

SINGULAR-HYPERBOLIC ATTRACTORS ARE CHAOTIC

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ABSTRACT. We prove that a singular-hyperbolic (or Lorenz-like) attractor of a 3-dimensional flow is chaotic, in two strong different senses. Firstly, the flow is expansive: if two points remain close for all times, possibly with time reparametrization, then their orbits coincide. Secondly, there exists a physical (or Sinai-Ruelle-Bowen) measure supported on the attractor whose ergodic basin covers a full Lebesgue (volume) measure subset of the topological basin of attraction. Moreover this measure has absolutely continuous conditional measures along the center-unstable direction, is a u -Gibbs state and an equilibrium state for the logarithm of the Jacobian of the time one map of the flow along the strong-unstable direction. In particular these results show that both *the flow defined by the Lorenz equations and the geometric Lorenz flows are expansive and have physical measures which are u -Gibbs states.*

1. INTRODUCTION

The uniform hyperbolic theory of Dynamical Systems was introduced in the 60's by Smale [36] and became the cornerstone of the development of the Qualitative Theory of Differential Equations since. However this theory leaves out large families of systems which do not conform with the basic assumptions of uniform hyperbolicity. The most representative example of such systems are arguably the Hénon map [14], for the discrete time case, and the Lorenz flow [22], for the continuous time case.

The properties displayed by the flow of the Lorenz equations prompted the search for an extension of the notion of uniform hyperbolicity encompassing time continuous systems presenting equilibria accumulated in a robust way by regular orbits. This led to the concept of singular-hyperbolic sets [30, 26].

The notion of singular-hyperbolic attractor is motivated by the classical construction of so-called geometric Lorenz models in [12, 1, 13, 41]. It was shown in [30, 26] that any attractor of a 3-dimensional flow containing in a robust fashion equilibria together with regular orbits must be *singular-hyperbolic*, that is, it must admit an invariant splitting $E^s \oplus E^{cu}$ of the tangent bundle into a 1-dimensional uniformly contracting sub-bundle and a 2-dimensional volume-expanding sub-bundle.

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Once this was settled, it is natural to ask how much of the properties shared by hyperbolic systems can be pushed for this new class of sets. Some of the topological features of hyperbolic systems were already extended for this new class [27, 9, 25, 29, 30, 28, 5].

Here we complement the previous conclusions by showing that singular-hyperbolicity implies the main geometric and ergodic features of the classical models. Namely, we are able to construct convenient cross-sections and invariant contracting foliations for a corresponding Poincaré map, that allow us to reduce the fbw dynamics to a 1-dimensional transformation. This transformation is uniformly expanding, due to volume-expansiveness of the fbw along E^{cu} , and so its geometric and ergodic properties can be well understood.

As a consequence, we prove that the fbw is expansive on the attractor, in a sense introduced by Komuro [20], that we recall in the sequel. Another main result, extending [10], is that typical orbits in the basin of the attractor have a well-defined statistical behavior: for Lebesgue almost every point the forward Birkhoff time average converges, and is given by a physical probability measure. Moreover we show that this measure admits absolutely continuous conditional measures along the center-unstable directions on the attractor. As a consequence this measure is a u -Gibbs state and an equilibrium state for the logarithm of the unstable Jacobian on the attractor.

The fbw of the Lorenz equation [22] and the geometric Lorenz fbws [12, 1, 13, 41] are known to display singular-hyperbolic attractors [37, 27, 38]. Hence in particular we show that *the attractors for the geometric Lorenz fbw and for the Lorenz equations are expansive and have a physical measure whose ergodic basin is a full Lebesgue measure subset of the topological basin of attraction*. Let us give the precise statements of these results.

1.1. Singular-hyperbolicity. Throughout, M is a compact boundaryless 3-dimensional manifold and $X^1(M)$ is the set of C^1 vector fields on M , endowed with the C^1 topology. From now on we fix some smooth Riemannian structure on M and an induced normalized volume form m that we call Lebesgue measure. We write also dist for the induced distance on M . Given $X \in X^1(M)$, we denote by X_t , $t \in \mathbb{R}$ the fbw induced by X , and if $x \in M$ and $[a, b] \subset \mathbb{R}$ then $X_{[a, b]}(x) = \{X_t(x), a \leq t \leq b\}$.

Let Λ be a compact invariant set Λ of $X \in X^1(M)$. We say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$$

If U above can be chosen such that $X_t(U) \subset U$ for $t > 0$, we say that Λ is an *attracting set*. The *topological basin* of an attracting set Λ is the set

$$W^s(\Lambda) = \{x \in M : \lim_{t \rightarrow +\infty} \text{dist}(X_t(x), \Lambda) = 0\}.$$

We say that an attracting set Λ is *transitive* if it coincides with the ω -limit set of a regular X -orbit. An *attractor* is a transitive attracting set, and a *repeller* is an attractor for the reversed vector field $-X$. An attractor, or repeller, is *proper* if it is not the whole manifold. An invariant set of X is *non-trivial* if it is neither a periodic orbit nor a singularity.

Definition 1.1. Let Λ be a compact invariant set of $X \in X^r(M)$, $c > 0$, and $0 < \lambda < 1$. We say that Λ has a (c, λ) -dominated splitting if the bundle over Λ can be written as a continuous

DX_t -invariant sum of sub-bundles

$$T_\Lambda M = E^1 \oplus E^2,$$

such that for every $t > 0$ and every $x \in \Lambda$, we have

$$(1) \quad \|DX_t | E_x^1\| \cdot \|DX_{-t} | E_{X_t(x)}^2\| < c\lambda^t.$$

The domination condition (1) implies that the direction of the fbw is contained in one of the sub-bundles.

We stress that we only deal with fbws in dimension 3. In all that follows, the first sub-bundle E^1 will be one-dimensional, and the fbw direction will be contained in the second sub-bundle E^2 , that we call *central direction* and denote by E^{cu} .

We say that a X -invariant subset Λ of M is *partially hyperbolic* if it has a (c, λ) -dominated splitting, for some $c > 0$ and $\lambda \in (0, 1)$, such that the sub-bundle $E^1 = E^s$ is uniformly contracting: for every $t > 0$ and every $x \in \Lambda$ we have

$$\|DX_t | E_x^s\| < c\lambda^t.$$

For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J_t^c(x)$ be the absolute value of the determinant of the linear map

$$DX_t | E_x^{cu} : E_x^{cu} \rightarrow E_{X_t(x)}^{cu}.$$

We say that the sub-bundle E_Λ^{cu} of the partially hyperbolic invariant set Λ is *volume expanding* if $J_t^c(x) \geq c e^{-\lambda t}$ for every $x \in \Lambda$ and $t \geq 0$. In this case we say that E_Λ^{cu} is (c, λ) -*volume expanding* to indicate the dependence on c, λ .

Definition 1.2. Let Λ be a compact invariant set of $X \in \mathcal{X}^r(M)$ with singularities. We say that Λ is a *singular-hyperbolic set* for X if all the singularities of Λ are hyperbolic, and Λ is partially hyperbolic with volume expanding central direction.

1.2. Expansiveness. The fbw is *sensitive to initial data* if there is $\delta > 0$ such that, for any $x \in M$ and any neighborhood N of x , there is $y \in N$ and $t \geq 0$ such that $\text{dist}(X_t(x), X_t(y)) > \delta$.

We shall work with a much stronger property, called *expansiveness*. Denote by $S(\mathbb{R})$ the set of surjective increasing continuous functions $h : \mathbb{R} \rightarrow \mathbb{R}$. We say that the fbw is *expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ such that, for any $h \in S(\mathbb{R})$, if

$$\text{dist}(X_t(x), X_{h(t)}(y)) \leq \delta \quad \text{for all } t \in \mathbb{R},$$

then $X_{h(t_0)}(y) \in X_{[t_0-\varepsilon, t_0+\varepsilon]}(x)$, for some $t_0 \in \mathbb{R}$. We say that an invariant compact set Λ is *expansive* if the restriction of X_t to Λ is an expansive fbw.

This notion was proposed by Komuro in [20], and he called it K^* -expansiveness. He proved that a geometric Lorenz attractor is expansive in this sense. Our first main result generalizes this to any singular-hyperbolic attractor.

Theorem A. *Let Λ be a singular-hyperbolic attractor of $X \in \mathcal{X}^1(M)$. Then Λ is expansive.*

An immediate consequence of this theorem is the following

Corollary 1. *A singular-hyperbolic attractor of a 3-fbw is sensitive to initial data.*

A stronger notion of expansiveness has been proposed by Bowen-Walters [8]. In it one considers continuous maps $h : \mathbb{R} \rightarrow \mathbb{R}$ with $h(0) = 0$, instead. This turns out to be unsuitable when dealing with singular sets, because it implies that all singularities are isolated [8, Lemma 1]. An intermediate definition was also proposed by Keynes-Sears [18]: the set of maps is the same as in [20], but they require $t_0 = 0$. Komuro [20] shows that a geometric Lorenz attractor does not satisfy this condition.

1.3. Physical or SRB measures. An invariant probability μ is a *physical* measure for the flow X_t , $t \in \mathbb{R}$ if the set $B(\mu)$ of points $z \in M$ satisfying

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \varphi(X_t(z)) dt = \int \varphi d\mu \quad \text{for all continuous } \varphi : M \rightarrow \mathbb{R}$$

has positive Lebesgue measure: $m(B(\mu)) > 0$. In that case, $B(\mu)$ is called the *basin* of μ .

Physical measures for singular-hyperbolic attractors were recently constructed by Colmenárez. Here we need to assume that $(X_t)_{t \in \mathbb{R}}$ is a flow of class C^2 since for the construction of physical measures a bounded distortion property for one-dimensional maps is needed. These maps are naturally obtained as quotient maps over the set of stable leaves, which form a $C^{1+\alpha}$ foliation of a finite number of cross-sections associated to the flow if the flow is C^2 , see Section 4.

Theorem B. *Let Λ be a singular-hyperbolic attractor. Then Λ supports a unique physical probability measure μ which is ergodic, hyperbolic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attraction, i.e. $B(\mu) = W^s(\Lambda)$, m mod 0.*

This statement extends the main result in Colmenárez [10], where hyperbolicity of the physical measure was not proved and the author assumed that periodic orbits in Λ exist and are dense. However in another recent work, Arroyo and Pujals [4] show that every singular-hyperbolic attractor has a dense set of periodic orbits, so the denseness assumption is no restriction. Here we give an independent proof of the existence of SRB measures which does not use denseness of periodic orbits and that enables us to obtain the hyperbolicity of the SRB measure.

Here hyperbolicity means *non-uniform hyperbolicity*: the tangent bundle over Λ splits into a sum $T_z M = E_z^s \oplus E_z^X \oplus F_z$ of three one-dimensional invariant subspaces defined for μ -a.e. $z \in \Lambda$ and depending measurably on the base point z , where μ is the physical measure in the statement of Theorem B, E_z^X is the flow direction (with zero Lyapunov exponent) and F_z is the direction with positive Lyapunov exponent, that is, for every non-zero vector $v \in F_z$ we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \|DX_t(z) \cdot v\| > 0.$$

We note that the invariance of the splitting implies that $E_z^{cu} = E_z^X \oplus F_z$ whenever F_z is defined. For a proof of non-uniform hyperbolicity without using the existence of invariant measures, but assuming density of periodic orbits, see [10].

Theorem B is another statement of sensitiveness, this time applying to the whole open set $B(\Lambda)$. Indeed, since non-zero Lyapunov exponents express that the orbits of infinitesimally close-by points tend to move apart from each other, this theorem means that most orbits in the basin of attraction separate under forward iteration. See Kifer [19], and Metzger [24], and references

therein, for previous results about invariant measures and stochastic stability of the geometric Lorenz models.

1.4. The physical measure is a u-Gibbs state. In the uniformly hyperbolic setting it is well known that physical measures for hyperbolic attractors admit a disintegration into conditional measures along the unstable manifolds of almost every point which are absolutely continuous with respect to the induced Lebesgue measure on these sub-manifolds, see [6, 7, 34, 39].

Here the existence of unstable manifolds is guaranteed by the hyperbolicity of the physical measure: the strong-unstable manifolds $W^{uu}(z)$ are the “integral manifolds” in the direction of the one-dimensional sub-bundle F , tangent to F_z at almost every $z \in \Lambda$. The sets $W^{uu}(z)$ are embedded sub-manifolds in a neighborhood of z which, in general, depend only measurably (including its size) on the base point $z \in \Lambda$. The *strong-unstable manifold* is defined by

$$W^{uu}(z) = \{y \in M : \lim_{t \rightarrow -\infty} \text{dist}(X_t(y), X_t(z)) = 0\}$$

and exists for almost every $z \in \Lambda$ with respect to the physical and hyperbolic measure obtained in Theorem B. We remark that since Λ is an attracting set, then $W^{uu}(z) \subset \Lambda$ whenever defined.

The tools developed to prove Theorem B enable us to prove that the physical measure obtained there has absolutely continuous disintegration along the center-unstable direction. To state this result precisely we need the following notations.

The uniform contraction along the E^s direction ensures the existence of *strong-stable one-dimensional manifolds* $W^{ss}(x)$ through every point $x \in \Lambda$, tangent to $E^s(x)$ at x . Using the action of the fbw we define the *stable manifold* of $x \in \Lambda$ by

$$W^s(x) = \bigcup_{t \in \mathbb{R}} X_t(W^{ss}(x)).$$

Analogously for μ -a.e. z we can define the *unstable-manifold* of z by

$$W^u(z) = \bigcup_{t \in \mathbb{R}} X_t(W^{uu}(z)).$$

We note that E_z^{cu} is tangent to $W^u(z)$ at z for μ -a.e. z . Given $x \in \Lambda$ let S be a smooth surface in M which is everywhere transverse to the vector field X and $x \in S$, which we call a *cross-section of the fbw at x* . Let ξ_0 be the connected component of $W^s(x) \cap S$ containing x . Then ξ_0 is a smooth curve in S and we take a parametrization $\psi : [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \rightarrow S$ of a compact neighborhood S_0 of x in S , for some $\varepsilon > 0$, such that

- $\psi(0, 0) = x$ and $\psi((-\varepsilon, \varepsilon) \times \{0\}) \subset \xi_0$;
- $\xi_1 = \psi(\{0\} \times (-\varepsilon, \varepsilon))$ is transverse to ξ_0 at x : $\xi_0 \pitchfork \xi_1 = \{x\}$.

We consider the family $\Pi(S_0)$ of connected components ζ of $W^u(z) \cap S_0$ containing $z \in S_0$ which *cross* S_0 . We say that a *curve* ζ *crosses* S_0 if it can be written as the graph of a map $\xi_1 \rightarrow \xi_0$.

Given $\delta > 0$ we let $\Pi_\delta(x) = \{X_{(\delta, \delta)}(\zeta) : \zeta \in \Pi(S_0)\}$ be a family of surfaces inside unstable leaves in a neighborhood of x crossing S_0 . The volume form m induces a volume form m_γ on each $\gamma \in \Pi_\delta(x)$ naturally. Moreover, since $\gamma \in \Pi_\delta(x)$ is a continuous family of curves (S_0 is compact

and each curve is tangent to a continuous sub-bundle E^{cu} , it forms a measurable partition of $\hat{\Pi}_\delta(x) = \cup\{\gamma : \gamma \in \Pi_\delta(x)\}$. We say that $\Pi_\delta(x)$ is a δ -adapted foliated neighborhood of x .

Hence $\mu \mid \hat{\Pi}_\delta(x)$ can be disintegrated along the partition $\Pi_\delta(x)$ into a family of measures $\{\mu_\gamma\}_{\gamma \in \Pi_\delta(x)}$ such that

$$\mu \mid \hat{\Pi}_\delta(x) = \int \mu_\gamma d\hat{\mu}(\gamma),$$

where $\hat{\mu}$ is a measure on $\Pi_\delta(x)$ defined by

$$\hat{\mu}(A) = \mu(\cup_{\gamma \in A} \gamma) \quad \text{for all Borel sets } A \subset \Pi_\delta(x).$$

We say that μ has an absolutely continuous disintegration along the center-unstable direction if for every given $x \in \Lambda$, each δ -adapted foliated neighborhood $\Pi_\delta(x)$ of x induces a disintegration $\{\mu_\gamma\}_{\gamma \in \Pi_\delta(x)}$ of $\mu \mid \hat{\Pi}_\delta(x)$, for all small enough $\delta > 0$, such that $\mu_\gamma \ll m_\gamma$ for $\hat{\mu}$ -a.e. $\gamma \in \Pi_\delta(x)$.

Theorem C. *Let Λ be a singular-hyperbolic attractor for a C^2 three-dimensional flow. Then the physical measure μ supported in Λ has a disintegration into absolutely continuous conditional measures μ_γ along center-unstable surfaces $\gamma \in \Pi_\delta(x)$ such that $\frac{d\mu_\gamma}{dm_\gamma}$ is uniformly bounded from above, for all δ -adapted foliated neighborhoods $\Pi_\delta(x)$ and every $\delta > 0$. Moreover $\text{supp}(\mu) = \Lambda$.*

Remark 1.3. It follows from our arguments that the densities of the conditional measures μ_γ are bounded from below away from zero on $\Lambda \setminus B$, where B is any neighborhood of the singularities $\text{Sing}(X \mid \Lambda)$. In particular the densities tend to zero as we get closer to the singularities of Λ .

The absolute continuity property along the center-unstable sub-bundle given by Theorem C ensures that

$$h_\mu(X_1) = \int \log |\det(DX_1 \mid E^{cu})| d\mu,$$

by the characterization of probability measures satisfying the Entropy Formula [21]. The above integral is the sum of the positive Lyapunov exponents along the sub-bundle E^{cu} by Oseledets Theorem [23, 40]. Since in the direction E^{cu} there is only one positive Lyapunov exponent along the one-dimensional direction F_z , μ -a.e. z , the ergodicity of μ then shows that the following is true.

Corollary 2. *If Λ is a singular-hyperbolic attractor for a C^2 three-dimensional flow X_t , then the physical measure μ supported in Λ satisfies the Entropy Formula*

$$h_\mu(X_1) = \int \log \|DX_1 \mid F_z\| d\mu(z).$$

Again by the characterization of measures satisfying the Entropy Formula we get that μ has absolutely continuous disintegration along the strong-unstable direction, along which the Lyapunov exponent is positive, thus μ is a u -Gibbs state [34]. This also shows that μ is an equilibrium state for the potential $-\log \|DX_1 \mid F_z\|$ with respect to the diffeomorphism X_1 . We note that the entropy $h_\mu(X_1)$ of X_1 is the entropy of the flow X_t with respect to the measure μ [40].

Hence we are able to extend most of the basic results on the ergodic theory of hyperbolic attractors to the setting of singular-hyperbolic attractors.

1.5. Application to the Lorenz and geometric Lorenz flows. It is well known that geometric Lorenz flows are transitive and it was proved in [27] that they are singular-hyperbolic attractors. Then as a consequence of our results we get the following corollary.

Corollary 3. *A geometric Lorenz flow is expansive and has a unique physical invariant probability measure whose basin covers Lebesgue almost every point of the topological basin of attraction. Moreover this measure is a u -Gibbs state satisfying the Entropy Formula.*

Recently Tucker [38] proved that the flow defined by the Lorenz equations [22] exhibits a singular-hyperbolic attractor. In particular our results then show the following.

Corollary 4. *The flow defined by the Lorenz equations is expansive and has a unique physical invariant probability measure whose basin covers Lebesgue almost every point of the topological basin of attraction. Moreover this measure is a u -Gibbs state satisfying the Entropy Formula.*

This paper is organized as follows. In Section 2 we obtain adapted cross-sections for the flow near Λ and deduce some hyperbolic properties for the Poincaré return maps between these sections. Theorem A is proved in Section 3. In Section 4 we reduce the dynamics of the global Poincaré return map between cross-sections to a one-dimensional piecewise expanding map. In Section 5 we explain how to construct invariant measures for the Poincaré return map from invariant measures for the induced one-dimensional map, and also how to obtain invariant measures for the flow through invariant measures for the Poincaré return map. These abstract results are applied to obtain a physical measure for a singular-hyperbolic attractor in Section 6, proving Theorem B. In Section 7 we again use the one-dimensional dynamics and the notion of hyperbolic times for the Poincaré return map to prove that the physical measure is a u -Gibbs state and that $\text{supp}(\mu) = \Lambda$, concluding the proof of Theorem C and of Corollary 2.

2. CROSS-SECTIONS AND POINCARÉ MAPS

The proof of Theorem A is based on analyzing Poincaré return maps of the flow to a convenient cross-section. In this section we give a few properties of *Poincaré maps*, that is, continuous maps $R : \Sigma \rightarrow \Sigma'$ of the form $R(x) = X_{t(x)}(x)$ between cross-sections Σ and Σ' . We always assume that the Poincaré time $t(\cdot)$ is large (Section 2.2). Recall that we assume singular-hyperbolicity.

Firstly, we observe (Section 2.1) that cross-sections have co-dimension 1 foliations which are dynamically defined: the leaves $W^s(x, \Sigma) = W_{loc}^s(x) \cap \Sigma$ correspond to the intersections with the stable manifolds of the flow. These leaves are uniformly contracted (Section 2.2) and, assuming the cross-section is *adapted* (Section 2.3) the foliation is invariant:

$$R(W^s(x, \Sigma)) \subset W^s(R(x), \Sigma') \quad \text{for all } x \in \Lambda \cap \Sigma.$$

Moreover, R is uniformly expanding in the transverse direction (Section 2.2). In Section 2.4 we analyze the flow close to singularities, again by means of cross-sections.

2.1. Stable foliations on cross-sections. We begin by recalling a few classical facts about partially hyperbolic systems, especially existence of strong-stable and center-unstable foliations. The standard reference is [15].

Hereafter, Λ is a singular-hyperbolic attractor of $X \in \mathcal{X}^1(M)$ with invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ with $\dim E^{cu} = 2$. Let $\tilde{E}^s \oplus \tilde{E}^{cu}$ be a continuous extension of this splitting to a small neighborhood U_0 of Λ . For convenience, we take U_0 to be forward invariant. Then \tilde{E}^s may be chosen invariant under the derivative: just consider at each point the direction formed by those vectors which are strongly contracted by DX_t for positive t . In general, \tilde{E}^{cu} is not invariant. However, we can always consider a cone field around it on U_0

$$C_a^{cu}(x) = \{v = v^s + v^{cu} : v^s \in \tilde{E}_x^s \text{ and } v^{cu} \in \tilde{E}_x^{cu} \text{ with } \|v^s\| \leq a \cdot \|v^{cu}\|\}$$

which is forward invariant for $a > 0$:

$$(2) \quad DX_t(C_a^{cu}(x)) \subset C_a^{cu}(X_t(x)) \quad \text{for all large } t > 0.$$

Moreover, we may take $a > 0$ arbitrarily small, reducing U_0 if necessary. For notational simplicity, we write E^s and E^{cu} for \tilde{E}^s and \tilde{E}^{cu} in all that follows.

The next result asserts that there exist locally strong-stable and center-unstable manifolds, defined at every regular point $x \in U_0$, which are embedded disks tangent to $E^s(x)$ and $E^{cu}(x)$, respectively. The strong-stable manifolds are locally invariant. Given any $x \in U_0$, define

$$W^{ss}(x) = \{y \in M : \text{dist}(X_t(x), X_t(y)) \rightarrow 0 \text{ as } t \rightarrow +\infty\}$$

$$W^s(x) = \bigcup_{t \in \mathbb{R}} W^s(X_t(x)) = \bigcup_{t \in \mathbb{R}} X_t(W^s(x)).$$

Given $\varepsilon > 0$, denote $I_\varepsilon = (-\varepsilon, \varepsilon)$ and let $\mathcal{E}^1(I_1, M)$ be the set of C^1 embedding maps $f : I_1 \rightarrow M$ endowed with the C^1 topology.

