On the density of hyperbolicity and homoclinic bifurcations for 3D-diffeomorphism in attracting regions.

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Abstract

In the present paper it is proved that given a maximal invariant attracting homoclinic class for a smooth three dimensional Kupka-Smale diffeomorphism, either the diffeomorphisms is C^1-approximated by another one exhibiting either a homoclinic tangency or a heterodimensional cycle, or it follows that the homoclinic class is conjugate to a hyperbolic set (in this case we say that the homoclinic class is "topologically hyperbolic").

We also characterize, in any dimension, the dynamics of a topologically hyperbolic homoclinic class and we describe the continuation of this homoclinic class for a perturbation of the initial system.

Moreover, we prove that, under some topological conditions, the homoclinic class is contained in a two dimensional manifold and it is hyperbolic.

1 Introduction and statements.

For a long time (mainly after Poincaré) it has been a goal of the theory of dynamical systems to describe the dynamics from the generic viewpoint, that is, to describe the dynamics of "big sets" (residual, dense, etc.) within the space of all dynamical systems.

It was briefly thought in the sixties that this could be realized by the so-called hyperbolic ones: systems with the assumption that the tangent bundle over the limit set $L(f)$ (the closure of the accumulations points of any orbit) splits into two complementary subbundles $T_{L(f)}M = E^s \oplus E^u$ so that vectors in E^s (respectively E^u) are uniformly forward (respectively backward) contracted by the tangent map Df. Under this assumption, it was proved that the limit set decomposes into a finite number of disjoint transitive sets such that the asymptotic behavior of any orbit is described by the dynamics in the trajectories in those finite transitive sets (see [S]). In other words, hyperbolic dynamics on the tangent bundle characterizes the dynamics over the manifold from a geometrical, topological and statistical point of view.

Uniform hyperbolicity was soon realized to be a less universal property than was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. The initial mechanisms to show this nonhyperbolic robustness (see [AS], [Sh]) were the existence of open sets of diffeomorphisms exhibiting hyperbolic periodic points of different stable

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indices inside a transitive set (the stable index of a hyperbolic periodic point is the number of eigenvalues with modulus smaller than one counted with multiplicity).

Related to this, is the notion of *heterodimensional cycle* where two periodic points of different indices are linked through the intersection of their stable and unstable manifolds (notice that at least one of the intersections is non-transversal; a more precise definition will follow).

In all of the above examples the underlying manifolds must have dimension at least three, so the case of surfaces was still unknown at the time. It was through the seminal works of Newhouse (see [N1], [N2], [N3]) that hyperbolicity was shown not to be dense in the space of C^r diffeomorphisms ($r \geq 2$) of compact surfaces. The underlying mechanism here was the presence of a *homoclinic tangency*: non-transversal intersection of the stable and unstable manifold of a periodic point.

These results naturally suggested the following question:

1. *What mechanisms lead to generic (meaning generic perturbation of the initial system) non-hyperbolic behavior?*

2. *Is it possible to identify the dynamical mechanism underlying any generic nonhyperbolic behavior?*

We have mentioned two basic mechanisms which are obstruction to hyperbolicity, namely *heterodimensional cycles* and *homolicinic tangencies.* In the early 80's Palis conjectured (see [P] and [PT]) that these are very common in the complement of the hyperbolic systems:

1. *Every C^r diffeomorphism of a compact manifold M can be C^r approximated by one which is hyperbolic or by one exhibiting a heterodimensional cycle or by one exhibiting a homoclinic tangency.*

2. *When M is a two-dimensional compact manifold every C^r diffeomorphism of M can be C^r approximated by one which is hyperbolic or by one exhibiting a homoclinic tangency.*

This conjecture may be thought of as a starting point for obtaining a generic description of C^r-diffeomorphisms. If it turns out to be true we may focus on the two mechanisms mentioned above in order to understand the dynamics.

To be precise, let us introduce some definitions.

A hyperbolic diffeomorphism means a diffeomorphism such that its limit set is hyperbolic. The limit set is the closure of the forward and backward accumulation points of all orbits. A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \leq C\lambda^n$$

and

$$|Df^{-n}_{/E^u(x)}| \leq C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n.

Moreover, a diffeomorphism is called Axiom A, if the non-wandering set is hyperbolic and it is the closure of the periodic points.

We recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f. A point of intersection of these manifolds is called a homoclinic point.

Definition 1 Homoclinic tangency. *We say that f exhibits a homoclinic tangency if there is a periodic point p such that there is a point $x \in W^s(p) \cap W^u(p)$ with $T_xW^s(p) + T_xW^u(p) \neq T_xM$. Given an open set V, we say that the tangency holds in V if p and x belong to V.*

The above conjecture was proved to be true for the case of surfaces and the C^1 topology (see [PS1]).

Theorem ([PS1]): *Let M^2 be a two dimensional compact manifold. Every $f \in Diff^1(M^2)$ can be C^1-approximated either by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism.*

In dimensions higher than two, the theorem stated above is false, due to another kind of homoclinic bifurcation that breaks the hyperbolicity in a robust way: the so-called heterodimensional cycles (see [D1] and [D2]).

Definition 2 Heterodimensional cycle. *We say that f exhibits a heterodimensional cycle if there are two periodic points q and p of different stable index, such that $W^u(q) \cap W^s(p) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$. Given an open set V, we say that the cycle holds in V if p, q and the points where the stable and unstable manifolds intersect belong to V. In the case that the manifold is three dimensional then q and p has stable index 1 and 2 respectively.*

It is remarkable that for a compact manifold with dimension larger than and equal to three, there are C^1-open sets of diffeomorphisms containing a dense set of diffeomorphisms exhibiting a tangency and a dense set of diffeomorphisms exhibiting a heterodimensional cycle. On the other hand, the conjecture states that the systems exhibiting either a tangency or a heterodimensional cycle are dense in the complement of the hyperbolic ones.

The present paper goes in the direction to prove the conjecture formulated by Palis in the C^1 topology for an attracting homoclinic class of a three dimensional C^2-diffeomorphisms. Observe that the conjecture is stated for the whole Limit set and recall that this set is the closure of the accumulation points of any orbit. Roughly speaking, in this paper we deal with the "attracting region of the Limit set". To be precise, first we have to introduce more definitions.

Definition 3 Homoclinic class. *Given a periodic point p, we define the homoclinic class associated to p as the closure of the transversal intersection of the stable manifold of p with the unstable manifold of p.*

Definition 4 Attracting homoclinic class. *Given a homoclinic class H_p, we say that H_p is an attracting homoclinic class if there exists an open set U such that $H_p \subset U$ and $H_p = \cap_{n>0} f^n(U)$*

Different kinds of examples of three dimensional attracting homoclinic classes have been found: the solenoid attractor, the Plykin attractor (both hyperbolic), the Henon attractor (that it can be approximated by a map exhibiting a tangency; see [BeCa], [V] and [U]), or partially hyperbolic attractors (which can be approximated by a map exhibiting a heterodimensional cycle; see [M], [BD] and [BV] for this kind of examples).

Now we can reformulate the conjectured stated by Palis for the context of an attracting homoclinic class in dimension theree : *Let $f \in Diff^2(M^3)$. Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point p, such that all the periodic points in H_p are hyperbolic. Then, holds: either i-H_p is hyperbolic; or ii- there exists g C^1-arbitrarily close to f exhibiting a homoclinic tangency in U; or iii-there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U.*

This problem is discussed in the next subsection.

1.1 "Hyperbolicity" or homoclinic bifurcations.

To prove the reformulated conjecture mentioned above, we consider two cases: either the periodic point p has index one or it has index two.

Theorem A: *Let $f \in Diff^2(M^3)$. Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point p of index one and such that all the periodic points in H_p are hyperbolic. Then, one of the following options holds:*

1. *H_p is hyperbolic;*
2. *there exists g C^1-arbitrarily close to f exhibiting a homoclinic tangency in U;*
3. *there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U.*

Observe that under the hypothesis of the previous theorem, if the systems cannot be approximated by onother one exhibiting either heterodimensional cycles or tangencies, then the homoclinic class is actually hyperbolic. In other words, the statement in this context of homoclinic classes is stronger than what is stated in the global conjecture. The previous theorem can be improved in terms of the nature of dominated splitting that the homoclinic class can exhibit and the type of homoclinic tangency that can be created by perturbation (see theorem F in section 5).

Now we consider the case that p has index two. Before formulating the corresponding theorem, we need to introduce some definitions.

Definition 5 *An f-invariant set Λ is said to have a dominated splitting, if the tangent bundle over Λ is decomposed in two invariant subbundles $T_\Lambda M = E \oplus F$, such that there exist $C > 0$ and $0 < \lambda < 1$ with the following property:*

$$|Df^n_{|E(x)}||Df^{-n}_{|F(f^n(x))}| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

If the bundle $T_\Lambda M$ is decomposed in more than two directions, i.e.: if $T_\Lambda M = \oplus_{i=1}^k E_i$ then it is said that the decomposition is a dominated splitting if for any $1 \leq j \leq k-1$ follows that

$$|Df^n_{|\oplus_{i=1}^j E_i(x)}||Df^{-n}_{|\oplus_{i=j+1}^k E_i(f^n(x))}| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

This concept was introduced independently by Mañé, Liao and Pliss, as a first step toward proving that structurally stable systems satisfy a hyperbolic condition on the tangent map. In some sense, a dominated splitting is a natural way to relax hyperbolicity.

Related to the notion of dominated splitting, there is a well known result proved in [HPS] that states that for any point $x \in \Lambda$ there are manifolds $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$ (not dynamically defined) tangents to the subbundles E and F respectively (see subsection 2.3.1 for details). If more than two subbundles are involved in the splitting, it follows that for any subbundle E_i and any for point $x \in \Lambda$ there is a manifold $W^{E_i}_\epsilon(x)$ tangent to E_i in x. It is natural to ask which is the relation of this tangent submanifolds with the local stable and unstable manifolds.
To precise, let us first recall the definition of local stable and unstable manifold of size ϵ (where ϵ is a positive constant):

$$W^s_\epsilon(x) = \{y \in M : \; dist(f^n(y), f^n(x)) \to 0 \, n \to \infty, dist(f^n(y), f^n(x)) < \epsilon\},$$

$$W^u_\epsilon(x) = \{y \in M : \; dist(f^n(y), f^n(x)) \to 0 \, n \to -\infty, dist(f^n(y), f^n(x)) < \epsilon\}.$$

To be concise, $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are called the local stable and unstable manifold respectively.

Observe that if Λ is hyperbolic, then follows that the tangent manifolds to E and F are contained in the local stable and unstable manifold respectively. However, the converse is false: it may happen that the tangent manifolds are dynamically defined and Λ is not hyperbolic. Taking into account this observation, we introduce the next definition:

Definition 6 Topologically hyperbolic sets: *Given a compact invariant set Λ, it is said that Λ is topologically hyperbolic if it is maximal invariant in a neighborhood and exhibits a dominated splitting $E \oplus F$ such that the local tangent manifold to E, $W^E_\epsilon(x)$, is contained in the local stable manifold and the local tangent manifold to F, $W^F_\epsilon(x)$, is contained in the local unstable manifold. It this case, it is said that E is topologically contractive and F is topologically expansive.*

In other words, it is said that a compact invariant set Λ is topologically hyperbolic if it is maximal invariant (i.e.: $\Lambda = \cap_{\{n\in\mathbb{Z}\}} f^n(U)$ for some neighborhood U; observe that it is not assumed that Λ is an attractor) and for each point, the local stable and unstable manifolds are two complementary submanifolds of size independent of the point. In this case, we also say that the tangent manifolds are dynamically defined.

Later, in theorem E1, we show that if Λ is a transitive topologically hyperbolic set, then it is a homoclinic class and it is conjugated to a subshift of finite symbols. Moreover, in theorem E2 and E3 we describe the continuation of a topologically hyperbolic homoclinic class for diffeomorphisms nearby.

In the next theorem, replacing "hyperbolicity" by "topological hyperbolicity" we obtain a version of theorem A, for the case that p has index two.

Theorem B: *Let $f \in Diff^2(M^3)$. Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point p of index two and such that all the periodic points in H_p are hyperbolic. Then, one of the following options holds:*

1. *H_p is hyperbolic with a two dimensional contractive subbundle;*
2. *H_p is topologically hyperbolic and exhibits a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that, E_1 is contractive, $E_1 \oplus E_2$ is topologically contractive and E_3 is topologically expansive;*
3. *there exists g C^1-arbitrarily close to f exhibiting a homoclinic tangency in U;*
4. *there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U.*

The previous theorem can be improved in terms of the nature of dominated splitting that the homoclinic class can exhibit and the type of homoclinic tangency that can be created by perturbation. This result is formulated in theorem G of section 5.

To complete the proof of Palis's conjecture under the hypothesis of theorem B, it is necessary to show that if the second item of the thesis holds, then either H_p is hyperbolic or there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle. Therefore it raises the following question: *given a topologically hyperbolic attracting homoclinic class of a three dimensional Kupka-Smale diffeomorphisms, and exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$ such that $E_1 \oplus E_2$ is topologically contractive and E_3 is topologically expansive; is it true that either the homoclinic class is hyperbolic or there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle?*

To deal with this problem, first we have to distinguish the cases when the homoclinic class is genuinely three dimensional or essentially two dimensional. This problem is discussed in the next subsection.

1.2 Actual "two dimensional" situation.

Even though our ambient manifold is three dimensional, it may happen that the homoclinic class that we are considering are contained in a two dimensional submanifolds it could turn out that they are actually two dimensional. In fact, to get examples of this kind a situation, let us consider an attractor for a surface diffeomorphism f (for instance a Plykin attractor or a Henon attractor), and then, let us embed this surface inside a in three dimensional manifold in a such a way that the three dimensional diffeomorphism coincides with f on the surface and such that this surface is invariant and normally hyperbolic for the new dynamics (see section 3 for precise definitions).

To distinguish the "two dimensional case" from the genuinely higher dimensional case, recall that in theorem B, the homoclinic class exhibits a dominated splitting $E_1 \oplus E_2 \oplus E_3$, such that $E_1 \oplus E_2$ is topologically contractive. This implies that the subbundle E_1 is contractive: there exist $C > 0$ and $0 < \lambda < 1$ such that $|Df^n_{|E_1(x)}| \leq C\lambda^n$, for all $x \in \Lambda, n \geq 0$. In fact, in proposition 2.7 it is proved that if $E_1 \oplus E_2$ is topologically contractive and E_2 is one dimensional, then the subbundle E_1 is contractive.

This allow us to consider a more general situation that may happen in any dimension. In fact, in what follows we will discuss the previous problem in any dimension. To do that, we start with a topologically hyperbolic sets Λ that has a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles. In this case, the local invariant manifold tangent to the subbundles E_1^s and E_4^u (noted $W_\epsilon^{E_1^s}(x)$ and $W_\epsilon^{E_4^u}(x)$ respectively) are C^1-manifolds usually called strong stable and strong unstable manifolds. Using these manifolds we define the following sets:

$$\mathcal{T}^{ss} = \{x \in H_p : [W_\epsilon^{E_1^s}(x) \setminus \{x\}] \cap H_p \neq \emptyset\};$$

$$\mathcal{T}^{uu} = \{x \in H_p : [W_\epsilon^{E^u}(x) \setminus \{x\}] \cap H_p \neq \emptyset\}.$$

Observe on one hand that Smale's hyperbolic solenoid attractor is a three dimensional hyperbolic attractor (in particular, a topologically hyperbolic attractor) that verifies that $\mathcal{T}^{ss}$ is not empty. In dimension higher than three, it is possible to get examples where both $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ are not empty. On the other hand, the examples that we considered before (a Plykin attractor or a Henon attractor embedded in a higher dimensional manifold) are examples where the sets $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ is empty.

If $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ are empty, from a result proved in [BC], the set Λ is contained inside a two dimensional normally hyperbolic submanifold. Observe that in this case, there is no chance to perturb the system in a way to create a heterodimensional cycle. In fact, if the submanifold that contains the attractor is normally hyperbolic, it follows that the submanifold is robust by perturbation and the perturbed homoclinic class is contained in this submanifold. So, for any g close to f we also have that $\mathcal{T}^{ss} = \emptyset$ and $\mathcal{T}^{uu} = \emptyset$. Therefore, it is not possible to get a heterodimensional cycle for g. In particular, if we want to prove the conjecture in the present case, we need to show that H_p is hyperbolic. More precisely, it is proved the following:

Theorem C: *Let $f \in Diff^2(M)$ be a Kupka-Smale diffeomorphisms. Let Λ be a topologically hyperbolic set exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$, such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles. If $\mathcal{T}^{ss} = \emptyset$ and $\mathcal{T}^{uu} = \emptyset$ then H_p is hyperbolic.*

We want to point out that the thesis of previous theorem is false if we assume that only one of $\mathcal{T}^{ss}$ or $\mathcal{T}^{uu}$ is not empty (see subsection 3.1 for an example of this statement). This shows, that the situation changes qualitative when we move from dimension two to higher dimensions. In fact, on the one hand observe that any topologically hyperbolic set for a smooth Kupka-Smale surface diffeomorphisms is hyperbolic (see theorem B proved in [PS1] and its generalization given by theorem C). On the other hand, there are topologically hyperbolic homoclinic classes for smooth Kupka-Smale three dimensional diffeomorphisms which are not hyperbolic.

Due to the previous remark, if $\mathcal{T}^{ss}$ is not empty or $\mathcal{T}^{uu}$ is not empty, it is naturally to ask whether either the homoclinic class is hyperbolic or one can create a heterodimensional cycle by small perturbation. To deal with this situation first it is consider the case that either the interior of $\mathcal{T}^{ss}$ is not empty or the interior of $\mathcal{T}^{uu}$ is not empty; where the interior is taken in the topology restricted to the set H_p. Observe that there are solenoid attractor in dimension three, that verifies that the interior of $\mathcal{T}^{ss}$ is not empty.

Theorem D: *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles. If H_p is nonhyperbolic and either the interior of $\mathcal{T}^{ss} \neq \emptyset$ or the interior of $\mathcal{T}^{uu} \neq \emptyset$, then there exists g C^1-arbitrarily close to f exhibiting a heterodimensional cycle in U.*

In other words, given a topologically hyperbolic homoclinic class, two scenarios can occur: either we are dealing essentially with a two dimensional system, meaning that the attractor is contained in a two dimensional submanifold, or the system is essentially higher dimensional. This alternative is related to the fact that in some sense "the strong manifolds is or not involved in the dynamics". When we are dealing with an essential two dimensional situation we prove that the homoclinic class is hyperbolic. When we are dealing with a "higher dimensional system", assuming that the interior of $\mathcal{T}^{ss} \neq \emptyset$ or the interior of $\mathcal{T}^{uu} \neq \emptyset$, we prove that a heterodimensional cycle can be created by perturbation. I remains the situation that either $\mathcal{T}^{ss} \neq \emptyset$ and interior of $\mathcal{T}^{ss} = \emptyset$ or $\mathcal{T}^{uu} \neq \emptyset$ and interior of $\mathcal{T}^{uu} = \emptyset$ (again, we want to point out that it is possible to get examples of this type).

This case is analyzed in [Pu], where it is shown under certain conditopn on the rate of dissipation that *given a three dimensional topologically hyperbolic attracting homoclinic associated to a point of index two and exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ with $\mathcal{T}^{ss} \neq \emptyset$, then either H_p is hyperbolic or by a C^1-perturbation it is created a heterodimensional cycle.*

Before ending the subsection, observe that theorems C and D can be applied to the homoclinic classes that satisfy the second part of the conclusion of theorem B.

1.3 Dynamical properties of topologically hyperbolic maximal invariant sets.

In the last section, we give a better description of the dynamics and structure of topologically hyperbolic sets (without any restriction of the dimension of the manifold) and we show that even if the set is not hyperbolic, it is conjugated to a subshift of finite symbols. In fact, we can obtain the following theorem:

Theorem E1: *Let $f \in Diff^1(M)$. Let Λ be a transitive topologically hyperbolic set of f. Then, Λ is a homoclinic class conjugated to a subshift with finite many symbols.*

Recall that an important property of hyperbolic sets is that can be continued for nearby systems (this continuatio is usually called analytic continuation): if $\Lambda \subset \Omega(f)$ is a hyperbolic set then, for any nearby diffeomorphism g there is a set Λ_g homeomorphic to Λ and such that the dynamics of f/Λ and g/Λ_g are conjugated. We may wonder if topologically hyperbolic homoclinic classes also exhibit a unique continuation. In the full generality, the answer is no (see example at the end of section 3.1). However, some partial results can be obtained.

