GLOBAL WELL-POSEDNESS FOR A NLS-KDV SYSTEM ON $\mathbb{T}$

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ABSTRACT. We prove that the Cauchy problem of the Schrödinger - Korteweg - deVries (NLS-KdV) system on $\mathbb{T}$ is globally well-posed for initial data (u_0, v_0) below the energy space $H^1 \times H^1$. More precisely, we show that the non-resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ and the resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 19/21$. The idea of the proof of this theorem is to apply the I-method of Colliander, Keel, Staffilani, Takaoka and Tao in order to improve the results of Arbieto, Corcho and Matheus concerning the global well-posedness of the NLS-KdV on $\mathbb{T}$ in the energy space $H^1 \times H^1$.

1. INTRODUCTION

We consider the Cauchy problem of the Schrödinger-Korteweg-deVries system

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv + \beta |u|^2 u, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), \quad t \in \mathbb{R}. \end{cases} \tag{1.1}$$

The Schrödinger-Korteweg-deVries (NLS-KdV) system naturally appears in fluid mechanics and plasma physics as a model of interaction between a short-wave $u = u(x,t)$ and a long-wave $v = v(x,t)$.

In this paper we are interested in global solutions of the NLS-KdV system for rough initial data. Before stating our main results, let us recall some of the recent theorems of local and global well-posedness theory of the Cauchy problem (1.1).

For continuous spatial variable (i.e., $x \in \mathbb{R}$), Corcho and Linares [5] recently proved that the NLS-KdV system is locally well-posed for initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \geq 0$, $s > -3/4$ and

- $k - 1 \leq s \leq 2k - 1/2$ if $k \leq 1/2$,
- $k - 1 \leq s < k + 1/2$ if $k > 1/2$.

Furthermore, they were able to prove the global well-posedness of the NLS-KdV system in the energy $H^1 \times H^1$ using three conserved quantities discovered by M. Tsutsumi [7], whenever $\alpha\gamma > 0$.

Also, Pecher [6] improved this global well-posedness result by an application of the I-method of Colliander, Keel, Stafillani, Takaoka and Tao (for instance, see [3]) combined with some refined bilinear estimates. In particular, Pecher proved that, if $\alpha\gamma > 0$, the NLS-KdV

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system is globally well-posed for initial data $(u_0, v_0) \in H^s \times H^s$ with $s > 3/5$ in the resonant case $\beta = 0$ and $s > 2/3$ in the non-resonant case $\beta \neq 0$.

On the other hand, in the periodic setting (i.e., $x \in \mathbb{T}$), Arbieto, Corcho and Matheus [1] proved the local well-posedness of the NLS-KdV system for initial data $(u_0, v_0) \in H^k \times H^s$ with $0 \leq s \leq 4k-1$ and $-1/2 \leq k-s \leq 3/2$. Also, using the same three conserved quantities discovered by M. Tsutsumi, one obtains the global well-posedness of NLS-KdV on $\mathbb{T}$ in the energy space $H^1 \times H^1$ whenever $\alpha\gamma > 0$.

Motivated by this scenario, we combine the new bilinear estimates of Arbieto, Corcho and Matheus [1] with the I-method of Tao and his collaborators to prove the following result

Theorem 1.1. *The NLS-KdV system (1.1) on $\mathbb{T}$ is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ in the non-resonant case $\beta \neq 0$ and $s > 8/9$ in the resonant case $\beta = 0$, whenever $\alpha\gamma > 0$.*

The paper is organized as follows. In the section 2, we discuss the preliminaries for the proof of the theorem 1.1: Bourgain spaces and its properties, linear estimates, standard estimates for the non-linear terms $|u|^2 u$ and $\partial_x(v^2)$, the bilinear estimates of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$, the I-operator and its properties. In the section 3, we apply the results of the section 2 to get a variant of the local well-posedness result of [1]. In the section 4, we recall some conserved quantities of (1.1) and its modification by the introduction of the I-operator; moreover, we prove that two of these modified energies are almost conserved. Finally, in the section 5, we combine the almost conservation results in section 4 with the local well-posedness result in section 3 to conclude the proof of the theorem 1.1.

2. Preliminaries

A sucessful procedure to solve some dispersive equations (such as the nonlinear Schrödinger and KdV equations) is to use the Picard's fixed point method in the following spaces:

$$\begin{aligned}\|f\|_{X^{k,b}} &:= \left(\int \sum_{n\in\mathbb{Z}} \langle n \rangle^{2k} \langle \tau + n^2 \rangle^{2b} |\widehat{f}(n,\tau)| d\tau \right)^{1/2} \\ &= \|U(-t)f\|_{H_t^b(\mathbb{R}, H_x^k)}\end{aligned}$$

$$\begin{aligned}\|g\|_{Y^{s,b}} &:= \left(\int \sum_{n\in\mathbb{Z}} \langle n \rangle^{2s} \langle \tau - n^3 \rangle^{2b} |\widehat{g}(n,\tau)| d\tau \right)^{1/2} \\ &= \|V(-t)f\|_{H_t^b(\mathbb{R}, H_x^s)}\end{aligned}$$

where $\langle \cdot \rangle := 1 + |\cdot|$, $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t\partial_x^3}$. These spaces are called Bourgain spaces. Also, we introduce the restriction in time norms

$$\|f\|_{X^{k,b}(I)} := \inf_{\widetilde{f}|_I = f} \|\widetilde{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} := \inf_{\widetilde{g}|_I = g} \|\widetilde{g}\|_{Y^{s,b}}$$

where I is a time interval.

The interaction of the Picard method has been based around the spaces $Y^{s,1/2}$. Because we are interested in the continuity of the flow associated to (1.1) and the $Y^{s,1/2}$ norm do not control the $L_t^\infty H_x^s$ norm, we modify the Bourgain spaces as follows:

$$\|u\|_{X^k} := \|u\|_{X^{k,1/2}} + \|\langle n\rangle^k \widehat{u}(n,\tau)\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{Y^s} := \|v\|_{Y^{s,1/2}} + \|\langle n\rangle^s \widehat{v}(n,\tau)\|_{L_n^2 L_\tau^1}$$

and, given a time interval I, we consider the restriction in time of the X^k and Y^s norms

$$\|u\|_{X^k(I)} := \inf_{\widetilde{u}|_I = u} \|\widetilde{u}\|_{X^k} \quad \text{and} \quad \|v\|_{Y^s(I)} := \inf_{\widetilde{v}|_I = v} \|\widetilde{v}\|_{Y^s}$$

Furthermore, the mapping properties of $U(t)$ and $V(t)$ naturally leads one to consider the companion spaces

$$\|u\|_{Z^k} := \|u\|_{X^{k,-1/2}} + \left\| \frac{\langle n\rangle^k \widehat{u}(n,\tau)}{\langle \tau + n^2\rangle} \right\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{W^s} := \|v\|_{Y^{s,-1/2}} + \left\| \frac{\langle n\rangle^s \widehat{v}(n,\tau)}{\langle \tau - n^3\rangle} \right\|_{L_n^2 L_\tau^1}$$

In the sequel, ψ denotes a non-negative smooth bump function supported on $[-2,2]$ with $\psi = 1$ on $[-1,1]$ and $\psi_\delta(t) := \psi(t/\delta)$ for any $\delta > 0$.