Proposition 2.1. *(stable and center-unstable manifolds) There are continuous maps $\phi^{ss} : U_0 \rightarrow \mathcal{E}^1(I_1, M)$ and $\phi^{cu} : U_0 \rightarrow \mathcal{E}^1(I_1 \times I_1, M)$ such that given any $0 < \varepsilon < 1$ and $x \in U_0$, if we denote $W_\varepsilon^{ss}(x) = \phi^{ss}(x)(I_\varepsilon)$ and $W_\varepsilon^{cu}(x) = \phi^{cu}(x)(I_\varepsilon \times I_\varepsilon)$,*

- (a) $T_x W_\varepsilon^{ss}(x) = E^s(x)$;
- (b) $T_x W_\varepsilon^{cu}(x) = E^{cu}(x)$;
- (c) $W_\varepsilon^{ss}(x)$ is a neighborhood of x inside $W^{ss}(x)$;
- (d) $y \in W^{ss}(x) \Leftrightarrow$ there is $T \geq 0$ such that $X_T(y) \in W_\varepsilon^{ss}(X_T(x))$ (local invariance);
- (e) $d(X_t(x), X_t(y)) \leq c \cdot \lambda^t \cdot d(x, y)$ for all $t > 0$ and all $y \in W_\varepsilon^{ss}(x)$.

The constants $c > 0$ and $\lambda \in (0, 1)$ are taken as in Definition 1.1 and the distance $d(x, y)$ is the intrinsic distance between two points on the manifold $W_\varepsilon^{ss}(x)$, given by the length of the shortest smooth curve contained in $W_\varepsilon^{ss}(x)$ connecting x to y .

Denoting $E_x^{cs} = E_x^s \oplus E_x^X$, where E_x^X is the direction of the fbw at x , it follows that

$$T_x W^{ss}(x) = E_x^s \quad \text{and} \quad T_x W^s(x) = E_x^{cs}.$$

We fix ε once and for all. Then we call $W_\varepsilon^{ss}(x)$ the local *strong-stable manifold* and $W_\varepsilon^{cu}(x)$ the local *center-unstable manifold* of x .

Now let Σ be a *cross-section* to the fbw, that is, a C^2 embedded compact disk transverse to X at every point. For every $x \in \Sigma$ we define $W^s(x, \Sigma)$ to be the connected component of $W^s(x) \cap \Sigma$ that contains x . This defines a foliation \mathcal{F}_Σ^s of Σ into co-dimension 1 sub-manifolds of class C^1 .

Remark 2.2. Given any cross-section Σ and a point x in its interior, we may always find a smaller cross-section also with x in its interior and which is the image of the square $[0, 1] \times [0, 1]$ by a C^1 diffeomorphism that sends horizontal lines inside leaves of \mathcal{F}_Σ^s . So, in what follows we always assume cross-sections are of the latter kind, see Figure 1. We denote by $\text{int}(\Sigma)$ the image of $(0, 1) \times (0, 1)$ under the above-mentioned diffeomorphism, which we call the *interior* of Σ .

We also assume that each cross-section Σ is contained in U_0 , so that every $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$.

Remark 2.3. In general, we can not choose the cross-section such that $W^s(x, \Sigma) \subset W_\varepsilon^{ss}(x)$. The reason is that we want cross-sections to be C^1 , whereas $x \mapsto W_\varepsilon^{ss}(x)$ is usually not differentiable. On the other hand, assuming that the cross-section is small with respect to ε , and choosing any curve $\gamma \subset \Sigma$ crossing transversely every leaf of \mathcal{F}_Σ^s , we may consider a Poincaré map

$$R_\Sigma : \Sigma \rightarrow \Sigma(\gamma) = \bigcup_{z \in \gamma} W_\varepsilon^{ss}(z)$$

with Poincaré time close to zero, see Figure 1. This is a homeomorphism onto its image, close to the identity, such that $R_\Sigma(W^s(x, \Sigma)) \subset W_\varepsilon^{ss}(R_\Sigma(x))$. So, identifying the points of Σ with their images under this homeomorphism, we may pretend that indeed $W^s(x, \Sigma) \subset W_\varepsilon^{ss}(x)$. We shall often do this in the sequel, to avoid cumbersome technicalities.

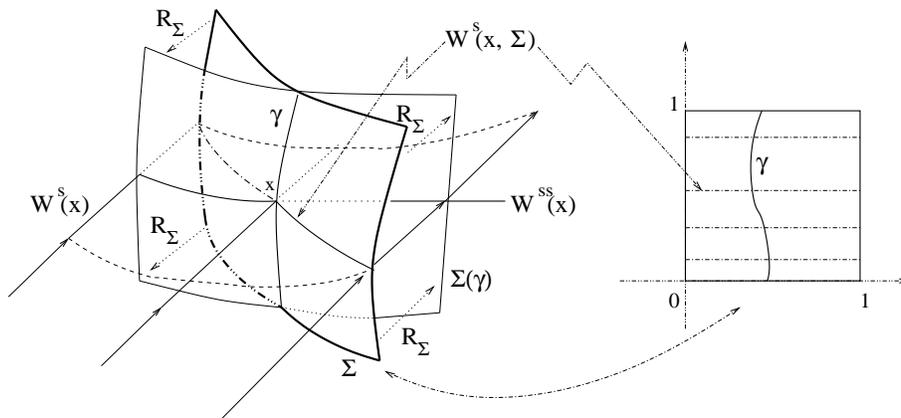


FIGURE 1. The sections Σ , $\Sigma(\gamma)$, the manifolds $W^s(x)$, $W^{ss}(x)$, $W^s(x, \Sigma)$ and the projection R_Σ , on the right. On the left, the square $[0, 1] \times [0, 1]$ identified with Σ , where \mathcal{F}_Σ^s becomes the horizontal foliation and the curve γ is transversal to the horizontal direction. Solid lines with arrows indicate the flow direction.

2.2. Hyperbolicity of Poincaré maps. Let Σ be a small cross-section to X and let $R : \Sigma \rightarrow \Sigma'$ be a Poincaré map $R(y) = X_{t(y)}(y)$ to another cross-section Σ' (possibly $\Sigma = \Sigma'$). Note that R needs not correspond to the first time the orbits of Σ encounter Σ' .

The splitting $E^s \oplus E^{cu}$ over U_0 induces a continuous splitting $E_\Sigma^s \oplus E_\Sigma^{cu}$ of the tangent bundle $T\Sigma$ to Σ (and analogously for Σ'), defined by

$$(3) \quad E_\Sigma^s(y) = E_y^{cs} \cap T_y \Sigma \quad \text{and} \quad E_\Sigma^{cu}(y) = E_y^{cu} \cap T_y \Sigma.$$

We are going to prove that if the Poincaré time $t(x)$ is sufficiently large then (3) defines a hyperbolic splitting for the transformation R on the cross-sections, at least restricted to Λ :

Proposition 2.4. *Let $R : \Sigma \rightarrow \Sigma'$ be a Poincaré map as before with Poincaré time $t(\cdot)$. Then $DR_x(E_\Sigma^s(x)) = E_\Sigma^s(R(x))$ at every $x \in \Sigma$ and $DR_x(E_\Sigma^{cu}(x)) = E_\Sigma^{cu}(R(x))$ at every $x \in \Lambda \cap \Sigma$.*

Moreover there exists $t_1 = t_1(\Sigma, \Sigma') > 0$ such that if $t(\cdot) > t_1$ at every point, then

$$\|DR | E_\Sigma^s(x)\| < 1/3 \quad \text{and} \quad \|DR | E_\Sigma^{cu}(x)\| > 3 \quad \text{at every } x \in \Sigma.$$

Remark 2.5. In what follows we use K as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction, and on upper and lower bounds for the norm of the vector field on the cross-sections. The conditions on t_1 in the proof of the proposition depend only on these bounds as well. In all our applications, all these angles and norms will be uniformly bounded from zero and infinity, and so both K and t_1 may be chosen uniformly.

Proof. The differential of the Poincaré map at any point $x \in \Sigma$ is given by

$$DR(x) = P_{R(x)} \circ DX_{t(x)} | T_x \Sigma,$$

where $P_{R(x)}$ is the projection onto $T_{R(x)} \Sigma'$ along the direction of $X(R(x))$. Note that $E_\Sigma^s(x)$ is tangent to $\Sigma \cap W^s(x) \supset W^s(x, \Sigma)$. Since the stable manifold $W^s(x)$ is invariant, we have invariance of the stable bundle: $DR(x)(E_\Sigma^s(x)) = E_\Sigma^s(R(x))$. Moreover for all $x \in \Lambda$ we have

$$DX_{t(x)}(E_\Sigma^{cu}(x)) \subset DX_{t(x)}(E_x^{cu}) = E_{R(x)}^{cu}.$$

Since $P_{R(x)}$ is the projection along the vector field, it sends $E_{R(x)}^{cu}$ to $E_{\Sigma'}^{cu}(R(x))$. This proves that the center-unstable bundle is invariant restricted to Λ , i.e. $DR(x)(E_\Sigma^{cu}(x)) = E_\Sigma^{cu}(R(x))$.

Next we prove the expansion and contraction statements. We start by noting that $\|P_{R(x)}\| \leq K$. Then we consider the basis $\{\frac{X(x)}{\|X(x)\|}, e_x^u\}$ of E_x^{cu} , where e_x^u is a unit vector in the direction of $E_\Sigma^{cu}(x)$. Since the flow direction is invariant, the matrix of $DX_t | E_x^{cu}$ relative to this basis is upper triangular:

$$DX_{t(x)} | E_x^{cu} = \begin{bmatrix} \frac{\|X(R(x))\|}{\|X(x)\|} & \star \\ 0 & \Delta \end{bmatrix}.$$

Moreover

$$\frac{1}{K} \cdot \det(DX_{t(x)} | E_x^{cu}) \leq \frac{\|X(R(x))\|}{\|X(x)\|} \Delta \leq K \cdot \det(DX_{t(x)} | E_x^{cu}).$$

Then

$$\begin{aligned} \|DR(x) e_x^u\| &= \|P_{R(x)}(DX_{t(x)}(x) \cdot e_x^u)\| = \|\Delta \cdot e_{R(x)}^u\| = |\Delta| \\ &\geq K^{-3} |\det(DX_{t(x)} | E_x^{cu})| \geq K^{-3} \lambda^{-t(x)} \geq K^{-3} \lambda^{-t_1}. \end{aligned}$$

Taking t_1 large enough we ensure that the latter expression is larger than 3.

To prove $\|DR | E_\Sigma^s(x)\| < 1/3$, let us consider unit vectors $e_x^s \in E_x^s$ and $\hat{e}_x^s \in E_\Sigma^s(x)$, and write

$$e_x^s = a_x \cdot \hat{e}_x^s + b_x \cdot \frac{X(x)}{\|X(x)\|}.$$

Since $\angle(E_x^s, X(x)) \geq \angle(E_x^s, E_x^{cu})$ and the latter is uniformly bounded from zero, we have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the fbw. Then

$$\begin{aligned}
(4) \quad \|DR(x) e_x^s\| &= \|P_{R(x)} \circ (DX_{t(x)}(x) \cdot e_x^s)\| \\
&= \frac{1}{|a_x|} \left\| P_{R(x)} \circ \left(DX_{t(x)}(x) \left(e_x^s - b_x \frac{X(x)}{\|X(x)\|} \right) \right) \right\| \\
&= \frac{1}{|a_x|} \|P_{R(x)} \circ (DX_{t(x)}(x) \cdot e_x^s)\| \leq \frac{K}{\kappa} \lambda^{t(x)} \leq \frac{K}{\kappa} \lambda^{t_1}.
\end{aligned}$$

Once more it suffices to take t_1 large to ensure that the right hand side is less than $1/3$. \square

Given a cross-section Σ , a positive number ρ , and a point $x \in \Sigma$, we define the unstable cone of width ρ at x by

$$(5) \quad C_\rho^u(x) = \{v = v^s + v^u : v^s \in E_\Sigma^s(x), v^u \in E_\Sigma^{cu}(x) \text{ and } \|v^s\| \leq \rho \|v^u\|\}$$

(we omit the dependence on the cross-section in our notations).

Let $\rho > 0$ be any small constant. In the following consequence of Proposition 2.4 we assume the neighborhood U_0 has been chose sufficiently small, depending on ρ and on a bound on the angles between the fbw and the cross-sections.

Corollary 2.6. *For any $R : \Sigma \rightarrow \Sigma'$ as in Proposition 2.4, with $t(\cdot) > t_1$, and any $x \in \Sigma$, we have*

$$DR(x)(C_\rho^u(x)) \subset C_{\rho/2}^u(R(x)) \quad \text{and} \quad \|DR_x(v)\| \geq \frac{5}{2} \cdot \|v\| \quad \text{for all } v \in C_\rho^u(x).$$

Proof. Proposition 2.4 immediately implies that $DR_x(C_\rho^u(x))$ is contained in the cone of width $\rho/4$ around $DR(x)(E_\Sigma^{cu}(x))$ relative to the splitting

$$T_{R(x)}\Sigma' = E_{\Sigma'}^s(R(x)) \oplus DR(x)(E_\Sigma^{cu}(x)).$$

(We recall that E_Σ^s is always mapped to $E_{\Sigma'}^s$.) The same is true for E_Σ^{cu} and $E_{\Sigma'}^{cu}$, restricted to Λ . So the previous observation already gives the conclusion of the first part of the corollary in the special case of points in the attractor. Moreover to prove the general case we only have to show that $DR(x)(E_\Sigma^{cu}(x))$ belongs to a cone of width less than $\rho/4$ around $E_{\Sigma'}^{cu}(R(x))$. This is easily done with the aid of the fbw invariant cone field C_a^{cu} in (2), as follows. On the one hand,

$$DX_{t(x)}(E_\Sigma^{cu}(x)) \subset DX_{t(x)}(E_x^{cu}) \subset DX_{t(x)}(C_a^{cu}(x)) \subset C_a^{cu}(R(x)).$$

We note that $DR(x)(E_\Sigma^{cu}(x)) = P_{R(x)} \circ DX_{t(x)}(E_\Sigma^{cu}(x))$. Since $P_{R(x)}$ maps $E_{R(x)}^{cu}$ to $E_{\Sigma'}^{cu}(R(x))$ and the norms of both $P_{R(x)}$ and its inverse are bounded by some constant K (see Remark 2.5), we conclude that $DR(x)(E_\Sigma^{cu}(x))$ is contained in a cone of width b around $E_{\Sigma'}^{cu}(R(x))$, where $b = b(a, K)$ can be made arbitrarily small by reducing a . We keep K bounded, by assuming the angles between the cross-sections and the fbw are bounded from zero and then, reducing U_0 if necessary, we can make a small so that $b < \rho/4$. This concludes the proof since the expansion estimate is a trivial consequence of Proposition 2.4. \square

By a *curve* we always mean the image of a compact interval $[a, b]$ by a C^1 map. We use $\ell(\gamma)$ to denote its length. By a *cu-curve* in Σ we mean a curve contained in the cross-section Σ and

whose tangent direction $T_z\gamma \subset C_\rho^u(z)$ for all $z \in \gamma$. The next lemma says that *cu-curves linking the stable leaves of nearby points must be short*.

Lemma 2.7. *Let us we assume that ρ has been fixed, sufficiently small. Then there exists a constant κ such that, for any pair of points $x, y \in \Sigma$, and any cu-curve joining x to some point of $W^s(y, \Sigma)$, we have $\ell(\gamma) \leq \kappa \cdot d(x, y)$.*

Here d is the intrinsic distance in the C^1 surface Σ .

Proof. We consider coordinates on Σ for which x corresponds to the origin, $E_\Sigma^{cu}(x)$ corresponds to the vertical axis, and $E_\Sigma^s(x)$ corresponds to the horizontal axis; through these coordinates we identify Σ with a subset of its tangent space at x , endowed with the Euclidean metric. In general this identification is not an isometry, but the distortion is uniformly bounded, and that is taken care of by the constants C_1 and C_2 in what follows. The hypothesis that γ is a cu-curve implies

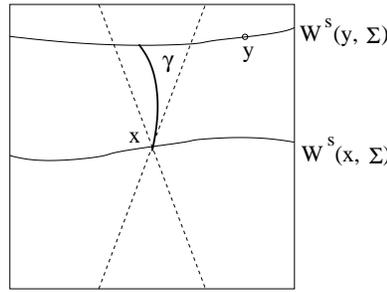


FIGURE 2. The stable manifolds on the cross-section and the *cu*-curve γ connecting them.

that it is contained in the cone of width $C_1 \cdot \rho$ centered at x . On the other hand, stable leaves are close to being horizontal. It follows (see Figure 2) that the length of γ is bounded by $C_2 \cdot d(x, y)$. This proves the lemma with $\kappa = C_2$. \square

2.3. Adapted cross-sections. The next step is to exhibit stable manifolds for Poincaré transformations $R : \Sigma \rightarrow \Sigma'$. The natural candidates are the intersections $W^s(x, \Sigma) = W_\varepsilon^s(x) \cap \Sigma$ we introduced previously. These intersections are tangent to the corresponding sub-bundle E_Σ^s and so, by Proposition 2.4, they are contracted by the transformation. For our purposes it is also important that the stable foliation be invariant:

$$(6) \quad R(W^s(x, \Sigma)) \subset W^s(R(x), \Sigma') \quad \text{for every } x \in \Lambda \cap \Sigma.$$

In order to have this we restrict somewhat our class of cross-sections whose center-unstable boundary is disjoint from Λ . Recall (Remark 2.2) that we are considering cross-sections Σ that are diffeomorphic to the square $[0, 1] \times [0, 1]$, with the horizontal lines $[0, 1] \times \{\eta\}$ being mapped to stable sets $W^s(y, \Sigma)$. The *stable boundary* $\partial^s \Sigma$ is the image of $[0, 1] \times \{0, 1\}$. The *center-unstable boundary* $\partial^{cu} \Sigma$ is the image of $\{0, 1\} \times [0, 1]$. The cross-section is δ -adapted if

$$d(\Lambda \cap \Sigma, \partial^{cu} \Sigma) > \delta,$$

where d is the intrinsic distance in Σ . We call *horizontal strip* of Σ the image $h([0, 1] \times I)$ for any compact subinterval I . Notice that every horizontal strip is a δ -adapted cross-section.

In order to prove that adapted cross-sections do exist, we need the following result.

Lemma 2.8. *If Λ is a singular-hyperbolic attractor, then every point $x \in \Lambda$ is in the closure of $W^{ss}(x) \setminus \Lambda$.*

Proof. The proof is by contradiction. Let us suppose that there exists $x \in \Lambda$ such that x is in the interior of $W^{ss}(x) \cap \Lambda$. Let $\alpha(x) \subset \Lambda$ be its α -limit set. Then

$$(7) \quad W^{ss}(z) \subset \Lambda \quad \text{for every } z \in \alpha(x),$$

since any compact part of the strong-stable manifold of z is accumulated by backward iterates of any small neighborhood of x inside $W^{ss}(x)$. It follows that $\alpha(x)$ does not contain any singularity: indeed, [30, Theorem B] proves that the strong-stable manifold of each singularity meets Λ only at the singularity. Therefore by [30, Proposition 1.8] the invariant set $\alpha(x) \subset \Lambda$ is hyperbolic. It also follows from (7) that the union

$$S = \bigcup_{y \in \alpha(x) \cap \Lambda} W^{ss}(y)$$

of the strong-stable manifolds through the points of $\alpha(x)$ is contained in Λ . By continuity of the strong-stable manifolds and the fact that $\alpha(x)$ is a closed set, we get that S is also closed. Using [30] once more, we see that S does not contain singularities and, thus, is also a hyperbolic set.

We claim that $W^u(S)$, the union of the unstable manifolds of the points of S , is an open set. To prove this, we note that S contains the whole stable manifold $W^s(z)$ of every $z \in S$: this is because S is invariant and contains the strong-stable manifold of z . Now, the union of the strong-unstable manifolds through the points of $W^s(z)$ contains a neighborhood of z . This proves that $W^u(S)$ is a neighborhood of S . Thus the backward orbit of any point in $W^u(S)$ must enter the interior of $W^u(S)$. Since the interior is, clearly, an invariant set, this proves that $W^u(S)$ is open, as claimed.

Finally, consider any backward dense orbit in Λ (we recall that for us an attractor is transitive by definition). On the one hand, its α -limit set is the whole Λ . On the other hand, this orbit must intersect the open set $W^u(S)$, and so the α -limit set must be contained in S . This implies that $\Lambda \subset S$, which is a contradiction, because Λ contains singularities. \square

Corollary 2.9. *For any $x \in \Lambda$ there exist points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in distinct connected components of $W^{ss}(x) \setminus \{x\}$.*

Proof. Otherwise there would exist a whole segment of the strong-stable manifold entirely contained in Λ . Considering any point in the interior of this segment, we would get a contradiction to Lemma 2.8. \square

Lemma 2.10. *Let $x \in \Lambda$ be a regular point, that is, such that $X(x) \neq 0$. Then there exists $\delta > 0$ for which there exists a δ -adapted cross-section Σ at x .*

Proof. Fix $\varepsilon > 0$ as in the stable manifold theorem. Any cross-section Σ_0 at x sufficiently small with respect to $\varepsilon > 0$ is foliated by the intersections $W_\varepsilon^s(x) \cap \Sigma_0$. By Corollary 2.9, we may find points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in each of the connected components of $W_\varepsilon^s(x) \cap \Sigma_0$. Since Λ

is closed, there are neighborhoods V^\pm of x^\pm disjoint from Λ . Let $\gamma \subset \Sigma_0$ be some small curve through x , transverse to $W_\varepsilon^s(x) \cap \Sigma_0$. Then we may find a continuous family of segments inside $W_\varepsilon^s(y) \cap \Sigma_0$, $y \in \gamma$ with endpoints contained in V^\pm . The union Σ of these segments is a δ -adapted cross-section, for some $\delta > 0$, see Figure 3. \square

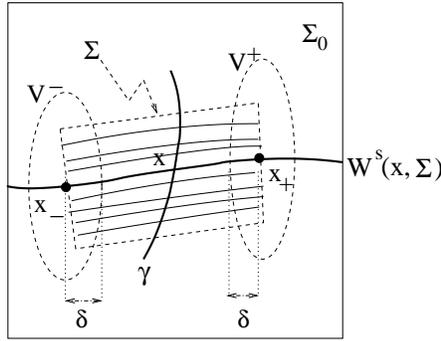


FIGURE 3. The construction of a δ -adapted cross-section for a regular $x \in \Lambda$.

We are going to show that if the cross-sections are adapted, then we have the invariance property (6). Given $\Sigma, \Sigma' \in \Xi$ we set $\Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\}$ the domain of the return map from Σ to Σ' .

Lemma 2.11. *Given $\delta > 0$ and δ -adapted cross-sections Σ and Σ' , there exists $t_2 = t_2(\Sigma, \Sigma') > 0$ such that if $R : \Sigma(\Sigma') \rightarrow \Sigma'$ defined by $R(z) = R_{t(z)}(z)$ is a Poincaré map with time $t(\cdot) > t_2$, then*

- (1) $R(W^s(x, \Sigma)) \subset W^s(R(x), \Sigma')$ for every $x \in \Sigma(\Sigma')$, and also
- (2) $d(R(y), R(z)) \leq \frac{1}{2}d(y, z)$ for every $y, z \in W^s(x, \Sigma)$ and $x \in \Sigma(\Sigma')$.

Proof. This is a simple consequence of the relation (4) from the proof of Proposition 2.4: the tangent direction to each $W^s(x, \Sigma)$ is contracted at an exponential rate λ

$$\|DR(x) e_x^s\| \leq \frac{K}{\kappa} \lambda^{t(x)}.$$

Choosing t_2 sufficiently large we ensure that

$$\frac{1}{\kappa} \lambda^{t_2} \cdot \sup\{\ell(W^s(x, \Sigma)) : x \in \Sigma\} < \delta.$$

In view of the definition of δ -adapted cross-section this gives part (1) of the lemma. Part (2) is entirely analogous: it suffices that $(K/\kappa) \cdot \lambda^{t_2} < 1/2$. \square

Lemma 2.12. *Let Σ be a δ -adapted cross-section. Then, given any $r > 0$ there exists ρ such that*

$$d(y, z) < \rho \quad \Rightarrow \quad \text{dist}(X_s(y), X_s(z)) < r$$

for all $s > 0$, every $y, z \in W^s(x, \Sigma)$, and every $x \in \Lambda \cap \Sigma$.

Remark 2.13. Clearly we may choose $t_2 > t_1$. Remark 2.5 applies to t_2 as well.