Let V be a neighborhood of H_p, we define for diffeomorphisms g nearby f the following set:

$$\Lambda_g(V) = Closure(\cap_{\{n\in\mathbb{Z}\}} g^n(V)).$$

Theorem E2: *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class. There exists a neighborhood $\mathcal{U}$ of f and V of H_p such that for any $g \in \mathcal{U}$ there is a continuous map*

$$h_g : \Lambda_g(V) \to H_p$$

such that

$$h_g \circ g = f \circ h_g.$$

Moreover, the map $g \to h_g$ is continuous with g, relative to the uniform topology.

Assuming that Λ has a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles, then it is possible to show that the map h_g is onto and it is possible to get a better description of the continuation of the homoclinic class for perturbations of the initial system, as we are going to see.

Given a periodic point q, we take $\lambda_2(q)$ and $\lambda_3(q)$ to be the eigenvalues of $D_q f^{n_q}$ (n_q being the period of q) associated to the subbundles $E_2(q)$ and $E_3(q)$ respectively. Given λ_2 and λ_3 such that $\lambda_2 < 1 < \lambda_3$, we take the set of periodic point

$$Per_{\lambda_2\lambda_3}(f/V) = \{q \in Per(f) : \, |\lambda_2(q)| < \lambda_2, \, |\lambda_3(q)| > \lambda_3\}.$$

Theorem E3: *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class. If H_p has a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles, then there exists λ_2^0 and λ_3^0 with $0 < \lambda_2^0 < 1 < \lambda_3^0$ such that for any λ_2, λ_3 with $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$, there exist a neighborhood $\mathcal{U}$ of f, λ_2^1, λ_3^1 with $\lambda_2 < \lambda_2^1 < 1 < \lambda_3^1 < \lambda_3$ and a neighborhood V of H_p such that*

$$H_p = Closure(Per_{\lambda_2\lambda_3}(f/V)) \text{ and,}$$

$$h_g : Per_{\lambda_2^1\lambda_3^1}(g/V) \to Per_{\lambda_2\lambda_3}(f/V)$$

is a homeomorphism. In particular, it follows that h_g is onto.

To get this result, it is proved that the periodic points with eigenvalue exponentially far from 1 are dense in H_p and it is shown that those points have a well defined dynamical continuation for any g in a uniform neighborhood of f.

At the end of the last section of this paper, is given a better description of the map h_g for the case that $f \in Diff^2(M^3)$ and H_p is a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that E_2, E_3 are one dimensional subbundles.

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2 Proofs of Theorem A and B.

The strategy of the proofs consists first in showing that if f cannot be approximated by another system exhibiting a homoclinic tangency, then in the case p has index one, H_p exhibits either a dominated splitting $E \oplus F^u$ with the property that $dim(F^u) = 2$ and F^u is an expansive subbundle or $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. In the case that p has index two, H_p exhibits either a dominated splitting $E^s \oplus F$ with the property that $dim(E^s) = 2$ and E^s is a contractive subbundle or $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. This is done in subsection 2.1.

In subsection 2.2 it is stated a result proved in [PS3] that shows that if H_p exhibits either a dominated splitting $E \oplus F^u$ with the property that $dim(F^u) = 2$ and F^u is an expansive subbundle or a dominated splitting $E^s \oplus F$ with the property that $dim(E^s) = 2$ and E^s is a contractive subbundle, then the splitting is hyperbolic.

In subsection 2.3 it is considered the case that H_p has a dominated splitting with three subbundles. For the case that p has index one, that under the assumption that H_p exhibits a dominated splitting decomposed in three direction then either the homoclinic class is hyperbolic , or it is created a heterodimensional cycle by small perturbations. For the case that p has index two, it is shown that either the homoclinic class is topologically hyperbolic, or it is created a heterodimensional cycle.

To prove these results, it is shown that under the assumption of dominated splitting, then the local manifolds tangent to E_1 and E_3 are dynamically defined embedded manifolds being contained in the local stable and unstable manifold respectively. Later, it is shown that the local manifold tangent to the subbundle E_2 is either dynamically defined (stable one in the case p has index two and unstable one in the case that p has index one) or a heterodimensional cycle can be created by small perturbations. In the case that p has index one, it is actually proved that all the subbundles are contractive or expansive.

2.1 Systems far from tangencies.

In the sequel, given two diffeomorphisms f and g we say that g is $C^r - \delta-$close to f if $|f - g|_r < \delta$ where $|.|_r$ is the usual norm in the C^r-topology.

We start by assuming that it is not possible to create a tangency by a C^1-perturbation.

Definition 7 *Let p be a saddle periodic point and let $H_p = \cap_{\{n > \in\}} f^n(U)$ be a maximal invariant homoclinic class. We say that the homoclinic class is C^1-far from tangencies, if there is a neighborhood $\mathcal{U} \subset Diff^1(M^3)$ of f such that any $g \in \mathcal{U}$ does not exhibit a tangency in U.*

Definition 8 *Given an f-invariant set Λ exhibiting a dominated splitting $T_\Lambda M = E \oplus F$, it is said that E (F) is contractive (expansive) if there exist $C > 0$ and $0 < \lambda < 1$ such that $|Df^n_{|E(x)}| \leq C\lambda^n$, for all $x \in \Lambda, n \geq 0$ ($|Df^{-n}_{|E(x)}| \leq C\lambda^n$, for all $x \in \Lambda, n \geq 0$).*

In the case that H_p is C^1-far from tangencies, we show that H_p exhibits a dominated splitting. More precisely, we show that the tangent bundle is either decomposed in two subbundles $E \oplus F$

such that either E or F has dimension two and they are contractive or expansive respectively, or it is decomposed in three subbundles $E_1 \oplus E_2 \oplus E_3$. In what follows, any decomposition is assumed to be dominated.

Theorem 2.1 *Let us assume that H_p is C^1-far from tangencies.*
If the point p has stable index one, then one of the next options holds:

1. *$T_{H_p}M = E \oplus F^u$ with the property that $dim(F^u) = 2$ and F^u is an expansive subbundle;*
2. *$T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.*

If the point p has stable index two, then one of the next options holds:

1. *$T_{H_p}M = E^s \oplus F$ with the property that $dim(E^s) = 2$ and E^s is a contractive subbundles;*
2. *$T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.*

This result follows from techniques introduced in [PS1], [PPV] and in [LW]. First we recall some definitions: It is said that a hyperbolic periodic point has stable index d if the number of stable eigenvalues (or eigenvalues with modulus smaller than one) counted with multiplicity is d. It is said that a dominated splitting $E \oplus F$ is a $d-$dominated splitting if $dim(E) = d$. The d-preperiodic set of a C^1 diffeomorphism f, is the set of points q for which there is a diffeomorphisms g C^1 close to f such that q is a periodic point of g with stable index d.

Theorem 2.2 *([LW]) Let $f \in Diff^1(M)$. The following assertions are equivalent:*

1. *f cannot be C^1 approximated by a diffeomorphism exhibiting homoclinic tangencies associated to a periodic point of stable index d.*
2. *The closure of the periodic set of f with stable index equal to d, has a d-dominated splitting.*
3. *The d-preperiodic set of f has a d-dominated splitting.*

In our context, if the homoclinic class is associated to a periodic point of stable index one and by a C^1-perturbation it cannot be created a homoclinic tangency, follows from the theorem 2.2 that H_p has dominated splitting $E \oplus F$ with dimension of F equal to two. If the homoclinic class is associated to a periodic point of stable index two and by a C^1-perturbation it cannot be created a homoclinic tangency, follows from theorem 2.2 that H_p has dominated splitting $E \oplus F$ with dimension of E equal to two. However, using that we are dealing with a homoclinic class, this result can be improved. In fact, it is proved that if the subbundle E cannot exhibit a dominated splitting $E_1 \oplus E_2$ with two subbundles and f is C^1-far from tangencies then E is contractive. The strategy to prove that goes by contradiction: if the subbundle E cannot be splitted in two one dimensional subbundles $E_1 \oplus E_2$ exhibiting a dominated splitting and E is not contractive, then it can be created a tangency associated to a periodic point with stable index one; i.e.: a tangency associated to point with one dimensional stable manifold and a two dimensional unstable manifold.

To precise, we say that E *cannot exhibit a dominated splitting with two subbundles* or also that E *cannot be decomposed in two subbundles exhibiting domination*, if it follows that any decomposition of E in two subbundles is not a dominated splitting. Related to this, it is proved the following proposition:

Proposition 2.1 *Let $f \in Diff^1(M^3)$. Let $H_p = \cap_{n\in\mathbb{Z}} f^n(U)$ be a maximal invariant homoclinic class associated to a periodic point of stable index two. Let us assume that $T_{H_p}M = E \oplus F$ with $dim(F) = 1$ such that E cannot be decomposed into two invariant subbundles exhibiting domination and f is C^1-far from tangencies in U. Then follows that E is contractive*

A similar result can be stated for the case that p has stable index one:

Proposition 2.2 *Let $f \in Diff^1(M^3)$. Let $H_p = \cap_{n\in\mathbb{Z}} f^n(U)$ be a maximal invariant homoclinic class associated to a periodic point of stable index one. Let us assume that $T_{H_p}M = E \oplus F$ with $dim(E) = 1$ such that F cannot be decomposed into two invariant subbundles exhibiting domination and f is C^1-far from tangencies in U. Then follows that F is expansive.*

Assuming the previous proposition, now we can prove the theorem 2.1.

Proof of theorem 2.1:

To prove theorem 2.1, first observe that theorem 2.2 implies that if p has stable index two, then

$$T_{H_p}M = E \oplus F \text{ with } dim(F) = 1.$$

If p has stable index one, then

$$T_{H_p}M = E \oplus F \text{ with } dim(E) = 1.$$

To conclude, if $dim(F) = 1$ and E cannot be decomposed in two other subbundles exhibiting domination, by the proposition 2.1 follows that E is contractive. The case that $dim(E) = 1$ is similar.

■

2.1.1 Proofs of proposition 2.1 and 2.2.

We give the proof of proposition 2.1; the proof of proposition 2.2 is similar. We prove the proposition 2.1 assuming that the thesis is false. The goal is to show that if the thesis is false then we can create a homoclinic tangency. First we introduce the notion of angle of two vectors:

Definition 9 *Let v and w be two vectors of $\mathbb{R}^d$. It is defined the angle $\alpha(v, w)$ as the unique positive number in $[0, \frac{\pi}{2}]$ such that*

$$\cos(\alpha(v, w)) = \frac{< v, w >}{|v||w|}$$

where $< ., . >$ is the internal product induced by the riemanniam metric. Given two one-dimensional subspaces, it is defined the angle between them as the angle between two generators.

It is used the following lemma, which is a simple yet powerful perturbation technique (in the C^1 topology). This results says, for instance, that any small perturbation of the linear maps along a periodic orbit can be realized through a diffeomorphism C^1-nearby:

Lemma 2.1.1 *[Fr, Lemma 1.1] Let M be a closed n-manifold and $f : M \to M$ be a C^1 diffeomorphism, and let $\mathcal{U}(f)$ a neighborhood of f. Then, there exist $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S \subset M$ is a finite set, $S = \{p_1, p_2, \dots p_m\}$ and L_i, $i = 1, \dots, m$ are linear maps $L_i : T_{p_i}M \to T_{f(p_i)}M$ satisfying $\|L_i - D_{p_i}g\| \leq \delta$, $i = 1, \dots, m$ then there exists $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$, $i = 1, ..., m$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2 \dots p_m\} \cup (M \backslash U)$.*

A s a consequences of this result, given a periodic point p and a perturbation of its derivative along the orbit of p, follows that the perturbation of the derivativer can be realized as the derivative as a perturbation of the initial map and keeping the orbit of p.

Now we introduce a lemma that states that if the thesis of proposition 2.1 is false, then for small C^1-perturbation it can be obtained a periodic point such that its derivative has two contractive eigenspaces exhibiting a small angle and with one contractive eigenvalue of modulus close to one.

Lemma 2.1.2 *Let us assume that the thesis of proposition 2.1 is false. Then, given $\gamma > 0$, $\delta_1 > 0$, $\delta_2 > 0$ there exists a saddle periodic point q of f and a diffeomorphism g $C^1 - \delta_1-$close to f such that q is a periodic point for g such that*

1. *q has two different real contractive eigenvalues;*
2. *$(1 - \delta_2)^{n_q} < |Df^{n_q}_{|E^s_2(q)}| < 1$;*
3. *$\alpha(E^s_1(q), E^s_2(q)) < \gamma$;*

where n_q is the period of q and $E^s_1(q), E^s_2(q)$ are the two stable eigenspace associated to the two real contractive eigenvalues of $D_q f^{n_q}$.

We postpone the proof of the lemma to the next subsection. The following lemma states that assuming the thesis of the previous lemma we can create a tangency by a small C^1-perturbation.

Lemma 2.1.3 *Let us assume that the thesis of lemma 2.1.2 holds. Then, there is g C^1- close to f exhibiting a homoclinic tangency associated to a periodic point q with stable index one.*

Proof: If there is a point q as in the thesis of lemma 2.1.2, using the lemma 2.1.1 we can perform a C^1-perturbation to get a new system g such that q remains periodic for it and such that for $D_q g^{n_q}$ it is verified that:

1. the directions $E^s_2(q)$ and $E^s_1(q)$ remains invariant,
2. the modulus of the eigenvalue associated to the direction $E^s_2(q)$ become larger than one,

3. the modulus of the eigenvalue associated to $E_1^s(q)$ is smaller than one.

So, the periodic point q for g has a local stable manifold of dimension one and a local unstable manifold of dimension two such that the angle between both local manifold is small. By lemma 2.2.2 proved in [PS1], it is possible to create with a new perturbation, a tangency between the mentioned manifolds.

■

So, to finish the proof of proposition 2.1 is enough to prove the lemma 2.1.2.

Proof of lemma 2.1.2.

To prove the lemma, we state a result proved in [PPV] (see proposition 2.3). Roughly speaking, this result states that if E is not uniformly contractive then there is a periodic point in the homoclinic class with rate of contraction close to one.

Definition 10 *Given two hyperbolic periodic points, it is said that they are homoclinically related (or homoclinically connected) if the stable manifold of each point intersects transversally the unstable manifold of the other periodic point.*

Proposition 2.3 *([PPV]) Let $f \in Diff^2(M^3)$ and H_p a homoclinic class associated to a periodic point of stable index two and such that $T_{H_p}M = E \oplus F$ with $dim(F) = 1$. If E is not uniformly contractive then for any $\delta > 0$ and $m > 0$ follows that there is a periodic point $q \in H_p$ with period n_q such that*

1. *q is homoclinically related with p,*
2. *$(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$,*
3. *$n_q > m$,*

where $E^s(q)$ is the stable eigenspace associated to $D_qf^{n_q}$.

Now we continue with the proof of lemma 2.1.2. Let us consider the set of periodic points such that they have two contractive real eigenvalues. Let us call $E_1^s(q)$ and $E_2^s(q)$ the two eigenspace associated to the two contractive eigenvalues, and let us assume that the absolute value of the eigenvalue associated to $E_1^s(q)$ is smaller and equal than the absolute value of the eigenvalue associated to $E_2^s(q)$.

Given $\delta > 0$, let us consider the set $\mathcal{P}_\delta$ formed by periodic points $q_1 \in H_p$ such that

$$(1-\delta')^{n_{q_1}} < |Df^{n_{q_1}}_{|E^s(q_1)}| < 1 \text{ for some } 0 < \delta' < \delta,$$

where n_{q_1} is the period of q_1.

We have to consider three different situations:

- **Case 1.** For any δ small $\mathcal{P}_\delta$ is infinite and for any $0 < \lambda < 1$, any positive integer n_0 and $\delta > 0$ there is $q_1 \in \mathcal{P}_\delta$ and $m > n_0$ such that

$$\frac{|Df^m(E_1^s(q_1))|}{|Df^m(E_2^s(q_1))|} > \lambda$$

- **Case 2.** For any δ small $\mathcal{P}_\delta$ is infinite and there is $0 < \lambda < 1$ and a positive integer n_0 such that for every $\delta > 0$ and $q_1 \in \mathcal{P}_\delta$ follows that $E_1^s(q_1)$ $(\lambda, n_0)-$dominates $E_2^s(q_1)$; i.e.:

$$\frac{|Df^{n_0}(E_1^s(f^j(q_1)))|}{|Df^{n_0}(E_2^s(f^j(q_1)))|} < \lambda$$

for any $j > 0$.

- **Case 3.** There is δ_0 such that for any $\delta < \delta_0$ the set $\mathcal{P}_\delta$ either is finite or empty.

Case 1.

In the first case, it is proved that after a C^1-perturbation we can get a new periodic point exhibiting two subbundles with small angle and one eigenvalue close to one. In fact, first it is used the following folklore lemma (the proof it can be found in [M1]):

Lemma 2.1.4 *Let us assume that for any δ the set $\mathcal{P}_\delta$ is infinite and does not exhibit a dominated splitting. Then, for any $\gamma > 0$ there is g C^1-arbitrarily close to f exhibiting a periodic point q with arbitrarily large period n_q and such that*

1. $(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$ *and*
2. $\alpha(E_1^s(q), E_2^s(q)) < \gamma$.

Observe that it may happen that $(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$ and $|Df^{n_q}_{|E_2^s(q)}| < \lambda_s^{n_q}$ for some $\lambda_s < 1$. In other words, the eigenvalues in the stable direction are much smaller than the norm in this direction. In this case, we perform another perturbation to conclude the proof of lemma 2.1.2.

Lemma 2.1.5 *Let us assume that the thesis of lemma 2.1.4 holds. Then there is g C^1-close to f exhibiting a periodic point q with large period such that $(1-\delta)^{n_q} < |Df^{n_{q_1}}_{|E_2^s(q)}| < 1$ and $\alpha(E_1^s(q), E_2^s(q)) < \gamma$.*

As a consequences of the previous lemma, follows that lemma 2.1.2 is proved in the case that $\mathcal{P}_\delta$ is infinite and it has not dominated splitting. So, to finish in this case, we have to give the proof of lemma 2.1.5.

Proof of lemma 2.1.5:

Let us consider the basis $\mathcal{B}$ in $E^s(q)$ given by two orthonormal vectors v_1, v_2 such that $v_1 \in E_1^s(q)$. Let $\mathcal{B}_i$ basis in $E^s(f^i(q)$ given by $\frac{Df^i(v_1)}{|Df^i(v_1)|}$ and an orthonormal vector to it.

Let $A_i = Df : E^s(f^{i-1}(q)) \to E^s(f^i(q))$ and in theses basis we can assume that:

$$A_i = \begin{bmatrix} \alpha_i & k_i \\ 0 & \beta_i \end{bmatrix}$$

Observe that

$$Df^{n_q}_{|E^s(q)} = \Pi_{i=1}^{n_q} A_i = \begin{bmatrix} \alpha & k \\ 0 & \beta \end{bmatrix}$$

since we are assuming that there exists $\lambda_s < 1$ such that for any q follows that $|Df^{n_q}_{|E^s_1(q)}| < |Df^{n_q}_{|E^s_2(q)}| < \lambda_s^{n_q}$ (otherwise there is nothing to prove) then

$$|\alpha| < \lambda_s^{n_q}, |\beta| < \lambda_s^{n_q}, (1-\delta)^{n_q} < |k| < 1$$

Let us take $\epsilon > 0$ and $\delta > 0$ small. Let us consider the following linear maps which are small perturbations of the maps A_i:

$$B_i = \begin{bmatrix} \alpha_i & k_i \frac{1+\delta}{1-\delta} \\ 0 & \beta_i \end{bmatrix} \quad 1 \leq i \leq n-2,$$

$$B_{n-1} = \begin{bmatrix} \alpha_{n-1} & k_{n-1} \frac{1+\delta}{1-\delta} \\ \epsilon & \beta_{n-1} \end{bmatrix}.$$

So,

$$B = \Pi_{i=1}^{n_q} B_i = \begin{bmatrix} \alpha & \hat{k} \\ \alpha\epsilon & \hat{k}\epsilon + \beta \end{bmatrix}$$

where

$$(1+\delta)^{n_q} < \hat{k} < (\frac{1+\delta}{1-\delta})^{n_q}.$$

Then, taking

$$\epsilon < (\frac{1-\delta}{1+\delta})^n$$

holds that one of the eigenvalues of B is close to one and the eigenspaces has small angle. By lemma 2.1.1, the linear maps B_i can be realized as the derivative along the orbit of q of a perturbation of f. This conclude the proof of lemma 2.1.5.

■

Case 2.

