{1+i\sqrt{3}}{2}$. Then $X^2+3Y^2 = A$ if and only if $X+i\sqrt{3}Y$ divides A in $\mathbb{Z}+\rho\mathbb{Z}$. Since $\mathbb{Z}+\rho\mathbb{Z}$ is an Euclidean domain, the number of divisors of A is at most $\exp c\frac{\log A}{\log\log A} < \exp c\frac{\log N}{\log\log N}$.

Because $(3m_1 - 2n, m_2)$ defines (n_1, n_2), this concludes the proof. ☐

2.3 Some higher dimensional estimates

For positive integers K, N, consider the sets

$$\Lambda_{A,N} = \{\zeta = (\xi, \lambda) \in \mathbb{Z}^n \times \mathbb{R} : N \leq |\xi| < 2N, \quad A \leq |\lambda - |\xi|^2| < 2A\}.$$

For an interval I of $\mathbb{Z}^n$, we define

$$\Lambda_{K,I} = \{\zeta \in I \times \mathbb{R} : A \leq |\lambda - |\xi|^2| < 2A\}.$$

Definition 2.6. Given a function $u \in L^2(\mathbb{T}^n \times \mathbb{R})$,

$$u = \sum_{\xi\in\mathbb{Z}^n} \int d\lambda\ \hat{u}(\zeta) e^{2\pi i(<x,\xi>+\lambda t)},$$

we define

$$|||u||| = \sup_{A,N} (A + 1)^{1/2} (N + 1)^s \left(\int_{\Lambda_{A,N}} |\hat{u}(\zeta)|^2 d\zeta \right)^{1/2}. \tag{14}$$

Fix an interval $[-\delta,\delta]$. We will consider the restriction norm $|||u||| = \inf |||\widetilde{u}|||$, where the infimum is taken over all $\widetilde{u}$ coinciding with u on $\mathbb{T}^n \times [0,\delta]$.

Define $S_{d,N} = \{(n_1,\dots,n_{d-1},|\overline{n}|^2) : n_j \in \mathbb{Z}, |n_j| < N\}$, $\overline{n} = (n_1,\dots,n_{d-1})$ and $|\overline{n}|^2 = n_1^2 + \cdots + n_{d-1}^2$.

Definition 2.7. A number p is called an *admissible exponent* if

$$p \geq \frac{2(d+1)}{d-1} \quad \text{and} \quad K_p(S_{d,N}) \ll N^{\varepsilon} N^{\frac{d-1}{2} - \frac{d+1}{p}}, \tag{15}$$

Concerning the existence and the properties of admissible exponents we have three important results:

Proposition 2.8 (Proposition 3.6 of [B]). *For $n = 2,3,4$, the exponent 4 is admissible, i.e.,*

- $K_4(S_{3,N}) \ll N^{\varepsilon}$
- $K_4(S_{4,N}) \ll N^{\frac{1}{4}+\varepsilon}$
- $K_4(S_{5,N}) \ll N^{\frac{1}{2}+\varepsilon}$

Proposition 2.9 (Proposition 3.110 of [B]). *For $n \geq 4$, $p \geq \frac{2(n+4)}{n}$,*

$$K_p(S_{d,N}) < cN^{\frac{d-1}{2} - \frac{d+1}{p}}.$$

Proposition 2.10 (Proposition 3.113 of [B]). *If $p_2 > p_1 \geq p_0 = \frac{2(d+1)}{d-1}$ and $K_{p_1}(S_{d,N}) \ll N^{\frac{d-1}{2} - \frac{d+1}{p_1} + \varepsilon}$, then $K_{p_2}(S_{d,N}) \leq C_{p_2} N^{\frac{d-1}{2} - \frac{d+1}{p_1}}$.*

The reason for the introduction of admissible exponents is explained by the good properties (with respect to the Fourier transform) below.