Proof. Let y and z be as in the statement. As in Remark 2.3, we may find $z' = X_\tau(z)$ in the intersection of the orbit of z with the strong-stable manifold of y satisfying

$$\frac{1}{K} \leq \frac{\text{dist}(y, z')}{d(y, z)} \leq K \quad \text{and} \quad |\tau| \leq K \cdot d(y, z).$$

Then, given any $s > 0$,

$$\begin{aligned} \text{dist}(X_s(y), X_s(z)) &\leq \text{dist}(X_s(y), X_s(z')) + \text{dist}(X_s(z'), X_s(z)) \\ &\leq C \cdot e^{\gamma s} \cdot \text{dist}(y, z') + \text{dist}(X_{s+\tau}(z), X_s(z)) \\ &\leq KC \cdot e^{\gamma s} \cdot d(y, z) + K|\tau| \leq (KC + K^2) \cdot d(y, z). \end{aligned}$$

Taking $\rho < r/(KC + K^2)$ we get the statement of the lemma. \square

2.4. Flow boxes of singularities. In this section we collect some known facts about the dynamics near the singularities of the fbw. It is known [27, Theorem A] that *each singularity of a singular-hyperbolic attracting set, accumulated by regular orbits of a 3-dimensional fbw, must be Lorenz-like.* In particular every singularity σ_k of a singular-hyperbolic attractor, as in the setting of Theorem A, is Lorenz-like, that is, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the derivative $DX(\sigma_k)$ are all real and satisfy

$$\lambda_1 > 0 > \lambda_2 > \lambda_3 \quad \text{and} \quad \lambda_1 + \lambda_2 > 0.$$

In particular, the unstable manifold $W^u(\sigma_k)$ is one-dimensional, and there is a one-dimensional strong-stable manifold $W^{ss}(\sigma_k)$ contained in the two-dimensional stable manifold $W^s(\sigma_k)$. Most important for what follows, the attractor intersects the strong-stable manifold at the singularity only [27, Theorem A].

Then for some $\delta > 0$ we may choose δ -adapted cross-sections contained in U_0

- $\Sigma^{o,\pm}$ at points y^\pm in different components of $W_{loc}^u(\sigma_k) \setminus \{\sigma_k\}$
- $\Sigma^{i,\pm}$ at points x^\pm in different components of $W_{loc}^s(\sigma_k) \setminus W_{loc}^{ss}(\sigma_k)$

and Poincaré maps $R^\pm : \Sigma^{i,\pm} \setminus \ell^\pm \rightarrow \Sigma^{o,-} \cup \Sigma^{o,+}$, where $\ell^\pm = \Sigma^{i,\pm} \cap W_{loc}^s(\sigma_k)$, satisfying (see Figure 4)

- (1) every orbit in the attractor passing through a small neighborhood of the singularity σ_k intersects some of the incoming cross-sections $\Sigma^{i,\pm}$;
- (2) R^\pm maps each connected component of $\Sigma^{i,\pm} \setminus \ell^\pm$ diffeomorphically inside a different outgoing cross-section $\Sigma^{o,\pm}$, preserving the corresponding stable foliations and unstable cones.

These cross-sections may be chosen to be planar relative to some linearizing system of coordinates near σ_k , e.g. for a small $\zeta > 0$

$$\Sigma^{i,\pm} = \{(x_1, x_2, \pm 1) : |x_1| \leq \zeta, |x_2| \leq \zeta\} \quad \text{and} \quad \Sigma^{o,\pm} = \{(\pm 1, x_2, x_3) : |x_2| \leq \zeta, |x_3| \leq \zeta\},$$

where the x_1 -axis corresponds to the unstable manifold near σ_k , the x_2 -axis to the strong-stable manifold and the x_3 -axis to the weak-stable manifold of the singularity which, in turn, is at the origin, see Figure 4.

Reducing the cross-sections if necessary, i.e. taking $\delta > 0$ small enough, we ensure that the Poincaré times are larger than t_2 , so that the same conclusions as in the previous sections apply

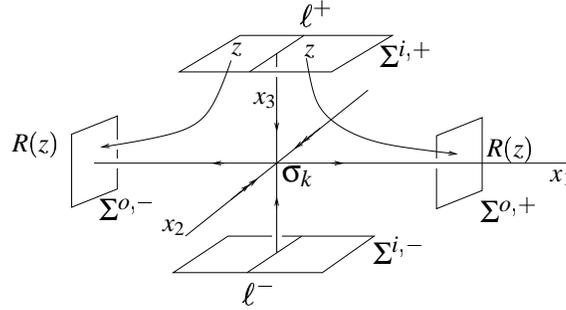


FIGURE 4. Ingoing and outgoing adapted cross-sections near a singularity.

here. Indeed using linearizing coordinates it is easy to see that for points $z = (x_1, x_2, \pm 1) \in \Sigma^{i,\pm}$ the time τ^\pm it takes the fbw starting at z to reach one of $\Sigma^{o,\pm}$ depends on x_1 only and is given by

$$\tau^\pm(x_1) = -\frac{\log x_1}{\lambda_1}.$$

We then fix these cross-sections once and for all and define for small $\varepsilon > 0$ the *flow-box*

$$U_{\sigma_k} = \bigcup_{x \in \Sigma^{i,\pm} \setminus \ell^\pm} X_{(-\varepsilon, \tau^\pm(x) + \varepsilon)}(x) \cup (-\delta, \delta) \times (-\delta, \delta) \times (-1, 1)$$

which is an open neighborhood of σ_k with σ_k the unique zero of $X|_{U_{\sigma_k}}$. We note that the function $\tau^\pm : \Sigma^{i,\pm} \rightarrow \mathbb{R}$ is integrable with respect to the Lebesgue (area) measure over $\Sigma^{i,\pm}$: we say that *the exit time function in a flow box near each singularity is Lebesgue integrable*.

In particular we can determine the expression of the Poincaré maps between ingoing and outgoing cross-sections easily thought linearized coordinates

$$(8) \quad \Sigma^{i,+} \cap \{x_1 > 0\} \rightarrow \Sigma^{o,+}, \quad (x_1, x_2, 1) \mapsto (1, x_2 \cdot x_1^{-\lambda_3/\lambda_1}, x_1^{-\lambda_2/\lambda_1}).$$

This shows that the map obtained identifying points with the same x_2 coordinate, i.e. points in the same stable leaf, is simply $x_1 \mapsto x_1^\beta$ where $\beta = -\lambda_2/\lambda_1 \in (0, 1)$. For the other possible combinations of ingoing and outgoing cross-sections the Poincaré maps have a similar expression. This will be useful to construct physical measures for the fbw.

3. PROOF OF EXPANSIVENESS

Here we prove Theorem A. The proof is by contradiction: let us suppose that there exist $\varepsilon > 0$, a sequence $\delta_n \rightarrow 0$, a sequence of functions $h_n \in S(\mathbb{R})$, and sequences of points $x_n, y_n \in \Lambda$ such that

$$(9) \quad d(X_t(x_n), X_{h_n(t)}(y_n)) \leq \delta_n \quad \text{for all } t \in \mathbb{R},$$

but

$$(10) \quad X_{h_n(t)}(y_n) \notin X_{[t-\varepsilon, t+\varepsilon]}(x_n) \quad \text{for all } t \in \mathbb{R}.$$

3.1. Proof of Theorem A. The main step in the proof is a reduction to a forward expansiveness statement about Poincaré maps which we state in Theorem 3.1 below.

We are going to use the following observation: there exists some regular (i.e. non-equilibrium) point $z \in \Lambda$ which is accumulated by the sequence of ω -limit sets $\omega(x_n)$. To see that this is so, start by observing that accumulation points do exist, since the ambient space is compact. Moreover, if the ω -limit sets accumulate on a singularity then they also accumulate on at least one of the corresponding unstable branches which, of course, consists of regular points. We fix such a z once and for all. Replacing our sequences by subsequences, if necessary, we may suppose that for every n there exists $z_n \in \omega(x_n)$ such that $z_n \rightarrow z$.

Let Σ be a δ -adapted cross-section at z , for some small δ . Reducing δ (but keeping the same cross-section) we may ensure that z is in the interior of the subset

$$\Sigma_\delta = \{y \in \Sigma : d(y, \partial\Sigma) > \delta\}.$$

By definition, x_n returns infinitely often to the neighborhood of z_n which, on its turn, is close to z . Thus dropping a finite number of terms in our sequences if necessary, we have that the orbit of x_n intersects Σ_δ infinitely many times. Let t_n be the time corresponding to the first intersection. Replacing x_n, y_n, t , and h_n by $x'_n = X_{t_n}(x_n), y'_n = X_{h_n(t_n)}(y_n), t' = t - t_n$, and $h'_n(t') = h_n(t' + t_n) - h_n(t_n)$, we may suppose that $x_n \in \Sigma_\delta$, while preserving all the relations (9) and (10). Moreover there exists a sequence $\tau_{n,j}, j \geq 0$ with $\tau_{n,0} = 0$ such that

$$(11) \quad x_{n,j} = X_{\tau_{n,j}}(x_n) \in \Sigma_\delta \quad \text{and} \quad \tau_{n,j} - \tau_{n,j-1} > \max\{t_1, t_2\}$$

for all $j \geq 1$, where t_1 is given by Proposition 2.4 and t_2 is given by Lemma 2.11.

Theorem 3.1. *Given $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $x \in \Sigma_\delta$ and $y \in \Lambda$ satisfy*

(a) *there exist τ_j such that*

$$x_j = X_{\tau_j}(x) \in \Sigma_\delta \quad \text{and} \quad \tau_j - \tau_{j-1} > \max\{t_1, t_2\} \quad \text{for all } j \geq 1;$$

(b) *$\text{dist}(X_t(x), X_{h(t)}(y)) < \delta_0$, for all $t > 0$ and some $h \in S(\mathbb{R})$;*

then there exists $s \in \mathbb{R}$ such that $X_{h(s)}(y) \in W_{\varepsilon_0}^{ss}(X_{[s-\varepsilon_0, s+\varepsilon_0]}(x))$.

We postpone the proof of Theorem 3.1 until the next section and explain first why it implies Theorem A. We are going to use the following observation.

Lemma 3.2. *There exist $c > 0$ and $\rho > 0$, depending only on the fbw, such that if z_1, z_2, z_3 are points in Λ satisfying $z_3 \in X_{[-\rho, \rho]}(z_2)$ and $z_2 \in W_\rho^{ss}(z_1)$, then*

$$\text{dist}(z_1, z_3) \geq c \cdot \max\{\text{dist}(z_1, z_2), \text{dist}(z_2, z_3)\}.$$

Proof. This is a direct consequence of the fact that the angle between E^{ss} and the fbw direction is bounded from zero which, on its turn, follows from the fact that the latter is contained in the center-unstable sub-bundle E^{cu} . \square

We fix $\varepsilon_0 = \varepsilon$ as in (10) and then consider δ_0 as given by Theorem 3.1. Next, we fix n such that $\delta_n < \delta_0$ and $\delta_n < c\rho$, and apply Theorem 3.1 to $x = x_n$ and $y = y_n$ and $h = h_n$. Hypothesis (a) in the theorem corresponds to (11) and, with these choices, hypothesis (b) follows from (9). Therefore we obtain that $X_{h(s)}(y) \in W_\varepsilon^{ss}(X_{[s-\varepsilon, s+\varepsilon]}(x))$. In other words, there exists $|\tau| \leq \varepsilon$

such that $X_{h(s)}(y) \in W_\varepsilon^{ss}(X_{s+\tau}(x))$. Hypothesis (10) implies that $X_{h(s)}(y) \neq X_{s+\tau}(x)$. Hence since strong-stable manifolds are expanded under backward iteration, there exists $\theta > 0$ maximum such that

$$X_{h(s)-t}(y) \in W_\rho^{ss}(X_{s+\tau-t}(x)) \quad \text{and} \quad X_{h(s+\tau-t)}(y) \in X_{[-\rho,\rho]}(X_{h(s)-t}(y))$$

for all $0 \leq t \leq \theta$, see Figure 5.

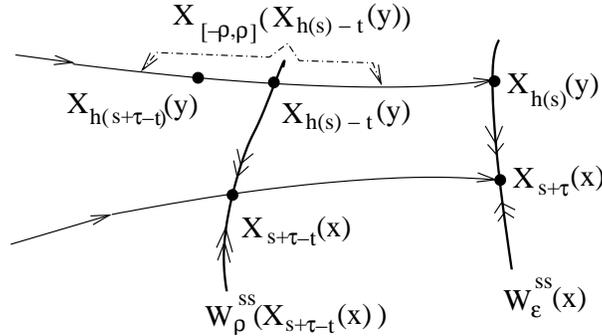


FIGURE 5. Sketch of the relative positions of the strong-stable manifolds and orbits in the argument reducing Theorem A to Theorem 3.1.

Using Lemma 3.2, we conclude that

$$\text{dist}(X_{s+\tau-t}(x), X_{h(s+\tau-t)}(y)) \geq c\rho > \delta_n$$

which contradicts (9). This contradiction reduces the proof of Theorem A to that of Theorem 3.1.

3.2. Infinitely many coupled returns. We start by outlining the proof of Theorem 3.1. There are three steps. The first one, which we carry out in the present section, is to show that to each return x_j of the orbit of x to Σ there corresponds a nearby return y_j of the orbit of y to Σ . The precise statement is in Lemma 3.3 below. The second, and most crucial step, is to show that there exists a smooth Poincaré map, with large return time, defined on the whole strip of Σ in between the stable manifolds of x_j and y_j . This is done in Section 3.3. The last step, Section 3.3.4, is to show that these Poincaré maps are uniformly hyperbolic, in particular, they expand cu -curves uniformly (recall the definition of cu -curve in Subsection 2.2). The theorem is then easily deduced: to prove that $X_{h(s)}(y)$ is in the orbit of $W_\varepsilon^{ss}(x)$ it suffices to show that $y_j \in W^s(x_j, \Sigma)$, by Remark 2.3. The latter must be true, for otherwise, by hyperbolicity of the Poincaré maps, the stable manifolds of x_j and y_j would move apart as $j \rightarrow \infty$, and this would contradict condition (b) of Theorem 3.1. See Subsection 3.3.4 for more details.

Lemma 3.3. *There exists $K > 0$ such that, in the setting of Theorem 3.1, there exists a sequence $(v_j)_{j \geq 0}$ such that*

- (1) $y_j = X_{v_j}(y)$ is in Σ for all $j \geq 0$.
- (2) $|v_j - h(\tau_j)| < K \cdot \delta_0$ and $d(x_j, y_j) < K \cdot \delta_0$.

Proof. By assumption $d(x_j, X_{h(\tau_j)}(y)) < K \cdot \delta_0$ for all $j \geq 0$. In particular $y'_j = X_{h(\tau_j)}(y)$ is close to Σ . Using a fbw box in a neighborhood of Σ we obtain $X_{\varepsilon_j}(y'_j) \in \Sigma$ for some $\varepsilon_j \in (-K \cdot \delta_0, K \cdot \delta_0)$. The constant K depends only on the vector field X and the cross-section Σ (more precisely, on the angle between Σ and the fbw direction). Taking $\upsilon_j = h(\tau_j) + \varepsilon_j$ we get the first two claims in the lemma. The third one follows from the triangle inequality; it may be necessary to replace K by a larger constant, still depending on X and Σ only. \square

3.3. Semi-global Poincaré map. Since we took the cross-section Σ to be adapted, we may use Lemma 2.11 to conclude that there exist Poincaré maps R_j with $R_j(x_j) = x_{j+1}$ and $R_j(y_j) = y_{j+1}$ and sending $W_\varepsilon^{ss}(x_j, \Sigma)$ and $W_\varepsilon^{ss}(y_j, \Sigma)$ inside $W_\varepsilon^{ss}(x_{j+1}, \Sigma)$ and $W_\varepsilon^{ss}(y_{j+1}, \Sigma)$, respectively. The goal of this section is to prove that R_j extends to a smooth Poincaré map on the whole strip Σ_j of Σ bounded by the stable manifolds of x_j and y_j .

We first outline the proof. For each j we choose a curve η_j transverse to the stable foliation of Σ , connecting x_j to y_j and such that η_j is disjoint from the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$. Using Lemma 2.11 in the same way as in the last paragraph, we see that it suffices to prove that R_j extends smoothly to η_j . For this purpose we consider a tube-like domain \mathcal{T}_j consisting of local stable manifolds through a surface S_j whose boundary is formed by η_j and η_{j+1} and the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$, see Figure 6. We will prove that the orbit of any point in η_j must leave the tube through η_{j+1} in finite time.

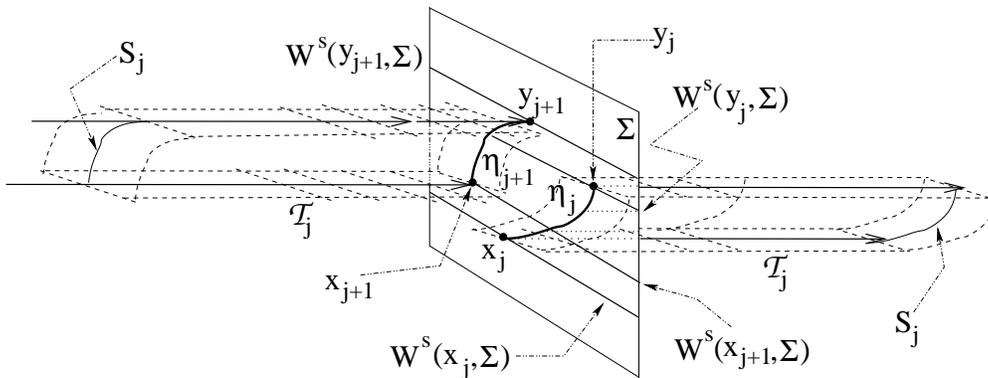


FIGURE 6. A tube-like domain.

We begin by showing that the tube contains no singularities. This uses hypothesis (b) together with the local dynamics near Lorenz-like singularities. Next, using hypothesis (b) together with a Poincaré-Bendixson argument on the surface S_j , we conclude that the forward orbit of any point in \mathcal{T}_j must leave the tube. Another argument, using hyperbolicity properties of the Poincaré map, shows that orbits through η_j must leave \mathcal{T}_j through η_{j+1} . In the sequel we detail these arguments.

3.3.1. A tube-like domain without singularities. Since we took η_j and η_{j+1} disjoint from the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$, the union of these four curves is an embedded circle. We

recall that the two orbit segments are close to each other, by hypothesis (b)

$$d(X_t(x), X_{h(t)}(y)) < \delta_0 \quad \text{for all } t \in [t_j, t_{j+1}].$$

Assuming that δ_0 is smaller than the radius of injectiveness of the ambient manifold, there exists a unique geodesic linking each $X_t(x)$ to $X_{h(t)}(y)$, and it varies continuously (even smoothly) with t . Using these geodesics we easily see that the union of $[y_j, y_{j+1}]$ with η_j and η_{j+1} is homotopic to a curve inside the orbit of x , with endpoints x_j and x_{j+1} , and so it is also homotopic to the segment $[x_j, x_{j+1}]$. This means that the previously mentioned embedded circle is homotopic to zero. It follows that there is a smooth immersion $\phi : [0, 1] \times [0, 1] \rightarrow M$ such that

- $\phi(\{0\} \times [0, 1]) = \eta_j$ and $\phi(\{1\} \times [0, 1]) = \eta_{j+1}$
- $\phi([0, 1] \times \{0\}) = [y_j, y_{j+1}]$ and $\phi([0, 1] \times \{1\}) = [x_j, x_{j+1}]$.

Moreover $S_j = \phi([0, 1] \times [0, 1])$ may be chosen such that, see Figure 7

- all the points of S_j are at distance less than δ_1 from the orbit segment $[x_j, x_{j+1}]$, for some uniform constant $\delta_1 > \delta_0$ which can be taken arbitrarily close to zero, reducing δ_0 if necessary;
- the intersection of S_j with an incoming cross-section of any singularity (Section 2.4) is transverse to the corresponding stable foliation.

Then we define \mathcal{T}_j to be the union of the local stable manifolds through the points of that disk.

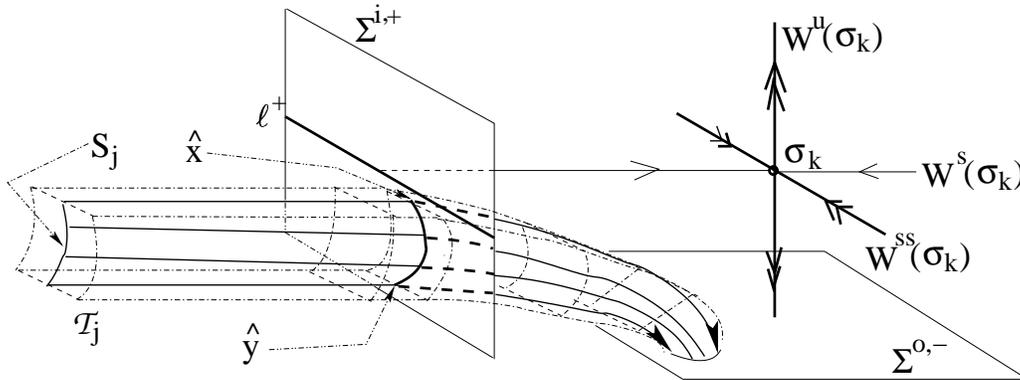


FIGURE 7. Entering the fbw box of a singularity.

Proposition 3.4. *The domain \mathcal{T}_j contains no singularities of the flow.*

Proof. By construction, every point of \mathcal{T}_j is at distance $\leq \varepsilon$ from S_j and, consequently, at distance $\leq \varepsilon + \delta_1$ from $[x_j, x_{j+1}]$. So, taking ε and δ_0 much smaller than the sizes of the cross-sections associated to the singularities (Section 2.4), we immediately get the conclusion of the proposition in the case when $[x_j, x_{j+1}]$ is disjoint from the incoming cross-sections of all singularities. In the general case we must analyze the intersections of the tube with the fbw boxes at the singularities. The key observation is in the following statement whose proof we postpone.

Lemma 3.5. *Suppose $[x_j, x_{j+1}]$ intersects an incoming cross-section Σ_k^i of some singularity σ_k at some point \hat{x} with $d(\hat{x}, \partial\Sigma_k^i) > \delta$. Then $[y_j, y_{j+1}]$ intersects Σ_k^i at some point \hat{y} with $d(\hat{x}, \hat{y}) < K \cdot \delta_0$ and, moreover \hat{x} and \hat{y} are in the same connected component of $\Sigma_k^i \setminus W_{loc}^s(\sigma_k)$.*

Let us recall that by construction the intersection of S_j with the incoming cross-section Σ_k^i is transverse to the corresponding stable foliation, see Figure 7. By the previous lemma this intersection is entirely contained in one of the connected components of $\Sigma_k^i \setminus W_{loc}^s(\sigma_k)$. Since \mathcal{T}_j consists of local stable manifolds through the points of S_j its intersection with Σ_k^i is contained in the region bounded by the stable manifolds $W^s(\hat{x}, \Sigma_k^i)$ and $W^s(\hat{y}, \Sigma_k^i)$, and so it is entirely contained in a connected component of $\Sigma_k^i \setminus W_{loc}^s(\sigma_k)$. In other words, the crossing of the tube \mathcal{T}_j through the fbw box is disjoint from $W_{loc}^s(\sigma_k)$, in particular, it does not contain the singularity. Repeating this argument for every intersection of the tube with a neighborhood of some singularity, we get the conclusion of the proposition. \square

Proof of Lemma 3.5. The first part is proved in exactly the same way as Lemma 3.3. We have

$$\hat{x} = X_{r_0}(x) \quad \text{and} \quad \hat{y} = X_{s_0}(y)$$

with $|s_0 - h(r_0)| < K\delta_0$. The proof of the second part is by contradiction and relies, fundamentally, on the local description of the dynamics near the singularity. Associated to \hat{x} and \hat{y} we have the points $\tilde{x} = X_{r_1}(x)$ and $\tilde{y} = X_{s_1}(y)$, where the two orbits leave the fbw box associated to the singularity. If \hat{x} and \hat{y} are in opposite sides of the local stable manifold of σ_k , then \tilde{x} and \tilde{y} belong to different outgoing cross-sections of σ_k . Our goal is to find some $t \in \mathbb{R}$ such that

$$\text{dist}(X_t(x), X_{h(t)}(y)) > \delta_0,$$

thus contradicting hypothesis (b).