Next, we recall some properties of the Bourgain spaces:

Lemma 2.1. $X^{0,3/8}([0,1]), Y^{0,1/3}([0,1]) \subset L^4(\mathbb{T} \times [0,1])$. *More precisely,*

$$\|\psi(t) f\|_{L_{xt}^4} \lesssim \|f\|_{X^{0,3/8}} \quad \textit{and} \quad \|\psi(t) g\|_{L_{xt}^4} \lesssim \|g\|_{Y^{0,1/3}}.$$

Proof. See [2]. □

Another basic property of these spaces are their stability under time localization:

Lemma 2.2. *Let* $X_{\tau=h(\xi)}^{s,b} := \{f : \langle \tau - h(\xi)\rangle^b \langle \xi\rangle^s |\widehat{f}(\tau,\xi)| \in L^2\}$. *Then,*

$$\|\psi(t) f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_{\psi,b} \|f\|_{X_{\tau=h(\xi)}^{s,b}}$$

for any $s, b \in \mathbb{R}$. *Moreover, if* $-1/2 < b' \leq b < 1/2$, *then for any* $0 < T < 1$, *we have*

$$\|\psi_T(t) f\|_{X_{\tau=h(\xi)}^{s,b'}} \lesssim_{\psi,b',b} T^{b-b'} \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Proof. First of all, note that $\langle \tau - \tau_0 - h(\xi)\rangle^b \lesssim_b \langle \tau_0\rangle^{|b|} \langle \tau - h(\xi)\rangle^b$, from which we obtain

$$\|e^{it\tau_0} f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_b \langle \tau_0\rangle^{|b|} \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Using that $\psi(t) = \int \widehat{\psi}(\tau_0) e^{it\tau_0} d\tau_0$, we conclude

$$\|\psi(t) f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_b \left(\int |\widehat{\psi}(\tau_0)| \langle \tau_0\rangle^{|b|} \right) \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Since ψ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s = 0$ and, by composition it suffices to treat the cases $0 \leq b' \leq b$ or $\leq b' \leq b \leq 0$. By duality, we may take

$0 \leq b' \leq b$. Finally, by interpolation with the trivial case $b' = b$, we may consider $b' = 0$. This reduces matters to show that

$$\|\psi_T(t) f\|_{L^2} \lesssim_{\psi,b} T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

for $0 < b < 1/2$. Partitioning the frequency spaces into the cases $\langle \tau - h(\xi)\rangle \geq 1/T$ and $\langle \tau - h(\xi) \leq 1/T$, we see that in the former case we'll have

$$\|f\|_{X^{0,0}_{\tau=h(\xi)}} \leq T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

and the desired estimate follows because the multiplication by ψ is a bounded operation in Bourgain's spaces. In the latter case, by Plancherel and Cauchy-Schwarz

$$\begin{aligned} \|f(t)\|_{L^2_x} \lesssim \|\widehat{f(t)}(\xi)\|_{L^2_\xi} &\lesssim \left\| \int_{\langle \tau - h(\xi)\rangle \leq 1/T} |\widehat{f}(\tau,\xi)| d\tau) \right\|_{L^2_\xi} \\ &\lesssim_b T^{b-1/2} \left\| \int \langle \tau - h(\xi)\rangle^{2b} |\widehat{f}(\tau,\xi)|^2 d\tau)^{1/2} \right\|_{L^2_\xi} = T^{b-1/2} \|f\|_{X^{s,b}_{\tau=h(\xi)}}. \end{aligned}$$

Integrating this against ψ_T concludes the proof of the lemma. □

Also, we have the following duality relationship between X^k (resp., Y^s) and Z^k (resp., W^s):

Lemma 2.3. *We have*

$$\left| \int \chi_{[0,1]}(t) f(x,t) g(x,t) dx dt \right| \lesssim \|f\|_{X^s} \|g\|_{Z^{-s}}$$

and

$$\left| \int \chi_{[0,1]}(t) f(x,t) g(x,t) dx dt \right| \lesssim \|f\|_{Y^s} \|g\|_{W^{-s}}$$

for any s and any f, g on $\mathbb{T} \times \mathbb{R}$.

Proof. See [4, p. 182–183] (note that, although this result is stated only for the spaces Y^s and W^s, the same proof adapts for the spaces X^k and Z^k). □

Now, we recall some linear estimates related to the semigroups $U(t)$ and $V(t)$:

Lemma 2.4 (Linear estimates). *It holds*

- $\|\psi(t) U(t) u_0\|_{Z^k} \lesssim \|u_0\|_{H^k}$ *and* $\|\psi(t) V(t) v_0\|_{W^s} \lesssim \|v_0\|_{H^s}$;
- $\|\psi_T(t) \int_0^t U(t-t') F(t') dt'\|_{X^k} \lesssim \|F\|_{Z^k}$ *and* $\|\psi_T(t) \int_0^t V(t-t') G(t') dt'\|_{Y^s} \lesssim \|G\|_{W^s}$.

Proof. See [3], [4] or [1]. □

Furthermore, we have the following well-known multiinear estimates for the cubic term $|u|^2 u$ of the nonlinear Schrödinger equation and the nonlinear term $\partial_x(v^2)$ of the KdV equation:

Lemma 2.5. $\|uv\overline{w}\|_{Z^k} \lesssim \|u\|_{X^{k,\frac{3}{8}}} \|v\|_{X^{k,\frac{3}{8}}} \|w\|_{X^{k,\frac{3}{8}}}$ *for any $k \geq 0$.*

Proof. See [2] and [1]. □

Lemma 2.6. $\|\partial_x(v_1v_2)\|_{W^s} \lesssim \|v_1\|_{Y^{s,\frac{1}{3}}}\|v_2\|_{Y^{s,\frac{1}{2}}} + \|v_1\|_{Y^{s,\frac{1}{2}}}\|v_2\|_{Y^{s,\frac{1}{3}}}$ *for any* $s \geq -1/2$, *if* $v_1 = v_1(x,t)$ *and* $v_2 = v_2(x,t)$ *are* x*-periodic functions having zero* x*-mean for all* t.

Proof. See [2], [3] and [1]. □

Next, we revisit the bilinear estimates of mixed Schrödinger-Airy type of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$ of the NLS-KdV system.

Lemma 2.7. $\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,\frac{3}{8}}}\|v\|_{Y^{s,\frac{1}{2}}} + \|u\|_{X^{k,\frac{1}{2}}}\|v\|_{Y^{s,\frac{1}{3}}}$ *whenever* $s \geq 0$ *and* $k - s \leq 3/2$.

