A BERNSTEIN-TYPE THEOREM FOR RIEMANNIAN MANIFOLDS WITH A KILLING FIELD

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Abstract. The classical Bernstein theorem asserts that any complete minimal surface in Euclidean space $\mathbb{R}^3$ that can be written as the graph of a function on $\mathbb{R}^2$ must be a plane. In this paper, we extend Bernstein's result to complete minimal surfaces in (maybe non complete) ambient spaces of non-negative Ricci curvature carrying a Killing field. This is done under the assumption that the sign of the angle function between a global Gauss map and the Killing field remains unchanged along the surface. In fact, our main result only requires the presence of a homothetic Killing field.

1. Introduction

The classical Bernstein theorem asserts that any complete minimal surface in Euclidean space $\mathbb{R}^3$ that can be written as the graph of a function on $\mathbb{R}^2$ must be a plane. H. Rosenberg observed in [16] that the following extension holds:

> *Any entire minimal graph in* $\mathbb{M}^2 \times \mathbb{R}$ *over a complete two-dimensional Riemannian manifold* $\mathbb{M}^2$ *with non-negative Gaussian curvature is a totally geodesic surface.*

According to his reasoning this follows since such a graph is necessarily stable, and a well-known result of Schoen [19] proves that a complete stable minimal surface in any three-dimensional Riemannian manifold with non-negative Ricci curvature must be totally geodesic

If $\mathbb{M}^2$ is complete there is an abundance of complete totally geodesic surfaces in $\mathbb{M}^2 \times \mathbb{R}$. First, there are the slices $\mathbb{M}^2 \times \{t\}$ for any $t \in \mathbb{R}$ that are stable. Then, there are the cylinders $\{\gamma\} \times \mathbb{R}$ for any complete geodesic γ in $\mathbb{M}^2$ and, as seen below, depending on γ these may or may not be stable. Slices are (entire) graphs over $\mathbb{M}^2$ and cylinders certainly not. But surfaces in both classes share the property that the

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angle function between the Gauss map and the (Killing) vector field $T = \partial/\partial t$ does not change sign.

In this paper, we further extend Bernstein's result to complete minimal surfaces in (maybe non complete) ambient spaces of non-negative Ricci curvature carrying a Killing field. This is done under the assumption that the sign of the angle function between a global Gauss map and the Killing field remains unchanged along the surface. In fact, our main result only requires the presence of a homothetic Killing field.

Recall that a vector field T on a Riemannian manifold $\mathbb{N}$ is called a *conformal Killing field* if the Lie derivative of the metric tensor $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\mathbb{N}}$ with respect to T satisfies $\mathcal{L}_T\langle\ ,\ \rangle = 2\phi\langle\ ,\ \rangle$ for some function $\phi \in \mathcal{C}^\infty(\mathbb{N})$. Equivalently, for any $U, V \in T\mathbb{N}$ we have that

$$\langle\bar{\nabla}_U T, V\rangle + \langle\bar{\nabla}_V T, U\rangle = 2\phi\langle U, V\rangle,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection in $\mathbb{N}^3$. Then T is called a *homothetic Killing field* if the function ϕ is constant, and just a Killing field whenever that constant vanishes.

In the following and main result in this paper, let $\mathbb{N}^3$ denote a three-dimensional (maybe non complete) Riemannian manifold endowed with a homothetic Killing field T. Then $\Sigma = \Sigma^2$ is assumed to be a *two-sided* minimal surface in $\mathbb{N}^3$. The latter condition means that there is a globally defined unit normal vector field η and, therefore, the function $\Theta \in \mathcal{C}^\infty(\Sigma)$ given by

$$\Theta(p) = \langle\eta(p), T(p)\rangle$$

is also globally defined. In the sequel, we denote by $K_{\mathbb{N}}$ and $\mathrm{Ric}_{\mathbb{N}}$ the sectional and Ricci curvature of $\mathbb{N}^3$, respectively.

Theorem 1. *Let Σ be a two sided complete minimal surface in $\mathbb{N}^3$ such that the function Θ does not change sign. Then either Σ is invariant by the one-parameter subgroup of homotheties generated by T or it is stable. In the latter case, we have:*

a) *If $\mathrm{Ric}_{\mathbb{N}} \geq 0$ then Σ is totally geodesic.*
b) *If $K_{\mathbb{N}} \geq 0$ then also Θ is constant and $\mathrm{Ric}_{\mathbb{N}}(\eta) = 0$ everywhere.*

In the preceding result the surface Σ may be invariant by T and, simultaneously, stable. For instance, in [17, Theorem 1.1] it was shown that a properly embedded totally geodesic surface of the form $\gamma \times \mathbb{R}$ in a Riemannian product $\mathbb{M}^2 \times \mathbb{R}$ with $\mathbb{M}^2$ compact is stable only if γ is a simply closed stable geodesic.