Let p_0 be admissible. By proposition 2.10, for $p > p_0$,

$$\left\| \sum_{|\xi| \leq N} a_\xi e^{2\pi i(<x,\xi>+t|\xi|^2)} \right\|_{L^p(\mathbb{T}^d)} \leq N^{\frac{d-1}{2} - \frac{d+1}{p}} \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{1/2}.$$

Let I be a $(d-1)$-interval of size N in $\mathbb{Z}^{d-1}$ centered at ξ_0. Writing

$$< x,\xi > + t|\xi|^2 = < x,\xi_0 > + t|\xi_0|^2 + < x + 2t\xi_0, \xi - \xi_0 > + t|\xi - \xi_0|^2.$$

The change of variables $x' = x + 2t\xi_0$, $t' = t$ implies that also

$$\left\| \sum_{\xi \in I} a_\xi e^{2\pi i(<x,\xi>+t|\xi|^2)} \right\|_{L^p(\mathbb{T}^d)} \leq N^{\frac{d-1}{2} - \frac{d+1}{p}} \left(\sum_{\xi \in I} |a_\xi|^2 \right)^{1/2}.$$

It follows that (writing $\lambda = |\xi|^2 + k$, $|k| < A$) the map

$$L^2_{\Lambda_{A,I}} \longrightarrow L^p(\mathbb{T}^n \times \mathbb{R}_{loc}) \tag{16}$$

$$\{a_\zeta\}_{\zeta\in\Lambda_{A,I}} \to \int_{\Lambda_{A,I}} a_\zeta e^{2\pi i(<x,\xi>+t\lambda)} d\zeta$$

has norm bounded by $A^{1/2} N^{\frac{d-1}{2}-\frac{d+1}{p}}$.

Since the map (16) from $L^2_{\Lambda_{A,I}}$ to $L^2(\mathbb{T}^n \times \mathbb{R}_{loc})$ has also bounded norm, by interpolation we obtain the following lemma:

Lemma 2.11. *Let $p_1 > p_0,\ p_1 > p_2 > 2,\ \frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{2}$. Then the map (16) ranging into $L^{p_2}(\mathbb{T}^n \times \mathbb{R}_{loc})$ has norm bounded by*

$$A^{\frac{1}{2}(1-\theta)} N^{(\frac{d-1}{2}-\frac{d+1}{p})(1-\theta)}.$$

To finish these preliminaries, we introduce the notation: for a dyadic M,

$$u_M = \sum_{|\xi|\le M} e^{2\pi i<x,\xi>} \int \hat{u}(\xi,\lambda) e^{2\pi i\lambda t} d\lambda,$$

$$\Delta_M u = u_M - u_{\frac{M}{2}}.$$

If I is an interval of $\mathbb{Z}^n$,

$$\begin{aligned} \Delta_I u &= \sum_{\xi\in I} e^{2\pi i<x,\xi>} \int \hat{u}(\xi,\lambda) e^{2\pi i\lambda t} d\lambda \\ &= \sum_{K\ dyadic} \int_{\Lambda_{A,N}} \hat{u}(\zeta) e^{2\pi i(<x,\xi>+\lambda t)} d\zeta. \end{aligned} \tag{17}$$

This dyadic localization will be helpful in the analysis of the Bourgain norm of the nonlinearity of the equation (1).

For later use, we observe that, using lemma 2.11

$$\|\Delta_I u\|_{p_2} \le c \sum_{A\ dyadic} A^{\frac{1}{2}(1-\theta)} M^{(\frac{d-1}{2}-\frac{d+1}{p})(1-\theta)} \left(\int_{\Lambda_{A,I}} |\hat{u}(\zeta)|^2 d\zeta \right)^{1/2}, \tag{18}$$

if the size of I is M.

Similarly,

$$\begin{aligned} \|\Delta_M u\|_{p_2} &\le c \sum_{A\ dyadic} A^{\frac{1}{2}(1-\theta)} M^{(\frac{d-1}{2}-\frac{d+1}{p})(1-\theta)} \left(\int_{\Lambda_{A,M}} |\hat{u}(\zeta)|^2 d\zeta \right)^{1/2} \\ &\le c M^{(\frac{d-1}{2}-\frac{d+1}{p})(1-\theta)-s} \, |||u|||. \end{aligned} \tag{19}$$

3 Local well-posedness for the periodic SDE

The basic lemma in the proof of our local well-posedness result is:

Lemma 3.1. *If* $s < \frac{d-1}{2}$, $\alpha < \frac{4}{d-1-2s}$ *and* $p_0 < \frac{2(d+1)}{d-1-\frac{2}{3}s}$, *where* p_0 *is an* admissible exponent, *then*

$$A^{-1/2} N^s \left(\int_{\Lambda_{A,N}} |\hat{w}(\zeta)|^2 d\zeta \right)^{1/2} \leq c A^{-\theta} N^{-\theta} (\|v_0\|_{H^s} |||u||| + |||u|||^{1+\alpha}), \quad (20)$$

for some $\theta > 0$.

