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HAUSDORFF DIMENSION FOR NON-HYPERBOLIC REPELLERS II: DA DIFFEOMORPHISMS

Vanderlei Horita and Marcelo Viana

Departamento de Matemática, IBILCE/UNESP
Rua Cristóvão Colombo, 2265
15055-S. J. Rio Preto, SP, Brazil

IMPA, Estrada Dona Castorina 110
22460-320 Rio de Janeiro, RJ, Brazil

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Abstract. We study non-hyperbolic repellers of diffeomorphisms derived from transitive Anosov diffeomorphisms with unstable dimension 2 through a Hopf bifurcation. Using some recent abstract results about non-uniformly expanding maps with holes, by ourselves and by Dysman, we show that the Hausdorff dimension and the limit capacity (box dimension) of the repeller are strictly less than the dimension of the ambient manifold.

1. **Introduction and Statement of Results.** In this paper we study the Hausdorff dimension of a class of non-hyperbolic repellers constructed by deformation of globally hyperbolic (Anosov) diffeomorphisms in dimension 3 or higher. A pattern we have in mind is described in Figure 1. A fixed point p of some Anosov diffeomor-

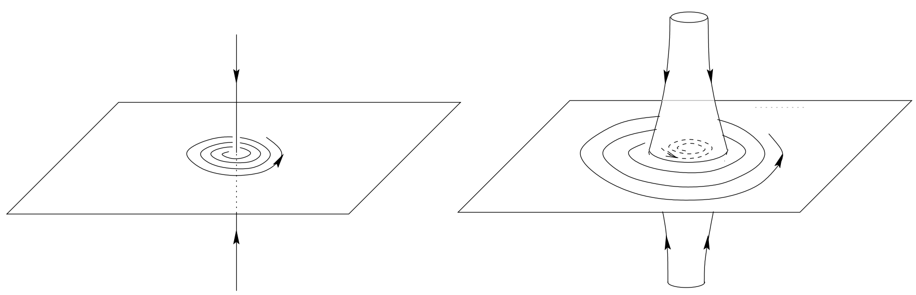

Figure 1. A Hopf bifurcation

phism goes through a Hopf bifurcation, and becomes an attractor. The complement Λ of the basin of attraction $W^s(p)$ is a repeller for the new diffeomorphism. *Does Λ have zero Lebesgue measure (volume)?* Even more, *is the Hausdorff dimension of the repeller strictly less than the dimension of the ambient manifold?*

Fractals invariants such as the Hausdorff dimension and limit capacity play an important role in various areas of Dynamical Systems, and have attracted a great deal of attention. We refer the reader to Falconer [8], Palis-Takens [15], Pesin [16]

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for an updated panorama of the theory. Computing these fractal invariants is usually difficult, because they depend on the microscopic structure of the set. Not surprisingly, most methods require the set to be self-similar, meaning that small pieces of it look very much like the whole. And self-similarity often arises from the dynamical system being uniformly hyperbolic (contracting and or expanding) and conformal, possibly after some dimension reduction.

Neither of these properties holds in the present setting. On the one hand, the repeller contains an invariant circle that is produced by the Hopf bifurcation, and so it can never be hyperbolic. On the other hand, conformality being a non-generic property, these diffeomorphisms are usually not conformal, nor can they be reduced to conformal maps. Nevertheless, we are able to give a positive answer to the questions raised above: *the Hausdorff dimension and the limit capacity of the repeller are strictly less than the dimension of the ambient manifold;* in particular, Λ has volume zero.

These conclusions arise from applying to our diffeomorphisms general results of our own [12] (for Hausdorff dimension) and Dysman [7] (for limit capacity), proved for an abstract class of systems that we called maps with holes. Actually, we are able to prove that in the present setting the boundaries of the smoothness domains of the relevant maps with holes have limit capacity strictly smaller than the ambient dimension. This allows us to use a relatively simple version of our criterion in [12] (no need for "extended rectangles").

An interesting open question is whether the dimension of the repeller converges the ambient dimension as the map approaches the Hopf bifurcation. Dysman [7] does prove a similar continuity fact for maps with holes, when the diameter of the hole goes to zero. However, she requires uniform bounds, for instance on the expansion rate, that do not hold in our situation, and so those results do not apply immediately here.

Let us describe precisely the families of diffeomorphisms for which we claim these results. For the sake of clearness we focus on one such family: our arguments apply, in particular, to any other one in a C^5 neighborhood.

1.1. **Diffeomorphisms derived from Anosov.** Let $M = \mathbb{T}^3$ be the 3-dimensional torus. We consider a linear Anosov diffeomorphism G on M having one real eigenvalue $\lambda \in (0,1)$ and two complex conjugate eigenvalues $\sigma e^{\pm i\alpha}$ with $\sigma > 3$. Assume $k\alpha \notin 2\pi\mathbb{Z}$ for $k = 1,2,3,4$. In cylindrical coordinates (ρ,θ,z) defined close to the fixed point $(0,0,0)$, we have

$$G(\rho,\theta,z) = (\sigma\rho, \theta+\alpha, \lambda z).$$

We are going to derive from G a family of diffeomorphisms $\hat{g}_\mu$, $\mu \in [-1,1]$ going through a Hopf bifurcation at $\mu = 0$. For this purpose, we consider a C^∞ real valued function $\Phi(\mu,w,z)$ defined on the unit cube $[-1,1]^3$ such that, for some $C_0 > 0$ and some small $\delta_0 > 0$,

- (C_1) $\Phi(\mu,0,0) = 1-\mu \le \Phi(\mu,w,z)$ for all $w \ge 0$.
- (C_2) $\Phi(\mu,w,z) = \sigma$ when either $w \ge \delta_0$ or $|z| \ge \delta_0$.
- (C_3) $0 < \partial_w\Phi(\mu,w,z) \le C_0/\delta_0$ when $0 \le w < \delta_0$ and $|z| < \delta_0$.
- (C_4) There exist $\sigma_1 \in (1,\sigma)$ and $\delta_1 \in (0,\delta_0)$ such that $\Phi(\mu,w,z) > \sigma_1$ for all $w \ge \delta_1$ and $\partial_w\Phi(\mu,w,z) \ge \partial_w\Phi(\mu,0,0)$ for all $w \in [0,\delta_1]$.

See Figure 2. We take $\delta_0 > 0$ to be small enough so that the domain

$$\{(\rho,\theta,z) : \rho^2 \le \delta_0 \text{ and } |z| \le \delta_0\}$$

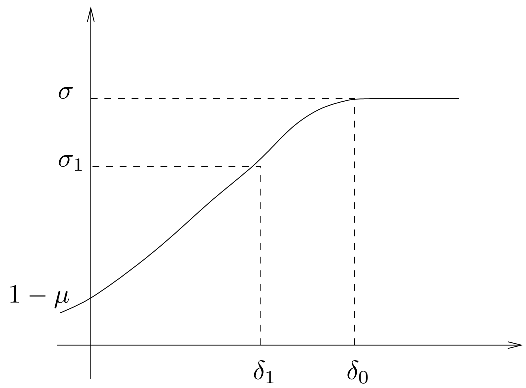

FIGURE 2. Graphic of $\Phi(\mu, \cdot, z)$

is contained in a small open neighborhood $V \subset M$ of the origin, such that the closure of V is itself contained in the interior of some rectangle of a Markov partition $\mathcal{S}$ for G. We deform G inside V to obtain a 1-parameter family $\hat{g}_\mu$ of diffeomorphisms coinciding with G outside V and whose restriction to V is given, in cylindrical coordinates, by

$$\hat{g}_\mu(\rho, \theta, z) = (\Phi(\mu, \rho^2, z)\rho, \theta + \alpha, \lambda z). \tag{1}$$

It is clear that the origin is a fixed point of $\hat{g}_\mu$ for all μ. We shall see that this fixed point goes through a Hopf bifurcation at $\mu = 0$: for $\mu > 0$ the fixed point becomes an attractor.

It follows that any family of diffeomorphisms $(g_\mu)_\mu$ close to $(\hat{g}_\mu)_\mu$ has a unique curve of fixed points p_μ close to the origin, and these fixed points also go through a Hopf bifurcation; the Hopf bifurcation parameter μ_* is close to zero and depends continuously on the family. Here closeness should be in the C^5 topology or higher; see Marsden-McCracken [14] and Section 2.2. Our first main result is

Theorem A. *Let $(g_\mu)_\mu$ be a family of diffeomorphisms in a C^5-neighborhood of $(\hat{g}_\mu)_\mu$. For each $\mu > \mu_*$ let Λ_μ be the complement of the basin of attraction $W^s(p_\mu)$ of the attracting fixed point p_μ. Then the limit capacity of this repeller satisfies*

$$\mathrm{c}(\Lambda_\mu) < 3 \quad \textit{for all } \mu > \mu_* \textit{ close to } \mu_*.$$

In particular, the Hausdorff dimension $\mathrm{HD}(\Lambda_\mu) < 3$ for all $\mu > \mu_*$ close to μ_*.

Recall that the *Hausdorff dimension* of a compact metric space X is the unique real number $\mathrm{HD}(X)$ such that $m_\alpha(X) = \infty$ for any $\alpha < \mathrm{HD}(X)$ and $m_\alpha(X) = 0$ for any $\alpha > \mathrm{HD}(X)$, where m_α is the Hausdorff α-measure:

$$m_\alpha(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{U \in \mathcal{U}} (\mathrm{diam}\ U)^\alpha : \mathcal{U} \text{ is an open covering of } X \text{ with} \right.$$
$$\left. \mathrm{diam}\ U \leq \varepsilon \text{ for all } U \in \mathcal{U} \right\}.$$

One always has $\mathrm{HD}(X) \leq \mathrm{c}(X)$, where the *limit capacity*, or *box dimension*, $\mathrm{c}(X)$ is defined by

$$\mathrm{c}(X) = \limsup_{\varepsilon \to 0} \frac{\log n(X, \varepsilon)}{|\log \varepsilon|}\,,$$

where $n(X, \varepsilon)$ is the smallest number of ε-balls needed to cover X.

The same arguments give a higher-dimensional version of this theorem, for families of diffeomorphisms derived from Anosov diffeomorphisms on the D-dimensional torus with two-dimensional expanding direction. The conclusion is that the limit capacity of Λ_μ is strictly smaller than D for all parameter values immediately after the bifurcation and, consequently, so is the Hausdorff dimension of Λ_μ.

We shall see in Section 2.1 that these diffeomorphisms g_μ are partially hyperbolic and admit a global central foliation $\mathcal{F}^c_\mu$, close to the unstable foliation of the Anosov diffeomorphism G. Our methods also yield an upper bound for the Hausdorff dimension of Λ_μ intersected with any leaf $\mathcal{F}^c_x$ of this central foliation.

Theorem B. *Let $(g_\mu)_\mu$ be a family of diffeomorphisms in a C^5-neighborhood of $(\hat{g}_\mu)_\mu$. Then*

$$\mathrm{c}(\Lambda_\mu \cap \mathcal{F}^c_x) < 2 \qquad (\textit{and so } \mathrm{HD}(\Lambda_\mu \cap \mathcal{F}^c_x) < 2)$$

for every $\mu > \mu_*$ close to μ_* and every central leaf $\mathcal{F}^c_x$.

It is a classical fact that uniformly hyperbolic repellers that do not coincide with the whole manifold have zero Lebesgue measure, if the map is C^2. See Bowen [3, Theorem 4.11]. It is also well-known that, in fact, their Hausdorff dimension is smaller than the dimension of the ambient manifold. See Shafikov, Wolf [17] for recent results on the dimension of hyperbolic sets and their stable and unstable sets. As we have already mentioned, our repellers Λ_μ are never hyperbolic, due to the presence of the invariant circle created in the center-unstable manifold $W^{cu}(p_\mu)$ of p_μ by the Hopf bifurcation. Indeed, hyperbolic sets of diffeomorphisms are expansive, and there are no expansive homeomorphisms on the circle; see Walters [19, Theorem 5.27].

Theorems A and B may also be thought of as generalizations of results of Díaz-Viana [6], who studied attractors of maps derived from Anosov diffeomorphisms in 2 dimensions. They exhibited an open set of families $(f_t)_t$ of diffeomorphisms of $\mathbb{T}^2$ crossing the boundary of Anosov systems through a saddle-node bifurcation, such that the Hausdorff dimension of the attractor varies discontinuously at the parameter of bifurcation. In contrast, they also showed that for other families of diffeomorphisms of $\mathbb{T}^2$, with a more degenerate bifurcation, the Hausdorff dimension may depend continuously on the parameter. Their situation is much simpler than the one here, because the problem can be reduced to dimension 1, by projecting along an invariant foliation, and also because their attractors are hyperbolic already immediately after the bifurcation.

Let us mention a few other results in a related context. Carvalho, in [5], considered a situation similar to ours giving rise to non-hyperbolic attractors Λ_μ (roughly speaking, we consider the inverses of her maps), and she proved that there exists a unique Sinai-Ruelle-Bowen measure supported on Λ_μ. This was extended by Bonatti-Viana [2] to a large class of partially hyperbolic attractors. These authors also observed in [2, Section 6.1] that the attractors in [5] have empty interior. Moreover, strong-unstable leaves are dense in them and, consequently, these attractors are transitive.