Theorem 1 can be applied to warped product manifolds $\mathbb{N}^3 = \mathbb{I}\times_t\mathbb{M}^2$, where $\mathbb{I}$ is an open interval in $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{N}^3$ is endowed with the Riemannian metric

$$\langle\ ,\ \rangle_{\mathbb{N}} = \pi_{\mathbb{I}}^*(dt^2) + t^2\pi_{\mathbb{M}}^*(\langle\ ,\ \rangle_{\mathbb{M}}).$$

Here $\pi_{\mathbb{I}}$ and $\pi_{\mathbb{M}}$ denote the projection onto the first and second factor of $\mathbb{N}^3$, respectively. It is easy to see that $t\partial/\partial t$ is a homothetic Killing field, as required by our Theorem 1. Moreover, the condition $\mathrm{Ric}_{\mathbb{N}} \geq 0$ in this case becomes $K_{\mathbb{M}} \geq 1$.

Any positive function $u \in \mathcal{C}^\infty(\Omega)$ defined on a domain $\Omega \subset \mathbb{M}^2$ determines a *radial graph* $G(u)$ over Ω in $\mathbb{R}_+ \times_t \mathbb{M}^2$ given by $p \in \Omega \mapsto (u(p), p)$. In particular, we have the following consequence of Theorem 1.

Corollary 2. *Let Σ be a complete minimal radial graph in $\mathbb{N}^3 = \mathbb{R}_+ \times_t \mathbb{M}^2$ over a domain $\Omega \subset \mathbb{M}^2$. If $K_{\mathbb{M}} \geq 1$ then Σ is totally geodesic.*

In particular, the only complete minimal radial graphs over a domain in $\mathbb{S}^2$ in $\mathbb{R}^3$ are the planes defined over an open hemisphere of the sphere. This Bernstein type result for radial graphs in $\mathbb{R}^3$ also follows from the results in [15].

Theorem 1 can also be applied to warped product manifolds $\mathbb{N}^3 = \mathbb{M}^2 \times_\varrho \mathbb{R}$, that is, the product manifold $\mathbb{M}^2 \times \mathbb{R}$ endowed with the Riemannian metric

$$\langle\ ,\ \rangle_{\mathbb{N}} = \pi_{\mathbb{M}}^*(\langle\ ,\ \rangle_{\mathbb{M}}) + \varrho^2 \pi_{\mathbb{R}}^*(dt^2)$$

where $\varrho \in \mathcal{C}^\infty(\mathbb{M})$. In this case $T = \partial/\partial t$ is a Killing field in $\mathbb{N}^3$, and the condition $\mathrm{Ric}_{\mathbb{N}} \geq 0$ is simply $K_{\mathbb{M}} \geq 0$.

It is a standard fact that $\mathbb{N}^3 = \mathbb{M}^2 \times_\varrho \mathbb{R}$ is complete if and only if $\mathbb{M}^2$ is complete. If $\mathbb{N}^3$ is complete and $K_{\mathbb{N}} \geq 0$, then ϱ must be constant. This is because $K_{\mathbb{M}} \geq 0$ and

$$\mathrm{Ric}_{\mathbb{N}}(T, T) = -\varrho \Delta_{\mathbb{M}} \varrho \geq 0,$$

where $\Delta_{\mathbb{M}}$ is the Laplacian on $\mathbb{M}^2$. Thus ϱ is a positive superharmonic function on $\mathbb{M}^2$. Since $\mathbb{M}^2$ is complete and $K_{\mathbb{M}} \geq 0$, then $\mathbb{M}^2$ is parabolic and hence ϱ is constant.

In the following result, by a *cylinder* in $\mathbb{N}^3 = \mathbb{M}^2 \times_\varrho \mathbb{R}$ over a unit speed curve $\gamma\colon \mathbb{R} \to \mathbb{M}^2$ we mean the equivariant surface in $\mathbb{M}^2 \times_\varrho \mathbb{R}$ obtained by acting on γ the one-parameter subgroup of vertical translations. A cylinder is minimal if $k_g = \varrho\, \partial\varrho/\partial\nu$, where $k_g = \langle \nabla_{\gamma'}\gamma', \nu\rangle$ is the geodesic curvature of γ and ν the unit normal field to γ in $\mathbb{M}^2$. In particular, a cylinder is totally geodesic if and only if γ is a geodesic and $\partial\varrho/\partial\nu = 0$.