Proof. Write $w = F_0(u) + F_1(u)$ with $F_0(u) := \mu u$ and $F_1(u) := \eta u$, where $\mu = e^{-t/K} v_0$ and $\eta = \frac{\epsilon}{K} \int_0^t e^{-(t-\tau)/K} |u(\tau)|^\alpha d\tau$. This reduces our goal to prove the estimates

$$A^{-1/2} N^s \left(\int_{\Lambda_{A,N}} |\hat{F}_0(\zeta)|^2 d\zeta \right)^{1/2} \leq c A^{-\theta} N^{-\theta} \|v_0\|_{H^s} |||u|||, \quad (21)$$

and

$$A^{-1/2} N^s \left(\int_{\Lambda_{A,N}} |\hat{F}_1(\zeta)|^2 d\zeta \right)^{1/2} \leq c A^{-\theta} N^{-\theta} |||u|||^{1+\alpha}. \quad (22)$$

First we analyse the left-hand side of (21). Write

$$F_0 = u\mu = e^{-t/K} \sum \left(u_M (v_0)_M - u_{\frac{M}{2}} (v_0)_{\frac{M}{2}} \right).$$

Hence, it suffices to prove the bound (21) with F_0 replaced by

$$\Delta_M u \cdot e^{-t/K} (v_0)_M$$

and

$$u_{\frac{M}{2}} \cdot e^{-t/K} \Delta_M v_0.$$

with $M \geq N$.

Because $u_M = \sum_{M_1 \leq M} \Delta_{M_1} u$, $(v_0)_M = \sum_{M_1 \leq M} \Delta_{M_1} v_0$ and $\Delta_M u = \sum_I \Delta_I u$, where I is a decomposition of $\frac{M}{2} \leq |\xi| \leq M$ into intervals of size M_1, our task is to show that (21) holds with F_0 replaced by

$$(F_0^{(1)})_I := \Delta_I u \cdot e^{-t/K} \Delta_{M_1} v_0$$

and

$$(F_0^{(2)})_I := \Delta_{M_1} u \cdot e^{-t/K} \Delta_I v_0$$

Choose $p_1 > p_0$, $p_1 > p_2 > 2$, $\frac{1}{p_2} = \frac{1-\theta_2}{p_1} + \frac{\theta_2}{2}$.

The dual form of lemma 2.11 gives

$$\left(\int_{\Lambda_{A,I}} |\widehat{F_0^{(1)}}_I(\zeta)|^2 d\zeta\right)^{1/2} \leq cA^{\frac{1}{2}(1-\theta_2)} M^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \|(F_0^{(1)})_I\|_{p_2'}$$

and

$$\left(\int_{\Lambda_{A,I}} |\widehat{F_0^{(2)}}_I(\zeta)|^2 d\zeta\right)^{1/2} \leq cA^{\frac{1}{2}(1-\theta_2)} M^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \|(F_0^{(2)})_I\|_{p_2'}$$

Therefore, by Hölder's inequality

$$\|(F_0^{(1)})_I\|_{p_2'} \leq \frac{1}{K} \|\Delta_I u\|_{p_2} \|e^{-t/K} \Delta_{M_1} v_0\|_{\frac{p_2-p_2'}{p_2 p_2'}}.$$

and

$$\|(F_0^{(2)})_I\|_{p_2'} \leq \frac{1}{K} \|\Delta_I v_0\|_{p_2} \|e^{-t/K} \Delta_{M_1} u\|_{\frac{p_2-p_2'}{p_2 p_2'}}.$$