We assume by contradiction that \hat{x}, \hat{y} are in different connected components of $\Sigma_k^{i,\pm} \setminus \ell^\pm$. There are two cases to consider. We suppose first that $h(r_1) > s_1$ and note that $s_1 \gg s_0 \approx h(r_0)$, so that $s_1 > h(r_0)$. It follows that there exists $t \in (r_0, r_1)$ such that $h(t) = s_1$ since h is non-decreasing and continuous. Then $X_t(x)$ is on one side of the fbw box of σ_k , whereas $X_{h(t)}(y)$ belongs to the outgoing cross-section at the other side of the fbw box. Thus $\text{dist}(X_t(x), X_{h(t)}(y))$ has the order of magnitude of the diameter of the fbw box, which we may assume to be much larger than δ_0 .

Now we suppose that $s_1 \geq h(r_1)$ and observe that $h(r_1) > h(r_0)$, since h is increasing. We recall also that $X_{h(r_0)}(y)$ is close to \hat{y} , near the incoming cross-section, so that the whole orbit segment from $X_{h(r_0)}(y)$ to $X_{s_1}(y)$ is contained in (a small neighborhood of) the fbw box, to one side of the local stable manifold of σ_j . The previous observation means that this orbit segment contains $X_{h(r_1)}(y)$. However $X_{r_1}(x)$ belongs to the outgoing cross-section at the opposite side of the fbw box, and so $\text{dist}(X_{r_1}(x), X_{h(r_1)}(y))$ has the order of magnitude of the diameter of the fbw box, which is much larger than δ_0 . \square

3.3.2. Every orbit leaves the tube. Our goal in this section is to show that the forward orbit of every point $z \in \mathcal{T}_j$ leaves the tube in finite time: The proof is based on a Poincaré-Bendixson argument applied to the fbw induced by X^t on the disk S_j .

We begin by defining this induced fbw. For the time being, we make the following simplifying assumption:

(H) $S_j = \phi([0, 1] \times [0, 1])$ is an embedded disk and the stable manifolds $W_\varepsilon^s(\xi)$ through the points $\xi \in S_j$ are pairwise disjoint.

This condition provides a well-defined continuous projection $\pi : \mathcal{T}_j \rightarrow S_j$ by assigning to each point $z \in \mathcal{T}_j$ the unique $\xi \in S_j$ whose local stable manifold contains z . The (not necessarily complete) fbw ψ_t induced by X_t on S_j is given by $\psi_t(\xi) = \pi(X_t(\xi))$ for the largest interval of values of t for which this is defined. It is clear, just by continuity, that given any subset E of S_j at a positive distance from ∂S_j , there exists $\varepsilon > 0$ such that $\psi_t(\xi)$ is defined for all $\xi \in E$ and $t \in [0, \varepsilon]$. In fact this remains true even if E approaches the curve η_j (since Σ is a cross-section for X_t , the fbw at η_j points inward S_j) or the X_t -orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$ on the boundary of S_j (because they are also ψ_t -orbit segments). Thus we only have to worry with the distance to the remaining boundary segment:

(U) given any subset E of S_j at positive distance from η_{j+1} , there exists $\varepsilon > 0$ such that $\psi_t(\xi)$ is defined for all $\xi \in E$ and $t \in [0, \varepsilon]$.

We observe also that for points ξ close to η_{j+1} the fbw $\psi_t(\xi)$ must intersect η_{j+1} , after which it is no longer defined.

Now we explain how to remove condition (H). In this case, the induced fbw is naturally defined on $[0, 1] \times [0, 1]$ rather than S_j , as we now explain. We recall that $\phi : [0, 1] \times [0, 1] \rightarrow M$ is an immersion. So given any $w \in [0, 1] \times [0, 1]$ there exist neighborhoods U of w and V of $\phi(w)$ such that $\phi : U \rightarrow V$ is a diffeomorphism. Moreover, just by continuity of the stable foliation, choosing V sufficiently small we may ensure that each stable manifold $W_\varepsilon^s(\xi)$, $\xi \in V$, intersects V only at the point ξ . This means that we have a well-defined projection π from $\cup_{\xi \in V} W_\varepsilon^s(\xi)$ to V associating to each point z in the domain the unique element of V whose stable manifold contains z . Then we may define $\psi_t(w)$ for small t , by

$$\psi_t(w) = \phi^{-1}(\pi(X_t(\phi(w)))).$$

As before, we extend ψ_t to a maximal domain. This defines a (partial) fbw on the square $[0, 1] \times [0, 1]$, such that both $[0, 1] \times \{i\}$, $i \in \{0, 1\}$ are trajectories. Notice also that forward trajectories of points in $\{0\} \times [0, 1]$ enter the square. Hence, the only way trajectories may exit is through $\{1\} \times [0, 1]$. So, we have the following reformulation of property (U):

(U) given any subset E of $[0, 1] \times [0, 1]$ at positive distance from $\{1\} \times [0, 1]$, there exists $\varepsilon > 0$ such that $\psi_t(w)$ is defined for all $w \in E$ and $t \in [0, \varepsilon]$.

Moreover for points w close to $\{1\} \times [0, 1]$ the fbw $\psi_t(\xi)$ must intersect $\{1\} \times [0, 1]$, after which it is no longer defined.

Proposition 3.6. *Given any point $z \in \mathcal{T}_j$ there exists $t > 0$ such that $X_t(z) \notin \mathcal{T}_j$.*

Proof. The proof is by contradiction. First, we assume condition (H). Suppose there exists $z \in \mathcal{T}_j$ whose forward orbit remains in the tube for all times. Let $\xi = \pi(z)$. Then $\psi_t(\xi)$ is defined for all $t > 0$, and so it makes sense to speak of the ω -limit set $\omega(\xi)$. The orbit $\psi_t(\xi)$ can not accumulate on η_{j+1} for otherwise it would leave S_j . Therefore $\omega(\xi)$ is a compact subset of S_j at positive distance from η_{j+1} . Using property (U) we can find a uniform constant $\varepsilon > 0$ such that $\psi_t(\eta)$ is defined for every $t \in [0, \varepsilon]$ and every $\eta \in \omega(\xi)$. Since $\omega(\xi)$ is an invariant set, we can extend ψ_t to a complete fbw on it.

In particular we may fix $\eta \in \omega(\xi)$, $\zeta \in \omega(\eta)$ and apply the arguments in the proof of the Poincaré-Bendixson Theorem. On the one hand, if we consider a cross-section S to the fbw at ζ , the forward orbits of ξ and η must intersect it on monotone sequences; on the other hand, every intersection of the orbit of η with S is accumulated by points in the orbit of ξ . This implies that ζ is in the orbit of η and, in fact, that the later is periodic.

We consider the disk $D \subset S_j$ bounded by the orbit of η . The fbw ψ_t is complete restricted to D and so we may apply Poincaré-Bendixson's Theorem (see [31]) once more, and conclude that ψ_t has some singularity ξ_0 inside D . This implies that X_t has a singularity in the local stable manifold of ξ_0 , which contradicts Proposition 3.4. This contradiction completes the proof of the proposition, under assumption (H). The general case is treated in the same way, just dealing with the fbw induced on $[0, 1] \times [0, 1]$ instead of on S_j . \square

3.3.3. *The Poincaré map is well-defined on Σ_j .* We have shown that for the induced fbw ψ_t on S_j (or, more generally, on $[0, 1] \times [0, 1]$) every orbit must eventually cross η_{j+1} (respectively, $\{1\} \times [0, 1]$). Hence there exists a continuous Poincaré map

$$r : \eta_j \rightarrow \eta_{j+1}, \quad r(\xi) = \psi_{\theta(\xi)}(\xi).$$

By compactness the Poincaré time $\theta(\cdot)$ is bounded. We are going to deduce that every forward X_t -orbit eventually leaves the tube \mathcal{T}_j through Σ_{j+1} , which proves that R_j is defined on the whole strip of Σ_j between the manifolds $W^s(x_j, \Sigma_j)$ and $W^s(y_j, \Sigma_j)$, as claimed in Subsection 3.2.

To this end, for each $\xi \in \eta_j$, let $t(\xi)$ be the smallest positive time for which $X_{t(\xi)}$ is on the boundary of \mathcal{T}_j . The crucial observation is that, in view of the construction of ψ_t , *each* $X_{t(\xi)}(\xi)$ belongs to the (global) stable manifold of $\psi_{t(\xi)}(\xi)$. We observe also that for $\xi = x_j$ we have $\psi_t(\xi) = X_t(\xi)$ and so $t(\xi) = \theta(\xi)$.

Now we take ξ close to x_j . Just by continuity, the X_t -trajectory of ξ remains close to the segment $[x_j, x_{j+1}]$. Moreover this orbit cannot leave the tube through the union of the local strong stable manifolds passing through $[x_j, x_{j+1}]$, for otherwise it would contradict the definition of ψ_t . Hence the trajectory must leave the tube through Σ_{j+1} . In other words $X_{t(\xi)}(\xi)$ is a point of Σ_{j+1} , close to x_{j+1} .

Let $\hat{\eta} \subset \eta_j$ be the largest connected subset containing x_j such that $X_{t(\xi)}(\xi) \in \Sigma_{j+1}$ for all $\xi \in \hat{\eta}$. We want to prove that $\hat{\eta} = \eta_j$ since this implies that R_j extends to the whole η_j and so, using Lemma 2.11, to the whole Σ_j .

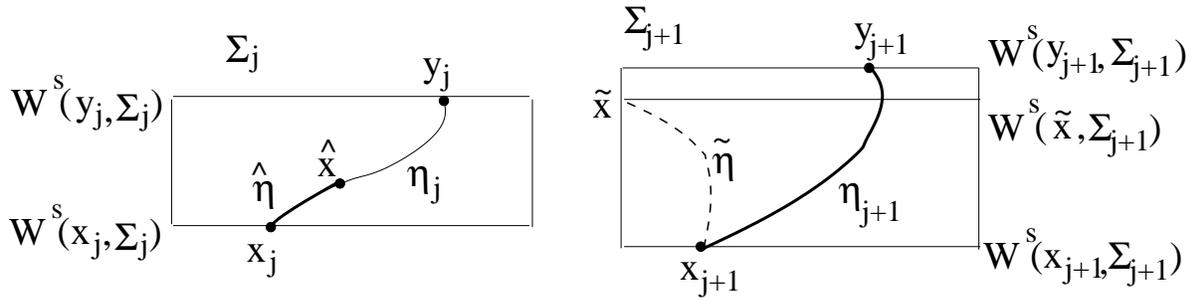
The proof is by contradiction. We assume $\hat{\eta}$ is not the whole η_j , and let \hat{x} be the endpoint different from x_j . Then by definition of \mathcal{F}_Σ^s and of ϕ_t from Subsection 3.3.2 $\tilde{x} = X_{t(\hat{x})}(\hat{x})$ is on the center-unstable boundary $\partial^{cu}\Sigma_{j+1}$ of the cross-section Σ_{j+1} , between the stable manifolds $W^s(x_{j+1}, \Sigma_{j+1})$ and $W^s(y_{j+1}, \Sigma_{j+1})$, see Figure 8.

By Corollary 2.6,

$$\tilde{\eta} = \{X_{t(\xi)}(\xi) : \xi \in \hat{\eta}\}$$

is a *cu*-curve. On the one hand, by Lemma 2.7, the distance between \tilde{x} and x_{j+1} dominates the distance between their stable manifolds and $\ell(\tilde{\eta})$

$$\ell(\tilde{\eta}) \leq \kappa \cdot d(x_{j+1}, \tilde{x}) \leq \kappa \cdot d(W^s(x_{j+1}, \Sigma_{j+1}), W^s(\tilde{x}, \Sigma_{j+1})).$$

FIGURE 8. Exiting the tube at Σ_{j+1} .

We note that $\ell(\tilde{\eta})$ is larger than δ , since x_{j+1} is in Λ and the section Σ_{j+1} is adapted. On the other hand, the distance between the two stable manifolds is smaller than the distance between the stable manifold of x_{j+1} and the stable manifold of y_{j+1} , and this is smaller than $K \cdot \delta_0$. Since δ_0 is much smaller than δ , this is a contradiction. This proves the claim that $X_{t(\xi)}(\xi) \in \Sigma$ for all $\xi \in \hat{\eta}$.

3.3.4. Conclusion of the proof of Theorem 3.1. We have shown that there exists a well defined Poincaré return map R_j on the whole strip between the stable manifolds of x_j and y_j inside Σ . By Proposition 2.4 and Corollary 2.6 we know that the map R_j is hyperbolic where defined and, moreover, that the length of each cu -curve is expanded by a factor of $3/2$ by R_j . Hence the distance between the stable manifolds $R_j(W^s(x_j, \Sigma))$ and $R_j(W^s(y_j, \Sigma))$ is increased by a factor strictly larger than one. This contradicts item (2) of Lemma 3.3 since this distance will eventually become larger than $K \cdot \delta_0$. Thus y_j must be in the stable manifold $W^s(x_j, \Sigma)$. Since the strong-stable manifold is locally fbw-invariant and $X_{h(\tau_j)}(y)$ is in the orbit of $y_j = X_{\nu_j}(y)$, then $X_{h(\tau_j)}(y) \in W^s(x_j) = W^s(X_{\tau_j}(x))$, see Lemma 3.3.

According to Lemma 3.3 we have $|\nu_j - h(\tau_j)| < K \cdot \delta_0$ and, by Remark 2.3, there exists a small $\varepsilon_1 > 0$ such that

$$R_\Sigma(y_j) = X_t(y_j) \in W_\varepsilon^{ss}(x_j) \quad \text{with} \quad |t| < \varepsilon_1.$$

Therefore the piece of orbit $\mathcal{O}_y = X_{[\nu_j - K \cdot \delta_0 - \varepsilon_1, \nu_j + K \cdot \delta_0 + \varepsilon_1]}(y)$ contains $X_{h(\tau_j)}(y)$. We note that this holds for all sufficiently small values of $\delta_0 > 0$ fixed from the beginning.

Now let $\varepsilon_0 > 0$ be given and let us consider the piece of orbit $\mathcal{O}_x = X_{[\tau_j - \varepsilon_0, \tau_j + \varepsilon_0]}(x)$ and the piece of orbit of x whose strong-stable manifolds intersect \mathcal{O}_y , i.e.

$$\mathcal{O}_{xy} = \{X_s(x) : \exists \tau \in [\nu_j - K \cdot \delta_0 - t, \nu_j + K \cdot \delta_0 + t] \text{ such that } X_\tau(y) \in W_\varepsilon^{ss}(X_s(x))\}.$$

Since $y_j \in W^s(x_j)$ we conclude that \mathcal{O}_{xy} is a neighborhood of $x_j = X_{\tau_j}(x)$ which can be made as small as we want taking δ_0 and ε_1 small enough. In particular we can ensure that $\mathcal{O}_{xy} \subset \mathcal{O}_x$ and so $X_{h(\tau_j)}(y) \in W_\varepsilon^{ss}(X_{[\tau_j - \varepsilon_0, \tau_j + \varepsilon_0]}(x))$. This finishes the proof of Theorem 3.1.

4. GLOBAL POINCARÉ MAPS AND REDUCTION TO ONE-DIMENSIONAL MAP

Here we construct a global Poincaré map for the fbw near the singular-hyperbolic attractor Λ . We then use the hyperbolicity properties of this map to reduce the dynamics to a one-dimensional piecewise expanding map through a quotient map over the stable leaves.

4.1. Cross-sections and invariant foliations. We observe first that by Lemma 2.10 we can take a δ -adapted cross-section at each non-singular point $x \in \Lambda$. We know also that near each singularity σ_k there is a fbw-box U_{σ_k} as in Subsection 2.4, see Figure 4.

Using a tubular neighborhood construction near any given adapted cross-section Σ , we linearize the fbw in an open set $U_\Sigma = X_{(-\varepsilon, \varepsilon)}(\text{int}(\Sigma))$ for a small $\varepsilon > 0$, containing the interior of the cross-section. This provides an open cover of the compact set Λ by fbw-boxes near the singularities and tubular neighborhoods around regular points.

We let $\{U_{\Sigma_i}, U_{\sigma_k} : i = 1, \dots, l; k = 1, \dots, s\}$ be a finite cover of Λ , where $s \geq 1$ is the number of singularities in Λ , and we set $t_3 > 0$ to be an upper bound for the time it takes any point $z \in U_{\Sigma_i}$ to leave this tubular neighborhood under the fbw, for any $i = 1, \dots, l$. We assume without loss that $t_2 > t_3$.

To define the Poincaré map R , for any point z in one of the cross-sections in

$$\Xi = \{\Sigma_j, \Sigma_{\sigma_k}^{i, \pm}, \Sigma_{\sigma_k}^{o, \pm} : j = 1, \dots, l; k = 1, \dots, s\},$$

we consider $\hat{z} = X_{t_2}(z)$ and wait for the next time $t(z)$ the orbit of \hat{z} hits again one of the cross-sections. Then we define $R(z) = X_{t_2+t(z)}(z)$ and say that $\tau(z) = t_2 + t(z)$ is the *Poincaré time* of z . If the point z never returns to one of the cross-sections, then the map R is not defined at z (e.g. at the lines ℓ^\pm in the fbw-boxes near a singularity). Moreover by Lemma 2.11, if R is defined for $x \in \Sigma$ on some $\Sigma \in \Xi$, then R is defined for every point in $W^s(x, \Sigma)$. Hence *the domain of $R \mid \Sigma$ consists of strips of Σ* . The smoothness of $(t, x) \mapsto X_t(x)$ ensures that the strips

$$(12) \quad \Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\}$$

have non-empty interior in Σ for every $\Sigma, \Sigma' \in \Xi$. When R maps to an outgoing strip near a singularity σ_k , there might be a boundary of the strip corresponding to the line ℓ_k^\pm of points which fall in the stable manifold of σ_k .

4.1.1. Finite number of strips. We show that fixing a cross-section $\Sigma \in \Xi$ the family of all possible strips as in (12) covers Σ except for finitely many stable leaves $W^s(x_i, \Sigma)$, $i = 1, \dots, m = m(\Sigma)$. Moreover we also show that each strip given by (12) has finitely many connected components. Thus the number of strips in each cross-section is finite.

We first recall that each $\Sigma \in \Xi$ is contained in U_0 , so $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$. Note that R is locally smooth for all points $x \in \text{int}(\Sigma)$ such that $R(x) \in \text{int}(\Xi)$ by the fbw box theorem and the smoothness of the fbw, where $\text{int}(\Xi)$ is the union of the interiors of each cross-section of Ξ . Let $\partial^s \Xi$ denote the union of all the leaves forming the stable boundary of every cross-section in Ξ .

Lemma 4.1. *The set of discontinuities of R in $\Xi \setminus \partial^s \Xi$ is contained in the set of points $x \in \Xi \setminus \partial^s \Xi$ such that:*

- (1) *either $R(x)$ is defined and belongs to $\partial \Xi$;*

(2) or there is some time $0 < t \leq t_2$ such that $X_t(x) \in W_{loc}^s(\sigma)$ for some singularity σ of Λ . Moreover this set is contained in a finite number of stable leaves of the cross-sections $\Sigma \in \Xi$.

Proof. We divide the proof into several steps.

Step 1: Cases (1) and (2) in the statement of the lemma correspond to all possible discontinuities of R in $\Xi \setminus \partial^s \Xi$.

Let x be a point in $\Sigma \setminus \partial^s \Sigma$ for some $\Sigma \in \Xi$, *not* satisfying any of the conditions in items (1) and (2). Then $R(x)$ is defined and $R(x)$ belongs to the interior of some cross-section Σ' . By the smoothness of the fbw and by the fbw box theorem we have that R is smooth in a neighborhood of x in Σ . Hence any discontinuity point for R must be in one the situations (1) or (2).

Step 2: Points satisfying item (2) are contained in finitely many stable leaves in each $\Sigma \in \Xi$.

Indeed if we set $W = X_{[-t_2, 0]}(\cup_{\sigma} W_{loc}^s(\sigma))$, where the union above is taken over all singularities σ of Λ , then W is a compact sub-manifold of M with boundary, tangent to the center-stable sub-bundle $E^s \oplus E^X$. This means that W is transversal to any cross-section of Ξ .

Hence the intersection of W with any $\Sigma \in \Xi$ is a one-dimensional sub-manifold of Σ . Thus the number of connected components of the intersection is finite in each Σ . This means that there are finitely many points $x_1, \dots, x_k \in \Sigma$ such that

$$W \cap \Sigma \subset W^s(x_1, \Sigma) \cup \dots \cup W^s(x_k, \Sigma).$$

Step 3: Points satisfying item (1) are contained in a finite number of stable leaves of each $\Sigma \in \Xi$.

We argue by contradiction. Assume that the set of points D of Σ sent by R into stable boundary points of some cross-section of Ξ is such that

$$L = \{W^s(x, \Sigma) : x \in D\}$$

has *infinitely many lines*. Note that D in fact equals L by Lemma 2.11. Then there exists an accumulation line $W^s(x_0, \Sigma)$. Since the number of cross-sections in Ξ is finite we may assume that $W^s(x_0, \Sigma)$ is accumulated by *distinct* $W^s(x_i, \Sigma)$ with $x_i \in D$ satisfying $R(x_i) \in W^s(z, \Sigma') \subset \partial^s \Sigma'$ for a fixed $\Sigma' \in \Xi$, $i \geq 1$. We may assume that x_i tends to x_0 when $i \rightarrow \infty$, that x_0 is in the interior of $W^s(x_0, \Sigma)$ and that the x_i are all distinct — in particular the points x_i do not belong to any periodic orbit of the fbw.

As a preliminary result we show that $R(x_i) = X_{s_i}(x_i)$ is such that s_i is a bounded sequence in the real line. For otherwise $s_i \rightarrow \infty$ and this means, by definition of R , that the orbit of $X_{t_2}(x_i)$ is very close to the local stable manifold of some singularity σ of Λ and that $R(x_i)$ belongs to the outgoing cross-section near this singularity: $R(x_i) \in \Sigma_{\sigma}^{o, \pm}$. Hence we must have that $X_{s_i}(x_i)$ tends to the stable manifold of σ when $i \rightarrow \infty$ and that $R(x_i)$ tends to the stable boundary of $\Sigma_{\sigma}^{o, \pm}$. Since no point in any cross-section in Ξ is sent by R into this boundary line, we get a contradiction.

Now the smoothness of the fbw and the fact that $W^s(z, \Sigma')$ is closed imply that $R(x_0) \in W^s(z, \Sigma')$ also since we have the following

$$R(x_0) = \lim_{i \rightarrow \infty} R(x_i) = \lim_{i \rightarrow \infty} X_{s_i}(x_i) = X_{s_0}(x_0) \quad \text{and} \quad \lim_{i \rightarrow \infty} s_i = s_0.$$

Moreover $R(W^s(x_0, \Sigma)) \subset W^s(z, \Sigma')$ and $R(x_0)$ is in the interior of $R(W^s(x_0, \Sigma))$, then $R(x_i) \in R(W^s(x_0, \Sigma))$ for all i big enough. This means that there exists a sequence $y_i \in W^s(x_0, \Sigma)$ and a sequence of real numbers τ_i such that $R(y_i) = R(x_i)$ for all sufficiently big integers i . By construction we have that $x_i \neq y_i$ and both belong to the same orbit. Since x_i, y_i are in the same cross-section we get that $x_i = X_{\alpha_i}(y_i)$ with $|\alpha_i| \geq t_3$ for all big i .

However we also have that $\tau_i \rightarrow s_0$ because $R(y_i) = R(x_i) \rightarrow R(x_0)$, $y_i \in W^s(x_0, \Sigma)$ and $R|_{W^s(x_0, \Sigma)}$ is smooth. Thus $|s_i - \tau_i| \rightarrow 0$. But $|s_i - \tau_i| = |\alpha_i| \geq t_3 > 0$. This is a contradiction.

This proves that D is contained in finitely many stable leaves.

Combining the three steps above we conclude the proof of the lemma. \square

Let Γ be the finite set of stable leaves of Ξ provided by Lemma 4.1 together with $\partial^s \Xi$. Then the complement $\Xi \setminus \Gamma$ of this set is formed by finitely many open strips where R is smooth. Each of these strips is then a connected component of the sets $\Sigma(\Sigma')$ for $\Sigma, \Sigma' \in \Xi$.