Lemma 2.8. $\|\partial_x(u_1\overline{u_2})\|_{W^s} \lesssim \|u_1\|_{X^{k,3/8}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,3/8}}$ *whenever* $1 + s \leq 4k$ *and* $k - s \geq -1/2$.

Remark 2.1. *Although the lemmas 2.7 and 2.8 are not stated as above in* [1], *it is not hard to obtain them from the calculations of Arbieto, Corcho and Matheus.*

Finally, we introduce the I-operator: let $m(\xi)$ be a smooth non-negative symbol on $\mathbb{R}$ which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For any $N \geq 1$ and $\alpha \in \mathbb{R}$, denote by I_N^α the spatial Fourier multiplier

$$\widehat{I_N^\alpha f}(\xi) = m\left(\frac{\xi}{N}\right)^\alpha \widehat{f}(\xi).$$

For latter use, we recall the following general interpolation lemma:

Lemma 2.9 (Lemma 12.1 of [4]). *Let* $\alpha_0 > 0$ *and* $n \geq 1$. *Suppose* $Z, X_1, X_2, \ldots, X_n$ *are translation-invariant Banach spaces and* T *is a translation invariant* n*-linear operator such that*

$$\|I_1^\alpha T(u_1, \ldots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_1^\alpha u_j\|_{X_j},$$

for all $u_1, \ldots, u_n$ *and* $0 \leq \alpha \leq \alpha_0$. *Then,*

$$\|I_N^\alpha T(u_1, \ldots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j}$$

for all $u_1, \ldots, u_n$, $0 \leq \alpha \leq \alpha_0$ *and* $N \geq 1$. *Here the implicit constant is independent of* N.

After these preliminaries, we can proceed to the next section where a variant of the local well-posedness of Arbieto, Corcho and Matheus is obtained.

In the sequel we take $N \gg 1$ a large integer and denote by I the operator $I = I_N^{1-s}$ for a given $s \in \mathbb{R}$.

3. A variant local well-posedness result

This section is devoted to the proof of the following proposition:

Proposition 3.1. *For any* $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ *with* $\int_{\mathbb{T}} v_0 = 0$ *and* $s \geq 1/3$, *the periodic NLS-KdV system (1.1) has a unique local-in-time solution on the time interval* $[0, \delta]$ *for some* $\delta \leq 1$ *and*

$$\delta \sim \begin{cases} (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-\frac{16}{3}-}, & \text{if } \beta \neq 0, \\ (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-8-}, & \text{if } \beta = 0. \end{cases} \tag{3.1}$$

Moreover, we have $\|Iu\|_{X^1} + \|Iv\|_{Y^1} \lesssim \|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1}$.

Proof. We apply the I-operator to the NLS-KdV system (1.1) so that

$$\begin{cases} iIu_t + Iu_{xx} = \alpha I(uv) + \beta I(|u|^2 u), \\ Iv_t + Iv_{xxx} + I(vv_x) = \gamma I(|u|^2)_x, \\ Iu(0) = Iu_0, \; Iv(0) = Iv_0. \end{cases}$$

To solve this equation, we seek for some fixed point of the integral maps

$$\Phi_1(Iu, Iv) := U(t)Iu_0 - i\int_0^t U(t-t')\{\alpha I(u(t')v(t')) + \beta I(|u(t')|^2 u(t'))\}dt',$$

$$\Phi_2(Iu, Iv) := V(t)Iv_0 - \int_0^t V(t-t')\{I(v(t')v_x(t')) - \gamma I(|u(t')|^2)_x\}dt'.$$

The interpolation lemma 2.9 applied to the linear and multilinear estimates in the lemmas 2.4, 2.5, 2.6, 2.7 and 2.8 yields, in view of the lemma 2.2,

$$\|\Phi_1(Iu, Iv)\|_{X^1} \lesssim \|Iu_0\|_{H^1} + \alpha\delta^{\frac{1}{8}-}\|Iu\|_{X^1}\|Iv\|_{Y^1} + \beta\delta^{\frac{3}{8}-}\|Iu\|_{X^1}^3,$$

$$\|\Phi_2(Iu, Iv)\|_{Y^1} \lesssim \|Iv_0\|_{H^1} + \delta^{\frac{1}{6}-}\|Iv\|_{Y^1}^2 + \gamma\delta^{\frac{1}{8}-}\|Iu\|_{X^1}^2.$$

In particular, these integrals maps are contractions provided that $\beta\delta^{\frac{3}{8}-}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^2 \ll 1$ and $\delta^{\frac{1}{8}-}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) \ll 1$. This completes the proof of the proposition 3.1. □

4. Modified energies

Consider the following three quantities:

$$M(u) := \|u\|_{L^2}, \tag{4.1}$$

$$L(u, v) := \alpha\|v\|_{L^2}^2 + 2\gamma\int \Im(u\overline{u_x})dx, \tag{4.2}$$

$$E(u, v) := \alpha\gamma\int v|u|^2 dx + \gamma\|u_x\|_{L^2}^2 + \frac{\alpha}{2}\|v_x\|_{L^2}^2 - \frac{\alpha}{6}\int v^3 dx + \frac{\beta\gamma}{2}\int |u|^4 dx. \tag{4.3}$$

In the sequel, we suppose $\alpha\gamma > 0$. Note that

$$|L(u, v)| \lesssim \|v\|_{L^2}^2 + M\|u_x\|_{L^2} \tag{4.4}$$

and

$$\|v\|_{L^2}^2 \lesssim |L| + M\|u_x\|_{L^2}. \tag{4.5}$$

Also, the Gagliardo-Nirenberg and Young inequalities implies

$$\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 \lesssim |E| + |L|^{\frac{5}{3}} + M^8 + 1 \tag{4.6}$$

and

$$|E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + |L|^{\frac{5}{3}} + M^8 + 1 \tag{4.7}$$

In particular, combining the bounds (4.4) and (4.7),

$$|E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v\|_{L^2}^{\frac{10}{3}} + M^{10} + 1. \tag{4.8}$$

Moreover, from the bounds (4.5) and (4.6),

$$\|v\|_{L^2}^2 \lesssim |L| + M|E|^{1/2} + M^6 + 1 \tag{4.9}$$

and hence

$$\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \lesssim |E| + |L|^{5/3} + M^8 + 1 \tag{4.10}$$

$$\begin{aligned} \frac{d}{dt}L(Iu, Iv) &= 2\alpha \int Iv(IvIv_x - I(vv_x))dx + 2\alpha\gamma \int Iv(I(|u|^2) - |Iu|^2)_x dx \\ &\quad + 4\alpha\gamma\Re \int I\overline{u}_x(IuIv - I(uv))dx + 4\beta\gamma\Re \int ((Iu)^2 I\overline{u} - I(u^2\overline{u}))I\overline{u}_x dx \\ &=: \sum_{j=1}^{4} L_j. \end{aligned} \tag{4.11}$$