Using (18), we obtain

$$\left(\sum_I \|\Delta_I u\|_{p_2}^2\right) \leq c M_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \cdot M^{-s} |||u|||.$$

and

$$\left(\sum_I \|\Delta_I v_0\|_{p_2}^2\right) \leq c M_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \cdot M^{-s} \|v_0\|_{H^s}.$$

Also, taking

$$p_3 > p_0, p_3 > p_4 > 2, \frac{1}{p_4} = \frac{1-\theta_4}{p_3} + \frac{\theta_4}{2} \text{ and } 1 > \frac{2}{p_2} + \frac{1}{p_4}$$

then, the estimate (19) implies

$$\|e^{-t/K} \Delta_{M_1} u\|_{\frac{p_2-p_2'}{p_2 p_2'}} \leq \|\Delta_{M_1} u\|_{p_4} \leq M_1^{(\frac{d-1}{2}-\frac{d+1}{p_3})(1-\theta_4)-s} |||u|||$$

and

$$\|e^{-t/K} \Delta_{M_1} v_0\|_{\frac{p_2-p_2'}{p_2 p_2'}} \leq \|\Delta_{M_1} v_0\|_{p_4} \leq M_1^{(\frac{d-1}{2}-\frac{d+1}{p_3})(1-\theta_4)-s} \|v_0\|_{H^s}.$$

Thus, after performing the summations over $M_1 \leq M$ and $M \geq N$ in the previous estimates, we get the desired bounds on $F_0^{(1)}$ and $F_0^{(2)}$. In particular, (21) is proved, if there are numbers $p_1, \ldots, p_4$ verifying the relations above.

Next, we show the estimate (22). Write

$$F_1 = u\eta = \frac{\varepsilon}{K}\sum\left(u_M\int_0^t e^{-(t-\tau)/K}|u_M|^\alpha d\tau - u_{\frac{M}{2}}\int_0^t e^{-(t-\tau)/K}|u_{\frac{M}{2}}|^\alpha d\tau\right).$$

So, we have to evaluate (22) with F_1 replaced by

$$\frac{\varepsilon}{K}\Delta_M u\cdot\int_0^t e^{-(t-\tau)/K}|u_M|^\alpha d\tau,$$

and

$$\frac{\varepsilon}{K}u_{\frac{M}{2}}\int_0^t e^{-(t-\tau)/K}(|u_M|^\alpha - |u_{\frac{M}{2}}|^\alpha)d\tau.$$

with $M \geq N$.

Since for $\alpha \geq 2$ and complex numbers z, w,

$$|z|^\alpha - |w|^\alpha = (z-w)\phi_1(z,w) + (\overline{z}-\overline{w})\phi_2(z,w),$$

where $|\phi_1|, |\phi_2| \leq c(|z|+|w|)^{\alpha-1}$, if we write $u_M = \sum\limits_{M_1\leq M}\Delta_{M_1}u$, it is sufficient to estimate (22) with F_1 replaced by

$$\frac{\varepsilon}{K}\Delta_M u\cdot\int_0^t e^{-(t-\tau)/K}\Delta_{M_1}u\cdot\phi(u_{M_1}, u_{\frac{M_1}{2}})d\tau \tag{23}$$

and

$$\frac{\varepsilon}{K}\Delta_{M_1} u\cdot\int_0^t e^{-(t-\tau)/K}\Delta_M u\cdot\phi(u_M, u_{\frac{M}{2}})d\tau \tag{24}$$

where $M_1 \leq M$, $M \geq N$. We subdivide $\frac{M}{2} < |\xi| \leq M$ in intervals I of size M_1 and write

$$\Delta_M u = \sum_I \Delta_I u.$$

Because the functions $\mathcal{A}_I = \frac{\varepsilon}{K}\Delta_I u\cdot\int_0^t e^{-(t-\tau)/K}\Delta_{M_1}u\cdot\phi(u_{M_1}, u_{\frac{M_1}{2}})d\tau$ (resp. $\mathcal{B}_I = \frac{\varepsilon}{K}\Delta_{M_1}u\cdot\int_0^t e^{-(t-\tau)/K}\Delta_I u\cdot\phi(u_M, u_{\frac{M}{2}})d\tau$) have essentially disjoint supports, the contributions of (23), (24) to (22) are

$$A^{-1/2}N^s\left(\sum_I\int_{\Lambda_{A,I}}|\hat{\mathcal{A}}_I(\zeta)|^2 d\zeta\right)^{1/2} \tag{25}$$

and

$$A^{-1/2}N^s\left(\sum_I\int_{\Lambda_{A,I}}|\hat{\mathcal{B}}_I(\zeta)|^2 d\zeta\right)^{1/2}. \tag{26}$$