4.1.2. Integrability of the global Poincaré return time. We claim that *the Poincaré time τ is integrable with respect to the Lebesgue area measure on Ξ* . Indeed given $z \in \Xi$, the point $\hat{z} = X_{t_2}(z)$ either is inside a fbw-box U_{σ_k} of a singularity σ_k , or not. In the former case, the time \hat{z} takes to reach an outgoing cross-section $\Sigma_{\sigma_k}^{\pm}$ is bounded by the exit time function $\tau_{\sigma_k}^{\pm}$ of the corresponding fbw-box, which is integrable, see Subsection 2.4. In the latter case, \hat{z} takes a time of at most $2 \cdot t_3$ to reach another cross-section, by definition of t_3 . Thus the Poincaré time on Ξ is bounded by $t_2 + 2 \cdot t_3$ plus a sum of finitely many integrable functions, one for each fbw-box near a singularity, by finiteness of the number of singularities, of the number of cross-sections in Ξ and of the number of strips at each cross-section. This proves the claim.

Remark 4.2. By Lemma 2.11 the Poincaré time τ is constant on stable leaves $W^s(x, \Sigma)$ for all $x \in \Sigma \in \Xi$. Thus there exists a return time function τ_I on I such that $\tau = \tau_I \circ p$, where $p : \Xi \rightarrow \gamma_{\Xi}$ is the joining of all p_{Σ} , $\Sigma \in \Xi$ and $\gamma_{\Xi} = \{\gamma_{\Sigma} : \Sigma \in \Xi\}$. The integrability of τ with respect to λ^2 (see Subsection 4.1) implies the λ -integrability of τ_I naturally since $(p_{\Sigma})_* \lambda^2 \ll \lambda$ and $\tau_I \circ p = \tau$.

Remark 4.3. Given $z \in \Sigma \in \Xi$ we write $\tau^k(z) = \tau(R^{k-1}(z)) + \dots + \tau(z)$ for $k \geq 1$ and so $\tau = \tau^1$. Since

$$R^k(W^s(z, \Sigma)) \subset X_{\tau^k(z)}(W^s(z, \Sigma)) \subset X_{\tau^k(z)}(U),$$

the length $\ell\left(R^k(W^s(z, \Sigma))\right)$ is uniformly contracted and $\tau^k(z) \rightarrow +\infty$ when $k \rightarrow +\infty$, we get that $R^k(W^s(z, \Sigma)) \subset \Sigma'$ for some $\Sigma' \in \Xi$ and

$$d\left(R^k(W^s(z, \Sigma)), \partial^{cu} \Sigma'\right) > \delta/2$$

for all big enough k , by the definition of U and of δ -adapted cross-section. (The distance $d(A, B)$ between two sets A, B means $\inf\{d(a, b) : a \in A, b \in B\}$.) We may assume that this property holds for all stable leaves $W^s(z, \Sigma)$, all $z \in \Sigma$ and every $\Sigma \in \Xi$ for all $k \geq k_0$, for some fixed big $k_0 \in \mathbb{N}$, by the uniform contraction property of R in the stable direction.

4.2. The first return map. We can also consider the Poincaré map given by the *first return map* $R_0 : \Xi \rightarrow \Xi$ defined simply as $R_0(z) = X_{T(z)}(z)$, where

$$T(z) = \inf\{t > 0 : X_t(z) \in \Xi\}$$

is the time the X -orbit of $z \in \Xi$ takes to arrive again at Ξ . This map R_0 is not defined on those points z which do not return and, moreover, R_0 might not satisfy the lemmas of Subsection 2.2, since we do not know whether the flow from z to $R_0(z)$ has enough time to gain expansion. However the stable manifolds are still well defined. By the definitions of R_0 and of R we see that R is induced by R_0 , i.e. if R is defined for $z \in \Xi$, then there exists an integer $r(x)$ such that

$$R(z) = R_0^{r(z)}(z).$$

We note that since the number of cross-sections in Ξ is finite and the Poincaré times τ for the points in the domain of R are also finite, then the function $r : \Xi \rightarrow \mathbb{N}$ is bounded: there exists $r_0 \in \mathbb{N}$ such that $r(x) \leq r_0$ for all $x \in \Xi$. Moreover $T \leq \tau$ thus T is Lebesgue integrable also.

Remark 4.4. The map $R_0 : \Sigma \cap R_0^{-1}(\Sigma') \rightarrow \Sigma'$, when defined (that is if $R_0^{-1}(\Sigma') \cap \Sigma \neq \emptyset$), can always be smoothly extended to the boundary of its domain on Σ , if the image cross-section is not an outgoing cross-section near a singularity. In fact, if $\Sigma' = \Sigma_\sigma^{o,\pm}$, then $\Sigma = \Sigma_\sigma^{i,\pm}$ and in this case there is no continuous extension of $R_0 | \Sigma_\sigma^{i,\pm}$ to the line ℓ_σ^\pm , for any singularity σ of X .

Otherwise we can always extend the first hitting map R_0 between Σ and Σ' until we reach the boundaries of Σ or of Σ' . Now we just have to recall that the cross-sections are C^2 embedded compact disks transversal to the C^1 flow X at every point.

In particular, the fact that $R(x)$ is a composition of R_0 with itself a finite number of times, dependent on the point x , together with Corollary 2.6 shows that whenever $R(x)$ is defined, we have $\|DR_0(x)v\| > c \cdot \|v\|$ for some $c = c(x) > 0$ for every $0 \neq v \in C_p^u(x)$.

By Remark 4.4 we see that for the first hitting map $R_0 : \Sigma \cap R_0^{-1}(\Sigma') \rightarrow \Sigma'$ between cross-sections of Ξ where Σ' is not an outgoing cross-section near any singularity, there is a positive lower bound $c(\Sigma, \Sigma') > 0$ for c restricted to $\Sigma \cap R_0^{-1}(\Sigma')$. In addition, for $R_0 = R | \Sigma_\sigma^{i,\pm} : \Sigma_\sigma^{i,\pm} \rightarrow \Sigma_\sigma^{o,\pm}$ we have that c tends to infinity near the singularity by the expression (8) of the map in linearizing coordinates, thus there also exists a positive lower bound for c on $\Sigma_\sigma^{i,\pm}$. Since the number of singularities of X in Λ and the number of cross-sections in Ξ are finite, we conclude that there exists c_0 such that

$$(13) \quad \frac{\|DR_0(x)v\|}{\|v\|} \geq c_0 \quad \text{for all } v \in C_\Sigma^u(x) \setminus \{0\}$$

and for every $x \in \Xi$ for which R_0 is defined.

Remark 4.5. After Remark 4.3 we have that $R^k(W^s(x, \Sigma))$ is well inside any cross section for $k \geq k_0$. Hence the orbits $X_{[0, \tau(x)]}(x), X_{[0, \tau(x)]}(y)$ of any two distinct points $x, y \in R^k(W^s(x, \Sigma))$ between consecutive images of R_0 must cut the same cross sections of Ξ . For otherwise some of the orbits must cut a cross-section at a distance smaller than $\delta/2$ from the center-stable boundary, because the distance between the orbits is bounded a constant factor of $d(x, y) \leq \ell(R^k(W^s(x, \Sigma)))$ due to the uniform contractive character of the stable foliation.

This means that $r \mid R^k(W^s(x, \Sigma))$ is constant for all $k \geq k_0$. In other words $R \mid R^k(W^s(x, \Sigma)) \equiv R_0^r$ for some constant r .

4.3. Reduction to the quotient leaf space. From now on we assume that the fbw $(X_t)_{t \in \mathbb{R}}$ is C^2 . Under this condition it is well known [32] that the stable leaf $W^s(x, \Sigma)$ for every $x \in \Sigma \in \Xi$ is a C^2 embedded disk and these leaves define a $C^{1+\alpha}$ foliation \mathcal{F}_Σ^s , $\alpha \in (0, 1)$, of each $\Sigma \in \Xi$.

We choose once and for all a C^2 curve γ_Σ transversal to \mathcal{F}_Σ^s in each $\Sigma \in \Xi$. Then the projection p_Σ along leaves of \mathcal{F}_Σ^s onto γ_Σ is a $C^{1+\alpha}$ map. We set

$$I = \bigcup_{\Sigma, \Sigma' \in \Xi} \text{int}(\Sigma(\Sigma')) \cap \gamma_\Sigma$$

and observe that by the properties of $\Sigma(\Sigma')$ obtained in Subsection 4.1 the set I is diffeomorphic to a finite union of non-degenerate open intervals I_1, \dots, I_m by a C^2 diffeomorphism and $p_\Sigma \mid p_\Sigma^{-1}(I)$ becomes a $C^{1+\alpha}$ submersion. In particular, denoting the Lebesgue area measure over Ξ by λ^2 and the Lebesgue length measure on I by λ , we have $(p_\Sigma)_* \lambda^2 \ll \lambda$.

According to Lemma 2.11, Proposition 2.4 and Corollary 2.6 the Poincaré map $R : \Xi \rightarrow \Xi$ takes stable leaves of \mathcal{F}_Σ^s inside stable leaves of the same foliation and is hyperbolic. Hence the map

$$f : I \rightarrow I \quad \text{given by} \quad I \ni z \mapsto p_{\Sigma'} \left(R(W^s(z, \Sigma) \cap \Sigma(\Sigma')) \right)$$

for $\Sigma, \Sigma' \in \Xi$ is a $C^{1+\alpha}$ map and $|f'| \geq 3/2$ when restricted to I_i , $i = 1, \dots, m$. Thus f is expanding. Moreover $|f'|^{-1} \mid I_j$ is a α -Hölder function since for all $x, y \in I_j$ we have

$$\frac{1}{|f'(x)|} - \frac{1}{|f'(y)|} \leq \frac{|f'(x) - f'(y)|}{|f'(x)f'(y)|} \leq \frac{C}{(3/2)^2} \cdot |x - y|^\alpha, \quad \text{for some } 0 < \alpha < 1.$$

Thus $f : I \rightarrow I$ is a $C^{1+\alpha}$ *piecewise expanding map*.

It is well known [39, 42, 16] that C^1 piecewise expanding maps f of the interval such that $1/|f'|$ is of bounded variation, have finitely many absolutely continuous invariant probability measures whose basins cover Lebesgue almost all points of I .

Using a generalization of the notion of bounded variation (defined below) it was shown in [17] that the results of existence and finiteness of absolutely continuous ergodic invariant measures can be extended to C^1 piecewise expanding maps f such that $1/|f'|$ is α -Hölder for some $0 < \alpha < 1$. Moreover the densities φ of the absolutely continuous invariant measures are functions of *universally bounded $1/\alpha$ -variation*, that is

$$\sup_{a=a_0 < a_1 < \dots < a_n = b} \left(\sum_{j=1}^n |\varphi(a_j) - \varphi(a_{j-1})|^{1/\alpha} \right)^\alpha < \infty,$$

where the supremum is taken over all finite partition of the interval $I = [a, b]$. In particular, the densities φ have at most countably many discontinuity points, which implies that the support of φ contains nonempty open intervals and φ is bounded from above.

In what follows we show how to use this to build physical measures for the fbw.

5. SUSPENDING INVARIANT MEASURES

Here we show how to construct an invariant measure for a transformation from an invariant measure for the quotient map obtained from a partition of the space. We show also that if the measure is ergodic on the quotient, then we also obtain ergodicity on the starting space.

Later we extend the transformation to a semi-fbw through a suspension construction and show that each invariant and ergodic measure for the transformation corresponds to a unique measure for the semi-fbw with the same properties.

5.1. Reduction to the quotient map. Let Ξ be a compact metric space, $\Gamma \subset \Xi$ and $F : (\Xi \setminus \Gamma) \rightarrow \Xi$ be a measurable map. We assume that there exists a partition \mathcal{F} of Ξ into measurable subsets, having Γ as an element, which is

- *invariant*: the image of any $\xi \in \mathcal{F}$ distinct from Γ is contained in some element η of \mathcal{F} ;
- *contracting*: the diameter of $F^n(\xi)$ goes to zero when $n \rightarrow \infty$, uniformly over all the $\xi \in \mathcal{F}$ for which $F^n(\xi)$ is defined.

We denote $p : \Xi \rightarrow \mathcal{F}$ the canonical projection, i.e. p assigns to each point $x \in \Xi$ the atom $\xi \in \mathcal{F}$ that contains it. By definition, $A \subset \mathcal{F}$ is measurable if and only if $p^{-1}(A)$ is a measurable subset of Ξ and likewise A is open if, and only if, $p^{-1}(A)$ is open in Ξ . The invariance condition means that there is a uniquely defined map

$$f : (\mathcal{F} \setminus \{\Gamma\}) \rightarrow \mathcal{F} \quad \text{such that} \quad f \circ p = p \circ F.$$

Clearly, f is measurable with respect to the measurable structure we introduced in \mathcal{F} . We assume from now on that the leaves are sufficiently regular so that Ξ/\mathcal{F} is a metric space with the topology induced by p .

Let μ_f be any probability measure on \mathcal{F} invariant under the transformation f . For any bounded function $\psi : \Xi \rightarrow \mathbb{R}$, let $\psi_- : \mathcal{F} \rightarrow \mathbb{R}$ and $\psi_+ : \mathcal{F} \rightarrow \mathbb{R}$ be defined by

$$\psi_-(\xi) = \inf_{x \in \xi} \psi(x) \quad \text{and} \quad \psi_+(\xi) = \sup_{x \in \xi} \psi(x).$$

Lemma 5.1. *Given any continuous function $\psi : \Xi \rightarrow \mathbb{R}$, both limits*

$$(14) \quad \lim_n \int (\psi \circ F^n)_- d\mu_f \quad \text{and} \quad \lim_n \int (\psi \circ F^n)_+ d\mu_f$$

exist, and they coincide.

Proof. Let ψ be fixed as in the statement. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|\psi(x_1) - \psi(x_2)| \leq \varepsilon$ for all x_1, x_2 with $d(x_1, x_2) \leq \delta$. Since the partition \mathcal{F} is assumed to be contractive, there exists $n_0 \geq 0$ such that $\text{diam}(F^n(\xi)) \leq \delta$ for every $\xi \in \mathcal{F}$ and any $n \geq n_0$. Let $n+k \geq n \geq n_0$. By definition,

$$(\psi \circ F^{n+p})_-(\xi) - (\psi \circ F^n)_-(f^p(\xi)) = \inf(\psi | F^{n+p}(\xi)) - \inf(\psi | F^n(f^p(\xi))).$$

Observe that $F^{n+p}(\xi) \subset F^n(f^p(\xi))$. So the difference on the right hand side is bounded by

$$\sup(\psi | F^n(f^k(\xi))) - \inf(\psi | F^n(f^k(\xi))) \leq \varepsilon.$$

Therefore

$$\left| \int (\psi \circ F^{n+k})_- d\mu_f - \int (\psi \circ F^n)_- \circ f^k d\mu_f \right| \leq \varepsilon.$$

Moreover, one may replace the second integral by $\int (\psi \circ F^n)_- d\mu_f$, because μ_f is f -invariant.

At this point we have shown that $\{\int (\psi \circ F^n)_- d\mu_f\}_{n \geq 1}$ is a Cauchy sequence in \mathbb{R} . In particular, it converges. The same argument proves that $\{\int (\psi \circ F^n)_+ d\mu_f\}_{n \geq 1}$ is also convergent. Moreover, keeping the previous notations,

$$0 \leq (\psi \circ F^n)_+(\xi) - (\psi \circ F^n)_-(\xi) = \sup(\psi | F^n(\xi)) - \inf(\psi | F^n(\xi)) \leq \varepsilon$$

for every $n \geq n_0$. So the two sequences in (14) must have the same limit. The lemma is proved. \square

Corollary 5.2. *There exists a unique probability measure μ_F on Ξ such that*

$$\int \psi d\mu_F = \lim \int (\psi \circ F^n)_- d\mu_f = \lim \int (\psi \circ F^n)_+ d\mu_f.$$

for every continuous function $\psi : \Xi \rightarrow \mathbb{R}$. Besides, μ_F is invariant under F . Moreover the correspondence $\mu_f \mapsto \mu_F$ is injective.

Proof. Let $\hat{\mu}(\psi)$ denote the value of the two limits. Using the expression for $\hat{\mu}(\psi)$ in terms of $(\psi \circ F^n)_-$ we immediately get that

$$\hat{\mu}(\psi_1 + \psi_2) \geq \hat{\mu}(\psi_1) + \hat{\mu}(\psi_2).$$

Analogously, the expression of $\hat{\mu}(\psi)$ in terms of $(\psi \circ F^n)_+$ gives the opposite inequality. So, the function $\hat{\mu}(\cdot)$ is additive. Moreover, $\hat{\mu}(c\psi) = c\hat{\mu}(\psi)$ for every $c \in \mathbb{R}$ and every continuous function ψ . Therefore, $\hat{\mu}(\cdot)$ is a linear real operator in the space of continuous functions $\psi : \Xi \rightarrow \mathbb{R}$.

Clearly, $\hat{\mu}(1) = 1$ and the operator $\hat{\mu}$ is non-negative: $\hat{\mu}(\psi) \geq 0$ if $\psi \geq 0$. By the Riesz-Markov theorem, there exists a unique measure μ_F on Ξ such that $\hat{\mu}(\psi) = \int \psi d\mu_F$ for every continuous ψ . To conclude that μ_F is invariant under F it suffices to note that

$$\hat{\mu}(\psi \circ F) = \lim_n \int (\psi \circ F^{n+1})_- d\mu_f = \hat{\mu}(\psi)$$

for every ψ .

To prove that the map $\mu_f \mapsto \mu_F$ is injective, we note that if $\mu_F = \mu'_F$ are obtained from μ_f and μ'_f respectively, then for any continuous function $\varphi : \mathcal{F} \rightarrow \mathbb{R}$ we have that $\psi = \varphi \circ p : \Xi \rightarrow \mathbb{R}$ is continuous. But

$$\mu_f((\psi \circ f^n)_\pm) = \mu_f((\varphi \circ p \circ F^n)_\pm) = \mu_f((\varphi \circ f^n \circ p)_\pm) = \mu_f(\varphi \circ f^n) = \mu_f(\varphi)$$

for all $n \geq 1$ by the f -invariance of μ_f . Hence by definition

$$\mu_f(\varphi) = \mu_F(\psi) = \mu'_F(\psi) = \mu'_f(\varphi)$$

and so $\mu_f = \mu'_f$. This finishes the proof of the corollary. \square

Remark 5.3. We note that $\int \psi d\mu_F = \lim_n \int (\psi \circ F^n)_\# d\mu_F$ for every continuous $\psi : \Xi \rightarrow \mathbb{R}$ and any choice of a sequence $(\psi \circ F^n)_\# : \mathcal{F} \rightarrow \mathbb{R}$ with

$$\inf(\psi | F^n(\xi)) \leq (\psi \circ F^n)_\#(\xi) \leq \sup(\psi | F^n(\xi)).$$

Moreover we can define $\int \psi d\mu_F$ for any measurable $\psi : \Xi \rightarrow \mathbb{R}$ such that

$$\lim_{n \rightarrow +\infty} (\sup(\psi | F^n(\xi)) - \inf(\psi | F^n(\xi))) = 0$$

uniformly in $n \in \mathbb{N}$ and in $\xi \in \mathcal{F}$. This will be useful in Subsection 6.2.

Lemma 5.4. *Let $\psi : \Xi \rightarrow \mathbb{R}$ be a continuous function and $\xi \in \mathcal{F}$ be such that*

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)_-(f^j(\xi)) = \int (\psi \circ F^k)_- d\mu_f$$

for every $k \geq 1$. Then $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \psi(F^j(x)) = \int \psi d\mu_F$ for every $x \in \xi$.

Proof. Let us fix ψ and ξ as in the statement. Then by definition of $(\psi \circ F^k)_\pm$ and by the properties of \mathcal{F} we have

$$(\psi \circ F^k)_-(f^j(\xi)) \leq (\psi \circ F^k)(F^j(x)) \leq (\psi \circ F^k)_+(f^j(\xi))$$

for all $x \in \xi$ and $j, k \geq 1$. Given $\varepsilon > 0$, by Corollary 5.2 there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\mu_F(\psi) - \frac{\varepsilon}{2} \leq \mu_f((\psi \circ F^k)_-) \leq \mu_f((\psi \circ F^k)_+) \leq \mu_F(\psi) + \frac{\varepsilon}{2}$$

and there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 = n_0(k)$

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)_-(f^j(\xi)) - \mu_f((\psi \circ F^k)_-) \right| < \frac{\varepsilon}{2}.$$

Hence we have that for all $n \geq n_0(k)$

$$\begin{aligned} \mu_F(\psi) - \varepsilon &\leq \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)(F^j(x)) \\ &= \frac{n+k}{n} \cdot \frac{1}{n+k} \sum_{j=0}^{n+k-1} (\psi \circ F^j)(x) - \frac{1}{n} \sum_{i=0}^{k-1} (\psi \circ F^j)(x) \leq \mu_F(\psi) + \varepsilon. \end{aligned}$$

Since n can be made arbitrarily big and $\varepsilon > 0$ can be taken as small as we want, we have concluded the proof of the lemma. \square

Corollary 5.5. *If μ_f is f -ergodic, then μ_F is ergodic for F .*

Proof. Since Ξ/\mathcal{F} is a metric space with the topology induced by p we have that $C^0(\mathcal{F}, \mathbb{R})$ is dense in $L^1(\mathcal{F}, \mathbb{R})$ for the L^1 -topology and $p : \Xi \rightarrow \mathcal{F}$ is continuous. Hence there exists a subset \mathcal{E} of \mathcal{F} with $\mu_f(\mathcal{E}) = 1$ such that the conclusion of Lemma 5.4 holds for a subset $E = p^{-1}(\mathcal{E})$ of Ξ . To prove the corollary it is enough to show that $\mu_F(E) = 1$.

Let $\varphi = \chi_E = \chi_{\mathcal{E}} \circ p$ and take $\psi_n : \mathcal{F} \rightarrow \mathbb{R}$ a sequence of continuous functions such that $\psi_n \rightarrow \chi_{\mathcal{E}}$ when $n \rightarrow +\infty$ in the L^1 topology with respect to μ_f . Then $\varphi_n = \psi_n \circ p$ is a sequence of continuous functions on Ξ such that $\psi_n \rightarrow \psi$ when $n \rightarrow +\infty$ in the L^1 norm with respect to μ_F .

Then it is straightforward to check that

$$\mu_F(\Psi_n) = \lim_{k \rightarrow +\infty} \mu_f\left((\Psi_n \circ F^k)_-\right) = \lim_{k \rightarrow +\infty} \mu_f(\varphi_n \circ f^k) = \mu_f(\varphi_n)$$

which converges to $\mu_f(\mathcal{E}) = 1$. Since $\mu_F(\Psi_n)$ tends to $\mu_F(E)$ when $n \rightarrow +\infty$, we conclude that $\mu_F(E) = 1$, as we wanted. \square

5.2. Suspension flow from the Poincaré map. Let Ξ be a measurable space, Γ be some measurable subset of Ξ , and $F : (\Xi \setminus \Gamma) \rightarrow \Xi$ be a measurable map. Let $\tau : \Xi \rightarrow (0, +\infty]$ be a measurable function such that $\inf \tau > 0$ and $\tau \equiv +\infty$ on Γ .

Let \sim be the equivalence relation on $\Xi \times [0, +\infty)$ generated by $(x, \tau(x)) \sim (F(x), 0)$, that is, $(x, s) \sim (\tilde{x}, \tilde{s})$ if and only if there exist

$$(x, s) = (x_0, s_0), (x_1, s_1), \dots, (x_N, s_N) = (\tilde{x}, \tilde{s})$$

in $\Xi \times (0, +\infty)$ such that, for every $1 \leq i \leq N$

$$\begin{aligned} \text{either } & x_i = F(x_{i-1}) \quad \text{and} \quad s_i = s_{i-1} - \tau(x_{i-1}); \\ \text{or } & x_{i-1} = F(x_i) \quad \text{and} \quad s_{i-1} = s_i - \tau(x_i). \end{aligned}$$

We denote by $V = \Xi \times [0, +\infty) / \sim$ the corresponding quotient space and by $\pi : \Xi \rightarrow V$ the canonical projection which induces on V a topology and a Borel σ -algebra of measurable subsets of V .