The second case (i.e.: the set $\mathcal{P}_\delta$ is infinite and it has dominated splitting) is more delicate. For that, we need another two lemmas that basically state that assuming that if the subbundle E is not contractive and it cannot be decomposed in two one dimensional subbundles with domination, follow that it is possible to get two periodic points q_2, q_3 homoclinically related such that:

1. the eigenvalue of $Df^{n_{q_2}}_{q_2}$ (where n_{q_2} is the period of q_2) associated to the subbundle E is a complex eigenvalue;

2. $Df^{n_{q_3}}_{q_3}$ (where n_{q_3} is the period of q_3) has two eigenspaces with small angle.

Observe that for the points q_2 and q_3 it may occur that the rate of contraction of Df in the subbundle E is exponentially far from one (i.e.: for any pair of points q_1 and q_2 as the one selected, the rate of contraction is smaller than some λ_s smaller than one). However, using that there is another periodic point q_1 such the rate of contraction of $D_{q_1}f^{n_{q_1}}$ in the direction E for q_1 is close to one (see proposition 2.3) and that the three periodic points (q_1, q_2, q_3) are homoclinically related, follows that we can get a another periodic points verifying the thesis of the lemma 2.1.2.

We start enunciating the following lemma which is the proposition 2.1 proved in [BDP] (page 376).

Lemma 2.1.6 *Let H_p be a homoclinic class exhibiting an splitting $E \oplus F$ with $dim(E) = 2$ and such that E does not exhibit a dominated splitting with two subbundles. Then there is a diffeomorphisms g C^1-close to f having a periodic point q with contractive complex eigenvalue and homoclinically related with p.*

The next lemma, is a folklore one and a proof of it can be found in [DPU].

Lemma 2.1.7 *Let q be a periodic point with complex eigenvalue and let us assume that there is a transversal intersection of the stable and unstable manifold of q. Then, for any $\gamma > 0$ then there is a diffeomorphisms g C^1-close to f, a periodic point q' of f such that q' is homoclinically related with q, q' is a periodic point for g and $\alpha(E^s_1(q'), E^s_2(q')) < \gamma$.*

First, we take a point q_2 in the condition of lemma 2.1.6, i.e.: q_2 has a complex contractive eigenvalue and exhibiting a transversal intersection of their invariant manifolds. To continue with the proof, we use the next lemma usually called C^1-connecting lemma:

Lemma ([H]): *(C^1- connecting lemma:) Let $f \in Diff^r(M^n)$ and let p be a periodic point such that there are a point x in the unstable manifold and y in the stable manifold, a sequence of points x_n accumulating in x and points $f^{k_n}(x_n)$ in the forward orbit of the sequences x_n accumulating on y. Then, there is a diffeomorphisms g C^1-close to f such that p remains periodic for g, x is in the unstable manifold, y is in the unstable manifold and y is in the forward orbit of x.*

Now, using the C^1-connecting lemma it is possible to perturb f in such way that q_1 and q_2 are homoclinically related (recall that q_1 is a point in $\mathcal{P}_\delta$). Then, also follows that the unstable manifold of q_2 intersects the stable manifold of q_2. Now, we introduce a second perturbation to get a point q_3 that verifies the thesis of lemma 2.1.7. Observe that the points q_1, q_2 and q_3 are homoclinically related. Using that, we prove that we can get a new periodic point q such that

1. $(1-\delta_2)^{n_q} < |Df^{n_q}_{|E^s_2(q)}| < 1$ and
2. $\alpha(E^s_1(q), E^s_2(q))$ is small.

In fact, we take three neighborhood V_1, V_2 and V_3 of the orbit of q_1 q_2 and q_3 respectively (in what follows we can assume that these points are fixed) and we can assume that for each neighborhood V_i follows that $Df_{|V_i} = Df(q_i)$. Using that the three periodic points q_1, q_2, q_3 are homoclinically related follows that we can get a periodic point q with period $n_3 + k_3 + n^1_2 + k^1_2 + n_1 + k_1 + n^2_2 + k^2_2$ such that

1. n_1, n^1_2, n_3, n^2_2 are arbitrarily large,

2. k_1, k_2^1, k_3, k_2^2 are bounded by some k_0 independently of the election of n_1, n_2^1, n_3, n_2^2,

3. $f^j(q) \in V_3$ for $0 \leq j \leq n_3$,

4. $f^j(q) \in V_2$ for $n_3 + k_3 \leq j \leq n_3 + k_3 + n_2^1$,

5. $f^j(q) \in V_1$ for $n_3 + k_3 + n_2^1 + k_2^1 \leq j \leq n_3 + k_3 + n_2^1 + k_2^1 + n_1$

6. and $f^j(q) \in V_2$ for $n_3 + k_3 + n_2^1 + k_2^1 + n_1 + k_1 \leq j \leq n_3 + k_3 + n_2^1 + k_2^1 + n_1 + k_1 + n_2^2$.

We consider the following linear maps

$$A_1 = Df(q_1) : T_{V_1}M \to T_{V_1}M; \; A_2 = Df(q_2) : T_{V_2}M \to T_{V_2}M$$

$$A_3 = Df(q_3) : T_{V_3}M \to T_{V_3}M$$

$$T_{32} = Df^{k_3} : T_{V_3}M \to T_{V_2}M; \; T_{21} = Df^{k_2^1} : T_{V_2}M \to T_{V_1}M$$

$$T_{12} = Df^{k_1} : T_{V_1}M \to T_{V_2}M; \; T_{23} = Df^{k_2^2} : T_{V_2}M \to T_{V_3}M$$

See figure 1.

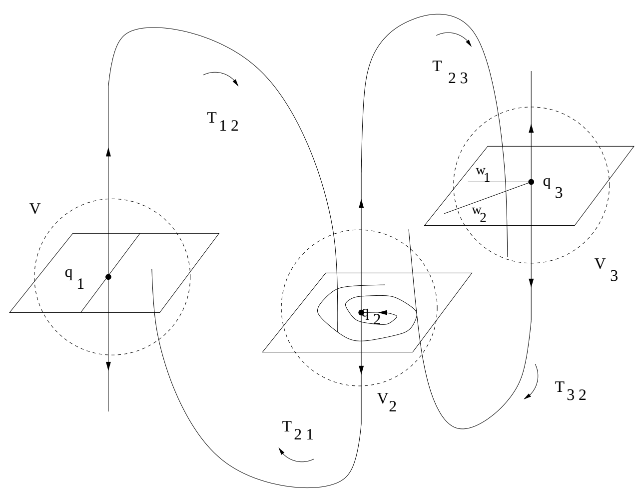

Figure 1

We consider the vectors w_1, w_2 such that $w_1 \in E_1^s(q_3)$, $w_2 \in E_2^s(q_3)$. Assuming that the complex eigenvalue has irrational imaginary part (if it is not the case with a small perturbation it would be the case), we can take n_2^1 and an small perturbation of T_{21} (we keep the same notation for the perturbation) such that

$$T_{21} A_2^{n_2^1} T_{32}(w_2) \in E_2^s(q_1)$$

Moreover, we can take an small perturbation of T_{23} (let us remain calling it T_{23}) and n_2^2 such that for any n_1 follows that

$$T_{23}A_2^{n_2^2}T_{12}A_1^{n_1}T_{21}A_2^{n_2^1}T_{32}(w_2) \in E_2^s(q_3)$$

In other word, follows that the direction $E_2^s(q)$ is invariant for

$$T_{23}A_2^{n_2^2}T_{12}A_1^{n_1}T_{21}A_2^{n_2^1}T_{32}$$

Observe that $\alpha(T_{21}A_2^{n_2^1}T_{32}(w_2), T_{21}A_2^{n_2^1}T_{32}(w_1))$ is small.
Since $T_{21}A_2^{n_2^1}T_{32}(w_2) \in E_2^s(q_1)$ and $E_2^s(q_1)$ is dominated by $E_1^s(q_1)$, follows that

$$\alpha(A_1^{n_1}T_{21}A_2^{n_2^1}T_{32}(w_1), A_1^{n_1}T_{21}A_2^{n_2^1}T_{32}(w_2)) < \gamma$$

with γ being small. Then we can get another small perturbation of T_{23} such that

$$T_{23}A_2^{n_2^2}T_{12}A_1^{n_1}T_{21}A_2^{n_2^1}T_{32}(w_1) \in E_2^s(q_1)$$

So, we obtain a new linear map close to the initial one such that has two eigenspaces with small angle. Moreover, if n_1 is chosen larger than the others, follows that the new linear map that has rate of contraction along w_1 and w_2 close to one. Again by lemma 2.1.1, the linear maps B_i can be realized as the derivative along the orbit of q of a perturbation of f.

Case 3.

If it follows that for some δ_0 holds that for any $\delta < \delta_0$ that $\mathcal{P}_\delta$ is empty or finite, by proposition 2.3 follows that there is a periodic point q having a contractive complex eigenvalue with modulus close to one. By lemma 2.1.7 we get a periodic point with two contractive real eigenvalues and such that their stable eigenspaces has small angle. Moreover, we can assume that this periodic point expends a large part of its orbit close to the periodic point q and so its the rate of contraction is also close to one. Then, we can apply the lemma 2.1.5 to conclude.

∎

2.2 Case that either $T_{H_p}M = E^s \oplus F$ or $T_{H_p}M = E \oplus F^u$.

First we consider the case that either $T_{H_p}M = E^s \oplus F$ or $T_{H_p}M = E \oplus F^u$. In these situations is proved that H_p is hyperbolic. To do that, we use a theorem proved in [PS4] that studies the dynamical consequences of a codimension one dominated splitting.

Theorem 2.3 *Let $f \in Diff^2(M^n)$ be a Kupka-Smale system. Let Λ be a compact invariant set contained in a homoclinic class such that exhibits a dominated splitting $T_{H_p}M = E^s \oplus F$ where E^s is contractive and $dim(F) = 1$. Then Λ is hyperbolic.*

The central argument follows from the fact that F has dimension one and the complementary subbundle is contractive. This allows to perform similar argument developed in [PS1]. In [Z] similar results was obtained: in the mentioned paper was proved that given a topological minimal compact invariant set Λ such that exhibits a dominated splitting $T_{H_p}M = E^s \oplus F$ where E^s is uniformly contracted and $dim(F) = 1$ follows that F is hyperbolic.

Applying theorem 2.3 to our context, we get the next two corollaries:

Corollary 2.1 *If $T_{H_p}M = E^s \oplus F$ with $dim(E^s) = 2$, then H_p is hyperbolic. If $T_{H_p}M = E \oplus F^u$ with $dim(F^u) = 2$, then H_p is hyperbolic.*

Corollary 2.2 *Let $f \in Diff^2(M^3)$ be such that $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. If E_2 is hyperbolic, then the homoclinic class is hyperbolic.*

The last corollary is immediate and holds in the following way: if E_2 is contractive, by domination holds that E_1 is also contractive. Then we are in the presence of a contractive codimension one dominated splitting and we can apply the theorem 2.3. If E_2 is expansive, by domination holds that E_3 is also expansive. Then we are in the presence of a expansive codimension one dominated splitting and we can apply the theorem 2.3.

2.3 Case that $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.

To finish with the proofs of theorem A and B we have to deal with the case i.e.: $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. The rest of the section is devoted to deal with this situation. The study in this case goes through different steps:

Step I: First, we conclude that under the assumption that H_p is a maximal invariant set, the local tangent manifold of the extremal subbundles (E_1 and E_3) are dynamically defined. More precisely we show that the local tangent manifold to the subbundle E_1 and E_3 are stable and unstable manifolds respectively. This is the statement of theorem 2.4 and it is formulated in the subsection 2.3.1.

Step II: Using that the local tangent manifold associated to the extremal subbundles are dynamically defined, it is proved that if the center subbundle is not hyperbolic, then there are periodic points homoclinically related with p such that the eigenvalue associated with the center subbundle (the subbundle E_2) is close to one. This is the theorem 2.5 and it is formulated in the subsection 2.3.2. This theorem is a reformulation of the proposition 2.3 stated in section 2.1.

Step III: We consider independently the case where the periodic point p has either stable index one or stable index two. In the case that p has stable index one, using the connecting lemma and the fact that we are dealing with an attractor, it follows that if H_p is not hyperbolic then it is possible to get an intersection between the tangent manifold to the extremal subbundles of a periodic point with central eigenvalue close to one. From this, the periodic point is bifurcated in a way to obtain a heterodimensional cycle. This is done in subsection 2.3.3. The next step deals with the case that p has stable index two.

Step IV: In the case that the periodic point p has stable index two (see subsection 2.3.4), first we study the dynamical behavior of the manifold tangent to the center subbundle. If the center manifold (the one tangent to E_2) is not a stable manifold then it is proved that by a C^1- perturbation it is obtained a periodic point with center eigenvalue close to one and such that the tangent manifold associated to the extremal subbundles (which are a local stable and unstable manifold) has an intersection. From this, the periodic point is bifurcated in a way to obtain a heterodimensional cycle (see subsection 2.3.4). If the local manifold tangent to the center subbundle is a stable manifold follows for every point there are two transversal complementary dynamically defined local manifold of uniform size: one is a two dimensional local stable manifold and the other is a one dimensional unstable manifold. In other words, we had proved that H_p is topologically hyperbolic.

2.3.1 Dynamical behavior of the tangent manifolds associated to the extremal subbundles.

First, we state the existences of manifolds tangent to each subbundle of the dominated splitting. Recall by [HPS] that there are 1−dimensional manifolds $W^{E_i}_\epsilon(x)$ tangents to each E_i. More precisely, let us define first $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^1(I_1, M)$ the set of C^1-embedding of I_1 on M.

Lemma 2.3.1 *For each subbundle E_i there exists a continuous functions $\phi^i : H_p \to Emb^1(I_1, M)$ such that for any $x \in H_p$ it is defined $W^{E_i}_\epsilon(x) = \phi^i(x)I_\epsilon$ and verifies:*

1. *$T_xW^{E_i}_\epsilon(x) = E_i(x)$,*
2. *if $f(W^{E_i}_\epsilon(x)) \subset B_\epsilon(f(x))$ then $f(W^{E_i}_\epsilon(x)) \subset W^{E_i}_\epsilon(f(x))$,*
3. *if $f^{-1}(W^{E_i}_\epsilon(x)) \subset B_\epsilon(f^{-1}(x))$ then $f^{-1}(W^{E_i}_\epsilon(x)) \subset W^{E_i}_\epsilon(f^{-1}(x))$.*

The previous lemma does not state any dynamical meaning for the tangent manifold. Recall the definition of local stable and unstable manifold of size ϵ given in the introduction.
With this definition in mind, we say that *the tangent manifold $W^{E_3}_\epsilon$ is dynamically defined* if there exists, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $x \in H_p$ follows that

$$W^{E_3}_{\epsilon_1}(x) \subset W^u_{\epsilon_2}(x).$$

In the same way, we say that $W^{E_1}_\epsilon$ *is dynamically defined* if there exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $x \in H_p$ follows that

$$W^{E_1}_{\epsilon_1}(x) \subset W^s_{\epsilon_2}(x).$$

In this case, we call $W^{E_3}_\epsilon$ and $W^{E_1}_\epsilon$ the strong unstable and strong stable manifolds respectively. Without loose of generality we can assume that $\epsilon_1 = \epsilon_2 = \epsilon$. Observe that in this case the tangent manifolds are unique.

The next theorem states that assuming that the system is C^2 and the homoclinic class is maximal invariant set follows that the tangent manifolds associated to the extremal subbundle are dynamically defined. The theorem is a consequences of a result stated in [PS4] and holds in any dimension assuming that the extremal subbundles are one dimensional. The precise statement of this theorem is formulated in next section.

Theorem 2.4 *Let $f \in Diff^2(M^3)$. If H_p is a maximal invariant homoclinic class exhibiting a dominated splitting with three subbundles $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$, then there exists $\epsilon > 0$ such that $W^{E_1}_\epsilon$ and $W^{E_3}_\epsilon$ are dynamical defined.*

Remark 2.1 *Observe that we are not assuming in this case that the homoclinic class is an attractor.*

As a consequences of the previous theorem we can get the next lemma that it shows that either the periodic points in the homoclinic class has the same stable index or it is possible to get a diffeomorphisms arbitrarily close to the initial one exhibiting a heterodimensional cycle.

Lemma 2.3.2 *Let $f \in Diff^2(M^3)$ and let $H_p = \cap_{\{n\in\mathbb{Z}\}} f^n(U)$ be a maximal invariant homoclinic class exhibiting a dominated splitting with three subbundles $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. Then, one of the following options holds:*

1. *there exists a neighborhood U of H_p such that all the periodic points in U has the same stable index;*
2. *there is g C^1 arbitrarily close to f exhibiting a heterodimensional cycle.*

Proof:
Let us assume that the point p in the homoclinic class has stable index two. We have to show that all the periodic points in the neighborhood U has stable index two. If it is not the case, we have to show that we can C^1-approximate f by another diffeomorphism g exhibiting a heterodimensional cycle. If there exists a periodic point q of stable index one in a small neighborhood of H_p, from the fact that the homoclinic class is maximal invariant, follows that it is contained in H_p. Since we are assuming that the homoclinic class exhibits three subbundles, follows that any intersection of the stable and unstable manifold of p is a transversal intersection. Then, there is a sequences of points q_n of stable index two homoclinically related to p and accumulating on q. Due to the fact that the strong stable manifold has uniform size for any point q_n close to q follows that the strong stable manifold of them intersects transversally the unstable manifold of q. Let us take a point q' of the sequences q_n. Observe that the intersection of the stable manifold of q' with the unstable manifold of q is robust by perturbation.

From the fact that q is in H_p and it has stable index one (the local stable manifold of q is one-dimensional), follows that there are points in the homoclinic class that accumulates in the stable manifold of the point q. Since H_p is an attractor, the unstable manifold of q' is contained in H_p and so there exist a point in H_p with orbit accumulating in the unstable manifold of q' and in the stable manifold of q. So, using the connecting lemma, we get that with a C^1-perturbation it is possible to connect the unstable manifold of q' with the stable manifold of q. Then it was created an heterodimensional cycle involving q and some q' close to q.
The case that p has stable index one is treated in the same way.

■

So, from now on, we assume that *all the periodic points in U has the same stable index.*

At this point, we consider two cases:

- **Theorem A:** The periodic point p has stable index one,
- **Theorem B:** The periodic point p has stable index two.

Before to deal with both situation, we need some results proved elsewhere. We enunciate these results in the next subsection, and in the subsections 2.3.3 and 2.3.4 we return to both cases enunciated above.

2.3.2 Previous results.

First, we start reformulating the proposition 2.3 to the case that the splitting has three subbundles. The present reformulation states that under the assumption of dominated splitting over a homoclinic class for a C^2 diffeomorphisms in a three dimensional manifold, holds that if the subbundle E_2 is not hyperbolic then there are periodic points contained in H_p and homoclinically related to p such that the eigenvalue associated to the center subbundle is close to one.

Theorem 2.5 *([PPV]) Let $f \in Diff^2(M^3)$ and let H_p be a homoclinic class exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.*

- *If p has stable index two and the subbundle E_2 is not hyperbolic, then for any $\delta > 0$ there exists a periodic point q with period n_q and homoclinically related to p such that $(1-\delta)^{n_q} < |Df^{n_q}_{|E_2(q)}| < 1$ (in this case we say that q has $\delta-$weak contraction along the center direction).*
- *If p has stable index one and the direction E_2 is not hyperbolic, then for any $\delta > 0$ there exists a periodic point q with period n_q and homoclinically related to p such that $1 < |Df^{n_q}_{|E_2(q)}| < (1+\delta)^{n_q}$ (in this case we say that q has $\delta-$weak expansion along the center direction).*

This version follows immediately from the proposition 2.3 and from the fact that we are assuming that all the periodic points in the homoclinic class has the same stable index. For instance, in the case that p has stable index two, and the homoclinic class is not hyperbolic, from proposition 2.3 follows that there is a periodic point with weak rate of contraction along the subbundle $E_1 \oplus E_2$. Since the angle between both subbundle is uniformly bounded from below and from the domination property, follows that

$$|Df_{E_1 \oplus E_2}| = \max\{|Df_{|E_1}|, |Df_{|E_2}|\} = |Df_{|E_2}|$$

and therefore follows the previous theorem.

It is important to remark that the previous theorem is not a perturbation theorem. More precisely, the theorem 2.5 shows that the obstruction of the hyperbolicity (in the context that we are considering) come from the existences of periodic points with eigenvalues close to one in the center direction.