A number of interesting papers on the dimension theory of non-conformal systems have appeared recently. Let us mention, in particular, Barreira [1], Shafikov-Wolf [17], and Luzia [13], dealing with hyperbolic systems, and Gelfert [9], where non-hyperbolic dynamics are also considered.

2. **Preliminaries.** In conditions (H_1)–(H_6) below we summarize a few consequences of hypotheses (C_1)–(C_4) that suffice for the proofs of Theorems A and B. Throughout, $(g_\mu)_\mu$ is a family of diffeomorphisms sufficiently close to $(\hat{g}_\mu)_\mu$, as in the statement of the theorems.

2.1. **Partial hyperbolicity and invariant foliations.** We begin by proving

- The diffeomorphisms g_μ are partially hyperbolic for every μ close to zero: there exists a splitting $TM = E^{ss}_\mu \oplus E^{cu}_\mu$ of the tangent bundle, invariant under the derivative Dg_μ, dominated, and such that E^{ss}_μ is uniformly contracting.

Since partial hyperbolicity is an open property, relative to the C^1 topology, it is enough to prove that $\hat{g}_\mu$ is partially hyperbolic for every small μ. This is done in the following proposition:

Proposition 2.1. *Fix $\varepsilon > 0$ and assume δ_0 is small. Then for every μ in a neighborhood of zero, there exists a $D\hat{g}_\mu$-invariant splitting $TM = E^{ss}_\mu \oplus E^{cu}_\mu$ of the tangent bundle, and there exist positive constants K_s, K_c, K such that*

(a) *E^{cu}_μ coincides with the (ρ, θ)-plane;*
(b) *$\|D\hat{g}^n_\mu | E^{ss}_\mu\| \le K_s \lambda^n$ for all $n \ge 1$ (uniform contraction);*
(c) *$\|D\hat{g}^n_\mu | E^{cu}_\mu\| \le K_c(\sigma + \varepsilon)^n$ for all $n \ge 1$;*
(d) *$\|D\hat{g}^{-n}_\mu | E^{cu}_\mu\| \le K(1-\mu)^{-n}$ for all $n \ge 1$;*
(e) *$\|D\hat{g}^{-n}_\mu | E^{cu}_\mu\| \, \|D\hat{g}^n_\mu | E^{ss}_\mu\| < 1/2$ for some $n \ge 1$ (domination).*

Proof. Up to changing the constants, the conclusions are independent of the choice of the norm. For simplicity, we consider the Euclidean norm $\|(\dot\rho, \dot\theta, \dot z)\|^2 = \dot\rho^2 + \rho^2\dot\theta^2 + \dot z^2$ (this extends smoothly to the origin). Define the cone field

$$C^{cu}_x = \big\{(\dot\rho, \dot\theta, \dot z) \in T_x M \text{ such that } |\dot z| \le \kappa \|(\dot\rho, \dot\theta)\|\big\},$$

where κ is some small positive constant to be chosen in the sequel, depending only on λ and Φ. In cylindrical coordinates,

$$D\hat{g}_\mu = \begin{pmatrix} \Phi + \partial_w\Phi \cdot 2\rho^2 & 0 & \partial_z\Phi \cdot \rho \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where Φ and its derivatives are computed at (μ, ρ^2, z). Hence,

$$D\hat{g}_\mu \begin{pmatrix} \dot\rho \\ \dot\theta \\ \dot z \end{pmatrix} = \begin{pmatrix} \Phi\dot\rho + \partial_w\Phi \cdot 2\rho^2\dot\rho + \partial_z\Phi \cdot \rho\dot z \\ \dot\theta \\ \lambda\dot z \end{pmatrix} =: \begin{pmatrix} \dot\rho_1 \\ \dot\theta_1 \\ \dot z_1 \end{pmatrix}$$

for every tangent vector $(\dot\rho, \dot\theta, \dot z)$. Recall that $\Phi \ge 1 - \mu$ and $\partial_w\Phi > 0$, by (C_1) and (C_3). Moreover, $|\partial_z\Phi|$ is bounded by some constant C, because Φ is C^∞. Therefore, for any tangent vector in C^{cu}_x we have

$$|\dot\rho_1| \ge (\Phi + \partial_w\Phi \cdot 2\rho^2)|\dot\rho| - |\partial_z\Phi \cdot \rho||\dot z| \ge (1-\mu)|\dot\rho| - C\kappa\|(\dot\rho, \dot\theta)\|.$$

$$\rho_1|\dot\theta_1| = \Phi\rho|\dot\theta| \ge (1-\mu)\rho|\dot\theta|$$

and so, assuming that μ and κ are sufficiently small,

$$\|(\dot\rho_1, \dot\theta_1)\|^2 \ge (1-\mu)^2\|(\dot\rho, \dot\theta)\|^2 - 2C\kappa|\dot\rho|\,\|(\dot\rho, \dot\theta)\| \ge \lambda^2\|(\dot\rho, \dot\theta)\|^2.$$

It follows that

$$|\dot z_1| = |\lambda\dot z| \le \lambda\kappa\|(\dot\rho, \dot\theta)\| \le \kappa\|(\dot\rho_1, \dot\theta_1)\|$$

and this proves that the cone field C^{cu} is positively invariant.

Consequently, the dual cone field

$$C_x^{ss} = \{(\dot\rho, \dot\theta, \dot z) \in T_x M \text{ such that } |\dot z| \ge \kappa \|(\dot\rho, \dot z)\|\},$$

is negatively invariant. Let us check that C^{ss} is a stable cone field, that is, its vectors are uniformly expanded by backward iterates. Note that for any vector $(\dot\rho, \dot\theta, \dot z) \in C^{ss}$

$$|\dot z| \le \|(\dot\rho, \dot\theta, \dot z)\| \le (1 + \kappa^{-1})|\dot z|\,. \tag{2}$$

By negative invariance, every $D\hat g_\mu^{-n} \cdot (\dot\rho, \dot\theta, \dot z) = (\cdots, \cdots, \lambda^{-n}\dot z)$ is also in the cone field. Applying (2) to each of the iterates we get that

$$\|D\hat g_\mu^{-n} \cdot (\dot\rho, \dot\theta, \dot z)\| \ge (1 + \kappa^{-1})^{-1}\lambda^{-n}\|(\dot\rho, \dot\theta, \dot z)\|$$

for all $n \ge 1$. This shows that the vectors in C^{ss} are expanded by negative iterates of the derivative, with expansion rate λ^{-1}.

Existence of these invariant cone fields implies that $D\hat g_\mu$ admits an invariant splitting $TM = E_\mu^{cu} \oplus E_\mu^{ss}$ with $E_\mu^{cu} \subset C^{cu}$ and $E_\mu^{ss} \subset C^{ss}$. In particular, E_μ^{ss} is uniformly contracting, as claimed in (b). Moreover, we may take E_μ^{cu} to coincide with the horizontal (ρ, θ)-direction, as stated in (a) since this is invariant under every $D\hat g_\mu$. We are left to prove the bounds in (c) and (d) concerning the behavior of the derivative along E_μ^{cu}.

Still in cylindrical coordinates, we have

$$D\hat g_\mu | E_\mu^{cu} = \begin{pmatrix} \Phi + \partial_w \Phi \cdot 2\rho^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that we consider $\|(\dot\rho, \dot\theta)\|^2 = \dot\rho^2 + \rho^2\dot\theta^2$ and that $\rho_1 = \Phi\rho$. Using conditions (C_2) and (C_3) we find

$$\|D\hat g_\mu | E_\mu^{cu}\| \le \max\{\Phi + 2\rho^2\partial_w\Phi, \Phi\} \le \sigma + 2\delta_0^2(C_0/\delta_0) \le \sigma + \varepsilon, \tag{3}$$

if δ_0 is chosen sufficiently small, and

$$\|(D\hat g_\mu | E_\mu^{cu})^{-1}\| \le \max\{(\Phi + 2\rho^2\partial_w\Phi)^{-1}, \Phi^{-1}\} = \Phi^{-1} \le (1-\mu)^{-1}. \tag{4}$$

This implies conclusions (c) and (d). The domination condition (e) is an immediate consequence of (b) and (d), assuming μ is small. ∎

Now let us define, for each $n \ge 1$ and $x \in M$,

$$a_\mu^n(x) = \big\|(Dg_\mu^n | E_\mu^{cu}(x))^{-1}\big\|,\ b_\mu^n(x) = \big\|Dg_\mu^n | E_\mu^{ss}(x)\big\|,\ c_\mu^n(x) = \big\|Dg_\mu^n | E_\mu^{cu}(x)\big\|.$$

We use the notations $\hat a_\mu^n$, $\hat b_\mu^n$, $\hat c_\mu^n$ when $g_\mu = \hat g_\mu$. We are going to prove

- There is $n \ge 1$ such that

$$\sup_{x \in M} \left(a_\mu^n\, b_\mu^n\, c_\mu^n\right) < 1 \quad \text{for all } \mu \text{ close to zero.} \tag{5}$$

Once again, since this is an open property (because the invariant subbundles E_μ^{ss} and E_μ^{cu} depend continuously on the map) in the C^1 topology, we only need to prove it for $g_\mu = \hat g_\mu$. This is done in

Corollary 2.2. *There is $n \ge 1$ such that $\sup_{x\in M}\left(\hat a_\mu^n\, \hat b_\mu^n\, \hat c_\mu^n\right) < 1$ for every μ close to zero.*

Proof. From the proposition we have

$$\left\|(D\hat{g}_\mu^n|E_\mu^{cu})^{-1}\right\| \cdot \left\|D\hat{g}_\mu^n|E_\mu^{ss}\right\| \cdot \left\|D\hat{g}_\mu^n|E_\mu^{cu}\right\| \leq K(1-\mu)^{-n}K_s\lambda^n K_c(\sigma+\varepsilon)^n.$$

Since G is a linear automorphism, we must have $\lambda\sigma^2 = \det DG = 1$. Hence, $(1-\mu)^{-1}\lambda(\sigma+\varepsilon) < 1$, as long as ε and μ are small enough. Therefore, the conclusion holds for every n sufficiently large. ∎

Remark 2.3. Since $\hat{g}_\mu$ coincides with G outside V it expands the center-unstable bundle E_μ^{cu} there:

$$\left\|(D\hat{g}_\mu(x)|E_\mu^{cu}(x))^{-1}\right\| \leq \sigma^{-1} \quad \text{for all } x \in M \setminus V.$$

Then the same is true for any nearby g_μ (reducing $\sigma > 3$ slightly, if necessary).

Let us comment on the significance of (5). By Hirsch-Pugh-Shub [10, 11], this property ensures that across the bifurcation the map g_μ has an invariant *center-unstable* (or *central*) *foliation* $\mathcal{F}_\mu^c$, whose tangent space E_μ^{cu} is everywhere contained in C_μ^{cu}, and an invariant *strong-stable foliation* $\mathcal{F}_\mu^{ss}$, whose tangent space E_μ^{ss} is everywhere contained in C_μ^{ss} and whose leaves are uniformly contracted by g_μ. In fact, this foliation is of class $C^{1+\delta}$ on M:

Corollary 2.4. *There exists $\delta > 0$ such that for every μ close to zero, the diffeomorphism g_μ admits a $C^{1+\delta}$ foliation $\mathcal{F}_\mu^{ss}$ of co-dimension 2, invariant by g_μ and whose tangent bundle is everywhere contained in the cone field C_μ^{ss}. Consequently, the leaves of $\mathcal{F}_\mu^{ss}$ are uniformly contracted by g_μ.*

Proof. This is an application of the Invariant Manifold Theorem 4.1 in [11]. Alternatively, a direct proof can be given following closely the arguments of Theorem 6.3 of Hirsch-Pugh [10]. We just sketch the main steps. For simplicity we suppose that $n = 1$ in Corollary 2.2. Consider the graph transform

$$T_\mu : \big(E_x\big)_x \mapsto \big(Dg_\mu^{-1}\, E_{g_\mu x}\big)_x$$

acting on the space of continuous vector bundles of codimension 2. This operator preserves the set of subbundles everywhere contained in the strong-stable cone field C_μ^{ss}, and is a contraction on that set, with respect to the uniform norm. It follows that E_μ^{ss} is the unique fixed point, and it attracts every orbit of T_μ. Using the fact that E_μ^{ss} is uniformly contracting one shows that it is uniquely integrable: the integral foliation $\mathcal{F}_\mu^{ss}$ is dynamically characterized by the property that its leaves are exponentially contracted by the forward iterates of g_μ.