We deal first with the contribution (25). Choose $p_1 > p_0$, $p_1 > p_2 > 2$, $\frac{1}{p_2} = \frac{1-\theta_2}{p_1} + \frac{\theta_2}{2}$.

The dual form of lemma 2.11 gives

$$\left(\int_{\Lambda_{A,I}} |\hat{\mathcal{A}}_I(\zeta)|^2 d\zeta\right)^{1/2} \leq cA^{\frac{1}{2}(1-\theta_2)} M^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \|\mathcal{A}_I\|_{p_2'} \tag{27}$$

and, by Hölder's inequality

$$\|\mathcal{A}_I\|_{p_2'} \leq \frac{1}{K}\|\Delta_I u\|_{p_2} \| \int_0^t e^{-(t-\tau)/K} \Delta_{M_1} u \cdot \phi(u_{M_1}, u_{\frac{M_1}{2}}) d\tau \|_{\frac{p_2-p_2'}{p_2 p_2'}}. \tag{28}$$

Using (18), we obtain

$$\left(\sum_I \|\Delta_I u\|_{p_2}^2\right) \leq cM_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \cdot M^{-s} |||u|||. \tag{29}$$

On the other hand, choosing

$$p_3 > p_0, p_3 > p_4 > 2, \frac{1}{p_4} = \frac{1-\theta_4}{p_3} + \frac{\theta_4}{2} \text{ and } 1 > \frac{2}{p_2} + \frac{1}{p_4} \tag{30}$$

then

$$\| \int_0^t e^{-(t-\tau)/K} \Delta_{M_1} u \cdot \phi(u_{M_1}, u_{\frac{M_1}{2}}) d\tau \|_{\frac{p_2-p_2'}{p_2 p_2'}} \leq \|\Delta_{M_1} u\|_{p_4} \cdot \|\phi\|_{(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} \tag{31}$$

Note that

$$\|\phi\|_{(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} \leq \|u_{M_1}\|^{\alpha-1}_{(\alpha-1)(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} \tag{32}$$

But, writing $u_{M_1} = \sum_{M_2 < M_1 \ dyadic} \Delta_{M_2} u$, if we choose $p_5 > p_0, p_5 > p_6 > 2$, $\frac{1}{p_6} = \frac{1-\theta_6}{p_5} + \frac{\theta_6}{2}$ and $\frac{\alpha-1}{p_6} \leq 1 - \frac{2}{p_2} - \frac{1}{p_4}$, then

$$(32) \leq c|||u|||^{\alpha-1}. \tag{33}$$

Putting together the estimates (33), (31), (19) and performing summations over $M_1 \leq M$ and $M \geq N$, we proved

$$(25) \leq cA^{-\theta} N^{-\theta} \cdot |||u|||^{1+\alpha} \tag{34}$$

for some $\theta > 0$, provided that we can assure the existence of $p_1, \ldots, p_6$ satisfying the relations above.

Similarly, the contribution of (26) can be analysed as follows. Keeping the same notation as above, the dual form of the lemma 2.11 still yields

$$\left(\int_{\Lambda_{A,I}} |\hat{\mathcal{B}}_I(\zeta)|^2 d\zeta\right)^{1/2} \leq cA^{\frac{1}{2}(1-\theta_2)} M^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \|\mathcal{B}_I\|_{p_2'} \tag{35}$$

and, by Hölder's inequality

$$\|\mathcal{B}_I\|_{p_2'} \leq \frac{1}{K}\|\Delta_{M_1} u\|_{p_4}\| \int_0^t e^{-(t-\tau)/K} \Delta_I u \cdot \phi(u_{M_1}, u_{\frac{M_1}{2}}) d\tau\|_{\widetilde{p}_4}, \tag{36}$$

where $\widetilde{p}_4 = (1 - \frac{1}{p_2} - \frac{1}{p_4})^{-1}$.