(4.12)
$$\begin{aligned}\frac{d}{dt}E(Iu,Iv) &= \alpha\int (I(vv_x)-IvIv_x)Iv_{xx}dx + \frac{\alpha}{2}\int (Iv)^2(I(vv_x)-IvIv_x)dx + \\ &+ 2\beta\gamma\Im\int (I(|u|^2u)_x - ((Iu)^2I\overline{u})_x)I\overline{u}_x dx \\ &+ \alpha\gamma\int |Iu|^2(IvIv_x - I(vv_x))dx + \alpha\gamma\int (|Iu|^2 - I(|u|^2))IvIv_x dx \\ &+ \alpha\gamma\int Iv_{xx}(|Iu|^2 - I(|u|^2))_x dx - 2\alpha\gamma\Im\int Iu_x(I(\overline{u}v) - I\overline{u}Iv)_x dx \\ &+ \alpha\gamma^2\int (I(|u|^2) - |Iu|^2)_x|Iu|^2 dx + 2\alpha^2\gamma\Im\int IvIu(I(\overline{u}v - I\overline{u}Iv))dx \\ &+ 2\beta^2\gamma\Im\int Iu(I\overline{u})^2(I(|u|^2u) - (Iu)^2I\overline{u})dx \\ &- 2\alpha\beta\gamma\Im\int IvIu(I(|u|^2\overline{u}) - Iu(I\overline{u}))^2dx - 2\alpha\beta\gamma\Im\int (Iu)^2I\overline{u}(I(\overline{u}v) - I\overline{u}Iv)dx \\ &=: \sum_{j=1}^{12} E_j\end{aligned}$$

4.1. **Estimates for the modified L-functional.**

Proposition 4.1. *Let (u,v) be a solution of (1.1) on the time interval $[0,\delta]$. Then, for any $N\geq 1$ and $s>1/2$,*

$$\begin{aligned}&|L(Iu(\delta),Iv(\delta)) - L(Iu(0),Iv(0))| \lesssim \\ &N^{-1+}\delta^{\frac{19}{24}-}(\|Iu\|_{X^{1,1/2}} + \|Iv\|_{Y^{1,1/2}})^3 + N^{-2+}\delta^{\frac{1}{2}-}\|Iu\|^4_{X^{1,1/2}}.\end{aligned} \tag{4.13}$$

Proof. Integrating (4.11) with respect to $t\in[0,\delta]$, it follows that we have to bound the (integral over $[0,\delta]$ of the) four terms on the right hand side. To simplify the computations, we assume that the Fourier transform of the functions are non-negative and we ignore the appearance of complex conjugates (since they are irrelevant in our subsequent arguments). Also, we make a dyadic decomposition of the frequencies $|n_i|\sim N_j$ in many places. In particular, it will be important to get extra factors N_j^{0-} everywhere in order to sum the dyadic blocks.

We begin with the estimate of $\int_0^\delta L_1$. It is sufficient to show that

$$\begin{aligned}&\int_0^\delta \sum_{n_1+n_2+n_3=0}\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right|\widehat{v_1}(n_1,t)|n_2|\widehat{v_2}(n_2,t)\widehat{v_3}(n_3,t)\lesssim \\ &N^{-1}\delta^{\frac{5}{6}-}\prod_{j=1}^3\|v_j\|_{Y^{1,1/2}}\end{aligned} \tag{4.14}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. In this case, note that

$$\begin{cases} |\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}| \lesssim |\frac{\nabla m(n_2)\cdot n_1}{m(n_2)}| \lesssim \frac{N_1}{N_2}, \text{ if } |n_1| \le N, \text{ and} \\ |\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}| \lesssim \left(\frac{N_1}{N}\right)^{1/2}, \text{ if } |n_1| \ge N. \end{cases}$$

Hence, using the lemmas 2.1 and 2.2 we obtain

$$|\int_0^\delta L_1| \lesssim \frac{N_1}{N_2}\|v_1\|_{L^4}\|(v_2)_x\|_{L^4}\|v_3\|_{L^2} \lesssim N^{-2+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^{3}\|v_i\|_{Y^{1,1/2}}$$

if $|n_1| \le N$, and

$$|\int_0^\delta L_1| \lesssim \left(\frac{N_1}{N}\right)^{1/2}\frac{1}{N_1N_3}\delta^{\frac{5}{6}-}\prod_{j=1}^{3}\|v_i\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^{3}\|v_i\|_{Y^{1,1/2}}.$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. The multiplier is bounded by

$$|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}| \lesssim \left(\frac{N_1}{N}\right)^{1-}.$$

In particular, using the lemmas 2.1 and 2.2,

$$|\int_0^\delta L_1| \lesssim \left(\frac{N_1}{N}\right)^{1-}\|v_1\|_{L^2}\|(v_2)_x\|_{L^4}\|v_3\|_{L^4} \lesssim N^{-1+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^{3}\|v_i\|_{Y^{1,1/2}}.$$

Now, we estimate $\int_0^\delta L_2$. Our task is to prove that

$$\int_0^\delta \sum_{n_1+n_2+n_3=0}\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right||n_1+n_2|\widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{v_3}(n_3,t) \lesssim \tag{4.15}$$
$$N^{-1+}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}$$

- $|n_2| \ll |n_1| \sim |n_3| \gtrsim N$. We estimate the multiplier by

$$\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \langle(\frac{N_2}{N})^{\frac{1}{2}}\rangle.$$

Thus, using $L^2_{xt}L^4_{xt}L^4_{xt}$ Hölder inequality and the lemmas 2.1 and 2.2

$$\int_0^\delta L_2 \lesssim \langle\left(\frac{N_2}{N}\right)^{\frac{1}{2}}\rangle\frac{1}{\langle N_2\rangle N_3}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}$$
$$\lesssim N^{-1+}\delta^{\frac{19}{24}-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}.$$

- $|n_1| \ll |n_2| \sim |n_3|$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Estimating the multiplier by

$$\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \left(\frac{N_2}{N}\right)^{1-}$$

we conclude

$$\int_0^\delta L_2 \lesssim \left(\frac{N_2}{N}\right)^{1-} \frac{1}{N_1 N_2} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}$$
$$\lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}.$$

Next, let us compute $\int_0^\delta L_3$. We claim that

$$\int_0^\delta \sum_{n_1+n_2+n_3=0} \left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \widehat{u_1}(n_1,t)\widehat{v_2}(n_2,t)|n_3|\widehat{u_3}(n_3,t) \tag{4.16}$$
$$\lesssim N^{-2+} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. The multiplier is bounded by

$$\begin{cases} \left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \left|\frac{\nabla m(n_1)\cdot n_2}{m(n_1)}\right| \lesssim \frac{N_2}{N_1}, \text{ if } |n_2| \leq N, \text{ and} \\ \left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \left(\frac{N_2}{N}\right)^{1/2}, \text{ if } |n_2| \geq N. \end{cases}$$