Corollary 3. *Let Σ be a two sided complete minimal surface in $\mathbb{N}^3 = \mathbb{M}^2 \times_\varrho \mathbb{R}$, where $\mathbb{M}^2$ is not necessarily complete. Assume that $K_{\mathbb{N}} \geq 0$ and that Θ does not change sign. Then, either*

(i) *Σ is a minimal cylinder, or*

(ii) Σ *is totally geodesic and* ϱ *is constant. Moreover, if* $K_{\mathbb{M}}(q) > 0$ *at a point* $q \in \pi_{\mathbb{M}}(\Sigma)$, *then* Σ *is a slice over a necessarily complete* $\mathbb{M}^2$.

If $\mathbb{M}^2 = \mathbb{R}^2$ and $\varrho = 1$ (i.e., $\mathbb{N}^3 = \mathbb{R}^3$) then any affine plane other than a horizontal or vertical one is an example of a totally geodesic surface with constant Θ that is neither a cylinder nor a slice. Moreover, the assumption $K_{\mathbb{M}} \geq 0$ is necessary since in $\mathbb{N}^3 = \mathbb{H}^2 \times \mathbb{R}$ there exist non-trivial entire minimal graphs (see [7] or [13]).

A given function $u \in \mathcal{C}^\infty(\mathbb{M})$ determines an entire (normal geodesic) graph Γ_u over $\mathbb{M}^2$ in $\mathbb{M}^2 \times \mathbb{R}$ given by

$$p \in \mathbb{M}^2 \mapsto \Gamma_u(p) = (p, u(p)).$$

In the special case of entire minimal graphs, the following result is slightly more general than Rosenberg's in [16]. Its proof makes use of a beautiful argument due to Salavessa [18] that shows that an entire constant mean curvature graph over $\mathbb{M}^2$ must be necessarily minimal if the Cheeger constant of $\mathbb{M}^2$ vanishes.

Theorem 4. *Let* $\mathbb{M}^2$ *be a complete surface with Gauss curvature* $K_{\mathbb{M}} \geq 0$.

(i) *Any entire constant mean curvature graph in* $\mathbb{M}^2 \times \mathbb{R}$ *is totally geodesic.*

(ii) *If, in addition,* $K_{\mathbb{M}}(q) > 0$ *at some point* $q \in \mathbb{M}^2$ *then the graph is a slice.*

In the special case of $\mathbb{N}^3 = \mathbb{M}^2 \times \mathbb{R}$, we give a proof of Corollary 3 by a direct elementary argument; see Proposition 7 below. The proof builds on an idea taken from Chern's proof [6] of classical Bernstein theorem, and is not only interesting by itself but also allows to prove the following half-space result for minimal graphs.

Theorem 5. *Let* $\mathbb{M}^2$ *be a complete surface satisfying*

$$\int_{\mathbb{M}} K_{\mathbb{M}}^- \, dA_{\mathbb{M}} < +\infty, \quad \text{where} \quad K_{\mathbb{M}}^-(q) = \max\{-K_{\mathbb{M}}(q), 0\}.$$

Then any entire minimal graph in a half-space $\mathbb{M}^2 \times [0, +\infty)$ *is a slice.*

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2. Proofs and further results

Before giving the proof of Theorem 1 we need to extend Proposition 1 in [10].

Proposition 6. *Let $\mathbb{N}^{n+1}$ be a $(n+1)$-dimensional Riemannian manifold endowed with a conformal Killing field T. For a two-sided hypersurface Σ^n in $\mathbb{N}^{n+1}$ the Laplacian of $\Theta \in \mathcal{C}^\infty(\Sigma)$ given by $\Theta = \langle \eta, T\rangle$ is*

$$\Delta\Theta = -n\langle \nabla H, T\rangle - (\|A\|^2 + \mathrm{Ric}_{\mathbb{N}}(\eta))\Theta - n(H\phi + \partial\phi/\partial\eta).$$

Here ∇H is the gradient of the mean curvature function H with respect to a unit normal vector field η and $\|A\|$ the norm of the second fundamental form of Σ^n.

