Density of hyperbolicity and homoclinic bifurcations for 3D-diffeomorphism in attracting regions.

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Abstract

In the present paper it is proved that given a maximal invariant attracting homoclinic class for a smooth three dimensional Kupka-Smale diffeomorphism, it follows that the homoclinic class is either hyperbolic or the diffeomorphisms is C^1- approximated by another map exhibiting a homoclinic tangency or a heterodimensional cycle.

1 Introduction and statements.

For a long time (mainly after Poincaré) it has been a goal of the theory of dynamical systems to describe the dynamics from the generic viewpoint, that is, to describe the dynamics of "big sets" (residual, dense, etc.) within the space of all dynamical systems.

It was briefly thought in the sixties that this could be realized by the so-called hyperbolic ones: systems with the assumption that the tangent bundle over the limit set $L(f)$ splits into two complementary subbundles $T_{L(f)}M = E^s \oplus E^u$ so that vectors in E^s (respectively E^u) are uniformly forward (respectively backward) contracted by the tangent map Df. Under this assumption, it is proved that the limit set decomposes into a finite number of disjoint transitive sets such that the asymptotic behavior of any orbit is described by the dynamics in the trajectories in those finite transitive sets (see [S]). Moreover, this topological description allows to get the statistical behavior of the system. In other words, hyperbolic dynamics on the tangent bundle characterizes the dynamics over the manifold from a geometrical, topological and statistical point of view.

Uniform hyperbolicity was soon realized to be a less universal property than was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. The initial mechanisms to show this nonhyperbolic robustness (see [AS], [Sh]) were the existence of open sets of diffeomorphisms exhibiting hyperbolic periodic points of different stable indices inside a transitive set (the stable index of a hyperbolic periodic point is the number of eigenvalues with modulus smaller than one counted with multiplicity). It is said that has .

Related to this is the notion of heterodimensional cycle where two periodic points of different indices are linked through the intersection of their stable and unstable manifolds (notice that at least one of the intersections is non-transversal; a more precise definition follows).

In all of the above examples the underlying manifolds must have dimension at least three, so the case of surfaces was still unknown at the time. It was through the seminal works of Newhouse (see [N1], [N2], [N3]) that hyperbolicity was shown not to be dense in the space of C^r diffeomorphisms ($r \geq 2$) of compact surfaces. The underlying mechanism here was the presence of a homoclinic tangency: non-transversal intersection of the stable and unstable manifold of a periodic point.

These results naturally suggested the following question:

1. *What mechanisms lead to generic (meaning generic perturbation of the initial system) nonhyperbolic behavior?*

2. *Is it possible to identify the dynamical mechanism underlying any generic nonhyperbolic behavior?*

We have mentioned two basic mechanisms which are obstruction to hyperbolicity, namely *heterodimensional cycles* and *homolicinic tangencies.* In the early 80's Palis conjectured (see [P] and [PT]) that these are very common in the complement of the hyperbolic systems:

1. *Every C^r diffemorphism of a compact manifold M can be C^r approximated by one which is hyperbolic or by one exhibiting a heterodimensional cycle or by one exhibiting a homoclinic tangency.*

2. *When M is a two-dimensional compact manifold every C^r diffemorphism of M can be C^r approximated by one which is hyperbolic or by one exhibiting a homoclinic tangency.*

This conjecture may be thought as a starting point to obtaining a generic description of C^r-diffeomorphisms. If it turns out to be true we may focus on the two mechanisms mentioned above in order to understand the dynamics. Nevertheless, the unfolding of these homoclinic bifurcations is mainly a local study.

To be precise, a hyperbolic diffeomorphism means a diffeomorphism such that its limit set is hyperbolic. A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$

invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n_{/E^s(x)}\| \le C\lambda^n$$

and

$$\|Df^{-n}_{/E^u(x)}\| \le C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n.

Moreover, a diffeomorphism is called Axiom A, if the non-wandering set is hyperbolic and it is the closure of the periodic points.

We recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f. A point of intersection of these manifolds is called a homoclinic point.

Definition 1 Homoclinic tangency. *We say that f exhibits a homoclinic tangency if there is a periodic point p such that there is a point $x \in W^s(p) \cap W^u(p)$ with $T_xW^s(p) + T_xW^u(p) \neq T_xM$. Given an open set V, we say that the tangency holds in V if p and x belongs to V.*

The above conjecture was proved to be true [PS1] for the case of surfaces and the C^1 topology. The case of higher topologies ($C^r, r \ge 2$) is, at this point, far from being solved:

Theorem ([PS1]): *Let M^2 be a two dimensional compact manifold. Every $f \in Diff^1(M^2)$ can be*

C^1-approximated either by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism

In dimensions higher than two, the theorem stated above is false, due to another kind of homoclinic bifurcation that breaks the hyperbolicity in a robust way: the so-called heterodimensional cycles (intersection of the stable and unstable manifolds of points of different indices, see [D1] and [D2]).

Definition 2 Heterodimensional cycle. *We say that f exhibits a heterodimensional cycle if there are two periodic points q and p of stable index 1 and 2 respectively, such that $W^u(q) \cap W^s(p) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$. Given an open set V, we say that the cycle holds in V if p, q and the points where the stable and unstable manifolds intersects belongs to V.*

The unfolding of these kinds of cycles implies the existence of striking dynamics: the appearance of non-hyperbolic robust transitive sets is more important. Moreover, any non-hyperbolic robust transitive set exhibits (generically) a heterodimensional cycle.

It is remarkable to say that for a compact manifold with dimension larger than and equal to three, there are C^1-open sets of diffeomorphisms containing a dense set of diffeomorphisms exhibiting either a tangency or a heterodimensional cycle. On the other hand, the conjecture states that the systems exhibiting either a tangency or a heterodimensional cycle are dense in the complement of the hyperbolic ones.

The present paper proves the conjecture formulated by Palis in the C^1 topology for an attracting homoclinic class of a three dimensional C^2-diffeomorphisms. Observe that the conjecture is stated for the whole Limit set and recall that this set is the closure of the accumulation points of any orbit. In this paper, we could say that we go in the direction to deal with the "attracting region of the Limit set". To be precise, first we have to introduce some definitions.

Definition 3 Homoclinic class. *Given a periodic point p, we define the homoclinic class associated to p as the closure of the set* $\{W^s(p) \cap W^u(p)\}$.

Definition 4 Attracting homoclinic class. *Given a homoclinic class we say that* H_p *is an attracting homoclinic class if there exists an open set* U *such that* $H_p \subset U$ *and* $H_p = \cap_{n>0} f^n(U)$

Different kind of examples of attracting homoclinic classes can be found: the solenoid attractor, the Henon attractor, the Plykin attractor or partially hyperbolic attractors (see [BD] and [BV] for the last kind of examples).

Main Theorem: *Let* $f \in Diff^2(M^3)$ *be a Kupka-Smale system. Let* $H_p = \cap_{n>0} f^n(U)$ *be an attracting homoclinic class associated to a periodic point p. Then, one of the following options holds:*

1. H_p *is hyperbolic;*

2. *there exists* g C^1-*arbitrarily close to* f *exhibiting a homoclinic tangency in* U*;*

3. *there exists* g C^1-*arbitrarily close to* f *exhibiting a heterodimensional cycle in* U.

Observe that under the hypothesis of the previous theorem, if the systems cannot be approximated by heterodimensional cycles or tangencies, then the homoclinic class is hyperbolic. In other words, the statement in the context of homoclinic classes is stronger than the goal of the conjecture.

Concerning the proof of Maim Theorem A, we show that if a diffeomorphism f cannot be approximated by one having a homoclinic tangency, then the homoclinic class is hyperbolic or it is approximated by a system exhibiting a heterodimensional cycle.

This is done in two steps. First, it is shown that if no tangencies can be created by C^1-perturbations, then it is possible to find a continuous splitting, namely dominated splitting (see next section for the definition) over the tangent bundle of the homoclinic class. Later, the possible dynamic scenarios under the assumption of dominated splitting are studied. It is shown that under the assumption of dominated splitting, the "strong stable" and "strong unstable" set of every point are embedded manifolds. Using this, the following two scenarios could hold: either we are dealing essentially with a two dimensional system, meaning that the attractor is contained in a two dimensional submanifold, or the system is essentially three dimensional. This alternative is related to the fact that if "the strong manifolds is or not involved in the dynamic". When we are dealing with "a two dimensional system" we prove that the homoclinic class is hyperbolic. When we are dealing with a "three dimensional system" we prove that if the homoclinic class is not hyperbolic, a heterodimensional cycle can be created by perturbation. To deal with this last situation, new perturbation arguments are developed.

The previous theorem can be improved in terms of the nature of the homoclinic tangency that can be created by perturbation. To clarify this, we need to recall some results about homoclinic tangencies and how tangencies are also related to the presence of heterodimensional cycles. For surfaces maps, the unfolding of a homoclinic tangencies leads to the nowadays so-called "Newhouse phenomena", i.e., residual subsets of diffeomorphisms displaying infinitely many periodic attractors. In particular, this shows that the unfolding of tangencies "destroys" transitive sets. This phenomena is not valid in higher dimension. In fact, robust transitive sets can coexist with the presence of an homoclinic tangency (see for instances the examples showed in [BV] of robust transitive systems). In these examples, tangencies and heteroclinic cycles are coexisting.

On the other hand, it was shown in [PV] that in dimension larger than two, the unfold of tangencies associated to sectional dissipative periodic points (the modulus of the product of any pair of eigenvalues is smaller than one) leads to the same Newhouse phenomena that holds in dimension two.

Regarding the previous comments, in the direction to improve the maim theorem, it would be useful to obtain a version that states that under the same hypothesis then *either the homoclinic*

class is hyperbolic, or it is C^1-approximated by a system exhibiting a heterodimensional cycle or by one exhibiting a sectional dissipative homoclinic tangency.

Unfortunately, this result is not completely obtained. However, some partial result are concluded. These results are stated in the last section.

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2 Systems far from tangencies.

In the sequel, given two diffeomorphisms f and g we say that g is $C^r-\delta-$close to f is $|f-g|_r<\delta$ where $|.|_r$ is the usual norm in the C^r-topology. To avoid notation sometimes, it is said that given f there is g C^r-arbitrarily close to f (or simply that there is g C^r-close to f) if for any δ there is g $C^r-\delta-$close to f.

We start assuming that it is not possible to create a tangency by a C^1-perturbation.

Definition 5 *Let p be a saddle periodic point and let $H_p=\cap_{\{n>0\}}f^n(U)$ be an attracting homoclinic class. We say that the homoclinic class is C^1-far from tangencies, if there is a neighborhood $\mathcal{U}\subset Diff^1(M^3)$ of f such that any $g\in\mathcal{U}$ does not exhibit a tangency in U.*

In the case that H_p is C^1-far from tangencies, we show that H_p exhibits a dominated splitting.

Definition 6 *An f-invariant set Λ is said to have a dominated splitting, if the tangent bundle is decomposed in two invariant subblundles $T_\Lambda M=E\oplus F$, and such that there exist $C>0$ and $0<\lambda<1$ with the following property:*

$$|Df^n_{/E(x)}||Df^{-n}_{/F(f^n(x))}|\leq C\lambda^n, \text{ for all } x\in\Lambda, n\geq 0.$$

Definition 7 *Given an f-invariant set Λ exhibiting a dominated splitting $T_\Lambda M=E\oplus F$, it is said that E (F) is contractive (expansive) if there exist $C>0$ and $0<\lambda<1$ such that $|Df^n_{|E(x)}|\leq C\lambda^n$, for all $x\in\Lambda, n\geq 0$ ($|Df^{-n}_{|E(x)}|\leq C\lambda^n$, for all $x\in\Lambda, n\geq 0$).*

In our context, we show that the tangent bundle is either decomposed in two directions $E\oplus F$ such that either E or F has dimension two and they are contractive or expansive respectively, or it is decomposed in three directions $E_1\oplus E_2\oplus E_3$.

In the first case, follows from [PS3] that the homoclinic class is hyperbolic (see subsection 3). In the later, it is shown that either the homoclinic class is hyperbolic or it is created a heterodimensional cycle (see subsection 4). In what follows, any decomposition is assumed to be dominated.

Theorem 2.1 *Let us assume that H_p is C^1-far from tangencies.*

If the point p has stable index one, then one of the next options holds:

1. *$T_{H_p}M=E\oplus F^u$ with the property that $dim(F^u)=2$ and F^u is an expansive subbundle;*

2. $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.

If the point p has stable index two, then one of the next options holds:

1. $T_{H_p}M = E^s \oplus F$ *with the property that* $dim(E^s) = 2$ *and* E^s *is a contractive subbundles;*

2. $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.

This result follows from techniques introduced in [PS1], [PPV] and in [LW].

Consider the set

$$\mathcal{U} = Diff^1(M^2) \setminus \{f \in Diff^1(M^2) : \text{exhibits a homoclinic tangency}\}$$

In [PS1] it is proved in Lemma 2.0.2 that generically in the set $\mathcal{U}$, the diffeomorphisms exhibit a finite number of sinks and repeller and its non-wandering set has dominated splitting.

This results was extended in [LW]. To state the result, first we recall some definitions: It is said that a hyperbolic periodic point has stable index d if the number of stable eigenvalues (or eigenvalues with modulus smaller than one) counted with multiplicity is d. It is said that a dominated splitting $E \oplus F$ is a $d-$dominated splitting if $dim(E) = d$. The d-preperiodic set of a C^1 diffeomorphism f, is the the set of points for which there is a diffeomorphisms g C^1 close to f such that p is a periodic point of g of stable index d.

Theorem 2.2 *([LW]) The following assertions are equivalent:*

1. *f cannot be C^1 approximated by a diffeomorphism exhibiting homoclinic tangencies associated to a periodic point of stable index d.*

2. *The closure of the periodic set of f with stable index equal to d, has a d-dominated splitting.*

3. *The d-preperiodic set of f has a d-dominated splitting.*

In our context, if the homoclinic class is associated to a periodic point of stable index one and by C^1-perturbation cannot be created a homoclinic tangency, follows from the theorem 2.2 that H_p has dominated splitting $E \oplus F$ with dimension of F equal to two. If the homoclinic class is associated to a periodic point of stable index two and by C^1-perturbation cannot be created a homoclinic tangency, follows from theorem 2.2 that H_p has dominated splitting $E \oplus F$ with dimension of E equal to two. However, using that we are dealing with a homoclinic class, this result can be improved. In fact, it is proved that if the direction E cannot be splitted in two direction $E_1 \oplus E_2$ exhibiting a dominated splitting and f is C^1-far from tangencies then E is contractive. The strategy to prove that goes by contradiction: if the direction E cannot be splitted in two direction $E_1 \oplus E_2$ exhibiting a dominated splitting and E is not uniformly contractive, then it can be created a tangency associated to a periodic point with stable index one; i.e.: a tangency associated to point with one dimensional stable manifold and a two dimensional unstable manifold.

To precise, we say that E cannot be decomposed in two subbundles exhibiting dominated splitting, if it follows that any decomposition of E in two subbundles is not a dominated splitting. Related to this, it is proved the following proposition:

Proposition 2.1 *Let $H_p = \cap_{n\in\mathbb{N}} f^n(U)$ be a maximal invariant homoclinic class associated to a periodic point of stable index two. Let us assume that $T_{H_p}M = E \oplus F$ with $dim(F) = 1$ such that E cannot be decomposed into two invariant subbundles exhibiting domination and f is C^1-far from tangencies in U. Then follows that E is contractive*

A similar result can be stated for the case that p has stable index one:

Proposition 2.2 *Let $H_p = \cap_{n\in\mathbb{N}} f^n(U)$ be a maximal invariant homoclinic class associated to a periodic point of stable index one. Let us assume that $T_{H_p}M = E \oplus F$ with $dim(E) = 1$ such that F cannot be decomposed into two invariant subbundles exhibiting domination and f is C^1-far from tangencies in U. Then follows that F is expansive.*

Assuming the previous proposition, now we can show how it is obtained the theorem 2.1.

Proof of theorem 2.1:

To prove theorem 2.1, first observe that from theorem 2.2 follows that if p has stable index two, then

$$T_{H_p}M = E \oplus F \text{ with } dim(F) = 1.$$

If p has stable index one, then

$$T_{H_p}M = E \oplus F \text{ with } dim(E) = 1.$$

To conclude the theorem 2.1 observe that if $dim(F) = 1$ and E cannot be decomposed in two other direction, then by the proposition 2.1 follows that E is uniformly contracted by Df. The case that $dim(E) = 1$ is treated in similar way.

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2.1 Proof of proposition 2.1 and 2.2.

We give the proof of proposition 2.1; the proof of proposition 2.2 is similar.

We prove the proposition 2.1 assuming that the thesis is false. The goal is to show that if the thesis is false then we can create an homoclinic tangency. First we introduce the notion of angle of two vectors:

Definition 8 *Let v and w be two vectors of $\mathbb{R}^d$. It is defined the angle $\alpha(v,w)$ as the unique positive number in $[0, \frac{\pi}{2}]$ such that*

$$\cos(\alpha(v,w)) = \frac{< v, w >}{|v||w|}$$

where $< .,. >$ is the internal product induced by the riemanniam metric. Given two one-dimensional subspaces, it is defined the angle between them as the angle between two generators.

It is used the following lemma, which is a simple yet powerful perturbation technique (in the C^1 topology):

Lemma 2.1.1 *[Fr, Lemma 1.1] Let M be a closed n-manifold and $f : M \to M$ be a C^1 diffeomorphism, and let a neighborhood of f, $\mathcal{U}(f)$ be given. Then, there exist $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S \subset M$ is a finite set, $S = \{p_1, p_2, \ldots p_m\}$ and L_i, $i = 1, \ldots, m$ are linear maps $L_i : T_{p_i}M \to T_{f(p_i)}M$ satisfying $\|L_i - D_{p_i}g\| \leq \delta$, $i = 1, \ldots, m$ then there exists $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$, $i = 1, ..., m$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2 \ldots p_m\} \cup (M \backslash U)$.*

This results says, for instance, that any small perturbation of the linear maps along a periodic orbit can be realized through a diffeomorphism C^1-nearby.

Lemma 2.1.2 *Let us assume that the thesis of proposition 2.1 is false. Then, given $\gamma > 0$, $\delta_1 > 0$, $\delta_2 > 0$ there exists a saddle periodic point q of f and a diffeomorphism g $C^1 - \delta_1-$close to f such that q is a periodic point for g such that*

1. *q has two different real contractive eigenvalues;*
2. *$(1-\delta_2)^{n_q} < |Df^{n_q}_{|E_2^s(q)}| < 1$;*
3. *$\alpha(E_1^s(q), E_2^s(q)) < \gamma$*

where n_q is the period of q and $E_1^s(q), E_2^s(q)$ are the two stable eigenspaces associated to the two real contractive eigenvalues of $Df^{n_q}(q)$.

We postpone the proof of the lemma to the next subsubsection. The following lemma states that assuming the thesis of the previous lemma we can create a tangency by a C^1-perturbation.

Lemma 2.1.3 *Let us assume that the thesis of lemma 2.1.2 holds. Then, there is g C^1- close to f exhibiting a homoclinic tangency associated to a periodic point q with stable index one.*

Proof: If there is a point q as in the thesis of lemma 2.1.2, using the lemma 2.1.1 we can perform a C^1-perturbation to get a new system g such that q remains periodic for it and such that

1. the directions $E_2^s(q)$ and $E_1^s(q)$ remains invariant,
2. the modulus of the eigenvalue associated to the direction $E_2^s(q)$ become larger than one,
3. the modulus of the eigenvalue associated to $E_1^s(q)$ is smaller than one.

So, the periodic point q for g has a stable manifold of dimension one and an unstable manifold of dimension two with small angle. By lemma 2.2.2 proved in [PS1], it is possible to create with a new perturbation, a tangency between the mentioned manifolds.

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So, to finish the proof of proposition 2.1 is enough to prove the lemma 2.1.2.

2.1.1 Proof of lemma 2.1.2.

To prove the lemma, we state a result proved in [PPV]. This result states that if E is not uniformly contractive then there is a periodic point in the homoclinic class with rate of contraction close to one.

Definition 9 *Given two hyperbolic periodic points, it is said that they are homoclinically related if the stable manifold of each point intersects transversally the unstable manifold of the other periodic point.*

Proposition 2.3 *([PPV]) Let $f \in Diff^2(M^3)$ and H_p a homoclinic class associated to a periodic point of stable index two and such that $T_{H_p}M = E \oplus F$ with $dim(F) = 1$. If E is not uniformly contractive then for any $\delta > 0$ and $m > 0$ follows that there is a periodic point $q \in H_p$ such that*

1. *q is homoclinically related with p,*
2. *$(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$,*

3. $n_q > m$

where n_q *is the period of* q *and* $E^s(q)$ *is the stable eigenspace associated to* Df^{n_q}*.*

Now we continue with the proof of lemma 2.1.2: Let us consider the set of periodic points such that they have two contractive real eigenvalues. Let us call $E_1^s(q)$ and $E_2^s(q)$ the two eigenspaces associated to the two contractive eigenvalues, and let us assume that the absolute value of the eigenvalue associated to $E_1^s(q)$ is smaller that the absolute value of the eigenvalue associated to $E_2^s(q)$.

Given $\delta > 0$, let us consider the set $\mathcal{P}_\delta$ formed by periodic points $q_1 \in H_p$ such that

$$(1-\delta')^{n_{q_1}} < |Df^{n_{q_1}}_{|E^s(q_1)}| < 1 \text{ for some } 0 < \delta' < \delta$$

We have to consider three different situations:

- **Case 1.** For every $\lambda < 1$, any positive integer n_0 and $\delta > 0$ there is $q_1 \in \mathcal{P}_\delta$ and $m > n_0$ such that
$$\frac{|Df^m(E_1^s(q_1))|}{|Df^m(E_2^s(q_1))|} > \lambda$$
- **Case 2.** There is $\lambda > 0$ and a positive integer n_0 such that for every $\delta > 0$ and $q_1 \in \mathcal{P}_\delta$ follows that $E_1^s(q_1)$ $(\lambda, n_0)-$dominates $E_2^s(q_1)$; i.e.:
$$\frac{|Df^n(E_1^s(f^j(q_1)))|}{|Df^n(E_2^s(f^j(q_1)))|} < \lambda$$
for every $n > n_0$ and any j.
- **Case 3.** There is δ_0 such that for any $\delta < \delta_0$ the set $\mathcal{P}_\delta$ either is finite or empty.

Case 1.

In the first case, it is proved that after a C^1-perturbation we can get a new periodic point exhibiting two directions with small angle and one eigenvalue close to one. In fact, first it is used the following folklore lemma and the proof it can be found [M1]:

Lemma 2.1.4 *Let us assume that the for any* δ *the set* $\mathcal{P}_\delta$ *does not exhibit a dominated splitting. Then, for any* $\gamma > 0$ *there is* g C^1-*arbitrarily close to* f *exhibiting a periodic point* q *with arbitrarily large period* n_q *and such that*

1. $(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$ *and*

2. $\alpha(E_1^s(q), E_2^s(q)) < \gamma$.

Observe that it could happen that $(1-\delta)^{n_q} < |Df^{n_q}_{|E^s(q)}| < 1$ and $|Df^{n_q}_{|E_2^s(q)}| < \lambda_s^{n_q}$ for some $\lambda_s < 1$, i.e.: the eigenvalues in the stable direction are much smaller than the norm in this direction. In this case, we perform another perturbation to get what we want.

Lemma 2.1.5 *Let us assume that the thesis of the previous lemma holds. Then there is g C^1-close to f exhibiting a periodic point q with large period such that* $(1-\delta)^{n_q} < |Df^{n_{q_1}}_{|E_2^s(q)}| < 1$ *and* $\alpha(E_1^s(q), E_2^s(q)) < \gamma$.

As a consequences of the previous lemma, follows that lemma 2.1.2 is proved in the case that $\mathcal{P}_\delta$ has not dominated splitting.

Proof of lemma 2.1.5: Let us consider the basis $\mathcal{B}$ in $E^s(q)$ given by two orthonormal vectors v_1, v_2 such that $v_1 \in E_1^s(q)$. Let $\mathcal{B}_i$ basis in $E^s(f^i(q)$ given by $\frac{Df^i(v_1)}{|Df^i(v_1)|}$ and an orthonormal vector to it.

Let $A_i = Df : E^s(f^{i-1}(q)) \to E^s(f^i(q))$ and in theses basis we can assume that:

$$A_i = \begin{bmatrix} \alpha_i & k_i \\ 0 & \beta_i \end{bmatrix}$$

Observe that

$$Df^{n_q} = \Pi_{i=1}^{n_q} A_i = \begin{bmatrix} \alpha & k \\ 0 & \beta \end{bmatrix}$$

with

$$|\alpha| < \lambda_s^{n_q}, |\beta| < \lambda_s^{n_q}, (1-\delta)^{n_q} < |k| < 1$$

Let us consider the following linear maps which are small perturbations of the maps A_i:

$$B_i = \begin{bmatrix} \alpha_i & k_i \frac{1+\delta}{1-\delta} \\ 0 & \beta_i \end{bmatrix} \quad 1 \le i \le n-2$$

$$B_{n-1} = \begin{bmatrix} \alpha_{n-1} & k_{n-1}\frac{1+\delta}{1-\delta} \\ \epsilon & \beta_{n_1} \end{bmatrix}$$

So,

$$\Pi_{i=1}^{n_q} B_i = \begin{bmatrix} \alpha & \hat{k} \\ \alpha\epsilon & \hat{k}\epsilon + \beta \end{bmatrix}$$

where

$$(1+\delta)^{n_q} < \hat{k} < (\frac{1+\delta}{1-\delta})^{n_q}$$

Then, for

$$\epsilon < (\frac{1-\delta}{1+\delta})^n$$

holds that one of the eigenvalues 0f B is close to one and the eigenspaces has small angle. By lemma 2.1.1, the linear maps can be realized as the derivative along the orbit of q of a perturbation of f.

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Case 2.

The second case (i.e.: the set $\mathcal{P}_\delta$ has dominated splitting) is more delicate. For that, we need another two lemmas that basically state that assuming that if the direction E is not contractive and it cannot be decomposed in two subbundles, follow that it is possible to get two periodic points q_2, q_3 homoclinically related such that:

1. the eigenvalue associated to the direction E for q_2 is a complex eigenvalues;
2. $Df^{n_{q_3}} : T_{q_3}M \to T_{q_3}M$ (where n_{q_3} is the period of q_3) has two eigenspaces with small angle.

Observe that for the points q_2 and q_3 it could occur that the rate of contraction of Df in the direction E is exponentially far from one. However, using that there is another periodic point q_1 such the rate of contraction of Df in the direction E for q_1 is close to one (see proposition 2.3) and that the three periodic points (q_1, q_2, q_3) are homoclinicaly related, follows that we can get a another periodic points verifying the thesis of the lemma 2.1.2.

We start enunciating the following lemma which is the proposition 2.1 proved in [BDP] (page 376).

Lemma 2.1.6 *Let H_p be a homoclinic class exhibiting an splitting $E \oplus F$ with dim$(E) = 2$ and such that E cannot be decomposed in two direction. Then for any $\delta > 0$ there exists a periodic point q of f and a diffeomorphisms g $C^1 - \delta-$close to f such that q is a periodic point for g with complex eigenvalue and homoclinically related with p.*

The next lemma, is a folklore one and a proof of it can be found in [DPU].