An immediate corollary is the following result:

Corollary 2.3 *Let $f \in Diff^2(M^3)$ and let H_p be a homoclinic class exhibiting a dominated splitting*

$$T_{H_p}M = E_1 \oplus E_2 \oplus E_3$$

- *If p has stable index two and the subbundle E_2 is not contractive, then for any $\delta > 0$ the periodic points exhibiting $\delta-$weak contraction are dense.*

- *If p has stable index one and the subbundle E_2 is not expansive, then for any $\delta > 0$ the periodic points exhibiting $\delta-$weak expansion are dense.*

In fact, to conclude this corollary from the theorem 2.5 it is enough to recall that the point point q with weak contraction (expansion) is homoclinically related to the point p. So, taking any point x in the homoclinic class, we can approximate it by a periodic point z homoclinically related to p and so homoclinically related to q. Then, we can take periodic points in horseshoes that contains z and q with the property that they accumulate on z but they expends more time close to q. So, these points have a weak contraction (expansion) along the center direction. This kind of arguments are folklore (see for instance [BDP]) and we state it here for sake of completeness. To be precise, we get the following lemma:

Lemma 2.3.3 *Let $f \in Diff^r(M)$ having two periodic points q and q_δ such that there are homoclinically connected and such that q_δ has $\frac{\delta}{2}-$weak contraction along the center direction. Then, for any $r > 0$ there is a periodic point q_δ^r homoclinically connected with q such that $dist(q, q_\delta^r) < r$ and q_δ has $\delta-$weak contraction along the center direction.*

For some periodic points in the homoclinic class follows that they exhibits a transverse intersection of its stable and unstable manifold. If this intersection holds along the strong stable and unstable manifolds we say that there is a strong homoclinic connection:

Definition 11 Strong homoclinic connection. *Given a periodic point q, we say that it has a strong homoclinic connection if the strong stable and strong unstable manifolds of q has an intersection.*

Now, let assume that there is a periodic point with weak contraction (expansion) along the center direction and also exhibiting a strong homoclinic connection. In this case, after a C^1 perturbation, it is possible to show that it is created a heterodimensional cycle.

Proposition 2.4 *Given $\delta_0 > 0$, there exists δ such that if there is a periodic point with $\delta-$weak contraction (expansion) along the central direction and exhibiting and strong homoclinic connection, then there is g $C^1 - \delta_0-$close to f exhibiting a heterodimensional cycle.*

The proof of this proposition is given in section 2.4.

Now we reformulate a lemma proved in [H] and already to stated in previous subsection, that allows to connect the strong stable and unstable manifolds when they are orbits that accumulates on both manifolds.

Lemma ([H]): *(C^1- connecting lemma:) Let $f \in Diff^r(M^n)$ and let p be a periodic point such that there are points x in the strong unstable manifold and y in the strong unstable manifold, a sequence of points x_n accumulating in x and points $f^{k_n}(x_n)$ in the forward orbit of the sequences x_n*

accumulating on y. Then, there is a diffeomorphisms g C^1-close to f such that p remains periodic for g, x is in the strong unstable manifold, y is in the strong unstable manifold and y is in the forward orbit of x.

2.3.3 Proof of Theorem A: p has stable index one.

In this case, we show that if the homoclinic class is not hyperbolic then we can get a heterodimensional cycle.
First, we get the following proposition (the proof of this proposition is given in section 2.4):

Proposition 2.5 *Let H_p be an attracting homoclinic class associated to a periodic point of stable index one and exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. If H_p is not hyperbolic, then for any $\delta > 0$ there exists g C^1-close to f and a periodic point q with $\delta-$weak expansion along E_2 such that q exhibits and strong homoclinic connection.*

Assuming proposition 2.5, we use proposition 2.4 to finish the proof of the theorem A.

2.3.4 Proof of Theorem B: p has stable index two.

We consider two situations: either the center manifold is dynamically defined or it is not the case.
More precisely, we say that $W^{E_2}_\epsilon$ is dynamically defined if there exist $\epsilon > 0$ and $\gamma > 0$ such that for any $x \in H_p$ follows that

1. $f^n(W^{E_2}_\epsilon(x)) \subset W^{E_2}_\gamma(f^n(x))$ for any $n \geq 0$,
2. $\ell(f^n(W^{E_2}_\epsilon(x))) \to 0$ as $n \to \infty$.

In other words, we are saying that $W^{E_2}_\epsilon(x)$ is dynamically defined if it is contained in $W^s_{\epsilon'}(x)$ for some $\epsilon' > 0$. We can assume that $\epsilon' = \gamma = \epsilon$.
Related to the previous option (if the center manifold is either dynamically defined or not) we get the following proposition (the proof of this proposition is given in section 2.4):

Proposition 2.6 *Let $f \in Diff^2(M^3)$. Let H_p be an attracting homoclinic class associated to a periodic point of stable index two and exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. Then, one of the following option holds*

1. *Case B.1: for any $\delta > 0$, there is a periodic point q with $\delta-$weak contraction along E_2 such that*
$$[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$$
In this case we say that the homoclinic class has a point in the strong stable manifold of the point q.
2. *Case B.2: the local manifold $W^{E_2}_\epsilon(x)$ tangent to E_2 is dynamical defined.*

It is important to remark that there exist robust example of both situations.

Case B.1.

In this case we have that for any $\delta > 0$ there exists a periodic point q homoclinically related to p such that

1. $(1-\delta)^{n_q} < |Dg^{n_q}_{|E_2(q)}| < 1$ where n_q is the period of q,
2. $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$ (i.e.: the homoclinic class has a point in the strong stable manifold of q).

Then for any $\delta > 0$ using the C^1-connecting lemma, we get that by a C^1-perturbation follows that there is g C^1- close to f with a periodic point q which $\delta-$weak contractive along E_2 and exhibiting a strong homoclinic connection.

Using the proposition 2.4 follows the existence of a heterodimensional cycle and so finishing the proof of theorem B in the case B1, i.e.: we finished showing the existence of a heterodimensional cycle in the case that the homoclinic class is not hyperbolic and the center subbundle is not dynamically defined.

Case B.2.: $W^{E_2}_\epsilon(x)$ is dynamically defined. H_p is topologically hyperbolic.

At this point the theorem B is concluded showing that H_p is topologically hyperbolic. We give other properties that the splitting satisfies.

Recall that in this case we are assuming that the center manifold tangent to E_2 is dynamically defined. Under this assumption, we get the following proposition (the proof of this proposition is given in section 2.4):

Proposition 2.7 *Let Λ be a compact invariant set exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3 \oplus E_4$. If $E_1 \oplus E_2$ is topologically contractive, and E_2 is one dimensional then it follows that E_1 is contractive. If $E_3 \oplus E_4$ is topologically expansive, and E_3 is one dimensional then it follows that E_4 is expansive.*

Applying the previous proposition to the case B.2., follows that taking $E = E_1 \oplus E_2$ and $F = E_3$ then for any point $x \in H_p$ there is a stable and unstable manifold of uniform size tangents to E and F respectively. The manifold tangent to E_1 is used to be called the strong stable manifold; the manifold tangent to E_2, the center manifold; the manifold tangent to $E_1 \oplus E_2$, the center-stable manifold; the manifold tangent to E_3, the unstable manifold; the manifold tangent to $E_2 \oplus E_3$, the center-unstable manifold. In the present context, follows that the center-stable manifold is contained in the local stable manifold.

Remark 2.2 *From the fact that the dominated splitting is decomposed in one dimensional subbundles, we can assume that there is an adapted metric such that the constant of domination is $\lambda < 1$ and $C = 1$. Moreover, we can assume that there is $\lambda_s < 1$ such that*

$$|Df_{|E_1}| < \lambda_s.$$

2.4 Proof of propositions and theorems of subsections 2.3.1, 2.3.3, 2.3.4.

First, we appeal to some results and definitions proved in [PS4] for "codimension one dominated splitting". It what follows with $\ell(I)$ it is denoted the usual length of an arc I.

Definition 12 *Let $f: M \to M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having dominated splitting $E \oplus F$ with $dim(F) = 1$. Let U be an open set containing Λ where is possible to extend the previous dominated splitting. We say that a C^2-arc I in M (i.e, a C^2-embedding of the interval (-1,1)) is a δ-E-arc provided the next two conditions holds:*

1. *$f^n(I) \subset U$, and $\ell(f^n(I)) \leq \delta$ for all $n \geq 0$.*
2. *$f^n(I)$ is always transverse to the E-subbundle.*

Related to this kind of arcs it is proved in [PS4] the following result.

Theorem 2.6 *There exists δ_0 such that if I is a δ-E-interval with $\delta \leq \delta_0$, then one of the following properties holds:*

1. *$\omega(I) = \cup_{\{x \in I\}} \omega(x)$ is a periodic simple closed curve and $f^m_{/\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ (where m is the period of $\mathcal{C}$) is conjugated to an irrational rotation,*
2. *$\omega(I) \subset J$ where J is a periodic arc.*

Now we can proceed to show how the theorem 2.4 follows from the previous result.

Proof of theorem 2.4:

First, we prove that the local manifold tangent to E_3 is an unstable manifold. We start showing that there exist $\epsilon > 0$ and $\gamma > 0$ such that $f^{-n}(W^{E_3}_\epsilon(x)) \subset W^{E_3}_\gamma(f^{-n}(x))$.

Let us assume that this is not the case. So it follows that there are a positive number γ, a sequences of positive numbers $\epsilon_n \to 0$, points x_n and a strictly increasing sequences of positive integers k_n such that

$$\ell(f^{-k_n}(W^{E_3}_{\epsilon_n}(x_n))) = \gamma$$

and

$$\ell(f^{-j}(W^{E_3}_{\epsilon_n}(x_n))) < \gamma \quad 0 \leq j \leq k_n.$$

Taking

$$I = \lim_{n \to +\infty} f^{-k_n}(W^{E_3}_{\epsilon_n}(x_n))$$

follows that I does not growth for positive iteration and it is transversal to $E_1 \oplus E_2$; i.e.: I is a $\gamma - E_1 \oplus E_2-$arc.

Then, we can apply theorem 2.6 and follows that either $\omega(I)$ is a periodic curve with dynamic conjugated to an irrational rotation or it is contained in a periodic arc. Both situation cannot hold inside a homoclinic class.

To prove that $\ell(f^{-n}(W^{E_3}_\epsilon(x))) \to 0$ as $n \to \infty$ we repeat the same argument. In fact, if it is not the case, we can find an arc I transversal to $E_1 \oplus E_2$ that does not growth by positive iterations and the same conclusion is obtained.

To show that W^{E_1} is dynamically defined, we take f^{-1} and it is done the same argument changing backward iterations by forward iterations.

■

Proof of lemma 2.3.3:

To prove the lemma, observe that using that q and q_δ are homoclinically related, follows that there is a horseshoes containing q and q_δ. Moreover, we can take two small neighborhoods W and W_δ of q and q_δ respectively such that there exists two positive integers k_1 and k_2 and for some positive integers n and n_δ arbitrarily large, there exists a periodic point z such that

1. the period of z is $n + k_1 + n_\delta + k_2$,
2. for any $0 \leq i \leq n$ follows that $f^i(z) \in W$ and we can assume that $|Df_{|E_2(f^i(z))}| < \lambda_q < 1$,
3. for any $0 \leq i \leq n_\delta$ follows that $f^{n+k_1+i}(z) \in W_\delta$ and we can assume that

$$1 - \frac{\delta}{2} < |Df_{|E_2(f^{n+k_1+i}(z))}| < 1$$

Observe that for any $r > 0$ there is $n = n(r)$ such that the corresponding periodic point z has an iterate such its distance to q is smaller than r.

To see that z is $\delta-$weakly contractive along the E_2 direction we proceed as follows:

Observe on one hand, that there is a positive constant C such that

$$C^{-1} < |Dg^{k_1}_{|E_2|}| < C, \quad C^{-1} < |Dg^{k_2}_{|E_2}| < C.$$

So, given the periodic point z follows that

$$C^{-2}|Df^n_{|E_2(z)}|(1 - \frac{\delta}{2})^{n_\delta} < |Df^{n+k_1+n_\delta+k_2}_{|E_2(z)}| < C^2|Df^n_{|E_2(z)}||Df^{n_\delta}_{|E_2(f^{n+k_1}(z))}| < C^2|Df^n_{|E_2(z)}|.$$

If n is large enough, follows that

$$C^2|Df^n_{|E_2(z)}| < 1.$$

Fixed n, we take n_δ large enough such that

$$(1-\delta)^{n+k_1+n_\delta+k_2} < C^{-2}|Df^n_{|E_2(z)}|(1 - \frac{\delta}{2})^{n_\delta}.$$

From both inequalities the lemma follows.

■

Proof of proposition 2.4.

The proof consist in to bifurcate the periodic point with center eigenvalue close to one in two periodic points of different stable index and to control the behavior of the strong unstable manifold and the strong stable manifold of the periodic point that it is created by the bifurcation. We use the lemma 2.1.1 to bifurcate the periodic point that has an eigenvalue close to one.

Let us take a point q with $\delta-$weak contraction along the direction and exhibiting a strong homoclinic connection. Let us take a point x contained in $[W^{E_3}_\epsilon(q) \setminus \{q\}] \cap [W^{E_1}_\epsilon(q) \setminus \{q\}]$ and let γ^u be a connected compact arc containing x and contained in a fundamental domain of $W^{E_3}(q)$. Let also takes γ^{ss} the compact arc contained in $W^{E_1}_\epsilon(q)$ that connects q with x. Using the lemma 2.1.1 we bifurcate q into three periodic points q_{-1}, q_0, q_1 for a diffeomorphisms g C^1-close to f such that q_{-1} and q_1 has stable index two, q_0 has stable index one and q_0 coincide with q. Observe that $W^s(q_1) \cap W^u(q_0) \neq \emptyset$. Moreover, the bifurcation can be done in a way that the arc γ^{ss} remains contained in $W^{E_1}_\epsilon(q_0) \setminus \{q_0\}$; i.e.: γ^{ss} remains contained in the local strong stable manifold of q_0. On the other hand, we can perform the bifurcation such that $g^{-n}(\gamma^u) \subset W^{E_3}_\epsilon(q_1)$ for some $n > 0$; i.e.: γ^u remains contained in the unstable manifold of q_1. So, follows that $W^u(q_1) \cap W^{E_1}_\epsilon(q_0) \neq \emptyset$; therefore, a heterodimensional cycle is created involving q_0 and q_1.

∎

Proof of proposition 2.5:

Let us take a point q with weak expansion along the center direction. On one hand, since q and p are homoclinically related and both points has stable index one, follows that $W^{E_1}_\epsilon(q) \cap W^{E_3}(p) \neq \emptyset$ and so $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$ (recall that $W^{E_1}_\epsilon(q)$ and $W^{E_3}_\epsilon(q)$ are the local strong stable and unstable manifold). On the other hand, again since H_p is a attractor, follows that $W^{E_3}_\epsilon(q) \subset H_p$. Then, there are orbits in H_p accumulating in $W^{E_3}_\epsilon(q) \setminus \{q\}$ with positive iterates also accumulating in $W^{E_1}_\epsilon(q) \setminus \{q\}$. By the connecting lemma, we can get a periodic point with weak expansion along the center direction and exhibiting a strong homoclinic connection.

∎

Proof of proposition 2.6:

Recall that all the periodic points has the same stable index. First, we start proving that either

1. there exist $\epsilon > 0$ and $\gamma > 0$ such that for any $x \in H_p$ $f^n(W^{E_2}_\epsilon(x)) \subset W^{E_2}_\gamma(f^n(x))$ for any $n > 0$ or,

2. there exists a periodic point q having weak contraction along the center direction and such that $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$.

Let us assume that the first option does not hold. Then follows that for any small positive number γ, there is a sequences of positive numbers ϵ_n, points x_n and an increasing sequences of positive integers k_n such that

$$\ell(f^{k_n}(W^{E_2}_{\epsilon_n}(x_n))) = \gamma$$

and

$$\ell(f^j(W^{E_2}_{\epsilon_n}(x_n))) < \gamma \quad 0 \leq j < k_n.$$

Taking

$$J = lim_{n \to +\infty} f^{k_n}(W^{E_2}_{\epsilon_n}(x_n))$$

follows that J does not growth for negative iteration; i.e.:

$$\ell(f^{-j}(J)) \leq \gamma \ \forall j > 0.$$

Then, if $\gamma > 0$ is small enough, it follows that

$$f^{-j}(J) \subset U \ \ \forall j > 0$$

and so J is contained in $H(p)$. So, it is approximated by periodic points and we can assume that these points have $\delta-$weak contraction along the direction E_2 (see lemma 2.3.3). Now we take

$$W^u_\epsilon(J) = \cup_{\{x \in J\}} W^{E_3}_\epsilon(x)$$

and a sequences of periodic point $\{q_n\}$ close to some point in the interior of J.

We have that either there are periodic points of the sequences $\{q_n\}$ which are not contained in $W^u_\epsilon(J)$ or they are contained in $W^u_\epsilon(J)$.

In the first case, we have that the strong stable manifold of some q of the sequences $\{q_n\}$, intersect $W^u_\epsilon(J)$ and $q \notin W^u_\epsilon(J)$. Taking $y = W^{E_1}_\epsilon(q) \cap W^u_\epsilon(J)$ follows that the backward orbit of the point y remains in U and so the point belongs to the homoclinic class and then we conclude that there is a periodic point such that its strong stable manifold intersects the homoclinic class; i.e.: we proved that there exists a periodic point q having weak contraction along the center direction and such that $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$.

So, to conclude the proof it is enough to conclude that the second case cannot occur. If a periodic point q of the sequences $\{q_n\}$ is contained in J, using the fact that q is periodic and J does not increase the size for negative iterates, we conclude that there is a point of different stable index in J which is an absurd. In fact, if q is contained in J observe that $J \subset W^{E_2}_\epsilon(q)$. Let r be the period of q, so $f^{-rk}(J) \subset W^{E_2}_\epsilon(q)$ for any positive k, and so taking $L = \cup_{k>0} f^{-rk}(J)$ follows that $L \subset W^{E_2}_\epsilon(q)$. Moreover, L is invariant by f^{-r} and $f^{-r}_{/L} : L \to L$ is an homeomorphism where q is a repelling fixed point. Taking $y' \in W^s(q) \cap L$ we get that there is $q' = \lim_{k\to\infty} f^{-kr}(y') \in L$ and so q' is an attracting fixed point for f^{-r}, i.e.: q' is a repelling periodic point for f in H_p. Which is an absurd because we are assuming that the periodic points in H_p has the same index .

In the case that a periodic point q of the sequences $\{q_n\}$ is not contained in J but contained in $W^u_\epsilon(J)$, we get that the unstable manifold of q intersects J. Using again that J does not increase the size by negative iteration, we conclude that there is an arc I contained in the center manifold of q such that does not increase the size by negative iterations, and again this implies that there is a periodic point of different stable index in U which is an absurd.

To finish, we have to prove that either:

1. $\ell(f^n(W^{E_2}_\epsilon(x))) \to 0$ as $n \to +\infty$ or
2. there exists a periodic point q having weak contraction along the center direction and such that $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$.

The argument to prove it, is similar to the one already performed and we leave it to the reader.

■

Proof of proposition 2.7:

To prove it, we start with the following lemma:

Lemma 2.4.1 *For any $\delta > 0$ small, there is $n_0 = n_0(\delta)$ such that for any $x \in \Lambda$ and any $n \geq n_0$ holds*

$$|Df^n_{|E_2(x)}| < (1+\delta)^n.$$

Observe that this lemma implies the proposition 2.7: In fact, since $\frac{|Df^n_{|E_1}|}{|Df^n_{|E_2}|} < C\lambda^n$ (recall that E_2 is one dimensional) then $|Df^n_{|E_1}| < C(\lambda(1+\delta))^n$ so if δ is small enough follows that $\lambda(1+\delta) < 1$.