To show that $\mathcal{F}_\mu^{ss}$ is a C^1 foliation and, even more, E_μ^{ss} is a C^1 bundle, one uses the graph transform $\mathcal{T}_\mu : (E, \mathcal{E}) \mapsto (T_\mu E, \mathcal{T}_\mu^E \mathcal{E})$ induced by T_μ on the level of 1'st jets: it satisfies

$$\mathcal{T}_\mu\big(E, DE\big) = \big(T_\mu E, D(T_\mu E)\big) \tag{6}$$

for every C^1 bundle E. Property (5) is designed to ensure that $\mathcal{T}_\mu$ is a contraction fiberwise: $\mathcal{T}_\mu^E$ is a contraction for every E, with uniform contraction rate. It follows that $\mathcal{T}_\mu$ has a unique fixed point $(E_\mu^{ss}, \mathcal{E}_\mu^{ss})$. Using relation (6) and the fact that the fixed point is a global attractor, one deduces that $\mathcal{E}_\mu^{ss} = DE_\mu^{ss}$. Hence, E_μ^{ss} is a C^1 bundle and, in particular, $\mathcal{F}_\mu^{ss}$ is a C^1 foliation. Finally, fixing $n \geq 1$ and $\delta > 0$ small enough so that

$$a_\mu^n b_\mu^n (c_\mu^n)^{1+\delta} < 1$$

we get that $\mathcal{E}_\mu^{ss}$ is δ-Hölder and so E_μ^{ss} is a $C^{1+\delta}$ bundle. ∎

Also by [10, 11], the center-unstable foliation is topologically conjugate to the unstable foliation $\mathcal{F}^u$ of G, in the sense that there is a homeomorphism sending leaves of one to leaves of the other: that is because $\mathcal{F}^c$ remains normally contracting throughout the isotopy (one must also check the technical plaque-expansiveness condition). In particular all the center-unstable leaves are dense in M.

2.2. **Hopf bifurcations.** Next we prove that any family of diffeomorphisms C^5 close to the family $(\hat{g}_\mu)_\mu$ satisfies

- There exists a curve $(p_\mu)_\mu$ of fixed points of $(g_\mu)_\mu$ and there exists μ_* close to zero such that p_μ is a hyperbolic saddle of g_μ for $\mu < \mu_*$, it goes through a generic Hopf bifurcation at $\mu = \mu_*$, and becomes an attractor for $\mu > \mu_*$, remaining all the time inside V.

First, let us recall the notion and some basic facts about Hopf bifurcations. See also Marsden-McCracken [14]. Let $\varphi_\mu : U \to \mathbb{R}^2$ be a C^1 one-parameter family of embeddings of some open subset U of $\mathbb{R}^2$. Assume there exists a curve p_μ of fixed (or periodic) points of each φ_μ, and there exists some parameter value μ_* such that

(a) $D\varphi_\mu(p_\mu)$ has complex eigenvalues $\lambda(\mu) \neq \overline{\lambda(\mu)}$ for all μ close to μ_*;
(b) $|\lambda(\mu_*)| = 1$ but $\lambda(\mu_*)^k \neq 1$ for all for $k = 1, 2, 3, 4$;
(c) the derivative of the norm $|\lambda(\mu)|$ is non-zero at $\mu = \mu_*$.

Depending on whether the derivative is negative or positive, the fixed point p_μ changes from a repeller to an attractor, or from an attractor to a repeller, as the parameter crosses μ_*.

Assuming the family φ_μ is of class C^5, this bifurcation admits the following normal form in convenient polar coordinates (see [14]):

$$\begin{aligned} &\varphi_\mu(\rho,\theta) = F_\mu(\rho,\theta) + O(\rho^5) \quad \text{with} \\ &F_\mu(\rho,\theta) = \big(a(\mu)\rho + b_1(\mu)\rho^3, \theta + \phi(\mu) + b_2(\mu)\rho^2\big), \end{aligned} \tag{7}$$

where $a(\mu)$ and $\phi(\mu)$ are, respectively, the norm and the argument of $\lambda(\mu)$. Assume, furthermore, that

(d) the coefficient $b_1(\mu)$ is non-zero at $\mu = \mu_*$.

Then we say that $(\varphi_\mu)_\mu$ *unfolds a generic Hopf bifurcation* at the parameter value $\mu = \mu_*$. A distinctive feature of this bifurcation is the formation of a φ_μ-invariant circle $\mathcal{C}(\varphi_\mu)$, close to the F_μ-invariant circle

$$\mathcal{C}(F_\mu) = \{(\rho,\theta) : b_1(\mu)\rho^2 = 1 - a(\mu)\}. \tag{8}$$

These invariant circles are defined for either $\mu > \mu_*$ or $\mu < \mu_*$ (close to μ_*) depending on the signs of $da/d\mu$ and $b_1(\mu)$ at $\mu = \mu_*$.

Now let $(\psi_\mu)_\mu$ be a family of diffeomorphisms on some manifold, admitting a smooth family of fixed (or periodic) points p_μ and a parameter value μ_* such that

(i) for $\mu = \mu_*$ the derivative $D\psi_\mu(p_\mu)$ has a pair of complex conjugate eigenvalues with norm 1, and all the other eigenvalues are all inside or all outside the unit circle;
(ii) there exists a smooth family N_μ, defined for all μ close to μ_*, of ψ_μ-invariant (central) manifolds through p_μ, of dimension 2 and tangent to the eigenspace associated to the pair of complex eigenvalues in (i);
(iii) the restriction $\varphi_\mu = \psi_\mu \mid N_\mu$ unfolds a generic Hopf bifurcation at the parameter μ_*, in the sense of conditions (a)-(d).

Then we also say that $(\psi_\mu)_\mu$ *unfolds a generic Hopf bifurcation* at the parameter μ_*. Notice that the fixed point changes from an attractor/repeller to a codimension-2 saddle, or just the other way around, as μ crosses the bifurcation parameter. According to the previous comments, as this happens an invariant circle $\mathcal{C}(\psi_\mu) = \mathcal{C}(\varphi_\mu)$ is formed inside the central manifold N_μ either for $\mu > \mu_*$ or for $\mu < \mu_*$.

Unfolding a generic Hopf bifurcation is an open property in the space of C^5 one-parameter families (the parameter μ_* depends on the family, of course); see [14]. Thus, in order to justify the statement made at the beginning of this section it is enough to prove the corresponding fact for the family $g_\mu = \hat{g}_\mu$:

Proposition 2.5. *The family of diffeomorphisms $(\hat{g}_\mu)_\mu$ unfolds a generic Hopf bifurcation, with $p_\mu = 0$, $\mu_* = 0$, $a(\mu) = 1 - \mu$, $b_1(\mu) > 0$ and $b_2(\mu) = 0$.*

Proof. Take N_μ to be the horizontal (ρ, θ)-plane through the fixed point $p_\mu = 0$. By construction and Proposition 2.1(a) this is a $\hat{g}_\mu$-invariant surface. The definition (1) gives that the restriction φ_μ of $\hat{g}_\mu$ to N_μ is described in polar coordinates by

$$\varphi_\mu(\rho, \theta) = (\Phi(\mu, \rho^2, 0)\rho, \theta + \alpha).$$

Conditions (i) and (ii) in the definition of Hopf bifurcation are clear from these remarks, so let us now check φ_μ satisfies conditions (a)-(d). Expanding $\Phi(\mu, w, 0) = (1 - \mu) + b_1(\mu)w + O(w^2)$ (recall (C_1)), we get

$$\varphi_\mu(\rho, \theta) = ((1 - \mu)\rho + b_1(\mu)\rho^3, \theta + \alpha) + O(\rho^5).$$

This expression shows that $a(\mu) = 1 - \mu$ and $b_2(\mu)$ is identically zero. Moreover, $b_1(\mu) = \partial_w \Phi(\mu, 0, 0) > 0$ by condition (C_3). From $a(0) = 1$ and $\phi(0) = \alpha$ we get that the eigenvalues of $D\varphi_0(0)$ are $e^{\pm i\alpha}$, and this gives (a). We have assumed that $k\alpha \notin 2\pi\mathbb{Z}$ for $k = 1, 2, 3, 4$ and this corresponds precisely to (b). Condition (c) is contained in $da/d\mu = -1$, and we already checked that $b_1(\mu) \neq 0$, as requested in (d). ■

In the next proposition we analyze the local dynamics of $f_\mu = g_\mu \mid \{z = 0\}$. Let us consider the neighborhood $V_1 = \{(\rho, \theta) : \rho^2 \leq \delta_1\}$ of $p_\mu = 0$ restricted to $\{z = 0\}$, where $\delta_1 > 0$ is as in condition (C_4).

Proposition 2.6. *For μ close to zero we have*

1. *f_μ is expanding outside V_1 : $\|Df_\mu^{-1}\|^{-1} \geq \sigma_1$*
2. *$b_1(\mu)\rho^2 \leq 1/1000$ for all $(\rho, \theta) \in V_1$*
3. *$\|Df_\mu^{-1} - DF_\mu^{-1}\| \leq b_1(\mu)\rho^2/10$ inside V_1*
4. *$|\det Df_\mu - \det DF_\mu| \leq b_1(\mu)\rho^2/100$ inside V_1*
5. *and the normal form F_μ satisfies $b_1(\mu) > 2|b_2(\mu)|$.*

Proof. Throughout we assume that μ is close enough to zero for the arguments to go through. By (C_2) and (C_4), the point p_μ has a neighborhood $\{(\rho, \theta, z) : \rho^2 \leq \delta_1$ and $|z| \leq \delta_0|\} \subset V$ outside of which $\hat{g}_\mu$ expands the center-unstable direction uniformly:

1. $\|(D\hat{g}_\mu \mid E_\mu^{cu})^{-1}\|^{-1} > \sigma_1$ outside that neighborhood.

Clearly, the same remains true for any small perturbation g_μ. Recall that $Df_\mu = Dg_\mu \mid E_\mu^{cu}$. Combining (4) and our choice of σ_1 we find that

$$\left\|(D\hat{g}_\mu | E_\mu^{cu})^{-1}\right\|^{-1} \geq \Phi(\mu, \rho^2, z) > \sigma_1$$

whenever $\rho^2 \geq \delta_1$. Clearly, this remains true for any g_μ close to $\hat{g}_\mu$. This proves part 1. Taking δ_1 sufficiently small from the beginning we get part 2. Parts 3 and

4 follow directly from the fact that f_μ and its normal form differ by $O(\rho^5)$. Finally, part 5 follows from the fact that $b_1(\mu) > 0$ and $b_2(\mu) = 0$ for every μ. ∎

Note that in this setting the fixed point p_μ changes from a saddle (with two expanding directions) to an attractor, as μ increases past $\mu_* = 0$. Since $b_1(\mu) > 0$ and $da/d\mu < 0$, the invariant circle $\mathcal{C}(g_\mu) = \mathcal{C}(\varphi_\mu)$ is defined for $\mu > \mu_*$. Relation (8) says that $\mathcal{C}(F_\mu)$ for the normal form F_μ of $g_\mu = \hat{g}_\mu$ is the round circle of radius

$$\rho_0 = \left(\frac{\mu}{b_1(\mu)}\right)^{1/2}.$$

In general, the invariant circle $\mathcal{C}(g_\mu)$ is described by

$$a(\mu)\rho + b_1(\mu)\rho^3 + O(\rho^5) = \rho.$$

Assuming g_μ is close to $\hat{g}_\mu$ and μ is close to the bifurcation parameter μ_*, the circle is close to the origin. Then the third term is negligible, relative to the second one, and we conclude that $\mathcal{C}(g_\mu)$ is contained in a corona bounded by circles of radii $\rho_1 < \rho_2$ with $\rho_i \approx \big((1 - a(\mu)/b_1(\mu)\big)^{1/2}$ for $i = 1, 2$, where $\approx$ means equality up to a factor which can be made arbitrarily close to 1 if g_μ is close to $\hat{g}_\mu$ and μ is close to the bifurcation parameter. Moreover, $1 - a(\mu) \approx (\mu - \mu_*)$, since $da/d\mu \approx 1$. Thus,

$$\rho_i \approx \left(\frac{\mu - \mu_*}{b_1(\mu)}\right)^{1/2} \quad \text{for } i = 1, 2. \tag{9}$$

Remark 2.7. Up to reparametrizing the family of diffeomorphisms, we may always suppose that $\mu_* = 0$ and $a(\mu) = 1 - \mu$, and we do so in what follows.

2.3. **Hyperbolic deformation and Markov partitions.** For any family $(g_\mu)_\mu$ of diffeomorphisms close to $(\hat{g}_\mu)_\mu$ we also check that

- There exists a smooth family $(G_\mu)_\mu$ of Anosov diffeomorphisms of $\mathbb{T}^3$ close to G and such that $g_\mu = G_\mu$ outside V for all μ.

Proof. Note that when $g_\mu = \hat{g}_\mu$ it suffices to consider $G_\mu = G$ for all μ. To treat the general case, consider a C^∞ function $\omega : \mathbb{T}^3 \to \mathbb{R}$ such that $\omega = 0$ on a neighborhood of the origin and $\omega = 1$ on the complement of V. Then define

$$G_\mu(x) = G(x) + \omega(x)(g_\mu(x) - \hat{g}_\mu(x))$$

(the sum is in the torus). It is clear that $G_\mu = g_\mu$ outside V. Moreover, $g_\mu - \hat{g}_\mu$ has small C^1 norm. This means that G_μ is uniformly close to G in the C^1 topology, and so it is an Anosov diffeomorphism. ∎

The main application of this property is to ensure that the maps g_μ have (not necessarily generating) Markov partitions:

- There exists a Markov partition $\mathcal{S}_\mu$ for g_μ such that V is contained in some of the Markov rectangles and the image of any Markov rectangle intersects less than $\eta \leq 1000\sigma^2$ Markov rectangles.

Let us recall the notions involved and comment on this property. See also [3]. A *Markov rectangle* for an Anosov diffeomorphism $f : M \to M$ is a (small) compact domain $S_i \subset M$ which coincides with the closure of its interior, and such that for every $x, y \in S_i$ the unique point $[x, y]$ in $W^u_{loc}(x) \cap W^s_{loc}(y)$ is also in S_i. For each $x \in S_i$ let $W^s_i(x)$ and $W^u_i(x)$ denote the connected components of, respectively, $W^s(x) \cap S_i$ and $W^u(x) \cap S_i$ that contain x.