Lemma 2.1.7 *Let q be a periodic point with complex eigenvalue and let us assume that there is a transversal intersection of the stable and unstable manifold of q. Then, for any $\delta > 0$ and $\gamma > 0$ there exists a periodic point q' of f and a diffeomorphisms g $C^1 - \delta-$close to f such that q' is homoclinically related with q, q' is a periodic point for g and $\alpha(E^s_1(q), E^s_2(q)) < \gamma$.*

First, we take a point q_2 in the condition of lemma 2.1.6, i.e.: q_2 has a complex contractive eigenvalue and exhibiting a transversal intersection of their invariant manifolds. Using the C^1-connecting lemma it is possible to perturb f in a way such that q_1 and q_2 are homoclinically related (recal that q_1 is a point in $\mathcal{P}_\delta$). Then, also follows that the unstable manifold of q_2 intersect the stable manifold of q_2. Now, we introduce a second perturbation to get a point q_3 that verifies the thesis of lemma 2.1.7. Observe that the points q_1, q_2 and q_3 are homoclinically related. Using that, we can get a new periodic point q such that

1. $(1-\delta_2)^{n_q} < |Df^{n_q}_{|E^s_2(q)}| < 1$ and
2. $\alpha(E^s_1(q), E^s_2(q))$ is small.

In fact, we take three neighborhood V_1, V_2 and V_3 of the orbit of q_1 q_2 and q_3 respectively (in what follows we can assume that these points are fixed) and we can assume that for each neighborhood V_i follows that $Df_{|V_i} = Df(q_i)$. Using that the therre periodic points q_1, q_2, q_3 are homoclinically related follows that we can get a periodic point q with period $n_3 + k_3 + n_2^1 + k_2^1 + n_1 + k_1 + n_2^2 + k_2^2$ such that

1. n_1, n_2^1, n_3, n_2^2 arbitrarially large,
2. k_1, k_2^1, k_3, k_2^2 uniformly bounded by some k_0,
3. $f^j(q) \in V_3$ for $0 \leq j \leq n_3$,
4. $f^j(q) \in V_2$ for $n_3 + k_3 \leq j \leq n_3 + k_3 + n_2^1$,
5. $f^j(q) \in V_1$ for $n_3 + k_3 + n_2^1 + k_2^1 \leq j \leq n_3 + k_3 + n_2^1 + k_2^1 + n_1$
6. and $f^j(q) \in V_2$ for $n_3+k_3+n_2^1+k_2^1+n_1+k_1 \leq j \leq n_3+k_3+n_2^1+k_2^1+n_1+k_1+n_2^2$.

We consider the following linear maps

$$A_1 = Df(q_1) : T_{V_1}M \to T_{V_1}M; \ A_2 = Df(q_2) : T_{V_2}M \to T_{V_2}M$$

$$A_3 = Df(q_3) : T_{V_3}M \to T_{V_3}M$$

$$T_{32} = Df^{k_3} : T_{V_3}M \to T_{V_2}M; \ T_{21} = Df^{k_2^1} : T_{V_2}M \to T_{V_1}M$$

$$T_{12} = Df^{k_1} : T_{V_1}M \to T_{V_2}M; \ T_{23} = Df^{k_2^2} : T_{V_2}M \to T_{V_3}M$$

See figure 1.

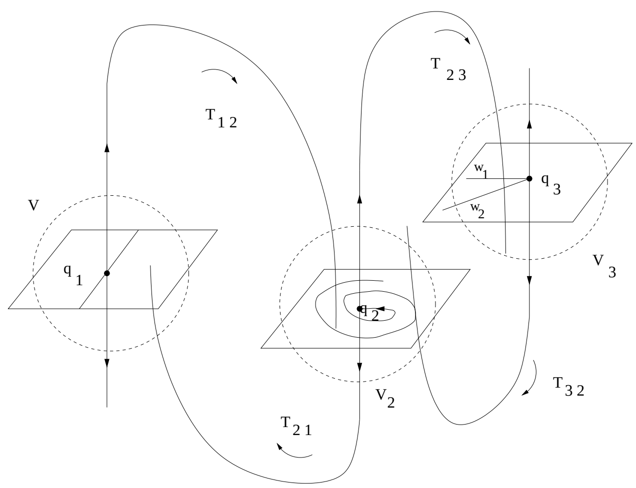


Figure 1

We consider the vectors w_1, w_2 such that $w_1 \in E_1^s(q_3)$, $w_2 \in E_2^s(q_3)$. Assuming that the complex eigenvalue has irrational imaginary part (if it is not the case with after a small perturbation it would b the case), we can take n_2^1 and an small perturbation of T_{21} (we keep the same notation for the perturbation) such that

$$T_{21} A_2^{n_2^1} T_{32}(w_2) \in E_2^s(q_1)$$

Moreover, we can take an small perturbation of T_{23} (let us call it T_{23}) and n_2^2 such that for any n_1 follows that

$$T_{23} A_2^{n_2^2} T_{12} A_1^{n_1} T_{21} A_2^{n_2^1} T_{32}(w_2) \in E_2^s(q_3)$$

In other word, follows that the direction $E_2^s(q)$ is invariant for

$$T_{23} A_2^{n_2^2} T_{12} A_1^{n_1} T_{21} A_2^{n_2^1} T_{32}$$

Observe that $\alpha(T_{21} A_2^{n_2^1} T_{32}(w_2), T_{21} A_2^{n_2^1} T_{32}(w_1))$ is small.
Since $T_{21} A_2^{n_2^1} T_{32}(w_2) \in E_2^s(q_1)$ and $E_2^s(q_1)$ is dominated by $E_1^s(q_1)$, follows that

$$\alpha(A_1^{n_1} T_{21} A_2^{n_2^1} T_{32}(w_1), A_1^{n_1} T_{21} A_2^{n_2^1} T_{32}(w_2)) < \gamma$$

with γ being small. Then we can get another small perturbation of T_{23} such that

$$T_{23} A_2^{n_2^2} T_{12} A_1^{n_1} T_{21} A_2^{n_2^1} T_{32}(w_1) \in E_2^s(q_1)$$

So, we obtain a new linear map close to the initial one such that has two eigenspaces with small angle. Moreover, if n_1 is chosen larger than the others, follows that the new linear map along w_1 and w_2 is weak contractive.

Case 3.

If it follows that for some δ_0 holds that for any $\delta < \delta_0$ that $\mathcal{P}_\delta$ is empty or finite, by proposition 2.3 follows that there is a peridic point having a contractive complex eigenvalue with modulus close to one. By lemma 2.1.7 we get a periodic point with two contractive real eigenvalues and such that teir stable eigenspacer has small angle. Moreover, we can assume that this periodic point expends a large part of its orbit close to the periodic point with complex eigenvalue and so its the rate of contraction is also close to one. Then, we can apply the lemma 2.1.5.

3 Case that either $T_{H_p}M = E^s \oplus F$ or $T_{H_p}M = E \oplus F^u$.

First we will consider the case such that either $T_{H_p}M = E^s \oplus F$ or $T_{H_p}M = E \oplus F^u$. To do that, we use a theorem proved in [PS4] that studies the dynamical consequences of a codimension one dominated splitting in any dimension.

Theorem 3.1 *Let $f \in Diff^2(M^n)$ be a Kupka-Smale system. Let Λ be a compact invariant set contained in an homoclinic class such that exhibits a dominated splitting $T_{H_p}M = E^s \oplus F$ where E^s is uniformly contracted and $dim(F) = 1$. Then Λ is hyperbolic.*

The central argument follows from the fact that F has dimension one and the complementary direction are uniformly contractive. This allows to perform similar argument developed in [PS1].

In [Z] similar results was obtained: in the mentioned paper was proved that given a topological minimal compact invariant set Λ such that exhibits a dominated splitting $T_{H_p}M = E^s \oplus F$ where E^s is uniformly contracted and $dim(F) = 1$ follows that F is hyperbolic.

In our context we get the next two corollaries:

Corollary 3.1 *If $T_{H_p}M = E^s \oplus F$ with $dim(E^s) = 2$, then H_p is hyperbolic. If $T_{H_p}M = E \oplus F^u$ with $dim(F^u) = 2$, then H_p is hyperbolic.*

Corollary 3.2 *Let $f \in Diff^2(M^3)$ be such that $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. If E_2 is hyperbolic, then the homoclinic class is hyperbolic.*

The last corollary is immediate and holds in the following way: if E_2 is uniformly contractive, by domination holds that E_1 is also uniformly contractive. Then we are in the presence of a contractive codimension one dominated splitting and we can apply the theorem 3.1. if E_2 is uniformly expansive, by domination holds that E_3 is also uniformly expansive. Then we are in the presence of a expansive codimension one dominated splitting and we can apply the theorem 3.1.

4 Case that $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.

To finish with the maim theorem we have to deal with the most difficult case i.e.: $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. In fact, the rest of the paper is devoted to deal with this situation. The study in this case goes through different steps:

Step I: First, we conclude that under the assumption of attracting set, the local tangent manifold of the extremal directions (E_1 and E_2) are dynamically defined. More precisely we show that the local tangent manifold to the direction E_1 and E_3 are stable and unstable manifolds respectively. This is the statement of theorem 4.1 and it is formulated in the subsection 4.1.

Step II: Using that the local tangent manifold associated to the extremal directions are dynamically defined, it is proved that if the center direction is not hyperbolic, then there are periodic points homoclinically related with p such that the eigenvalue associated with the central direction is close to one. This is the theorem 4.2 and it is formulated in the subsection 4.2. This theorem is a reformulation of the proposition 2.3 stated in section 2.

Step III: We consider independently the case where the periodic point p has either stable index one or stable index two (where stable index means the number of eigenvalues smaller than one). In the case that p has stable index one, using the connecting lemma and the fact that we are dealing with an attractor, it is possible to get an intersection between the tangent manifold to the extremal direction of a periodic point with central eigenvalue close to one. From this, the periodic point is bifurcated in a way to obtain a heterodimensional cycle. This is done in subsection 4.3. The rest of the steps deals with the case that p has stable index two.

Step IV: In the case that the periodic point p has stable index two (see subsection 4.4), first we study the dynamical behavior of the manifold tangent to the center direction. If the center manifold is not a stable manifold then it is proved that by a C^1- perturbation it is obtained a periodic point with center eigenvalue close to one and such that the tangent manifold associated to the extremal direction has an intersection. From this, the periodic point is bifurcated in a way to obtain a heterodimensional cycle (see subsection 4.4.1). If the center manifold is a stable manifold we proceed with the next step.

Step V: At this point, we are dealing with a homoclinic class such that for every point there are two transversal dynamically defined local manifold of uniform size: one is a two dimensional local stable manifold and the other is a one dimensional unstable manifold. Observe that the local stable manifold contains a strong stable one tangent to E_1. We consider two situation: either each strong stable manifold intersects the attractor in only one point, or it is not the case (see subsection 4.4.2). In the former, by a result proved in [BC] follows that there exists a two dimensional normally hyperbolic submanifold containing the attractor; i.e: we are dealing with a two dimensional system normally hyperbolic (see section 6) and from there, using a result similar to the second theorem proved in [PS1], it is shown that the homoclinic class is hyperbolic. In the latter, a series of different suitable perturbation are performed, with the goal to make a connection between the tangent manifold to the extremal direction of a periodic point with central eigenvalue close to one, and again the periodic point is bifurcated in a way to obtain a heterodimensional cycle. Large part of the paper is devoted to introduce new perturbation techniques that allows to obtain a intersection between the strong manifolds associated to a periodic points. These perturbation performed in this context are possible using the structure that was obtained for the homoclinic class. This is done in the sections 7 and 8.

4.1 Dynamical behavior of the tangent manifolds associated to the extremal directions.

First, we state the existences of center manifold tangent to each subbundle of the dominated splitting. Recall by [HPS] that there are $1-$dimensional manifolds $W^{E_i}_\epsilon(x)$ tangents to each E_i. More precisely, let us define first $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^1(I_1, M)$ the set of C^1-embedding of I_1 on M.

Lemma 4.1.1 *For each subbundle E_i there exists a continuous functions ϕ^i : $H_p \to Emb^1(I_1, M)$ such that for any $x \in H_p$ it is defined $W^{E_i}_\epsilon(x) = \phi^i(x)I_\epsilon$ and verifies:*

1. $T_x W^{E_i}_\epsilon(x) = E_i(x)$,
2. *if* $f(W^{E_i}_\epsilon(x)) \subset B_\epsilon(f(x))$ *then* $f(W^{E_i}_\epsilon(x)) \subset W^{E_i}_\epsilon(f(x))$,
3. *if* $f^{-1}(W^{E_i}_\epsilon(x)) \subset B_\epsilon(f^{-1}(x))$ *then* $f^{-1}(W^{E_i}_\epsilon(x)) \subset W^{E_i}_\epsilon(f^{-1}(x))$.

The previous lemma does not state any dynamical meaning for the tangent manifold. Later it is proved that under some assumption, these manifolds can be dynamically defined.
To precise, first recall the definition of local stable and unstable manifold of size ϵ:

$$W^s_\epsilon(x) = \{y \in M :\ dist(f^n(y), f^n(p)) \to 0\, n \to \infty, dist(f^n(y), f^n(p)) < \epsilon\},$$

$$W^u(p) = \{y \in M :\ dist(f^n(y), f^n(p)) \to 0\, n \to -\infty, dist(f^n(y), f^n(p)) < \epsilon\}$$

With this definition in mind, we say that *the tangent manifold $W^{E_3}_\epsilon$ is dynamically defined* if there exists, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $x \in H_p$ follows that

$$W^{E_3}_{\epsilon_1}(x) \subset W^u_{\epsilon_2}(x)$$

In the same way, we say that $W_\epsilon^{E_1}$ *is dynamically defined* if there exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $x \in H_p$ follows that

$$W_{\epsilon_1}^{E_1}(x) \subset W_{\epsilon_2}^{s}(x)$$

In this case, we call $W_\epsilon^{E_3}$ and $W_\epsilon^{E_1}$ the strong unstable and strong stable manifolds respectively. Observe that in this case the tangent manifold are unique.

The next theorem states that assuming that the system is C^2 and the homoclinic class being a maximal invariant set follows that the tangent manifolds associated to the extremal direction are dynamically defined (see figure 2). The theorem is a consequences of a result stated in [PS4] and holds in any dimension assuming that the extremal directions are one dimensional. The precise statement of this theorem

is formulated in next section.

Theorem 4.1 *Let $f \in Diff^2(M^3)$. If H_p is a maximal invariant homoclinic class exhibiting a dominated splitting with three directions $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$, then there exists $\epsilon.0$ such that $W_\epsilon^{E_1}$ and $W_\epsilon^{E_3}$ are dynamical defined (see figure 2).*

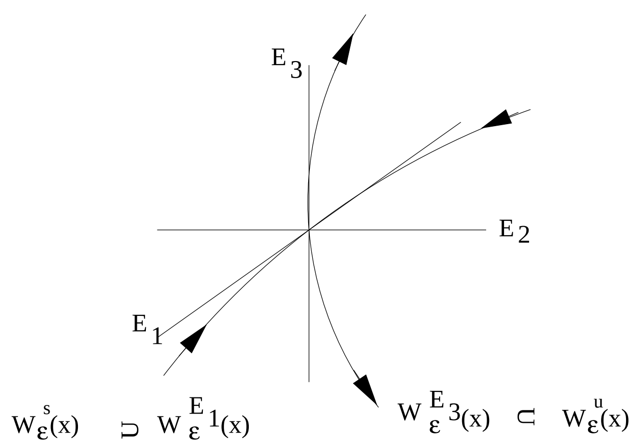


Figure 2

Remark 4.1 *Observe that we are not assuming in this case that the homoclinic class is an attractor. It is only assumed that the homoclinic class is maximal invariant; i.e.: $H_p = \cap_{\{n \in \mathbb{Z}\}} f^n(U)$ for some open neighborhood U.*

As a consequences of the previous theorem we can get the next lemma that it shows that either the periodic points in the homoclinic class has the same stable index or it is possible to get a diffeomorphisms arbitrarily close to the initial one exhibiting a heterodimensional cycle.

Lemma 4.1.2 *Let $f \in Diff^2(M^3)$ and let $H_p = \cap_{\{n \in \mathbb{Z}\}} f^n(U)$ be a maximal invariant homoclinic class exhibiting a dominated splitting with three directions $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. Then, one of the following options holds:*

1. *there exists a neighborhood U of H_p such that all the periodic points in U has the same stable index;*

2. *there is g C^1 arbitrarily close to f exhibiting a heterodimensional cycle.*

Proof:

Let us assume that the point p in the homoclinic class has stable index two. We have to show that all the periodic points in the neighborhood U has stable index two. If it is not the case, we have to show that we can C^1-approximate f by another diffeomorphism g exhibiting an heterodimensional cycle. If there exists a periodic point q of stable index one in a small neighborhood of H_p, from the fact that the homoclinic class is maximal invariant, follows that it is contained in H_p. Since we are assuming that the homoclinic class exhibits three directions, follows that any intersection of the stable and unstable manifold of p is a transversal intersection. Then, there is a sequences of points q_n of stable index two homoclinically related to p and accumulating on q. Due to the fact that the strong stable manifold has uniform size for any point q_n close to q follows that the strong stable manifold of them intersects transversally the unstable manifold of q. Let us take a point q' of the sequences q_n. Observe that the intersection of the stable manifold of q' with the unstable manifold of q is robust by perturbation.

¿From the fact that q is in H_p and it has stable index one *the local stable manifold of q is one-dimensional), follows that there are points in the homoclinic class that accumulates in the stable manifold of the point q. Since H_p is an attractor, the unstable manifold of q' is contained in H_p and so there exist a point in H_p with orbit accumulating in the unstable manifold of q' and in the stable manifold of q. So, using the connecting lemma, we get that with a C^1-perturbation it is possible to connect the unstable manifold of q' with the stable manifold of q. Then it was created an heterodimensional cycle involving q and some q' close to q.

The case that p has stable index one is treated in the same way.

■

So, from now on, we assume that *all the periodic points in U has the same stable index.*

At this point, we split the proof of the maim theorem in two cases:

- **Case A:** The periodic point p has stable index one,
- **Case B:** The periodic point p has stable index two.

Before to deal with both situation, we need some results proved elsewhere. We enunciate these results in the next subsection, and in the subsections 4.3 and 4.4 we return to both cases enunciated above.

4.2 Previous results.

First, we start reformulating the proposition 2.3 to the case that the splitting has three directions. The present reformulation states that under the assumption of dominated splitting over a homoclinic class for a C^2 diffeomorphisms in a three dimensional manifold, holds that if the direction E_2 is not hyperbolic then there are periodic points contained in H_p and homoclinically related to p such that the eigenvalue associated to the center direction is close to one.

Theorem 4.2 *([PPV]) Let $f \in Diff^2(M^3)$ and let H_p be a homoclinic class exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$.*

- *If p has stable index two and the direction E_2 is not hyperbolic, then for any $\delta > 0$ there exists a periodic point q with period n_q and homoclinically related to p such that $(1-\delta)^{n_q} < |Df^{n_q}_{|E_2(q)}| < 1$ (in this case we say that q has $\delta-$weak contraction along the center direction).*

- *If p has stable index one and the direction E_2 is not hyperbolic, then for any $\delta > 0$ there exists a periodic point q with period n_q and homoclinically related to p such that $1 < |Df^{n_q}_{|E_2(q)}| < (1+\delta)^{n_q}$ (in this case we say that q has $\delta-$weak expansion along the center direction).*

This version follows immediately from the proposition 2.3 and from the fact that we are assuming that all the periodic points in the homoclinic class has the same stable index. For instance, in the case that p has stable index two, and the homoclinic class is not hyperbolic, from proposition 2.3 follows that there is a periodic point with weak rate of contraction along the direction $E_1 \oplus E_2$. Since the angle between both direction is uniformly bounded from
below and from the domination property, follows that

$$|Df_{E_1 \oplus E_2}| = \max\{|Df_{|E_1}|, |Df_{|E_2}|\} = |Df_{|E_2}|$$

and therefore follows the previous theorem.

It is important to remark that the previous theorem is not a perturbation theorem. More precisely, the theorem 4.2 shows that the obstruction of the hyperbolicity (in the context that we are considering) come from the fact that there are periodic points with eigenvalues close to one in the center direction.

An immediate corollary is the following result:

Corollary 4.1 *Let H_p be a homoclinic class exhibiting a dominated splitting*

$$T_{H_p}M = E_1 \oplus E_2 \oplus E_3$$

- *If the direction E_2 is not contractive and p has stable index two, then for any $\delta > 0$ the periodic points exhibiting $\delta-$weak contraction are dense.*
- *If the direction E_2 is not expansive and p has stable index two, then for any $\delta > 0$ the periodic points exhibiting $\delta-$weak expansion are dense.*

In fact, to conclude this corollary from the theorem 4.2 it is enough to recall that the point point q with weak contraction (expansion) is homoclinically related to the point p. So, taking any point x in the homoclinic class, we can approximate it by a periodic point z homoclinically related to p and so homoclinically related to q. Then, we can take periodic points in horseshoes that contains z and q with the property that they accumulate on z but they expends more time close to q. So, these points have a weak contraction (expansion) along the center direction. This kind of arguments are folklore (see for instance [BDP]) and we state it

here for sake of completeness. To be precise, we get the following lemma:

Lemma 4.2.1 *Let $f \in Diff^r(M)$ having two periodic points q and q_δ such that there are homoclinically connected and such that q_δ has $\frac{\delta}{2}-$weak contraction along the center direction. Then, for any $r > 0$ there is a periodic point q_δ^r homoclinically connected with q such that $dist(q, q_\delta^r) < r$ and q_δ has $\delta-$weak contraction along the center direction.*

For some periodic points in the homoclinic class follows that they exhibits a transverse intersection of its stable and unstable manifold. If this intersection holds along the strong stable and unstable manifolds we say that there is a strong homoclinic connection:

Definition 10 Strong homoclinic connection. *Given a periodic point q, we say that it has a strong homoclinic connection if the strong stable and strong unstable manifolds of q has an intersection.*

Now, let assume that there is a periodic point with weak contraction (expansion) along the center direction and also exhibiting a strong homoclinic connection. In this case, after a C^1 perturbation, it is possible to show that it is created a heterodimensional cycle.

Proposition 4.1 *Given $\delta_0 > 0$, there exists δ such that if there is a periodic point with $\delta-$weak contraction (expansion) along the central direction and exhibiting and strong homoclinic connection, then there is g $C^1 - \delta_0-$close to f exhibiting a heterodimensional cycle.*

The proof of this proposition is given in section 5.

Now we formulate a lemma proved in [H] that allows to connect the strong stable and unstable manifolds when they are orbits that accumulates on both manifolds.

Lemma ([H]): *(C^1- connecting lemma:) Let $f \in Diff^r(M^n)$ and let p be a periodic point such that there are points x in the strong unstable manifold and y in the strong unstable manifold, a sequence of points x_n accumulating in x and points $f^{k_n}(x_n)$ in the forward orbit of the sequences x_n accumulating on y. Then, there is a diffeomorphisms g C^1-close to f such that p remains periodic for g, x is in the strong unstable manifold, y is in the strong unstable manifold and y is in the forward orbit of x.*

4.3 Case A: p has stable index one.

In this case, we show that if the homoclinic class is not hyperbolic then we can get a heterodimensional cycle.

First, we get the following proposition (the proof of this proposition is given in section 5):

Proposition 4.2 *Let H_p be an attracting homoclinic class associated to a periodic point of of stable index one and exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. If H_p is not hyperbolic, then for any $\delta > 0$ there exists g C^1-close to f and a periodic point q with $\delta-$weak expansion along E_2 such that q exhibits and strong homoclinic connection.*

After that, we use proposition 4.1 finishing the proof of the Maim Theorem when the periodic point p has stable index one.

4.4 Case B.

To continue, we consider two cases: either the center manifold is dynamically defined or it is not the case.

More precisely, we say that $W^{E_2}_{\epsilon}$ is dynamically defined if there exist $\epsilon > 0$ and $\gamma > 0$ such that for any $x \in H_p$ follows that

1. $f^n(W^{E_2}_{\epsilon}(x)) \subset W^{E_2}_{\gamma}(f^n(x))$ for any $n \geq 0$,
2. $\ell(f^n(W^{E_2}_{\epsilon}(x))) \to 0$ as $n \to \infty$.

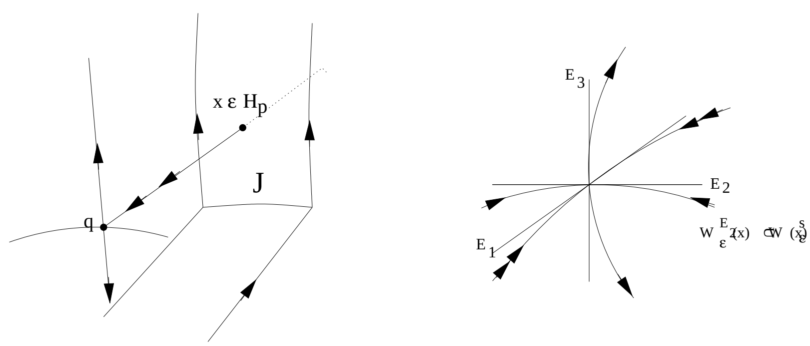
In other words, we are saying that $W^{E_2}_{\epsilon}(x)$ is dynamically defined if it is contained in $W^s_{\epsilon'}(x)$ for some $\epsilon' > 0$.

Related to the previous option (if the center manifold is either dynamically defined or not) we get the following proposition (the proof of this proposition is given in section 5):

Proposition 4.3 *Let $f \in Diff^2(M^3)$. Let H_p be an attracting homoclinic class associated to a periodic point of of stable index two and exhibiting a dominated splitting $T_{H_p}M = E_1 \oplus E_2 \oplus E_3$. Then, one of the following option holds (see figure 3):*

1. *Case B.1: for any $\delta > 0$, there is a periodic point q with $\delta-$weak contraction along E_2 such that*
$$[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$$
In this case we say that the homoclinic class has a point in the strong stable manifold of the point q.

2. *Case B.2: the center manifold $W^{E_2}_\epsilon(x)$ tangent to E_2 is dynamical defined.*



CASE B1

CASE B2

Figure 3

ε

The first case is similar to the case A and it is treated similarly. This is done in the next subsection.

4.4.1 Case B.1.: the center is not a stable manifold.

In this case we have that for any $\delta > 0$ there exists a periodic point q homoclinically related to p such that

1. $(1-\delta)^{n_q} < |Dg^{n_q}_{|E_2(q)}| < 1$ where n_q is the period of q,

2. the homoclinic class has a point in the strong stable manifold of q (i.e.: $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$).

Then for any $\delta > 0$ using the C^1-connecting lemma, we get that by a C^1-perturbation follows that there is g C^1- close to f with a periodic point q which $\delta-$weak contractive along E_2 and exhibiting a strong homoclinic connection (see figure 4).

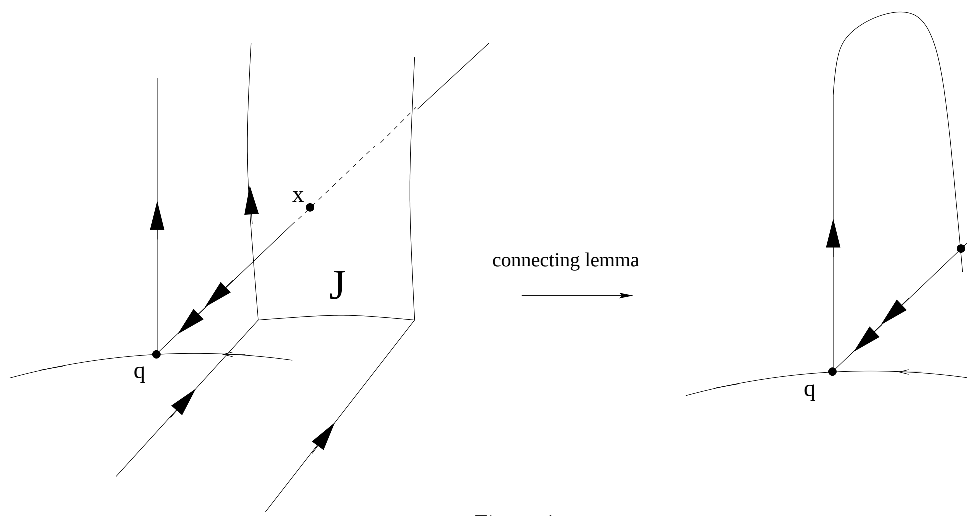


Figure 4

Using the proposition 4.1 follows the existence of a heterodimensional cycle and so finishing the proof of the Maim Theorem in the case B1, i.e.: we finished showing the existence of a heteroclinic cycle in the case that the homoclinic class is not hyperbolic and it has stable index one and in the case that the homoclinic class has stable index two and the center direction is not dynamically defined.