Proof: Set $T = T^\top + \Theta\eta$, where $(\ )^\top$ denotes the tangential component of a vector field in $T\mathbb{N}$ along Σ^n. Being T conformal, we have

$$X(\Theta) = -\langle AX, T^\top\rangle - \langle \bar\nabla_\eta T, X\rangle$$

for any $X \in T\Sigma$. It follows that

$$\nabla\Theta = -AT^\top + (\bar\nabla_\eta T)^\top. \tag{1}$$

The Codazzi equation for Σ^n is

$$(\nabla_X A)Y = (\nabla_Y A)X - (R_{\mathbb{N}}(X,Y)\eta)^\top \tag{2}$$

where $R_{\mathbb{N}}$ is the curvature tensor of $\mathbb{N}^{n+1}$. From (2) for $Y = T^\top$ and

$$\nabla_X T^\top = (\bar\nabla_X T)^\top + \Theta AX \tag{3}$$

we easily obtain that

$$\mathrm{div}(AT^\top) = \mathrm{tr}(\nabla_{T^\top} A) + nH\phi + \Theta\|A\|^2 - \mathrm{Ric}_{\mathbb{N}}(T^\top, \eta),$$

where div denotes the divergence on Σ^n. Using that the trace commutes with the covariant derivative, we have

$$\mathrm{div}(AT^\top) = n\langle \nabla H, T\rangle + nH\phi + \Theta\|A\|^2 - \mathrm{Ric}_{\mathbb{N}}(T^\top, \eta). \tag{4}$$

On the other hand, a straightforward computation yields

$$\mathrm{Ric}_{\mathbb{N}}(T,\eta) = \mathrm{div}((\bar\nabla_\eta T)^\top) - n\partial\phi/\partial\eta, \tag{5}$$

and the result follows from (1), (4) and (5). ∎

We are now in condition to prove our main result.

Proof of Theorem 1. Choose the orientation of η such that $\Theta \geq 0$. Then Θ is a Jacobi field since the Jacobi operator $J = \Delta + \|A\|^2 + \mathrm{Ric}_{\mathbb{N}}$ satisfies

$$J\Theta = \Delta\Theta + (\|A\|^2 + \mathrm{Ric}_{\mathbb{N}}(\eta))\Theta = 0 \tag{6}$$

by Proposition 6. Suppose that $\Theta(p_0) = 0$ at some point $p_0 \in \Sigma$. For a sufficiently small neighborhood U of p_0 the first eigenvalue for the Dirichlet problem of J satisfies $\lambda_1(J) > 0$ on U. Therefore, by Theorem 1 in [9] there exists a positive solution g of $Jg = 0$ on U. Setting $\Theta = \alpha g$, then $\alpha \geq 0$ and it follows from $J\Theta = 0$ that $\mathrm{div}(g^2\nabla\alpha) = 0$

on U. The maximum principle applies to this equation since it is of divergence form. Thus, α attains its minimum at an interior point of U. Then α, and hence Θ, must vanish. We conclude that either $\Theta = 0$ or $\Theta > 0$ on Σ.

If $\Theta = 0$ the surface is invariant by the one parameter subgroup of homotheties generated by T. Otherwise Θ is a positive Jacobi field, and the surface is stable by a well-known result due to Fischer-Colbrie and Schoen [9, Corollary 1]. For the sake of completeness we give the following standard argument that proves this fact in our case. An arbitrary function ψ with compact support on Σ can be written as $\psi = \varphi\Theta$ where φ has compact support. From $J\Theta = 0$ we have

$$J\psi = \Theta\Delta\varphi + 2\langle\nabla\varphi, \nabla\Theta\rangle$$

and

$$\psi J\psi = \Theta^2\varphi\Delta\varphi + \frac{1}{2}\langle\nabla\varphi^2, \nabla\Theta^2\rangle = \mathrm{div}(\Theta^2\varphi\nabla\varphi) - \Theta^2\|\nabla\varphi\|^2.$$

Thus

$$-\int_\Sigma \psi J\psi dA_\Sigma = \int_\Sigma \Theta^2\|\nabla\varphi\|^2 dA_\Sigma \geq 0,$$

and the surface is stable.