To proceed with the proof of the lemma, we have to state a lemma due to Pliss:

Pliss's Lemma: Given $0 < \gamma_0 < \gamma_1$ and $a > 0$, there exist $N_1 = N_1(\gamma_0, \gamma_1, a)$ and $l = l(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $\{a_i\}_{0 \leq i \leq n}$ with $n > N_1$, $a^{-1} < a_i < a$ and $\Pi_{i=0}^n a_i < \gamma^n$ then there exist n_0 with $n_0 < ln$ such that

$$\Pi_{i=n_0}^j a_i < \gamma_1^{j-n_0} \quad n_0 < j < n.$$

So, if the lemma 2.4.1 is not true, we get that there is a sequences of points x_n and an increasing sequences of positive integers k_n such that $|Df^{k_n}_{|E_2(x_n)}| > (1+\delta)^{k_n}$, i.e.: $|Df^{-k_n}_{|E_2(f^{k_n}(x_n))}| < (1+\delta)^{-k_n}$. Using Pliss's lemma holds that there exist points y_n, and integer n_0 and an increasing sequences of positive integers j_n such that $|Df^{-j}_{|E_2(y_n)}| < (1+\frac{\delta}{2})^{-j}$ for $n_0 < j < j_n$. Taking an accumulation point x of the sequences y_n follows that

$$|Df^{-n}_{|E_2(x)}| < (1+\frac{\delta}{2})^{-n} \;\; \forall\, n > n_0.$$

Then, the center manifold along x is stable for f^{-1}, which is a contradiction. In fact, to see that it is proved a folklore claim that we repeat here for completeness. The claim states the following

Claim 1 *Let $g \in Diff^r(M)$ having a dominated splitting $T_\Lambda M = \oplus_{i=1}^k E_i$ on a compact invariant set Λ follows that if for some subbundle $E = E_i$ and some $x \in \Lambda$ holds that there exists $\delta > 0$ and n_0 such that*

$$\Pi_{i=0}^{n-1} |Dg_{|E(g^i(x))}| < (1-\delta)^n \;\; \forall n > n_0$$

then there exists δ_0 such that

1. $g^n(W^E_{\delta_0}(x)) \subset W^E_{\delta_0}(g^n(x))$ *for any* $n > n_0$,
2. $\ell(g^n(W^E_{\delta_0}(x))) \to 0$ *as* $n \to +\infty$.

To get that observe that given $\delta_2 > 0$ there exists $\delta_3 > 0$ such that for any $y \in \Lambda$ and $z \in W^E_{\delta_3}(y)$ follows that

$$\frac{|Dg_{|\tilde{E}(z)}|}{|Dg_{|E(y)}|} < 1 + \delta_2 \text{ where } \tilde{E}(z) = T_z W^E_{\delta_3}(y).$$

Then, it is taken δ_2 such that $(1-\delta)(1+\delta_2) < \gamma < 1$ for some $\gamma < 1$. Then we can take $\delta_0 > 0$ such that $\delta_0 < \delta_3$ and

$$\ell(g^k(W^E_{\delta_0}(x))) \subset W^E_{\delta_3}(g^k(x)) \; 1 \leq k \leq n_0.$$

From that, follows if $z \in W^E_{\delta_0}(x)$ then

$$\frac{|Dg^{n_0}_{|\tilde{E}(z)}|}{|Dg^{n_0}_{|E(y)}|} < (1+\delta_2)^{n_0}$$

and so

$$\Pi_{i=0}^{n_0-1}|Dg_{|\tilde{E}(g^i(z))}| < |Dg_{|E(g^i(y))}|(1+\delta_2)^{n_0} < ((1-\delta)(1+\delta_2))^{n_0} < \gamma^{n_0}$$

and so

$$g^{n_0}(W^E_{\delta_0}(x)) \subset W^E_{\delta_0}(g^{n_0}(x)).$$

Making and inductive argument, the claim follows.

■

Coming back to the proof of the lemma, we can apply the previous claim to the subbundle E_2 using that E_2 is one dimensional and so

$$|Df^n_{|E_2(x)}| = \Pi_{i=0}^n |Df_{|E_2(f^i(x))}|$$

■

3 Proof of Theorem C.

In the proof we use a theorem in [BC] that allows us to reduce the problem to a problem for surfaces diffeomorphisms. To conclude, we adapt a theorem proved in [PS1].

First we start recalling the definition of normally hyperbolic submanifold.

Definition 13 *We say that an invariant submanifold S is normally hyperbolic if there is a splitting $T_S M = E^s \oplus F \oplus E^u$ such that*

1. *E^s is contractive;*
2. *there is $\lambda < 1$ such that $|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}| < \lambda$ for any $x \in S$;*
3. *E^u is expansive;*
4. *there is $\lambda < 1$ such that $|Df_{|F(x)}||Df^{-1}_{|E^u(f(x))}| < \lambda$ for any $x \in S$;*
5. *$T_x S = F(x)$ for any $x \in S$.*

If it holds that $f \in Diff^r(M)$ and

$$|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}|^r < \lambda < 1 \quad |Df_{|F(x)}|^r |Df^{-1}_{|E^u(f(x))}| < \lambda < 1$$

it is said that S is $r-$normally hyperbolic and follows that S is C^r (see [HPS]).

Theorem 3.1 *([BC]) Let $f \in Diff^r(M)$ $(r \geq 1)$ be a diffeomorphism on a compact manifold M. Let Λ be a compact maximal invariant set exhibiting a dominated splitting $T_\Lambda = E^s \oplus F \oplus E^u$ where E^s is contractive and E^u is expansive. Let also assume that for every $x \in \Lambda$ holds that $W^{ss}_\epsilon(x) \cap \Lambda = \{x\}$ (where $W^{ss}_\epsilon(x)$ is the local strong stable manifold tangent to E^s) and $W^{uu}_\epsilon(x) \cap \Lambda = \{x\}$ (where $W^{uu}_\epsilon(x)$ is the local strong unstable manifold tangent to E^u) . Then, there exist two C^1-submanifold normally hyperbolic S and $\hat{S}$ such that,*

1. *$T_x S = F(x)$,*
2. *$S \subset \hat{S}$,*
3. *$\Lambda \subset S$, $f(S) \subset \hat{S}$ and $f^{-1}(S) \subset \hat{S}$.*

Applying the previous theorem to the homoclinic class H_p follows the next corollary:

Corollary 3.1 *Let H_p be a topological hyperbolic homoclinic class exhibiting a dominated splitting $E^s_1 \oplus E_2 \oplus E_3 \oplus E^u_4$ such that $E^s_1 \oplus E_2$ is topologically contractive, $E_3 \oplus E^u_4$ is topologically expansive, E_2, E_3 are one dimensional subbundle, and $\mathcal{T}^{ss} = \emptyset$ and $\mathcal{T}^{uu} = \emptyset$. Then there is a C^1-submanifold S containing H_p and such that $f_{|S}$ is a C^1-surface map exhibiting a dominated splitting.*

Even f is C^2, the manifold obtained in theorem 3.1 it could be only C^1. In fact, if there is a periodic point q in H_p with stable eigenvalues λ_1 and λ_2 such that $0 < \lambda_1 < \lambda_2$ but $\lambda_2^2 < \lambda_1$ follows that S cannot be C^2.

On the other hand, for C^2-surfaces maps exhibiting a dominated splitting it is possible to obtain a well description of the limit set:

Theorem 3.2 *([PS1]) Let $f \in Diff^2(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting such that any periodic point is a hyperbolic saddle periodic point. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

Due to the fact that S is C^1, the restriction of f to the submanifold S is only C^1 (even f is C^2). So, the two dimensional result stated above cannot be directly applied. However, using that H_p is topologically hyperbolic, we have some extra properties associated to f: *the manifold tangent to E_2 and E_3 are dynamically defined, being stable and unstable respectively.* So, we are in a situation that we have more information for the map f restricted to S. We use this extra information to show that another extra property holds along the stable and unstable manifold (called bounded distortion). This extra property, allows to get a generalization of the theorem 3.2 for C^1-diffeomorphism, and this generalization finish the proof of theorem C.

To be more precise, we have to introduce some definitions for two dimensional diffeomorphisms.

Let S be a surface and $f \in Diff^1(S)$. Let us assume that f has an invariant set Λ exhibiting a two dimensional dominated splitting $E \oplus F$. Recall that for each subbundle and for every point $x \in \Lambda$ we have associated the tangent manifolds $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$.

Definition 14 *We say that $W^F_\epsilon(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W^F_\epsilon(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^{-i}(J)) \leq \delta$ for $0 \leq i \leq n$ then*

1. $\frac{|Df^{-n}_{/\tilde{F}(y)}|}{|Df^{-n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$,

2. $|Df^{-n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J))) \quad \tilde{F}(y) = T_y W^F_\epsilon(x)$.

We say that $W^E_\epsilon(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W^E_\epsilon(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^i(J)) \leq \delta$ for $0 \leq i \leq n$ then

1. $\frac{|Df^{n}_{/\tilde{F}(y)}|}{|Df^{n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J)))$,

2. $|Df^{n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J))) \quad \tilde{F}(y) = T_y W^E_\epsilon(x)$.

With this definition in mind, it is possible to get the following result which is a generalization of the theorem 3.2 for C^1-maps on surfaces:

Theorem 3.3 *([PS1]) Let $f \in Diff^1(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting $E \oplus F$ such that any periodic point is a hyperbolic saddle periodic point. Moreover, assume that $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$ has bounded distortion. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ...\mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

The proof of theorem 3.3 is similar to the proof of the theorem 3.2. In fact, in the proof of theorem 3.2 it is only used that f is C^2 to show that the center manifolds are C^2 (see lemma 3.0.3 in [PS1]) and as a consequences of it is proved that the center manifolds have bounded distortion property (see lemma 3.5.1 in [PS1]). In the theorem 3.3, the distortion property are taken for grant. For details we refer to [PS1].

Therefore, to apply theorem 3.3 to the map $f_{/S}$ where S is the submanifold given by proposition 3.1, it is necessary to show that along the local center-unstable manifold and the local center-stable manifold hold the bounded distortion property. The center manifold are not unique so it could happen that the one chosen are not contained in S. However, if we take the manifold defined as $W^{cs}_\epsilon(x) \cap S$ (where $W^{cs}_\epsilon(x) = W^{E^s_1 \oplus E_2}_\epsilon(x)$) follows that this manifolds are invariant by f, $T_x(W^{cs}_\epsilon(x) \cap S) = E_2(x)$ and they are stable. On the same way, if we take the manifold defined as $W^{cu}_\epsilon(x) \cap S$ (where $W^{cs}_\epsilon(x) = W^{E^s_1 \oplus E_2}_\epsilon(x)$) follows that this manifolds are invariant by f, $T_x(W^{cu}_\epsilon(x) \cap S) = E_3(x)$ and they are unstable.

Proposition 3.1 *Let $f \in Diff^2(M)$ and let H_p be a topologically hyperbolic homoclinic class. Let us assume that there exists a two dimensional C^1-normally submanifold S such $H_p \subset S$. Then, the tangent manifolds $W^{cu}_\epsilon(x) \cap S$ and $W^{cs}_\epsilon(x) \cap S$ have bounded distortion property.*

Proof:

First, we start proving that for f, the stable discs and the unstable manifold are C^2. At this point, it is used that the manifold are dynamically defined. For that, we need the following lemma:

Lemma 3.0.2 *There exist a constant $C > 0$ and $0 < \sigma < 1$ such that for every $x \in \Lambda$ and for all positive integer n the following holds:*

$$|Df^n_{|E^s_1 \oplus E_2(x)}|^2 |Df^{-n}_{|E_3 \oplus E^u_4(f^n(x))}| = \frac{|Df^n_{|E_2(x)}|^2}{|Df^n_{|E_3(x)}|} < C\sigma^n,$$

$$|Df^n_{|E^s_1 \oplus E_2(x)}| |Df^{-n}_{|E_3 \oplus E^u_4(f^n(x))}|^2 = \frac{|Df^n_{|E_2(x)}|}{|Df^n_{|E_3(x)}|^2} < C\sigma^n.$$

Proof of the lemma:

Let us start with the first inequality. Recall that the manifold tangent to E_3 is a unstable manifold. From this, we claim that for any $\delta > 0$ there is $n_0 = n_0(\delta)$ such that for any $n \geq n_0$ holds

$$|Df^n_{|E_3}| > (1-\delta)^n$$

So, if the claim is not true, we get that there is a sequences of points x_n and an increasing sequences of positive integers k_n such that $|Df^{k_n}_{|E_3(x_n)}| < (1-\delta)^{k_n}$. Using Pliss's lemma and that E_3 is a one dimensional subbundle, holds that there exist points y_n, and integer n_0 and an increasing sequences of positive integers j_n such that

$$|Df^j_{|E_3(y_n)}| < (1-\frac{\delta}{2})^j \quad n_0 < j < j_n.$$

Taking an accumulation point x of the sequences y_n follows that,

$$|Df^n_{|E_3(x)}| < (1-\frac{\delta}{2})^n \quad n > n_0.$$

Then, using the claim 1, follows that the manifold $W^u_\epsilon(x)$ is a stable manifold for f, which is a contradiction.

Then,

$$\frac{|Df^n_{|E_2}|}{|Df^n_{|E_3}|^2} = \frac{|Df^n_{|E_2}|}{|Df^n_{|E_3}|}\frac{1}{|Df^n_{|E_3}|} < \lambda^n \frac{1}{(1-\delta)^n} = (\frac{\lambda}{1-\lambda})^n$$

for $n > n_0$; so if δ is small enough the first part of the lemma follows.

For the second inequality, we repeat a similar argument using that the manifold tangent to E_2 is a stable manifold and arguing as in lemma 2.4.1.

■

Now, we can apply a result in [HPS] that establish that if the inequality stated in the previous proposition, then the manifold tangent to $E_1^s \oplus E_2$ and to $E_3 \oplus E_4^u$ are C^2 subbundles and so the local stable and unstable manifold are C^2.

Observe that even the map is C^2, the central leaves $W^{cs}_\epsilon(x) \cap S$ and $W^{cu}_\epsilon(x) \cap S$ could be only C^1 inside the stable discs. However, we can show that they have distortion property:

Lemma 3.0.3 *There exists a constant K such that*

1. *if $y \in W^{cs}_\epsilon(x) \cap S$ follows that*

$$\frac{|Df^n_{|E_2(x)}|}{|Df^n_{|E_2(y)}|} \leq \exp(K\sum_{i=0}^{n-1}|f^i(x) - f^i(y)|).$$

2. *if $y \in W^{cu}_\epsilon(x) \cap S$ follows that*

$$\frac{|Df^{-n}_{|E_3(x)}|}{|Df^{-n}_{|E_3(y)}|} \leq \exp(K\sum_{i=0}^{n-1}|f^{-i}(x) - f^{-i}(y)|).$$

Proof of the lemma:

Let us start with the first inequality (the second is similar). We want to control

$$\frac{|Df^n_{|E_2(x)}|}{|Df^n_{|E_2(y)}|}$$

where $x, y \in J$.

Observe that we can assume that $|Df_{|E}| = |Df_{|E_2}|$ where $E = E^s_1 \oplus E_2$. So

$$\frac{|Df^n_{|E_2(x)}|}{|Df^n_{|E_2(y)}|} = \frac{|Df^n_{|E(x)}|}{|Df^n_{|E(y)}|}$$

Moreover

$$|Df^n_{|E(x)}| = |Df^n_{|E_2(x)}| = \Pi^n_{i=0}|Df_{|E_2(f^i(x))}| = \Pi^n_{i=0}|Df_{|E(f^i(x))}|$$

For each x we have defined the map $y \in W^s_\epsilon(x) \to log|Df_{|E(y)}|$ and recalling that the discs $W^s_\epsilon(x)$ are C^2 follows that the maps $y \in log|Df_{|E(y)}|$ are Lipschitz. Since Λ is compact follows that there is a constant K independent of the discs such that

$$|log(|Df_{|E(x)}|) - log(|Df_{|E(y)}|)| < K|x - y| \;\; \forall\, y \in W^{cs}_\epsilon(x)$$

where K is the Lipschitz constant for $log(|Df_{|E}|)$.

So

$$log(\frac{|Df^n_{|E(x)}|}{|Df^n_{|E(y)}|}) = \sum log(|Df_{|E(f^i(x))}|) - log(|Df_{|E(f^i(x))}|) < K \sum |f^i(x) - f^i(y)|.$$

Then we get that

$$\frac{|Df^n_{|E_2(x)}|}{|Df^n_{|E_2(y)}|} \leq \exp(K \sum_{i=0}^{n-1} |f^i(x) - f^i(y)|).$$

The proof of the second inequality is similar replacing $E^s_1 \oplus E_2$ by $E_3 \oplus E^u_4$. and f by f^{-1}.

■

To finish showing that the manifold has the bounded distortion property, we have to show that they verifies the two second items. This is immediately since the submanifold are C^1.

■

3.1 Topologically hyperbolic sets of Kupka-Smale diffeomorphisms are not necessary hyperbolic.

Before to end the section we would like to make some remarks. Observe that when $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ are empties, it was proved that the homoclinic class is hyperbolic. It is natural to ask if it is possible to get a similar result when either $\mathcal{T}^{ss}$ or $\mathcal{T}^{uu}$ are not empty. In other words, *given a Kupka-Smale* *topologically hyperbolic class such that the strong subbundle is involved in the dynamic, is it true that the homoclinic class is hyperbolic?*

The answer is no and it is easy to construct a counterexample:

Let $H_p = \cap_{\{n>0\}} f^n(U)$ be a hyperbolic attracting homoclinic class for a surface diffeomorphism f. Let $\mathcal{M}$ be a minimal set contained in H_p.
Let $h : U \to R$ be a C^∞ function such that

1. $0 < h(x) \leq 1$ for all $x \in U$,
2. $h_{|\mathcal{M}} = 1$,
3. $h_{|\mathcal{M}^c} < 1$.

Let $F : U \times [-1, 1] \to U \times [-1, 1]$ defined as

$$F(x, y) = (f(x), h(x)y - y^3)$$

Observe that the set

$$H_p \times \{0\}$$

is an attracting homoclinic class. In fact,

$$H_p \times \{0\} = \cap_{\{n>0\}} F^n(U \times [\frac{-1}{2}, \frac{1}{2}]).$$

Moreover, the homoclinic class has dominated splitting $E_1^s \oplus E_2 \oplus E_3^u$ and $F_{|U\times[-1,1]}$ is a Kupka-Smale system. This follows from the fact that the periodic points of F are contained in the complement of $\mathcal{M}$ and in this set the center subbundle is contractive from the fact that $|DF_{|E_2(x)}| = |h(x)| < 1$ for any $x \in \mathcal{M}^c$. However, F is not hyperbolic from the fact that $|DF_{|E_2(x)}|$ is equal to one when $x \in \mathcal{M}$.

4 Proof of theorem D:

In this section we assume that either the interior of $\mathcal{T}^{ss}$ is not empty or $\mathcal{T}^{uu}$ is not empty. First we need a lemma equivalent to theorem 2.3 for the topologically hyperbolic sets considered in the hypothesis of theorem D. The prof is immediately and it is similar to the proof of the classical Anosov closing lemma for hyperbolic sets:

Lemma 4.0.1 *Let $f \in Diff^1(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive, E_2 and E_3 are one dimensional. Then it follows that:*

1. *if E_2 is not contractive, then for any δ there is a periodic point in q having $\delta-$weak contraction along the direction E_2;*
2. *if E_3 is not expansive, then for any δ there is a periodic point in q having $\delta-$weak expansion along the direction E_3.*

Proof:

If E_2 is not contractive, using the Pliss's lemma, the fact that E_2 is one dimensional and the lemma 2.4.1 follows that there is a point x and sequences of iterates $\{f^{k_n}(x)\}$ such that

$$(1-\delta)^m < |Df^n_{|E_2(f^{k_n}(x))}| < (1+\delta)^m, \quad \forall\, n > n_0 = n_0(\delta).$$

Taking $f^{k_{n_1}}(x)$ and $f^{k_{n_2}}(x)$ close enough, and using that the manifold tangent to $E = E_1^s \oplus E_2$ and $F = E_3 \oplus E_4^u$ are dynamically defined, follows that we can get a periodic point q shadowing the orbit

$$\{f^{k_{n_1}+j}(x)\}_{\{0 \le j \le n_2 - n_1\}}.$$

For details about that see lemma 6.2.1. Using that the subbundle E_2 on the orbit of q is close to the subbundle E_2 on the orbit of $f^{k_{n_1}}(x)$ up to the iterate $n_2 - n_1$, follows the conclusion of the lemma.

The proof of the second item is similar.

■

Let us start considering the case that the interior of $\mathcal{T}^{ss}$ is not empty. From the fact that the periodic points with weak contraction along the direction E_2 are dense (in fact, a similar lemma to lemma 6.2.4 can be obtained for the present situation), follows immediately that there exists a periodic point q with weak contraction along the center direction such that $[W^{ss}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$. Then, applying the C^1-connecting lemma and proposition 2.4, we conclude the theorem D. The proof of the case that $\mathcal{T}^{uu}$ is not empty, is similar.

Before to finish the section, we would like to make some remarks. If we want to get a complete results, it remains to consider the case that $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ are not empty but the interior of $\mathcal{T}^{ss}$ and $\mathcal{T}^{uu}$ are empty.