4.4.2 Case B.2.: the center manifold is a stable manifold.

Recall that in this case we are assuming that the central manifold tangent to E_2 is dynamically defined. Under this assumption, we get the following proposition (the proof of this proposition is given in section 5):

Proposition 4.4 *If the center manifold tangent to E_2 is dynamically defined then follows that E_1 is uniformly contracted.*

¿From the fact that the dominated splitting is decomposed in one dimensional subbundles, we can assume that there is an adapted metric such that the constant of domination is $\lambda < 1$ and $C = 1$. Moreover, we can assume that there is $\lambda_s < 1$ such that

$$|Df_{|E_1}| < \lambda_s$$

Now we get that taking $E = E_1 \oplus E_2$ and $F = E_3$ follows that for any point $x \in H_p$ there is a stable and unstable manifold of uniform size tangents to E

and F respectively. From now on, we call the manifold tangent to E_1, the strong stable manifold; the manifold tangent to E_2, the center manifold; the manifold tangent to $E_1 \oplus E_2$, the center-stable manifold; the manifold tangent to E_3, the unstable manifold; the manifold tangent to $E_2 \oplus E_3$, the center-unstable manifold. In the present context, follows that the center-stable manifold is contained in the local stable manifold.
More precisely: there exist continuous functions

$$\phi^{cs} : H_p \to Emb^1(D_1, M)$$

$$\phi^{ss} : H_p \to Emb^1(I_1, M)$$
$$\phi^{c} : H_p \to Emb^1(I_1, M)$$
$$\phi^{u} : H_p \to Emb^1(I_1, M)$$

where $I_1 = (-1, 1)$, $I_\epsilon = (-\epsilon, \epsilon)$; $D_1 = \{z \in R^2 : ||z|| < 1\}$; $D_\epsilon = \{z \in R^2 : ||z|| < \epsilon\}$ such that for any $x \in H_p$ it is defined

$$W^{cs}_\epsilon(x) = \phi^{cs}(x)D_\epsilon;\ \ W^{ss}_\epsilon(x) = \phi^{ss}(x)I_\epsilon;\ \ W^{c}_\epsilon(x) = \phi^{c}(x)I_\epsilon;\ \ W^{u}_\epsilon(x) = \phi^{u}(x)I_\epsilon$$

and verifying

1. $T_xW^{cs}_\epsilon(x) = E(x)$, $T_xW^{ss}_\epsilon(x) = E_1(x)$, $T_xW^{c}_\epsilon(x) = E_2(x)$, $T_xW^{u}_\epsilon(x) = F(x)$
2. $W^{cs}_\epsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0, dist(f^n(x), f^n(y)) < \epsilon\}$,
3. $W^{ss}_\epsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) < \lambda_s^n, dist(f^n(x), f^n(y)) < \epsilon\}$
4. $W^{c}_\epsilon(x) \subset W^{cs}_\epsilon(x) = W^{s}_\epsilon(x)$
5. $W^{u}_\epsilon(x) = \{y \in M : dist(f^{-n}(x), f^{-n}(y)) \to 0, dist(f^{-n}(x), f^{-n}(y)) < \epsilon\}$.

Definition 11 Topologically hyperbolic homoclinic class:
Given a maximal invariant homoclinic class exhibiting a dominated splitting $E_1 \oplus E_2 \oplus E_3$, it is said that the homoclinic class is a topologically hyperbolic homoclinic class if the direction E_1 is contractive, the local tangent manifold to E_2 is contained in the local stable manifold and the local tangent manifold to E_3 is contained in the the local unstable manifold.

To study this case we consider the next obvious alternative:

1. **Case B.2.1. The strong stable direction is not involved:** *there exists $\epsilon > 0$ such that for any $x \in H_p$ follows that $[W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p = \emptyset$.*

2. **Case B.2.2. The strong stable direction is involved:** *there is a pair of points x, y in the homoclinic class such that $y \in W^{ss}_\epsilon(x)$.*

In other words, we consider the set

$$\mathcal{T} = \{x \in H_p : [W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset\}$$

and in the case B.2.1 we assume that $\mathcal{T}$ is empty and in the case B.2.2 we suppose that $\mathcal{T}$ is not empty.

The two previous cases are possible. For instances, in the case of the three dimensional solenoid we get that the strong stable direction is involved in the dynamic. To get an example of the second situation, consider a two dimensional attractor for a two dimensional diffeomorphisms f (for instance a Plykin's attractor) in a two dimensional manifold, and then, embeds this manifold in three dimensional manifold in a way that the three dimensional diffeomorphism coincide with f in the two submanifold and such that this submanifold is invariant and normally hyperbolic for the new dynamic (see section 6 for precise definitions).

The rough idea to deal with the two previous case is the following;

1. *In the case that $\mathcal{T} = \emptyset$, observe that projecting along the strong stable manifold we get a* two dimensional diffeomorphisms *exhibiting dominated splitting.*

2. *In the case that $\mathcal{T} \neq \emptyset$, assuming that the direction E_2 is not hyperbolic, using a suitable perturbation argument we get a periodic point q with a weak contraction along the direction E_2 and exhibiting a connection along the strong directions. After that, again it is created a heterodimensional cycle.*

Case B.2.1. The strong stable direction is not involved ($\mathcal{T} = \emptyset$).

In this case, observe that "$\Pi^{ss} \circ f$ is a two dimensional diffeomorphism", where Π^{ss} is the projection along the strong stable manifold over some center-unstable manifold. More precisely, from a result proved in [BC], the attractor is contained inside a two dimensional normally hyperbolic submanifold. Observe that in this case, there is no chance to perturb the system in a way to create a heterodimensional cycle. In fact, if the submanifold that contains the attractor is normally hyperbolic, follows that it is robust by perturbation and the perturbed homoclinic class will be contained in this submanifold. So, for any g close to f follows that for any $x \in \Lambda_g = \cap_{\{n>0\}} g^n(U)$ we get that $W^{ss}_\epsilon(x, g) \cap \Lambda_g = \{x\}$. Therefore, there is not possible to get an heterodimensional cycle for g.

Therefore, to prove the maim theorem in this case, we have to prove that the homoclinic class is hyperbolic. To conclude that, we prove the following theorem.

Theorem 4.3 *Let $f \in Diff^2(M^3)$ and let H_p be an attracting topologically hyperbolic homoclinic class such that $\mathcal{T} = \emptyset$. Then, the homoclinic class is hyperbolic.*

The proof of this theorem is given in section 6.

This finish the maim theorem in the case B.2.1.

Case B.2.2. The strong stable direction is involved ($\mathcal{T} \neq \emptyset$).

Here we are assuming that there exist $\epsilon > 0$ such that for some $x \in H_p$ follows that $[W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset$. We do not know a priori if the point x is periodic. If it is the case, we could do the same kind of argument done in the case A and in case B.1. If x is not periodic (in particular, it could happen that the orbit of x is dense or at least it could accumulate in the point y) it is not possible to apply the connecting lemma kind of arguments. However, using the geometrical structure of the system, it is possible to perform a series of suitable perturbation to get a periodic point with a strong homoclinic connection.

Theorem 4.4 *Let H_p be a topologically hyperbolic attracting homoclinic class. Let also assume that H_p is not hyperbolic. If the strong stable manifold is involved in the dynamic (i.e.: $\mathcal{T} \neq \emptyset$), then for every $\delta > 0$ there is g C^1 close to f such that it has a periodic point q having a $\delta-$weak contraction along E_2 and exhibiting a strong homoclinic connection.*

The proof of this theorem is given in section 7. In fact, to conclude the existence of a heterodimensional cycle we apply again the proposition 4.1. This finish the maim theorem in the case B.2.2.

To prove the previous theorem, we consider either if the interior of $\mathcal{T}$ is empty or if it is not; where the interior is taken in the topology restricted to the set H_p.

If the interior is not empty, from the density of the periodic points and corollary 4.1 follows that there is a periodic point with weak contraction along the center direction contained in $\mathcal{T}$. Then, it is applied the connecting lemma to obtain a periodic point with weak contraction along the center direction and exhibiting a strong connection. Using proposition 4.1, we obtain a heterodimensional cycle. If the interior of $\mathcal{T}$ is empty, we cannot assume that x is a

periodic point, and so it is performed a perturbation in a way to be back to a similar situation to the one previously considered.

Before to end the section we would like to make some remarks. Observe that in the case B.2.1 (when $\mathcal{T}$ is empty) it was proved that the homoclinic class is hyperbolic. It could be asked if it is possible to get a similar result when $\mathcal{T}$ is not empty: *given a Kupka-Smale topologically hyperbolic class such that the strong direction is involved in the dynamic, is it true that the homoclinic class is hyperbolic?*

The answer is no and it is easy to construct a counterexample:

Let $H_p = \cap_{\{n>0\}} f^n(U)$ be a hyperbolic attracting homoclinic class for a surface diffeomorphism f.
Let $\mathcal{M}$ be a minimal set contained in H_p.
Let $h : U \to R$ be a C^∞ function such that

1. $0 < h(x) \leq 1$ for all $x \in U$,
2. $h_{|\mathcal{M}} = 1$,
3. $h_{|\mathcal{M}^c} < 1$.

Let $F : U \times [-1, 1] \to U \times [-1, 1]$ defined as

$$F(x, y) = (f(x), h(x)y - y^3)$$

Observe that the set

$$H_p \times \{0\}$$

is an attracting homoclinic class. In fact, $H_p \times \{0\} = \cap_{\{n>0\}} F^n(U \times [\frac{-1}{2}, \frac{1}{2}])$. Moreover, the homoclinic class has dominated splitting $E_1^s \oplus E_2 \oplus E_3^u$ and $F_{|U\times[-1,1]}$ is a Kupka-Smale system. This follows from the fact that the periodic points of F are contained in the complement of $\mathcal{M}$ and in this set the center direction is contractive from the fact that $|DF_{E_2(x)}| = h(x)$. However, F is not hyperbolic from the fact that $|DF_{E_2(x)}|$ is equal to one when $x \in \mathcal{M}$.

5 Proof of propositions and theorems of subsections 4.1, 4.3, 4.4.

First, we appeal to some results and definitions proved in [PS4] for "codimension one dominated splitting".

Definition 12 *Let $f : M \to M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having dominated splitting $E \oplus F$ with $dim(F) = 1$. Let U be an open set containing Λ where is possible to extend the previous dominated splitting. We say that a C^2-arc I in M (i.e, a C^2-embedding of the interval (-1,1)) is a δ-E-arc provided the next two conditions holds:*

1. *$f^n(I) \subset U$, and $|f^n(I)| \leq \delta$ for all $n \geq 0$.*
2. *$f^n(I)$ is always transverse to the E-direction.*

Related to this kind of arcs it is proved in [PS4] the following result.

Theorem 5.1 *There exists δ_0 such that if I is a δ-E-interval with $\delta \leq \delta_0$, then one of the following properties holds:*

1. *$\omega(I) = \cup_{\{x\in I\}}\omega(x)$ is a periodic simple closed curve $\mathcal{C}$ normally hyperbolic and $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (where m is the period of $\mathcal{C}$) is conjugated to an irrational rotation,*
2. *$\omega(I) \subset J$ where J is the a periodic arc normally hyperbolic.*

Now we can proceed to show how the theorem 4.1 follows from the previous result.

Proof of theorem 4.1:

First, we prove that the manifold tangent to E_3 is an unstable manifold. We start showing that there exist $\epsilon > 0$ and $\gamma > 0$ such that $f^{-n}(W^{E_3}_\epsilon(x)) \subset W^{E_3}_\gamma(f^{-n}(x))$.

Let us assume that this is not the case. So it follows that there are a positive number γ, a sequences of positive numbers $\epsilon_n \to 0$, points x_n and a strictly increasing sequences of positive integers k_n such that

$$\ell(f^{-k_n}(W^{E_3}_{\epsilon_n}(x_n))) = \gamma$$

and

$$\ell(f^{-j}(W^{E_3}_{\epsilon_n}(x_n))) < \gamma \quad 0 \leq j \leq k_n$$

Taking

$$I = \lim_{n\to+\infty} f^{-k_n}(W^{E_3}_{\epsilon_n}(x_n))$$

follows that I does not growth for positive iteration and it is transversal to $E_1 \oplus E_2$; i.e.:

$$\ell(f^j(I)) \leq \gamma \ \forall j > 0$$

Then, we can apply theorem 5.1 and follows that $\omega(I)$ is a periodic normally hyperbolic curve with dynamic conjugated to an irrational rotation or it is contained in a periodic arc. Both situation cannot hold inside a homoclinic class. To prove that $\ell(f^{-n}(W^{E_3}_\epsilon(x))) \to 0$ as $n \to \infty$ we repeat the same argument. In fact, if it is not the case, we can find an arc I transversal to $E_1 \oplus E_2$ that does not growth by positive iterations and the same conclusion is obtained.

To show that W^{E_1} is dynamically defined, we take f^{-1} and it is done the same argument changing backward by forward iterations.

■

Proof of lemma 4.2.1:

To prove the lemma, observe that using that q and q_δ are homoclinically connected, follows that there is a horseshoes containing q and q_δ. Moreover, we can take two small neighborhoods W and W_δ of q and q_δ respectively such that there exists two positive integers k_1 and k_2 and for some positive integer n and n_δ arbitrarily large, there exists a periodic point z such that

1. the period of z is $n + k_1 + n_\delta + k_2$,
2. for any $0 \leq i \leq n$ follows that $f^i(z) \in W$,
3. for any $0 \leq i \leq n_\delta$ follows that $f^{n+k_1+i}(z) \in W_\delta$.

Observe that for any $r > 0$ there is $n = n(r)$ such that the corresponding periodic point z has an iterate such its distance to q is smaller than r.

To see that z is $\delta-$weakly contractive along the E_2 direction we proceed as follows:

Observe on one hand that in the neighborhood W of q_δ follows that if $y \in W$ then

$$(1 - \frac{\delta}{2}) < |Dg_{|E_2(y)}| < 1$$

(we can suppose for simplicity that q_δ is fixed). On the other hand, observe that there is a positive constant C such that

$$C^{-1} < |Dg^{k_1}_{|E_2|}| < C \quad C^{-1} < |Dg^{k_2}_{|E_2}| < C$$

So, given the periodic point z

$$C^{-2}|Dg^n_{|E_2(z)}|(1 - \frac{\delta}{2})^{n_\delta} < |Dg^{n+k_1+n_\delta+k_2}_{|E_2(z)}| < C^2|Dg^n_{|E_2(z)}|$$

If n is large enough, follows that

$$C^2|Dg^n_{|E_2(z)}| < 1$$

Fixed n, we take n_δ large enough such that

$$(1-\delta)^{n+k_1+n_\delta+k_2} < C^{-2}|Dg^n_{|E_2(z)}|(1-\frac{\delta}{2})^{n_\delta}$$

¿From both inequalities the lemma follows.

■

Proof of proposition 4.1.

The proof consist in to bifurcate the periodic point with center eigenvalue close to one in two periodic points of different stable index and to control the behavior of the unstable manifold of the periodic point that is created by the bifurcation. We use the lemma 2.1.1 to bifurcate the periodic point that has an eigenvalue close to one.

Let us take a point q with $\delta-$weak contraction along the direction and exhibiting a strong homoclinic connection. Let us take a point x contained in $[W^{E_3}_\epsilon(q)\setminus\{q\}]\cap[W^{E_1}_\epsilon(q)\setminus\{q\}]$ and let γ^u be a connected compact arc containing x and contained in a fundamental domain of $W^{E_3}(q)$. Let also takes γ^{ss} the compact arc contained in $W^{E_1}_\epsilon(q)$ that connects q with x. Using the lemma 2.1.1 we bifurcate q into three periodic points q_{-1}, q_0, q_1 for a diffeomorphisms g C^1-close to f such that q_{-1} and q_1 has stable index two and q_0 has stable index one. On one hand observe that $W^s(q_0)\cap W^u(q_1)\neq\emptyset$. Observe also that for each point q_i there is an arc γ^{ss}_i contained in $W^{E_1}_\epsilon(q_i)\setminus\{q_i\}$ which remains close to γ^{ss}. On the other hand, we can perform the bifurcation such that $g^{-1}(\gamma^u)\subset W^{E_3}_\epsilon(q_{-1})$. So, a heterodimensional cycle is created (see figure 5).

■

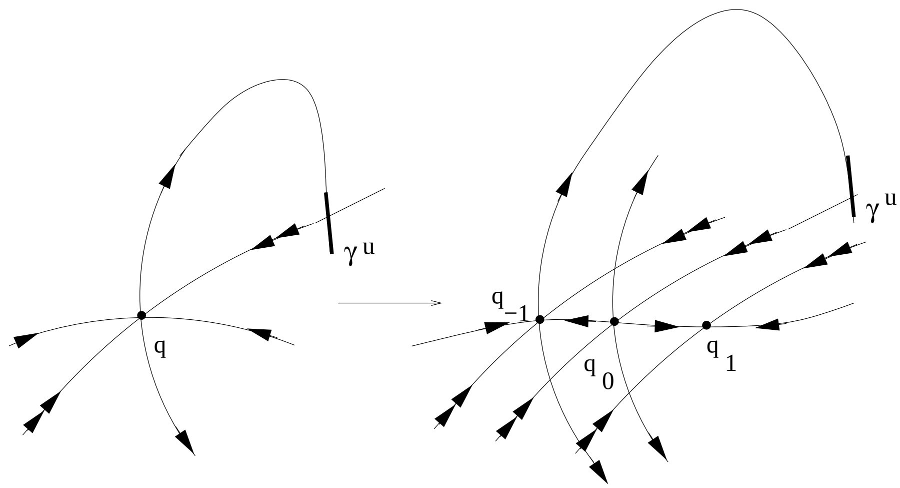


Figure 5

Proof of proposition 4.2.

Let us take a point q with weak expansion along the center direction.

On one hand, since q and p are homoclinically related and both points has stable index one, follows that $W^{E_1}_\epsilon(q) \cap W^{E_3}(p) \neq \emptyset$ and so $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$ (recall that $W^{E_1}_\epsilon(q)$ and $W^{E_3}_\epsilon(q)$ are the local strong stable and unstable manifold). On the other hand, again since H_p is a attractor, follows that $W^{E_3}_\epsilon(q) \subset H_p$. Then, there are orbits in H_p accumulating in $W^{E_3}_\epsilon(q) \setminus \{q\}$ with positive iterates also accumulating in $W^{E_1}_\epsilon(q) \setminus \{q\}$. By the connecting lemma, we can get a periodic point with weak expansion along the center direction and exhibiting a strong homoclinic connection.

■

Proof of proposition 4.3

Recall that all the periodic points has the same stable index.

First, we start proving that either

1. there exist $\epsilon > 0$ and $\gamma > 0$ such that $f^n(W^{E_2}_\epsilon(x)) \subset W^{E_2}_\gamma(f^n(x))$ for any $n > 0$ or,
2. there exists a periodic point q having weak contraction along the center direction and such that $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$.

Let us assume that the first option does not hold. Then follows that for any small positive number γ, there is a sequences of positive numbers ϵ_n, points x_n and an increasing sequences of positive integers k_n such that

$$\ell(f^{k_n}(W^{E_2}_{\epsilon_n}(x_n))) = \gamma$$

and

$$\ell(f^j(W^{E_2}_{\epsilon_n}(x_n))) < \gamma \quad 0 \leq j < k_n$$

Taking

$$J = lim_{n\to+\infty} f^{k_n}(W^{E_2}_{\epsilon_n}(x_n))$$

follows that J does not growth for negative iteration; i.e.:

$$\ell(f^{-j}(J)) \leq \gamma \;\; \forall j > 0$$

Then, it follows that

$$f^{-j}(J) \subset U \;\; \forall j > 0$$

If $\gamma > 0$ is small enough then

$$f^{-j}(J) \subset U \;\; \forall j > 0$$

and so J is contained in $H(p)$. So, it is approximated by periodic points and we can assume that these points have $\delta-$weak contraction along the direction E_2. Now we take

$$W^u_\epsilon(J) = \cup_{\{x\in J\}} W^{E_3}_\epsilon(x)$$

and a sequences of periodic point q_n close to some point in the interior of J.

We have that either the periodic points are not contained in $W^u_\epsilon(J)$ or they are contained in $W^u_\epsilon(J)$. In the first case, we have that the strong stable manifold of some q of the sequences q_n, intersect $W^u_\epsilon(J)$ and $q \notin W^u_\epsilon(J)$. Taking $y = W^{E_1}_\epsilon(q) \cap W^u_\epsilon(J)$ follows that the backward orbit of the point y remains in U and so the point belongs to the homoclinic class and then we conclude that there is a periodic point such that its strong stable manifold intersects the homoclinic class.

So, to conclude the proof it is enough to conclude that the second case cannot occur. If the periodic point q is contained in J, using the fact that q is periodic and J does not increase the size for negative iterates, we conclude that there is a point of different stable index in J which is an absurd. In fact, if q is contained in J observe that $J \subset W^{E_2}_\epsilon(q)$. Let r be the period of q, so $f^{-rk}(J) \subset W^{E_2}_\epsilon(q)$ for any positive k, and so taking $L = \cup_{k>0} f^{-rk}(J)$ follows that $L \subset W^{E_2}_\epsilon(q)$. Moreover, L is invariant by f^{-r} and $f^{-r}_{/L} : L \to L$ is an homeomorphism where q is a repelling fixed point. Taking $y' \in W^s(q) \cap L$ we get that there is $q' = \lim_{k\to\infty} f^{-kr}(y') \in L$ and y is an attracting fixed point for f^{-r}, i.e.: q' is a repelling periodic point for f. Which is an absurd.

In the case that the periodic point q is not contained in J but contained in $W^u_\epsilon(J)$, we get that the unstable manifold of q intersects J. Using again that J does not increase the size by negative iteration, we conclude that there is an arc I contained in the center manifold of q such that does not increase the size by negative iterations, and again this implies that there is a periodic point of different stable index in U which is an absurd.

To finish, we prove that either

1. $\ell(f^n(W^{E_2}_\epsilon(x))) \to 0$ as $n \to +\infty$ or
2. there exists a periodic point q having weak contraction along the center direction and such that $[W^{E_1}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$.

The argument to prove it, is similar to the one already performed and we leave it to the reader.

■

Proof of proposition 4.4: To prove it, we start with the following lemma:

Lemma 5.0.1 *For any $\delta > 0$ small, there is $n_0 = n_0(\delta)$ such that for any $n \geq n_0$ holds*

$$|Df^n_{|E_2}| < (1+\delta)^n$$

Observe that this lemma implies the proposition 4.4: In fact, since $\frac{|Df_{|E_1}|}{|Df_{|E_2}|} < \lambda$ then $|Df^n_{|E_1}| < (\lambda(1+\delta))^n$ so if δ is small enough follows that $\lambda(1+\delta) < 1$.

To proceed with the proof of the lemma, we have to state a lemma due to Pliss:

Pliss's Lemma: Given $0 < \gamma_0 < \gamma_1$ and $a > 0$, there exist $N_1 = N_1(\gamma_0, \gamma_1, a)$ and $l = l(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $\{a_i\}_{0 \leq i \leq n}$ with $n > N_1$, $a^{-1} < a_i < a$ and $\Pi^n_{i=0} a_i < \gamma^n$ then there exist n_0 with $n_0 < ln$ such that

$$\Pi^j_{i=n_0} a_i < \gamma_1^{j-n_0} \quad n_0 < j < n$$

So, if the lemma is not true, we get that there is a sequences of points x_n and an increasing sequences of positive integers k_n such that $|Df^{k_n}_{|E_2(x_n)}| > (1+\delta)^{k_n}$, i.e.: $|Df^{-k_n}_{|E_2(f^{k_n}(x_n))}| < (1+\delta)^{-k_n}$. Using Pliss's lemma holds that there exist points y_n, and integer n_0 and an increasing sequences of positive integers j_n such that $|Df^{-j}_{|E_2(y_n)}| < (1+\frac{\delta}{2})^{-j}$ for $n_0 < j < j_n$. Taking an accumulation point x of the sequences y_n follows that

$$|Df^{-n}_{|E_2(x)}| < (1+\frac{\delta}{2})^{-n} \ \forall \ n > n_0$$

Then, the center manifold along x is stable for f^{-1}, which is a contradiction. In fact, to see that it is proved a folklore claim that we repeat here for completeness. The claim states the following

Claim 1 *Let $g \in Diff^r(M)$ having a dominated splitting $T_\Lambda M = \oplus^k_{i=1} E_i$ on a compact invariant set Λ follows that if for some direction $E = E_i$ and some $x \in \Lambda$ holds that there exists $\delta > 0$ and n_0 such that*

$$\Pi^{n-1}_{i=0} |Dg_{|E(g^i(x))}| < (1-\delta)^n \ \ \forall n > n_0$$

then there exists δ_0 such that

1. $g^n(W^E_{\delta_0}(x)) \subset W^E_{\delta_0}(g^n(x))$ *for any* $n > n_0$,

2. $\ell(g^n(W^E_{\delta_0}(x))) \to 0$ *as* $n \to +\infty$.

To get that observe that given $\delta_2 > 0$ there exists $\delta_3 > 0$ such that for any $y \in \Lambda$ and $z \in W^E_{\delta_3}(y)$ follows that

$$\frac{|Dg_{|\tilde{E}(z)}|}{|Dg_{|E(y)}|} < 1 + \delta_2 \text{ where } \tilde{E}(z) = T_z W^E_{\delta_3}(y)$$

Then, it is taken δ_2 such that $(1-\delta)(1+\delta_2) < \gamma < 1$ for some $\gamma < 1$. Then we can take $\delta_0 > 0$ such that $\delta_0 < \delta_3$ and

$$\ell(g^k(W^E_{\delta_0}(x))) \subset W^E_{\delta_3}(g^k(x)) \; 1 \leq k \leq n_0$$

¿From that, follows if $z \in W^E_{\delta_0}(x)$ then

$$\frac{|Dg^{n_0}_{|\tilde{E}(z)}|}{|Dg^{n_0}_{|E(y)}|} < (1+\delta_2)^{n_0}$$

and so

$$\Pi_{i=0}^{n_0-1}|Dg_{|\tilde{E}(g^i(z))}| < |Dg_{|E(g^i(y))}|(1+\delta_2)^{n_0} < ((1-\delta)(1+\delta_2))^{n_0} < \gamma^{n_0}$$

and so

$$g^{n_0}(W^E_{\delta_0}(x)) \subset W^E_{\delta_0}(g^{n_0}(x))|$$

Making and induction argument, the claim follows.

∎

Coming back to the proof of the lemma, we can apply the previous claim to the direction E_2 because E_2 is one dimensional and so $|Df^n_{|E_2}| = \Pi_{i=0}^{n_1}|Df_{|E_2(f^i(x))}$.
∎

6 Case B.2.1. Proof of Theorem 4.3

In this case, we use a theorem proved in [BC] that allows us to reduce the problem to a problem for surfaces diffeomorphisms. First we start recalling the definition of normally hyperbolic submanifold. We say that an invariant manifold S is normally hyperbolic there is an splitting $T_S M = E \oplus F$ such that

1. E^s is uniformly contractive (or expansive);
2. $|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}| < \lambda < 1$ for any $x \in S$
3. $T_x S = F$ for any $x \in S$.