Using (18), we obtain again

$$\left(\sum_I \|\Delta_I u\|_{p_2}^2\right) \leq cM_1^{(\frac{d-1}{2}-\frac{d+1}{p_1})(1-\theta_2)} \cdot M^{-s}|||u|||. \tag{37}$$

On the other hand, choosing

$$\frac{1}{\widetilde{p}_4} = \frac{1}{p_2} + \frac{1}{\widehat{p}_4} \tag{38}$$

then, since $\widehat{p}_4 = (1 - \frac{2}{p_2} - \frac{1}{p_4})^{-1}$,

$$\| \int_0^t e^{-(t-\tau)/K} \Delta_{M_1} u \cdot \phi(u_{M_1}, u_{\frac{M_1}{2}}) d\tau\|_{\widetilde{p}_4} \leq \|\Delta_I u\|_{p_2} \cdot \|\phi\|_{(1-\frac{2}{p_2}-\frac{1}{p_4})^{-1}} \tag{39}$$

Thus, we can apply the same arguments used in the treatment of (25) to get

$$(26) \leq cA^{-\theta} N^{-\theta} |||u|||^{1+\alpha}. \tag{40}$$

Finally, it remains only to justify the existence of the numbers $p_1, \ldots, p_6$ satisfying the claimed relations. However, it is not difficult to prove (see [B, p.149]) that these numbers exist if $s < \frac{d-1}{2}$, $\alpha < \frac{4}{d-1-2s}$ and $p_0 < \frac{2(d+1)}{d-1-\frac{2}{3}s}$. $\square$

Once this lemma is proved, it is a standard matter to get the local well-posedness statements in the theorems A, B, C and D. Indeed, the lemma 3.1 can be applied to give the estimate

$$\|w\|_{X^{s,-1/2+}} \leq c(\|v_0\|_{H^s}\|u\|_{X^{s,1/2}} + \|u\|_{X^{s,1/2}}^{1+\alpha}).$$

In particular, this estimate can be combined with the bounds (10) and (11) to obtain that the integral formulation of the SDE (1) is a contraction of a large ball in the space $X^{s,b}$ into itself. This completes the proof of the local well-posedness thereoms A, B, C and D.

4 Global well-posedness for the periodic SDE

We start with the case of cubic nonlinearity in dimensions $n = 1, 2, 3$: the proof of the theorem E clearly follows from the conservation of the L^2-norm of u, if we can prove the estimate

$$\|w\|_{X^{s,-1/2+}} \leq c(\|v_0\|_{H^s}\|u\|_{X^{s,1/2}} + \|u\|^2_{X^{0,1/2}}\|u\|_{X^{s,1/2}}). \tag{41}$$

Similarly, the proof of theorems F and G follows from the estimate

$$\|w\|_{X^{s,-1/2+}} \leq c(\|v_0\|_{H^s}\|u\|_{X^{s,1/2}} + \|u\|^2_{X^{1,1/2}}\|u\|_{X^{s,1/2}}). \tag{42}$$

However, the bound in (41) is easily obtained via a simple modification of the calculations in [B3, p.110–114] using the Strichartz estimate in lemma 2.3. Analogously, the bound (42) follows from simple modifications of the calculations in [B3, p.115–118] (along the lines of the proof of the lemma 3.1) using the Strichartz bounds in propositions 2.8 and 2.9.

Next, we study the variation of the H^1-norm of u (see the proposition below). Using this, we will derive an *a priori estimate* for the solution.