So, it is not hard to see that

$$\int_0^\delta L_3 \lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. This case is completely similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Since the multiplier is bounded by N_2/N, we get

$$\int_0^\delta L_3 \lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

Finally, it remains to estimate the contribution of $\int_0^\delta L_4$. It suffices to see that

$$\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} \left|\frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)}\right| |n_4| \prod_{j=1}^4 \widehat{u_j}(n_j,t) \lesssim \tag{4.17}$$
$$N^{-2+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}$$

- $N_1, N_2, N_3 \gtrsim N$. Since the multiplier verifies

$$\left|\frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)}\right| \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\frac{N_3}{N}\right)^{\frac{1}{2}},$$

the application of $L^4_{xt}L^4_{xt}L^4_{xt}L^4_{xt}$ Hölder inequality and the lemmas 2.1, 2.2 yields

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\frac{N_3}{N}\right)^{\frac{1}{2}} \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-3+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \gtrsim N$ and $N_3, N_4 \ll N_1, N_2$. Here the multiplier is bounded by $\left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}\rangle$. Hence,

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}\rangle \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 \langle N_3\rangle} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_4 \gtrsim N$ and $N_2, N_3 \ll N_1, N_4$. In this case we have the following estimates for the multiplier

$$\left|\frac{m(n_1+n_2+n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)}\right| \lesssim \begin{cases} \left|\frac{\nabla m(n_1)(n_2+n_3)}{m(n_1)}\right| \lesssim \frac{N_2+N_3}{N_1}, & \text{if } N_2, N_3 \leq N \\ \left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}\rangle, & \text{if } N_2 \geq N, \\ \left(\frac{N_1}{N}\frac{N_3}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_2}{N}\right)^{\frac{1}{2}}\rangle, & \text{if } N_3 \geq N. \end{cases}$$

Therefore, it is not hard to see that, in any of the situations $N_2, N_3 \leq N$, $N_2 \geq N$ or $N_3 \geq N$, we have

$$\int_0^\delta L_4 \lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \sim N_4 \gtrsim N$ and $N_3 \ll N_1, N_2, N_4$. Here we have the following bound

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}\rangle \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

At this point, clearly the bounds (4.14), (4.15), (4.16) and (4.17) concludes the proof of the proposition 4.1. □

4.2. **Estimates for the modified E-functional.**

Proposition 4.2. *Let (u, v) be a solution of (1.1) on the time interval $[0, \delta]$ such that $\int_{\mathbb{T}} v = 0$. Then, for any $N \geq 1$, $s > 1/2$,*

$$\begin{aligned} &|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| \lesssim \\ &\left(N^{-1+}\delta^{\frac{1}{6}-} + N^{-\frac{2}{3}+}\delta^{\frac{3}{8}-} + N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}\right)(\|Iu\|_{X^1} + \|Iv\|_{Y^1})^3 + \\ &N^{-1+}\delta^{\frac{1}{2}-}(\|Iu\|_{X^1} + \|Iv\|_{Y^1})^4 + N^{-2+}\delta^{\frac{1}{2}-}\|Iu\|_{X^1}^4(\|Iu\|_{X^1}^2 + \|Iv\|_{Y^1}). \end{aligned} \tag{4.18}$$

Proof. Again we integrate (4.12) with respect to $t \in [0, \delta]$, decompose the frequencies into dyadic blocks, etc., so that our objective is to bound the (integral over $[0, \delta]$ of the) E_j for each $j = 1, \ldots, 12$.

For the expression $\int_0^\delta E_1$, apply the lemma 2.3. We obtain

$$\left|\int_0^\delta E_1\right| \lesssim \|Iv_{xx}\|_{Y^{-1}} \|IvIv_x - I(vv_x)\|_{W^1} \lesssim \|Iv\|_{Y^1} \|IvIv_x - I(vv_x)\|_{W^1}$$

Writing the definition of the norm W^1, it suffices to prove the bound

$$\left\|\frac{\langle n_3\rangle}{\langle\tau_3-n_3^3\rangle^{\frac12}}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}}+$$
$$\left\|\frac{\langle n_3\rangle}{\langle\tau_3-n_3^3\rangle}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}}\lesssim \tag{4.19}$$
$$N^{-1+}\delta^{\frac16-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}.$$

Recall that the dispersion relation $\sum\limits_{j=1}^{3}\tau_j-n_j^3=-3n_1n_2n_3$ implies that, since $n_1n_2n_3\neq0$, if we put $L_j:=|\tau_j-n_j^3|$ and $L_{\max}=\max\{L_j;j=1,2,3\}$, then $L_{\max}\gtrsim\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle$.

- $|n_2|\sim|n_3|\gtrsim N$, $|n_1|\ll|n_2|$. The multiplier is bounded by

$$\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right|\lesssim\begin{cases}\frac{N_1}{N_2}, & \text{if } |n_1|\le N,\\ \left(\frac{N_1}{N}\right)^{\frac12}, & \text{if } |n_1|\ge N.\end{cases}$$

Thus, if $|\tau_3-n_3^3|=L_{\max}$, we have

$$\left\|\frac{\langle n_3\rangle}{\langle\tau_3-n_3^3\rangle^{\frac12}}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}}$$
$$\lesssim\begin{cases}\frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{\frac12}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}}\lesssim N^{-1+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1|\le N,\\ \left(\frac{N_1}{N_2}\right)^{\frac12}\frac{N_3}{N_1}\frac{1}{(N_1N_2N_3)^{\frac12}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}}\lesssim N^{-\frac32+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}, & \text{if } |n_1|\ge N.\end{cases}$$

and

$$\left\|\frac{\langle n_3\rangle}{\langle\tau_3-n_3^3\rangle}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}}$$
$$\lesssim\begin{cases}\frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{\frac12-}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}}\lesssim N^{-1+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1|\le N,\\ \left(\frac{N_1}{N_2}\right)^{\frac12}\frac{N_3}{N_1}\frac{\delta^{\frac13-}}{(N_1N_2N_3)^{\frac12-}}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}\lesssim N^{-\frac32+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}, & \text{if } |n_1|\ge N.\end{cases}$$

If either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$, we have

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3,\tau_3}}$$
$$\lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{6}-}}{N_1} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-1+}\delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1,1/2}} \|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{1}{(N_1N_2N_3)^{\frac{1}{2}}} \delta^{\frac{1}{6}-} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{3}-} N_{\max}^{0-} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases}$$

and

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3} L^1_{\tau_3}}$$
$$\lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{6}-}}{N_1} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-1+}\delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1,1/2}} \|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{\delta^{\frac{1}{6}-}}{(N_1N_2N_3)^{\frac{1}{2}-}} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases}$$