Definition 5.6. The *suspension of F with return-time τ* is the semi-fbw $(X^t)_{t \geq 0}$ defined on V by

$$X^t(\pi(x, s)) = \pi(x, s + t) \quad \text{for every } (x, s) \in \Xi \times [0, +\infty) \text{ and } t > 0.$$

It is easy to see that this is indeed well defined.

Remark 5.7. If F is injective then we can also define

$$X^{-t}(\pi(x, s)) = \pi(F^{-n}(x), s + \tau(F^{-n}(x)) + \dots + \tau(F^{-1}(x)) - t)$$

for every $x \in F^n(\Xi)$ and $0 < t \leq s + \tau(F^{-n}(x)) + \dots + \tau(F^{-1}(x))$. The expression on the right does not depend on the choice of $n \geq 1$. In particular, the restriction of the semi-fbw $(X_t)_{t \geq 0}$ to the maximal invariant set

$$\Lambda = \left\{ (x, t) : x \in \bigcap_{n \geq 0} F^n(\Xi) \text{ and } t \geq 0 \right\}$$

extends, in this way, to a fbw $(X^t)_{t \in \mathbb{R}}$ on Λ .

Let μ_F be any probability measure on Ξ that is invariant under F . Then the product $\mu_F \times dt$ of μ_F by Lebesgue measure on $[0, +\infty)$ is an infinite measure, invariant under the trivial fbw

$(x, s) \mapsto (x, s + t)$ in $\Xi \times [0, +\infty)$. In what follows we assume that the return time is integrable with respect to μ_F , i.e.

$$(15) \quad \mu_F(\tau) = \int \tau d\mu_F < \infty.$$

In particular $\mu_F(\Gamma) = 0$. Then we introduce the probability measure μ_X on V defined by

$$\int \varphi d\mu_X = \frac{1}{\mu_F(\tau)} \int \int_0^{\tau(x)} \varphi(\pi(x, t)) dt d\mu_F(x)$$

for each bounded measurable $\varphi : V \rightarrow \mathbb{R}$.

We observe that the correspondence $\mu_F \mapsto \mu_X$ defined above is injective. Indeed for any bounded measurable $\psi : \Xi \rightarrow \mathbb{R}$, defining φ on $\{x\} \times [0, \tau(x))$ to equal $\psi(x)$ gives a bounded measurable map $\varphi : V \rightarrow \mathbb{R}$ such that $\mu_X(\varphi) = \mu_F(\psi)$. Hence if $\mu_X = \mu'_X$ then $\mu_F = \mu'_F$.

Lemma 5.8. *The measure μ_X is invariant under the semi-flow $(X^t)_{t \geq 0}$.*

Proof. It is enough to show that $\mu_X((X^t)^{-1}(B)) = \mu_X(B)$ for every measurable set $B \subset V$ and any $0 < t < \inf \tau$. Moreover, we may suppose that B is of the form $B = \pi(A \times J)$ for some $A \subset \Xi$ and J a bounded interval in $[0, \inf(\tau | A))$. This is because these sets form a basis for the σ -algebra of measurable subsets of V .

Let B be of this form and (x, s) be any point in Ξ with $0 \leq s < \tau(x)$. Then $X^t(x, s) \in B$ if and only if $\pi(x, s + t) = \pi(\tilde{x}, \tilde{s})$ for some $(\tilde{x}, \tilde{s}) \in A \times J$. In other words, $(x, s) \in (X^t)^{-1}(B)$ if and only if there exists some $n \geq 0$ such that

$$\tilde{x} = F^n(x) \quad \text{and} \quad \tilde{s} = s + t - \tau(x) - \dots - \tau(F^{n-1}(x)).$$

Since $s < \tau(x)$, $t < \inf \tau$, and $\tilde{s} \geq 0$, it is impossible to have $n \geq 2$. So,

- either $\tilde{x} = x$ and $\tilde{s} = s + t$ (corresponding to $n = 0$),
- or $\tilde{x} = F(x)$ and $\tilde{s} = s + t - \tau(x)$ (corresponding to $n = 1$)

The two possibilities are mutually exclusive: for the first one (x, s) must be such that $s + t < \tau(x)$, whereas in the second case $s + t \geq \tau(x)$. This shows that we can write $(X^t)^{-1}(B)$ as a disjoint union $(X^t)^{-1}(B) = B_1 \cup B_2$, with

$$B_1 = \pi\{(x, s) : x \in A \text{ and } s \in (J - t) \cap [0, \tau(x))\}$$

$$B_2 = \pi\{(x, s) : F(x) \in A \text{ and } s \in (J + \tau(x) - t) \cap [0, \tau(x))\}.$$

Since $t > 0$ and $\sup J < \tau(x)$, we have $(J - t) \cap [0, \tau(x)) = (J - t) \cap [0, +\infty)$ for every $x \in A$. So, by definition, $\mu_X(B_1)$ equals

$$\int_A \ell\left((J - t) \cap [0, \tau(x))\right) d\mu_F(x) = \mu_F(A) \cdot \ell\left((J - t) \cap [0, +\infty)\right).$$

Similarly $\inf J \geq 0$ and $t < \tau(x)$ imply that

$$(J + \tau(x) - t) \cap [0, \tau(x)) = \tau(x) + (J - t) \cap (-\infty, 0).$$

Hence $\mu_X(B_2)$ is given by

$$\int_{F^{-1}(A)} \ell\left((J - t) \cap (-\infty, 0)\right) d\mu_F(x) = \mu_F(F^{-1}(A)) \cdot \ell\left((J - t) \cap (-\infty, 0)\right).$$

Since μ_F is invariant under F , we may replace $\mu_F(F^{-1}(A))$ by $\mu_F(A)$ in the last expression. It follows that

$$\mu_X((X^t)^{-1}(B)) = \mu_X(B_1) + \mu_X(B_2) = \mu_F(A) \cdot \ell((J-t)).$$

Clearly, the last term may be written as $\mu_F(A) \cdot \ell(J)$ which, by definition, is the same as $\mu_X(B)$. This proves that μ_X is invariant under the semi-fbw and ends the proof. \square

Given a bounded measurable function $\varphi : V \rightarrow \mathbb{R}$, let $\hat{\varphi} : \Xi \rightarrow \mathbb{R}$ be defined by

$$(16) \quad \hat{\varphi}(x) = \int_0^{\tau(x)} \varphi(\pi(x,t)) dt.$$

Observe that $\hat{\varphi}$ is integrable with respect to μ_F and by the definition of μ_X

$$\int \hat{\varphi} d\mu_F = \mu_F(\tau) \cdot \int \varphi d\mu_X.$$

Lemma 5.9. *Let $\varphi : V \rightarrow \mathbb{R}$ be a bounded function, and $\hat{\varphi}$ be as above. We assume that $x \in \Xi$ is such that $\tau(F^j(x))$ and $\hat{\varphi}(F^j(x))$ are finite for every $j \geq 0$, and also*

$$(a) \quad \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \tau(F^j(x)) = \int \tau d\mu_F, \text{ and}$$

$$(b) \quad \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \hat{\varphi}(F^j(x)) = \int \hat{\varphi} d\mu_F.$$

Then $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \varphi(\pi(x,s+t)) dt = \int \varphi d\mu_X$ for every $\pi(x,s) \in V$.

Proof. Let x be fixed, satisfying (a) and (b). Given any $T > 0$ we define $n = n(T)$ by

$$T_{n-1} \leq T < T_n \quad \text{where} \quad T_j = \tau(x) + \dots + \tau(F^j(x)) \text{ for } j \geq 0$$

Then using $(y, \tau(y)) \sim (F(y), 0)$ we get

$$(17) \quad \begin{aligned} \frac{1}{T} \int_0^T \varphi(\pi(x,s+t)) dt &= \frac{1}{T} \left[\sum_{j=0}^{n-1} \int_0^{\tau(F^j(x))} \varphi(\pi(F^j(x),t)) dt \right. \\ &\quad \left. + \int_0^{T-T_{n-1}} \varphi(\pi(F^n(x),t)) dt - \int_0^s \varphi(\pi(x,t)) dt \right]. \end{aligned}$$

Using the definition of $\hat{\varphi}$, we may rewrite the first term on the right hand side as

$$(18) \quad \frac{n}{T} \cdot \frac{1}{n} \sum_{j=0}^{n-1} \hat{\varphi}(F^j(x)).$$

Now we fix $\varepsilon > 0$. Assumption (a) and the definition of n imply that,

$$n \cdot \left(\int \tau d\mu_F - \varepsilon \right) \leq T_{n-1} \leq T \leq T_n \leq (n+1) \cdot \left(\int \tau d\mu_F + \varepsilon \right),$$

for every large enough n . Observe also that n goes to infinity as $T \rightarrow +\infty$, since $\tau(F^j(x)) < \infty$ for every j . So, for every large T ,

$$\mu_F(\tau) - \varepsilon \leq \frac{T}{n} \leq \frac{n+1}{n} \mu_F(\tau) + \varepsilon \leq \mu_F(\tau) + 2\varepsilon.$$

This proves that T/n converges to $\mu_F(\tau)$ when $T \rightarrow +\infty$. Consequently, assumption (b) implies that (18) converges to

$$\frac{1}{\mu_F(\tau)} \int \hat{\varphi} d\mu_F = \int \varphi d\mu_X.$$

Now we prove that the remaining terms in (17) converge to zero when T goes to infinity. Since φ is bounded

$$(19) \quad \left| \frac{1}{T} \int_0^{T-T_{n-1}} \varphi(\pi(F^n(x), t)) dt \right| \leq \frac{T - T_{n-1}}{T} \sup |\varphi|.$$

Using the definition of n once more,

$$T - T_{n-1} \leq T_n - T_{n-1} \leq (n+1) \left(\int \tau d\mu_F + \varepsilon \right) - n \left(\int \tau d\mu_F - \varepsilon \right)$$

whenever n is large enough. Then

$$\frac{T - T_{n-1}}{T} \leq \frac{\int \tau d\mu_F + (2n+1)\varepsilon}{n(\int \tau d\mu_F - \varepsilon)} \leq \frac{4\varepsilon}{\int \tau d\mu_F - \varepsilon}$$

for all large enough T . This proves that $(T - T_{n-1})/T$ converges to zero, and then so does (19). Finally, it is clear that

$$\frac{1}{T} \int_0^s \varphi(\pi(x, t)) dt \rightarrow 0 \quad \text{when } T \rightarrow +\infty.$$

This completes the proof of the lemma. \square

Corollary 5.10. *If μ_F is ergodic then μ_X is ergodic.*

Proof. Let $\varphi : V \rightarrow \mathbb{R}$ be any bounded measurable function, and $\hat{\varphi}$ be as in (16). As already noted, $\hat{\varphi}$ is μ_F -integrable. It follows that $\hat{\varphi}(F^j(x)) < \infty$ for every $j \geq 0$, at μ_F -almost every point $x \in \Xi$. Moreover, by the Ergodic Theorem, condition (b) in Lemma 5.9 holds μ_F -almost everywhere. For the same reasons, $\tau(F^j(x))$ is finite for all $j \geq 0$, and condition (a) in the lemma is satisfied, for μ_F -almost all $x \in \Xi$.

This shows that Lemma 5.9 applies to every point x in a subset $A \subset \Xi$ with $\mu_F(A) = 1$. It follows that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \varphi(X^t(z)) dt = \int \varphi d\mu_X$$

for every point z in $B = \pi(A \times [0, +\infty))$. Since the latter has $\mu_X(B) = 1$, we have shown that the Birkhoff average of φ is constant μ_X -almost everywhere. Then the same is true for any integrable function, as bounded functions are dense in $L^1(\mu_X)$. Thus μ_X is ergodic and the corollary is proved. \square

6. CONSTRUCTION OF PHYSICAL MEASURES

Here we complete the proof of Theorem B. We unwind the reductions made in Sections 4 and 5 one step at a time and obtain physical measures for the original fbw at the end.

According to Section 4, by choosing a *global Poincaré section* Ξ , we can reduce to the quotient over the stable leaves, with the exception of finitely many leaves Γ , corresponding to the points whose orbit falls into the local stable manifold of some singularity or are sent into the stable boundary $\partial^s \Sigma$ of some $\Sigma \in \Xi$ by R , where the return time function τ is discontinuous.

As explained in Subsection 4.3, the global Poincaré map $R : \Xi \rightarrow \Xi$ induces a map $f : \mathcal{F} \setminus \Gamma \rightarrow \mathcal{F}$ on the leaf space, diffeomorphic to a finite union of open intervals I , which is piecewise expanding and admits finitely many ν_1, \dots, ν_l ergodic absolutely continuous (with respect to λ) invariant probability measures (acim) whose basins cover Lebesgue almost all points of I .

Moreover the Radon-Nikodym derivatives (densities) $\frac{d\nu_k}{d\lambda}$ are functions of universally bounded $1/\alpha$ -variation [17] where α is the Hölder exponent of f^l . Hence *these densities are bounded from above and the support of each ν_k contains nonempty open intervals*, so the basin $B(\nu_k)$ contains nonempty open intervals Lebesgue modulo zero, $k = 1, \dots, l$.

6.1. Physical measure for the global Poincaré map. We are now in the setting of the results of Subsection 5.1. Hence the finitely many acim ν_1, \dots, ν_l for f uniquely induce R -invariant ergodic probability measures $\tilde{\eta}_1, \dots, \tilde{\eta}_l$ on Ξ .

We claim that the basins of each $\tilde{\eta}_1, \dots, \tilde{\eta}_l$ have positive Lebesgue area λ^2 on Ξ and cover λ^2 almost every point of $p^{-1}(I)$. Indeed the uniform contraction of the leaves $\mathcal{F}_\Sigma \setminus \Gamma$ provided by Lemma 2.11, implies that the forward time averages of any pair x, y of points in $\xi \in \mathcal{F} \setminus p(\Gamma)$ on continuous functions $\varphi : \Xi \rightarrow \mathbb{R}$ are equal

$$\lim_{n \rightarrow +\infty} \left[\frac{1}{n} \sum_{j=0}^{n-1} \varphi(R^j(x)) - \frac{1}{n} \sum_{j=0}^{n-1} \varphi(R^j(y)) \right] = 0.$$

Hence $B(\tilde{\eta}_i) \subset p^{-1}(B(\nu_i))$, $i = 1, \dots, l$. This shows that $B(\tilde{\eta}_i)$ contains an entire strip except for a subset of λ^2 -null measure, because $B(\nu_i)$ contains some open interval λ modulo zero. Since $p_*(\lambda^2) \ll \lambda$ we get in particular

$$\lambda^2(B(\tilde{\eta}_i)) > 0 \quad \text{and} \quad \lambda^2\left(p^{-1}(I) \setminus \bigcup_{i=1}^l B(\tilde{\eta}_i)\right) = p_*(\lambda^2)\left(I \setminus \bigcup_{i=1}^l B(\nu_i)\right) = 0,$$

which shows that $\tilde{\eta}_1, \dots, \tilde{\eta}_l$ are physical measures whose basins cover $p^{-1}(I)$ Lebesgue almost everywhere. We observe that $p^{-1}(I) \subset \Xi$ is forward invariant under R , thus it contains $\Lambda \cap \Xi$.

6.2. Physical measures for the global first return map. Now we pass from R -invariant measures to R_0 -invariant measures. Recalling that R is induced by R_0 and the boundedness of the inducing time $r(x)$ on Ξ we set

$$(20) \quad \eta_i = \sum_{j=1}^{r_0} \sum_{k=0}^{j-1} (R_0^k)_* (\tilde{\eta}_i | r^{-1}(\{j\})), \quad i = 1, \dots, l.$$

It is easy to see that the mass of each η_i is given by

$$\eta_i(\Xi) = \sum_{j=1}^{r_0} j \cdot \tilde{\eta}_i(r^{-1}(\{j\})) = \int r d\tilde{\eta}_i = \tilde{\eta}_i(r), i = 1, \dots, l.$$

From now on we fix $\eta = \eta_i$ and $\tilde{\eta} = \tilde{\eta}_i$. We first show that η is R_0 -invariant and R_0 -ergodic. Then we prove that η is a physical measure for R_0 .

For the invariance under R_0 we write

$$(R_0)_* \eta = \sum_{j=1}^{r_0} \sum_{k=1}^{j-1} (R_0^k)_* (\tilde{\eta} | r^{-1}(\{j\})) + \sum_{j=1}^{r_0} R_* (\tilde{\eta} | r^{-1}(\{j\})) = \eta,$$

since $\sum_{j=1}^{r_0} R_* (\tilde{\eta} | r^{-1}(\{j\})) = \tilde{\eta} = \sum_{j=1}^{r_0} \tilde{\eta} | r^{-1}(\{j\})$ because $\tilde{\eta}$ is R -invariant.

For the ergodicity we normalize η and let $A \subset \Xi$ be such that $R_0^{-1}(A) = A$. We note that

$$R^{-1}(A) = \bigcup_{j=1}^{r_0} (R_0^{-j}(A) \cap r^{-1}(\{j\})) = \bigcup_{j=1}^{r_0} (A \cap r^{-1}(\{j\})) = A$$

thus by ergodicity of η with respect to R , either $\tilde{\eta}(A)$ or $\tilde{\eta}(\Xi \setminus A)$ is zero. Since for $k = 0, \dots, j-1$ and $j = 1, \dots, r_0$ we have

$$(R_0^k)_* (\tilde{\eta} | r^{-1}(\{j\})) (A) = \tilde{\eta} (R_0^{-k}(A) \cap r^{-1}(\{j\})) = \tilde{\eta} (A \cap r^{-1}(\{j\})) \leq \tilde{\eta}(A),$$

we see that $\eta(A) \cdot \eta(\Xi \setminus A) = 0$ if $\tilde{\eta}(A) \cdot \tilde{\eta}(\Xi \setminus A) = 0$. Thus η is R_0 -ergodic.

To prove that η is physical for R_0 , we recall that for any $x \in W^s(z, \Sigma)$ we have $r(R^k(x)) = r(R^k(z))$ for all $k \geq k_0$, by Remarks 4.3 and 4.5. This together with the uniform contraction along the stable leaves implies that there exists a uniform constant $C > 0$ such that

$$d(R_0^j(R^k(x)), R_0^j(R^k(z))) \leq C \cdot d(R^k(x), R^k(z)), \quad j = 1, \dots, r_0.$$

Hence, given a continuous function $\varphi : \Xi \rightarrow \mathbb{R}$, if we define

$$(21) \quad \tilde{\varphi}(w) = \sum_{j=0}^{r(w)-1} \varphi(R_0^j(w)), \quad \text{then} \quad \lim_{k \rightarrow \infty} |\tilde{\varphi}(R^k(x)) - \tilde{\varphi}(R^k(z))| = 0.$$

Now we assume in addition that $z \in B(\tilde{\eta})$ and write $r^k(x) = r(R^{k-1}(x)) + \dots + r(x)$ for all $k \geq 1$. We have for big n

$$\sum_{j=0}^{n-1} \varphi(R_0^j(x)) = \sum_{j=0}^{r^{k-1}(x)} \varphi(R_0^j(x)) + \sum_{j=r^{k-1}(x)}^{n-1} \varphi(R_0^j(x)) = A + B,$$

for the unique $k \geq 2$ such that $r^{k-1}(x) \leq n < r^k(x)$. Since $r^k(x) - r^{k-1}(x) \leq r_0$ we get $|B|/n \rightarrow 0$ when $n \rightarrow +\infty$. By analogous arguments to the ones proving Lemma 5.9 we deduce that

$$(22) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \cdot A = \lim_{n \rightarrow +\infty} \frac{k}{n} \cdot \frac{1}{k} \cdot A = \frac{1}{\tilde{\eta}(r)} \cdot \tilde{\eta}(\tilde{\varphi}) = \eta(\varphi).$$

This shows that $B(\eta) \supset B(\tilde{\eta})$ and so the basin has positive area in Ξ . Now we detail the arguments proving the convergence in (22).

First, $r \circ R^k$ is constant on stable leaves for all $k \geq k_0$ by Remark 4.5, thus $r \circ R^k = r_I \circ p$ for a map $r_I : \mathcal{F} \rightarrow \{1, \dots, r_0\}$. Hence $\tilde{\eta}(r) = \tilde{\eta}(r \circ R^k) = \tilde{\eta}(r_I \circ p) = \nu(r_I) < \infty$ is well defined, where ν is the f -invariant ergodic measure which induces $\tilde{\eta}$ (recall Subsection 6.1). The construction of $B(\tilde{\eta})$ given in Subsection 6.1 shows that $p(z) \in B(\nu)$ and so the time averages of r on the entire stable leaf $W^s(z, \Sigma)$ are the same. In particular since $x \in W^s(z, \Sigma)$ and by the expression above for r^k we get $\frac{1}{k} \cdot r^k(x) \rightarrow \tilde{\eta}(r)$. Moreover for any given $\varepsilon > 0$ we have

$$(k-1)(\tilde{\eta}(r) - \varepsilon) \leq r^{k-1}(x) \leq n < r^k(x) \leq k(\tilde{\eta}(r) + \varepsilon)$$

for all big enough n , which shows that $k/n \rightarrow 1/\tilde{\eta}(r)$.

Secondly, we note that $A/k = k^{-1} \sum_{j=0}^{k-1} \tilde{\varphi}(R^j(x))$. Moreover since the limit in (21) holds for every point x in $W^s(z, \Sigma)$, we are in the setting of Remark 5.3 and thus $\lim_{k \rightarrow +\infty} \int (\tilde{\varphi} \circ R^k)_+ d\nu$ exists. We are left to show that this limit equals $\tilde{\eta}(\tilde{\varphi})$.

The continuity of $\tilde{\varphi} \circ R^k$ along every stable leaf shows that, given $\varepsilon > 0$, we may find a compact $K \subset I$ with $\nu(I \setminus K) < \varepsilon$ such that $\varphi_k = \tilde{\varphi} \circ R^k | p^{-1}(K)$ is continuous. Let $\psi_\varepsilon : \Xi \rightarrow \mathbb{R}$ be a continuous extension of φ_k to Ξ with the same norm: $\sup_{w \in \Xi} |\psi_\varepsilon(w)| = \sup_{w \in \Xi} |\varphi_k(w)| \leq r_0 \cdot \sup |\varphi|$. Then for all $j \geq 1$ we have

$$\begin{aligned} \left| \int ((\tilde{\varphi} \circ R^k) \circ R^j)_+ d\nu - \int (\psi_\varepsilon \circ R^j)_+ d\nu \right| &= \left| \int_{f^{-k}(I \setminus K)} [((\tilde{\varphi} \circ R^k) \circ R^j)_+ - (\psi_\varepsilon \circ R^j)_+] d\nu \right| \\ &\leq 2r_0 \cdot \sup |\varphi| \cdot \nu(I \setminus K) \leq 2r_0 \cdot \sup |\varphi| \cdot \varepsilon. \end{aligned}$$

Letting $j \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$ we get

$$|\tilde{\eta}(\tilde{\varphi}) - \tilde{\eta}(\psi_\varepsilon)| = |\tilde{\eta}(\tilde{\varphi} \circ R^k) - \tilde{\eta}(\psi_\varepsilon)| \rightarrow 0, \text{ which shows that } \lim_{k \rightarrow +\infty} \int (\tilde{\varphi} \circ R^k)_+ d\nu = \tilde{\eta}(\tilde{\varphi}).$$

This finishes the proof of (22).

6.3. Physical measure for the suspension. Using the results from Subsection 5.2 it is straightforward to obtain ergodic probability measures ν_1, \dots, ν_l invariant under the suspension $(X^t)_{t \geq 0}$ of R_0 with return time T , corresponding to η_1, \dots, η_l respectively.

Now we use Lemma 5.9 to show that each ν_i is a physical measure for $(X^t)_{t \geq 0}$, $i = 1, \dots, l$. Let $x \in \Sigma \cap B(\nu_i)$ for a fixed $\Sigma \in \Xi$ and $i \in \{1, \dots, l\}$. According to Remark 4.2 the return time τ_I on I is Lebesgue integrable, thus ν_i -integrable also since $\frac{d\nu_i}{d\lambda}$ is bounded. Hence τ is $\tilde{\eta}_i$ -integrable by the construction of $\tilde{\eta}_i$ from ν_i (see Subsection 5.1). Moreover we have

$$\tau(x) = T(R_0^{r(x)-1}(x)) + \dots + T(x),$$

so $\eta_i(T) = \tilde{\eta}_i(\tau)$ (see Subsection 6.2). Thus T is η_i -integrable.