Then a *Markov partition* for $f : M \to M$ is a finite covering $\mathcal{S}$ of M by Markov rectangles with pairwise disjoint interiors, such that the following crucial condition (10) is satisfied:

$$f(W_i^s(x)) \subset W_j^s(f(x)) \quad \text{and} \quad f(W_i^u(x)) \supset W_j^u(f(x)) \tag{10}$$

for all $x \in S_i \cap f^{-1}(S_j)$. See [3, 18] for a proof that Anosov diffeomorphisms (and, more generally, hyperbolic basic sets) always admit Markov partitions with arbitrarily small diameter. Moreover, such partitions are *generating*: there is at most one point with any given itinerary relative to the partition.

The stable boundary $\partial^s S_i$ of a Markov rectangle is the set of points x which are not interior to $W_i^u(x)$, relative to the local unstable manifold. The dual notion of unstable boundary $\partial^u S_i$ is defined similarly. Using the definition of Markov rectangle, and the fact that $(x, y) \mapsto [x, y]$ is a continuous map, one easily sees that $\partial^s S_i$ is a union of stable sets $W_i^s(z)$ and $\partial^u S_i$ is a union of unstable sets $W_i^u(z)$. We denote $\partial^s \mathcal{S} = \cup_i \partial^s S_i$ and $\partial^u \mathcal{S} = \cup_i \partial^u S_i$. The Markov property (10) implies that $f(\partial^s \mathcal{S}) \subset \partial^s \mathcal{S}$ and $f^{-1}(\partial^u \mathcal{S}) \subset \partial^u \mathcal{S}$.

Using a Markov partition, we may associate to f a quotient map ϕ in the space of all stable leaves $W_i^s(x)$, sending each $W_i^s(x)$ to the stable leaf $W_j^s(y)$ that contains it (ϕ may be multivalued at the boundaries of the Markov rectangles). This quotient map is uniformly expanding and has a Markov property: denoting $R_i = \{W_i^s(x) : x \in S_i\}$,

$$\phi(\text{int}(R_i)) \cap \text{int}(R_j) \neq \emptyset \quad \Rightarrow \quad \phi(R_i) \supset R_j\,. \tag{11}$$

Going back to our setting, recall that we have fixed some Markov partition $\mathcal{S}$ for the Anosov diffeomorphism G such that the closure of V is contained in the interior of some Markov rectangle S_0. By construction, every G_μ is C^1 close to G. Since Anosov diffeomorphisms are structurally stable, it follows that the two maps must be topologically conjugate: there exists some homeomorphism $h_\mu : M \to M$ such that $G_\mu \circ h_\mu = h_\mu \circ G$. Then we may consider the family

$$\mathcal{S}_\mu = \left\{ S_{i,\mu} = h_\mu(S_i) \ : \ S_i \in \mathcal{S} \right\}$$

and it is a Markov partition for G_μ. Moreover, the conjugacy h_μ is C^0 close to the identity if G_μ is close to G.

Assuming $(g_\mu)_\mu$ is close enough to $(\hat{g}_\mu)_\mu$ we may ensure that the closure of V is contained in the interior of $S_{0,\mu} = h_\mu(S_0)$ for every μ. Since g_μ and G_μ coincide outside V, it follows that $g_\mu(S_{i,\mu}) = G_\mu(S_{i,\mu})$ for every i (including $i = 0$). Observe also that, since the forward iterates of all points in $\partial^s \mathcal{S}_\mu$ under G_μ never pass through the region V, they coincide with the corresponding iterates under g_μ.

Then the same is true for the partially hyperbolic diffeomorphisms g_μ just with strong-stable leaves in the place of stable manifolds. That is because the forward iterates of the points in $\partial^s(h_\mu(S_i))$ never pass through the perturbation region V, and so their local strong-stable leaves for g_μ coincide with their local stable manifolds for G_μ. It follows that we may still define a quotient map in the space of local strong-stable leaves of g_μ, and this is still Markov, in the sense of (11): the domains $R_{i,\mu}$ for the quotient maps of g_μ and G_μ coincide, and so do their images under those quotient maps.

It is in this sense that we say that $\mathcal{S}_\mu$ is also a Markov partition for our partially hyperbolic diffeomorphisms g_μ. Observe, however, that these Markov partitions are generally *not* generating.

Finally, the statement on the number of Markov rectangles intersecting the image of any of them is a direct consequence of the construction of $\mathcal{S}_\mu$ and the corresponding fact for the initial Anosov diffeomorphism G. Let us comment on this property. Roughly, the "average" number of rectangles intersected by each Markov rectangle is given by the Jacobian in the unstable/central direction, which for G is σ^2. Of course, there may be oscillations around this average, e.g. if some rectangles are much bigger than others. The factor 1000, which is certainly not optimal, provides room for such oscillations.

2.4. **Summary.** Let us summarize the conclusions of this section. The proofs of our theorems rely on properties (H_1)–(H_6) below only.

We are considering one-parameter families $g_\mu : M \to M$, $-1 \le \mu \le 1$, of C^r diffeomorphisms, $r \ge 5$, of the 3-dimensional torus such that

(H_1) There exists an open set $V \subset M$ and a continuous family $(G_\mu)_\mu$ of transitive Anosov diffeomorphisms with
$$\|DG_\mu^{-1} \mid E_\mu^u\|^{-1} \ge \sigma > 3$$
such that g_{-1} coincides with G_{-1}, and $g_\mu = G_\mu$ outside V, for all μ. Moreover, for all μ, the set V is contained in some rectangle of a Markov partition $\mathcal{S}_\mu$ of G_μ.

(H_2) There exists a curve $(p_\mu)_\mu$ of fixed (or periodic) points of $(g_\mu)_\mu$ and there exists μ_* such that p_μ is a hyperbolic saddle of g_μ for $\mu < \mu_*$, it goes through a generic Hopf bifurcation at $\mu = \mu_*$, and becomes an attractor for $\mu > \mu_*$, remaining all the time inside V.

(H_3) The diffeomorphisms g_μ are partially hyperbolic, for all μ close to zero: there exists a splitting $TM = E_\mu^{ss} \oplus E_\mu^{cu}$ of the tangent bundle, invariant under the derivative Dg_μ, dominated, and such that E_μ^{ss} is uniformly contracting.

(H_4) There is $n \ge 1$ such that, for every μ close to zero,
$$\sup_M \left\|(Dg_\mu^n|E_\mu^{cu})^{-1}\right\| \cdot \left\|Dg_\mu^n|E_\mu^{ss}\right\| \cdot \left\|Dg_\mu^n|E_\mu^{cu}\right\| < 1\,.$$

(H_5) There exists a neighborhood V_1 of p_μ restricted to $W_\mu^{cu}(p_\mu)$ with fixed radius and which satisfies conclusions 1–5 of Proposition 2.6.

(H_6) The Markov partition $\mathcal{S}_\mu$ for g_μ may be taken such that the maximum number η of rectangles the image of a rectangle intersects is less than $1000\sigma^2$.

3. **Proof of Theorems A and B.** In this section we prove the main theorems of this paper. We derive Theorem B from Theorem 1 of [12], Theorem 1 of Dysman [7] and our key Proposition 3.9. Then we prove Theorem A from Theorem B.

3.1. **Maps with holes.** The results of [12] and Dysman [7] are for an abstract model, called maps with holes. We begin by describing this model, and recalling the precise statements in those previous papers.

Let $f : M \to M$ be a map on a d-dimensional Riemannian manifold, $d \ge 1$ such that

(A_1) there exist domains $R_1, \ldots, R_m$ in M, with pairwise disjoint interiors, such that the restriction of f to each R_i is a $C^{1+\varepsilon}$ diffeomorphism onto some domain W_i that contains $R_1 \cup \cdots \cup R_m$. Moreover, the difference $H_i = W_i \setminus (R_1 \cup \cdots \cup R_m)$ has non-empty interior, the inner diameter of R_i is finite, and the boundaries $\partial R_1, \ldots, \partial R_m$ have limit capacity less than d.

By *domain* we always mean a compact path-connected subset. Figure 3 describes an example where $H_i = H$ and $W_i = W$ are the same for all i. The *repeller* of f in $R_1 \cup \cdots \cup R_m$ is the set of points Λ whose forward orbits never fall into the H_i, that is,

$$\Lambda = \{x : f^n(x) \in R_1 \cup \cdots \cup R_m \text{ for every } n \geq 0\}.$$

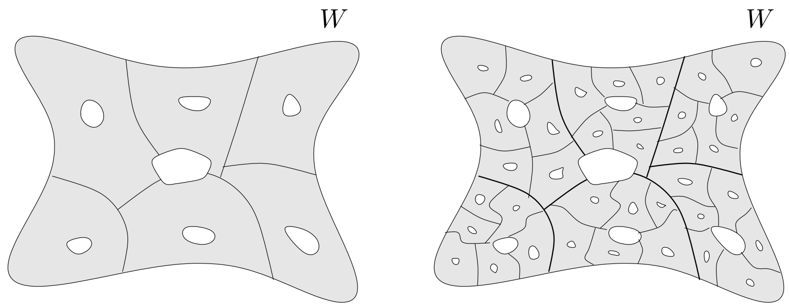

FIGURE 3. The repeller of a map with holes

The R_i need not be disjoint. The condition on the inner diameter means that there exists $L > 0$ such that any two points in R_i may be joined by a piecewise C^1 curve inside R_i and with length less than L.

Remark 3.1. It suffices to suppose that every W_i contains the union of the R_j over some subset of indices j, and the iterates of R_i eventually hit a hole, that is, they eventually contain some W_k such that $H_k = W_k \setminus \cup_j R_j$ has non-empty interior. See Remark 1 of [12].

Given $n \geq 1$, we call *n-cylinder* any set of the form

$$C(\alpha_1, \ldots, \alpha_n) = R_{\alpha_1} \cap f^{-1}(R_{\alpha_2}) \cap \cdots \cap f^{-n+1}(R_{\alpha_n})$$

with $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\{1, \ldots, m\}$. That is, an n-cylinder consists of all the points remaining in $R_1 \cup \cdots \cup R_m$, and sharing a given itinerary with respect to the family $\{R_1, \ldots, R_m\}$, up to time n. Clearly, n-cylinders form a covering of the repeller Λ, for each $n \geq 1$.

For each $n \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\{1, \ldots, m\}$, we consider the *average least expansion*

$$\phi_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \inf_{x \in C_j} \log \left\| Df^{-1}\big(f^j(x)\big) \right\|^{-1},$$

where the infimum is over all x in $C_j = C(\alpha_1, \ldots, \alpha_j)$. Throughout, $Df^{-i}(f^j(y))$ is to be understood as the inverse of the derivative $Df^i(f^{j-i}(y))$, for any y and $j \geq i$. Note that $\phi_n(\alpha_1, \alpha_2, \ldots, \alpha_n) > c > 0$ implies that the derivative Df^n expands every vector:

$$\|Df^{-n}(f^n(x))\| \leq \prod_{j=1}^{n} \|Df^{-1}(f^j(x))\| \leq e^{-cn} \quad \text{for all } x \text{ in } C(\alpha_1, \ldots, \alpha_n).$$

Denote by $\mathcal{Q}_n(c)$ the union of all length-n cylinders $C(\alpha_1, \ldots, \alpha_n)$ such that $\phi_n(\alpha_1, \ldots, \alpha_n) \leq c$. We also assume

(A_2) There exist $c>0$ and $c_1>0$ such that $\sum_{C\in\mathcal{Q}_n(c)}\operatorname{Leb}(C)\le e^{-c_1 n}$ for every n sufficiently large.

Theorem 3.2 (Theorem 1 in [12]). *Let $f:M\to M$ and Λ be as above, satisfying (A_1) and (A_2). Then* $\operatorname{HD}(\Lambda)<d$.

Condition (A_2) implies assumption (NU_2) of Dysman [7]

(NU_2) There exists $c_1>0$ such that, for every large n, we have $\operatorname{Leb}(\mathcal{B}_n(c))\le e^{-c_1 n}$,

where $\mathcal{B}_n(c)$ is the set of points that belong to some cylinder $C(\alpha_1,\dots,\alpha_n)$ such that $\phi_j(\alpha_1,\dots,\alpha_j)\le c$ for all $j\in\{1,\dots,n\}$. Indeed, this set $\mathcal{B}_n(c)$ is contained in the union of the cylinders $C\in\mathcal{Q}_n(c)$. Therefore, we have the following stronger result:

Theorem 3.3 (Theorem 1 in Dysman[7]). *Under the assumptions of Theorem 3.2, we have* $\mathrm{c}(\Lambda)<d$.