If the surface is stable and the Ricci curvature satisfies $\mathrm{Ric}_{\mathbb{N}} \geq 0$, then a result of Schoen [19] implies that Σ is totally geodesic. Moreover, if $K_{\mathbb{N}} \geq 0$, then Σ is totally geodesic and, from the Gauss equation, has non-negative Gauss curvature. By a classical result by Ahlfors [1] and Blanc-Fiala-Huber [8] a complete surface of non-negative Gaussian curvature is parabolic in the sense that any non-negative superharmonic function on the surface must be constant. Then, it follows from (6) that Θ is constant, and hence, $\mathrm{Ric}_{\mathbb{N}}(\eta) = 0$ unless $\Theta = 0$. ∎

Proof of Corollary 3. It is a standard fact that the covariant derivative in $\mathbb{N}^3$ satisfies

$$\bar\nabla_Z T = \varrho^{-1}\langle\nabla\varrho, Z\rangle T \quad \text{and} \quad \bar\nabla_T T = -\varrho\bar\nabla\varrho, \tag{7}$$

where $Z \in T\mathbb{M}$ and $\bar\nabla\varrho$ denotes the gradient of ϱ as a function in $\mathbb{N}^3$. Hence $\bar\nabla\varrho$ is the lift of the gradient of ϱ in $\mathbb{M}^2$. In particular, it follows easily that $T = \partial/\partial t$ is a Killing field in $\mathbb{N}^3$.

Theorem 1 applies, and thus either Σ is a minimal cylinder or Σ is totally geodesic. In the latter case, we have that Θ is a nonzero constant and $\mathrm{Ric}_{\mathbb{N}}(\eta) = 0$ everywhere. We prove now that in the latter case ρ =constant. Given any $Y \in T\Sigma$, we set

$$Y = Z + \varrho^{-2}\langle Y, T\rangle T,$$

where $Z \in T\mathbb{M}$. We have,

$$Y(\Theta) = \langle \eta, \bar{\nabla}_Y T \rangle = \varrho^{-1}(\Theta \langle \bar{\nabla} \varrho, Y \rangle - \langle T, Y \rangle \langle \bar{\nabla} \varrho, \eta \rangle).$$

Thus the gradient $\nabla\Theta$ of Θ along Σ is

$$\nabla\Theta = \varrho^{-1} \left(\Theta \bar{\nabla} \varrho - \langle \bar{\nabla} \varrho, \eta \rangle T \right).$$

In particular,

$$\|\nabla\Theta\|^2 = \varrho^{-2} \left(\Theta^2 \|\bar{\nabla} \varrho\|^2 + \varrho^2 \langle \bar{\nabla} \varrho, \eta \rangle^2 \right).$$

Since Θ is constant and does not vanish it follows that ϱ is constant. Therefore, if at some point $K_{\mathbb{M}}(p_0) > 0$, then $\mathrm{Ric}_{\mathbb{N}}(\eta(p_0)) = 0$ is possible only if $\Theta = 1$. ∎

Proof of Theorem 4. Recall that the differential equation for the mean curvature function H is

$$\mathrm{div}_{\mathbb{M}} \left(\frac{Du}{\sqrt{1 + \|Du\|^2}} \right) = 2H, \tag{8}$$

where Du denotes the gradient of $u \in \mathcal{C}^\infty(\mathbb{M})$ and $\mathrm{div}_{\mathbb{M}}$ the divergence on $\mathbb{M}^2$.

It suffices to show that $H = 0$ and the proof follows from Corollary 3. If $\mathbb{M}^2$ is compact this is a consequence of the divergence theorem applied to (8). In the non-compact case, we first argue that its Cheeger constant $\mathfrak{h}(\mathbb{M})$ vanishes. Recall that

$$\mathfrak{h}(\mathbb{M}) = \inf_D \frac{\mathrm{length}\,(\partial D)}{\mathrm{area}\,(D)}$$

where $D \subset \mathbb{M}^2$ is any compact domain with smooth boundary. Let $B_p(r) \subset \mathbb{M}^2$ denote the geodesic disk of center p and radius r. Since $K_{\mathbb{M}} \geq 0$ we know from Theorem 1.1 of [5] that the first eigenvalue of the Dirichlet problem on $B_p(r)$ satisfies

$$\lambda_1(B_p(r)) \leq \frac{c}{r^2}, \quad 0 < r < +\infty,$$

for a positive constant c. On the other hand, by a result of Cheeger [4] (cf. Theorem 3 p. 95 in [3]) we have that $\lambda_1(B_p(r)) \geq \mathfrak{h}^2(B_p(r))/4$. We obtain that

$$\mathfrak{h}^2(\mathbb{M}) \leq \mathfrak{h}^2(B_p(r)) \leq 4\lambda_1(B_p(r)) \leq \frac{4c}{r^2}$$

for any $0 < r < +\infty$, and hence $\mathfrak{h}(\mathbb{M}) = 0$.