If it holds that $f \in Diff^r(M)$ and

$$|Df_{|E^s(x)}||Df^{-1}_{|F(f(x))}|^r < \lambda < 1$$

it is said that S is $r-$normally hyperbolic and follows that S is C^r.

Theorem 6.1 *([BC]) Let $f \in Diff^r(M)$ $(r \geq 1)$ be a diffeomorphism on a compact manifold M. Let Λ be a compact invariant set exhibiting a dominated splitting $T_\Lambda = E^s \oplus F$ where E^s is uniformly contractive. Let also assume that for every $x \in \Lambda$ holds that $W^{ss}_\epsilon(x) \cap \Lambda = \{x\}$ (where $W^{ss}_\epsilon(x)$ is the local strong stable manifold tangent to E^s). Then, there exist two C^1-submanifold normally hyperbolic S and $\hat{S}$ such that,*

1. *$T_x S = F(x)$,*
2. *$S \subset \hat{S}$,*
3. *$\Lambda \subset S$, $f(S) \subset \hat{S}$ and $f^{-1}(S) \subset \hat{S}$.*

Applying the previous theorem to the homoclinic class H_p follows the next corollary:

Corollary 6.1 *Let H_p be a topological hyperbolic homoclinic class such that $\mathcal{T} = \emptyset$. Then there is a C^1-submanifold S containing H_p and such that $f_{|S}$ is a surface map exhibiting a dominated splitting.*

Even f is C^2, the manifold obtained by 6.1 it could be only C^1. In fact, if there is a periodic point q in H_p with stable eigenvalues λ_1 and λ_2 such that $\lambda_1 < \lambda_2$ but $\lambda_2^2 < \lambda_1$ follows that S can not be $2-$normally hyperbolic.

For surfaces maps exhibiting dominated splitting it is possible to obtain a well description of the limit set:

Theorem 6.2 *([PS1]) Let $g \in Diff^2(M^2)$ and assume that $\Lambda \subset \Omega(g)$ is a compact invariant set exhibiting a dominated splitting such that any periodic point is a hyperbolic saddle periodic point. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

Due to the fact that S is C^1, the restriction of f to the submanifold S is only C^1 (even f is C^2). So, the two dimensional result stated above cannot be directly applied. However, we have some extra properties associated to f: *the manifold tangent to E_2 and E_3 are dynamically defined, being stable and unstable respectively.* So, we are in a situation that we have more information for the two dimensional system. Later, using an extra property that holds along the stable and unstable manifold and called bounded distortion, we can use a generalization of the theorem 6.2 for C^1- diffeomorphism.

To be more precise, we have to introduce some definitions for two dimensional diffeomorphisms.

Let S be a 2-dimensional manifold and $g \in Diff^1(S)$. Let us assume that g has an invariant set Λ exhibiting a two dimensional dominated splitting $E \oplus F$. Recall that for each direction and for every point $x \in \Lambda$ we have associated the tangent manifolds $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$.

Definition 13 *We say that $W^F_\epsilon(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W^F_\epsilon(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^{-i}(J)) \leq \delta$ for $0 \leq i \leq n$ then*

1. $\frac{|Df^{-n}_{/\tilde{F}(y)}|}{|Df^{-n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$

2. $|Df^{-n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$ $\quad \tilde{F}(y) = T_y W^F_\epsilon(x)$

We say that $W^E_\epsilon(x)$ has bounded distortion property if there exists K_0 and $\delta > 0$ such that for all $x \in \Lambda$ and $J \subset W^E_\epsilon(x)$ we have for all $z, y \in J$ and $n \geq 0$, if $\ell(f^i(J)) \leq \delta$ for $0 \leq i \leq n$ then

1. $\frac{|Df^{n}_{/\tilde{F}(y)}|}{|Df^{n}_{/\tilde{F}(z)}|} \leq \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J)))$

2. $|Df^{n}_{/\tilde{F}(x)}| \leq \frac{\ell(f^{n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{i}(J)))$ $\quad \tilde{F}(y) = T_y W^E_\epsilon(x)$

With this definition in mind, it is possible to get the following result which is a generalization of the theorem 6.2:

Theorem 6.3 *([PS1]) Let $g \in Diff^1(M^2)$ and assume that $\Lambda \subset \Omega(g)$ is a compact invariant set exhibiting a dominated splitting $E \oplus F$ such that any periodic point is a hyperbolic saddle periodic point. Moreover, assume that $W^E_\epsilon(x)$ and $W^F_\epsilon(x)$ has bounded distortion. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ...\mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

The proof is similar to the one done for the theorem 6.2. In fact, in the proof it is used that f is C^2 to show that the center manifolds are C^2 (see lemma 3.0.3 in [PS1]) and as a consequences of it it is proved that the center manifolds have bounded distortion property (see lemma 3.5.1 in [PS1]). In the theorem 6.3, the distortion property are taken for grant. For details we refer to [PS1].

Therefore, to apply theorem 6.3 to the map $f_{/S}$ where S is the submanifold given by proposition 6.1, it is necessary to show that along the local unstable manifold and the local center-stable hold the bounded distortion property. Actually, in the case of the unstable arcs (which are contained in the attractor and so they are contained in S), it is proved that they are C^2. On the other hand, the center manifold are not unique so it could happen that the one chosen are not contained in S. However, if we take the manifold defined as $W^{cs}_\epsilon(x) \cap S$ follows that this manifolds are invariant by f, $T_x(W^{cs}_\epsilon(x) \cap S) = E_2(x)$ and they are stable.

Proposition 6.1 *Let $f \in Diff^2(M^3)$ and let H_p be a topologically hyperbolic homoclinic class. Let us assume that there exists a two dimensional C^1-normally submanifold S such $H_p \subset S$. Then, the tangent manifolds $W^u_\epsilon(x)$ and $W^{cs}_\epsilon(x) \cap S$ have bounded distortion property.*

First, we start proving that for f, the stable discs and the unstable manifold are C^2. At this point, it is used that the manifold are dynamically defined. For that, we need the following lemma:

Lemma 6.0.2 *There exist a constant $C > 0$ and $0 < \sigma < 1$ such that for every $x \in \Lambda$ and for all positive integer n the following holds:*

$$\frac{|Df^n_{|E_1 \oplus E_2}|}{|Df^n_{|E_3}|^2} = \frac{|Df^n_{|E_2}|}{|Df^n_{|E_3}|^2} < C\sigma^n$$

$$\frac{|Df^n_{|E_1 \oplus E_2}|^2}{|Df^n_{|E_3}|} = \frac{|Df^n_{|E_2}|^2}{|Df^n_{|E_3}|} < C\sigma^n$$

Proof:

Recall that the manifold tangent to E_3 is a unstable manifold. ¿From this, we claim that for any $\delta > 0$ there is $n_0 = n_0(\delta)$ such that for any $n \geq n_0$ holds

$$|Df^n_{|E_3}| > (1-\delta)^n$$

So, if the claim is not true, we get that there is a sequences of points x_n and an increasing sequences of positive integers k_n such that $|Df^{k_n}_{|E_3(x_n)}| < (1-\delta)^{k_n}$. Using Pliss's lemma holds that there exist points y_n, and integer n_0 and an increasing sequences of positive integers j_n such that $|Df^{j}_{|E_3(y_n)}| < (1-\frac{\delta}{2})^j$ for $n_0 < j < j_n$. Taking an accumulation point x of the sequences y_n follows that $|Df^n_{|E_3}| < (1-\frac{\delta}{2})^n$ for $n > n_0$. Then, using the observation 1, follows that the manifold $W^u_\epsilon(x)$ is a stable manifold for f, which is a contradiction.

Then,

$$\frac{|Df^n_{|E_2}|}{|Df^n_{|E_3}|^2} = \frac{|Df^n_{|E_2}|}{|Df^n_{|E_3}|}\frac{1}{|Df^n_{|E_3}|} < \lambda^n \frac{1}{(1-\delta)^n} = (\frac{\lambda}{1-\lambda})^n$$

for $n > n_0$; so if δ is small enough follows the desired property.

For the second inequality, we repeat a similar argument using that the manifold tangent to E_2 is a stable manifold and arguing as in lemma 5.0.1.

■

Now, we can apply a result in [HPS] that establish that if the inequality stated in the previous proposition, then the manifold tangent to $E_1 \oplus E_2$ and to E_3 are C^2.

As a consequences of the previous lemma, follows that along the unstable manifold the bounded distortion property holds. To get the bounded distortion property along the center-stable we have to be more careful. Observe that even the map is C^2, the central leaves could be only C^1 inside the stable discs. However, we can show that they have distortion property:

Lemma 6.0.3 *There exists a constant K such that if $y \in W^c_\epsilon(x) = W^{cs}_\epsilon(x) \cap S$ follows that*

$$\frac{|Df^n_{E_2(x)}|}{|Df^n_{E_2(y)}|} \leq \exp(K \sum_{i=0}^{n-1} |f^i(x) - f^i(y)|)$$

Proof:

We want to control

$$\frac{|Df^n_{E_2(x)}|}{|Df^n_{E_2(y)}|}$$

where $x, y \in J$.

Observe that we can assume that $|Df_{|E}| = |Df_{|E_2}|$ where $E = E_1 \oplus E_2$. So

$$\frac{|Df^n_{E_2(x)}|}{|Df^n_{E_2(y)}|} = \frac{|Df^n_{E(x)}|}{|Df^n_{E(y)}|}$$

Moreover

$$|Df^n_{E(x)}| = |Df^n_{E_2(x)}| = \Pi^n_{i=0}|Df_{E_2(f^i(x))}| = \Pi^n_{i=0}|Df_{E(f^i(x))}|$$

For each x we have defined the map $y \in W^s_\epsilon(x) \to log|Df_{|E(y)}|$ and recalling that the discs $W^s_\epsilon(x)$ are C^2 follows that the maps $y \in log|Df_{|E(y)}|$ are Lipschitz. Since Λ is compact follows that there is a constant K independent of the discs such that

$$|log(|Df_{|E(x)}|) - log(|Df_{|E(y)}|)| < K|x-y| \;\; \forall\, y \in W^{cs}_\epsilon(x)$$

where K is the Lipschitz constant for $log(|Df_{|E}|)$.

So

$$log(\frac{|Df^n_{E(x)}|}{|Df^n_{E(y)}|}) = \sum log(|Df_{E(f^i(x))}|) - log(|Df_{E(f^i(x))}|) < K \sum |f^i(x) - f^i(y)|$$

Then we get that

$$\frac{|Df^n_{E_2(x)}|}{|Df^n_{E_2(y)}|} \leq \exp(K \sum_{i=0}^{n-1} |f^i(x) - f^i(y)|)$$

■

To finish showing that the manifold has the bounded distortion property, we have to show that they verifies the two second items. This is immediately since the submanifold are C^1.

7 Case B.2.2. Proof of Theorem 4.4.

In this section we assume that

$$\mathcal{T} = \{x \in H_p : [W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset\} \neq \emptyset$$

Observe that the set $\mathcal{T}$ is not necessary either open or close.

The proof of the theorem 4.4 goes considering different situations related to the set $\mathcal{T}$. More precisely, we consider if *the interior of $\mathcal{T}$ in the homoclinic class is or not empty*, where the topology is the restricted topology to H_p.

In the case that the interior of $\mathcal{T}$ is not empty, from the fact that the periodic points with weak contraction are dense, follows immediately that there exists a periodic point q with weak contraction along the center direction such that $[W^{ss}_\epsilon(q) \setminus \{q\}] \cap H_p \neq \emptyset$. Then, applying the C^1-connecting lemma, we conclude the theorem 4.4 in this case. In other words, we have proven the followinf proposition:

Proposition 7.1 *Let H_p be an attracting topological hyperbolic homoclinic class. If the interior of $\mathcal{T}$ is not empty then the thesis of the theorem 4.4 follows.*

If the interior of $\mathcal{T}$ is empty we show that:
there is a C^1 suitable perturbations of f, exhibiting a pair of periodic points q_1 and q_2 homoclinically related and such that $W^u(q_2) \cap W^{ss}_\epsilon(q_1) \neq \emptyset$.
After that, we can produce a C^k perturbation to get a periodic point with weak contraction along the center direction and also exhibiting an strong homoclinic connection. More precisely, we get the following proposition:

Proposition 7.2 *Let $g \in Diff^r(M^3)$ and $\delta > 0$ such that*

1. *g has two hyperbolic periodic points q_1 and q_2 such that*
   - *(a) q_1 and q_2 are homoclinically connected,*
   - *(b) $W^u(q_2) \cap W^{ss}_\epsilon(q_1) \neq \emptyset$;*
2. *there exists a periodic points q_δ with $\frac{\delta}{2}-$weak contraction along the center direction and homoclinically related with q_1.*

Then, there is $\hat{g}$ arbitrarily C^k-close to g and a periodic point $\hat{q}_\delta$ with $\delta-$weak contraction along the center direction and exhibiting a strong homoclinic connection.

Observe that the previous proposition implies the theorem 4.4. So the goal is to show that if the interior of $\mathcal{T}$ is empty, for any $\delta > 0$ we can get by perturbation a diffeomorphisms g C^1-arbitrarily close to f verifying the hypothesis of proposition 7.2.

Before to do that, let us show the proof of proposition 7.2.

Proof of propositions 7.2:

By lemma 4.2.1 we get a sequences of periodic points q_δ^n such that q_δ^n accumulates on q_1, they are homoclinically connected with q_1 and they have $\delta-$weak

contraction along the center direction. Moreover, we can suppose that the orbits of this point does not accumulate in q_2 and so they do not accumulate over the point of intersection between the unstable manifold of q_2 and the strong stable manifold of q_1. Observe that the strong stable manifold of the points q_δ^n accumulates over the local strong stable manifold of q_1. Since the points q_δ^n are homoclinically connected with q_1 and so with q_2, follows that their unstable manifolds accumulate over the connected arc of the unstable manifold of q_2 that contains q_2 and a point $z \in W^u(q_2) \cap W^{ss}_\epsilon(q_1)$. Then, it is possible to unfold the intersection of the unstable manifold of q_2 with the strong stable manifold of q_1 in a way that the unstable manifold of some q_δ^n intersect the local strong stable manifold of the same q_δ^n.

More precisely, we can do that in two steps: First, it is performed an arbitrarily small perturbation such that the unstable manifold of q_2 intersect the strong stable manifold of same q_δ^n sufficiently close to q_1. Since q_δ^n remains homoclinically connected with q_2, follows that they are arc contained in the unstable manifold of q_δ^n that accumulates over the connected arc of the unstable manifold of q_2 that contains q_2 and a point $z \in W^u(q_2) \cap W^{ss}_\epsilon(q_1)$. The second perturbation consist in unfolding the intersection of the unstable manifold of q_2 with the strong stable manifold of q_δ^n in a way that the unstable manifold of q_δ^n intersect the local strong stable manifold of the same point.

The first perturbation is straightforward from the fact that the orbits of the points q_δ^n do not accumulate over q_2 For the second one, we take a sequences of compacts arcs $\{l_m\}_m$ contained in the unstable manifold of q_δ^n such that:

1. the arcs $\{l_m\}$ accumulates in a compact arc l which is contained in the unstable manifold of q_2 and it contains the point q_2 and a point in $W^u(q_2) \cap W^{ss}_\epsilon(q_\delta^n)$;
2. for each m follows that $\{f^{-i}(l_m)\}_{\{i>0\}}$ does not accumulate on l.

So, perturbing g in a way to unfold the intersection of l with $W^{ss}_\epsilon(q_\delta^n)$ and at the same time not perturbing q_δ^n, follows that the unstable manifold of q_δ^n intersects $W^{ss}_\epsilon(q_\delta^n)$ and this conclude the proof of the proposition 7.2.

■

To get the pair of periodic points in the hypothesis of the proposition 7.2, we have to consider two alternatives: In one alternative, we are able to show that there exist two periodic points q_1 and q_2 such that there are x and y in the unstable manifold of q_1 and q_2 respectively and such that also holds that $y \in W^{ss}_\epsilon(x)$ (see proposition 7.3). Then, by a C^kperturbation ($k \geq 1$ see proposition 7.4), which consist essentially to move the unstable manifold of one of the periodic point, we obtain a pair of points in the hypothesis of proposition 7.2. In the second case, we have two points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$ but we cannot guarantee that the points x and y belongs to the unstable manifold of some periodic points. Therefore, the perturbation considered before it does not work in the present situation. However, we can get some extra properties in the system that allows us to introduce another kind of perturbation that get the pair of periodic points verifying the hypothesis of the proposition 7.2 (see proposition 7.5).

The mentioned alternative depends on the "joint integrability of the stable and unstable manifolds".

Let us consider the pair of points x, y such that $y \in W^{ss}_\epsilon(x)$. Let us take

$$W^{cu}_\epsilon(x) = \cup_{\{z \in W^c_\epsilon(x)\}} W^u_\epsilon(z)$$

and observe that $W^u_\epsilon(x)$ splits $W^{cu}_\epsilon(x)$ in two connected components. Let us consider

$$\Pi^{ss} : B(x) \to W^{cu}_\epsilon(x)$$

where $B(x)$ is a neighborhood of x contained in U and such that contains y and the local strong unstable manifold of both points. With Π^{ss} we denote the projection induces by the strong stable foliation. To say that there is pair of points x, y such that $y \in W^{ss}_\epsilon(x)$ is equivalent to assert that

$$\Pi^{ss}(W^u(y)) \cap W^u(x) \neq \emptyset$$

To avoid notation, in some cases, we denote the set $\Pi^{ss}(W^u_\epsilon(x)) \cap W^u_\epsilon(y)$ with $W^u_\epsilon(x) \cap_s W^u_\epsilon(y)$ and if $\Pi^{ss}(W^u_\epsilon(x)) \cap W^u_\epsilon(y) \neq \emptyset$ we say that $W^u_\epsilon(x)$ s-intersects $W^u_\epsilon(y)$.

The strategy of the proof of theorem 4.4, splits in different parts related to the kind of intersection of $\Pi^{ss}(W^u_\epsilon(x))$ with $W^u_\epsilon(y)$.
As we already said, it depends on the "joint integrability of the strong manifolds".

Definition 14 Joint integrability: *We say that strong stable foliation and the strong unstable foliation are jointly integrable if there exist $0 < \epsilon_1 < \epsilon_2$ such that for any x and y in the homoclinic class with $y \in W^{ss}_\epsilon(x)$ holds*

$$\forall\, z \in W^u_\epsilon(x) \text{ then } W^{ss}_\epsilon(z) \cap W^u_\epsilon(y) \neq \emptyset$$

in other words, for all $x, y \in \Lambda$ such that $y \in W^{ss}_{\epsilon}(x)$ follows that

$$\Pi^{ss}(W^u_{\epsilon_1}(y)) \subset W^u_{\epsilon_2}(x).$$

Without loss of generality, we can assume that $\epsilon = \epsilon_1 = \epsilon_2$ and

$$\Pi^{ss}(W^u_{\epsilon}(y)) = W^u_{\epsilon}(x).$$

Another equivalent way to formulate the previous definition, is to consider the following two sets

$$W^{su}_{\epsilon}(x) = \cup_{\{z\in W^u_{\epsilon}(x)\}}[W^{ss}_{\epsilon}(z) \cap \Lambda] \quad W^{us}_{\epsilon}(x) = \cup_{\{z\in W^{ss}_{\epsilon}(x)\cap\Lambda\}} W^u_{\epsilon}(z)$$

To assert that the strong foliation are jointly integrable is equivalent to assert that for any x, y such that $y \in W^{ss}_{\epsilon}(x)$ holds that

$$W^{su}_{\epsilon}(x) = W^{us}_{\epsilon}(x)$$

We consider independently *the case that the strong foliation are jointly integrable and the case that this does not happen.* In the case that the strong foliation are not jointly integrable, we can conclude there are a pair of periodic points p_x, p_y such that the local unstable manifold of p_x s-intersect the local unstable manifold of p_y. Later, performing a suitable perturbation it is concluded the existence of a new diffeomorphisms verifying the hypothesis of proposition 7.2. In the case that the strong foliation are jointly integrable, it is necessary to perform another perturbation different that the one done in the previous case. The goal of the next subsection are devoted to consider both situations.

7.1 The strong foliations are not jointly integrable.

In the case that the strong foliation are not jointly integrable we get the following proposition (the proof is given in subsection 7.3):

Proposition 7.3 *Let H_p be a topologically hyperbolic attracting homoclinic class such that $\mathcal{T} \neq \emptyset$ and the strong foliations are not jointly integrable. Then, there are a pair of points x, y in the homoclinic class and a pair of periodic points p_x, p_y also in the homoclinic class such that*

1. $y \in [W^{ss}_{\epsilon}(x) \setminus \{x\}]$,
2. $x \in W^u(p_x)$ *and*
3. $y \in W^u(p_y)$.

We do not know if the hypothesis of proposition 7.3 imply that there are two periodic points in the hypothesis of proposition 7.2. However, if the thesis of proposition 7.3 it is possible to get a diffeomorphisms g C^k-close to f, that verifies the hypothesis of proposition 7.2 for a perturbation

of the initial map. This is the purpose of the following proposition. Observe that the perturbation is C^k for any $k < \infty$. It remains the question if the previous proposition is true assuming that the strong foliations are jointly integrable.

To conclude the proof of theorem 4.4 in the present situation we get the next proposition (the proof is given in subsection 7.4):

Proposition 7.4 *Let $f \in Difff^r(M^3)$. Let H_p be a non-hyperbolic topologically hyperbolic attracting homoclinic class verifying the thesis of proposition 7.3. Then, for any $\delta > 0$ there exists a diffeomorphisms g arbitrarily C^k-close to f and periodic points q_δ, q_1, q_2 of g verifying the hypothesis of proposition 7.2.*

Observe that in the two previous proposition was not assumed that the interior of $\mathcal{T}$ is empty. Moreover, it is not difficult to show also that the proposition can be generalized to higher dimension assuming that the center direction is one dimensional.

This finished the proof of theorem 4.4 in the case that the strong foliations are not jointly integrable.

7.2 The strong foliations are jointly integrable.

Now we have to address the case that the strong foliations are jointly integrable. It is not clear if under the hypothesis of joint integrability it is possible to get two points as in the proposition 7.3. Moreover, we do not know if in the present case it could occur that $\mathcal{T} \cap Per(f) = \emptyset$. If it was the case, the present case would be treated similarly as the the case that the interior of $\mathcal{T}$ is not empty.

However, it is possible to perform a C^1-perturbation to get two periodic points as in the proposition 7.2. To perform such perturbation, it is necessary the following theorem that state if the interior of $\mathcal{T}$ is empty, then there is a subset Λ containing a pair of points x, y such that $y \in W^{ss}_\epsilon(x)$ and E_3 is uniformly expansive on Λ. Observe that in this theorem it is not assumed that the strong foliations are jointly integrable (the proof is given in section 8).

Theorem 7.1 *Let H_p be a topologically hyperbolic attracting homoclinic class such that $\mathcal{T} \neq \emptyset$ and the interior of $\mathcal{T}$ is empty. Then, there is a compact transitive invariant subset Λ such that*

1. *there is a pair of points $x, y \in \Lambda$ such that $y \in W^{ss}_\epsilon(x)$,*

2. *E_3 is uniformly expansive in Λ.*

Corollary 7.1 *For any g C^1-close to f and any compact set Λ_g close to Λ follows that the direction $E_3(g)$ is uniformly hyperbolic.*

Observe that in section 6 was proved that if $\mathcal{T} = \emptyset$ (i.e.: for any $x \in H_p$ follows that $[W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset$) then H_p is hyperbolic. In the context of theorem 7.1 it is only assumed that the interior of $\mathcal{T}$ is empty. Even with this weak hypothesis we can manage to guarantee that E_3 is hyperbolic in same set Λ which also has the property that $\mathcal{T}_\Lambda \neq \emptyset$, where

$$\mathcal{T}_\Lambda = \{x \in \Lambda : [W^{ss}_\epsilon(x) \setminus \{x\}] \cap \Lambda \neq \emptyset\}.$$

The previous theorem is necessary to control the perturbation done in the next proposition (the proof is given in section 7.5).

Proposition 7.5 *Let H_p be a topologically hyperbolic attracting homoclinic class such that the strong foliations are jointly integrable and the interior of $\mathcal{T}$ is empty. Then, for any $\delta > 0$ there exists a diffeomorphisms g arbitrarily C^1-close to f and periodic points q_δ, q_1, q_2 of g verifying the hypothesis of proposition 7.2.*

We want to remark, that the previous proposition is also true if it is not assumed that the interior of $\mathcal{T}$ is empty but assuming that there is a compact invariant subset Λ with $\mathcal{T}_\Lambda \neq \emptyset$ and verifying that E_3 is uniformly expansive in Λ. Actually, it is used that the interior of $\mathcal{T}$ is empty to use the theorem 7.1 which it guarantees the existence of a set Λ with the above property mentioned.

Again we conclude the theorem 4.4 when the strong foliation are jointly integrable.

7.3 Proof of proposition 7.3.

To prove the proposition, it is equivalent to show that they are a pair of periodic points p_1, p_2 such that their local unstable manifold s-intersect each other.

Definition 15 *We say that $\Pi^{ss}(W^u_\epsilon(x))$* **intersect transversally** *$W^u_\epsilon(y)$ if $\Pi^{ss}(W^u_\epsilon(x))$ intersect both components of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$.*

To prove the proposition 7.3, basically it is done the following: If there is a pair of points x, y such that $y \in W^{ss}_\epsilon(x)$ and the intersection of their unstable manifolds is s-transversal, then it follows from the continuity of the local unstable manifolds that there are two local unstable manifold of two periodic points with the property that they s-intersect each other (see item 1 in the following). If the intersection is not transversal, it can be introduced the notion of boundary point (see item 2). Using the fact that the interior of $\mathcal{T}$ is empty the it is shown the existence of these boundary points (see lemma 7.3.1). Related to this notion it is proved that either the boundary point are contained in the unstable manifold of some periodic points or again we can find two periodic points such that their local unstable manifolds intersect each other (see item 2.1). Assuming that the boundary points belong to the unstable manifold of some periodic points, it is repeated a similar argument as the one done when $W^u_\epsilon(x)$ s-intersect transversally $W^u_\epsilon(y)$ to show that they are two periodic points such that their local unstable manifolds s-intersect each other (see item 2.2. for this last part).

1. *There exists x, y such that $\Pi^{ss}(W^u_\epsilon(x))$ intersect transversally $W^u_\epsilon(y)$.*

Observe that the unstable manifold of the periodic points accumulate over $W^u_\epsilon(x)$ and $W^u_\epsilon(y)$ and since $W^u_\epsilon(x)$ and $W^u_\epsilon(y)$ s-intersects transversally, follows that there are two periodic points such that their unstable manifold s-intersect transversally. In fact, let us consider the map Π^{ss} defined from a neighborhood of x to the center-unstable manifold of x. Let us a take a periodic point p_x close to x and a periodic point p_y close to y. So, the local unstable manifold of p_x and p_y are closed to the local unstable manifold of x and y respectively. So, follows that $\Pi^{ss}(W^u_\epsilon(p_x))$ and $\Pi^{ss}(W^u_\epsilon(p_y))$ are closed to $W^u_\epsilon(x)$ and $\Pi^{ss}(W^u_\epsilon(y))$ respectively and therefore, $\Pi^{ss}(W^u_\epsilon(p_x))$ and $\Pi^{ss}(W^u_\epsilon(p_y))$ intersects transversally (see figure 6).