3.2. **Constructing a map with holes.** The first step is to associate to each diffeomorphism g_μ a 2-dimensional map with holes f_μ . Let $\mathcal{S}_\mu=\{S_0\,,S_1\,,\dots,S_m\}$ be a Markov partition for g_μ as in Section 2.3. Recall that, for each $x\in S_i$ we denote by $W_i^s(x)$ the connected component of $W^{ss}(x)\cap S_i$ that contains x. For each $i\ge 1$, fix a domain $R_{i,\mu}\subset W^c(p_\mu)$ that intersects each stable leaf $W_i^s(x)$ at exactly one point. Analogously for $i=0$, except that in this case we denote the domain by $R^*_{0,\mu}$ and we ask that $p_\mu\in R^*_{0,\mu}$. Let

$$W_\mu=R^*_{0,\mu}\cup R_{1,\mu}\cup\cdots\cup R_{m,\mu}\,.$$

Note that W_μ needs not be connected. Let $H_\mu=R^*_{0,\mu}\cap W^s_{loc}(p_\mu)$ and $R_{0,\mu}=R^*_{0,\mu}\setminus H_\mu$. Define $\pi_\mu:M\to W_\mu$ to be the projection along the leaves $W_j^s(x)$ inside each S_j . By Corollary 2.4, π_μ is of class $C^{1+\delta}$ restricted to each rectangle. Then define $f_\mu:W_\mu\to W_\mu$ by

$$f_\mu=\pi_\mu\circ g_\mu\,.$$

Note that π_μ is multi-valued on the boundaries of the rectangles, and so f_μ is also a multi-valued $C^{1+\delta}$ map. This f_μ is a map with hole, in the sense of (A_1), (A_2), and Remark 3.1, as we are going to explain.

3.3. **Checking the Markov and hyperbolicity conditions.** Let us check the Markov condition (A_1) for $\mu>0$ close enough to zero. For simplicity we often omit reference to the parameter in the subscripts of the Markov rectangles. Take $H_0=H_\mu$ and $H_i=\emptyset$ for $i\ge 1$. By construction, H_0 has non-empty interior. Since $\mathcal{S}_\mu$ is a Markov partition, every image $f_\mu(R_i)$ is a union of domains R_j over some subset $J(i)$ of indices j. Recall also that the Anosov maps G_μ are transitive, because they are conjugate to a linear Anosov diffeomorphism. Then, for every $0\le i\le m$, there exists $\ell\ge 0$ such that $f_\mu^\ell(R_i)$ contains R_0. Furthermore $H_0=f_\mu(R_0)\setminus\{R_0,\dots,R_m\}$ has non-empty interior. This gives the Markov property, in the formulation of Remark 3.1.

To complete the verification of condition (A_1), it remains to prove that the Markov rectangles have bounded inner diameter, and the limit capacity of their boundaries is less than the ambient dimension $\dim W_\mu$. The first property is dealt with by Proposition 3.4, and the second one is covered by Proposition 3.5. In both cases, we give somewhat more general statements, in terms of Markov rectangles of Anosov diffeomorphisms.

Proposition 3.4. *Given an Anosov diffeomorphism $\Phi : M \to M$ with stable index 1, and a generating Markov partition $\mathcal{R}$, there exists $L > 0$ such that for every x, y in some Markov rectangle, there exists a piecewise smooth curve $\gamma \subset R$ connecting x and y with* $\text{length}(\gamma) < L$.

Proof. It is suffices to prove the lemma for the quotient map ϕ of the Anosov diffeomorphism in the space of local stable leaves, since the stable leaves are smooth curves with bounded length. For each $j = 0, \ldots, m$, fix some point $a_j \in \text{int}(R_j)$. Moreover, fix any upper bound $K > 0$ for the inner distances

$$\text{dist}_{in}(\phi(a_r), a_s),$$

over all r, s such that $\phi(a_r)$ and a_s are in the same rectangle. We are going to prove that any point in R_j can be joined to a_j by a piecewise curve with length less than $L/2$, for a convenient $L = L(\phi, K) > 0$. Since $\mathcal{R}$ is a generating partition, given any $x \in R_j$, we have

$$\{x\} = R_{i_0} \cap \phi^{-1}(R_{i_1}) \cap \cdots \cap \phi^{-k}(R_{i_k}) \cap \cdots ,$$

for some sequence $(i_k)_k$ in $\{0, \ldots, m\}^{\mathbb{N}}$. Let $x_0 = a_{i_0}$ and for $k = 1, 2, \ldots$ denote by x_k the pre-image of a_{i_k} by ϕ^k in the same connected component of $\phi^{-k}(R_{i_k})$ that contains x. The fact that ϕ is a uniformly expanding map implies that the diameters of these connected components go to zero when k goes to infinity. Therefore, x_k goes to x when k goes to infinity. Moreover, for every $j \geq 0$, we have

$$\begin{aligned} \text{dist}_{in}(x_j, x_{j+1}) &= \text{dist}_{in}(\phi^{(-j)}(a_{i_j}), \phi^{(-(j+1))}(a_{i_{j+1}})) \\ &\leq \rho^{-(j+1)} \, \text{dist}_{in}(\phi(a_{i_j}), a_{i_{j+1}}) \leq K\rho^{-(j+1)}, \end{aligned}$$

where $\phi^{(-n)}$ denotes the appropriate inverse branch of ϕ^n and $\rho^{-1} < 1$ is the supremum of $\|D\phi^{-1}\|$. Then,

$$\text{dist}_{in}(x, x_0) \leq \sum_{j=0}^{\infty} \text{dist}_{in}(x_j, x_{j+1}) \leq K \sum_{j=0}^{\infty} \rho^{-(j+1)} \leq L/2,$$

if L is chosen sufficiently large. This proves the proposition. ■

Proposition 3.5. *Given an Anosov diffeomorphism $\Phi : M \to M$ and a Markov partition $\mathcal{R}$ for Φ, the limit capacity $c(\partial^u \mathcal{R})$ of the union of the unstable boundaries of the Markov rectangles is strictly smaller than the unstable dimension of Φ.*

Proof. As in the previous proposition, we may consider the quotient map ϕ induced by Φ in the space of local stable leaves, which is a uniformly expanding map. The Markov partition of Φ gives rise to a Markov partition of ϕ, that we also denote by $\mathcal{R}$. Then it suffices to prove that the limit capacity of the union Γ of the boundaries of these Markov rectangles for this map ϕ is strictly smaller than the corresponding ambient dimension (the unstable dimension of Φ). The argument combines estimates from [3] with the following general fact (Proposition 3.2 of [8]):

Proposition 3.6. *The limit capacity of a set $\Gamma \subset \mathbb{R}^d$ is given by*

$$\text{c}(\Gamma) = d - \liminf_{\delta \to 0} \frac{\log \text{Leb}(\Gamma_\delta)}{\log \delta}$$

where Γ_δ is the δ-neighborhood of Γ.

In view of this proposition, to prove Proposition 3.5 we only have to show that the lim inf is positive, that is, the volume of the δ-neighborhood of $\Gamma = \partial^u \mathcal{R}$ decays, at least, as fast as a positive power of δ. The first step is the following version of the volume Lemma 4.3 in [4]:

Lemma 3.7. *Given $\varepsilon > 0$ and $\delta > 0$, there exists $\beta = \beta(\varepsilon, \delta) > 0$ such that for any $x \in \Gamma$ and any $y \in B(x, \varepsilon, n)$, we have*

$$\operatorname{Leb}(B(y, \delta, n)) \geq \beta \operatorname{Leb}(B(x, \varepsilon, n))$$

for every $n \geq 1$.

Proof. By the mean value theorem, there exist η in $B(y, \delta, n)$ and ξ in $B(x, \varepsilon, n)$ such that

$$\operatorname{Leb}(B(y, \delta, n)) = \frac{\operatorname{Leb}(B(\phi^n(y), \delta))}{|\det D\phi^n(\eta)|}$$

and

$$\operatorname{Leb}(B(x, \varepsilon, n)) = \frac{\operatorname{Leb}(B(\phi^n(n), \varepsilon))}{|\det D\phi^n(\xi)|}.$$

Since $\operatorname{Leb}(B(w, \delta)) / \operatorname{Leb}(B(z, \varepsilon))$ is bounded away from zero, by a constant that depends only on δ and ε, it is sufficient to show that

$$\frac{|\det D\phi^n(\eta)|}{|\det D\phi^n(\xi)|}$$

is bounded from infinity. Notice that

$$\phi^j(B(x, \varepsilon, n)) \subset B(\phi^j(x), \varepsilon\rho^{j-n}) \quad \text{and} \quad \phi^j(B(y, \delta, n)) \subset B(\phi^j(x), (\varepsilon + \delta)\rho^{j-n}).$$

Recall that $\rho =$ infimum of $\|D\phi^{-1}\|^{-1}$. Therefore, since $\log |\det D\phi|$ is Hölder continuous,

$$\begin{aligned}
\log \frac{|\det D\phi^n(\eta)|}{|\det D\phi^n(\xi)|} &= \sum_{j=0}^{n-1} \log \frac{|\det D\phi(\phi^j(\eta))|}{|\det D\phi(\phi^j(\xi))|} \leq C \sum_{j=0}^{n-1} \operatorname{dist}(\phi^j(\eta), \phi^j(\xi))^\nu \\
&\leq C \sum_{i=0}^{\infty} (2(\varepsilon + \delta)\rho^{-i})^\nu < +\infty.
\end{aligned}$$

This proves the volume lemma. ∎

Let $B(\Gamma, \varepsilon, n)$ represent the dynamical ε-neighborhood of Γ of length n, that is, the set of points that remain within distance ε from the forward invariant Γ from time 0 to time $n - 1$. Using the previous lemma we are going to prove

Lemma 3.8. *There exist $p \geq 1$ and $\theta < 1$ such that, for every $n \geq 1$,*

$$\operatorname{Leb}(B(\Gamma, \varepsilon, n + p)) \leq \theta \operatorname{Leb}(B(\Gamma, \varepsilon, n)).$$

Proof. This is similar to Theorem 4.11 of [3]. Given any $\varepsilon > 0$, let us fix $\gamma > 0$ such that the ball of radius ε around any point $z \in \Gamma$ contains some point w such that $\operatorname{dist}(w, \Gamma) > 4\gamma$. Moreover, let us fix $p \geq 1$ such that

$$\phi^p(B(z, \gamma)) \text{ contains } B(\phi^p(z), \varepsilon) \tag{12}$$

for every $z \in \Gamma$. Now let E be a $(4\gamma, n)$-separated subset of Γ: no element of E belongs to the dynamical 4γ-neighborhood of length n of any other element. Consequently,

(a) the dynamical 2γ-neighborhoods of length n of the elements of E are pairwise disjoint.

Moreover, take E to be maximal: this implies that

(b) the dynamical 2γ-neighborhood of length n of Γ is contained in the union of the dynamical 6γ-neighborhood of length n of all the elements of E.

Consider any $x \in E$. By (12), we have

$$\phi^{n+p}(B(x,\gamma,n)) = \phi^p(B(\phi^n(x),\gamma)) \supset B(\phi^{n+p}(x),\varepsilon)$$

and so, there exists $y = y(x) \in B(x,\gamma,n)$ such that $\operatorname{dist}(\phi^{n+p}(y),\Gamma) > 4\gamma$. Fix $\delta \in (0,\gamma)$ such that

$$\operatorname{dist}(z,w) < \delta \Rightarrow \operatorname{dist}(\phi^p(z),\phi^p(w)) < 2\gamma.$$

Then $B(y,\delta,n) \subset B(x,2\gamma,n)$ and

$$\phi^{n+p}(B(y,\delta,n)) = \phi^p(B(\phi^n(y),\delta)) \subset B(\phi^{n+p}(y),2\gamma).$$

Since $\operatorname{dist}(\phi^{n+p}(y),\Gamma) > 4\gamma$, this implies that $B(y,\delta,n)$ is disjoint from $B(\Gamma,2\gamma,n+p)$. That is, we have show that each $B(y,\delta,n)$ is contained in $B(\Gamma,2\gamma,n)$ but is disjoint from $B(\Gamma,2\gamma,n+p)$. In view of property (a), the $B(y(x),\delta,n)$ are pairwise disjoint. So, we conclude that

$$\operatorname{Leb}(B(\Gamma,2\gamma,n+p)) \le \operatorname{Leb}(B(\Gamma,2\gamma,n)) - \sum_{x\in E}\operatorname{Leb}(B(y(x),\delta,n)).$$

Using Lemma 3.7, we deduce

$$\operatorname{Leb}(B(\Gamma,2\gamma,n+p)) \le \operatorname{Leb}(B(\Gamma,2\gamma,n)) - \beta\sum_{x\in E}\operatorname{Leb}(B(x,6\gamma,n)),$$

where $\beta = \beta(6\gamma,\delta) > 0$. Then, using property (b),

$$\operatorname{Leb}(B(\Gamma,2\gamma,n+p)) \le (1-\beta)\operatorname{Leb}(B(\Gamma,2\gamma,n)),$$

and this proves the claim, with $\theta = 1-\beta$. ■

Proposition 3.5 is now an easy consequence. Indeed, this last lemma implies that

$$\operatorname{Leb}(B(\Gamma,\varepsilon,n)) \le \theta^q\operatorname{Leb}(B(\Gamma,\varepsilon,r))$$

for every $n \ge 1$, where q is the largest integer such that $n \ge pq$ and $r = n - pq$. This implies that the volume of the dynamical neighborhoods decays exponentially fast with the length n:

$$\operatorname{Leb}(B(\Gamma,\varepsilon,n)) \le C\theta^{n/p}$$

for a conveniently chosen constant C. On the other hand, $\Gamma_\delta \subset B(\Gamma,\varepsilon,n)$ as long as

$$\delta(\sup\|D\phi\|)^n \le \varepsilon.$$

Let $\varepsilon > 0$ be fixed. For each $\delta > 0$, take n largest such that this inequality holds. Then

$$\operatorname{Leb}(\Gamma_\delta) \le \operatorname{Leb}(B(\Gamma,\varepsilon,n)) \le C'\delta^\alpha$$

where $\alpha = -\log\theta/(p\log\sup\|D\phi\|)$ and C' is another convenient constant, independent of δ. This gives that

$$\liminf_{\delta\to 0}\frac{\log\operatorname{Leb}(\Gamma_\delta)}{\log\delta} \ge \alpha > 0$$

and so the limit capacity of Γ is at most $d-\alpha$. The proof of Proposition 3.5 is complete. ■

The hyperbolicity condition (A_2) is contained in the following crucial result:

Proposition 3.9. *For families $\left(f_\mu\right)_\mu$ of maps as above, there exist $c_0 > 0$, $\mu_0 > 0$, $n_0 \geq 1$, and $c_1 > 0$ such that for every $0 < \mu \leq \mu_0$,*

$$\sum_{C \in \mathcal{Q}_{\mu,n}(c_0\mu)} \operatorname{Leb}_2(C) \leq e^{-c_1 n\mu} \quad \textit{for all } n \geq n_0$$

where Leb_2 *represents the normalized Lebesgue measure induced by the Riemannian metric.*

The proof of this proposition is long, and we postpone it to Section 4.