If for instance $\mathcal{T}^{ss}$ is not empty, follows that there exists a pair of points in the homoclinic class x, y such that $y \in W^{ss}_{\epsilon}(x)$. On one hand, observe that x is accumulated by a sequence $\{q_n\}$ of periodic points and so it follows that there is a sequences of points $\{q_n^*\}$ such that $q_n^* \in W^{ss}_{\epsilon}(q_n)$ and $q_n^* \to y$. Moreover, we can assume that the periodic points q_n have weak contraction along the center direction. On the other hand, the unstable manifold of p accumulates on y and therefore, the unstable manifold of the points q_n also accumulates on y. Observe that if it holds that for some q_n holds that $q_n^* \in H_p$ the we can apply the C^1-connecting lemma. However, even if for any q_n holds that $[W^{ss}_{\epsilon}(q_n) \setminus \{q_n\}] \cap H_p = \emptyset$, since $y \in H_p$ and $q_n^* \to y$, it is natural to try to perform some kind of connecting lemma argument's with the goal to connect the unstable manifold of one of the points q_n with the local strong stable manifold of the same point. If this type of perturbation can be done, then a heterodimensional cycle is created.

However, to use the connecting lemma it is necessary to assume some restrictions over the orbits of the periodic points $\{q_n\}$. For instance, if the periodic points $\{q_n\}$ do not accumulate on y then it can be applied the connecting lemma. On the other hands, if it occurs that the periodic points $\{q_n\}$ do accumulate on y, then connecting lemma argument's can not be performed. In fact, if the pair of points x and y belongs to a minimal invariant set contained in H_p, then the situation mentioned above holds. Therefore, it is necessary to develop other techniques to deal with these type of situation.

The paper [Pu] is devoted to overcome these difficulties.

5 Extended versions of Theorem A and B.

The next two theorems give a better description of the kind of homoclinic bifurcation that may occur in the case that the homoclinic class is not hyperbolic. This description it is related to the different kind of splitting that the attractor may exhibit. We state different theorems for the case that the point p has stable index either one or two.

To clarify this, we need to recall some results about homoclinic tangencies and the relation between tangencies and the presence of heterodimensional cycles. For surfaces maps, the unfolding of a homoclinic tangencies leads to the nowadays so-called "Newhouse phenomena", i.e., residual subsets of diffeomorphisms displaying infinitely many periodic attractors. In particular, this shows that the unfolding of tangencies "destroys" transitive sets. This phenomena is not valid in higher dimension. In fact, robust transitive sets can coexist with the presence of an homoclinic tangency (see for instances the examples showed in [BV] of robust transitive systems). In these examples, tangencies and heterodimensional cycles are coexisting.

On the other hand, it was shown in [PV] that in dimension larger than two, the unfold of tangencies associated to sectional dissipative periodic points (the modulus of the product of any pair of eigenvalues is smaller than one) leads to the same Newhouse phenomena that holds in dimension two.

Regarding the previous remarks, some partial result are concluded.

Theorem F: *Let $f \in Diff^2(M^3)$ be a Kupka-Smale system.*
Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point of stable index one. Then, the following options holds:

1. *If H_p does not exhibit any dominated splitting, then there exists a g C^1- close to f such that g has a homoclinic tangency and a heterodimensional cycle in U.*

2. *If H_p exhibits a dominated splitting but it does not exhibit any dominated splitting $E \oplus F$ with $dim(F) = 2$; then there exists a g C^1- close to f having a heterodimensional cycle and a homoclinic tangency in U.*

3. *If H_p has a dominated splitting $E \oplus F$ with $dim(F) = 2$ and F cannot be decomposed in two subbundles with domination, then follows that either*

    - *H_p is hyperbolic or*
    - *there exists a g C^1- close to f exhibiting a homoclinic tangency associated to a point of stable index one and exhibiting a heterodimensional cycle in U.*

4. *If H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that either*

    - *H_p is hyperbolic or*
    - *there exists a g C^1- close to f exhibiting a heterodimensional cycle in U.*

Remark 5.1 *Observe that in the previous theorem, any time that it can be created a tangency by a C^1-perturbation it also can be created a heterodimensional cycle.*

Let us assume now that p has stable index two. In this case it is not possible to get a strong version as in theorem F.

Theorem G: *Let $f \in Diff^2(M^3)$ be a Kupka-Smale system. Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point of stable index two. Then, the following options holds:*

1. *If H_p does not exhibit any dominated splitting, then there exists a g C^1- close to f such that g exhibits a homoclinic tangency and a heterodimensional cycle in U.*
2. *If H_p has a dominated splitting $E \oplus F$ with $dim(E) = 2$ and E cannot be decomposed in two subbundles then follows that either*
   - *H_p is hyperbolic or*
   - *there exists a g C^1- close to f exhibiting a homoclinic tangency associated to a point of stable index one and exhibiting a heterodimensional cycle in U.*
3. *If H_p has a dominated splitting $E \oplus F$ with $dim(E) = 1$ such that F cannot be decomposed in two subbundles. Then follows that either:*
   (a) *there is a g C^1- close to f exhibiting a homoclinic tangency and a heterodimensional cycle in U;*
   (b) *all the periodic points in H_p has stable index two, E is uniformly contractive and one of the following options holds:*
      - *there exists a g C^1- close to f exhibiting a sectional dissipative homoclinic tangency in U and the set H_p is contained in a normally hyperbolic submanifold;*
      - *the set $\mathcal{T}$ is not empty, (i.e.: there exists x such that $[W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset$) and there exists a g C^1- close to f exhibiting a homoclinic tangency in U;*
4. *If H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that either*
   - *H_p is topologically hyperbolic or*
   - *there exists a g C^1- close to f exhibiting a heterodimensional cycle in U.*

Remark 5.2 *To get a better description it remains the question that if in the case* 3.*b when $\mathcal{T}$ is not empty it also follows that a heterodimensional cycle can be created.*

Observe that from theorem A and B follows the last case of theorem F and G. In fact, if H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that either H_p is hyperbolic (or topologically hyperbolic in case of theorem B) or there exists a g C^1- close to f exhibiting a heterodimensional cycle.

5.1 Proof of Theorem F:

5.1.1 There is not a dominated splitting.

Following the techniques in [BDP] it can be proved that if H_p has not a dominated splitting then there is a diffeomorphisms g close to f and a periodic point q with orbit arbitrarily close to H_p such that $D_q g^{n_q}$ has three eigenspaces with arbitrarily small angle, having only real eigenvalues and at most one eigenvalue with modulus smaller than one.

From the fact that the angle between all the eigenspaces is small, follows that by a C^1 perturbation, a tangency can be created between the strong subbundles. From there, follows that we get a periodic point having a strong homoclinic connection. Therefore, from the fact the angle between the eigenspaces is small, by another C^1-perturbation follows that the center eigenvalue has a weak expansion. Therefore, we a get a periodic point with weak expansion along the center subbundle and exhibiting a strong homoclinic connection. Then, applying proposition 2.4 follows the existences of a heterodimensional cycle.

5.1.2 There is not a dominated splitting $T_{H_p}M = E \oplus F$ with $dim(F) = 2$.

In the present case, we want to show that by C^1-perturbations we can create a tangency and a heterodimensional cycle. First we state a proposition whic is usefull in what follows and shows a general mechanism to create heterodimensional cycles.

Proposition 5.1 *Let $g \in Diff^r(M^3)$ and $\delta > 0$ such that*

1. *g has two hyperbolic periodic points q_1 and q_2 verifying*
   - *(a) q_1 and q_2 are homoclinically connected,*
   - *(b) $W^u(q_2) \cap W^{ss}_\epsilon(q_1) \neq \emptyset$;*
2. *there exists a periodic points q_δ with $\frac{\delta}{2}-$weak contraction along the center direction and homoclinically related with q_1.*

Then, there is $\hat{g}$ arbitrarily C^k-close to g and a periodic point $\hat{q}_\delta$ with $\delta-$weak contraction along the center direction and exhibiting a strong homoclinic connection.

Observe that the previous proposition implies the theorem F. So the goal is to show that if the interior of $\mathcal{T}$ is empty, for any $\delta > 0$ we can get by perturbation a diffeomorphisms g C^1-arbitrarily close to f verifying the hypothesis of proposition 5.1. Before to do that, let us show the proof of proposition 5.1.

Proof of propositions 5.1:

By lemma 2.3.3 we get a sequences of periodic points q^n_δ such that q^n_δ accumulates on q_1, they are homoclinically connected with q_1 and they have $\delta-$weak contraction along the center direction. Moreover, we can suppose that the orbits of these points do not accumulate in q_2 and so they do not accumulate over the point of intersection between the unstable manifold of q_2 and the strong stable manifold of q_1. Observe that the strong stable manifold of the points q_δ^n accumulate over the local strong stable manifold of q_1. Since the points q_δ^n are homoclinically connected with q_1 and so with q_2, follows that their unstable manifolds accumulate over the connected arc of the unstable manifold of q_2 that contains q_2 and a point $z \in W^u(q_2) \cap W_\epsilon^{ss}(q_1)$. Then, it is possible to unfold the intersection of the unstable manifold of q_2 with the strong stable manifold of q_1 in such a way that the unstable manifold of some q_δ^n intersects the local strong stable manifold of the same q_δ^n.

More precisely, we can do that performing two perturbation: First, it is performed an arbitrarily small perturbation such that the unstable manifold of q_2 intersect the strong stable manifold of same q_δ^n sufficiently close to q_1. Since q_δ^n remains homoclinically connected with q_2, follows that they are arcs contained in the unstable manifold of q_δ^n that accumulates over the connected arc of the unstable manifold of q_2 that contains q_2 and a point $z \in W^u(q_2) \cap W_\epsilon^{ss}(q_1)$. The second perturbation consists in unfolding the intersection of the unstable manifold of q_2 with the strong stable manifold of q_δ^n in a way that the unstable manifold of q_δ^n intersect the local strong stable manifold of the same point.

The first perturbation is straightforward from the fact that the orbits of the points q_δ^n do not accumulate over q_2 For the second one, we take a sequences of compacts arcs $\{l_m\}_m$ contained in the unstable manifold of q_δ^n such that:

1. the arcs $\{l_m\}$ accumulates in a compact arc l which is contained in the unstable manifold of q_2 and it contains the point q_2 and a point in $W^u(q_2) \cap W_\epsilon^{ss}(q_\delta^n)$;

2. for each m follows that $\{f^{-i}(l_m)\}_{\{i>0\}}$ does not accumulate on l.

So, perturbing g in a way to unfold the intersection of l with $W_\epsilon^{ss}(q_\delta^n)$ and at the same time not perturbing q_δ^n, follows that the unstable manifold of q_δ^n intersects $W_\epsilon^{ss}(q_\delta^n)$ and this conclude the proof of the proposition 5.1.

■

First we state a lemma that shows that in the hypothesis of the present case,then it can be created a heterodimensional cycle.

Proposition 5.2 *Let us assume that H_p is an attractor. If p has stable index one, then there exist g arbitrarily C^1-close to f and a periodic point q of f such that the analytic continuation q_g is homoclinically connected with p_g and it exhibits a strong homoclinic intersection.*

Proof:

Let us assume first that p has real eigenvalues. In this case, we can show that the point p verifies the thesis of the lemma. In fact, from the fact that H_p is an attractor follows that the strong unstable manifold of p is contained in the homoclinic class. On the other hand, there are orbits in the homoclinic class accumulating in the stable manifold of p, which is one dimensional, so its coincide with the strong stable manifold. Then, using the C^1-connecting lemma we can perturb the systems in a way to connect the strong stable and unstable manifold of p.

If p has complex expanding eigenvalue we use the following lemma which is a consequences of lemma 2.1.7:

Lemma 5.1.1 *Let p be a periodic point with complex eigenvalue and stable index one. Let us assume that there is a transversal intersection of the stable and unstable manifold of p. Then, there exists a periodic point q of f and a diffeomorphisms g C^1-close to f such that q has real eigenvalues and the strong unstable manifold of q intersect the stable manifold of p.*

The proof of the lemma is similar to the proof of lemma 5.1.1.

To conclude the proof of proposition 5.2, if p has an expanding complex eigenvalue, first it is created a transversal homoclinic intersection and later it is applied the previous lemma. Now, observe that by theorem 2.1, if there is not a splitting $T_{H_p}M = E \oplus F$ with $dim(F) = 2$ then it implies that there exist g C^1-close to f and a periodic point q' of f such that the analytic continuation q'_g has a tangency between the stable and unstable manifold and q'_g. Using that we are dealing with a homoclinic classes, it can be proved that q'_g is homoclinically connected with p_g. In particular, it is connected with the point q_g obtained in the previous lemma.

Unfolding the tangency, we can get another tangency and another periodic point $\hat{q}_g$ homoclinically connected with q'_g and q_g, and exhibiting a weak expansion along the center direction (see for instance [PT]). Then, we have a periodic point with real eigenvalues having a weak expansion along the center direction and homoclinically connected with a periodic point having a strong homoclinic connection. In fact, to finish the proof of proposition 5.2, we use proposition 5.1.
Therefore, to finish the proof of theorem F in the present case, we need to create a tangency by a small perturbation. To do that, we apply the first part of theorem 2.1.

■

5.1.3 There is a dominated splitting $T_{H_p}M = E \oplus F$ with $dim(F) = 2$.

From theorem 2.1 follows that if F cannot be decomposed in two subbundle then either F is expansive or it can be created a tangency. Moreover, by lemma 2.1.2 applied to f^{-1}, follows that there is a periodic point of stable index one with real eigenvalues and having a weak expansion along the center direction. On the other hand, by lemma 5.1.1, we get a periodic point with real eigenvalues and exhibiting a strong homoclinic connection. Using proposition 5.1 it is conclude the existence of a heterodimensional cycle for a perturbation of the initial map.

To deal with the situation that the splitting is decomposed in three subbundle we proceed as in the proof of the main theorem.

5.2 Proof of theorem G.

In the case that there are not a dominated splitting we proceed as in the previous theorem.

5.2.1 There is a dominated $E \oplus F$ with $dim(E) = 2$, and E cannot be decomposed in two subbundles.

In this case, by theorem 2.1 follows that either E is contractive or there exists a g C^1- close to f exhibiting a homoclinic tangency. To finish the proof, remains to show that it can also be created a heterodimensional cycle. Since we are assuming that E cannot be decomposed in two invariant subbundles and E is not contractive, by lemma 2.1.2 follows that for any $\gamma > 0$ and $\delta > 0$ there exists a periodic point q for a diffeomorphisms g C^1-close to f such that

1. q has two real contractive eigenvalues;
2. $(1-\delta_2)^{n_q} < |Df^{n_q}_{|E^s_2(q)}| < 1$;
3. $\alpha(E^s_1(q), E^s_2(q)) < \gamma$
4. q has a transversal homoclinic point.

Then, using that the angle between $E^s_1(q)$ and $E^s_2(q)$ is small it can be shown that after a second perturbation it is possible to get a strong homoclinic connection. Since one of the stable eigenvalues is close to one, then it can be performed a perturbation to get a heterodimensional cycle.

5.2.2 $T_{H_p}M = E \oplus F$ with $dim(E) = 1$.

Let us assume that there is a periodic point q in H_p with stable index one. using that E is one dimensional and therefore the local manifold tangent to E is dynamically defined, we can argue as in lemma 2.3.2 and it is proved that by a C^1-perturbation it is created a heterodimensional cycle. From the fact that F cannot be decomposed in two subbundle, follows also the existences of a tangency.

Now we deal with the case that all the periodic points in H_p has stable index two. First it is proved that in this case, the subbundle E is contractive.

Lemma 5.2.1 *Let us assume that all the periodic points in H_p has stable index two. Then follows that E_1 is contractive.*

Proof:

The proof is similar to the proof of the lemma 2.4.1. In fact, if the thesis does not hold, for any $\delta > 0$, using the Pliss's lemma and that the subbundle E_1 is one dimensional, it is possible to find a point x such that

$$\Pi^n_{i=0}|Df_{E(f^i(x)))}| > (1-\delta)^n$$

for any $n > n_0$ and some n_0. Again, by the Pliss's lemma, we can get a sequences n_k converging to infinity such that

$$\Pi^n_{i=0}|Df_{E(f^{n_k}(x)))}| > (1-\delta)^n.$$

By the domination property, follows that

$$\Pi^n_{i=0}|Df^{-1}_{F(f^{-n_k}(x)))}| < (\frac{\lambda}{1-\delta})^n.$$

Observe that this implies that there exists $\epsilon_0 = \epsilon_0(\delta, f, n_0) > 0$ for any $f^{-n_k}(x)$ follows that

$$W^F_{\epsilon_0}(f^{-n_k}(x)) \subset W^u_{\epsilon_0}(f^{-n_k}(x))$$

where$W^F_{\epsilon_0}(y)$ is the manifold tangent to the subbundle F. Taking two integers n_{k_1} and n_{k_2} such that $f^{-n_{k_1}}(x)$ and $f^{-n_{k_2}}(x)$ are close, using that the manifold tangent to the subbundle E is dynamically defined (because the subbundle E is one dimensional), then we can find a periodic point q of stable index one close to $f^{-n_{k_1}}(x)$ which is a contradiction with the hypothesis.

■

Then, we have two options: either E is involved in the dynamics or it is not the case; ie.: either $\mathcal{T}^{ss}$ is not empty or it is empty. In the second case, we can apply the theorem 3.1 and follows that there exists a C^1 two dimensional normally hyperbolic submanifold S such that the homoclinic class is contained in S. Since H_p is an attractor, follows that $Df_{|S}$ cannot be volume expanding. Moreover, follows that there is a periodic point q which is dissipative restricted to S. Since the subbundle E is contractive, follows that q is dissipative. From the fact that F cannot be decomposed in two subbundle having a dominated splitting, follows that there is g C^1-close to f exhibiting a tangency. Using that we are dealing with a homoclinic class, it is possible to show that the tangency is associated to a periodic point q' that remains homoclinically connected with q. Therefore, we can get a tangency associated to q, which is a sectional dissipative periodic point.

6 Topological hyperbolic sets.

In the first subsection, we give some dynamical properties of the topological hyperbolic sets. This is done basically using the notion of adapted metric for expansive maps introduced by Fathi in [Fa]. In the second subsection, it is proved theorem E2 where it is characterized the continuation of a topological hyperbolic set. In the third subsection, proving theorem E3 we analyze the particular case that Λ exhibits a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles. In the last subsection, it is studied the case that $E_4^u = \{0\}$.

6.1 Dynamical properties of a topological hyperbolic set. Proof of theorem E1.

Recall that we are assuming that Λ is a maximal compact invariant that exhibits a dominated splitting $E \oplus F$ and that there are continuous functions

$$\phi^s : \Lambda \to Emb^1(D_1^s, M) \quad \phi^u : H_p \to Emb^1(D_1^u, M),$$

where

1. $D_1^s = \{z \in R^s : ||z|| < 1\}$; $D_\epsilon^s = \{z \in R^s : ||z|| < \epsilon\}$ with $s = dim(E)$,
2. $D_1^u = \{z \in R^s : ||z|| < 1\}$; $D_\epsilon^u = \{z \in R^s : ||z|| < \epsilon\}$ with $u = dim(F)$,

and such that for any $x \in \Lambda$ it is defined

$$W_\epsilon^s(x) = \phi^s(x) D_\epsilon^s; \;\; W_\epsilon^u(x) = \phi^u(x) D_\epsilon^u$$

verifying

1. $T_x W_\epsilon^s(x) = E(x)$, $T_x W_\epsilon^u(x) = F(x)$
2. $W_\epsilon^s(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0, dist(f^n(x), f^n(y)) < \epsilon\}$,
3. $W_\epsilon^u(x) = \{y \in M : dist(f^{-n}(x), f^{-n}(y)) \to 0, dist(f^{-n}(x), f^{-n}(y)) < \epsilon\}$.

From the above properties, it follows immediately that if Λ is a transitive topologically hyperbolic set then it is a homoclinic class. Moreover, it can be also proved that topologically hyperbolic set contained in the Limit set, they have local product structure and they exhibit a spectral decomposition as it holds for hyperbolic sets. The proof of this facts goes in the same way that goes for hyperbolic sets. We refer to [Sh] for proofs and definitions.

The next proposition states that for topologicaly hyperbolic set it is possible to get a hyperbolic metric (not necessarily coherent with a riemannian structure).

Proposition 6.1 *Given a topologically hyperbolic set, follows that there exists an adapted metric dist compatible with the topology, and there exist constants* $\epsilon > 0$ *and* $0 < \lambda_0 < 1$ *such that*

1. if $y \in W^s_\epsilon(x)$ *then*
$$dist(f^n(x), f^n(y)) < \lambda_0^n dist(x, y).$$

2. if $y \in W^u_\epsilon(x)$ *then*
$$dist(f^{-n}(x), f^{-n}(y)) < \lambda_0^n dist(x, y).$$

Before to give the proof, we state an easy lemma that says that a transitive topologically hyperbolic set is expansive.