Proposition 4.1.

$$\frac{d}{dt}\left(\int_{\mathbb{T}^n}|\nabla u(t)|^2 - \int_{\mathbb{T}^n}|u(t)|^2 v(t)\right) = \frac{1}{K}\cdot\left(\int_{\mathbb{T}^n}|u(t)|^2 v(t) - \varepsilon\int_{\mathbb{T}^n}|u(t)|^p\right). \tag{43}$$

Proof. Write $u = a + ib$. The equation (1) implies that

$$\begin{cases} \partial_t a = -\Delta b + bv, \\ \partial_t b = \Delta a - av \end{cases} \tag{44}$$

But,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}|\nabla u(t)|^2 &= \int_{\mathbb{T}^n} < \nabla a, \nabla\partial_t a > + \int_{\mathbb{T}^n} < \nabla b, \nabla\partial_t b >= \\ &= \int_{\mathbb{T}^n}(\partial_t a\Delta a + \partial_t b\Delta b)\,. \end{aligned}$$

Hence by equation (44),

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}|\nabla u(t)|^2 = \int_{\mathbb{T}^n}(b\Delta av - a\Delta bv)\,. \tag{45}$$

On the other hand, the equation (1) also implies

$$\partial_t v = -\frac{1}{K}v + \frac{\varepsilon}{K}|u|^\alpha \tag{46}$$

However,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}|u(t)|^2 v(t) = \frac{1}{2}\int_{\mathbb{T}^n} v(t)\partial_t|u|^2 + \frac{1}{2}\int_{\mathbb{T}^n}|u(t)|^2\partial_t v.$$

So using equations (44), (46), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^n}|u(t)|^2v(t)=\int_{\mathbb{T}^n}(b\Delta av-a\Delta bv)-\frac{1}{2K}\int_{\mathbb{T}^n}|u(t)|^2v(t)+\frac{\varepsilon}{2K}\int_{\mathbb{T}^n}|u(t)|^p. \tag{47}$$

Then, if we subtract the equations (45) and (47), the proof is complete. □

Integrating the equation of proposition 4.1, we obtain

$$\int_{\mathbb{T}^n}|\nabla u(t)|^2=\int_{\mathbb{T}^n}|u(t)|^2v(t)-\int_{\mathbb{T}^n}|\nabla u_0|^2+\int_{\mathbb{T}^n}|u_0|^2v_0+\frac{1}{K}\int|u|^2v-\frac{\varepsilon}{K}\int|u|^p. \tag{48}$$

We recall the following basic inequality

$$\|f\|_{L^p(\mathbb{T}^n)}\leq c\|f\|_2^{1-\theta}\|f\|_{H^1}^{\theta}, \tag{49}$$

where $\theta:=n\left(\frac{1}{2}-\frac{1}{p}\right)<1$.

Then, by Hölder inequality,

$$\int_{\mathbb{T}^n}|u(t)|^2v(t)\leq\|u(t)\|_4^2\|v(t)\|_2,$$

$$\int|u|^2v\leq\|u\|_4^2\|v\|_2$$

But, by (49), since $\|u(t)\|_2=\|u_0\|_2$,

$$\|u(t)\|_4\leq c\|u_0\|_2^{1-\theta_0}\|u(t)\|_{H^1}^{\theta_0},$$

$$\|u\|_4\leq cT^{1/4}\|u_0\|_2^{1-\theta_0}\sup_{t\in[0,T]}\|u(t)\|_{H^1}^{\theta_0},$$

$$\int|u|^p\leq cT\|u_0\|_2^{p(1-\theta)}\sup_{t\in[0,T]}\|u(t)\|_{H^1}^{p\theta}.$$

with $\theta_0=n(\frac{1}{2}-\frac{1}{4})=\frac{n}{4}<1$, $\theta=n(\frac{1}{2}-\frac{1}{p})<1$. Moreover,

$$\|v(t)\|_2\leq\|v_0\|_2+\frac{c}{K}\|u\|_{2\alpha}^{\alpha}\leq$$
$$\|v_0\|_2+\frac{c}{K}T^{1/2}\|u_0\|_2^{\alpha(1-\theta_1)}\sup_{t\in[0,T]}\|u(t)\|_{H^1}^{\alpha\theta_1},$$

$$\|v\|_2\leq T^{1/2}\|v_0\|_2+\frac{c}{K}T\|u_0\|_2^{\alpha(1-\theta_1)}\sup_{t\in[0,T]}\|u(t)\|_{H^1}^{\alpha\theta_1}.$$

where $\theta_1 = n(\frac{1}{2} - \frac{1}{2\alpha}) < 1$.