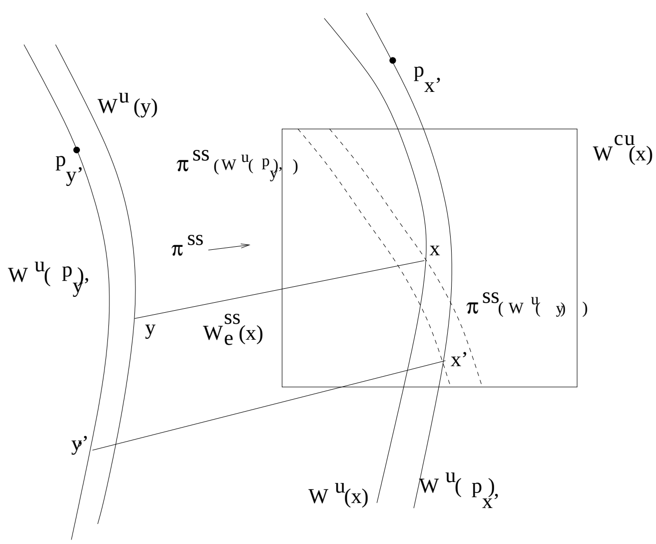


Figure 6

2. *$\forall\, x, y \in H_p$, $\Pi^{ss}(W^u_\epsilon(x))$ does not intersect transversally $W^u_\epsilon(y)$.*

Now, let us assume that for any pair of points x, y in the homoclinic class such that $y \in W^{ss}_\epsilon(x)$ then $\Pi^{ss}(W^u_\epsilon(x))$ and $W^u_\epsilon(y)$ do not intersect transversally.

Observe that in the present situation, given $x, y \in H_p$ with $y \in W^{ss}_\epsilon(x)$ if we take

$$\Pi^{ss} : B(x) \to W^{cu}_\epsilon(x)$$

follows that $\Pi^{ss}_\epsilon(W^u_\epsilon(y))$ is contained in the closure of one of the connected components of $W^{cu}_\epsilon(x) \setminus W^u_e(x)$.

Observe that given a point in the homoclinic class, the local strong stable manifold of it, splits the local stable manifold in two disjoint sets; i.e.: $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$ has two disjoint connected components. Using this, we introduce the following definition:

Definition 16 Stable boundary point: *We say that a point x is a stable boundary point, if there are point of the homoclinic class contained in the local stable manifold of x and accumulating on x from only one connected component of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$.*

Now we prove that under the assumption of nonintegrability of the strong foliation and that there are not transversal intersection then there are stable boundary points.

Lemma 7.3.1 *If for any pair of points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$ follows that $\Pi^{ss}(W^u_\epsilon(x))$ does not intersect transversally $W^u_\epsilon(y)$ and the strong foliations are not jointly integrable, then there exist stable boundary points.*

Proof: Let us assume that there are not stable boundary points. If there are not stable boundary points we have two situations to consider: either for every $x \in H_p$ follows that $[W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)] \cap H_p = \emptyset$ or for every $x \in H_p$, the homoclinic class intersect both components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$.

In the first case, follows that the the strong foliations are jointly integrable. In fact, recall that there are not isolated point in the homoclinic class. So, given x, and a point x' close to x and contained in the stable manifold of x, follows that it is contained in the strong stable manifold of x. Now we take the local unstable manifold of x and the local unstable manifold of x'. It follows that for any x'' in the local unstable manifold of x holds that the strong stable manifold of x'' has to intersect the local unstable manifold of x': if it is not the case, it would follows that the local unstable manifold of x' would intersect the stable manifold of x'' out of the strong stable manifold of x'', which is a contradiction.

To finish the proof, it is enough to show that if for every $x \in H_p$, the homoclinic class intersect both components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$ and the strong foliation are not jointly integrable then we can find a pair of points such that their local unstable manifold s-intersect transversally. Which is a contradiction with the hypothesis of the lemma.

To prove that, let us consider a pair of point x and y such that they belong to the same strong stable manifold. Moreover, since that we are assuming that the strong foliations are not jointly integrable, we can suppose that $\Pi^{ss}(W^u_\epsilon(x))$ does not coincide with $W^u_\epsilon(y)$. Then $\Pi^{ss}(W^u_\epsilon(x))$ is contained in the closure of one of the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ and there is a point x' in $\Pi^{ss}(W^u_\epsilon(x))$ which is properly contained in $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$. This is equivalent to say that, there is a point $y' \in W^u_\epsilon(y)$ such that is properly contained in $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$; i.e. $dist(y', \Pi^{ss}(W^u_\epsilon(x))) > r_0 > 0$. Since y is not a boundary points, we can take a point z close to y contained in $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$ such that $\Pi^{ss}(z)$ is contained in the same connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that contains x'. Moreover, follows that $\Pi^{ss}(z)$ is contained in the connected component of $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$ that does not contain y'. Therefore, since there are not transversal intersections, follows that $\Pi^{ss}(W^u_\epsilon(z))$ is contained in the closure of the connected component of $W^{cu}_\epsilon(y) \setminus \Pi^{ss}(W^u_\epsilon(x))$ that does not contain y' and so $dist(y', \Pi^{ss}(W^u_\epsilon(z))) > r_0 > 0$. However, if z is close enough to y follows that $\Pi^{ss}(W^u_\epsilon(z))$ is close to $W^u_\epsilon(y)$ and in particular $\Pi^{ss}(W^u_\epsilon(z))$ is arbitrarily close to y' which is a contradiction.

■

Now, we have consider two cases: either there is a boundary point which is not contained in the unstable manifold of any periodic point or any boundary points is contained in the unstable manifold of some periodic point.

2.1 There is a boundary point which is not contained in the unstable manifold of any periodic point.

Let x be a boundary point not contained in the unstable manifold of any periodic point. Let us take the sequence $\{f^{-n}(x)\}_{n>0}$ and take n_1, n_2, n_3 arbitrarily large such that the points $f^{-n_1}(x), f^{-n_2}(x), f^{-n_3}(x)$ are close enough one to each other. Observe that $f^{-n_i}(x) \notin W^u_\epsilon(f^{-n_j}(x))$ for $i \neq j$ $i, j = 1, 2, 3$. If it is not the case, follows that $f^{-n_i}(x)$ is contained in the local unstable manifold of a periodic point.

Lemma 7.3.2 *The local unstable manifold of at least two of the three points $f^{-n_1}(x), f^{-n_2}(x), f^{-n_3}(x)$ s-intersects each other.*

Proof:

Assume now that the local unstable manifold of the three points do not s-intersect each other. In this case, follows that there is one of the three points, for instance $f^{-n_2}(x)$ such that the unstable manifold of $f^{-n_1}(x)$ and $f^{-n_3}(x)$ intersects the stable manifold of $f^{-n_2}(x)$ on opposite connected components of $W^s_\epsilon(f^{-n_2}(x)) \setminus W^{ss}_\epsilon(f^{-n_2}(x))$ of it.
Now, taking

$$z_{n_2} = W^u_\epsilon(f^{-n_1}(x)) \cap W^s_\epsilon(f^{-n_2}(x)) \quad z'_{n_2} = W^u_\epsilon(f^{-n_3}(x)) \cap W^s_\epsilon(f^{-n_2}(x))$$

follows that they are in different components of $W^s_\epsilon(f^{-n_2}(x)) \setminus W^{ss}_\epsilon(f^{-n_2}(x))$. Then, using that n_1, n_2, n_3 are arbitrarily large and for each point there is defined a local stable manifold of uniform size, follows that $f^{n_2}(z_{n_2}) \to x$ and $f^{n_2}(z'_{n_2}) \to x$ as $n_2 \to +\infty$ accumulating on x from different components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$, which is a contradiction since we are assuming that x is a boundary point.

■

Let us suppose without lose of generality that the local unstable manifold of $f^{-n_1}(x)$ s-intersects the local unstable manifold of $f^{-n_2}(x)$, with n_1 and n_2 arbitrarily large. We prove that in this case the proposition 7.3 follows.

We consider the following two obvious cases:

$$i - \ W^{ss}_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(f^{-n_2}(x)) = \emptyset \ \text{ or}$$

$$ii - \ W^{ss}_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(f^{-n_2}(x)) \neq \emptyset.$$

In the first situation, observe that the unstable manifold of $f^{-n_2}(x)$ intersect the stable manifold of $f^{-n_1}(x)$ on one component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$.

We claim that the point $f^{-n_1}(x)$ is accumulated by points of the homoclinic class only in the same component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$ where the unstable manifold of $f^{-n_2}(x)$ intersects $W^s_\epsilon(f^{-n_1}(x))$. In fact, if this is not the case, using that n_1 and n_2 are arbitrary large follows that x is not a boundary point; i.e.: if there are points $z \in H_p$ close to $f^{-n_1}(x)$ in the opposite component of $W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$ that contains $z_{n_1} = W^u_\epsilon(f^{-n_2}(x)) \cap W^s_\epsilon(f^{-n_1}(x))$ follows that x is accumulated by $f^{n_1}(z)$ and by $f^{n_1}(z_{n_1})$ from different connected components of $W^s_\epsilon(x) \setminus W^{ss}_\epsilon(x)$, which is an absurd since we are assuming that x is a boundary point.

As a consequences of this, we can take any periodic point q such that its unstable manifold intersect the stable manifold of $f^{-n_1}(x)$ in the same connected component where the unstable manifold of $f^{-n_2}(x)$ intersect the stable manifold of $f^{-n_1}(x)$ and such that the distance from $W^u_\epsilon(q) \cap W^s_\epsilon(f^{-n_1}(x))$ to $f^{-n_1}(x)$ is smaller than the distance

from $W^u_\epsilon(f^{-n_2}(x)) \cap W^s_\epsilon(f^{-n_1}(x))$ to $f^{-n_1}(x)$.

Since there are not transversal intersection, follows that $W^u_\epsilon(q)$ s-intersects $W^u_\epsilon(f^{-n_1}(x))$ at least in the points contained in $W^u_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(f^{-n_2}(x))$.

In fact, taking

$$\Pi^{ss} : B_\epsilon(f^{-n_1}(x)) \to W^{cu}_\epsilon(f^{-n_1}(x))$$

follows that $\Pi^{ss}(W^u_\epsilon(q))$ and $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ are in the same component of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus W^u_\epsilon(f^{-n_1}(x))$.

Since there are not transversal intersection and $\Pi^{ss}(W^u_\epsilon(q) \cap W^s_\epsilon(f^{-n_1}(x)))$ is contained in the region bounded by $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and $W^u_\epsilon(f^{-n_1}(x))$ follows that $\Pi^{ss}(W^u_\epsilon(q))$ intersects the points contained in $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x))) \cap W^u_\epsilon(f^{-n_1}(x))$.

Taking another periodic points q' in the same way that we have chosen q, also follows that $\Pi^{ss}(W^u_\epsilon(q'))$ intersects the points contained in $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x))) \cap W^u_\epsilon(f^{-n_1}(x))$. Therefore we conclude that there are two periodic points such that their local unstable manifold intersects (see figure 7) and the proof of the proposition is finished in this case..

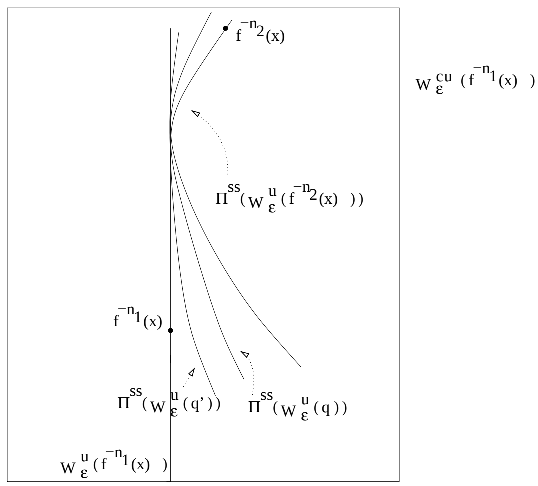


Figure 7

In the second case $(W^{ss}_\epsilon(f^{-n_1}(x)) \cap W^u_\epsilon(f^{-n_2}(x)) \neq \emptyset)$, again we have to consider two situations:

$$ii.i - \quad W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x)) = W^u_\epsilon(f^{-n_1}(x))$$

$$ii.ii - \quad W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x)) \text{ is properly contained in } W^u_\epsilon(f^{-n_1}(x))$$

We show that in the case $ii.i$ the thesis of the proposition holds and latter it is proved that the case $ii.ii$ can not occur.

Observe that the first situation can hold even we are assuming that the strong foliation are not jointly integrable. In the first case, we can assume without lose of generality, that $n_2 < n_1$ and $n_1 - n_2$ is arbitrarily large. We take an arc l containing $f^{-n_1}(x)$ and such that $l \subset W^u_\epsilon(f^{-n_1}(x)) \cap_s W^u_\epsilon(f^{-n_2}(x))$. Then we take $f^k(l)$ where $k = n_1 - n_2$ and observe that $f^k(l) \subset W^u(f^{-n_2}(x))$ and $\Pi^{ss}(f^k(l))$ contains l (where Π^{ss} projects over $W^{cu}_\epsilon(f^{-n_1}(x))$). So, there is a periodic point q such that $W^u_\epsilon(q)$ contains $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and $\Pi^{ss}(W^u_\epsilon(f^{-n_1}(x)))$ (where the projection is done over the center unstable manifold of q).

We can also assume that for another pair of integer $n_2 < n_2' < n_1'$ the situation that we are considering also holds;

i.e.: $W^u_\epsilon(f^{-n_1'}(x)) \cap_s W^u_\epsilon(f^{-n_2'}(x)) = W^u_\epsilon(f^{-n_1}(x))$. Repeating the argument, follows that there is another periodic point q' such that the strong stable manifold of $W^u_\epsilon(f^{-n_2'}(x))$ intersect the local unstable manifold of q' and so its unstable manifold s-intersect the strong stable manifold of $f^{n_2'-n_2}(f^{-n_2'}(x)) = f^{n_2}(x)$ and this implies that the unstable manifold of q' s-intersect the unstable manifold of q. Therefore, the thesis of the proposition holds.

Now we show that the second situation can not occur. First observe that changing $f^{-n_1}(x)$ with $f^{-n_2}(x)$ also follows that

$$W^{ss}_\epsilon(f^{-n_2}(x)) \cap_s W^u_\epsilon(f^{-n_1}(x)) \neq \emptyset$$

In fact if it is not the case, we can repeat the arguments done in case i. If $f^{-n_1}(x) \in W^{ss}_\epsilon(f^{-n_2}(x))$ follows that x is periodic and we are assuming that it is not the case.

So $W^{ss}_\epsilon(f^{-n_2}(x)) \cap_s W^u_\epsilon(f^{-n_1}(x))$ and $W^{ss}_\epsilon(f^{-n_2}(x)) \cap_s W^u_\epsilon(f^{-n_1}(x))$ are different points.

Let us take $\Pi^{ss} : B(f^{-n_1}(x)) \to W^{cu}_\epsilon(f^{-n_1}(x))$. Let us take $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and $W^u_\epsilon(f^{-n_1}(x))$ and points $z_1 \in W^u_\epsilon(f^{-n_1}(x))$, $z_2 \in W^u_\epsilon(f^{-n_2}(x))$ such that $\Pi^{ss}(z_2) \notin W^u_\epsilon(f^{-n_1}(x))$, $z_1 \notin \Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$.

Let us take the connected component of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus W^u_\epsilon(f^{-n_1}(x))$ and we named $L^+_{f^{-n_1}(x)}$ the one that contains $\Pi^{ss}_\epsilon(z_2)$. Let us take the connected component of $W^{cu}_\epsilon(f^{-n_1}(x)) \setminus \Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ and we named $L^-_{f^{-n_2}(x)}$ the one that contains z_1; with $L^+_{f^{-n_2}(x)}$ we named the other component. Related to this components, observe that $L^+_{f^{-n_2}(x)} \subset L^+_{f^{-n_1}(x)}$ and $L^-_{f^{-n_1}(x)} \subset L^-_{f^{-n_2}(x)}$.

Let q_2 be a periodic point close to z_2; observe that $\Pi^{ss}(W^u_\epsilon(q_2))$ is contained in the closure of $L^+_{f^{-n_2}(x)}$: in fact, if it is not the case, since $\Pi^{ss}(W^u_\epsilon(q_2))$ is close to $\Pi^{ss}(W^u_\epsilon(f^{-n_2}(x)))$ follows that $\Pi^{ss}(W^u_\epsilon(q_1))$ intersect transversally $W^u_\epsilon(f^{-n_1}(x))$ which is an absurd. Now, if it happens that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_2))$ we take another periodic point q_2' and we do the same analysis; if it happens again that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_2'))$ we are in the case that we have two periodic points such that their local unstable manifold s-intersect each other and the proof is finished. In the case that $f^{-n_1}(x) \notin \Pi^{ss}(W^u_\epsilon(q_2))$, this implies that $\Pi^{ss}(W^u_\epsilon(q_2))$ intersect $L^+(f^{-n_1}(x)) \cap W^s_\epsilon(f^{-n_1}(x))$. Let q_1 be a periodic point close to z_1; observe that $\Pi^{ss}(W^u_\epsilon(q_1))$ is contained in the closure of $L^-_{f^{-n_1}(x)}$: in fact, if it is not the case, since $\Pi^{ss}(W^u_\epsilon(q_1))$ is close to $W^u_\epsilon(f^{-n_1}(x))$ follows that $\Pi^{ss}(W^u_\epsilon(q_1))$ intersect transversally $\Pi^{ss}(W^u_\epsilon(f^{-n_1}(x)))$ which is an absurd. Now, if it happens that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_1))$ we take another periodic point q_1' and we do the same analysis; if it happens again that $f^{-n_1}(x) \in \Pi^{ss}(W^u_\epsilon(q_1'))$ we are in the case that we have two periodic points such that their local unstable manifold s-intersect each other, and again the proof is finished. In the case that $f^{-n_1}(x) \notin \Pi^{ss}(W^u_\epsilon(q_1))$, this implies that $\Pi^{ss}(W^u_\epsilon(q_2))$ intersect $L^-(f^{-n_1}(x)) \cap W^s_\epsilon(f^{-n_1}(x))$. Therefore $f^{-n_1}(x)$ is accumulated by points on both connected components of

$W^s_\epsilon(f^{-n_1}(x)) \setminus W^{ss}_\epsilon(f^{-n_1}(x))$, which is an absurd because we are assuming that they are not transversal intersections (see figure 8).

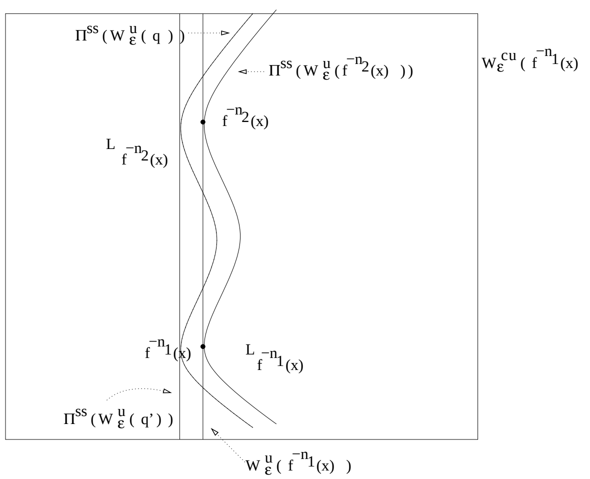


Figure 8

Now, we have to consider the case that the boundary points are contained in the unstable manifolds of periodic points

2.2. The boundary points are contained in the unstable manifolds of some periodic points.

In this case, we can prove the following the lemma:

Lemma 7.3.3 *Let us assume that there are not transversal intersection and the stable boundary points are contained in the unstable manifold of some periodic points. Then, there exists a pair of points x, y such that $y \in W^{ss}_\epsilon(x)$ and such that x and y are boundary points.*

Proof:

Let us assume that the lemma is false. Then we can take x, y such that $y \in W^{ss}_\epsilon(x)$. and for instances y is not a boundary point. Moreover, if the unstable foliations are not jointly integrable, we can assume that $\Pi^{ss}(W^u_\epsilon(y))$ intersect the interior of one of the connected components of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$.
Let us take a periodic point q close to y such that $\Pi^{ss}(q)$ is in the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that its closure contains $\Pi^{ss}(W^u_\epsilon(x))$. Then, since $\Pi^{ss}(W^u_\epsilon(q))$ is close to $W^u_\epsilon(y)$, $\Pi^{ss}(W^u_\epsilon(q))$ is in the connected component of $W^{cu}_\epsilon(y) \setminus W^u_\epsilon(y)$ that its closure contains $\Pi^{ss}(W^u_\epsilon(x))$. Since $\Pi^{ss}(W^u_\epsilon(x))$ and $W^u_\epsilon(y)$ do not coincide, follows that $\Pi^{ss}(W^u_\epsilon(q))$ intersects $W^u_\epsilon(x)$ transversally (see figure 9). Which is an absurd because we are assuming that they are not transversal intersection.

∎

This finish the proof of proposition 7.3.

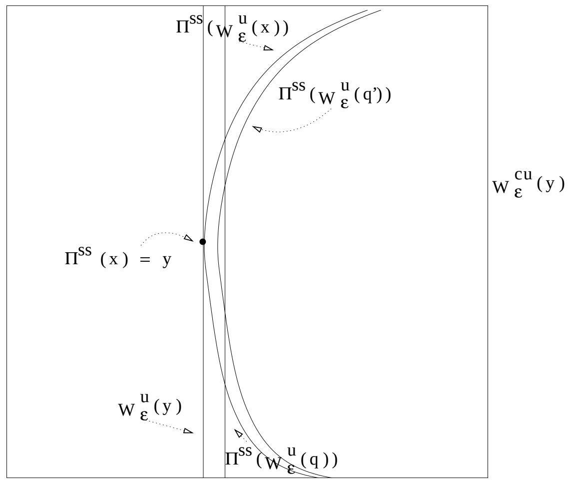


Figure 9

7.4 Proof of proposition 7.4:

To prove the proposition, we introduce a one parameter family of diffeomorphisms (see lemma 7.4.2). Before to do that, we need to chhose some constants, some neighborhood of the initial map f and some definitions. We list these selestions:

1. Let us take $\delta > 0$ and let q_δ be a periodic point with $\frac{\delta}{3}-$weak contraction along the center direction. We consider an arbitrarily small open neighborhood $\mathcal{U} = \mathcal{U}(\delta) \subset Diff^1(M^3)$ of f such that for any $g \in \mathcal{U}$ follows that the periodi points p_x and p_y has analytic continuation and q_δ remains $\frac{\delta}{2}-$weak contractive and homoclinically related to p_x.

2. Let η_0 be a positive number arbitrarily small such that if $|g - f|_k < \eta_0$ then $g \in \mathcal{U}$.

3. Let us take the point x and related to it we take a pair of neighborhood $B_{\eta_0}(x), B_{\eta_0}(f^{-1}(x))$ of size η_0 around x and $f^{-1}(x)$ respectively.

4. Given a point z we take a rectangle inside $W^{cu}_\epsilon(z)$ of size η_0 defined as

$$R_{\eta_0}(z) = \cup_{\{y \in W^c_{\eta_0}(z)\}} W^u_{\eta_0}(y)$$

and observe that $W^{cu}_\epsilon(z) \cap B_{\eta_0}(z) \subset R_{\eta_0}(z)$.

Now, let us take the rectangles $R_{\eta_0}(x)$ and $R_{\eta_0}(f^{-1}(x))$ of size η_0 around x and $f^{-1}(x)$ respectively inside the local center-unstable manifold that contains x and $f^{-1}(x)$ respectively. The same is done for $\eta < \eta_0$. Now, let us take the point y

such that $y \in W^{ss}_\epsilon(x)$ and let us take a rectangle $R_{\eta_0}(y)$ of size η_0 around y inside the local center-unstable manifold that contains y.
5. Let us consider the projection Π^{ss} induces by of the strong stable manifold from $R_{\eta_0}(x)$ to $R_{\eta_0}(y)$. Recall the following folklore lemma (see [HPS])

Lemma 7.4.1 *The strong stable foliation $W^{ss}_\epsilon(x) = \phi^{ss}_f(x)I_\epsilon$ is C^r-Holder ($r < 1$) respect to x for some $r > 0$. Moreover, if g is C^k-close enough to f and U is an small neighborhood of H_p follows that $\phi^{ss}_g(x)$ is C^r-Holder respect to x in $\Lambda_g = \cap_{\{n>0\}} g^n(U)$.*

Now, for each $g \in \mathcal{U}$ we take

$$\Pi^{ss}_g : W^{cu}_\epsilon(x) \to W^{cu}_\epsilon(y)$$

Using that the strong stable foliation is C^r-Holder, taking $\gamma = 1/r$ follows that for every $g \in \mathcal{U}$ holds

$$R_{\eta_0{}^{(k+1)\gamma}}(y) \subset \Pi^{ss}_g(R_{\eta_0{}^{k+1}}(x))$$

where $R_{\eta_0{}^{(k+1)\gamma}}(y)$ is a rectangle of size $\eta_0{}^{(k+1)\gamma}$ around y.
6. Let us also consider the rectangle

$$R_{\eta_0{}^{2(k+1)\gamma}}(y)$$

of size $\eta_0{}^{2(k+1)\gamma}$ around y (see figure 10) and we take

$$l_y = W^u_{\eta^{2(k+1)\gamma}}(y) = W^u_\epsilon(y) \cap R_{\eta_0{}^{2(k+1)\gamma}}(y)$$

7.

Remark 7.1 *From the fact that H_p is topologically hyperbolic, follows that there exists $L > 0$ such that for any z, $z' \in H_p$*

$$W^s_L(z) \cap W^u_\epsilon(z') \neq \emptyset$$

where with $W^s_L(z)$ we denote $f^{-L}(W^s_\epsilon(z))$.

Now, let us take $N_0 = N(\eta_0)$ such that

$$f^{N_0}(W^u_{\eta_0^{2\gamma(k+1)}}(y)) \cap W^s_L(p_y) \neq \emptyset, \text{ and}$$

$$f^k(W^u_{\eta_0^{2\gamma(k+1)}}(y)) \cap W^s_L(p_y) = \emptyset \;\; 0 \leq k < N_0$$

¿From the previous remark, observe that

$$\ell(f^{N_0}(W^u_{\eta^{2(k+1)\gamma}}(y))) \leq \epsilon$$

7. Let us consider the discs $\hat{D}_1, ..., \hat{D}_m$ given by

$$\hat{D}_i = f^{-N_0}(W^s_L(p_y)) \cap R_{\eta_0}(f^{-1}(x))$$

For each $\hat{D}_i$ let z_i be a point in $\hat{D}_i$. Let us take a disc

$$D_i = W^{cs}_\epsilon(z_i).$$

8. We also consider the rectangle $R_{\eta_0{}^{k+1}}(f^{-1}(x))$ of size ${\eta_0}^{k+1}$ around $f^{-1}(x)$, and we take

$$l = W^u_\epsilon(f^{-1}(x)) \cap R_{\eta_0{}^{k+1}}(f^{-1}(x))$$

In the next lemma it is introduced a one parameter family of diffeomorphisms $\mathcal{F} = \{g_\eta\}_{\eta\in[-\eta_0,\eta_0]}$ of C^k perturbation of f. This family is used to prove the proposition 7.4.

Lemma 7.4.2 *There exists a one parameter family $\mathcal{F} = \{g_\eta\}_{\eta\in[-\eta_0,\eta_0]}$ such that for $g = g_\eta \in \mathcal{F}$ the following properties hold (see figure 10):*

1. *$|g - f|_k < \eta$ where $|.|_k$ is the C^k-norm,*
2. *$g_{|B_{\eta_0}(f^{-1}(x))^c} = f$,*
3. *$g(l) \subset R_{\eta_0}(x)$ and $g(l)$ moves continuously with g,*
4. *for any $g \in \mathcal{F}$ follows that $g(D_i) \subset f(D_i)$,*
5. *the arc $g_{\eta_0}(l)$ and $g_{-\eta_0}(l)$ are on opposite side of the rectangle $R_{\eta_0{}^{k+1}}(x) \setminus W^u_\epsilon(x)$; moreover, $dist(g_{\eta_0}(l), l) > 0$ and $dist(g_{-\eta_0}(l), l) > 0$.*

Proof:

We take a rectangle R in $\mathbb{R}^3$ given by

$$R = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| < \eta_0, |\bar{y}| < \eta_0, |\bar{z}| < \eta_0, \}$$

We take a map

$$C : R_{\eta_0}(x) \to R$$

such that

1. $C(W^u_\epsilon(x) \cap R_{\eta_0}(x)) = \{\bar{x} = 0, \bar{y} = 0\}$
2. for any discs $\hat{D}_i$ holds that $H(\hat{D}_i \cap R_{\eta_0}(x))$ is a plane parallel to $\bar{z} = 0$.