To conclude the proof we use an argument due to Salavessa [18] to show that if $\mathbb{M}^2$ satisfies that $\mathfrak{h}(\mathbb{M}) = 0$ then any entire graph in $\mathbb{M}^2 \times \mathbb{R}$ with constant mean curvature H is necessarily minimal. If $u \in \mathcal{C}^\infty(\mathbb{M})$ determines an arbitrary entire graph in $\mathbb{M}^2 \times \mathbb{R}$, then integrating (8) over a compact domain $D \subset \mathbb{M}^2$ and using the divergence theorem we have

$$2\min_D H\,\mathrm{area}\,(D) \leq 2\int_D H dA_{\mathbb{M}} = \oint_{\partial D} \frac{\langle Du, \nu\rangle}{\sqrt{1+\|Du\|^2}} ds \leq \mathrm{length}\,(\partial D)$$

and, similarly,

$$2\max_D H\,\mathrm{area}\,(D) \geq -\mathrm{length}\,(\partial D).$$

Therefore, for any compact domain $D \subset \mathbb{M}^2$, we have

$$\inf_{\mathbb{M}} H \leq \frac{1}{2}\frac{\mathrm{length}\,(\partial D)}{\mathrm{area}\,(D)} \quad \text{and} \quad \sup_{\mathbb{M}} H \geq -\frac{1}{2}\frac{\mathrm{length}\,(\partial D)}{\mathrm{area}\,(D)},$$

and hence,

$$\inf_{\mathbb{M}} H \leq \frac{1}{2}\mathfrak{h}\,(\mathbb{M}) \quad \text{and} \quad \sup_{\mathbb{M}} H \geq -\frac{1}{2}\mathfrak{h}\,(\mathbb{M}).$$

In particular, when $\mathfrak{h}(\mathbb{M}) = 0$ we obtain

$$\inf_{\mathbb{M}} H \leq 0 \leq \sup_{\mathbb{M}} H,$$

and if H is constant we conclude that it must vanish. ∎

The second statement in Theorem 4 says that globally defined solutions of (8) over $\mathbb{M}^2$ for constant H exist only for $H = 0$, and that they are the constant functions. The latter is no longer true if $\mathbb{M}^2$ is flat since non-horizontal planes in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ correspond to non-constant linear solutions.

For Riemannian products $\mathbb{M}^2 \times \mathbb{R}$ the following result generalizes Rosenberg's referred at the beginning of the paper. In the sequel, by the *height function* h of an immersed surface Σ in $\mathbb{M}^2 \times \mathbb{R}$ we mean the projection $h : \Sigma \to \mathbb{R}$ onto the second factor. Observe that $\nabla h = T^\top$ and that $\|\nabla h\|^2 = 1 - \Theta^2$. Besides, since T is parallel in $\mathbb{M}^2 \times \mathbb{R}$ we also have from (3) that $\nabla_X \nabla h = \Theta A X$, and hence

$$\Delta h = 2H\Theta. \tag{9}$$

Proposition 7. *Let Σ be a two-sided complete minimal surface in $\mathbb{M}^2 \times \mathbb{R}$. Assume that Θ does not change sign.*

(i) *If $K_{\mathbb{M}} \geq 0$ along $\pi_{\mathbb{M}}(\Sigma)$ then Σ is totally geodesic.*
(ii) *If, in addition, $K_{\mathbb{M}}(q) > 0$ at some point $q \in \pi_{\mathbb{M}}(\Sigma)$ then either Σ is a cylinder over a complete geodesic of $\mathbb{M}^2$, or $\mathbb{M}^2$ is necessarily complete and Σ is a slice.*

Proof: We choose η such that $\Theta \geq 0$. Since the surface is minimal, we have

$$A^2 = \frac{1}{2}\|A\|^2 I$$

where I stands for the identity map on $T\Sigma$. Then using (1) we obtain

$$\|\nabla\Theta\|^2 = \frac{1}{2}\|A\|^2(1-\Theta^2).$$

Therefore,

$$\Delta \log(1+\Theta) = \frac{\Delta\Theta}{1+\Theta} - \frac{\|\nabla\Theta\|^2}{(1+\Theta)^2} = -\frac{1}{2}\|A\|^2 - \Theta(1-\Theta)K_{\mathbb{M}}(\pi). \tag{10}$$