- $|n_1| \sim |n_2| \gtrsim N$. Estimating the multiplier by

$$\left| \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_1}{N} \right)^{1-},$$

we have that, if $|\tau_3 - n_3^3| = L_{\max}$,

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3,\tau_3}} +$$
$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3} L^1_{\tau_3}}$$
$$\lesssim \left\{ \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{3}-}}{N_1} + \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{3}-}}{N_1} \right\} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}$$
$$\lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{3}-} N_{\max}^{0-} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}$$

and, if either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$,

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1) \ n_2 \ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3,\tau_3}} +$$
$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1) \ n_2 \ \widehat{v_2}(n_2,\tau_2) \right\|_{L^2_{n_3} L^1_{\tau_3}}$$
$$\lesssim \left\{ \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{6}-}}{N_1} + \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1N_2N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{6}-}}{N_1} \right\} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}$$
$$\lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}.$$

For the expression $\int_0^\delta E_2$, it suffices to prove that

$$(4.20) \quad \begin{aligned} &\left| \int_0^\delta \sum \frac{m(n_3+n_4) - m(n_3)m(n_4)}{m(n_3)m(n_4)} \widehat{v_1}(n_1,t)\widehat{v_2}(n_2,t)\widehat{v_3}(n_3,t) \ n_4 \ \widehat{v_4}(n_4,t) \right| \lesssim \\ &N^{-2+} \delta^{\frac{2}{3}-} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}. \end{aligned}$$

Since at least two of the N_i are $\gtrsim N$, we can assume that $N_1 \geq N_2 \geq N_3$ and $N_1 \gtrsim N$. Hence,

$$\int_0^\delta E_2 \lesssim$$
$$\begin{cases} \left(\frac{N_1}{N}\right)^{1-} \frac{\delta^{\frac{2}{3}-}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}, \text{ if } |n_3| \sim |n_4| \gtrsim N, \\ \frac{N_3}{N_4} \frac{\delta^{\frac{2}{3}-}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}, \text{ if } |n_3| \ll |n_4|, |n_3| \leq N |n_4| \gtrsim N, \\ \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \frac{\delta^{\frac{2}{3}-}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-2+}\delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,\frac{1}{2}}}, \text{ if } |n_3| \ll |n_4|, |n_3| \geq N, |n_4| \gtrsim N. \end{cases}$$

Next, we estimate the contribution of $\int_0^\delta E_3$. We claim that

$$(4.21) \quad \begin{aligned} &\int_0^\delta \sum \frac{m(n_1n_2n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{u_3}(n_3,t) \ |n_4|^2 \ \widehat{u_4}(n_4,t) \lesssim \\ &N^{-1+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}. \end{aligned}$$

- $|n_1| \sim |n_2| \sim |n_3| \sim |n_4| \gtrsim N$. Since the multiplier satisfies

$$\frac{m(n_1n_2n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \left(\frac{N_1}{N}\right)^{\frac{3}{2}}$$

we obtain

$$\int_0^\delta E_3 \lesssim \left(\frac{N_1}{N}\right)^{\frac{3}{2}} \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly two frequencies are $\gtrsim N$. We consider the most difficult case $|n_4| \gtrsim N$, $|n_1| \sim |n_4|$ and $|n_2|, |n_3| \ll |n_1|, |n_4|$. The multiplier is estimated by

$$\frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \begin{cases} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \rangle \left(\frac{N_2}{N}\right)^{\frac{1}{2}}, & \text{if } |n_2| \geq N, \\ \langle \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \rangle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}, & \text{if } |n_3| \geq N, \\ \frac{N_2+N_3}{N_1}, & \text{if } |n_2|, |n_3| \leq N. \end{cases}$$

Thus,

$$\int_0^\delta E_3 \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly three frequencies are $\gtrsim N$. The most difficult case is $|n_1| \sim |n_2| \sim |n_4| \gtrsim N$ and $|n_3| \ll |n_1|, |n_2|, |n_4|$. Here the multiplier is bounded by

$$\frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \rangle.$$

Hence,

$$\int_0^\delta E_3 \lesssim \left(\frac{N_1}{N}\frac{N_2}{N}\right)^{\frac{1}{2}} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \rangle \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

The contribution of $\int_0^\delta E_4$ is controlled if we are able to show that

$$\int_0^\delta \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,t)\ |n_2|\ \widehat{v_2}(n_2,t)\widehat{u_3}(n_3,t)\widehat{u_4}(n_4,t) \lesssim \tag{4.22}$$
$$N^{-1+} \delta^{\frac{7}{12}-} \prod_{j=1}^2 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}.$$

We crudely bound the multiplier by

$$\left|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \left(\frac{N_{\max}}{N}\right)^{1-}.$$

The most difficult case is $|n_2| \geq N$. We have two possibilities:

- Exactly two frequencies are $\gtrsim N$. We can assume $N_3 \ll N_2$. In particular,

$$\int_0^\delta E_4 \lesssim \left(\frac{N_{\max}}{N}\right)^{1-} \frac{\delta^{\frac{7}{12}}}{N_1 N_3 N_4} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-1+} \delta^{\frac{7}{12}-} N_{\max}^{0-} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}}.$$

- At least three frequencies are $\gtrsim N$. In this case,

$$\int_0^\delta E_4 \lesssim N^{-2+}\delta^{\frac{7}{12}-}N_{\max}^{0-}\prod_{j=1}^{2}\|u_j\|_{X^{1,\frac{1}{2}}}\|v_j\|_{Y^{1,\frac{1}{2}}}.$$

The expression $\int_0^\delta E_5$ is controlled if we are able to prove

$$\begin{aligned}&\int_0^\delta \sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{v_3}(n_3,t)\ |n_4|\ \widehat{v_4}(n_4,t) \lesssim \\ &N^{-1+}\delta^{\frac{7}{12}-}\prod_{j=1}^{2}\|u_j\|_{X^{1,1/2}}\|v_j\|_{Y^{1,1/2}}.\end{aligned} \tag{4.23}$$

This follows directly from the previous analysis for (4.22).

For the term $\int_0^\delta E_6$, we apply the lemma 2.3 to obtain

$$\int_0^\delta E_6 \lesssim \|(Iv)_{xx}\|_{Y^{-1}}\|(|Iu|^2 - I(|u|^2))_x\|_{W^1} \lesssim \|Iv\|_{Y^1}\|(|Iu|^2 - I(|u|^2))_x\|_{W^1}.$$

So, the definition of the W^1 norm means that we have to prove

$$\begin{aligned}&\left\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}} + \\ &\left\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}} \lesssim \\ &\left\{N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-} + N^{-\frac{2}{3}}\delta^{\frac{3}{8}-}\right\}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.\end{aligned} \tag{4.24}$$