Now we take a map

$$\Delta_0 : \mathbb{R} \to \mathbb{R}$$

and η_0^* smaller than η_0 such that

1. $|\Delta_0|_k < \eta_0^*$
2. $\Delta_{0|[-\eta_0^{k+1}, \eta_0^{k+1}]} = \eta_0^*$
3. $\Delta_{0|[-\eta_0, \eta_0]^c} = 0$

For each η with $-\eta_0 < \eta < \eta_0$ we take the map

$$T_\eta(\bar{z}, \bar{y}, \bar{z}) = (\bar{x}, \bar{y} + \eta\Delta_0(\bar{x}), \bar{z})$$

Now we take η^* smaller than η_0 such that for each η follows that

$$|C^{-1} \circ T_\eta \circ C|_k < \eta_0$$

Now we take

$$g_\eta = H^{-1} \circ T_\eta \circ H \circ f$$

It is not difficult to verify that the family $\mathcal{F} = \{g_\eta\}_{\eta \in [-\eta_0, \eta_0]}$ verifies the thesis of the lemma.

■

Remark 7.2 *From the construction of the family $\mathcal{F}$ follows that*

$$\Pi^{ss}_{g_{\eta_0}}(g_{\eta_0}(l)) \text{ and } \Pi^{ss}_{g_{-\eta_0}}(g_{-\eta_0}(l))$$

are in the opposite sides of $R_{\eta_0^{(k+1)\gamma}}(y) \setminus W^u_\epsilon(y)$. Moreover, there is a positive constant s_0 such that

$$dist(\Pi^{ss}_{g_{\eta_0}}(g_{\eta_0}(l)), W^u_{\eta_0^{2(k+1)\gamma}}(y)) > s_0 > 0$$

$$dist(\Pi^{ss}_{g_{-\eta_0}}(g_{-\eta_0}(l)), W^u_{\eta_0^{2(k+1)\gamma}}(y)) > s_0 > 0$$

See figure 10.

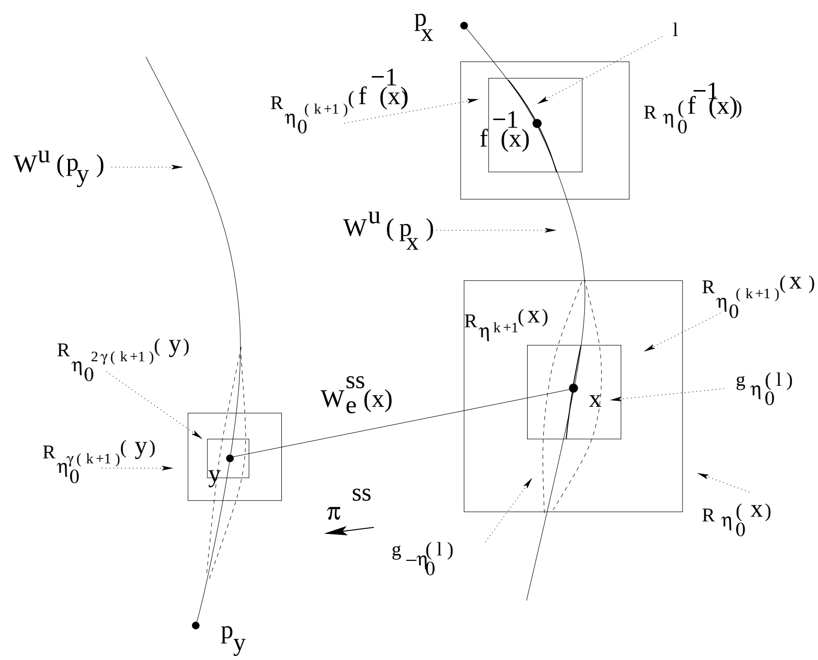


Figure 10

Related to the parameter family $\mathcal{F}$, we prove the following lemma that implies proposition 7.4.

Lemma 7.4.3 *For any positive s smaller than s_0, there is a periodic point q_f of f, such that for any $g \in \mathcal{F}$ follows that:*

1. *there exists the analytic continuation q_g of q_f,*
2. *$dist(l_y, q_g) < s$ where $l_y = W^u_{\eta_0^{2\gamma(k+1)}}(y)$,*
3. *q_g is homoclinically related to p_y,*

Before to give the proof we show how the previous lemma implies the proposition 7.4.

Lemma 7.4.3 implies proposition 7.4.

First observe that for any $g \in \mathcal{F}$ follows that p_x and p_y remains periodic, $l_y \subset W^u(p_y)$ and $g(l) \subset W^u(p_x)$. The last two facts follows because $f^{-n}(x) \notin R_{\eta_0}(f^{-1}(x))$ and $f^{-n}(y) \notin R_{\eta_0}(f^{-1}(x))$ for any $n > 1$. In particular, $f^{-n}(l) \notin R_{\eta_0}(f^{-1}(x))$ and $f^{-n}(l_y) \notin R_{\eta_0}(f^{-1}(x))$for any $n > 0$. Then, for any $g \in \mathcal{F}$ follows that g coincides with f along the backward orbit of l and l_y.

Then, take $s < {\eta_0}^{2(k+1)\gamma}$ and s smaller by the constant s_0 given by remark 7.2. By lemma 7.4.3 follows that there is a periodic point g_f such that for any $g \in \mathcal{F}$

the analytic continuation q_g verifies that $dist(l_y, q_g) < s$ and if s is sufficientely small also holds that

$$dist(l_y, W^{ss}_\epsilon(q_g) \cap W^{cu}_\epsilon(y)) < s_0.$$

Now, for each $g \in \mathcal{F}$ consider $\hat{l}_g = \Pi^{ss}_g(l_g)$ where $l_g = g(l)$. Observe that $g \to \hat{l}_g$ and $g \to q_g$ move continuously and recall that $\hat{l}_{g_{\eta_0}}$ and $\hat{l}_{g_{-\eta_0}}$ are in the extremal opposite components of $R_{\eta_0^{(k+1)\gamma}}(y) \setminus W^u_\epsilon(y)$. So, by continuity follows that there is $g \in \mathcal{F}$ such that $W^u(p_x) \cap W^{ss}_\epsilon(q_g) \neq \emptyset$. From the election of η_0 follows that q_δ remains periodic for any $g \in \mathcal{F}$ and homoclinically connected with p_y; so, it is homoclinically connected with q_g. And this conclude the proof of the proposition 7.4.

∎

Now we proceed to prove the previous lemma.

Proof of lemma 7.4.3.

The proof is done performing a geometrical construction: it is taken a small cylinder T containing the point y such that there is a positive integer M and for any $g \in \mathcal{F}$ follows that

1. $g^M(T)$ intersects T and $g^M(T) \setminus T$ has two connected components;
2. $g^i(T)$ remains close to $f^i(T)$ for $0 \leq i \leq M$;
3. T is expanded along the direction E_3 and it is contracted along the direction $E_1 \oplus E_2$.

This implies that for any $g \in \mathcal{F}$ there is a unique periodic point q_g with period M which is the analytic continuation of the unique periodic point of f of period M contained in $f^M(T) \cap T$.

To perform the pevious sketched construction we start with an easy claim.

Claim 2 *It follows that either $x \notin W^s(p_y)$ or $y \notin W^s(p_x)$.*

If not, since $y \in W^{ss}_\epsilon(x)$ and $W^{ss}_\epsilon(x) \subset W^s_\epsilon(x)$ follows that $W^s(p_y)$ intersect $W^s(p_x)$; which is absurd because p_x and p_y are different points.

As a consequences of the previous claim, we get that taking η_0 small enough then

$$W^s_L(p_y) \cap B_{\eta_0}(f^{-1}(x)) = \emptyset,$$

Moreover, there is a connected arc $l(p_y)$ contained in $W^u(p_y)$ and containing p_y and y, such that

$$l(p_y) \cap B_{\eta_0}(f^{-1}(x)) = \emptyset$$

So there is a neighborhoods $V^s(p_y)$ of $W^s_L(p_y)$, $V^u(p_y)$ of $l(p_y)$ such that

$$V^s(p_y) \cap B_{\eta_0}(f^{-1}(x)) = \emptyset, \text{ and } V^u(p_y) \cap B_{\eta_0}(f^{-1}(x)) = \emptyset$$

Let us take the integer $N_0 = N(\eta_0)$ chosen before such that it is the first positive integer verifying that $f^{N_0}(W^u_{\eta^{2(k+1)\gamma}}(y)) \cap W^s_L(p_y)$ and let

$$z_0 \in W^u_{\eta^{2(k+1)\gamma}}(y) \text{ such that } f^{N_0}(z_0) \in W^s_L(p_y)$$

Let us take $g \in \mathcal{F}$ and let us consider the orbit of z_0 by f and g up to the iterate N_0. We need to compute the distance between $f^k(z_0)$ and $g^k(z_0)$ for $k \leq N_0$. Observe that it could happen that $N_0 \to \infty$ as $\eta_0 \to 0$ and so the number of iterates $0 \leq i \leq N_0$ such that $f^i(z_0) \in R_{\eta_0}(f^{-1}(x))$ could also growth as $\eta_0 \to 0$ and so it could happen that $g^{N_0}(z_0)$ is far away from $f^{N_0}(z_0)$. In the next lemma we show that this is not the case.

Lemma 7.4.4 *There is γ_0 smaller than ϵ such that for η_0 small, and for any $g \in \mathcal{F}$ follows that*

$$dist(g^i(z_0), f^i(z_0)) < \gamma_0 \quad 0 \leq i \leq N_0$$

¿From this lemma follows the next obvious corollary

Corollary 7.2 *There exists a connected arc l_0 contained in $W^u_{\eta^{2(k+1)\gamma}}(y)$ such that $z_0 \in l_0$ and for any $g \in \mathcal{F}$ holds that $dist(g^i(l_0), f^i(l_0)) < \gamma_0$ for $0 \leq i \leq N_0$*

Before to give the proof of the previous lemma, let us continue proving lemma 7.4.3.

Let us take the connected arc l_0 given by the previous corollary and let $d > 0$ be a positive number smaller than s where s is the positive constant in the hypothesis of the lemma 7.4.3. Let us take T_d a cylinder of size d containing l_0, i.e.:

$$T_d = \cup_{\{z \in l_0\}} B_d(z)$$

The cylinder mentioned in the beginning of the present proof it is a cylinder contained in T_d.

Observe that

1. $f^{N_0}(T_d) \cap W^s_L(p_y) \neq \emptyset$
2. $f^{N_0}(T_d) \setminus W^s_L(p_y)$ splits $f^{N_0}(T_d)$ in two connected components.

As a consequences of lemma 7.4.3 and if d is small enough, follows that for any $z \in T_d$ and for any $g \in \mathcal{F}$ holds that

$$dist(f^{N_0}(z), g^{N_0}(z)) < \gamma$$

Therefore for any $g \in \mathcal{F}$ follows that

1. $g^{N_0}(T_d) \cap W^s_L(p_y) \neq \emptyset$
2. $g^{N_0}(T_d) \setminus W^s_L(p_y)$ has two connected components.

Now observe that there exists k_0 such that for any $k > k_0$ follows that

1. $W^s_L(f^{-k}(y))$ is contained in $V^s(p_y)$ and
2. for any $g \in \mathcal{F}$ follows that $g^j(W^s_L(f^{-k}(y))) \in V^s(p_y) \cup V^u(p_y)$ for $0 \leq j \leq k$.

Claim 3 *There exists N_1 such that for any $N > N_1$ there exists a cylinder $T^1 = T^1(N)$ such that for any $g \in \mathcal{F}$*

1. *$g^{N_0}(T_d) \setminus T^1$ splits $g^{N_0}(T_d)$ in two connected components,*
2. *for $k > k_0$, $g^N(T^1) \cap W^s_L(f^{-k}(y)) \neq \emptyset$ and $g^N(T^1) \setminus W^s_L(f^{-k}(y))$ has two connected components,*
3. *$g^i(T^1) \subset V^s(p_y) \cup V^u(p_y)$ for any $0 \leq i \leq N$,*
4. *$g^N(T^1) \setminus T_d$ has two connected components.*

In fact, taking N_1 large enough follows that if $N > N_1$ then $W^s_L(f^{-N}(y)) \cap g^{N_0}(T_d) \neq \emptyset$ for any $g \in \mathcal{F}$. Then, taking an small rectangle T^1 containing a discs contained $W^s_L(f^{-N}(y))$ follows that $f^N(T^1)$ intersects $W^s_{\epsilon'}(y)$ with ϵ' smaller than $\frac{d}{2}$. Moreover, it follows that $f^N(T^1)$ is C^r-close to a fixed arc contained in the local unstable manifold of p_y and containing y.

Now, let us take

$$\hat{T}_d(g) = g^{-N_0}(T^1),$$

related to it, we get the following claim:

Claim 4 *As a consequences of the election of $\hat{T}_d$ observe that for any $g \in \mathcal{F}$ follows that*

1. *$\hat{T}_d(g) \subset T_d$,*
2. *$\hat{T}_d(g)$ splits $g^{N_0+N}(\hat{T}_d(g))$ in two connected components,*

3. for any $z \in g^{N_0+N}(\hat{T}_d(g))$ *follows that* $dist(z, l(p_y)) < d/2$. *See figure 11.*

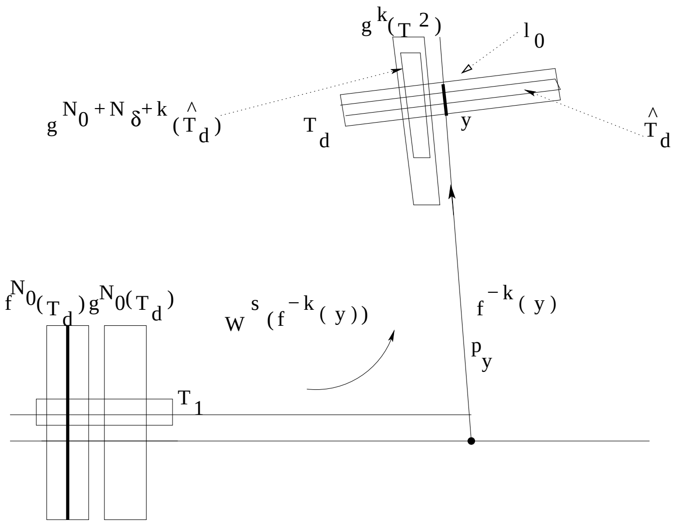


Figure 11

To conclude the proof of lemma 7.4.3 it is enough to show that we can choose N such that for any $g \in \mathcal{F}$ there is a unique periodic point q_g such that

1. $q_g \in \hat{T}_d(g)$,
2. the period of q_g is $N_0 + N$,
3. q_g is contractive along the direction E_2 and it is homoclinically related with p_x.

To get the periodic points q_g, the goal is to show that we keep the expansion along the unstable direction and contraction along the stable manifolds inside $\hat{T}_d(g)$. Therefore, from the properties stated in claim 4 about $\hat{T}_d$ and $g^{N_0+N}(\hat{T}_d)$, we conclude the existence of a periodic point q_g for g nearby y which is the analytic continuation of q. To precise, we formulate the following claim:

Claim 5 *There is* $\lambda_1 < 1$ *such that for any* $g \in \mathcal{F}$ *follows that for any* $z \in \hat{T}_d(g)$

1. $dist(f^i(q_f), g^i(q_g)) < \gamma_0$ *for* $0 \leq i \leq N_0 + N$,
2. $|Dg^{N_0+N}_{|E_1 \oplus E_2(z)}| < \lambda_1^{N_0+N}$.

3. $|Dg^{N_0+N}_{|E_3(z)}| > \lambda_1^{-(N_0+N)}$.

In fact, if the previous assertion holds and from claim 4 follows that there is a unique periodic point contained in $g^{N_0+N}(\hat{T}_d)(g)$ of period $N_0 + N$.

To prove claim 4 observe that the first item is immediate by the construction. The contraction stated in the second item is obtained, because for any $z \in \hat{T}_d$ follows that any expansion along the center direction that could appear along the piece of orbit $\{g^i(z)\}_{\{0\leq i\leq N_0\}}$ is compensated with contraction in the piece of orbit $\{g^i(z)\}_{\{N_0\leq i\leq N_0+N\}}$. The expansion stated in the third item follows with similar argument.

This finish the proof of proposition 7.4 and now we proceed to prove lemma 7.4.4.

Proof of lemma 7.4.4.

First, observe that f is expansive in the neighborhood U that contains H_p; i.e.: there exists $r > 0$ such that if $dist(f^n(x), f^n(y)) < r$ for any integer n then $x = y$. This follows immediately from the fact that H_p is topologically hyperbolic. It was proved in [Fa] that for expansive homeomorphisms, it is possible to obtain an hyperbolic adapted metric, not necessarily coherent with a riemannian structure.

Lemma 7.4.5 *([Fa])*

Given a expansive homeomorphisms f in a metric compact invariant set, there exists an adapted metric dist compatible with the topology, and there exist constants $r > 0$ and $0 < \lambda < 1$ such that if

$$dist(f^n(x), f^n(y)) < r \text{ then } dist(f^n(x), f^n(y)) < \lambda^n dist(x, y).$$

In the case that we are dealing, a topologically hyperbolic attracting homoclinic class, follows from lemma 7.4.5 the next corolary

Corollary 7.3 *Let us consider dist, λ and r, the distance and constant given by lemma 7.4.5. Then, for any $x \in H_p$ follows that if $y \in W^{cs}_\epsilon(x)$ then*

$$dist(f^n(x), f^n(y)) < \lambda^n dist(x, y).$$

Let us consider the distance given by lemma 7.4.5 and let us take r_0 smaller than r and sufficiently small such that if

$$dist(z, f^{-1}(x)) < r_0 \text{ then } dist(f(z), f^{-1}(x)) > 2r_0$$

In particular, if $dist(z, f^{-1}(x)) < r$ and $dist(f^n(z), f^{-1}(x)) < r$ then $n \geq 2$.

Let us η_0 and γ_0 smaller than r, and λ be the constant given by lemma 7.4.5. Let us take η_0 small such that

$$\lambda(\gamma_0 + \eta_0^{k+1}) < \gamma_0$$

.

Let us take now the sequences of positive integer k_i such that

$$W^{cs}_{\gamma_0}(f^{k_i}(z_0)) \cap R_{\eta_0}(f^{-1}(x)) \neq \emptyset$$

Observe that if $f^j(W^{cs}_{\gamma_0}(z)) \cap R_{\eta_0}(f^{-1}(x)) = \emptyset$ for any $0 < j < n$ then for any $g \in \mathcal{F}$ and $z' \in W^{cs}_{\gamma_0}(z)$ holds that

$$dist(g^j(z'), f^j(z)) = dist(f^j(z'), f^j(z)) < \lambda^j \gamma_0$$

To conclude the lemma, first we prove that

$$dist(f^{k_i}(z_0), g^{k_i}(z_0)) < \gamma_0 \text{ and } g^{k_i}(z_0) \in W^{cs}_{\gamma_0}(f^{k_i}(z_0))$$

We prove it by induction: first observe that it is true for k_1; in fact, $g^j(z_0) = f^j(z_0)$ for $j \leq k_1$. Now, let us assume that it is true for k_i. ¿From the fact that

$$W^{cs}_{\gamma_0}(f^{k_i}(z_0)) \subset D_{j_i}$$

for some j_i and from the construction of the family $\mathcal{F}$ follows that

$$dist(f^{k_i+1}(z_0), g^{k_i+1}(z_0)) < \gamma_0 + \eta_0^{k+1} \text{ and } g^{k_i+1}(z_0) \in W^{cs}_{2\gamma_0}(f^{k_i+1}(z_0))$$

Since $W^{cs}_{2\gamma_0}(f^{k_i+1}(z_0)) \cap R_{\eta_0}(f^{-1}(x)) = \emptyset$ follows that

$$g_{|W^{cs}_{2\gamma_0}(f^{k_i+1}(z_0))} = f_{|W^{cs}_{2\gamma_0}(f^{k_i+1}(z_0))}$$

Then,

$$dist(f(f^{k_i+1}(z_0)), g(g^{k_i+1}(z_0))) = dist(f(f^{k_i+1}(z_0)), f(g^{k_i+1}(z_0))) <$$
$$< \lambda dist(f^{k_i+1}(z_0)), g^{k_i+1}(z_0)) < \lambda(\gamma_0 + \eta_0^{k+1}) < \gamma_0$$

Therefore

$$g^{k_i+2}(z_0) \in W^{cs}_{\gamma_0}(f^{k_i+2}(z_0))$$

Since $k_{i+1} - k_i \geq 2$ and $W^{cs}_{\gamma_0}(f^{k_i+j}(z_0)) \cap R_{\eta_0}(f^{-1}(x)) = \emptyset$ for $0 < j < n_{i+1} - n_i$, follows that

$$dist(f^{k_{i+1}}(z_0), g^{k_{i+1}}(z_0)) < \gamma_0$$

Now, we can show that for any j follows that

$$dist(f^j(z_0), g^j(z_0)) < 2\gamma_0$$

From the fact that $dist(f^{k_i}(z_0), g^{k_i}(z_0)) < \gamma_0$ and arguing as before, follows that for $j < k_{i+1} - k_i$ then $dist(f^{k_i+j}(z_0), g^{k_i+j}(z_0)) < \lambda^j(\gamma_0 + \eta_0^{k+1}) < 2\gamma_0$ ■

So, we have concluded the proof of Proposition 4.4 and we are finished with the case B.2.2. when the strong foliations are not jointly integrable.

7.5 Proof of proposition 7.5 (joint integrable case):

Let us take the points $x, y \in H_p$ such that $y \in W^{ss}_\epsilon(x)$. If x and y are stable boundary point (recall the definition given in last section) follows that x and y belongs to the unstable manifold of some periodic points and in this case the proposition is again proved (see subsection 7.5.2).

In this case we perform a suitable perturbation taking in account the geometry that follows from the joint integrability.

Observe that it could happen that the homoclinic class H_p could be destroyed by the perturbation. Moreover, it could occur that the set $\Lambda_g(U) = \cap_{\{n>0\}} g^n(U)$ is not topologically hyperbolic. In fact, it is not clear that the center direction remains a stable direction and that the unstable foliation remains unstable for the perturbed map.

Observe also that it could happen that the orbits of the points x and y do not remain in the neighborhood U. Even if it is the case, it could happen that the backward orbit of x and the backward orbit of y could intersect the domain of perturbation (this is not the situation for the perturbation introduced when it was considered the non-integrable case). So their local unstable manifolds do not necessary remains the same after the perturbation.

To overcome this difficulty, first it is proved that the points x and y has a well defined "continuation" (their existences is proved in lemma 7.6). If for some g close to f holds that the "continuation" of the points x and y do not belong to the same strong stable leaf, then considering an isotopy between the initial map and the perturbation, follows that for some map of the isotopy holds that there are two periodic points as in the thesis of proposition 7.2. If it occurs that for any g close to f holds that the "continuation" of the points x and y have the property that they belong to the same strong stable leaf, then it is performed a perturbation such that the local unstable manifold of the "continuation" of the points x and y are not jointly integrable, and this allows to find two periodic points as in the hypothesis of proposition 7.2.

In the next sub-subsection 7.5.1 it is shown that the set Λ_f obtained from the theorem 7.1 has some well controlled continuation for any perturbation. In the last sub-subsection, it is concluded the proof of the proposition 7.5.

7.5.1 Continuation of the set Λ_f.

To understand how the dynamic changes for the perturbed map, first we shows that the points x and y has a well defined continuation. Before to do that, we state a well known lemma about the continuation of a dominated splitting for perturbation of an initial map.

Lemma 7.5.1 *Let $f \in Diff^r(M)$ $(r \leq 1)$ and Λ be a compact invariant set of f exhibiting a dominated splitting $T_\Lambda M = E \oplus F$. There exists an open neighborhood $\mathcal{U}$ of f in $Diff^r(M)$ and an open neighborhood U of Λ such that for each $g \in \mathcal{U}$ there exist two continuous function, $T_g : \Lambda_g \to T_{\Lambda_g} M$ and $\phi_g : \Lambda_g \times Diff(M) \to Emb^1(D, M)$ such that for any $g \in \mathcal{U}$and $x \in \Lambda_g$ it is defined the*

dominated splitting $E(g) \oplus F(g)$ and the center manifold $W^E_\epsilon(x, g) = \phi_g(x) D_\epsilon$ and verifying

1. $T_x W^{E(g)}_\epsilon(x, g) = E(g, x)$,
2. *if* $g(W^{E(g)}_\epsilon(x, g)) \subset B_\epsilon(g(x))$ *then* $g(W^{E(g)}_\epsilon(x, g)) \subset W^{E(g)}_\epsilon(g(x), g)$,
3. *if* $g^{-1}(W^{E(g)}_\epsilon(x)) \subset B_\epsilon(g^{-1}(x))$ *then* $g^{-1}(W^{E(g)}_\epsilon(x, g)) \subset W^{E(g)}_\epsilon(g^{-1}(x), g)$.
4. *the maps* $g \in \mathcal{U} \to T_g$ *and* $g \in \mathcal{U} \to Emb^1(D, M)$ *are continuous.*

Remark 7.3 *If one of the previous direction is hyperbolic, then it remains hyperbolic*

after a C^r-perturbation of the system.

Let us take the set Λ_f given by theorem 7.1. We take two small compact neighborhood $V_1 \subset V_2$ of Λ_f and we consider the set

$$\Lambda_f(V_i) = Closure(\cap_{\{n \in \mathbb{Z}\}} f^n(V_i)) \ i = 1, 2$$

The neighborhood V_1, V_2 are taken sufficiently small such that the direction E_3 remains hyperbolic over $\Lambda_f(V_i)$, $i = 1, 2$. Observe that this set is a compact invariant topologically hyperbolic set such that the closure of the periodic points in $\Lambda_f(V_1)$ contains Λ_f; i.e.:

$$\Lambda_f \subset Closure\{Per(f_{|\Lambda_f(V_1)})\}$$

Moreover, the periodic points have a homoclinic intersections with orbits inside V_2. Using this, we can also prove that the closure of the periodic points with good rate of contraction along the center direction and orbit in V_2 contains Λ_f. This is the statement of the next lemma.

Lemma 7.5.2 *There exists an small neighborhood V of Λ_f and a positive constant $\lambda_c < 1$ such that the closure of the periodic points of f with orbit in V and center eigenvalue smaller than λ_c contains Λ_f*

Proof: Let $V_1 \subset V$ be two small neighborhood of Λ_f. Since Λ_f is topologically hyperbolic transitive set, follows that the periodic points contained in $\Lambda_f(V_1)$ are dense in Λ_f. Since we are assuming that f is Kupka-Smale, then the periodic points in $\Lambda_f(V_1)$ are hyperbolic. Let us take a hyperbolic periodic point p_0 and let $\lambda_c^1 < 1$ the center eigenvalue. Moreover, again from the fact that Λ_f is topologically hyperbolic and transitive, follows that for any $z \in \Lambda_f$ there is $z' \in W^s(p_0) \cap W^u(p_0)$ arbitrarily close to z with the property that the orbit of z' is contained in V. Then we can take λ_c such that $0 < \lambda_c^1 < \lambda_c < 1$ such that associated to the transversal intersection z' of the stable and unstable manifold of p_0 it is possible to get a periodic point with center eigenvalue smaller than λ_c and orbit in V. To do that, it is only necessary to get a periodic point that expends large part of the orbit close enough to the orbit of p_0. For more details see the proof of lemma 4.2.1.