On the other hand, the Gauss equation gives

$$K_\Sigma = \bar{K}_\Sigma + \det A$$

where K_Σ denotes the Gauss curvature of (Σ, ds^2) and $\bar{K}_\Sigma$ the sectional curvature in $\mathbb{M}^2\times\mathbb{R}$ of the plane tangent to Σ. The latter is given by

$$\bar{K}_\Sigma = K_{\mathbb{M}}(\pi)(1-\|T^\top\|^2) = \Theta^2 K_{\mathbb{M}}(\pi).$$

Thus, the Gauss equation becomes

$$K_\Sigma = \Theta^2 K_{\mathbb{M}}(\pi) - \frac{1}{2}\|A\|^2,$$

and (10) reduces to

$$\Delta\log(1+\Theta) = K_\Sigma - \Theta K_{\mathbb{M}}(\pi). \tag{11}$$

Next we introduce on Σ the complete metric $d\tilde{s}^2 = (1+\Theta)^2 ds^2$. It is a standard fact that the Gauss curvature $\tilde{K}$ of $(\Sigma, d\tilde{s}^2)$ is given by

$$(1+\Theta)^2\tilde{K} = K_\Sigma - \Delta\log(1+\Theta). \tag{12}$$

We conclude from (11) and (12) that

$$\tilde{K} = \frac{\Theta}{(1+\Theta)^2}K_{\mathbb{M}}(\pi). \tag{13}$$

In particular, if $K_{\mathbb{M}} \geq 0$ on $\pi(\Sigma)$ then $\tilde{K} \geq 0$ on Σ. From a classical result by Ahlfors [1] and Blanc-Fiala-Huber [8] a complete surface of non-negative Gaussian curvature is parabolic in the sense that any non-negative superharmonic function on the surface must be constant. Since superharmonic is preserved under a conformal change of metric, then $(\Sigma, d\tilde{s}^2)$ and (Σ, ds^2) are both parabolic.

We have that $\log(1+\Theta) \geq 0$, and from (10) we know that

$$\Delta\log(1+\Theta) = -\frac{1}{2}\|A\|^2 - \Theta(1-\Theta)K_{\mathbb{M}}(\pi) \leq 0.$$

Then $\Theta = \Theta_0$ is constant, $\|A\| = 0$ and $\Theta_0(1-\Theta_0)K_{\mathbb{M}}(\pi) = 0$. It follows that the surface is totally geodesic and, if $K_{\mathbb{M}} > 0$ somewhere on $\pi(\Sigma)$, then either $\Theta_0 = 0$ or $\Theta_0 = 1$. The case $\Theta_0 = 0$ means that T is tangent to the surface and, then, the surface must be a cylinder over a complete geodesic of $\mathbb{M}^2$. If $\Theta_0 = 1$, then $\nabla h = 0$ and it follows that the surface is a slice over a necessarily complete $\mathbb{M}^2$. ∎

If Σ is compact a stronger version of Corollary 3 holds true without assumptions neither on the immersion nor on the Gauss curvature of $\mathbb{M}^2$. In fact, if Σ is a compact minimal surface in $\mathbb{M}^2 \times \mathbb{R}$, then $\mathbb{M}^2$ is necessarily compact and Σ a slice since its height function must be harmonic on Σ, and thus constant.

Proof of Theorem 5. As before, we orient the graph Γ_u of u such that $\Theta > 0$. If ds^2 denotes the complete metric on $\mathbb{M}^2$ induced by Γ_u, then (13) becomes

$$\tilde{K} = \frac{\Theta}{(1+\Theta)^2} K_{\mathbb{M}},$$

where $\tilde{K}$ is the Gauss curvature of the complete conformal metric $d\tilde{s}^2 = (1+\Theta)^2 ds^2$. Observe that the area elements of ds^2 and $d\tilde{s}^2$ are related by $d\tilde{A} = (1+\Theta)^2 dA$. Since $\Theta > 0$, we have

$$\tilde{K}^- d\tilde{A} = \Theta K_{\mathbb{M}}^- \, dA. \tag{14}$$

On the other hand, from

$$\eta = \frac{1}{\sqrt{1+\|Du\|^2}}\,(T - Du)$$

we obtain that

$$\Theta = \frac{1}{\sqrt{1+\|Du\|^2}}.$$

Since $dA = \sqrt{1+\|Du\|^2}\, dA_{\mathbb{M}}$, then (14) becomes $\tilde{K}^- d\tilde{A} = K_{\mathbb{M}}^- dA_{\mathbb{M}}$. Therefore,

$$\int_{\mathbb{M}} \tilde{K}^- d\tilde{A} < +\infty.$$

Then, the classical result of Huber [8, Theorem 15] (see Section 10 in [11]), implies that $(\mathbb{M}^2, d\tilde{s}^2)$ is parabolic. Hence, also $(\mathbb{M}^2, ds^2)$ is parabolic. Since the height function u is harmonic on $(\mathbb{M}^2, ds^2)$ it must be constant. ∎

The following result relates to Theorem 4 in the introduction.