Note that $\sum \tau_j = 0$ and $\sum n_j = 0$. In particular, we obtain the dispersion relation

$$\tau_3 - n_3^3 + \tau_2 + n_2^2 + \tau_1 + n_1^2 = -n_3^3 + n_1^2 + n_2^2.$$

- $|n_1| \gtrsim N$, $|n_2| \ll |n_1|$. Denoting by $L_1 := |\tau_1 + n_1^2|$, $L_2 := |\tau_2 + n_2^2|$ and $L_3 := |\tau_3 - n_3^3|$, the dispersion relation says that in the present situation $L_{\max} := \max\{L_j\} \gtrsim N_3^3$. Since the multiplier is bounded by

$$\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right| \lesssim \begin{cases}\frac{\nabla m(n_1)n_2}{m(n_1)} \lesssim \frac{N_2}{N_1}, & \text{if } |n_2| \le N,\\ \left(\frac{N_2}{N}\right)^{\frac{1}{2}}, & \text{if } |n_2| \ge N,\end{cases}$$

we deduce that

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}}+$$

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}}\lesssim$$

$$\frac{N_3^2}{N_3^{\frac{3}{2}-}}\frac{\delta^{\frac{1}{8}-}}{NN_1}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.$$

- $|n_1|\sim|n_2|\gtrsim N$, $|n_3|^3\gg|n_2|^2$. In the present case the multiplier is bounded by $\left(\frac{N_1}{N}\right)^{1-}$ and the dispersion relation says that $L_{\max}\gtrsim N_3^3$. Thus,

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}}+$$

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}}\lesssim$$

$$\frac{N_3^2}{N_3^{\frac{3}{2}-}}\left(\frac{N_1}{N}\right)^{1-}\frac{\delta^{\frac{1}{8}-}}{N_1N_2}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.$$

- $|n_1|\sim|n_2|\gtrsim N$ and $|n_3|^3\lesssim|n_2|^2$. Here the dispersion relation does not give useful information about $L_{\max}$. Since the multiplier is estimated by $\left(\frac{N_2}{N}\right)^{\frac{1}{2}}$, we obtain the crude bound

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3,\tau_3}}+$$

$$\left\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{\frac{1}{2}}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\right\|_{L^2_{n_3}L^1_{\tau_3}}\lesssim$$

$$N_3^2\left(\frac{N_2}{N}\right)^{\frac{1}{2}}\frac{\delta^{\frac{3}{8}-}}{N_1N_2}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\lesssim N^{-\frac{2}{3}+}\delta^{\frac{3}{8}-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.$$

Next, the desired bound related to $\int_0^\delta E_7$ follows from

$$\int_0^\delta\sum\left|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\right||n_1+n_2|\widehat{u_1}(n_1,t)\widehat{v_2}(n_2,t)|n_3|\widehat{u_3}(n_3,t)\lesssim \tag{4.25}$$

$$N^{-1+}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}$$

- $|n_1| \ll |n_2| \gtrsim N$. The multiplier is $\lesssim (|n_2|/N)^{1/2}$ so that

$$\int_0^\delta E_7 \lesssim \frac{1}{N^{1/2}} \int_0^\delta \sum |n_1+n_2| \widehat{u_1}(n_1,t) |n_2|^{1/2} \widehat{v_2}(n_2,t) |n_3| \widehat{u_3}(n_3,t) \lesssim N^{-1} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

- $|n_1| \sim |n_2| \gtrsim N$. The multiplier is $\lesssim |n_2|/N$. Hence,

$$\int_0^\delta E_7 \lesssim N^{-1} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

- $|n_1| \gtrsim N$, $|n_2| \leq N$. The multiplier is again $\lesssim N_2/N$, so that it can be estimated as above.

Now we turn to the term $\int_0^\delta E_8$. The objective is to show that

$$\int_0^\delta \left| \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1+n_2| \prod_{j=1}^4 \widehat{u_j}(n_j,t) \lesssim N^{-1+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \tag{4.26}$$

- At least three frequencies are $\gtrsim N$. We can assume $|n_1| \geq |n_2|$. The multiplier is bounded by $N_{\max}/N$ so that

$$\int_0^\delta E_8 \lesssim \frac{N_{\max}}{N} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly two frequencies are $\gtrsim N$. Without loss of generality, we suppose $|n_1| \sim |n_2| \gtrsim N$ and $|n_3|, |n_4| \ll N$. Since the multiplier satisfies

$$\left| \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_{\max}}{N} \right)^{1-},$$

we get the bound

$$\int_0^\delta E_8 \lesssim \left(\frac{N_{\max}}{N} \right)^{1-} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

The contribution of $\int_0^\delta E_9$ is estimated if we prove that

$$\int_0^\delta \left| \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \widehat{u_1}(n_1,t) \widehat{v_2}(n_2,t) \widehat{u_3}(n_3,t) \widehat{v_4}(n_4,t) \lesssim N^{-2+} \delta^{\frac{7}{12}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}. \tag{4.27}$$

This follows since at least two frequencies are $\gtrsim N$ and the multiplier is always bounded by $(N_{\max}/N)^{1-}$, so that

$$\int_0^\delta E_9 \lesssim \left(\frac{N_{\max}}{N}\right)^{1-} \|u_1\|_{L^4}\|v_2\|_{L^4}\|u_3\|_{L^4}\|v_4\|_{L^4} \lesssim$$
$$\left(\frac{N_{\max}}{N}\right)^{1-} \frac{\delta^{\frac14+\frac13-}}{N_1N_2N_3N_4}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}\|v_4\|_{Y^{1,1/2}} \lesssim$$
$$N^{-2+}\delta^{\frac{7}{12}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}\|v_4\|_{Y^{1,1/2}}.$$

Now, we treat the term $\int_0^\delta E_{10}$. It is sufficient to prove

$$\int_0^\delta \sum \Big|\frac{m(n_4+n_5+n_6)-m(n_4)m(n_5)m(n_6)}{m(n_4)m(n_5)m(n_6)}\Big| \prod_{j=1}^6 \widehat{u_j}(n_j,t) \lesssim \tag{4.28}$$
$$N^{-2+}\delta^{\frac12-}\prod_{j=1}^6 \|u_j\|_{X^1}$$

However, this follows easily from the facts that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$, say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$, the Strichartz bound $X^{0,3/8} \subset L^4$ and the inclusion[1] $X^{\frac12+} \subset L^\infty_{xt}$. Indeed, if we combine these informations, it is not hard to get

$$\int_0^\delta E_{10} \lesssim \left(\frac{N_{\max}}{N}\right)^{\frac32} \frac{1}{N_{i_1}N_{i_2}N_{i_3}N_{i_4}}\delta^{\frac12-}\frac{1}{(N_{i_5}N_{i_6})^{1/2-}}\prod_{j=1}^6 \|u_j\|_{X^1} \lesssim N^{-2+}\delta^{\frac12-}N_{\max}^{0-}\prod_{j=1}^6\|u_j\|_{X^1}$$

For the expression $\int_0^\delta E_{11}$, we use again that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$ (say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$), the Strichartz bounds in lemma 2.1 and the inclusions $X^{\frac12+}, Y^{\frac12+} \subset L^\infty_{xt}$ to obtain

$$\int_0^\delta \sum \Big|\frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)}\Big| \prod_{j=1}^4 \widehat{u_j}(n_j,t)\widehat{v_5}(n_5,t) \lesssim \tag{4.29}$$
$$\left(\frac{N_{\max}}{N}\right)^{\frac32}\frac{1}{N_{i_1}N_{i_2}N_{i_3}N_{i_4}}\frac{\delta^{\frac12-}}{N_{i_5}^{1/2-}}\prod_{j=1}^4\|u_j\|_{X^1}\|v_5\|_{Y^1} \lesssim$$
$$N^{-2+}\delta^{\frac12-}\prod_{j=1}^4\|u_j\|_{X^1}\|v_5\|_{Y^1}.$$

The analysis of $\int_0^\delta E_{12}$ is similar to the $\int_0^\delta E_{11}$. This completes the proof of the proposition 4.2. □

[1] This inclusion is an easy consequence of Sobolev embedding.