3.4. **Proof of Theorems A and B.** First we prove Theorem B. Having checked all the hypotheses, we are in a position to apply Theorems 3.2 and 3.3 to the map $f = f_\mu$ constructed previously. We get that, for every small $\mu > 0$, the set of points in W^c_μ that never fall into H_μ has Hausdorff dimension and limit capacity strictly less than 2. Consequently,

$$\operatorname{HD}(\Lambda_\mu \cap W^c_\mu) \leq \operatorname{c}(\Lambda_\mu \cap W^c_\mu) < 2.$$

Now observe that the repeller Λ_μ is saturated by the stable foliation, that is, it consists of entire leaves. Recall that the holonomy maps of the stable foliation are $C^{1+\delta}$ and, in particular, Lipschitz. Consequently, they preserve Hausdorff dimension and limit capacity. It follows that the previous statement remains true for the intersection of the repeller with any other central leaf:

$$\operatorname{HD}(\Lambda_\mu \cap \mathcal{F}^c_x) \leq \operatorname{c}(\Lambda_\mu \cap \mathcal{F}^c_x) < 2 \quad \text{for every } x,$$

as claimed. This completes the proof of Theorem B.

Now we prove Theorem A using Theorem B. Let $\mu > 0$ be small, as previously. Since $\operatorname{c}(\Lambda_\mu \cap W_\mu) < 2$, given $\operatorname{c}(\Lambda_\mu \cap W_\mu) < \alpha < 2$, there exists $\varepsilon > 0$ such that the smallest number of ε-balls needed to cover $\Lambda_\mu \cap W_\mu$ satisfies

$$n(\Lambda_\mu \cap W_\mu, \varepsilon) < \varepsilon^{-\alpha}.$$

The fact that the holonomy along leaves of $\mathcal{F}^{ss}_\mu$ defined in Section 2.1 are $C^{1+\delta}$, consequently Lipschitz, implies that there exists a constant $K \geq 1$ such that for every Markov rectangle and every $x' \in \pi^{-1}(x)$ and $y' \in \pi^{-1}(y)$ with $x, y \in R_j$, we have $\operatorname{dist}(x', y') \leq K \operatorname{dist}(x, y)$. Besides the fact that the Markov rectangles have compact closure and they are finitely many, we get a constant $K' > 0$, depending only on the metric, such that Λ_μ can be covered by

$$n(\Lambda_\mu \cap W_\mu, \varepsilon) K' \varepsilon^{-D+2}$$

D-dimensional ε-balls. Therefore, since $\alpha < 2$, we have

$$\begin{aligned}
\operatorname{HD}(\Lambda_\mu) \leq \operatorname{c}(\Lambda_\mu) &= \limsup_{\varepsilon \to 0} \frac{\log n(\Lambda_\mu, \varepsilon)}{|\log \varepsilon|} \\
&\leq \limsup_{\varepsilon \to 0} \frac{\log[n(\Lambda_\mu \cap W_\mu, \varepsilon) K' \varepsilon^{-D+2}]}{|\log \varepsilon|} < D.
\end{aligned}$$

The proof of Theorem A is complete.

4. **Central Lyapunov Exponents.** We are left to prove Proposition 3.9. The proof, to be given in Section 4.3, combines estimates of the derivative and Jacobian of f_μ near the bifurcation point p_μ (Section 4.1) with a combinatorial analysis of the visits of orbits to the neighborhood of p_μ (Section 4.2).

The intuition behind it is the following. For $\mu > 0$, inside the Markov rectangle $S_0 \supset V$ the derivative of g_μ may contract the central bundle in some directions, opposite to what occurs outside S_0, where the derivative always expands the central bundle uniformly. In principle, an orbit may spend a lot of time in S_0 accumulating contraction along the central bundle, and the time spent outside S_0 may not be sufficient to compensate these contractions. However, for a full Lebesgue measure subset and every small $\mu > 0$, expansion does prevail, as required by (A_2).

Our estimates involve a large number of constants. Rather than having to keep track of the relations between all those constants, we have chosen to use explicit values whenever possible. However, these values are usually technical, and not meant to be optimal.

4.1. **Iterates in the non-hyperbolic region.** Lemma 4.1 says that f_μ is not too contracting in the central direction, outside a small neighborhood of p_μ. Lemma 4.2 asserts that f_μ is volume expanding outside H_μ. For simplicity of notation, we denote $W_\mu \cap V$ also as V. Recall that $V_1 \subset V$ is the neighborhood of p_μ introduced in (H_5)–Proposition 2.6.

Lemma 4.1. *Let $(g_\mu)_\mu$ satisfy properties (H_1)–(H_6). Then there exists a constant $\mu_1 > 0$ such that for every $0 < \mu \le \mu_1$ we have $H_\mu \subset V_1$ and there exists a neighborhood $V_2(\mu) \subset H_\mu$ of p_μ such that*

(a) $\log \left\| (Df_\mu(x))^{-1} \right\|^{-1} \ge -\frac{33}{32}\mu + \frac{31}{32} b_1(\mu)\rho^2$ *for every x in V_1.*

(b) $\log \left\| (Df_\mu(x))^{-1} \right\|^{-1} \ge -\frac{3}{32}\mu$ *for every x outside $V_2(\mu)$.*

Proof. For clearness, we begin by proving (a) in the much simpler case when $g_\mu = \hat{g}_\mu$ is of the form (1). By condition (C_4), we have $\partial_w \Phi(\mu, w, z) > \partial_w \Phi(\mu, 0, 0) = b_1(\mu)$ for $\rho^2 \le \delta_1$ and so

$$\Phi(\mu, \rho^2, z) > \Phi(\mu, 0, z) + b_1(\mu)\rho^2 \ge (1-\mu) + b_1(\mu)\rho^2$$

whenever $\rho^2 \le \delta_1$. Using the elementary fact

$$\log(1 - a + b) \ge -\frac{33a}{32} + \frac{31b}{32} \quad \text{if } 0 \le a, b \le \frac{1}{32}, \tag{13}$$

we get that

$$\begin{aligned} \log \left\| (D\hat{g}_\mu | E^{cu}_\mu)^{-1} \right\|^{-1} &\ge \log \Phi(\mu, \rho^2, z) > \log(1 - \mu + b_1(\mu)\rho^2) \\ &\ge -\frac{33}{32}\mu + \frac{31}{32} b_1(\mu)\rho^2. \end{aligned}$$

For proving part (a) in the general situation of conditions (H_1)–(H_6), let

$$\left\{ \frac{\partial}{\partial \rho}\bigg|_{(\rho,\theta)}, \frac{\partial}{\partial \theta}\bigg|_{(\rho,\theta)} \right\},$$

be the image of the canonical frame in $\mathbb{R}^2$ under the change of coordinates $\varphi(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$. The frame

$$\left\{ \frac{\partial}{\partial \rho}\bigg|_{(\rho,\theta)}, \frac{1}{\rho}\frac{\partial}{\partial \theta}\bigg|_{(\rho,\theta)} \right\}, \tag{14}$$

is orthonormal relative to the Euclidean metric. From the expression of the normal form (7) (recall Remark 2.7)

$$F_\mu(\rho,\theta) = \big((1-\mu)\rho + b_1(\mu)\rho^3, \theta + \phi(\mu) + b_2(\mu)\rho^2\big)$$

we obtain

$$DF_\mu \frac{\partial}{\partial\rho}\bigg|_{(\rho,\theta)} = \big(1-\mu+3b_1(\mu)\rho^2\big)\frac{\partial}{\partial\rho}\bigg|_{F_\mu(\rho,\theta)} + 2b_2(\mu)\rho\frac{\partial}{\partial\theta}\bigg|_{F_\mu(\rho,\theta)}$$

and

$$DF_\mu \frac{\partial}{\partial\theta}\bigg|_{(\rho,\theta)} = \frac{\partial}{\partial\theta}\bigg|_{F_\mu(\rho,\theta)}.$$

Hence, the matrix of DF_μ relative to the frame (14) is

$$DF_\mu(\rho,\theta) = \begin{pmatrix} (1-\mu)+3b_1(\mu)\rho^2 & 0 \\ 2b_2(\mu)\rho\big((1-\mu)\rho+b_1(\mu)\rho^3\big) & (1-\mu)+b_1(\mu)\rho^2 \end{pmatrix}. \tag{15}$$

Therefore, the inverse $DF_\mu^{-1}(\rho,\theta)$ is given by

$$\begin{pmatrix} ((1-\mu)+3b_1(\mu)\rho^2)^{-1} & -2b_2(\mu)\rho^2((1-\mu)+3b_1(\mu)\rho^2)^{-1} \\ 0 & ((1-\mu)+b_1(\mu)\rho^2)^{-1} \end{pmatrix}.$$

It follows that, for every unit vector $(u,v)\in T_{(\rho,\theta)}M$,

$$\begin{aligned}
&\|DF_\mu^{-1}(\rho,\theta)(u,v)\| = \\
&= \big\|\big(((1-\mu)+3b_1(\mu)\rho^2)^{-1}(u-2b_2(\mu)\rho^2 v), ((1-\mu)+b_1(\mu)\rho^2)^{-1}v\big)\big\| \\
&\le \big\|((1-\mu)+3b_1(\mu)\rho^2)^{-1}\big(u-2b_2(\mu)\rho^2 v, v\big)\big\| + \\
&\qquad\qquad + \big\|\big(0, 2b_1(\mu)\rho^2((1-\mu)+b_1(\mu)\rho^2)^{-1}v\big)\big\| \\
&\le ((1-\mu)+3b_1(\mu)\rho^2)^{-1}(u^2+v^2-4b_2(\mu)\rho^2uv+4b_2(\mu)^2\rho^4v^2)^{1/2}.
\end{aligned}$$

Note that $\|(u,v)\|=1$ implies $|uv|\le 1/2$. Therefore, using also part (5) of (H_5)–Proposition 2.6, the last factor is bounded above by

$$(1+2|b_2(\mu)|\rho^2+4b_2(\mu)^2\rho^4)^{1/2} \le 1+2|b_2(\mu)|\rho^2 \le 1+b_1(\mu)\rho^2.$$

Since μ is small, (H_5)–Proposition 2.6(2) implies $(1-\mu)+3b_1(\mu)\rho^2 < 2$ for every point $(\rho,\theta)\in V_1$. Then, by (H_5)–Proposition 2.6(3),

$$\begin{aligned}
\|Df_\mu^{-1}\| &\le \|DF_\mu^{-1}\| + b_1(\mu)\rho^2/10 \\
&\le ((1-\mu)+3b_1(\mu)\rho^2)^{-1}(1+b_1(\mu)\rho^2) + b_1(\mu)\rho^2/10 \\
&\le ((1-\mu)+3b_1(\mu)\rho^2)^{-1}(1+3b_1(\mu)\rho^2/2).
\end{aligned}$$

Using (13) once more, we deduce that

$$\begin{aligned}
\log\|Df_\mu^{-1}\|^{-1} &\ge \log(1-\mu+3b_1(\mu)\rho^2) - \log(1+3b_1(\mu)\rho^2/2) \\
&\ge -\frac{33}{32}\mu + \frac{31}{32}3b_1(\mu)\rho^2 - \frac{3}{2}b_1(\mu)\rho^2 \\
&\ge -\frac{33}{32}\mu + \frac{31}{32}b_1(\mu)\rho^2,
\end{aligned} \tag{16}$$

as claimed in (a).