Lemma 5.9 together with the fact that η_i is physical for R_0 , ensure that $B(\nu_i)$ contains the positive X^t orbit of almost every point $(x, 0), x \in B(\nu_i)$, with respect to λ^2 on $B(\eta_i)$. If we denote by $\lambda^3 = \pi_*(\lambda^2 \times dt)$ a natural volume measure on V , then we get $\lambda^3(B(\nu_i)) > 0$.

The basins $B(\nu_1), \dots, B(\nu_l)$ cover λ^3 -almost every point in $V_0 = \pi(p^{-1}(I) \times [0, +\infty))$ and this subset is a neighborhood of the suspension $\pi((\Lambda \cap \Xi \setminus \Gamma) \times [0, +\infty))$ of $\Lambda \cap \Xi \setminus \Gamma$.

6.4. Physical measure for the flow. Finally to extend the previous conclusions to the original fbw, we relate the suspension $(X^t)_{t \geq 0}$ of R_0 with return time τ to $(X_t)_{t \geq 0}$ in U as follows. We define

$$\Phi : \Xi \times [0, +\infty) \rightarrow U \quad \text{by} \quad (x, t) \mapsto X_t(x)$$

and since $\Phi(x, \tau(x)) = (R_0(x), 0) \in \Xi$, this map naturally defines a quotient map

$$(23) \quad \phi : V \rightarrow U \quad \text{such that} \quad \phi \circ X^t = X_t \circ \phi, \quad \text{for all } t \geq 0,$$

through the identification \sim from Subsection 5.2. The map ϕ is a homeomorphism into its image $V_0 = \phi(V) \subset U$ (since Φ is obviously a quotient map onto its image because we have defined the relation \sim using the first return time τ) and a diffeomorphism when restricted to $V \setminus \pi(\Gamma)$, which is a full Lebesgue (λ^3) measure subset of V .

Therefore the measures ν_i define physical measures $\mu_i = \phi_*(\nu_i)$, $i = 1, \dots, l$, whose basins cover a full Lebesgue (m) measure subset of V_0 , which is a neighborhood of Λ . Since $V_0 \subset U$ we have

$$W^s(\Lambda) = \bigcup_{t < 0} X_t(V_0).$$

Moreover X_t is a diffeomorphism for all $t \in \mathbb{R}$, thus preserves subsets of zero m measure. Hence $\bigcup_{t < 0} X_t(B(\mu_1) \cup \dots \cup B(\mu_l))$ has full Lebesgue measure in $W^s(\Lambda)$. In other words, Lebesgue (m) almost every point x in the basin $W^s(\Lambda)$ of Λ is such that $X_t(x) \in B(\mu_i)$ for some $t > 0$ and $i = 1, \dots, l$.

6.4.1. Uniqueness of the physical measure. Let us assume that there exists $z_0 \in \Lambda$ such that $\{X_t(z_0) : t > 0\}$ is dense regular orbit in Λ and that the number l of distinct physical measures is bigger than one. Then we can take distinct physical measures η_1, η_2 for R_0 on Ξ associated to distinct physical measures μ_1, μ_2 for $X \upharpoonright \Lambda$. Then there are open sets $U_1, U_2 \subset \Xi$ such that

$$U_1 \cap U_2 = \emptyset \quad \text{and} \quad \lambda^2(B(\eta_i) \setminus U_i) = 0, \quad i = 1, 2.$$

For a very small $\zeta > 0$ we consider the open subsets $V_i = X_{(-\zeta, \zeta)}(U_i)$, $i = 1, 2$ of U such that $V_1 \cap V_2 = \emptyset$. According to the construction of μ_i we have $\mu_i(B(\mu_i) \setminus V_i) = 0$, $i = 1, 2$.

The transitivity assumption ensures that there are positive times $T_1 < T_2$ (exchanging V_1 and V_2 if needed) such that $X_{T_i}(z_0) \in V_i$, $i = 1, 2$. Since V_1, V_2 are open sets and $g = X_{T_2 - T_1}$ is a diffeomorphism, there exists a small open set $W_1 \subset V_1$ such that $g \upharpoonright W_1 : W_1 \rightarrow V_2$ is a C^1 diffeomorphism into its image $W_2 = g(W_1) \subset V_2$.

Now the C^1 smoothness of $g \upharpoonright W_1$ ensures that a full Lebesgue (m) measure subset of W_1 is sent into a full Lebesgue measure subset of W_2 . By the definition of g and the choice of V_1, V_2 , there exists a point in $B(\mu_1) \cap W_1$ whose positive orbit contains a point in $B(\mu_2) \cap W_2$, thus $\mu_1 = \mu_2$. Hence *singular-hyperbolic attractors have a unique physical probability measure* μ .

6.4.2. Hyperbolicity of the physical measure. For the hyperbolicity of the measure μ we note that

- the sub-bundle E^s is one-dimensional and uniformly contracting, thus on the E^s -direction the Lyapunov exponent is negative for every point in U ;

- the sub-bundle E^{cu} is two-dimensional, dominates E^s , contains the fbw direction and is volume expanding, thus by Oseledets Theorem [23, 40] the sum of the Lyapunov exponents on the direction of E^{cu} is given by $\mu_i(\log |\det DX_1 | E^{cu}|) > 0$. Hence there is a positive Lyapunov exponent for μ_i -almost every point on the direction of E^{cu} , $i = 1, \dots, l$.

We will show that the expanding direction in E^{cu} does not coincide with the fbw direction $E_z^X = \{s \cdot X(z) : s \in \mathbb{R}\}$, $z \in \Lambda$. Indeed, the invariant direction given by E_z^X cannot have positive Lyapunov exponent, since for all $t > 0$ and $z \in U$

$$(24) \quad \frac{1}{t} \log \|DX_t(z) \cdot X(z)\| = \frac{1}{t} \log \|X(X_t(z))\| \leq \frac{1}{t} \log \|X\|_0,$$

where $\|X\|_0 = \sup\{\|X(z)\| : z \in U\}$ is a constant. Analogously this direction cannot have positive exponent for negative values of time, thus the Lyapunov exponent along the fbw direction must be zero at regular points.

This shows that at μ -almost every point z the Oseledets splitting of the tangent bundle has the form

$$T_z M = E_z^s \oplus E_z^X \oplus F_z,$$

where F_z is the one-dimensional measurable sub-bundle of vectors with positive Lyapunov exponent. The proof of Theorem B is complete.

7. ABSOLUTELY CONTINUOUS DISINTEGRATION OF THE PHYSICAL MEASURE

Here we prove Theorem C. We let μ be a physical ergodic probability measure for a singular-hyperbolic attractor Λ of a C^2 -fbw in an open subset $U \subset M^3$, obtained through the sequence of reductions of the dynamics of the fbw X_t to the suspension fbw X^t of the Poincaré map R_0 and return time function T , with corresponding X^t -invariant measure ν obtained from the R_0 -invariant measure η . In addition η is obtained through the ergodic invariant measure ν of the one-dimensional map $f : I \rightarrow I$. This is explained in Section 6 and Subsection 6.2. We know that μ is hyperbolic as explained in Subsection 6.4.

Let us fix $\delta_0 > 0$ small. Then by Pesin's non-uniformly hyperbolic theory [33, 11, 35] we know that there exists a compact subset $K \subset \Lambda$ such that $\mu(\Lambda \setminus K) < \delta_0$ and there exists $\delta_1 > 0$ for which every $z \in K$ admits a strong-unstable manifold $W_{\delta_1}^{uu}(z)$ with inner radius δ_1 . We refer to this kind of sets as *Pesin's sets*. The *inner radius* of $W_{\delta_1}^{uu}(z)$ is defined as the length of the shortest smooth curve in this manifold from z to its boundary. Moreover $K \ni z \mapsto W_{\delta_1}^{uu}(z)$ is a continuous map $K \rightarrow \mathcal{E}^1(I_1, M)$ (recall the notations in subsection 2.1).

The suspension fbw X^t defined on V in Subsection 5.2 is conjugated to the X_t -fbw on an open subset of U through a homeomorphism ϕ , defined in Subsection 6.4, which takes orbits to orbits and preserves time as in (23). Hence there exists a corresponding set $K' = \phi^{-1}(K)$ satisfying the same properties of K with respect to X^t , where the constants δ_0, δ_1 are changed by at most a constant factor due to ϕ^{-1} by the compactness of K . In what follows we use the measure $\nu = (\phi^{-1})_* \mu$ instead of μ and write K for K' .

We fix a density point $x_0 \in K$ of $\nu | K$. We may assume that $x_0 \in \Sigma$ for some $\Sigma \in \Xi$. Otherwise if $x_0 \notin \Xi$, since $x_0 = (x, t)$ for some $x \in \Sigma$, $\Sigma \in \Xi$ and $0 < t < T(x)$, then we use $(x, 0)$ instead of x_0 in the following arguments, but we still write x_0 . Clearly the length of the unstable manifold

through $(x, 0)$ is unchanged due to the form of the suspension flow, at least for small values of δ_1 . Since ν is given as a product measure on the quotient space V (see Subsection 6.3), we may assume without loss that x_0 is a density point of η on $\Sigma \cap K$.

We set $W^u(x, \Sigma)$ to be the connected component of $W^u(x) \cap \Sigma$ that contains x , for $x \in K \cap \Sigma$, where $W^u(x)$ is defined in Subsection 1.4. Then $W^u(x, \Sigma)$ has inner radius bigger than some positive value $\delta_2 > 0$ for $x \in K \cap \Sigma$, which depends only on δ_1 and the angle between $W_{\delta_1}^{uu}(x)$ and $T_x \Sigma$. Since this angle is uniformly bounded (because Ξ and K are compact) we have that $\text{dist}(R_0^{-n}(y), R_0^{-n}(x)) \rightarrow 0$ for all $y \in W^u(\Sigma, x)$ when $n \rightarrow \infty$ (we recall that R_0 is injective, see Section 4, and that by definition of $W^u(x)$ its points must have well defined pre-images under R_0). Thus $W^u(\Sigma, x) \subset W^u(R_0, x)$, the unstable manifold of R_0 through $x \in \Sigma$.

Let $\mathcal{F}^s(x_0, \delta_2) = \{W^s(x, \Sigma) : x \in W^u(x_0, \Sigma)\}$ and $F^s(x_0, \delta_2) = \cup_{\gamma \in \mathcal{F}^s(x_0, \delta_2)} \gamma$ be a horizontal strip in Σ . Points $z \in F^s(x_0, \delta_2)$ can be specified using coordinates $(x, y) \in W^u(x_0, \Sigma) \times \mathbb{R}$, where x is given by $W^u(x_0, \Sigma) \cap W^s(z, \Sigma)$ and y is the length of the shortest smooth curve connecting x to z in $W^s(z, \Sigma)$. Let us consider

$$\mathcal{F}^u(x_0, \delta_2) = \{W^u(z, \Sigma) : z \in \Sigma \text{ and } W^u(z, \Sigma) \text{ crosses } F^s(x_0, \delta_2)\},$$

where we say that a curve γ crosses $F^s(x_0, \delta_2)$ if the trace of γ can be written as the graph of a map $W^u(x_0, \Sigma) \rightarrow W^s(x_0, \Sigma)$ using the coordinates outlined above. We stress that $\mathcal{F}^u(x_0, \delta_2)$ is not restricted to leaves through points of K .

We may assume that $F^u(x_0, \delta_2) = \cup \mathcal{F}^u(x_0, \delta_2)$ satisfies $\eta(F^u(x_0, \delta_2)) > 0$ up to taking a smaller $\delta_2 > 0$, since x_0 is a density point of $\eta \mid K \cap \Sigma$. Let $\hat{\eta}$ be the measure on $\mathcal{F}^u(x_0, \delta_2)$ given by

$$\hat{\eta}(A) = \eta\left(\bigcup_{\gamma \in A} \gamma\right) \text{ for every measurable set } A \subset \mathcal{F}^u(x_0, \delta_2).$$

Proposition 7.1. *The measure $\eta \mid F^u(x_0, \delta_2)$ admits a disintegration into conditional measures η_γ along $\hat{\eta}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$ such that $\eta_\gamma \ll \lambda_\gamma$, where λ_γ is the measure (length) induced on γ by the natural Riemannian measure λ^2 (area) on Σ . Moreover there exists $D_0 > 0$ such that*

$$\frac{1}{D_0} \leq \frac{d\eta_\gamma}{d\lambda_\gamma} \leq D_0, \quad \eta_\gamma\text{-almost everywhere for } \hat{\eta}\text{-almost every } \gamma.$$

This is enough to conclude the proof of Theorem C since both δ_0 and δ_2 can be taken arbitrarily close to zero, so that all unstable leaves $W^u(x, \Sigma)$ through almost every point with respect to η will support a conditional measure of η .

Indeed, to obtain the disintegration of ν along the center-unstable leaves that cross any small ball around a density point x_0 of K , we project that neighborhood of x_0 along the flow in negative time on a cross section Σ . Then we obtain the family $\{\eta_\gamma\}$, the disintegration of η along the unstable leaves $\gamma \in \mathcal{F}^u$ on a strip F^s of Σ , and consider the family $\{\eta_\gamma \times dt\}$ of measures on $\mathcal{F}^u \times [0, T]$ to obtain a disintegration of ν , where $T > 0$ is a fixed time slightly smaller than the return time of the points in the strip F^s .

In fact, $\eta_\gamma \times dt \ll \lambda_\gamma \times dt$ and $\lambda_\gamma \times dt$ is the induced (area) measure on the center-unstable leaves by the volume measure λ^3 on V , and it can be given by restricting the volume form λ^3 to the surface $\gamma \times [0, T]$ which we write λ_γ^3 , for $\gamma \in \mathcal{F}^u$. Thus by Proposition 7.1 and by the definition

of ν , we have

$$\nu_\gamma = \eta_\gamma \times dt = \frac{d\eta_\gamma}{d\lambda_\gamma} \cdot \lambda_\gamma^3, \quad \gamma \in \mathcal{F}^u$$

and the densities of the conditional measures $\eta_\gamma \times dt$ with respect to λ_γ^3 are also uniformly bounded from above and from below away from zero – we have left out the constant factor $1/\mu(\tau)$ to simplify the notation.

Since $\mu = \phi_* \nu$ and ϕ is a diffeomorphism when restricted to $V \setminus \pi(\Gamma)$, then μ also has an absolutely continuous disintegration along the center-unstable leaves. The densities are related by the expression

$$\mu_\gamma = \phi_*(\nu_\gamma) = \phi_* \left(\frac{d\eta_\gamma}{d\lambda_\gamma} \cdot \lambda_\gamma^3 \right) = \left(\frac{1}{\det D(\phi | \gamma \times [0, T])} \cdot \frac{d\eta_\gamma}{d\lambda_\gamma} \right) \circ \phi^{-1} \cdot m_\gamma, \quad \gamma \in \mathcal{F}^u$$

which implies that the densities along the center-unstable leaves are uniformly bounded from above. (Here m_γ denotes the area measure on the center-unstable leaves induced by the volume form m .) Indeed, the tangent bundle of $\gamma \times [0, T]$ is sent by $D\phi$ into the bundle E^{cu} by construction and recalling that $\phi(x, t) = X_t(x)$ then, if e_1 is a unit tangent vector at $x \in \gamma$, \hat{e}_1 is the unit tangent vector at $\phi(x, 0) \in W^u(x, \Sigma)$ and e_2 is the fbw direction at (x, t) we get

$$D\phi(x, t)(e_1) = DX_t(X_t(x))(\hat{e}_1) \quad \text{and} \quad D\phi(x, t)(e_2) = DX_t(X_t(x))(X(x, 0)) = X(X_t(x)).$$

Hence $D(\phi | \gamma \times [0, T])(x, t) = DX_t | E_{\phi(x, t)}^{cu}$ for $(x, t) \in \gamma \times [0, T]$ and so

$$|\det D(\phi | \gamma \times [0, T])(x, t)| = J_t^c(x).$$

Now the volume expanding property of X_t along the center-unstable sub-bundle, together with the fact that the return time function τ is not bounded from above near the singularities, show that the densities of μ_γ are uniformly bounded from above throughout Λ but not from below. In fact, this shows that these densities will tend to zero close to the singularities of X in Λ .

This finishes the proof of Theorem C except for the proof of Proposition 7.1 and of $\text{supp}(\mu) = \Lambda$, which we present in Subsection 7.1.

7.1. Constructing the disintegration. Here we prove Proposition 7.1. We split the proof into several lemmas keeping the notations of the previous sections. Let $R_0 : p^{-1}(I) \rightarrow \Xi$, λ^2 , $\mathcal{F}^u(x_0, \delta_2)$, $F^u(x_0, \delta_2)$ and η be as before, where $x_0 \in K \cap \Sigma$ is a density point of $\eta | K$ and K is a compact Pesin set. We write $\{\eta_\gamma\}$ and $\{\lambda_\gamma^2\}$ for the disintegrations of $\eta | F^u(x_0, \delta_2)$ and λ^2 along $\gamma \in \mathcal{F}^u(x_0, \delta_2)$.

Lemma 7.2. *Either $\eta_\gamma \ll \lambda_\gamma^2$ for $\hat{\eta}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$, or $\eta_\gamma \perp \lambda_\gamma^2$ for $\hat{\eta}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$.*

Proof. We start by assuming that the first item in the statement does not hold and proceed to show that this implies the second item. We write η for $\eta(F^u(x_0, \delta_2))^{-1} \cdot \eta | F^u(x_0, \delta_2)$ to simplify the notation in this proof.

Let us suppose that there exists $A \subset F^u(x_0, \delta_2)$ such that $\eta(A) > 0$ and $\lambda_\gamma^2(A) = 0$ for $\hat{\eta}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$. Let $B = \cup_{k \geq 0} R_0^k(A)$. We claim that $\eta(B) = 1$.

Indeed, we have $R_0(B) \subset B$, then $B \subset R_0^{-1}(B)$ and $(R_0^{-k}(B))_{k \geq 0}$ is a nested increasing family of sets. Since η is R_0 -ergodic we have for any measurable set $C \subset \Xi$

$$(25) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta(C \cap R_0^{-j}(B)) = \eta(C) \cdot \eta(B).$$

But $\eta(\cup_{k \geq 0} R_0^{-k}(B)) = 1$ because this union is R_0 -invariant and $\eta(B) = \eta(R_0^{-k}(B)) > 0$ by assumption, for any $k \geq 0$. Because the sequence is increasing and nested we have $\eta(R_0^{-k}(B)) \nearrow 1$. Hence from (25) we get that $\eta(C) = \eta(C) \cdot \eta(B)$ for all sets $C \subset X$. Thus $\eta(B) = 1$ as claimed.

Therefore $1 = \eta(B) = \int \eta_\gamma(B) d\hat{\eta}(\gamma)$ and so $\eta_\gamma(B) = 1$ for $\hat{\eta}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$ since every measure involved is a probability measure.

We now claim that $\lambda_\gamma(B) = 0$ for $\hat{\mu}$ -a.e. $\gamma \in \mathcal{F}^u(x_0, \delta_2)$. For if $R_0(A) \cap \gamma \neq \emptyset$ for some $\gamma \in \mathcal{F}^u(x_0, \delta_2)$, then $A \cap R_0^{-1}(\gamma) \cap F^u(x_0, \delta_2) \neq \emptyset$ and so it is enough to consider only $A \cap F_1^u$, where $F_1^u = R_0^{-1}(F^u(x_0, \delta_2)) \cap F^u(x_0, \delta_2)$. But $\lambda_\gamma(A \cap F_1^u) \leq \lambda_\gamma(A) = 0$ thus

$$0 = \lambda_\gamma(R_0(A \cap F_1^u)) \geq \lambda_\gamma(R_0(A) \cap F^u(x_0, \delta_2)) = \lambda_\gamma(R_0(A))$$

for $\hat{\eta}$ -a.e. γ since R_0 is piecewise smooth, hence a regular map. Therefore we get $\lambda_\gamma(R_0^k(A)) = 0$ for all $k \geq 1$ implying that $\lambda_\gamma(B) = 0$ for $\hat{\eta}$ -a.e. γ .

This shows that η_γ is singular with respect to λ_γ for $\hat{\eta}$ -a.e. γ . The proof is finished. \square

7.1.1. Existence of hyperbolic times for f_0 and consequences to R_0 . Now we show that a positive measure subset of $\mathcal{F}^u(x_0, \delta_2)$ has absolutely continuous disintegrations, which is enough to conclude the proof of Proposition 7.1 by Lemma 7.2, except for the bounds on the densities.

We start by noting that since vectors in center-unstable cones C_Σ^{cu} for $\Sigma \in \Xi$ have angles with the horizontal (stable) directions bounded away from zero, then there exists $\beta_0 > 0$ such that for any cu -curve $\gamma: (a, b) \rightarrow \Sigma$ in a cross-section Σ we have the following bound on the derivative

$$(26) \quad \frac{1}{\beta_0} \leq (p | \gamma)' \leq \beta_0.$$

We need the notion of *hyperbolic time* for the one-dimensional map f [2]. We know that this map is piecewise $C^{1+\alpha}$ and the boundaries Γ_0 of the intervals I_1, \dots, I_n can be taken as a *singular set* for f (where the map is not defined or is not differentiable) which behaves like a *power of the distance to Γ_0* , as follows. Denoting by d the usual distance on the intervals I , there exist $B > 0$ and $\beta > 0$ such that

- $\frac{1}{B} \cdot d(x, \Gamma_0)^\beta \leq |f'| \leq B \cdot d(x, \Gamma_0)^{-\beta}$;
- $|\log |f'(x)| - \log |f'(y)|| \leq B \cdot d(x, y) \cdot d(x, \Gamma_0)^{-\beta}$,

for all $x, y \in I$ with $d(x, y) < d(x, \Gamma_0)/2$. This is true of f since in Subsection 4.3 it was shown that $f' | I_j$ either is bounded from above and below away from zero, or else is of the form x^β with $\beta \in (0, 1)$.

Given $\delta > 0$ we define $d_\delta(x, \Gamma_0) = d(x, \Gamma_0)$ if $d(x, \Gamma_0) < \delta$ and 1 otherwise.

Definition 7.3. Given $b, c, \delta > 0$ we say that $n \geq 1$ is a (b, c, δ) -hyperbolic time for $x \in I$ if

$$(27) \quad \prod_{j=n-k}^{n-1} |f'(f^j(x))|^{-1} \leq e^{-ck} \quad \text{and} \quad \prod_{j=n-k}^{n-1} d_\delta(f^j(x), \Gamma_0) \geq e^{-bk}$$

for all $k = 0, \dots, n-1$.

Since f has positive Lyapunov exponent ν -almost everywhere, i.e.

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log |(f^n)'(x)| > 0 \quad \text{for } \nu\text{-almost all } x \in I,$$

and $\frac{d\nu}{d\lambda}$ is bounded from above (where λ is the Lebesgue length measure on I), thus $|\log d(x, \Gamma_0)|$ is ν -integrable and for any given $\varepsilon > 0$ we can find $\delta > 0$ such that for ν -a.e. $x \in I$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log d_\delta(f^j(x), \Gamma_0) = \int -\log d_\delta(x, \Gamma_0) d\nu(x) < \varepsilon.$$

This means that f is *non-uniformly expanding* and has *slow recurrence to the singular set*. Hence we are in the setting of the following result.

Theorem 7.4 (Existence of a positive frequency of hyperbolic times). *Let $f : I \rightarrow I$ be a $C^{1+\alpha}$ map, behaving like a power of the distance to a singular set Γ_0 , non-uniformly expanding and with slow recurrence to Γ_0 with respect to an absolutely continuous invariant probability measure ν . Then for $b, c, \delta > 0$ small enough there exists $\theta = \theta(b, c, \delta) > 0$ such that ν -a.e. $x \in I$ has infinitely many (b, c, δ) -hyperbolic times. Moreover if we write $0 < n_1 < n_2 < n_3 < \dots$ for the hyperbolic times of x then their asymptotic frequency satisfies*

$$\liminf_{N \rightarrow \infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \theta \quad \text{for } \nu\text{-a.e. } x \in I.$$

Proof. A complete proof can be found in [2, Section 5] with weaker assumptions corresponding to Theorem C in that paper. □

From now on we fix values of (b, c, δ) so that the conclusions of Theorem 7.4 are true.