5. Global well-posedness below the energy space

In this section we combine the variant local well-posedness result in proposition 3.1 with the two almost conservation results in the propositions 4.1 and 4.2 to prove the theorem 1.1.

Remark 5.1. *Note that the spatial mean $\int_{\mathbb{T}} v(t,x)dx$ is preserved during the evolution (1.1). Thus, we can assume that the initial data v_0 has zero-mean, since otherwise we make the change $w = v - \int_{\mathbb{T}} v_0 dx$ at the expense of two harmless linear terms (namely, $u\int_{\mathbb{T}} v_0 dx$ and $\partial_x v \int_{\mathbb{T}} v_0$).*

The definition of the I-operator implies that the initial data satisfies $\|Iu_0\|_{H^1}^2 + \|Iv_0\|_{H^1}^2 \lesssim N^{2(1-s)}$ and $\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \lesssim 1$. By the estimates (4.4) and (4.8), we get that $|L(Iu_0, Iv_0)| \lesssim N^{1-s}$ and $|E(Iu_0, Iv_0)| \lesssim N^{2(1-s)}$.

Also, any bound for $L(Iu, Iv)$ and $E(Iu, Iv)$ of the form $|L(Iu, Iv)| \lesssim N^{1-s}$ and $|E(Iu, Iv)| \lesssim N^{2(1-s)}$ implies that $\|Iu\|_{L^2}^2 \lesssim M$, $\|Iv\|_{L^2}^2 \lesssim N^{1-s}$ and $\|Iu\|_{H^1}^2 + \|Iv\|_{H^1}^2 \lesssim N^{2(1-s)}$.

Given a time T, if we can uniformly bound the H^1-norms of the solution at times $t = \delta$, $t = 2\delta$, etc., the local existence result in proposition 3.1 says that the solution can be extended up to any time interval where such a uniform bound holds. On the other hand, given a time T, if we can interact $T\delta^{-1}$ times the local existence result, the solution exists in the time interval $[0, T]$. So, in view of the propositions 4.1 and 4.2, it suffices to show

$$(N^{-1+}\delta^{\frac{19}{24}-}N^{3(1-s)} + N^{-2+}\delta^{\frac{1}{2}-}N^{4(1-s)})T\delta^{-1} \lesssim N^{1-s} \tag{5.1}$$

and

(5.2)

$$\left\{(N^{-1+}\delta^{\frac{1}{6}-} + N^{-\frac{2}{3}+}\delta^{\frac{3}{8}-} + N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-})N^{3(1-s)} + N^{-1+}\delta^{\frac{1}{2}-}N^{4(1-s)} + N^{-2+}\delta^{\frac{1}{2}-}N^{6(1-s)}\right\}\frac{T}{\delta} \lesssim N^{2(1-s)}$$

At this point, we recall that the proposition 3.1 says that $\delta \sim N^{-\frac{16}{3}(1-s)-}$ if $\beta \neq 0$ and $\delta \sim N^{-8(1-s)-}$ if $\beta = 0$. Hence,

- $\beta \neq 0$. The condition (5.1) holds for

$$-1 + \frac{5}{24}\frac{16}{3}(1-s) + 3(1-s) < (1-s), \text{ i.e. }, s > 19/28$$

and

$$-2 + \frac{1}{2}\frac{16}{3}(1-s) + 4(1-s) < (1-s), \text{ i.e. }, s > 11/17;$$

Similarly, the condition (5.2) is satisfied if

$$-1 + \frac{5}{6}\frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. }, s > 40/49;$$

$$-\frac{2}{3} + \frac{5}{6}\frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. }, s > 11/13;$$

$$-\frac{3}{2}+\frac{7}{8}\frac{16}{3}(1-s)+3(1-s)<2(1-s),\ \text{i.e.}\ ,s>25/34;$$

$$-1+\frac{1}{2}\frac{16}{3}(1-s)+4(1-s)<2(1-s),\ \text{i.e.}\ ,s>11/14$$

and

$$-2+\frac{1}{2}\frac{16}{3}(1-s)+6(1-s)<2(1-s),\ \text{i.e.}\ ,s>7/10.$$

Thus, we conclude that the non-resonant NLS-KdV system is globally well-posed for any $s>11/13$.

- $\beta=0$. The condition (5.1) is fulfilled when

$$-1+\frac{5}{24}8(1-s)+3(1-s)<(1-s),\ \text{i.e.}\ ,s>8/11$$

and

$$-2+\frac{1}{2}8(1-s)+4(1-s)<(1-s),\ \text{i.e.}\ ,s>5/7;$$

Similarly, the condition (5.2) is verified for

$$-1+\frac{5}{6}8(1-s)+3(1-s)<2(1-s),\ \text{i.e.}\ ,s>20/23;$$

$$-\frac{2}{3}+\frac{5}{6}8(1-s)+3(1-s)<2(1-s),\ \text{i.e.}\ ,s>8/9;$$

$$-\frac{3}{2}+\frac{7}{8}8(1-s)+3(1-s)<2(1-s),\ \text{i.e.}\ ,s>13/16;$$

$$-1+\frac{1}{2}8(1-s)+4(1-s)<2(1-s),\ \text{i.e.}\ ,s>5/6$$

and

$$-2+\frac{1}{2}8(1-s)+6(1-s)<2(1-s),\ \text{i.e.}\ ,s>3/4.$$

Hence, we obtain that the resonant NLS-KdV system is globally well-posed for any $s>8/9$.

References

[1] A. Arbieto, A. Corcho and C. Matheus, *Rough solutions for the periodic Schrödinger - Korteweg - deVries system,* Preprint (2005).

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations,* Geometric and Functional Anal., **3** (1993), 107–156, 209–262.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$,* J. Amer. Math. Soc., **16** (2003), 705–749.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Multilinear estimates for periodic KdV equations, and applications,* J. Funct. Analysis, **211** (2004), 173–218.

[5] A. J. Corcho, and F. Linares, *Well-posedness for the Schrödinger - Kortweg-de Vries system,* Preprint (2005).

[6] H. Pecher, *The Cauchy problem for a Schrödinger - Kortweg - de Vries system with rough data,* Preprint (2005).

[7] M. Tsutsumi, *Well-posedness of the Cauchy problem for a coupled Schrödinger-KdV equation,* Math. Sciences Appl., **2** (1993), 513–528.

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