Part (b) is an easy consequence. Indeed, recall that $\rho_0(\mu) = (\mu/b_1(\mu))^{1/2}$ is the radius of the F_μ-invariant circle created by the Hopf bifurcation. We define $V_2(\mu)$ to be the neighborhood of p_μ given in polar coordinates by

$$\left\{(\rho,\theta) : \rho < \left(\frac{127}{128}\right)^{1/2} \rho_0(\mu)\right\}.$$

Since f_μ is $O(\rho^5)$-close to F_μ, we have that, as long as μ is sufficiently small, the domain $V_2(\mu)$ is indeed contained in the disk H_μ bounded by the invariant circle of f_μ, as claimed. For $(\rho,\theta) \notin V_2(\mu)$ and μ small, we have

$$b_1(\mu)\rho^2 \geq \frac{127}{128} b_1(\mu)\rho_0(\mu)^2 = \frac{127}{128}\mu \tag{17}$$

and so the inequality (16) gives

$$\log \|(Df_\mu)^{-1}\|^{-1} \geq -\frac{33}{32}\mu + \frac{31}{32}\frac{127}{128}\mu \geq -\frac{3}{32}\mu.$$

This proves the lemma. ∎

Now we estimate the Jacobian $\operatorname{Jac} f_\mu(x) = |\det Df_\mu(x)|$ of f_μ.

Lemma 4.2. *For every* $0 < \mu \leq \mu_1$ *we have* $\log \operatorname{Jac} f_\mu(x) \geq 2\log\sigma_1 > 0$ *for every* x *outside* V_1. *Moreover,*

(a) $\log \operatorname{Jac} f_\mu(x) \geq -\frac{65}{32}\mu + \frac{127}{32} b_1(\mu)\rho^2$ *for every* x *in* V_1

(b) $\log \operatorname{Jac} f_\mu(x) \geq \frac{61}{32}\mu$ *for every* x *outside* $V_2(\mu)$.

Proof. The first statement follows directly from $\|Df_\mu^{-1}\|^{-1} \geq \sigma_1 > 1$, which is (H_5)–Proposition 2.6(1). To proof part (a) we note that the form (15) of DF_μ together with part (4) of (H_5)-Proposition 2.6 give

$$\begin{aligned}\operatorname{Jac} f_\mu &\geq ((1-\mu) + 3b_1(\mu)\rho^2)((1-\mu) + b_1(\mu)\rho^2) - b_1(\mu)\rho^2/100 \\ &\geq 1 - 2\mu + 4(1-\mu)b_1(\mu)\rho^2 - b_1(\mu)\rho^2/100 \\ &\geq 1 - 2\mu + \frac{510}{128} b_1(\mu)\rho^2,\end{aligned}$$

for every $0 < \mu \leq 1/100$ and $\rho \in V_1$. Therefore, using the elementary fact that

$$\log(1 - a + b) \geq -\frac{65a}{64} + \frac{999b}{1000} \quad \text{for all } 0 \leq a, b \leq \frac{1}{1000},$$

we obtain

$$\log \operatorname{Jac} f_\mu \geq -\frac{65}{32}\mu + \frac{127}{32} b_1(\mu)\rho^2.$$

To prove (b) it suffices to combine this inequality with (17): if $x \notin V_2(\mu)$ then

$$\log \operatorname{Jac} f_\mu(x) \geq -\frac{65}{32}\mu + \frac{127}{32}\frac{127}{128}\mu \geq \frac{61}{32}\mu.$$

This proves the lemma. ∎

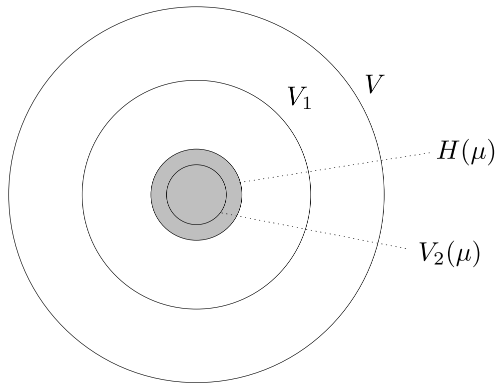

FIGURE 4. Neighborhoods of p_μ

4.2. **Visits to the non-hyperbolic region.** We are going to deduce more global versions of the estimates in the previous section, that apply to segments of orbits visiting the non-hyperbolic region several times.

Given integers $n \geq 1$, $t \geq 1$ and k_j, l_j, for $1 \leq j \leq t$, with

$$k_1 + l_1 + \cdots + k_t + l_t = n,$$

we denote as $\mathcal{C}(k_1, l_1, \ldots, k_t, , l_t)$ the set of all extended n-cylinders C_n which spend, alternately, k_i iterates outside R_0 and then l_i iterates inside R_0. More precisely, $C_n = C(\alpha_0, \ldots, \alpha_{n-1})$ is in $\mathcal{C}(k_1, l_1, \ldots, k_t, l_t)$ if and only if $\alpha_j > 0$ whenever j is in the time intervals

$$[0, k_1), \quad [k_1 + l_1 , k_1 + l_1 + k_2), \quad \ldots, \quad [k_1 + \cdots + l_{t-1} , k_1 + \cdots + l_{t-1} + k_t)$$

and $\alpha_j = 0$ for all j in the complement. Here $k_1 \geq 0$ and $l_t \geq 0$, otherwise all k_i and l_i are strictly positive. Given $0 \leq l \leq n$ we denote as $\mathcal{C}(n, l, t)$ the union of all $\mathcal{C}(k_1, l_1, \ldots, k_t, l_t)$ with $l_1 + \cdots + l_t = l$.

Recall that $\eta \geq 1$ is an upper bound for the number of Markov rectangles the image of any of them can intersect.

Lemma 4.3. *For every $n, l, t, k_1 , l_1 , \ldots, k_t , l_t$*

1. $\#\mathcal{C}(k_1 , l_1 , \ldots, k_t , l_t) \leq \eta^k$ *where* $k = k_1 + \cdots + k_t = n - l$
2. $\#\mathcal{C}(n, l, t) \leq \binom{l}{t-1}\binom{n-l}{t-1}\eta^k$

Proof. Each cylinder $C_n = C'(\alpha_0 , \ldots, \alpha_{n-1})$ is uniquely determined by the sequence α_j . Given any α_j there are at most η admissible values for α_{j+1} . Moreover, l symbols are equal to zero if C_n is in $\mathcal{C}(k_1 , l_1 , \ldots, k_t , l_t)$. Thus, there are at most η^k admissible sequences, as stated in part 1 of the lemma.

To prove part 2 we use the well known fact that, given integers $1 \leq t \leq m$ there are

$$\binom{m-1}{t-1}$$

ways one can decompose m as a sum of t strictly positive integers. In particular, there are

$$\binom{l-1}{t-1} + \binom{l-1}{t-2} = \binom{l}{t-1}$$

solutions to $l_1+\cdots+l_t=l$ with $l_i \geq 1$ for $i<t$ and $l_t \geq 0$. Together with a similar estimate for $k_1+\cdots+k_t=n-l$, this gives that $\mathcal{C}(n,l,t)$ is formed by not more than

$$\binom{l}{t-1}\binom{n-l}{t-1}$$

sets $\mathcal{C}(k_1,l_1,\ldots,k_t,l_t)$. Together with part 1, this completes the proof of the lemma. ∎

Lemma 4.4. *For every $k_1,l_1,\ldots,k_t,l_t$ and $C_n \in \mathcal{C}(k_1,l_1,\ldots,k_t,l_t)$ there are points $x_i \in f_\mu^{k_1+l_1+\cdots+k_i}(C_n) \subset R_0'$ so that*

$$\operatorname{Leb}_2(C_n) \leq \sigma^{-2(n-l)} \prod_{i=1}^{t} \operatorname{Jac} f_\mu^{l_i}(x_i)^{-1},$$

where $3<\sigma \leq \|Df_\mu^{-1}(x)\|^{-1}$ for x outside V.

Proof. Write $C_n = C(\alpha_0,\ldots,\alpha_{n-1})$. If $\alpha_j>0$ then $f_\mu^j(C_n)$ is in the hyperbolic region, where the Jacobian along the central direction is bounded below by σ^2. This happens for $n-l$ values of i and, for each one of them, we have

$$\operatorname{Leb}_2(f_\mu^{j+1}(C_n)) \geq \sigma^2 \operatorname{Leb}_2(f_\mu^j(C_n)). \tag{18}$$

Now consider $j=k_1+l_1+\cdots+k_i$, for each $1\leq i\leq t$. Then $f_\mu^j(C_n)$ is contained in R_0' and remains there for l_i iterates. Hence,

$$\operatorname{Leb}_2(f_\mu^{k_1+l_1+\cdots+k_i+l_i}(C_n)) \geq \operatorname{Jac} f_\mu^{l_i}(x_i) \operatorname{Leb}_2(f_\mu^{k_1+l_1+\cdots+k_i}(C_n)) \tag{19}$$

for some x_i as in the statement. The claim in the lemma follows, multiplying the inequalities (18) and (19) and taking into account that $\operatorname{Leb}_2(f_\mu^n(C_n)) \leq 1$.

∎

The next two lemmas give us estimates about the derivative and the Jacobian in central direction along an orbit.

Lemma 4.5. *For every $k_1,l_1,\ldots,k_t,l_t$ and $C_n \in \mathcal{C}(k_1,l_1,\ldots,k_t,l_t)$ we have*

$$\sum_{j=1}^{n} \inf_{x\in C_n} \log \left\|(Df_\mu(f_\mu^j(x)))^{-1}\right\|^{-1} \geq -\frac{3}{32} l\mu - \frac{13}{32} t \log\mu + (n-l)\log\sigma.$$

Proof. Given neighborhoods $U_1 \subset U_2$ of p_μ, we say that an orbit segment $\mathcal{O}$ *crosses* $U_2 \setminus U_1$ if the first iterate is in U_1, the last one is outside U_2, and all the others are in $U_2 \setminus U_1$. Let V_μ be the neighborhood of p_μ corresponding to $\rho = 2\rho_0(\mu)$ in local coordinates (in the context of Lemma 4.1). We are going to consider the worst possible case, namely, orbits that cross $V_1 \setminus V_\mu$ every time they visit R_0. The arguments are valid in general and, in fact, the estimates coming from Lemma 4.1 are better in the other cases. For simplicity of the presentation, let us assume that every crossing segment spends q_0 iterates in $R_0 \setminus V$, q_1 iterates in $V \setminus V_1$, and q_μ iterates in $V_1 \setminus V_\mu$: this simplification is harmless because these numbers of iterates may vary by, at most, a fixed finite amount.

For notational simplicity, let us write $\left\|(Df_\mu(f_\mu^j(x)))^{-1}\right\|^{-1} = \lambda_\mu(f_\mu^j(x))$. In each visit to R_0, the orbit crosses each region $R_0 \setminus V$, $V \setminus V_1$ and $V_1 \setminus V_\mu$ once only. If l_i

is the number of iterates of a orbit during a visit to R_0, we have

$$\begin{aligned}\sum_{j=1}^{l_i} \inf_{x\in C_n} \log \left\|(Df_\mu(f_\mu^j(x)))^{-1}\right\|^{-1} &= \sum_{j=1}^{l_i} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) \\ &\geq \sum_{j=1}^{q_0} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) + \sum_{j=q_0+1}^{q_0+q_1} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) + \\ &\quad + \sum_{j=q_0+q_1+1}^{q_0+q_1+q_\mu} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) + \sum_{j=q_0+q_1+q_\mu+1}^{l_i} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)).\end{aligned}$$

According to Lemma 4.1, we have

$$\begin{aligned}\sum_{j=1}^{l_i} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) &\geq q_0 \log \sigma + q_1\sigma_1 + \sum_{j=1}^{q_\mu}\Big(-\frac{33}{32}\mu + \frac{31}{32} b_1(\mu)\rho_j^2\Big) - \\ &\quad - \frac{3}{32}\mu(l_i - q_\mu - q_1 - q_0) \\ &\geq \sum_{j=1}^{q_\mu}\Big(-\frac{33}{32}\mu + \frac{31}{32} b_1(\mu)\rho_j^2\Big) - \frac{3}{32}\mu l_i,\end{aligned}$$

where ρ_j is the ρ-coordinate of $f_\mu^j(x)$, $x \in \partial V_\mu$, in the local system of coordinates.