We now outline the properties of these special times. For detailed proofs see [2, Proposition 2.8] and [3, Proposition 2.6, Corollary 2.7, Proposition 5.2].

Proposition 7.5. *There are constants $\beta_1, \beta_2 > 0$ depending on (b, c, δ) and f only such that, if n is (b, c, δ) -hyperbolic time for $x \in I$, then there are neighborhoods $W_k(x) \subset I$ of $f^{n-k}(x)$, $k = 1, \dots, n$, such that*

- (1) $f^k |_{W_k(x)}$ maps $W_k(x)$ diffeomorphically to the ball of radius β_1 around $f^n(x)$;
- (2) for every $1 \leq k \leq n$ and $y, z \in W_k(x)$

$$d(f^{n-k}(y), f^{n-k}(z)) \leq e^{-ck/2} \cdot d(f^n(y), f^n(z));$$

- (3) for $y, z \in W_n(x)$

$$\frac{1}{\beta_2} \leq \frac{|(f^n)'(y)|}{|(f^n)'(z)|} \leq \beta_2.$$

The conjugacy $p \circ R = f \circ p$ between the actions of the Poincaré map and the one-dimensional map on the space of leaves, together with the bounds on the derivative (26), enables us to extend the properties given by Proposition 7.5 to any cu -curve inside $B(\eta)$, as follows.

Let $\gamma: J \rightarrow \Xi$ be a cu -curve in $\Xi \setminus \Gamma$ such that $\gamma(s) \in B(\eta)$ for Lebesgue almost every $s \in J$, J a non-empty interval — such a curve exists since the basin $B(\eta)$ contains entire strips of some section $\Sigma \in \Xi$ except for a subset of zero area. Note that we have the following limit in the weak* topology

$$\lim_{n \rightarrow +\infty} \lambda_\gamma^n = \eta \quad \text{where} \quad \lambda_\gamma^n = \frac{1}{n} \sum_{j=0}^{n-1} (R_0)_*^j(\lambda_\gamma),$$

by the choice of γ and by an easy application of the Dominated Convergence Theorem.

Proposition 7.6. *There are constants $\kappa_0, \kappa_1 > 0$ depending on (b, c, δ) and $R_0, \beta_0, \beta_1, \beta_2$ only such that, if $x \in \gamma$ and n is big enough and a (b, c, δ) -hyperbolic time for $p(x) \in I$, then there are neighborhoods $V_k(x)$ of $R^{n-k}(x)$ on $R^{n-k}(x)(\gamma)$, $k = 1, \dots, n$, such that*

- (1) $R^k | V_k(x)$ maps $V_k(x)$ diffeomorphically to the ball of radius κ_0 around $R^n(x)$ on $R^n(\gamma)$;
- (2) for every $1 \leq k \leq n$ and $y, z \in V_k(x)$

$$d_{R^{n-k}(\gamma)}(R^{n-k}(y), R^{n-k}(z)) \leq \beta_0 \cdot e^{-ck/2} \cdot d_{R^n(\gamma)}(R^n(y), R^n(z));$$

- (3) for $y, z \in V_n(x)$

$$\frac{1}{\kappa_1} \leq \frac{|D(R^n | \gamma)(y)|}{|D(R^n | \gamma)(z)|} \leq \kappa_1;$$

- (4) the inducing time of R^k on $V_k(x)$ is constant, i.e. $r^{n-k} | V_k(x) \equiv \text{const.}$

Here d_γ denotes the distance along γ given by the shortest smooth curve in γ joining two given points and λ_γ denotes the normalized Lebesgue length measure induced on γ by the area form λ^2 on Ξ .

Remark 7.7. Item (4) of Proposition 7.6 ensures that in items (1) through (3) of this proposition we can replace R^k by $R^{r^k(x)}$ everywhere. So the statement of Proposition 7.6 may be read as a statement about certain iterates of the map R_0 .

Proof of Proposition 7.6. Let $x_0 = p(x)$ and $W_k(x_0)$ be given by Proposition 7.5, $k = 1, \dots, n$. We have that $p(\gamma)$ is an interval in I and that $p | \gamma: \gamma \rightarrow p(\gamma)$ is a diffeomorphism — we may take γ with smaller length if needed.

If n is big enough, then $W_n(x_0) \subset p(\gamma)$. Moreover the conjugacy implies that the following maps are all diffeomorphisms

$$\begin{array}{ccc} V_k(x) & \xrightarrow{R^k} & R^k(V_k(x)) \\ p \downarrow & & \downarrow p \\ W_k(x_0) & \xrightarrow{f^k} & B(f^k(x_0), \kappa_0) \end{array},$$

and the diagram commutes, where $V_k(x) = (p | R^k(\gamma))^{-1}(W_k(x_0))$, $k = 1, \dots, n$. Using the bounds (26) to compare derivatives we get $\kappa_0 = \beta_1/\beta_0$ and $\kappa_1 = \beta_0 \cdot \beta_2$.

To get item (4) we just note that by definition of (b, c, δ) -hyperbolic time none of the sets $W_k(x_0)$ may intersect Γ_0 . According to the definition of Γ_0 , this means that orbits through $x, y \in V_k(x)$ cannot cut different cross-sections in Ξ before the next return in time $\tau(x), \tau(y)$ respectively. Hence every orbit through $W_k(x_0)$ cuts the same cross-sections in its way to the next return cross-section. In particular the number of cross-section cuts is the same, i.e. $r \mid V_k(x)$ is constant, $k = 1, \dots, n$. Hence by definition of r^k we obtain the statement of item (4) since $R(V_k(x)) = V_{k-1}(x)$ by definition. This completes the proof of the proposition. \square

We define for $n \geq 1$

$$H_n = \{x \in \gamma : n \text{ is a } (b, c/2, \delta)\text{-hyperbolic time for } p(x)\}.$$

As a consequence of items (1-2) of Proposition 7.6, we have that H_n is an open subset of γ and for any $x \in \gamma \cap H_n$ we can find a connected component γ^n of $R^n(\gamma) \cap B(R^n(x), \kappa_0)$ containing x such that $R^n \mid V_n(x) : V_n(x) \rightarrow \gamma^n$ is a diffeomorphism. In addition γ^n is a *cu*-curve according to Corollary 2.6, and by item (3) of Proposition 7.6 we deduce that

$$(28) \quad \frac{1}{\kappa_1} \leq \frac{d(R_*^n(\lambda_\gamma) \mid B(R^n(x), \kappa_0))}{d\lambda_{\gamma^n}} \leq \kappa_1, \quad \lambda_{\gamma^n} - \text{a.e. on } \gamma^n,$$

where λ_{γ^n} is the Lebesgue induced measure on γ^n for any $n \geq 1$, if we normalize both measures so that $((R^n)_*(\lambda_\gamma) \mid B(R^n(x), \kappa_0))(\gamma^n) = \lambda_{\gamma^n}(\gamma^n)$, i.e. their masses on γ^n are the same.

Moreover the set $R^n(\gamma \cap H_n)$ has an at most countable number of connected components which are diffeomorphic to open intervals. Each of these components is a *cu*-curve with diameter bigger than κ_0 and hence we can find a *pairwise disjoint family* γ_i^n of κ_0 -neighborhoods around $R^n(x_i) = R^{n(x_i)}(x_i)$ in $R^n(\gamma)$, for some $x_i \in H_n$, with maximum cardinality, such that

$$(29) \quad \Delta_n = \bigcup_i \gamma_i^n \subset R^n(\gamma \cap H_n) \quad \text{and} \quad ((R^n)_*(\lambda_\gamma) \mid \Delta_n)(\Delta_n) \geq \frac{1}{2\kappa_1} \cdot \lambda_\gamma(H_n).$$

Indeed since $R^n(\gamma \cap H_n)$ is one-dimensional, for each connected component the family Δ_n may miss a set of points of length at most equal to the length of one γ_i^n , for otherwise we would manage to include an extra κ_0 -neighborhood in Δ_n . Hence we have in the worst case (assuming that there is only one set γ_i^n for each connected component)

$$\lambda_{\gamma^n}(R^n(\gamma \cap H_n) \setminus \Delta_n) \leq \lambda_{\gamma^n}\left(\bigcup_i \gamma_i^n\right) = \lambda_{\gamma^n}(\Delta_n) \quad \text{so that} \quad \lambda_{\gamma^n}(\Delta_n) \geq \frac{1}{2} \cdot \lambda_{\gamma^n}(R^n(\gamma \cap H_n))$$

and the constant κ_1 comes from (28).

For a fixed small $\rho > 0$ we consider $\Delta_{n,\rho}$ given by the balls γ_i^n with the same center $x_{n,i}$ but a reduced radius of $\kappa_0 - \rho$. Then the same bound in (29) still holds with $2\kappa_1$ replaced by $3\kappa_1$.

We re-index each γ_i^n to $\gamma_{r^n(x_{n,i})}$ and regroup the balls with the same index n in the class $D_{n,\rho}$, for all $n \geq 1$. The index now reflects the number of iterates of R_0 (instead of R) needed to get the uniform expansion with bounded distortion into a disk with uniform radius. We write D_n for the family of disks from $\bigcup_{j \geq 1} \Delta_j$ with the same expanding iterate (the disks with the same centers as the ones from $D_{n,\rho}$ but with their original size). Since $r^k \leq r_0 \cdot k$ we have the following relation

for all $k \geq 1$

$$(30) \quad \bigcup_{i=0}^{r_0 \cdot k} D_{i,\rho} \supset \bigcup_{j=0}^k \Delta_j.$$

We define the following sequences of measures

$$\omega_\rho^n = \frac{1}{n} \sum_{j=0}^{n-1} (R_0)_*^j(\lambda_\gamma) \mid D_{j,\rho} \quad \text{and} \quad \bar{\lambda}_\gamma^n = \lambda_\gamma^n - \omega_\rho^n, \quad n \geq 1.$$

Then any weak* limit point $\tilde{\eta} = \lim_k \omega_\rho^{n_k}$ for some subsequence $n_1 < n_2 < \dots$ and $\bar{\eta} = \lim_k \bar{\lambda}_\gamma^{n_k}$ (where n_k' may be taken as a subsequence of n_k), are R_0 -invariant measures which satisfy $\eta = \tilde{\eta} + \bar{\eta}$.

We claim that $\tilde{\eta} \neq 0$, thus $\eta = \tilde{\eta}$ as a consequence of the ergodicity of η . In fact, we can bound the mass of ω_ρ^n from below using the density of hyperbolic times from Theorem 7.4, the bound from (29) and the relation (30) through the following Fubini-Tonelli-type argument

$$\begin{aligned} \omega_\rho^n(M) &\geq \frac{1}{3\kappa_1 \cdot n} \sum_{j=0}^{\lfloor n/r_0 \rfloor - 1} \lambda_\gamma(H_j) = \frac{\lfloor n/r_0 \rfloor}{3\kappa_1 n} \int \int \chi_i(x) d\lambda_\gamma(x) d\#_{\lfloor n/r_0 \rfloor}(i) \\ &= \frac{\lfloor n/r_0 \rfloor}{3\kappa_1 n} \int \int \chi_i(x) d\#_{\lfloor n/r_0 \rfloor}(i) d\lambda_\gamma(x) \geq \frac{\theta}{6\kappa_1 r_0} > 0, \end{aligned}$$

for every n big enough by the choice of γ . Here $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ and $\#_n(J) = \#J/n$ for any $J \subset \{0, \dots, n-1\}$ is the uniform discrete measure on the first n integers. Also we set $\chi_i(x) = 1$ if $x \in H_i$ and zero otherwise, $i = 0, \dots, n-1$.

7.1.2. Approximating unstable curves by images of curves at hyperbolic times. We now observe that since $\eta(F^u(x_0, \delta_2)) > 0$ and x_0 is a density point of $\eta \mid F^u(x_0, \delta_2)$, then $\omega_\rho^n(F^u(x_0, \delta_2)) \geq c$ for some constant $c > 0$ for all big enough n . If we assume that $\delta_2 < \rho$, which poses no restriction, then we see that the cu -curves from $D_{j,\rho}$ intersecting $F^u(x_0, \delta_2)$ will cross this horizontal strip when we restore their original size. Thus the leaves $\bigcup_{j=0}^{\lfloor n/r_0 \rfloor - 1} D_j$ in the support of ω_ρ^n which intersect $F^u(x_0, \delta_2)$ cross this strip. Given any sequence γ^{n_k} of leaves in D_{n_k} crossing $F^u(x_0, \delta_2)$ with $n_1 < n_2 < n_3 < \dots$, then there exists a C^1 -limit leaf γ^∞ also crossing $F^u(x_0, \delta_2)$, by the Ascoli-Arzelà Theorem. We claim that this leaf coincides with the unstable manifold of its points, i.e. $\gamma^\infty = W^u(x, \Sigma)$ for all $x \in \gamma^\infty$. This shows that the accumulation curves γ^∞ are defined independently of the chosen sequence γ^{n_k} of curves in D_n .

To prove the claim let us fix $l > 0$ and take a big k so that $n_k \gg l$. We note that for any distinct $x, y \in \gamma^\infty$ there are $x_k, y_k \in \gamma^{n_k}$ such that $(x_k, y_k) \rightarrow (x, y)$ when $k \rightarrow \infty$. Then for x_k, y_k there exists a neighborhood V_{n_k} of a point $z \in \gamma$ such that $\gamma^{n_k} = R_0^{n_k}(V_{n_k})$ and $n_k = r^h(z)$.

We take $j \geq 1$ such that $r^j(z) \leq n_k - l < r^{j+1}(z)$. Setting $p = r^{j+1}(z) - (n_k - l)$ we can now write for some $u, v \in V_{n_k}$

$$\begin{aligned} d(x_k, y_k) &= d\left(R^{h-j+1}(R_0^p(u)), R^{h-j+1}(R_0^p(v))\right) \geq \frac{e^{\frac{c}{4}(h-j+1)}}{\beta_0} \cdot d(R_0^p(u), R_0^p(v)) \\ &\geq \frac{c_0}{\beta_0} e^{\frac{c}{4}(h-j+1)} \cdot d(u, v), \end{aligned}$$

where $c_0 > 0$ is a lower bound (13) for the least expansion of R_0 along the center-unstable cone, see Subsection 4.2. This shows that

$$d(R_0^{-l}(x_k), R_0^{-l}(y_k)) \leq \frac{\beta_0}{c_0} e^{-\frac{c}{4} \frac{h-j+1}{l}} \cdot d(x_k, y_k) \leq \frac{\beta_0}{c_0} e^{-\frac{c}{4} \cdot \frac{l}{r_0}} \cdot d(x_k, y_k).$$

By Proposition 7.5 and the C^1 -smoothness of R_0 away from Γ we get

$$d(R_0^{-l}(x), R_0^{-l}(y)) \leq \frac{\beta_0}{c_0} e^{-\frac{c}{4} \cdot \frac{l}{r_0}} \cdot d(x, y)$$

letting $k \rightarrow \infty$. Hence y is in the unstable manifold of x with respect to R_0 in $F(x_0, \delta)$, thus $y \in W^u(x, \Sigma)$ by construction, see the beginning of Section 7. This proves the claim.

We define \mathcal{F}_∞^u to be the family of all leaves γ^∞ obtained as C^1 accumulation points of leaves in

$$\mathcal{F}_n^u = \{\xi \in \cup_{j=0}^{n-1} D_j : \xi \text{ crosses } F^u(x_0, \delta_2)\}.$$

We note that $\mathcal{F}_\infty^u \subset \mathcal{F}^u(x_0, \delta_2)$. Since for all n we have $\omega_0^n \geq \omega_p^n$ and so $\omega_0^n(\cup \mathcal{F}_n^u) > c$, we get that $\eta(\cup \mathcal{F}_n^u) \geq c$. By definition of \mathcal{F}_n^u and by (28) we see that $\omega_0^n \lfloor F_n^u$ disintegrates along the partition \mathcal{F}_n^u of $F_n^u = \cup \mathcal{F}_n^u$ into measures ω_ξ^n having density with respect to λ_ξ uniformly bounded from above and below, for almost every $\xi \in \mathcal{F}_n^u$.

To take advantage of this in order to prove Proposition 7.1 we consider a sequence of increasing partitions $(\mathcal{V}_k)_{k \geq 1}$ of $W^s(x_0, \Sigma)$ whose diameter tends to zero. This defines a sequence \mathcal{P}_k of partitions of $\tilde{\mathcal{F}} = \cup_{0 \leq n \leq \infty} \mathcal{F}_n^u$ as follows: we fix $k \geq 1$ and say that two elements $\xi \in \mathcal{F}_i^u, \xi' \in \mathcal{F}_j^u, 0 \leq i, j \leq \infty$ are in the same atom of \mathcal{P}_k when both intersect $W^s(x, \Sigma)$ in the same atom of \mathcal{V}_k and either $i, j \geq k$ or $i = j < k$.

If q is the projection $q: \tilde{\mathcal{F}} \rightarrow W^s(x_0, \Sigma)$ given by the transversal intersection $\xi \cap W^s(x_0, \Sigma)$ for all $\xi \in \tilde{\mathcal{F}}$, then $\tilde{\mathcal{F}}$ can be identified with a subset of the real line. Thus we may assume without loss that the union $\partial \mathcal{P}_k$ of the boundaries of \mathcal{P}_k satisfies $\eta(\partial \mathcal{P}_k) = \hat{\eta}(\partial \mathcal{P}_k) = 0$ for all $k \geq 1$, by suitably choosing the sequence \mathcal{V}_k .

7.1.3. Upper and lower bounds for densities. Given $\zeta \in \tilde{\mathcal{F}}$ we write $p: F^u(x_0, \delta_2) \rightarrow \zeta$ the projection along stable leaves and ω for ω_0 . Writing $\mathcal{P}_k(\zeta)$ for the atom of \mathcal{P}_k which contains ζ , then since $\mathcal{P}_k(\zeta)$ is a union of leaves, for any given Borel set $B \subset \zeta$ and $n \geq 1$

$$(31) \quad \omega^n(\mathcal{P}_k(\zeta) \cap p^{-1}(B)) = \int \omega_\xi^n(\mathcal{P}_k(\zeta) \cap p^{-1}(B)) d\hat{\omega}^n(\xi)$$

through disintegration, where $\hat{\omega}^n$ is the measure on $\tilde{\mathcal{F}}$ induced by ω^n . Moreover by (28) and because each curve in $\tilde{\mathcal{F}}$ crosses $F^u(x_0, \delta_2)$

$$(32) \quad \frac{1}{\kappa_1 \kappa_2} \cdot \lambda_\zeta(B) \leq \frac{1}{\kappa_1} \cdot \lambda_\xi(p^{-1}(B)) \leq \omega_\xi^n(\mathcal{P}_k(\zeta) \cap p^{-1}(B)) \leq \kappa_1 \cdot \lambda_\xi(p^{-1}(B)) \leq \kappa_1 \kappa_2 \cdot \lambda_\zeta(B)$$

for all $n, k \geq 1$ and $\hat{\omega}^n$ -a.e. $\xi \in \tilde{\mathcal{F}}$, where $\kappa_2 > 0$ is a constant such that

$$\frac{1}{\kappa_2} \cdot \lambda_\zeta \leq \lambda_\xi \leq \kappa_2 \cdot \lambda_\zeta \quad \text{for all } \xi \in \tilde{\mathcal{F}},$$

which exists since the angle between the stable leaves in any $\Sigma \in \Xi$ and any cu -curve is bounded from below. Finally letting $\zeta \in \mathcal{F}_\infty^u$ and choosing B such that $\eta(\partial p^{-1}(B)) = 0$ (which poses no restriction), assuming that $\eta(\partial(\mathcal{P}_k(\zeta) \cap p^{-1}(B))) = 0$ we get from (31) and (32) for all $k \geq 1$

$$(33) \quad \frac{1}{\kappa_1 \kappa_2} \cdot \lambda_\zeta(B) \cdot \hat{\eta}(\mathcal{P}_k(\zeta)) \leq \eta(\mathcal{P}_k(\zeta) \cap p^{-1}(B)) \leq \kappa_1 \kappa_2 \cdot \lambda_\zeta(B) \cdot \hat{\eta}(\mathcal{P}_k(\zeta))$$

by the weak* convergence of ω^n to η . Thus to conclude the proof we are left to check that $\eta(\partial(\mathcal{P}_k(\zeta) \cap p^{-1}(B))) = 0$. For this we observe that $\mathcal{P}_k(\zeta) \cap p^{-1}(B)$ can be written as the product $q(\mathcal{P}_k(\zeta)) \times B$. Hence the boundary is equal to

$$(\partial q(\mathcal{P}_k(\zeta)) \times B) \cup (q(\mathcal{P}_k(\zeta)) \times \partial B) \subset q^{-1}(\partial q(\mathcal{P}_k(\zeta))) \cup p^{-1}(B)$$

and the right hand side has η -zero measure by construction.

This completes the proof of Proposition 7.1 since we have $\{\zeta\} = \bigcap_{k \geq 1} \mathcal{P}_k(\zeta)$ for all $\zeta \in \tilde{\mathcal{F}}$ and, by the Theorem of Radon-Nikodym, the bounds in (33) imply that the disintegration of $\eta \upharpoonright \cup \mathcal{F}_\infty^u$ along the curves $\zeta \in \mathcal{F}_\infty^u$ is absolutely continuous with respect to Lebesgue measure along these curves and with uniformly bounded densities from above and from below.

7.1.4. The support covers the whole attractor. Finally to conclude that $\text{supp}(\mu) = \Lambda$ it is enough to show that $\text{supp}(\mu)$ contains some cu -curve $\gamma: (a, b) \rightarrow \Sigma$ in some subsection $\Sigma \in \Xi$. Indeed, letting $x_0 \in \Lambda \cap \Sigma$ be a point of a forward dense regular X -orbit and fixing $c \in (a, b)$ and $\varepsilon > 0$ such that $a < c - \varepsilon < c + \varepsilon < b$, then for any $\rho > 0$ there exists $t > 0$ satisfying $\text{dist}(\gamma(c), X_t(x_0)) < \rho$. Since $W^s(X_t(x_0), \Sigma) \pitchfork (\gamma \mid (c - \varepsilon, c + \varepsilon)) = \{z\}$ (because γ is a cu -curve in Σ and $\rho > 0$ can be made arbitrarily small, where \pitchfork means transversal intersection), then, by the construction of the adapted cross-section Σ (see Section 2), this means that $z \in W^s(X_t(x_0))$. Hence the ω -limit sets of z and x_0 are equal to Λ . Thus $\text{supp}(\mu) \supseteq \Lambda$ because $\text{supp}(\mu)$ is X -invariant and closed, and $\Lambda \supseteq \text{supp}(\mu)$ because Λ is an attracting set.

We now use (33) to show that $\hat{\eta}$ -almost every $\gamma \in \tilde{\mathcal{F}}$ is contained in $\text{supp}(\eta)$, which is contained in $\text{supp}(\mu)$ by the construction of μ from η in Section 6. In fact, $\hat{\eta}$ -almost every $\zeta \in \tilde{\mathcal{F}}$ is a density point of $\hat{\eta} \upharpoonright \tilde{\mathcal{F}}$ and so for any one ζ of these curves we have $\hat{\eta}(\mathcal{P}_k(\zeta)) > 0$ for all $k \geq 1$. Fixing $z \in \zeta$ and choosing $\varepsilon > 0$ we may find $k \geq 1$ big enough and a small enough open neighborhood B of z in ζ such that

$$\mathcal{P}_k(\zeta) \cap p^{-1}(B) \subset B(z, \varepsilon) \cap \Sigma \quad \text{and} \quad \eta(\mathcal{P}_k(\zeta) \cap p^{-1}(B)) > 0,$$

by the left hand side inequality in (33). Since $\varepsilon > 0$ and $z \in \zeta$ where arbitrarily chosen, this shows that $\zeta \in \text{supp}(\eta) \subset \text{supp}(\mu)$ and completes the proof of Theorem C.

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