Applying these inequalities for equation (48), we get the following *a priori* estimate:

$$\sup_{t\in[0,T]} \|u(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + c\|v_0\|_2 \|u_0\|_2^{\frac{(4-n)}{2}} \sup_{t\in[0,T]} \|u(t)\|_{H^1}^{\frac{n}{2}} +$$
$$c\|u_0\|_2^{\frac{(4-n)}{2}} \mu_1(T) \sup_{t\in[0,T]} \|u(t)\|_{H^1}^{\frac{n}{2}+\alpha\theta_1} +$$
$$\frac{c}{K} T \|u_0\|_2^{\frac{(4-n)}{2}} \left(\|v_0\| + \mu_1(T) \sup_{t\in[0,T]} \|u(t)\|_{H^1}^{\alpha\theta_1} \right) \sup_{t\in[0,T]} \|u(t)\|_{H^1}^{\frac{n}{2}} +$$
$$\frac{c}{K} T \|u_0\|_2^{p(1-\theta)} \sup_{t\in[0,T]} \|u(t)\|_{H^1}^{p\theta}. \tag{50}$$

where $\mu_1(T) = \frac{c}{K} T^{1/2} \|u_0\|_2^{\alpha(1-\theta_1)}$. From the previous a priori estimate, using a standard argument, if $\theta_0, \theta_1, \theta < 1$, then we will obtain our global well-posedness results in the theorems E, F and G for $H^1 \times H^1$ data, as follows.

Note that $\theta_0 < 1 \iff n \leq 3$. Also, if $n = 1, 2$, $\theta_1 < 1$ for any $\alpha > 0$ (i.e., any p); if $n = 3$, $\theta_1 < 1 \iff \alpha < 3$ (i.e., $p < 5$). Finally, $\theta < 1 \iff p < \frac{2n}{n-2}$. These informations together clearly gives the desired results.

5 Concluding remarks

We finish this article with two questions motivated by the previous results. Firstly, in view of the global well-posedness theorem for the periodic NLS equation in dimension 4 proved by Bourgain in [B2], it is natural to ask:

Question 1. In dimension 4, is the periodic SDE (1) where the nonlinearity $|u|^\alpha$ is replaced by $f(|u|^2)$ with $f(t) = O''(t^{1/2})$ (i.e., $|f(t)| \leq ct^{1/2}$, $|f'(t)| \leq ct^{-1/2}$ and $|f''(t)| \leq ct^{-3/2}$) globally well-posed for $H^s \times H^s$ initial data satisfying $s \geq 2$?

Secondly, while our results are always stated for $H^s \times H^s$ initial data, Corcho and Linares [CL] were able to prove well-posedness for $H^k \times H^s$ initial data with $k \neq s$. Thus, a interesting question is:

Question 2. Is the periodic SDE (1) well-posed for $H^k \times H^s$ initial data with $k \neq s$?

We plan to attack these issues in forthcoming papers. At the present moment, we advance that some work in progress by Corcho and the second indicates the possibitity of a satisfactory answer for the second question in dimension 1.

References

[Bi1] B. Bidégaray, On the Cauchy problem for systems ocurring in nonlinear optics, *Adv. Diff. Equat.*, **3**, 473–496, 1998.

[Bi2] B. Bidégaray, The Cauchy Problem to Schrödinger-Debye equations, *Math. Models Methods Appl. Sci.*, **10**, 307–315, 2000.

[B] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part I: Schrödinger equations, *GAFA*, **3**, 107–156, 1993.

[B2] J. Bourgain, Exponential sums and nonlinear Schrödinger equations, *GAFA*, **3**, 157–178, 1993.

[B3] J. Bourgain, Global solutions of nonlinear Schrödinger equations, *AMS Colloquim Publications*, **46**, 1999.

[CL] A. Corcho and F. Linares, Well-Posedness for the Schrödinger-Debye equation, *Contemporary Math.*, **362**, 113–131, 2004.

Alexander Arbieto (alexande@impa.br)
Carlos Matheus (matheus@impa.br)
IMPA, Est. D. Castorina 110, Jardim Botânico, 22460-320
Rio de Janeiro, RJ, Brazil