If ρ_j is the ρ-coordinate of $f_\mu^j(2\rho_0)$, $j \geq 1$, then

$$\frac{\rho_{j+1}}{\rho_j} = \frac{(1-\mu)\rho_j + b_1(\mu)\rho_j^3}{\rho_j} = (1-\mu) + b_1(\mu)\rho_j^2.$$

Let $\hat{\rho} = f_\mu^{q_\mu+1}(2\rho_0)$ the ρ-coordinate of the first iterate of $2\rho_0$ outside V_1. The fact that for μ close to zero ρ_0 is close to zero, implies that

$$\frac{\rho_1}{\hat{\rho}} = \frac{(1-\mu)2\rho_0 + b_1(\mu)(2\rho_0)^3}{\hat{\rho}} \leq b_1(\mu)^{30/32}\rho_0^{15/32} = \mu^{15/32},$$

for μ sufficiently small. Hence,

$$1 = \frac{\rho_1}{\hat{\rho}} \prod_{j=1}^{q_\mu} \frac{\rho_{j+1}}{\rho_j} \leq \mu^{15/32} \prod_{j=1}^{q_\mu} \big((1-\mu) + b_1(\mu)\rho_j^2\big).$$

Using the elementary fact

$$\log(1-a+b) \leq -\frac{31}{32}a + \frac{33}{32}b \quad \text{for } 0 < a, b \leq \frac{1}{32},$$

we get, for every $\mu > 0$ sufficiently small,

$$\sum_{j=1}^{q_\mu}\Big(-\frac{31}{32}\mu + \frac{33}{32} b_1(\mu)\rho_j^2\Big) \geq -\frac{15}{32}\log \mu. \tag{20}$$

We have for every $\rho \geq 2\rho_0$, $b_1(\mu)\rho^2 \geq 4b_1(\mu)\rho_0^2 = 4\mu$. Then,

$$\sum_{j=1}^{q_\mu}\Big(-\frac{33}{32}\mu + \frac{31}{32} b_1(\mu)\rho_j^2\Big) \geq \frac{13}{15}\sum_{j=1}^{q_\mu}\Big(-\frac{31}{32}\mu + \frac{33}{32} b_1(\mu)\rho_j^2\Big) \geq -\frac{13}{32}\log \mu.$$

Hence,

$$\sum_{j=1}^{n} \inf_{x\in C_n} \log \left\|(Df_\mu(f_\mu^j(x)))^{-1}\right\|^{-1} =$$

$$\begin{aligned}
&= \sum_{i=1}^{t}\Big[\sum_{j=1}^{l_i} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x)) + \sum_{j=1}^{k_i} \inf_{x\in C_n} \log \lambda_\mu(f_\mu^j(x))\Big] \\
&\geq \sum_{i=1}^{t}\Big(\sum_{j=1}^{q_\mu}\Big(-\frac{33}{32}\mu + \frac{31}{32}b_1(\mu)\rho_j^2\Big) - \frac{3}{32}\mu l_i + k_i \log\sigma\Big) \\
&\geq -\frac{13}{32}t\log\mu - \frac{3}{32}\mu l + (n-l)\log\sigma.
\end{aligned}$$

We have assumed $\lambda_\mu(x) \geq \sigma$ outside V. This completes the proof. ■

4.3. **Proof of Proposition 3.9.** Let us first give some outline of the proof. In order to estimate the expression

$$\sum_{C\in\mathcal{Q}_{\mu,n}(c_0\mu)} \mathrm{Leb}_2(C)$$

we have to deal with two opposing effects: the exponential growth of the number of n-cylinders in $\mathcal{Q}_{\mu,n}$ versus the exponential decay of the volume of each one of these cylinders. We have to check that the latter prevails.

For this, we split the family of n-cylinders into subfamilies $\mathcal{Q}_{\mu,n,l,t}$ with a given type of trajectory: t visits to R_0 adding to a total of l iterates in there. On the one hand, we use the combinatorial bounds in Lemma 4.3 and the metric estimates in Lemma 4.4 to obtain some upper bound for the total volume of cylinders in each $\mathcal{Q}_{\mu,n,l,t}$. This is stated in (23) and (24) below. On the other hand, our definition of $\mathcal{Q}_{\mu,n}(c_0\mu)$

$$\phi_{\mu,n}(\alpha_1,\ldots,\alpha_n) \leq c_0\mu$$

(recall the statement of A_2) means that cylinders in $\mathcal{Q}_{\mu,n}(c_0\mu)$ exhibit weak growth of the derivative, at best. Combining this with Lemma 4.5 and Lemma 4.6, we deduce that for such cylinders the values of l and t must satisfy certain relations, depending on μ, that are collected in Lemma 4.7. Using these relations we conclude that the bound (24) on the total volume of the cylinders in each of the $\mathcal{Q}_{\mu,n,l,t}$ is small. This is stated in (25). To complete the proof of the proposition, we only need to sum this estimate over all l and t. Now we fill the details in this outline.

Lemma 4.6. *Given $\tau > 0$, there exists $l_0 \geq 1$ and $\kappa_0 > 0$ such that, for every $l \geq l_0$ and $0 < \kappa \leq \kappa_0$,*

$$\log\binom{l}{t} \leq l(1+\tau)\kappa\log\frac{1}{\kappa} \quad \textit{whenever } 0 \leq t \leq \kappa l.$$

Proof. Let $l \geq t \geq 1$ such that $t \leq l/2$. By Stirling's formula,

$$\binom{l}{t} = \frac{l!}{t!(l-t)!} \leq \frac{\sqrt{2\pi l}\; l^l e^{-l}(1+1/(4l))}{(\sqrt{2\pi t}\; t^t e^{-t})(\sqrt{2\pi(l-t)}\;(l-t)^{(l-t)}e^{-(l-t)})}.$$

Observe that,

$$\frac{\sqrt{2\pi l}}{\sqrt{2\pi t}\sqrt{2\pi(l-t)}} = \left[\frac{l}{2\pi t(l-t)}\right]^{1/2} \leq \left[\frac{l}{2\pi t(l-l/2)}\right]^{1/2} = \frac{1}{\sqrt{t\pi}}.$$

Then,

$$\binom{l}{t} \le \frac{1}{\sqrt{t\pi}}\left(1+\frac{1}{4l}\right)\frac{l^l}{t^t(l-t)^{l-t}}.$$

Therefore, there exists an integer $l_0 \ge 1$ such that

$$\frac{1}{\sqrt{t\pi}}\left(1+\frac{1}{4l}\right) \le 1,$$

for every $l \ge l_0$ and every $0 < t < l/2$. Thus,

$$\binom{l}{t} \le \frac{l^l}{t^t(l-t)^{l-t}} = \left(\frac{l}{t}\right)^t \left(\frac{l}{l-t}\right)^{l-t} = \left[\left(\frac{l}{t}\right)^{\frac{t}{l}} \left(\frac{l}{l-t}\right)^{\frac{l-t}{l}}\right]^l,$$

for every $l \ge l_0$ and every $0 < t < l/2$.

Since $t \le \kappa l$ is equivalent to

$$\frac{l}{t} \ge \frac{1}{\kappa} \quad \text{and} \quad \frac{l}{l-t} \le \frac{1}{1-\kappa},$$

and the function $x^{1/x}$ is a increasing function for x close to 1 and it is decreasing for x large, there exists $\kappa_1 > 0$ such that

$$\binom{l}{t} \le \left[\left(\frac{1}{\kappa}\right)^{\kappa}\left(\frac{1}{1-\kappa}\right)^{1-\kappa}\right]^l \quad \text{for every } 0 < \kappa \le \kappa_1.$$

Let τ be a positive constant. We define

$$h(\kappa) = \tau\kappa \log\frac{1}{\kappa} - (1-\kappa)\log\frac{1}{1-\kappa}.$$

Then, $h(\kappa)$ is a smooth function for $0 < \kappa < 1$. Moreover,

$$\lim_{\kappa\to 0^+} h(\kappa) = \lim_{\kappa \to 1^-} h(\kappa) = 0$$

and $h(\kappa)$ vanishes in some point of the interval $(0,1)$. Furthermore, the derivative of $h(\kappa)$ is given by

$$h'(\kappa) = \tau\log\frac{1}{\kappa} + \log\frac{1}{1-\kappa} - 1 - \tau.$$

Therefore, given any $\tau > 0$ there exists $0 < \kappa_0 = \kappa_0(\tau) \le \kappa_1$ such that $h'(\tau) > 0$, for every $0 < \kappa \le \kappa_0$, that is, h is increasing for every $0 < \kappa \le \kappa_0$. Hence $h(\kappa) \ge 0$ in this interval. Thus,

$$(1-\kappa)\log\frac{1}{1-\kappa} \le \tau\kappa\log\frac{1}{\kappa}, \quad \text{for every} \quad 0 < \kappa \le \kappa_0.$$

Then,

$$\log\binom{l}{t} \le l(1+\tau)\kappa\log\frac{1}{\kappa}, \quad \text{for every } 0 < \kappa \le \kappa_0.$$

This proves the lemma. ■

Fix $c_0 = 1/256$. Recall that $\mathcal{Q}_{\mu,n}(c_0\mu)$ is the set of cylinders C_n for which

$$\phi_{\mu,n}(\alpha_1,\dots,\alpha_n) \le c_0\mu.$$

Split $\mathcal{Q}_{\mu,n}(c_0\mu)$ as the disjoint union of all $\mathcal{Q}_{\mu,n,l,t} = \mathcal{Q}_{\mu,n}(c_0\mu) \cap \mathcal{C}(n,l,t)$ over all l and t. From Lemma 4.5 we have that

$$\sum_{j=1}^{n} \inf_{x\in C_n} \log\left\|Df_\mu^{-1}(f_\mu^j(x))\right\|^{-1} \ge -\frac{3}{32}l\mu - \frac{13}{32}t\log\mu + (n-l)\log\sigma.$$

for every $C_n \in \mathcal{C}(n,l,t)$. The expression on the left hand side coincides with $n\phi_{\mu,n}(\alpha_1,\ldots,\alpha_n)$. Thus, a necessary condition for a cylinder $C_n \in \mathcal{C}(n,l,t)$ to be in $\mathcal{Q}_{\mu,n,l,t}$ is

$$-\frac{3}{32}l\mu - \frac{13}{32}t\log\mu + (n-l)\log\sigma \le c_0\mu n. \tag{21}$$

Lemma 4.7. *If* $C_n \in \mathcal{Q}_{\mu,n,l,t}$ *then*

1. $\dfrac{n-l}{l} \le \dfrac{1}{8}\dfrac{\mu}{\log\sigma}$
2. $\dfrac{t}{l} \le \dfrac{1}{4}\dfrac{\mu}{-\log\mu}$.

Proof. Throughout the proof we assume that μ is sufficiently small. From (21) we obtain, noting that $t\log\mu < 0$,

$$-\frac{3}{32}l\mu + (n-l)\log\sigma \le c_0\mu n\,.$$

This inequality implies

$$\frac{n-l}{l} \le \frac{(3/32)\mu + c_0\mu}{\log\sigma - c_0\mu} \le \frac{\mu/8}{\log\sigma} \quad (\le 1). \tag{22}$$

This gives statement 1. Finally, since $\log\sigma > 0$, the relation (21) also implies

$$-\frac{13}{32}t\log\mu \le c_0\mu n + \frac{3}{32}l\mu\,.$$

The inequality (22) implies that $n \le 2l$. It follows that

$$-\frac{13}{32}t\log\mu \le \left(\frac{3}{32} + 2c_0\right)l\mu \le \frac{13}{128}l\mu$$

because $c_0 = 1/256$. This gives statement 2. ∎

By part 2 of Lemma 4.7, we only have to consider $t \le \kappa l$ with $\kappa = \mu/{-4\log\mu}$. Fix $\tau = 1/1000$, for instance, and assume μ is small enough so that $\kappa \le \kappa_0$ as given by Lemma 4.6. From Lemmas 4.3 and 4.4 we obtain

$$\log \sum_{C\in\mathcal{Q}_{\mu,n,l,t}} \mathrm{Leb}(C) \le \log \binom{l}{t-1}\binom{n-l}{t-1}\left(\frac{\eta}{\sigma^2}\right)^{n-l}\prod_{i=1}^{t}\mathrm{Jac}\, f_\mu^{l_i}(x_i)^{-1}. \tag{23}$$

Using Lemmas 4.6 and 4.2 part (b) and the elementary relation

$$\binom{m}{t} \le \sum_{j=0}^{m}\binom{m}{j} = 2^m \quad \text{for all } 0 \le t \le m,$$

we obtain

$$\begin{aligned}\log \sum_{C\in\mathcal{Q}_{\mu,n,l,t}} \mathrm{Leb}_2(C) \le\\ \le l(1+\tau)\kappa\log\frac{1}{\kappa} + (n-l)\log 2 + (n-l)\log\frac{\eta}{\sigma^2} - \frac{61}{32}\mu l.\end{aligned} \tag{24}$$

Using assumption (H_6) and part 1 of Lemma 4.7, we obtain

$$(n-l)\log 2 + (n-l)\log\frac{\eta}{\sigma^2} \le (n-l)\log 2000 \le 8(n-l) \le \frac{\mu l}{\log\sigma} \le \mu l.$$

For the last inequality, recall that $\sigma > 3$. Replacing this in the previous inequality, we get

$$\log \sum_{C\in\mathcal{Q}_{\mu,n,l,t}} \mathrm{Leb}_2(C) \le l(1+\tau)\kappa\log\frac{1}{\kappa} - \frac{29}{32}\mu l.$$

On the other hand,

$$\frac{1}{\mu}\kappa\log\frac{1}{\kappa} = \frac{\log(-4\log\mu)}{-4\log\mu} - \frac{\log\mu}{-4\log\mu}$$

converges to $1/4$ when μ goes to zero. Therefore, assuming μ is sufficiently small, we have that the first term in the previous inequality is less than $13/32$, say. It follows that

$$\log \sum_{C\in\mathcal{Q}_{\mu,n,l,t}} \mathrm{Leb}_2(C) \le -\frac{1}{2}\mu l \le -\frac{1}{4}\mu n. \tag{25}$$

The last inequality uses (22). To complete the proof of the proposition, we need to sum over all l and t. Since, $l \le n$ and $t \le \kappa l$, this sum involves less than κn^2 terms. Hence,

$$\sum_{C\in\mathcal{Q}_{\mu,n}(c_0\mu)} \mathrm{Leb}_2(C) \le \kappa n^2 e^{-\mu n/4} \le e^{-\mu n/8}$$

for all $n \ge n_0(\mu)$. So, we may take $c_1 = 1/8$. The proof of Proposition 3.9 is complete.

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E-mail address: `vhorita@ibilce.unesp.br; viana@impa.br`