Proposition 8. *Let $\mathbb{M}^2$ be a complete surface that satisfies*

$$\int_{\mathbb{M}} K_{\mathbb{M}}^- \, dA_{\mathbb{M}} < +\infty, \quad \textit{where} \quad K_{\mathbb{M}}^-(q) = \max\{-K_{\mathbb{M}}(q), 0\}.$$

Then, any entire graph Γ_u contained in a slab $\mathbb{M}^2 \times [a,b]$, $-\infty < a \leq b < +\infty$, with constant mean curvature and Gauss curvature bounded from below is a slice.

Proof: Since we are dealing with graphs we may take $\Theta > 0$. In what follows we see u as a function along Γ_u. We do not assume that the mean curvature H is constant yet. Since u and the Gauss curvature of Γ_u are both bounded from below, by Omori's lemma [14] there exists a sequence of points $\{q_j\} \in \Gamma_u$ such that

$$\lim_{j\to\infty} u(q_j) = \inf_{\Gamma} u, \quad \|\nabla u(q_j)\| < 1/j \quad \text{and} \quad \Delta u(q_j) > -1/j.$$

Thus by (1) we have

$$\|\nabla u(q_j)\|^2 = 1 - \Theta^2(q_j) < 1/j^2.$$

This implies that $\lim_{j\to+\infty} \Theta(q_j) = 1$, and using (9) that

$$\Delta u(q_j) = 2H(q_j)\Theta(q_j) > -1/j.$$

Hence, $\lim_{j\to+\infty} H(q_j) \geq 0$. Similarly, since u is also bounded from above there is a sequence of points such that $\lim_{j\to+\infty} H(p_j) \leq 0$. Thus,

$$\inf_{\Gamma} H \leq \lim_{j\to+\infty} H(p_j) \leq 0 \leq \lim_{j\to+\infty} H(q_j) \leq \sup_{\Gamma} H.$$

In particular, if H is constant we obtain that Γ_u must be minimal, and the proof follows from Theorem 5. ∎

After the statement of Theorem 1 we observed that a cylinder may not be stable. The following example shows that, in fact, the index may be infinity.

Example 9. Take a cylinder $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{S}^2 \times \mathbb{R}$ over an equator of the unit round sphere $\mathbb{S}^2$. The Jacobi operator is $J = \Delta + 1$, where Δ is the Laplacian operator on the cylinder. Since the subsets $\Omega_r = \mathbb{S}^1 \times (-r, r)$ with $r > 0$ form an exhaustion of $\mathbb{S}^1 \times \mathbb{R}$ by bounded domains with compact closure, we can compute

$$\operatorname{Ind}(\mathbb{S}^1 \times \mathbb{R}) = \lim_{r\to\infty} \operatorname{Ind}(\Omega_r).$$

For $k = 1, 2, \ldots$ the functions

$$\phi_{r,k}(x,t) = \begin{cases} \cos\frac{\pi k t}{2r}, & \text{if } k \text{ is odd} \\ \sin\frac{\pi k t}{2r}, & \text{if } k \text{ is even} \end{cases}$$

satisfy

$$\Delta\phi_{r,k} + \frac{\pi^2 k^2}{4r^2}\phi_{r,k} = 0$$

on Ω_r and $\phi_{r,k} = 0$ on $\partial\Omega_r$; that is, they are linearly independent eigenfunctions for the Dirichlet eigenvalue problem of the Laplacian on Ω_r. Thus

$$J\phi_{r,k} + \left(\frac{\pi^2 k^2}{4r^2} - 1\right)\phi_{r,k} = 0,$$

and therefore

$$\lambda_{r,k} = \frac{\pi^2 k^2}{4r^2} - 1$$

is an eigenvalue for the Dirichlet problem of J on Ω_r for every $k \geq 1$. Finally, for every $r > \pi/2$ we have that $\lambda_{r,k} < 0$ if $1 \leq k < 2r/\pi$, which implies that $\mathrm{Ind}(\Omega_r) \geq [2r/\pi]$, and, in particular, we conclude that $\mathrm{Ind}(\mathbb{S}^1 \times \mathbb{R}) = +\infty$.

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