■

We take the set of periodic point

$$Per_{\lambda_c}(f/V) = \{q \in Per(f) : O(q) \subset V, \, |\lambda_c(q)| < \lambda_c\}$$

i.e.: $Per_{\lambda_c}(f/V)$ is formed by the periodic points in V with center eigenvalue smaller than λ_c. By the previous lemma we get that $\Lambda_f \subset Closure(Per_{\lambda_c}(f/V))$.

For g C^k- close to f we take the sets

$$\Lambda_g(V_i) = Closure(\cap_{\{n\in\mathbb{Z}\}} g^n(V_i)) \;\; i = 1, 2$$

¿From lemma 7.5.1 follows that for any g there is a dominated splitting

$$E_1(g) \oplus E_2(g) \oplus E_3(g)$$

such that $E_1(g)$ and $E_3(g)$ are contractive and expansive respectively in $\Lambda_g(V_i)$, $i = 1, 2$.

Proposition 7.6 *There exists a neighborhood $\mathcal{U}$ of f and a pair of neighborhood $V \subset V_2$ of Λ such that for any $g \in \mathcal{U}$ follows that there is a continuous map*

$$h_g : Closure(Per_{\lambda_c}(f/V)) \to \Lambda_g(V_2)$$

that conjugate g with f ; i.e.: $h_g \circ f = g \circ h_g$. Moreover the map $g \to h_g$ is continuous.

To prove the proposition, we prove that: *orbits in Λ_f can be shadows by orbits in Λ_g.*

Observe that the set Λ_g is not necessary expansive; in fact it could happen that its center manifold is not necessary stable. However, it is possible to show the existence of the map. The proof, use strongly the fact that the direction E_3 is hyperbolic in the set Λ_f. The map h_g it is defined first on $Per_{\lambda_c}(f/V)$, i.e.: it is defined over the hyperbolic periodic points in a neighborhood of Λ_f that exhibits a good rate of contraction along the center direction and latter extended to the closure. The proof of the previous proposition follows from the lemma 7.5.3 that allows to define the maps h_g over periodic points. In what follows we denote $\Lambda_g(V_1)$ with Λ_g.

Lemma 7.5.3 *Let Λ_f be the set previously defined. Then, for any $\lambda_c < 1$ and $d_0 > 0$ there exists a neighborhood $\mathcal{U} = \mathcal{U}(\lambda_c, f)$ of f such that for every periodic point q in $Per_{\lambda_c}(f/V)$ and any $g \in \mathcal{U}$ follows that there exists the analytic continuation q_g of q and $dist(g^i(q_g), f^i(q)) < d_0$.*

Now we proceed to state a proposition that states the existences of a semi-conjugacy over H_p for any g close to f.

Proposition 7.7 *There exists a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{F}$ follows that there is a continuous map*

$$\hat{h}_g : \cap_{\{n>0\}} g^n(U) \to H_p$$

that conjugate g with f ; i.e.: $\hat{h}_g \circ g = f \circ \hat{g}_g$. Moreover the map $g \to \hat{h}_g$ is continuous.

Proof:

Recall that f is expansive and by lemma 7.4.5 there exists an adapted metric such that it is a hyperbolic metric for f. Using this, and the fact that we are dealing with a homoclinic class which is topologically hyperbolic, the proof of the shadowing lemma for hyperbolic sets with local product structure can be pushed in the present case.

In other words, there exist $\alpha > 0$ and $\beta > 0$ such that if $\{x_i\}$ is a $\beta-$pseudo orbit (meaning that for all i holds that $|x_{i+1} - f(x_i)| < \beta$) then there is a unique x such that $|f^n(x) - x_n| < \alpha$. Then, observe that for g close to f and U_0 a small neighborhood of H_p, follows that $\{g^n(z)\}$ is a $\beta-$pseudo orbit, then there is unique x such that the orbit of x by f shadows the orbit of z by g. We can define a map $\hat{h}_g(z)$ from $\cap_{\{n>0\}} g^n(U)$ (where U is an small neighborhood of H_p) to H_p as the unique point in H_p that shadows the orbit of z by g.

■

Observe that the map $\hat{h}_g$ is not necessary injective. In fact, it could happen for instance that after the perturbation, a periodic point q of f bifurcates either along the center manifold or along the unstable one, in two periodic points with orbits that remains close. In this case, the orbit of this two periodic points are shadowed by the orbit of q. In the proposition 7.6 we state that restricted to the set Λ_f, it is possible to define a continuous inverse to the map $\hat{h}_g$.

Now we are in condition to prove how lemma 7.5.3 and proposition 7.7 implies the proposition 7.6.

Proof of proposition 7.6. Lemma 7.5.3 and proposition 7.7 implies proposition 7.6:

Using the lemma 7.5.3 we define h_g over the set $Per_{\lambda_c}(f/V_1)$ in the way that giving $q \in Per_{\lambda_c}(f/V_1)$ it is taken $h_g(q)$ as the unique analytic continuation of q. Using lemma 7.5.2, the map is extended to the closure. To check that the map is continuous observe that

$$h_g = \hat{h}_g^{-1}/|Closure(Per_{\lambda_c}(f/V_1))$$

Since $\hat{h}_g$ is continuous and $Closure(Per_{\lambda_c}(f/V_1))$ is compact, follows that h_g is continuous.

■

To prove lemma 7.5.3 we start with the following lemma which is a weak version of a shadowing lemma. As we said above, it is possible to shadows pseudo-orbits for f in a neighborhood of H_p with real orbits of f in H_p. It could occur that after

f is perturbed to get a new map g, the homoclinic class H_p does not remain expansive. However, if we restrict g to Λ_g follows that g is partially hyperbolic in Λ_g and some stable properties remains along the center direction. We show that this properties allows to obtain the following shadowing lemma:

Lemma 7.5.4 *For any $\gamma_0 > 0$ there exists a neighborhood $\mathcal{U} = \mathcal{U}(\gamma, f)$ of f, there exist positive constants α_0, β_0 and r_0 such that for any $g \in \mathcal{U}$ and $\alpha < \alpha_0$, there exists $\beta < \beta_0$ such that if $\{x_i\}$ is a $\beta-$pseudo orbit and $dist(x_i, \Lambda_g) < r_0$ then there is $x \in B_{r_0}(\Lambda_g)$ such that*

$$dist(g^n(x), x_n) < \alpha + \gamma_0$$

Observe that if Λ_g is hyperbolic, then γ_0 is zero. In the situation that we are dealing, γ_0 could be considered as the "error" performed by the shadow orbit due to the fact that the direction E_2 is not hyperbolic.

In the proof of the previous lemma, it becomes clear the assumption that E_3 restricted to Λ_f is uniformly expansive. Before to give the proof we state another lemma and a easy claims that allows to conclude the lemma 7.5.3. The next lemma states that for g close to f, the set Λ_g does not collapse.

Lemma 7.5.5 *Let Λ_f be the set previously defined. Let us assume that $f_{|\Lambda_f(V_1)}$ is Kupka-Smale. Then, for every r_0 there exists a neighborhood $\mathcal{U} = \mathcal{U}(f)$ of f, such that for any $g \in \mathcal{U}$ and $x \in \Lambda_f$ there exists $x' \in \Lambda_g$ such that $dist(x, x') < r_0$.*

Proof:

Let us assume that lemma is false. Then there is a sequence of diffeomorphisms g_n converging to f and points $x_n \in \Lambda_f$ such that $dist(x_n, \Lambda_{g_n}) > r_0$. Taking an accumulation point x of x_n follows that $dist(x, \Lambda_{g_n}) > \frac{r_0}{2}$ for n large. Recall that the closure of periodic points in $\Lambda_f(V)$ contains Λ_f. Then, we take a periodic point q close to x. Since we are assuming that they are hyperbolic, for g close enough to f follows that q has a continuation for g close to f and this continuation is close to q and therefore close to x. Which is a contradiction if g is one of the diffeomorphisms of the sequence

g_n.

■

Claim 6 *Given* $\delta_0 > 0$ *there exists* $\gamma_0 = \gamma(\delta_0)$ *and* $\mathcal{U}$ *such that if* $g \in \mathcal{U}$ *and* $dist(x, y) < \gamma_0$ *then*

$$\frac{|Df_{|E_2(x,f)}|}{|Dg_{|E_2(y,g)}|} < 1 + \delta_0$$

The claim follows from the fact that the subbundles moves continuously with g.

Now we are in condition to show the proof of lemma 7.5.3.

Proof of lemma 7.5.3. Lemmas 7.5.4, 7.5.5 and claim 6 imply lemma 7.5.3:

Let λ_c given by the lemma 7.5.2. Let us take δ_0 and $\lambda_1 < 1$ such that $\lambda_c(1+\delta_0) < \lambda_1 < 1$. Now we take the neighborhood $\mathcal{U}_0$ and the constant γ_0 given by claim 6. Now we take $\gamma_1 < \gamma_0$ and let us take the neighborhood $\mathcal{U}_1$ and the constants α_0, β_0, r_0 given by lemma 7.5.4. Let us choose $\alpha < \alpha_0$ such $\gamma_1 + \alpha < \gamma_0$. Then, let us take $\beta = \beta(\alpha)$ given by lemma 7.5.4. Now, taking β as before, let us consider the neighborhood $\mathcal{U}_2$ given lemma 7.5.5.

Now we take $\mathcal{U} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2$.

Let $g \in \mathcal{U}$ and let q be a periodic point of f in a neighborhood of Λ_f with central eigenvalue small than λ_c. Using the Pliss's lemma, we can assume that $|Df^k_{|E_2(q',g)}| < \lambda_c^k$ for all $k > 0$.

By lemma 7.5.4 follows that there exists q' in a neighborhood of Λ_g such that $dist(f^i(q), g^i(q')) < \gamma_1 + \alpha < \gamma_0$ and so $|Dg^k_{|E_2(q',g)}| < \lambda_1^k$ for all $k > 0$.

We claim that

$$g^{n_q}(q') \in W^{cs}_\epsilon(q', g)$$

In fact, if $g^{n_q}(q') \notin W^{cs}_\epsilon(q', g)$ then $W^{cs}_\epsilon(q', g) \cap [W^u_\epsilon(g^{n_q}(q'), g) \setminus \{g^{n_q}(q')\}] \neq \emptyset$. Let $z = W^{cs}_\epsilon(q', g) \cap W^u_\epsilon(g^{n_q}(q'))$. Since Λ_g is partially hyperbolic follows that there is a positive integer m such that $dist(g^m(z), g^m(g^{n_q}(q'))) > \epsilon$. Since $dist(q', q) < \gamma_0$, $dist(g^{n_q}(q'), q) < \gamma_0$ and $|Dg^k_{|E_2(q',g)}| < \lambda_1^k$ follows that $dist(g^m(q'), g^m(z)) < 2\gamma_0$. So,

$$dist(g^m(g^{n_q}(q')), f^m(q)) > dist(g^m(g^{n_q}(q')), g^m(z)) -$$

$$-dist(g^m(z), g^m(q')) - dist(g^m(q'), f^m(q')) > \epsilon - 3\gamma_0$$

Taking γ_0 sufficiently small, we get a contradiction because also holds that $dist(g^m(g^{n_q}(q')), f^m(q)) = dist(g^{m+n_q}(q')), f^{m+n_q}(q)) < \gamma_0$.

Using that $|Dg^k_{|E_2(q',g)}| < \lambda_1^k$ for all $k > 0$ and that $g^{n_q}(q') \in W^{cs}_\epsilon(q', g)$ follows that $g^{n_q}(W^{cs}_\epsilon(q')) \subset W^{cs}_\epsilon(q')$ and $\ell(g^n(W^{cs}_\epsilon(q'))) \to 0$. Therefore, there is a periodic point of period smaller or equal to n_q contained in $W^{cs}_\epsilon(q')$. Observe that this periodic point also shadows the point q so without loosing of generality we can assume that the periodic point is equal to q'. To check that the period is equal to n_q we argue by contradiction. If the period is n with $n < n_q$ let us take the point $f^n(q)$ and observe that $f^n(q)$ is close to q (recall that q' shadows q). Since the period of q is n_q with $n_q > n$ follows that $W^{cs}_\epsilon(q, f) \cap [W^u_\epsilon(f^n(q, f)) \setminus \{f^n(q')\}] \neq \emptyset$. Arguing as before, replacing q' by q and $g^{n_q}(q')$ by $f^n(q)$ we get a contradiction.

■

Now we proceed to prove lemma 7.5.4

Proof of lemmas 7.5.4:

First, we have to study how the dynamic of a perturbed map behave related to the distance introduce in lemma 7.4.5. Observe that the adapted metric not necessary is coming from a riemannian metric so even the distance along the center manifold are contracted exponentially this does not imply the the derivative is contractive. In particular, we cannot expect that a perturbation of the initial map contracts distances along the center manifold. However, some contraction is kept when the points are not close enough one to each other. This is the statement of the next lemma and we give the proof before continuing with the proof of lemma 7.5.4.

Lemma 7.5.6 *Let dist, r and λ the distances and the constants introduced in lemma 7.4.5. Then, for any $\gamma < r$ there exist a neighborhood $\mathcal{U}$ of f and λ_1 with $\lambda < \lambda_1 < 1$ such that for any $g \in \mathcal{U}$ if $y \in W^c_\epsilon(x, g)$ follows that:*

1. *if $dist(x, y) > \gamma$ then $dist(g(x), g(y)) < \lambda_1 dist(x, y)$;*
2. *if $y \in W^c_\epsilon(x, g)$ and $dist(x, y) < \gamma$ then $dist(g(x), g(y)) < \gamma$.*

Moreover, the distance dist remains hyperbolic along $E_1(g)$ and $E_3(g)$.

Proof:

The proof of this lemma follows from the fact that the tangent manifolds associated to diffeomorphisms close to f are closed in the distance obtained in lemma 7.4.5. In fact, for g C^1-close to f follows that if $y \in W^{cs}_\epsilon(x, g)$ then

$$dist(g(x), g(y)) < \lambda dist(x, y) + r'$$

where $r' = r'(|g - f|_1) > 0$ and is arbitrarily small if g is sufficiently close to f. So, there exists $\gamma = \gamma(r')$ with γ small if r' is small such that if $dist(x, y) > \gamma$ follows that

$$\lambda dist(x, y) + r' < \lambda_1 dist(x, y)$$

for some λ_1 verifying $\lambda + r' < \lambda_1 < 1$

■

Continuing with the proof of lemma 7.5.4, given γ_0 small, we take $\gamma < \gamma_0$. Then, given γ we take $\mathcal{U}$ and $\lambda_1 < 1$ given by lemma 7.5.6. Let us note λ_1 with λ.

Now, given α smaller than ϵ (ϵ is the size of the Local stable manifolds), we take β small such that:

1. $\lambda(\gamma_0 + \beta) + \beta < \gamma_0 + \beta$,
2. $\beta + \beta \Sigma_{i=0}^{\infty} \lambda_u^{-i} < \alpha$ where λ_u is the rate of expansion along the unstable direction over Λ_g for any g close to f.

Given $g \in \mathcal{U}$ and a $\beta-$pseudo orbit $\{x_n\}$, first we construct by induction a sequences $\{y_n\}$ such that

1. $y_{n+1} \in W_\epsilon^u(g(y_n), g) \cap W_\epsilon^{cs}(x_{n+1}, g)$,
2. $dist(y_{n+1}, x_{n+1}) < \gamma_0 + \beta$

For $n = 0$ we take $y_0 = x_0$ and we take $y_1 \in W_\epsilon^u(g(x_0), g) \cap W_\epsilon^c(x_1, g)$. Since $dist(g(x_0), x_1) < \beta$ follows that $dist(g(x_0), x_1) < \beta < \gamma_0 + \beta$.

Assuming that we have chosen y_n, we take

$$y_{n+1} \in W_\epsilon^u(g(y_n), g) \cap W_\epsilon^c(x_{n+1}, g)$$

We need to prove that $dist(y_{n+1}, x_{n+1}) < \gamma_0 + \beta$. Observe that if $dist(y_n, x_n) > \gamma$ then

$$dist(g(x_n), g(y_n)) < \lambda dist(x_n, y_n) < \lambda(\gamma_0 + \beta)$$

Recalling that $dist(g(x_n), x_{n+1}) < \beta$, and from the election of β follows that

$$dist(y_{n+1}, x_{n+1}) < \lambda(\gamma_0 + \beta) + \beta < \gamma_0 + \beta.$$

In case that $dist(y_n, x_n) < \gamma$ then

$$dist(g(x_n), g(y_n)) < \gamma < \gamma_0$$

so, again follows that

$$dist(y_{n+1}, x_{n+1}) < \gamma_0 + \beta.$$

Now we define

$$z_n = g^{-n}(y_n)$$

and observe that $z_n \in W^u_\epsilon(x_0, g)$. In fact, $y_n \in W^u_\beta(g(y_{n-1}), g)$ so

$$dist(g^{-1}(y_n), y_{n-1}) < \lambda_u^{-1}\beta$$

Arguing by induction, follows that the $dist(g^{-n}(y_n), y_0) < \beta\Sigma_{i=0}^n \lambda_u^{-i} < \alpha$. Taking x as an accumulation point of z_n, it is concluded that

$$dist(g^n(x), x_n) < \gamma_0 + \beta + \beta\Sigma_{i=0}^\infty \lambda_u^{-i} < \gamma_0 + \alpha$$

■

7.5.2 End of the proof of proposition 7.5.

Given $\delta > 0$ we take a point q_δ such that q_δ has $\frac{\delta}{3}-$weak contraction along the center direction. We consider an arbitrarily small open neighborhood $\mathcal{U} = \mathcal{U}(\delta) \subset Diff^1(M^3)$ of f such that for any $g \in \mathcal{U}$ follows that q_δ remains $\frac{\delta}{2}-$weak contractive and homoclinically related to p. Moreover, we take $\mathcal{U}$ in such a way that for any $g \in \mathcal{U}$ is well defined the map h_g. Related to the map h_g and the periodic points in $Per_{\lambda_c}(f/V_1)$ follows the next lemma.

Lemma 7.5.7 *For any $g \in \mathcal{U}$ and any $q \in Per_{\lambda_c}(f/V_1)$ follows that the $h_g(q)$ is homoclinically related with q_δ.*

Proof:

Let us suppose that the lemma is false. Then, there exists a sequences of periodic points $\{q_n\}$ of f in $Per_{\lambda_c}(f/V_1)$, such that for each q_n there is a diffeomorphisms g_n such that $h_{g_n}(q_n)$ is not homoclinically related with q_δ. Let us assume first that for any $h_{g_n}(q_n)$, the unstable manifold of q_δ does not intersect the stable manifold of $h_{g_n}(q_n)$. Taking an iterate of each point q_n if it is necessary, we can assume that $|Df^i_{|E_2(q_n)}| < \lambda_c^i$ for every $i > 0$. From the fact that the orbit of $h_{g_n}(q_n)$ remains close to the orbit of q_n follows that $|Dg^i_{n|E_2(h_{g_n}(q_n))}| < \hat{\lambda}_c^i$ for every $i > 0$ with $\lambda_c < \hat{\lambda}_c < 1$. This implies that there is ϵ_0 such that $W^c_{\epsilon_0}(h_{g_n}(q_n), g_n)$ is contained in the stable manifold of $h_{g_n}(q_n)$. Let us take take z_0 an accumulation point of the points q_n. There is a connected compact arc γ contained in the unstable manifold of q_δ such that intersect $W^{cs}_{\frac{\epsilon_0}{2}}(z_0)$. If $\mathcal{U}$ is small, on one hand follows that for any $g \in \mathcal{U}$, there is an arc $\gamma(g)$ close to γ contained in the unstable manifold of q_δ; on the other hand, follows that $h_{g_n}(q_n)$ is close to q_n and the local center stable manifold of $h_{g_n}(q_n)$ is close to the local center manifold of q_n which is close to the one of z_0. So, the center stable manifold of $h_{g_n}(q_n)$ intersect the arc $\gamma(g_n)$. Since the local center stable manifold of $h_{g_n}(q_n)$ is contained in the stable manifold of $h_{g_n}(q_n)$ we get a contradiction since we are assuming that the unstable manifold of q_δ does not intersect the stable of q_n.

To show that the stable manifold of q_δ intersect the unstable one of q_n we argue in the same way, using that for any q_n the local unstable manifold has a uniform size (recall that in the set that we are considering the direction E_3 is expansive).

■

Let us take the points x and y in Λ_f such that $y \in W^{ss}_\epsilon(x)$. Given $g \in \mathcal{U}$ let us take the points

$$x_g = h_g(x) \text{ and } y_g = h_g(y)$$

Recall that $g \to x_g$ and $y \to y_g$ are continuous with g.

To finish the proof of proposition 7.3, we consider the following options:

1. either there exists $g \in \mathcal{U}$ such that the points x_g and y_g verifies that

$$W^u_\epsilon(y_g) \cap W^{ss}_\epsilon(x_g) = \emptyset$$

2. or for all $g \in \mathcal{U}$ follows that

$$W^u_\epsilon(y_g) \cap W^{ss}_\epsilon(x_g) \neq \emptyset$$

In the former, we show that for some other $g \in \mathcal{U}$ the thesis of the proposition holds. In the later, we perform a suitable perturbation such that the unstable manifold of x_g and the unstable manifold of y_g for some g are not jointly integrable and from there again we conclude the thesis of the proposition. Before to start, we show that the point x and y can be taken in such a way that either x or y is not a boundary point. More precisely, using that the strong foliations are jointly integrable, we can prove the following lemma.

Lemma 7.5.8 *Let H_p be a topologically hyperbolic homoclinic class. Let also assume that the strong foliations are jointly integrable. Then, for any $y \in H_p$ one of the next options holds:*

1. *for any positive integer n_0 and a positive constant r, there exist positive integers n_1, n_2, n_3 such that*
   (a) *$n_i > n_0$ for $i = 1, 2, 3$,*
   (b) *$dist(f^{-n_i}(y), f^{-n_j}(y)) < r$ for $i, j = 1, 2, 3$,*
   (c) *the local unstable manifold of $f^{-n_1}(y)$ and $f^{-n_3}(y)$ intersects different connected components of $W^s_\epsilon(f^{-n_2}(y)) \setminus W^{ss}_\epsilon(f^{-n_2}(y))$;*

2. *$y \in W^u(q)$ for some periodic point q;*

3. *there exits a a pair of periodic points q_1 and q_2 such that the local strong stable manifold of each point intersect the unstable manifold of the other point.*

Proof:

The proof is similar to some part of the proof of proposition 7.3. If the first item does not hold, then follows that there exist n_1 and n_2 arbitrarily large such that either $f^{-n_2}(y) \in W^u_\epsilon(f^{-n_1}(y))$ or $[W^{ss}_\epsilon(f^{-n_2}(y)) \setminus \{f^{-n_2}(y)\}] \cap W^u_\epsilon(f^{-n_1}(y)) \neq \emptyset$. In the first case, follows that y belong to the unstable manifold of some periodic point. In the second case, from the joint integrability follows that $W^u_\epsilon(f^{-n_2}(y)) \subset W^{su}_\epsilon(f^{-n_1}(y))$ arguing as item *ii.i* of point 2.1 of proposition 7.3, the third situation follows.

■

Observe that if both points x and y verify either the second or third item of the previous lemma, then we proceed using the proposition 7.3.

So, we can assume that at least one of the points x, y such that $y \in W^{ss}_\epsilon(x)$, verifies the first item of the previous lemma. Moreover, we assume that the points x, y are contained in the set Λ given by theorem 7.1. Let us denote with Λ_f the set Λ.

¿From the fact that at least the point $y \in \Lambda_f$ verifies the first item of lemma 7.5.8 we can show that the point y is accumulated by periodic points in a neighborhood of Λ_f converging on y from both connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$. This is the statement of the next lemma:

Lemma 7.5.9 *Let $y \in \Lambda_f$ such that verifies the first item of lemma 7.5.8. Then, for any small open neighborhood V of Λ_f follows that there are periodic points in $\cap_{n\in Z} f^n(V)$ such that the local unstable manifold of these periodic points intersects different connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$.*

Proof:

Let us take a periodic point q_{21} with orbit in V and close to $f^{-n_1}(y)$ such that the local unstable manifold of q_{21} intersect the same connected components of $W^s_\epsilon(f^{-n_2}(y)) \setminus W^{ss}_\epsilon(f^{-n_2}(y))$ where the local unstable manifold of $f^{-n_1}(y)$ intersects $W^s_\epsilon(f^{-n_2}(y)) \setminus W^{ss}_\epsilon(f^{-n_2}(y))$. Let us take a periodic point q_{23} with orbit in V and close to $f^{-n_3}(y)$ such that the local unstable manifold of q_{23} intersect the same connected components of $W^s_\epsilon(f^{-n_2}(y)) \setminus W^{ss}_\epsilon(f^{-n_2}(y))$ where the local unstable manifold of $f^{-n_3}(y)$ intersects $W^s_\epsilon(f^{-n_2}(y)) \setminus W^{ss}_\epsilon(f^{-n_2}(y))$. So, observe that there are arcs γ_{21} and γ_{32} of the local unstable manifold of q_{21} and q_{32} such that $f^i(\gamma_{21}) \subset V$, $f^i(\gamma_{32}) \subset V$ for $0 \leq i \leq n_2$, and intersecting different connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$. Using a dense orbit in Λ_f (recall that Λ_f

is transitive), follows that there are discs D_{21} and D_{32} of the local stable manifold of q_{21} and q_{32} such that $f^{-i}(D_{21}) \subset V$ for $0 \leq i \leq k_2$ and k_2 large, $f^{-i}(D_{32}) \subset V$ for $0 \leq i \leq k_3$ and k_3 large, such that $f^{-k_2}(D_{21})$ and $f^{-k_3}(D_{32})$ intersect the local unstable manifold of y. Moreover, we can suppose that $f^{n_2}(\gamma_{21})$ intersects $f^{-k_2}(D_{21})$ and $f^{n_2}(\gamma_{32})$ intersects $f^{-k_3}(D_{32})$. Then, there homoclinic points z_{21} and z_{32} of q_{21} and q_{32} respectively, with orbits in V and such that their local unstable manifolds intersects the local stable manifold of y in different connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$. Then, we can get a pair of periodic point, each one arbitrarily closed to each homoclinic point. This conclude the proof.

■

As a consequences of previous lemma and using lemma 7.5.2 follows that we can assume that there are periodic points in $Per_{\lambda_c}(f/V_1)$ accumulating on y such that the local unstable manifold of these periodic points intersects different connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$. Recall, that for any $g \in \mathcal{U}$ the map h_g is also well defined over the periodic points in $Per_{\lambda_c}(f/V_1)$.

Now we start analyzing the first case.