Lemma 6.1.1 *There exists* $r > 0$ *such that if* $dist(f^n(x), f^n(y)) < r$ *for any integer* n *then* $x = y$.

Proof:
The proof follows from the fact that for points nearby their local stable and unstable manifold intersect transversally.

■

Proof of proposition 6.1:
It was proved in [Fa] that for expansive homeomorphisms, it is possible to obtain an hyperbolic adapted metric, not necessarily coherent with a riemannian structure.

Lemma 6.1.2 *([Fa]) Given a expansive homeomorphisms* f *in a compact metric space, then there exists an adapted metric dist compatible with the topology, and there exist constants* $r > 0$ *and* $0 < \lambda_0 < 1$ *such that if*

$$dist(f^n(x), f^n(y)) < r \text{ then } dist(f^n(x), f^n(y)) < \lambda_0^n dist(x, y).$$

This lemma can be easily adapted to an expansive compact maximal invariant set, with the property that the metric is defined over a whole neighborhood of Λ.

So, to conclude the proposition it is enough to adapt the previous lemma and to apply it to the local stable manifold and to the local unstable manifold.

■

Using the hyperbolic metric, it is possible to repeat for topologically hyperbolic set the classic construction of Markov partition done for maximal invariant hyperbolic set (see [Bow]). Using the Markov partition, follows immediately that $f_{/\Lambda}$ is conjugated to a subshift of finite symbols.

6.2 Continuation of transitive topologically hyperbolic sets. Proof of theorem E2 and E3.

By theorem E2, we can assume in what follows that we are dealing with a homoclinic class. Observe that it may happen that the homoclinic class H_p does not remain maximal invariant by perturbations. Moreover, it may occur that the set $\Lambda_g(U) = \cap_{\{n\in\mathbb{Z}\}} g^n(U)$ is not topologically hyperbolic. In fact, it is not clear that the tangent manifolds remain a stable and unstable foliation for the perturbed map. However, it is possible to show that after perturbation the maximal invariant set $\Lambda_g(U)$ is semi conjugate to H_p.

6.2.1 Proof of theorem E2:

Using that there exists an adapted metric such that it is a hyperbolic metric for f, and the fact that we are dealing with a homoclinic class which is topologically hyperbolic, the proof of the shadowing lemma for hyperbolic sets with local product structure can be pushed in the present case (see [Fa]).

In other words, it s possible to prove the following lemma:

Lemma 6.2.1 *Given a topologically hyperbolic homoclinic class, there exist $\alpha > 0$ and $\beta > 0$ such that if $\{x_i\}$ is a $\beta-$pseudo orbit (meaning that for all integer i holds that $dist(x_{i+1} - f(x_i)) < \beta$) then there is a unique x such that $dist(f^n(x), x_n) < \alpha$.*

Then, observe that for g close to f and a small neighborhood V of H_p, follows that for any $z \in \cap_{n\in\mathbb{Z}} g^n(V)$, the orbit of z by g, $\{g^n(z)\}_{\{n\in\mathbb{Z}\}}$, is a $\beta-$pseudo orbit. Then there is unique x such that the orbit of x by f shadows the orbit of z by g. Therefore, we can define a map h_g from $\cap_{\{n\in\mathbb{Z}\}} g^n(V)$ to H_p such that for $z \in \cap_{\{n\in\mathbb{Z}\}} g^n(V)$ follows that $h_g(z)$ is the unique point in H_p that its orbits by f shadows the orbit of z by g.

■

6.2.2 Proof of theorem E3.

Observe that the map h_g introduced in theorem E2, is not necessary injective neither onto. In fact, it could happen that after the perturbation, a periodic point q of f bifurcates either along the stable manifold or along the unstable one, in two periodic points with orbits that remains close. In this case, the orbit of this two periodic points are shadowed by the orbit of q. In proposition 6.2 we state it is possible to show that the map is onto and that restricted to some set it is defined the inverse of the map h_g.

Recall that in the hypothesis of theorem E3, we are assuming that H_p has a dominated splitting $E_1^s \oplus E_2 \oplus E_3 \oplus E_4^u$ such that $E_1^s \oplus E_2$ is topologically contractive, $E_3 \oplus E_4^u$ is topologically expansive and E_2, E_3 are one dimensional subbundles. As a consequences of that, follows that:

1. there are continuous functions

$$\phi^{css} : \Lambda \to Emb^1(D_1^s, M) \quad \phi^{cuu} : H_p \to Emb^1(D_1^u, M),$$

   where

   (a) $D_1^s = \{z \in R^s : ||z|| < 1\}$; $D_\epsilon^s = \{z \in R^s : ||z|| < \epsilon\}$ with $s = dim(E_1^s \oplus E_2)$,

   (b) $D_1^u = \{z \in R^s : ||z|| < 1\}$; $D_\epsilon^u = \{z \in R^s : ||z|| < \epsilon\}$ with $u = dim(E_3 \oplus E_4^u)$;

2. there exist continuous functions

$$\phi^{cs} : H_p \to Emb^1(I_1, M) \quad \phi^{cu} : H_p \to Emb^1(I_1, M)$$

   where $I_1 = (-1, 1)$, $I_\epsilon = (-\epsilon, \epsilon)$;

3. there are continuous functions

$$\phi^{ss} : H_p \to Emb^1(D_1^{ss}, M) \quad \phi^{uu} : H_p \to Emb^1(D_1^{uu}, M),$$

where

   (a) $D_1^{ss} = \{z \in R^{ss} : ||z|| < 1\}$; $D_\epsilon^{ss} = \{z \in R^{ss} : ||z|| < \epsilon\}$ with $ss = dim(E_1^s)$,
   (b) $D_1^{uu} = \{z \in R^{uu} : ||z|| < 1\}$; $D_\epsilon^{uu} = \{z \in R^{uu} : ||z|| < \epsilon\}$ with $uu = dim(E_4^u)$;

such that for any $x \in H_p$ it is defined

$$W_\epsilon^{css}(x) = \phi^{css}(x)D_\epsilon^s; \;\; W_\epsilon^{ss}(x) = \phi^{ss}(x)D_\epsilon^{ss}; \;\; W_\epsilon^{cs}(x) = \phi^{cs}(x)I_\epsilon,$$

$$W_\epsilon^{cuu}(x) = \phi^{cuu}(x)D_\epsilon^s; \;\; W_\epsilon^{uu}(x) = \phi^{uu}(x)D_\epsilon^{uu}; \;\; W_\epsilon^{cu}(x) = \phi^{cu}(x)I_\epsilon,$$

and verifying

   (a) $T_xW_\epsilon^{css}(x) = E_1^s \oplus E_2(x)$, $T_xW_\epsilon^{ss}(x) = E_1^s(x)$, $T_xW_\epsilon^{cs}(x) = E_2(x)$,
   (b) $T_xW_\epsilon^{cuu}(x) = E_3 \oplus E_4^u(x)$, $T_xW_\epsilon^{ss}(x) = E_4^s(x)$, $T_xW_\epsilon^{cu}(x) = E_3(x)$,
   (c) $W_\epsilon^{css}(x) = W_\epsilon^s(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0, dist(f^n(x), f^n(y)) < \epsilon\}$,
   (d) $W_\epsilon^{cuu}(x) = W_\epsilon^u(x) = \{y \in M : dist(f^{-n}(x), f^{-n}(y)) \to 0, dist(f^{-n}(x), f^{-n}(y)) < \epsilon\}$,
   (e) $W_\epsilon^{ss}(x) = \{y \in M : dist(f^n(x), f^n(y)) < \lambda_s^n, dist(f^n(x), f^n(y)) < \epsilon\}$,
   (f) $W_\epsilon^{uu}(x) = \{y \in M : dist(f^{-n}(x), f^{-n}(y)) < \lambda_u^n, dist(f^{-n}(x), f^{-n}(y)) < \epsilon\}$,
   (g) $W_\epsilon^{cs}(x) \subset W_\epsilon^{css}(x) = W_\epsilon^s(x)$,
   (h) $W_\epsilon^{cu}(x) \subset W_\epsilon^{cuu}(x) = W_\epsilon^u(x)$.

Now, we need a result about the continuation of a dominated splitting.

Lemma 6.2.2 *Let $f \in Diff^r(M)$ $(r \leq 1)$ and Λ be a compact maximal invariant set of f exhibiting a dominated splitting $T_\Lambda M = \oplus_{i=1}^k E_i$. There exists an open neighborhood $\mathcal{U}$ of f in $Diff^r(M)$ and an open neighborhood U of Λ such that for each $g \in \mathcal{U}$ and any subbundle E_i there exists a continuous function, $T_g : \Lambda_g \to T_{\Lambda_g}M$ and $\phi_g^i : \Lambda_g \times Diff(M) \to Emb^1(D, M)$ such that for any $g \in \mathcal{U}$and $x \in \Lambda_g$ it is defined the dominated splitting $\oplus_{i=1}^k E_i(g)$ and the manifold tangent to $E_i(g)$ is given by $W_\epsilon^{E_i}(x, g) = \phi_g^i(x)D_\epsilon$ and verifying*

1. *$T_xW_\epsilon^{E_i(g)}(x, g) = E_i(g, x)$,*
2. *if $g(W_\epsilon^{E_i(g)}(x, g)) \subset B_\epsilon(g(x))$ then $g(W_\epsilon^{E_i(g)}(x, g)) \subset W_\epsilon^{E_i(g)}(g(x), g)$,*
3. *if $g^{-1}(W_\epsilon^{E_i(g)}(x)) \subset B_\epsilon(g^{-1}(x))$ then $g^{-1}(W_\epsilon^{E_i(g)}(x, g)) \subset W_\epsilon^{E_i(g)}(g^{-1}(x), g)$.*
4. *the maps $g \in \mathcal{U} \to T_g$ and $g \in \mathcal{U} \to Emb^1(D, M)$ are continuous.*

Remark 6.1 *If one of the subbundles of the dominated splitting is hyperbolic, then it remains hyperbolic after a C^r-perturbation of the system.*

We take a small neighborhood V of H_p and for g C^k- close to f we take the set

$$\Lambda_g = \Lambda_g(V) = Closure(\cap_{\{n\in\mathbb{Z}\}} g^n(V)).$$

From lemma 6.2.2 and previous remark, follows that for any g close to f there is a dominated splitting

$$E_1^s(g) \oplus E_2(g) \oplus E_3(g) \oplus E_4^u(g),$$

such the subbundle $E_1^s(g)$ is contractive in $\Lambda_g(V)$, and the subbundle $E_4^u(g)$ is expansive in $\Lambda_g(V)$. In the sequel, we denote with $W_\epsilon^{css}(x,g)$ the tangent manifold to $E_1(g) \oplus E_2(g)$, with $W_\epsilon^{cs}(x,g)$ the tangent manifold to $E_2(g)$ and with $W_\epsilon^{cu}(x,g)$ the tangent manifold to $E_3(g)$, $W_\epsilon^{cuu}(x,g)$ the tangent manifold to $E_3(g) \oplus E_4^u(g)$, and with $W_\epsilon^{ss}(x,g), W_\epsilon^{uu}(x,g)$ the tangent manifold to $E_1^s(g), E_4^u(g)$ respectively .

Now, we study how the dynamic of a perturbed map behave related to the distance introduce in lemma 6.1.2 and proposition 6.1. Observe that the adapted metric not necessary is coming from a riemannian metric so even the distance along the center manifold are contracted exponentially this does not imply that the derivative is either contractive or expansive along the respective subbundles. In particular, we cannot expect that a perturbation of the initial map contracts distances along the center manifold. However, some contraction along the center stable manifold is kept when the points are not close enough one to each other. This is the statement of the next lemma.

Lemma 6.2.3 *Let dist, r and λ the distances and the constants introduced in lemma 6.1.2. Then, for any $\gamma < r$ there exist a neighborhood $\mathcal{U}$ of f and λ_1 with $\lambda < \lambda_1 < 1$ such that for any $g \in \mathcal{U}$ holds:*

1. *if $y \in W_\epsilon^{cs}(x,g)$ follows that:*
   - *(a) if $dist(x,y) > \gamma$ then $dist(g(x), g(y)) < \lambda_1 dist(x,y)$,*
   - *(b) if $dist(x,y) < \gamma$ then $dist(g(x), g(y)) < \gamma$;*
2. *if $y \in W_\epsilon^{cu}(x,g)$ follows that:*
   - *(a) if $dist(x,y) > \gamma$ then $dist(g^{-1}(x), g^{-1}(y)) < \lambda_1 dist(x,y)$,*
   - *(b) if $dist(x,y) < \gamma$ then $dist(g^{-1}(x), g^{-1}(y)) < \gamma$.*

Moreover, the distance dist remains hyperbolic along $E_1^s(g)$ and $E_4^u(g)$, being contractive and expansive respectively.

Proof:

The proof of this lemma follows from the fact that the tangent manifolds associated to diffeomorphisms close to f are closed in the distance obtained in lemma 6.1.2. In fact, for g C^1-close to f follows that if $y \in W^{cs}_\epsilon(x,g)$ then

$$dist(g(x), g(y)) < \lambda_0 dist(x,y) + r'$$

where $r' = r'(|g-f|_1) > 0$. Moreover, r' is arbitrarily small if g is sufficiently close to f. So, there exists $\gamma = \gamma(r')$ with γ small if r' is small such that if $dist(x,y) > \gamma$ follows that

$$\lambda dist(x,y) + r' < \lambda_1 dist(x,y)$$

for some λ_1 verifying $\lambda_0 + r' < \lambda_1 < 1$

■

Corollary 6.1 *The same conclusion of lemma 6.2.3 holds for $W^{css}_\epsilon(.,g)$ and $W^{cuu}_\epsilon(.,g)$.*

Now, given a periodic point q, we take $\lambda_2(q)$ and $\lambda_3(q)$ the eigenvalues of $D_q f^{n_q}$ (n_q being the period of q) associated to the subbundles $E_2(q)$ and $E_3(q)$ respectively.

Lemma 6.2.4 *There exist positive constants $\lambda_2^0 < 1 < \lambda_3^0$ such that for any λ_2 and λ_3 such that $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$ follows that the periodic points of f with center eigenvalue smaller than λ_2 and unstable eigenvalue larger than λ_3, are dense in H_p.*

Proof: Since we are assuming that f is Kupka-Smale, then the periodic points in H_p are hyperbolic. Let us take a hyperbolic periodic point p_0 and let $\lambda_2^0 < 1 < \lambda_3^0$ be the center and unstable eigenvalue. Moreover, again from the fact that H_p is topologically hyperbolic and transitive, follows that for any $z \in H_p$ there is $z' \in W^s(p_0) \cap W^u(p_0)$ arbitrarily close to z. Then we can take λ_2, λ_3 such that $0 < \lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$ such that given a transversal intersection z' of the stable and unstable manifold of p_0 it is possible to get a periodic point z'' with eigenvalue $|\lambda_2(z'')|$ smaller than λ_2 and unstable eigenvalue $\lambda_3(z'')$ larger that λ_3. To do that, it is only necessary to get a periodic point that expends large part of the orbit close enough to the orbit of p_0. For more details see the proof of lemma 2.3.3.

■

Now, given λ_2 and λ_3 such that $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$, we take the set of periodic point

$$Per_{\lambda_2\lambda_3}(f/V) = \{q \in Per(f) : |\lambda_2(q)| < \lambda_2, \; |\lambda_3(q)| > \lambda_3\}.$$

By lemma 6.2.4 follows that

$$H_p = Closure(Per_{\lambda_2\lambda_3}(f/V))$$

Remark 6.2 *Given a hyperbolic periodic point q for f, there exists a neighborhood U of q and $\mathcal{U} = \mathcal{U}(q,f)$ of f, such that for any $g \in \mathcal{U}$ follows that g has a unique periodic point in U with the same period of q. This periodic point is called the analytic continuation of q and it is noted $q(g)$.*

Observe that $\mathcal{U}$ in the previous definition, depends on f and q. In the next proposition it is shown that for any point in $Per_{\lambda_2\lambda_3}(f/V)$ follows that the analytic continuation is defined in the whole neighborhood $\mathcal{U}$.

Proposition 6.2 *There exist positive constants $0 < \lambda_2^0 < 1 < \lambda_3^0$ and $d_0 > 0$ such that for any λ_2 and λ_3 such that $\lambda_2^0 < \lambda_2 < 1 < \lambda_3 < \lambda_3^0$ there exists a neighborhood $\mathcal{U} = \mathcal{U}(\lambda_2, \lambda_3, f)$ of f and λ_2^1, λ_3^1 verifying $\lambda_2 < \lambda_2^1 < 1 < \lambda_u^1 < \lambda_3$, such that for every periodic point q in $Per_{\lambda_2\lambda_3}(f)$ and any $g \in \mathcal{U}$ follows that*

1. *there exists the analytic continuation q_g of q,*
2. *$dist(g^i(q_g), f^i(q)) < d_0$,*
3. *$q_g \in Per_{\lambda_2^1\lambda_3^1}(g/V)$,*
4. *$h_g(q_g) = q$ and $h_g^{-1}(q) = q_g$ (where h_g is the map introduced in theorem E2).*

Corollary 6.2 *There exists a neighborhood $\mathcal{U} = \mathcal{U}(\lambda_2, \lambda_3, f)$ of f such that for any $g \in \mathcal{U}$ the map h_g introduced in theorem E2 is onto.*

Proof:

Let $x \in H_p$ and let a sequences of points $\{q_n\}$ in $Per_{\lambda_2,\lambda_3}(f/V)$ accumulating on x. Let us consider the points $h_g^{-1}(q_n)$ and let us take z an accumulation point of them. It follows that $q_n = h_g(h_g^{-1}(q_n))$ accumulates on $h_g(z)$ and therefore, $h_g(z) = x$.

■

Observe that the previous proposition and corollary finishes the proof of theorem E3.

To prove the proposition 6.2 we start with the next lemma which is a weak version of a shadowing lemma. Recall that in lemma 6.2.1, it is shown that pseudo-orbits for f in a neighborhood of H_p are shadowed with real orbits of f in H_p. Since it may occur that for small perturbations of f, the homoclinic class H_p does not remain expansive, then it is not expected to get a lemma 6.2.1 for maps close to f. However, we can obtain the following weak shadowing lemma:

Lemma 6.2.5 *For any $\gamma_0 > 0$ there exists a neighborhood $\mathcal{U} = \mathcal{U}(\gamma_0, f)$ of f, there exist positive constants α_0, β_0 and r_0 such that for any $g \in \mathcal{U}$ and $\alpha < \alpha_0$, there exists $\beta < \beta_0$ such that if $\{x_i\}$ is a $\beta-$pseudo orbit and $dist(x_i, \Lambda_g) < r_0$, then there is $x \in B_{r_0}(\Lambda_g)$ such that*

$$dist(g^n(x), x_n) < \alpha + 2\gamma_0.$$

Observe that if Λ_g is hyperbolic, then γ_0 is zero. In the present situation, γ_0 could be considered as the "error" performed by the shadow orbit due to the fact that the subbundle E_2 and E_3 are not hyperbolic. Before to give the proof we state another lemma and an easy claims that allows to conclude the proposition 6.2. The next lemma states that for g close to f, the set Λ_g does not collapse.

Lemma 6.2.6 *For every r_0 there exists a neighborhood $\mathcal{U} = \mathcal{U}(f, r_0)$ of f, such that for any $g \in \mathcal{U}$ and $x \in H_p$ there exists $x' \in \Lambda_g$ such that $dist(x, x') < r_0$.*

Proof:

Let us assume that lemma is false. Then there is a sequence of diffeomorphisms g_n converging to f and points $x_n \in H_p$ such that $dist(x_n, \Lambda_{g_n}) > r_0$. Taking an accumulation point x of x_n follows that $dist(x, \Lambda_{g_n}) > \frac{r_0}{2}$ for n large. Recall that the closure of periodic points in H_p contains H_p. Then, we take a periodic point q close to x. Since we are assuming that they are hyperbolic, for g close enough to f follows that q has a continuation for g close to f and this continuation is close to q and therefore close to x. Which is a contradiction if g is one of the diffeomorphisms of the sequence g_n.

■

Claim 2 *Given $\delta_0 > 0$ there exists $\gamma_0 = \gamma(\delta_0)$ and $\mathcal{U}$ such that if $g \in \mathcal{U}$ and $dist(x, y) < \gamma_0$ then*

$$1 - \delta_0 < \frac{|Df_{|E_2(x,f)}|}{|Dg_{|E_2(y,g)}|} < 1 + \delta_0 \quad 1 - \delta_0 < \frac{|Df_{|E_3(x,f)}|}{|Dg_{|E_3(y,g)}|} < 1 + \delta_0$$

The claim follows from the fact that the subbundles moves continuously with g.

Now we are in condition to show the proof of proposition 6.2.

Proof of proposition 6.2. Lemmas 6.2.5, 6.2.6 and claim 2 imply proposition 6.2:

Let λ_2 and λ_3 be the constants given by the lemma 6.2.4. Let us take δ_0 and $\lambda_2^1 < 1 < \lambda_3^1$ such that $\lambda_2(1 + \delta_0) < \lambda_2^1 < 1 < \lambda_3^1 < (1 - \delta_0)\lambda_3$. Now we take the neighborhood $\mathcal{U}_0$ and the constant γ_0 given by claim 2. Now we take $\gamma_1 < \gamma_0$ and let us take the neighborhood $\mathcal{U}_1$ and the constants α_0, β_0, r_0 given by lemma 6.2.5. Let us choose $\alpha < \alpha_0$ such $\gamma_1 + \alpha < \gamma_0$. Then, giving $\alpha < \alpha_0$ let us take $\beta = \beta(\alpha)$ given by lemma 6.2.5. Now, let us consider the neighborhood $\mathcal{U}_2$ given lemma 6.2.6. Now we take $\mathcal{U} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2$.

Let $g \in \mathcal{U}$ and let q be a periodic point of f $Per_{\lambda_2 \lambda_3}(f/V)$. Using the Pliss's lemma, there is a positive k_0 such that we can assume that $|Df^k_{|E_2(q,f)}| < \lambda_2^k$ for all $k > k_0$. Again, using Pliss's lemma and the fact that the subbundle is one dimensional follows that there is a positive k_0 and j_0 such that for any $m > 0$ holds that $|Df^{-k}_{|E_3(f^{mj_0}(q),f)}| < \lambda_3^{-k}$ for all $k_0 < k < mj_0$.
By lemma 6.2.5 follows that there exists q' in a neighborhood of Λ_g such that $dist(f^i(q), g^i(q')) < \gamma_1 + \alpha < \gamma_0$ for any $i \geq 0$, therefore, by claim 2 follows that

$$|Dg^k_{|E_2(q',g)}| < (\lambda_2^1)^k \ \ \forall\, k > k_0 \ \text{ and } \ |Dg^{-k}_{|E_3(g^{mj_0}(q'),g)}| < (\lambda_3^1)^{-k} \ \ \forall\, m > 0 \ k_0 < k < mj_0.$$

We claim that

$$g^{n_q}(q') \in W^{csss}_\epsilon(q', g) \quad (*).$$

In fact, if $g^{n_q}(q') \notin W^{csss}_\epsilon(q', g)$ then $W^{css}_\epsilon(q', g) \cap [W^{cuu}_\epsilon(g^{n_q}(q'), g) \setminus \{g^{n_q}(q')\}] \neq \emptyset$. Let $z = g^{n_q}(q')$ and $z' = W^{css}_\epsilon(q', g) \cap W^{cuu}_\epsilon(g^{n_q}(q'))$. We assert that there is a positive integer m such that

$dist(g^m(z), g^m(z')) > 2\gamma_0$. Observe that from the fact that $|Dg^{-k}_{|E_3(g^{mj_0}(q'),g)}| < (\lambda_3^1)^k$ for all $m > 0$ and $k_0 < k$, follows that for any $\gamma > 0$ then there is a positive integer n that

$$\ell(g^n(W^{cuu}_\gamma(g^{n_q}(q'), g)) > \epsilon;$$

therefore the assertion follows.
So to finish the proof of $(*)$,

$$dist(g^m(g^{n_q}(q')), f^m(q)) > dist(g^m(g^{n_q}(q')), g^m(z)) -$$

$$-dist(g^m(z), g^m(q')) - dist(g^m(q'), f^m(q')) > \epsilon - 3\gamma_0.$$

Taking γ_0 sufficiently small, we get a contradiction because also holds that

$$dist(g^m(g^{n_q}(q')), f^m(q)) = dist(g^{m+n_q}(q')), f^{m+n_q}(q)) < \gamma_0.$$

Using that $|Dg^k_{|E_2(q',g)}| < (\lambda_2^1)^k$ for all $k > k_0$ and that $g^{n_q}(q') \in W^{cs}_\epsilon(q', g)$ follows that $g^{n_q}(W^{cs}_\epsilon(q')) \subset W^{cs}_\epsilon(q')$ and $\ell(g^n(W^{cs}_\epsilon(q'))) \to 0$ (see claim 1 for details). Therefore, there is a periodic point of period smaller or equal to n_q contained in $W^{cs}_\epsilon(q')$. To check that the period is equal to n_q we argue by contradiction. If the period is n with $n < n_q$ let us take the point $f^n(q)$ and observe that $f^n(q)$ is close to q (recall that q' shadows q). Since the period of q is n_q with $n_q > n$ follows that $W^{css}_\epsilon(q, f) \cap [W^{cuu}_\epsilon(f^n(q, f)) \setminus \{f^n(q')\}] \neq \emptyset$. Arguing as before, replacing q' by q and $g^{n_q}(q')$ by $f^n(q)$ we get a contradiction. Therefore, follows that q' is the analytic continuation of q. ■

Now we proceed to prove lemma 6.2.5

Proof of lemma 6.2.5:

Given γ_0 small, we take $\gamma < \frac{\gamma_0}{2}$. Then, given γ we take $\mathcal{U}$ and $\lambda_1 < 1$ given by lemma 6.2.3. Let us note λ_1 with λ.

Observe that given β small follows that for any g close to f if $y \in W^{css}_\epsilon(x, g)$ and $dist(x, x') < \beta$, then $W^{cuu}_\epsilon(y, g) \cap W^{css}_\epsilon(x', g)$ is a unique point. Moreover, there exists $\beta_1 = \beta_1(\beta)$ such that for any g close to f follows that if $y \in W^{css}_\epsilon(x, g)$, $dist(x, x') < \beta$, $y' = W^{cuu}_\epsilon(y, g) \cap W^{css}_\epsilon(x', g)$ then follows that

$$dist(x', y') < dist(x, y) + \beta_1 \text{ and } dist(y, y') < \beta_1.$$

Given α smaller than ϵ (ϵ is the size of the local stable and local unstable manifolds), we take $\beta < \gamma$ small such that $\lambda(\gamma_0 + \beta) + \beta_1 < \gamma_0$ and $\beta_1 < \gamma$.

Given $g \in \mathcal{U}$ and a $\beta-$pseudo orbit $\{x_n\}$, first we claim that we can construct by induction a sequences $\{y_n\}$ such that

$$y_n = W^{cuu}_\epsilon(g(y_{n-1}), g) \cap W^{css}_\epsilon(x_n, g),$$

verifying:

1. $dist(y_n, x_n) < \gamma_0 + \beta$,

2. for any $k < n$ follows that $dist(y_k, g^{-(n-k)}(y_n)) < \gamma_0$.

The proof of the first item is done by induction in n. For $n = 0$ we take $y_0 = x_0$ and we take $y_1 = W^{cuu}_\epsilon(g(x_0), g) \cap W^{css}_\epsilon(x_1, g)$. Since $dist(g(x_0), x_1) < \beta$ follows that $dist(y_1, x_1) < \beta_1 < \gamma < \gamma_0 + \beta$.

Assuming that we have chosen y_n, we take

$$y_{n+1} = W^{cuu}_\epsilon(g(y_n), g) \cap W^{css}_\epsilon(x_{n+1}, g).$$

We need to prove that $dist(y_{n+1}, x_{n+1}) < \gamma_0 + \beta$. Observe that if $dist(y_n, x_n) > \gamma$ then

$$dist(g(x_n), g(y_n)) < \lambda dist(x_n, y_n) < \lambda(\gamma_0 + \beta).$$

Recalling that $dist(g(x_n), x_{n+1}) < \beta$, and from the election of β follows that

$$dist(y_{n+1}, x_{n+1}) < \lambda(\gamma_0 + \beta) + \beta_1 < \gamma_0 < \gamma_0 + \beta.$$

In case that $dist(y_n, x_n) < \gamma$ then

$$dist(g(x_n), g(y_n)) < \gamma < \gamma_0$$

so, again follows that

$$dist(y_{n+1}, x_{n+1}) < \gamma + \beta_1 < 2\frac{\gamma_0}{2} < \gamma_0 + \beta.$$

To check the second item we perform induction in k.

Let us assume that $dist(y_{n-k}, g^{-k}(y_n)) < \gamma_0$.

Then we have to check that $dist(y_{n-(k+1)}, g^{-(k+1)}(y_n)) < \gamma_0$.

From the following four facts

1. $y_{n-k} \in W^{cuu}_\epsilon(g(y_{n-(k+1)})) \cap W^{css}(x_{n-k})$,
2. $g(y_{n-(k+1)}) \in W^{css}_\epsilon(g(x_{n-(k+1)}))$,
3. $dist(g(y_{n-(k+1)}), g(x_{n-(k+1)})) < \gamma_0 + \beta$ and
4. $dist(g(x_{n-(k+1)}), x_{n-k}) < \beta$,

follows that

$$dist(y_{n-k}, g(y_{n-(k+1)})) < \beta_1.$$

Therefore,

$$dist(g^{-k}(y_n), g(y_{n-(k+1)})) < \gamma + \beta_1 < \gamma_0 + \beta_1.$$

Then, if $dist(g^{-k}(y_n), g(y_{n-(k+1)})) > \gamma$ follows that

$$dist(g^{-k+1}(y_n), y_{n-(k+1)}) < \lambda(\gamma_0 + \beta_1) < \gamma_0.$$

If $dist(g^{-k}(y_n), g(y_{n-(k+1)})) < \gamma$ follows that

$$dist(g^{-k+1}(y_n), y_{n-(k+1)}) < \gamma < \gamma_0.$$

Now we take

$$\{g^{-n}(y_n)\}_{n\in\mathbb{N}}$$

and observe that $g^{-n}(y_n) \in W^{cuu}_{\epsilon}(x_0, g)$. We take an accumulation point y^* of the sequences $g^{-n}(y_n)$ and we prove that

$$dist(g^n(y^*), x_n) < \alpha + 2\gamma_0.$$

In fact,

$$dist(g^n(y^*), x_n) < dist(g^n(y^*), y_n) + dist(y_n, x_n) <$$
$$< dist(g^n(y^*), g^n(g^{-m}(y_m))) + dist(y_n, g^n(g^{-m}(y_m))) + dist(y_n, x_n).$$

To conclude, observe that $dist(g^n(y^*), g^n(g^{-m}(y_m)))$ is small provided m large and if $m > n$ then

$$dist(y_n, g^n(g^{-m}(y_m))) = dist(y_n, g^{-(m-n)}(y_m)) < \gamma_0.$$

■

Now we formulate a series of lemma that improve the description of the semi conjugacy between H_p and Λ_g.

Lemma 6.2.7 *Let $g \in \mathcal{U}$ and let q be a periodic point of g such that there exists $q' \in Per_{\lambda_2\lambda_3}(f/V)$ with $q' = h_g(q)$. Then*

$$W^{cs}_{\epsilon}(q, g) \subset W^{s}_{\epsilon}(q, g) \text{ and } W^{cu}_{\epsilon}(q) \subset W^{u}_{\epsilon}(q).$$

Proof: On one hand, by proposition 6.2 follows that $|\lambda_2(q)| < (\lambda_2^{\frac{1}{2}})^{n_q}$, where n_q is the period of q; in particular, holds that there is an iterate of q, name $g^m(q)$ such that $W^{cs}_{\epsilon}(g^m(q), g) \subset W^{s}_{\epsilon}(g^m(q))$. On the other hand, by lemma 6.2.3 follows that $\ell(g^n(W^{cs}_{\epsilon}(q, g))) < \epsilon$ for any n . Therefore, $g^m(W^{cs}_{\epsilon}(q, g)) \subset W^{cs}_{\epsilon}(g^m(q), g) \subset W^{s}_{\epsilon}(q^m(q))$. So,

$$\ell(g^k(g^m(W^{cs}_{\epsilon}(q, g)))) \to 0 \text{ as } k \to \infty \;\; \ell(g^k(W^{cs}_{\epsilon}(q, g))) < \epsilon \;\forall\, k.$$

Then, $W^{cs}_{\epsilon}(q, g) \subset W^{s}_{\epsilon}(q, g)$. The proof for the center unstable manifold is similar, changing g by g^{-1}.

■

Lemma 6.2.8 *Let $g \in \mathcal{U}$ and let q be a periodic point of g and $q' \in Per_{\lambda_2\lambda_3}(f/V)$ such that $dist(g^n(q), f^n(q')) < \epsilon$ for all n. Then, $q' = h_g(q)$.*

Proof:

The proof is similar to the proof of proposition 6.2.

■

Lemma 6.2.9 *Let q_0 be a hyperbolic periodic point in H_p. Then, there exists $\mathcal{U} = \mathcal{U}(q_0, f)$ such that for any $g \in \mathcal{U}$ and any $q \in Per_{\lambda_2\lambda_3}(f/V)$ follows that the $h_g^{-1}(q)$ is homoclinically related with $h_g^{-1}(q_0)$.*

Proof:

Let us suppose that the lemma is false. Then, there exists a sequences of periodic points $\{q_n\}$ of f in $Per_{\lambda_2\lambda_3}(f/V)$, and a sequence of diffeomorphisms $\{g_n\}$ accumulating on f such that $h_{g_n}^{-1}(q_n)$ is not homoclinically related with $h_{g_n}^{-1}(q_0)$. Observe that $h_{g_n}^{-1}(q_n)$ is close to q_n and $h_{g_n}^{-1}(q_0)$ is close to q_0. By lemma 6.2.7 follows that $W_\epsilon^s(h_{g_n}^{-1}(q_n), g_n)$ is close to $W_\epsilon^s(q_n, f)$. Let us take z_0 an accumulation point of the points q_n. Observe that there is a connected compact arc γ contained in the unstable manifold of q_0 such that intersect $W_\epsilon^s(z_0, f)$ and so intersects $W_\epsilon^s(q_n, f)$. Moreover, if $\mathcal{U}$ is small, on one hand follows that for any $g \in \mathcal{U}$, there is an arc $\gamma(g)$ close to γ contained in the unstable manifold of $h_g^{-1}(q_0)$. Therefore, for g_n close to f follows that the unstable manifold of $h_{g_n}^{-1}(q_0)$ intersects $W_\epsilon^s(h_{g_n}^{-1}(q_n), g_n)$.
With a same argument it is shown that the stable manifold of $h_{g_n}^{-1}(q_0)$ intersects $W_\epsilon^u(h_{g_n}^{-1}(q_n), g_n)$, getting a contradiction.

■

6.2.3 Topologically hyperbolic sets with dominated splitting $E_1^s \oplus E_2 \oplus E_3$.

Now we consider a specific case: Λ exhibits a dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles. The following results can be applied to the topologically hyperbolic homoclinic classes obtained in theorem B.

Moreover, these results are used in [Pu] to show that given a topologically hyperbolic attracting homoclinic class is either hyperbolic or by perturbation it can be created a heterodimensional cycle.

In what follows, we consider neighborhood $\mathcal{U}^{1,2}$ of f given by C^2-maps that they are C^1-close to f. In the next propositions, it is characterized the pre image by h_g and it is considered the particular case that a point that does not belong to the stable manifold of a periodic point. In this case, it is proved that the pre image by h_g is contained in a center stable disc and the local center unstable manifold is contained in the unstable manifold. The proof of the proposition uses the theorem 2.6, which characterized the omega limit of a center unstable arc that does not increase it size by positive iteration. Observe that this result is only valid for at least C^2-maps.

Proposition 6.3 *Let $f \in Diff^2(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles. Let h_g be the semicongugacy introduced in theorem E2. Then there exists a neighborhood $\mathcal{U}^{1,2}$ of f such that for any $g \in \mathcal{U}^{1,2}$ it follows that given $z' \in h_g^{-1}(z)$ either*

1. *$h_g^{-1}(z) \cap W_\epsilon^{cu}(z', g)$ is a single point or*
2. *$W_\epsilon^{cu}(z', g) \cap h_g^{-1}(z)$ is a compact arc such that its $\omega-$limit is a periodic arc.*

Proposition 6.4 *Let $f \in Diff^2(M)$. Let H_p be a topologically hyperbolic homoclinic class exhibiting dominated splitting $E_1^s \oplus E_2 \oplus E_3$ such that $E_1^s \oplus E_2$ is topologically contractive, E_3 is topologically expansive and E_2, E_3 are one dimensional subbundles. Let h_g be the semicongugacy introduced in theorem E2. Then there exists a neighborhood $\mathcal{U}^{1,2}$ of f such that for any $g \in \mathcal{U}^{1,2}$ if $z \in H_p$ does not belong to the stable manifold of some periodic point then for any $z' \in h_g^{-1}(z)$ and for any $\delta > 0$ follows that there exists a positive integer n such that*

$$\ell(g^n(W_\delta^{cu}(z', g))) > \epsilon.$$

Moreover either

1. *$h_g^{-1}(z)$ is a single point or*
2. *$h_g^{-1}(z)$ is an arc contained in the center stable manifold of some point $z' \in \Lambda_g(V)$ such $\ell(g^{-n}(h_g^{-1}(z))) \leq \gamma_0$ for any $n > 0$.*

Corollary 6.3 *Let q be a hyperbolic periodic point of f in H_p. There exists a neighborhood $\mathcal{U}^{1,2} = \mathcal{U}^{1,2}(q, f)$ such that for any $g \in \mathcal{U}^{1,2}$ follows that if $z \in W^u(q)$ and z does not belong to the stable manifold of some periodic point, then $h_g^{-1}(z)$ is a single point.*

Proof: Let us take $\mathcal{U}^{1,2}(q, f)$ with the property that q remains hyperbolic for any $g \in \mathcal{U}^{1,2}(q, f)$. In particular, it follows that the center stable manifold of q has not bounded length.

On one hand, if $h_g^{-1}(z)$ is not a single point, by proposition 6.4, follows that $h_g^{-1}(z)$ is contained in the center stable arc of a point z' in the unstable manifold of q with the property that its length remains bounded by negative iteration. On the other hand, by negative iteration follows that $h_g^{-1}(z)$ converges to the center stable manifold of q, which has not bounded length. Therefore, $h_g^{-1}(z)$ is a single point

■

A similar result to the one obtained in propositions 6.3 and 6.4 can be stated for points that belong to the stable manifold of a periodic point.

Proposition 6.5 *Let q be a hyperbolic periodic point of f in H_p. There exists a neighborhood $\mathcal{U}^{1,2} = \mathcal{U}^{1,2}(q, f)$ such that for any $g \in \mathcal{U}^{1,2}$ follows that if $z \in W^s(q)$ then $h_g^{-1}(z)$ is contained in the stable manifold of $h_g^{-1}(q)$.*

Proof of proposition 6.3:

It follows easily that either $h_g^{-1}(z) \cap W_\epsilon^{cu}(z', g)$ is a single point or it is an arc l contained in $W_\epsilon^{cu}(z', g)$. In the last case, since for any $w \in l = h_g^{-1}(z) \cap W_\epsilon^{cu}(z', g)$ follows that $dist(g^n(w), g^n(z'))$ is small, then by theorem 2.6 the proposition follows.

■

Proof of proposition 6.3:

Let $z' \in h_g^{-1}(z)$ such that z does not belong to the stable manifold of some periodic point. If there is some $\delta > 0$ small such that for any positive integer n follows that $\ell(g^n(W_\delta^{cu}(z', g))) < \epsilon$ then it holds that $\omega(W_\delta^{cu}(z', g))$ is a periodic interval; therefore, $g^n(z')$ converge to a periodic point.

Since $g^n(z')$ and $f^n(z)$ remains close, follows that the orbit of z by f converges to a periodic point, and so z belongs to the stable manifold of some periodic point. Which is a contradiction.

For the last part, if there are two points z_1 and z_2 such that $z_2 \notin W^{cs}_\epsilon(z_1)$, it follows that $W^{cs}_\epsilon(z_2)$ intersect the local unstable manifold of z_1 in a point z_3 different than z_1. If we consider the arc I contained in the local unstable manifold of z_1 and bounded by z_1 and z_3, follows that the arc does not growth the length by positive iterations; which is contradiction with the previous item. The last item is immediately.

■

The proof of proposition 6.5 is similar to the proof of 6.4 and it is left for the reader.

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