1. There exists $g \in \mathcal{U}$ such that $W^u_\epsilon(y_g) \cap W^{ss}_\epsilon(x_g) = \emptyset$.

Lemma 7.5.10 *Let us assume that there exists $g \in \mathcal{U}$ such that $W^u_\epsilon(y_g) \cap W^{ss}_\epsilon(x_g) = \emptyset$ and $x_g \notin W^{su}_\epsilon(y_g)$. Then there exists $\hat{g} \in \mathcal{U}$ such that the thesis of proposition 7.5 holds for $\hat{g}$.*

Proof:

Let us consider a homotopy $\mathcal{F} = \{g_\eta\}_{0\leq\eta\leq 1}$ such that $g_\eta \in \mathcal{U}$ for any η, $g_0 = f$ and g_1 is the diffeomorphism in the hypothesis of the present lemma. For each $g \in \mathcal{F}$ let us consider x_g and let us take $W^{cu}_\epsilon(x_g)$. For each g follows that $W^{cu}_\epsilon(x_g) \setminus W^u_\epsilon(x_g)$ has two connected components that we note it as $L^+(x_g)$ and $L^-(x_g)$. Using that $W^{cu}_\epsilon(x_g)$ and $W^u_\epsilon(x_g)$ are continuous with g, for each g we can choose the connected components $L^\pm(x_g)$ in a way that they move continuously with g. We can suppose that

$$\Pi^{ss}_{g_1}(y_{g_1}) \in L^+(x_{g_1})$$

By lemma 7.5.9 there exist a pair of periodic points q_x and q_y of f such that q_x is close to x, q_y is close to y and such that

$$q_x \in Per_{\lambda_c}(f/V_1) \;\; q_y \in Per_{\lambda_c}(f/V_1)$$

Therefore, for each $g \in \mathcal{F}$ we have the points

$$h_g(q_x) \;\text{ and }\; h_g(q_y)$$

Moreover, we claim that these points can be choosen such that they verify:

1. $\Pi^{ss}(q_y) \in L^-(x)$;

2. for any $g \in \mathcal{F}$ follows that:

$$dist(\Pi^{ss}(q_x), x) < dist(\Pi^{ss}(q_y), \Pi^{ss}(y))$$

$$dist(x_g, \Pi_g^{ss}(h_g(q_x))) < \frac{dist(x_{g_1}, \Pi_{g_1}^{ss}(y_{g_1}))}{2}$$

$$dist(\Pi_g^{ss}(y_g), \Pi_g^{ss}(h_g(q_y))) < \frac{dist(x_{g_1}, \Pi_{g_1}^{ss}(y_{g_1}))}{2}$$

To check this election, recall that there are periodic points in $Per_{\lambda_c}(f/V_1)$ accumulating on y such their local unstable manifolds intersects different connected components of $W_\epsilon^s(y) \setminus W_\epsilon^{ss}(y)$. Then, we can take points q_x and q_y verifying the first item and the first inequality of the second item. To check the last one, recall that for each $g \in \mathcal{U}$ the map h_g is continuous, the map $g \to h_g$ and $g \to \Pi_g^{ss}$ are continuous so the family $\mathcal{F}$ is uniformly continuous.

Now, for each g we take

$$W_\epsilon^{cu}(x_g) \setminus \Pi_g^{ss}(W_\epsilon^u(h_g(q_x)))$$

and we note the both connected components with $L^+(h_g(q_x))$ and $L^-(h_g(q_x)))$. Again we can choose the connected components $L^\pm(h_g(q_x))$ in a way that they move continuously with g

¿From the second item follows that

$$\Pi_{g_1}^{ss}(h_{g_1}(q_y)) \in L^+(h_{g_1}(q_x))$$

¿From the first item and from the first inequality of second one, follows that

$$\Pi_{g_0}^{ss}(h_{g_0}(q_y)) \in L^-(h_{g_0}(q_x))$$

Using that the maps $g \to \Pi_g^{ss}(h_g(q_x))$ and $g \to \Pi_g^{ss}(h_g(q_y))$ move continuously with g follows that there is $\hat{g} \in \mathcal{F}$ such that

$$\Pi_{\hat{g}}^{ss}(q_y) \in \Pi_{\hat{g}}^{ss}(W_\epsilon^u(h_{\hat{g}}(q_x)))$$

i.e.:

$$W_\epsilon^{ss}(h_{\hat{g}}(q_y))) \cap W_\epsilon^u(h_{\hat{g}}(q_x)) \neq \emptyset$$

and so the lemma follows.

■

2. For every $g \in \mathcal{U}$ follows that $W^u_\epsilon(y_g) \cap W^{ss}_\epsilon(x_g) \neq \emptyset$.

Given the pair x, y, we can suppose that there is a periodic point p_0 close to them such that $W^u_\epsilon(x) \cap W^s_\epsilon(p_0) \neq \emptyset$ and $W^u_\epsilon(y) \cap W^s_\epsilon(p_0) \neq \emptyset$. We can assume that the point p_0 is fixed. We can take a disc D contained in $W^s_\epsilon(p_0)$ such that $W^u_\epsilon(x) \cap D \neq \emptyset$ and $W^u_\epsilon(y) \cap D \neq \emptyset$. We take the points

$$x^- \in W^u_\epsilon(x) \cap D, \text{ and } \quad y^- \in W^u_\epsilon(y) \cap D$$

Observe that it could occur that $x^- = x$ and $y^- = y$. We can also suppose that for any g close to f, the point p_0 remains fixed and the disc D remains contained in $W^s_\epsilon(p_0)$.

Now, for each $g \in \mathcal{U}$ we considerer the points

$$x^-_g = W^u_\epsilon(x_g, g) \cap D \text{ and } y^-_g = W^u_\epsilon(y_g, g) \cap D$$

If it holds that there is $g \in \mathcal{U}$ such that

$$y^-_g \notin W^{ss}_\epsilon(x^-_g)$$

then we prove proposition 7.8 that allows to prove proposition 7.5.

If it holds that for every $g \in \mathcal{U}$ holds that

$$y^-_g \in W^{ss}_\epsilon(x^-_g)$$

then there is performed a C^1-suitable perturbation (see proposition 7.10) to show that the strong foliation associated to these points are not jointly integrable and then we show that this implies the proposition 7.5.

2.1. There is $g \in \mathcal{U}$ such that $y^-_g \notin W^{ss}_\epsilon(x^-_g)$.

Proposition 7.8 *Let us assume that there exists $g \in \mathcal{U}$ such that $y^-_g \notin W^{ss}_\epsilon(x^-_g)$. Then, there exists $\hat{g} \in \mathcal{U}$ such that the proposition 7.5 holds for $\hat{g}$.*

Proof:

Let us consider a homotopy $\mathcal{F} = \{g_\eta\}_{0 \leq \eta \leq 1}$ such that $g_\eta \in \mathcal{U}$ for any η, $g_0 = f$ and g_1 is the diffeomorphism in the hypothesis of the present proposition. As in lemma 7.5.10 for each $g \in \mathcal{F}$ let us take $W^{cu}_\epsilon(x_g)$ and the connected components of $W^{cu}_\epsilon(x_g) \setminus W^u_\epsilon(x_g)$ that we note as $L^+(x_g)$ and $L^-(x_g)$.

By hypothesis, we are assuming that $y^-_{g_1} \notin W^{ss}_\epsilon(x^-_{g_1})$ and we can suppose that

$$y^-_{g_1} \in L^+_{g_1}(x_{g_1})$$

that implies that $dist(\Pi^{ss}_{g_1}(y^-_{g_1}), x^-_{g_1}) > 0$

Recalling that there are periodic points in $Per_{\lambda_c}(f/V_1)$ accumulating on y such that their local unstable manifolds intersect different connected components of $W^s_\epsilon(y) \setminus W^{ss}_\epsilon(y)$ follows that we can choose a periodic point q_y such that

$$\Pi^{ss}(q_y) \in L^-(x)$$

Now, we choose another periodic point q_x contained in $Per_{\lambda_c}(f/V_1)$ close to x

For each $g \in \mathcal{F}$ we take the point

$$h_g(q_x)^- = W^u_\epsilon(h_g(q_x), g) \cap D \text{ and } h_g(q_y)^- = W^u_\epsilon(h_g(q_y), g) \cap D$$

Moreover, we can chose the periodic points q_x and q_y such that

1. for any $g \in \mathcal{F}$ holds that

$$dist(\Pi^{ss}_g(h_g(q_y)^-), \Pi^{ss}_g(y^-_g)) < \frac{dist(\Pi^{ss}_{g_1}(y^-_{g_1}), x^-_{g_1})}{2}$$

2. for any $g \in \mathcal{F}$ holds that

$$dist(\Pi^{ss}_g(h_g(q_x)^-), x^-_g) < \frac{dist(\Pi^{ss}_g(h_g(q_y)^-), \Pi^{ss}_g(y^-_g))}{2}$$

For each $g \in \mathcal{F}$, we consider the point $h_g(q_x)$ and the connected components $L^+(\Pi^{ss}_g(h_g(q_x)))$ and $L^-(\Pi^{ss}_g(h_g(q_x)))$ of $W^{cu}_\epsilon(x_g) \setminus \Pi^{ss}_g(h_g(q_x))$

The election of the point implies that

$$\Pi^{ss}_{g_0}(h_{g_0}(q_y)^-) \in L^-(\Pi^{ss}_{g_0}(h_{g_0}(q_x)))$$

but

$$\Pi^{ss}_{g_1}(q^-_y) \in L^+(\Pi^{ss}_{g_1}(h_{g_1}(q_x)))$$

¿From the continuity of $g \to \Pi^{ss}_g$ follows that there is another $\hat{g}$ such that

$$\Pi^{ss}_{\hat{g}}(h_{\hat{g}}(q_y)^-) \in W^u_\epsilon(h_{\hat{g}}(q_x))$$

Using the definition of $h_g(q_y)^+$ and $h_g(q_x)^-$ follows that

$$h_{\hat{g}}(q_y)^- \in W^{ss}_\epsilon(h_{\hat{g}}(q_x)^-)$$

In other words, we get that there exists $\hat{g}$ such that the local unstable manifold of the periodic points $h_{\hat{g}}(q_x)$ and $h_{\hat{g}}(q_y)$ s-intersect each other.

To show that we can get a periodic point such that the local strong stable manifold intersect the local unstable of another periodic point, observe that there is a hyperbolic set $H_{h_{\hat{g}}(q_x)}$ that contains $h_{\hat{g}}(q_x)$, $h_{\hat{g}}(q_x)^- = D \cap W^u_\epsilon(h_{\hat{g}}(q_x), g)$, p_0, and $W^s_\epsilon(h_{\hat{g}}(q_x), g) \cap W^u_\epsilon(p)$. Moreover, it can also be assumed that $H_{h_{\hat{g}}(q_x)} \cap W^u_\epsilon(h_{\hat{g}}(q_y), g) = \emptyset$. So, it follows that there is a periodic point q_1 arbitrarily close to $h_{\hat{g}}(q_x)^-$ and with orbit uniform disjointed from $W^u_\epsilon(h_{\hat{g}}(q_y), g)$. So, it is possible to unfold the intersection between $W^u_\epsilon(h_{\hat{g}}(q_y), g)$ and $W^{ss}_\epsilon(h_{\hat{g}}(q_x)^-, g)$ in a such a way that for the perturbation follows that

$$W^u_\epsilon(h_{\hat{g}}(q_y), g) \cap W^{ss}_\epsilon(q_1)$$

Therefore, the proof of the proposition 7.5 is finished in this case.

■

2.2. For every $g \in \mathcal{U}$ holds that $y^-_g \in W^{ss}_\epsilon(x^-_g)$.

To deal with the case that for every $g \in \mathcal{U}$ follows that $y^-_g \in W^{ss}_\epsilon(x^-_g)$ we introduce some perturbations that allows to show that the strong manifolds associated to x^-_g and y^-_g are not jointly integrable. In other words, in this case we prove that there is g C^1-close to f such that

$$\Pi^{ss}_g(W^u_\epsilon(g(y_g))) \text{ does not coincide with } W^u_\epsilon(g(x_g))$$

After that, arguing in a similar way as in proposition 7.8 we consclude the proposition 7.5.

Proposition 7.9 *Let us assume that for every $g \in \mathcal{U}$ holds that $y^-_g \in W^{ss}_\epsilon(x^-_g)$ and there exists a diffeomorphisms g C^1-close to f and $r > 0$ such that for any $z \in W^u_r(g(y^-_g), g) \setminus \{g(y^-_g)\}$ follows that $W^{ss}_\epsilon(z, g) \cap W^u_r(g(x^-_g), g) = \emptyset$. Then, the proposition 7.5 follows.*

Proof:

Observe that the previous proposition implies that the local unstable manifold of y_g and x_g are not jointly integrable. To precise, the proposition 7.9 implies the next assertion:

Claim 7 *There is a compact disk D^* contained in the stable manifold of p_0 such that there exist x^*_g and y^*_g verifying:*

1. $x^*_g \in W^u_\epsilon(g(x_g), g) \cap D^*$, $y^*_g \in W^u_\epsilon(g(y_g), g) \cap D^*$
2. $y^*_g \notin W^{ss}_\epsilon(x^*_g, g)$.

In fact, recall that $W^u(p_0)$ intersect transversally the stable manifold of p_0 and so there are stable discs $\{D_n\}$ contained in the stable manifolds of p_0 such that the discs D_n converge to $W^s_\epsilon(p_0)$. Taking D^* close enough to $W^s_\epsilon(p)$ follows that for any $g \in \mathcal{U}$ follows that the disk $D^*(g)$ intersects the unstable manifold of size r of the points x_g^- and y_g^-.

To conclude the proof of the proposition, we repeat the proof of lemma 7.8 changing the points x_g^-, y_g^- by x_g^*, y_g^* and the disc D by D^*

■

Now, we have to show that there exists a diffeomorphisms g C^1-close to f Verifying the hypothesis of proposition 7.9. For that, first we introduce some coordinates nearby the point p_0.

Remark 7.4 Local coordinates. *Let us take the point p_0 such that $x^-, y^- \in D \subset W^s_\epsilon(p_0)$. We can assume that there is a C^k- map from a neighborhood B_0 of p_0 to a neighborhood $\tilde{B}_0$ of $(0,0,0)$ in R^3 such that*

1. *$H(W^s_\epsilon(p_0)) = \{\bar{z} = 0\}$,*
2. *$H(W^u_\epsilon(p_0)) = \{\bar{x} = 0, \bar{y} = 0\}$*
3. *given $v \in H(W^s_\epsilon(p_0))$ follows that $W^{ss}_\epsilon(v) = \{w : \bar{x}(w) = \bar{x}(v), \bar{z}(w) = 0\}$,*
4. *$H(D) \subset \{z = 0\}$;*
5. *given the point $f(x)$ then $H(W^u_\epsilon(f(x))) = \{z \in B_0 : \bar{x}(z) = \bar{x}(x), \bar{y}(z) = \bar{y}(x)\}$*
6. *given the point $f(y)$ then $H(W^u_\epsilon(y)) = \{z \in B_0 : \bar{x}(z) = \bar{x}(y), \bar{y}(z) = \bar{y}(y)\}$.*

We denote with $(\bar{x}(z), \bar{y}(z), \bar{z}(z))$ the $(\bar{x}, \bar{y}, \bar{z})$-coordinates of a point $z \in \tilde{B}_0$.

¿From now on, we fix constants $\eta_0^c > 0$, $\eta_0^{ss} > 0$ and for each $\eta_0^u > 0$ we consider two rectangles $R(\eta_0^u)$ and $\hat{R}(\eta_0^u)$ such that

$$H(f(x^-)) \in \hat{R} \subset R \subset B \,\text{and}\, H(f(y^-)) \notin R$$

We can assume that in the local coordinates

$$\hat{R}(\eta_0^u) = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x} - \bar{x}(H(f(x^-)))| < \frac{\eta_0^c}{2}; |\bar{y} - \bar{y}(H(f(x^-)))| < \frac{\eta_0^{ss}}{2}; |\bar{z}| < \frac{\eta_0^u}{2}\}$$

$$R(\eta_0^u) = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x} - \bar{x}(H(f(x^-)))| < \eta_0^c; |\bar{y} - \bar{y}(H(f(x^-)))| < \eta_0^{ss}; |\bar{z}| < \eta_0^u\}.$$

To avoid notation, we also note the rectangles $H^{-1}(R(\eta_0))$ and $H^{-1}(\hat{R}(\eta_0))$ with $R(\eta_0)$ and $\hat{R}(\eta_0)$.

Perturbation of the map f.

Now, given η_0, it is constructed a C^1-perturbation g of f with the property that $|f - g|_1 < \eta_0$ and the local unstable manifold of x_g and y_g are not jointly integrable.

Lemma 7.5.11 *Given $\eta_0 > 0$, follows that for any $\eta_0^u > 0$ there exists a C^1-diffeomorphism $g = g(\eta_0, \eta_0^u)$ such that the following properties hold:*

1. $|g - f|_1 = \eta_0$ *where* $|.|_1$ *is the* C^1-*norm,*

2. $g_{|R(\eta_0^u)^c} = f$,

3. *for every* $z \in \hat{R}(\eta_0^u)$ *follows that* $D_z g(0,0,1)$ *is collinear to the vector* $(0, \eta_0, 1)$

Proof:

Let us consider the map

$$H : \tilde{B}_0 \to B_0$$

given by the previous remark. First, we consider a perturbation of the identity map in B_0. We take the map

$$T(x, y, z) = (x, y + T_1(y)T_2(z), z)$$

for some appropriate maps T_1 and T_2 chosen latter.
Observe that

$$DT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + T_1'(y)T_2(z) & T_1(y)T_2'(z) \\ 0 & 0 & 1 \end{bmatrix}$$

We assume that

1. $T_1(y) = 0$ for $|y| > \eta_0^c$

2. $T_1(y) = 1$ for $|y| < \frac{\eta_0^c}{2}$

3. $|T_1'(y)| < \frac{2}{\eta_0^c}$

4. $|T_2'(z)| < \eta_0$ for any z

5. $T_2'(z) = 0$, $T_2(z) = 0$ for $|z| > \eta_0^u$

Observe that $|T_2(z)| < \eta_0^u \eta_0$ for any z. In fact $T_2(z) = \int_0^y T_2'(s)ds$ and the support of T_2' is contained in the interval $[-\eta_0^u, \eta_0^u]$.

So, taking η_0^u small enough, follows that $|T - I|_1 < \eta_0$.

To get the the map g it is enough to compose T with f; i.e.: it is taken the map

$$g = H^{-1} \circ T \circ H \circ f$$

■

Remark 7.5 *Let* $g = g(\eta_0, \eta_0^u)$ *as in the previous lemma and let* $z \in R$. *Then* $dist(g(z), f(z)) < \eta_0 \eta_0^u$

Proposition 7.10 *Given* $\eta_0 > 0$ *there exists* η_0^u *a diffeomorphisms* $g = g(\eta_0, \eta_0^u)$ *as in lemma 7.5.11 and* $r > 0$ *such that for every* $z \in W_r^u(g(y_g^-), g) \setminus \{g(y_g^-)\}$ *follows that* $W_\epsilon^{ss}(z, g) \cap W_r^u(g(x_g^-), g) = \emptyset$.

Observe that the proposition 7.10 implies the proposition 7.9.

To finish, we have to prove proposition 7.10. In this direction, first we need to compute how the strong stable manifold and unstable manifold changes for the perturbed maps $g = g(\eta_0, \eta_0^u)$ as the one in lemma 7.5.11. This is the goal of the next proposition. It states that the angle between the local unstable manifold of $f(x^-)$ and $g(x_g^-)$ is much larger than the angle between the local unstable manifold of $f(y^-)$ and $g(y_g^-)$. Moreover, it states, that the strong stable manifold remains close to the initial one.

Recall that using the map H introduced in remark 7.4 we can assume that we are in R^3. More precisely, we can do any computation for the map $H \circ g$, where $g = g(\eta_0, \eta_0^u)$ as in lemma 7.5.11.

Proposition 7.11 *Given* η_0 *small, there exists* η_0^u *and* $g = g(\eta_0, \eta_0^u)$ *as in lemma 7.5.11 such that there exists* θ_1, θ_2 *and* $r_0 > 0$ *such that*

1. $\theta_1 + \theta_2 < \frac{\eta_0}{2}$,
2. *for any* $z \in W_{r_0}^u(g(y_g^-), g)$ *follows that*

$$Slope((0,0,1), DH[E_3(z,g)]) < \theta_1$$

3. *for any* $z' \in W_\epsilon^{ss}(z, g)$ *with* $z \in\in W_{r_0}^u(g(y_g^-), g)$ *follows that*

$$Slope(DH[E_1(z', f)], (1,0,0)) < \theta_2$$

4. *for any* $z \in W_{r_0}^u(g(x_g^-), g)$ *follows that*

$$Slope((0,0,1), DH[E_3(z,g)]) \geq \frac{\eta_0}{2}$$

Proposition 7.11 implies proposition 7.10:

We consider

$$\hat{\Pi}_g^{ss} : B(f(x^-)) \to W_\epsilon^{cu}(f(x^-))$$

where $B(f(x^-))$ is a neighborhood of $f(x^-)$ that contains $f(y^-)$ and $\hat{\Pi}_g^{ss} = H \circ \Pi_g^{ss}$. Given a point $z \in R^3$ a vector $v \in R^3$ and a positive constante θ we define the cone in z, direction v and amplitude θ in the following way:

$$C(z,v,\theta)=\{w\in R^3 : |w-(z+v)|<\theta\}$$

Using that $H[W^u_r(f(y^-),f)]=\{z\in B : \bar{x}(z)=\bar{x}(f(y^-)), \bar{y}(z)=\bar{y}(f(x^-))\}$ and from the second item of proposition 7.11 follows that

$$H[W^u_r(g(y^-_g),g)]\subset C(g(y^-_g),(0,0,1),\theta_1)$$

From the third item of proposition 7.11 and that for any $v\in W^s_\epsilon(p_0)$ holds that
$W^{ss}_\epsilon(v)=\{w : \bar{x}(w)=\bar{x}(v), \bar{z}(w)=0\}$, follows that for any $z\in H[W^u_r(g(y^-_g),g)]$ holds that

$$H[W^{ss}_\epsilon(z,g)]\subset C(z,(1,0,0),\theta_2)$$

From the last item of proposition 7.11 and the definition of the map $g=(\eta_0,\eta^u_0)$ as in lemma 7.5.11 follows that

$$H[W^u_r(g(x^-_g),g)]\subset C(y^-g,(0,0,1),\frac{\eta_0}{2})^c$$

Then, follows that

$$\Pi^{ss}(H[W^u_r(g(y^-_g),g)])\subset C(y^-g,(0,0,1),\theta_1+\theta_2)\cap H[W^{cu}_\epsilon(f(x^-))]$$

$$\Pi^{ss}(H[W^u_r(g(x^-_g),g)])\subset C(y^-g,(0,0,1),\frac{\eta_0}{2})^c\cap W^{cu}_\epsilon(f(x^-))$$

and therefore, since $\theta_1+\theta_2<\frac{\eta_0}{2}$ follows that the cones $C(y^-g,(0,0,1),\theta_1+\theta_2)$ and $C(y^-g,(0,0,1),\theta_1+\theta_2)^c$ are disjoints and so the manifold cannot be jointly integrable.

■

To finish, we have to give the proof of proposition 7.11.

Proof of Proposition 7.11:

To prove it, we need a series of lemmas. The first one, it is a folklore results and it states that the strong stable foliation are Holder (see [HPS]). The second one estimates the distance between a point z and $h_g(z)$ for $z\in\Lambda_f$ and g as in lemma 7.5.11.

Lemma 7.5.12 *There exists $\alpha>0$ and a neighborhood $\mathcal{U}$ of f such that for any $g\in\mathcal{U}$, $z\in\Lambda_f(V)$ and $z'\in\Lambda_g(V)$ follows that*

$$Slope(E_3(z,f),E_3(z',g))<dist(z,z')^\alpha+|g-f|_1^\alpha$$

$$Slope(E_1(z,f),E_1(z',g))<dist(z,z')^\alpha+|g-f|_1^\alpha$$

Lemma 7.5.13 *Given η_0 follows that for any $\gamma_0 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ as in lemma 7.5.11 such that if $z \in \Lambda_f$ then*

$$dist(h_g(z), z) < \gamma_0$$

In few words, the previous lemma states that if the vertical size of the support of the perturbation is made extremely small (i.e.: η_0^u small), then the map h_g is extremely close to the identity, despite the fact the perturbation twist the vertical vector in a fix quantity (i.e.: η_0).

Proof of lemma 7.5.13:

We start with a claim that follows from the fact that f is expansive and topologically hyperbolic.

Claim 8 *For any $\gamma_1 > 0$ there exists $N = N(\gamma_1)$ such that for any $z \in H_p$ and $z' \in U$*

1. *if $dist(f^n(z), f^n(z')) < \epsilon$ for all $0 \leq n \leq N$ then $dist(z', W^{cs}_\epsilon(z)) < \gamma_1$;*
2. *if $dist(f^n(z), f^n(z')) < \epsilon$ for all $-N \leq n \leq 0$ then $dist(z', W^u_\epsilon(z)) < \gamma_1$.*

The next claim follows from the fact that the local unstable and local stable manifold are transversal:

Claim 9 *There is a constant c such that if $dist(z', W^{cs}_\epsilon(z)) < r$ and $dist(z', W^u_\epsilon(z)) < r$ with r small, then $dist(z, z') < c.r$. In what follows, we assume that $c = 1$*

The next lemma follows from the fact that if η_0^u is small, then the points in $R(\eta_0^u)$ are close to the stable manifold of p_0.

Claim 10 *For any positive integer M there exists η_0^u such that for $g = g(\eta_0, \eta_0^u)$ as in lemma 7.5.11 follows that if $z \in \Lambda_f$, $h_g(z) \in R(\eta_0^u)$ and $g^n(h_g(z)) \in R(\eta_0^u)$ then $n > M$.*

Now we continue with the proof of the lemma 7.5.13.

We take $\gamma_1 > 0$ smaller than γ_0. Let $N(\gamma_1)$ be the positive integer given by the first claim. Now we choose η_0^u such that

1. $\gamma_1 + \eta_0 \eta_0^u < \gamma_0$ and
2. if $h_g(z) \in R(\eta_0^u)$ and $g^n(h_g(z)) \in R(\eta_0^u)$ then $n > N(\gamma_1)$.

Let $z \in \Lambda_f$, then if n_z^+ and n_z^- are such that $g^{n_z^+}(h_g(z)) \in R(\eta_0^u)$ and $g^{-n_z^-}(h_g(z)) \in R(\eta_0^u)$ follows that either $n_z^+ > N(\gamma_1)$ or $n_z^- > N(\gamma_1)$. Let us suppose that $n_z^+ > N(\gamma_1)$. Observe that $g^i(h_g(z)) \notin R(\eta_0^u)$ for $0 \leq i < n_z^+$ so $g^i(h_g(z)) = f^i(h_g(z))$ and $dist(f^i(h_g(z)), f^i(z)) < \epsilon$. Then,

$$dist(h_g(z), W_\epsilon^{cs}(z)) < \gamma_1 < \gamma_0$$

Now, let us consider the points $f^{-n_z^-}(z)$ and $g^{-n_z^-}(h_g(z))$. Observe that from the fact $g^{-n_z^-}(h_g(z)) \in R(\eta_0^u)$ follows that the number of backward iterates to visit again $R(\eta_0^u)$ is larger than $N(\gamma_1)$ and therefore

$$dist(g^{-n_z^-}(h_g(z)), W_\epsilon^u(f^{-n_z^-}(z))) < \gamma_1$$

By remark 7.5 follows that

$$dist(g(g^{-n_z^-}(h_g(z))), W_\epsilon^u(f(f^{-n_z^-}(z))) < \gamma_1 + \eta_0\eta_0^u$$

So,

$$dist(h_g(z), W_\epsilon^u(z)) < \gamma_1 + \eta_0\eta_0^u < \gamma_0$$

Therefore, we conclude that the distance $dist(h_g(z), W_\epsilon^{cs}(z)) < \gamma_0$ and $dist(h_g(z), W_\epsilon^u(z)) < \gamma_0$, so by claim 9 follows that

$$dist(h_g(z), z) < \gamma_0$$

■

Lemma 7.5.14 *Given η_0 follows that for any $\gamma_1 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ such that if $z \notin f(B_0) \cap B_0$ then*

$$Slope(E_3(z, f), E_3(h_g(z), g)) < \gamma_1$$

Proof:

First observe that the splitting in H_p can be extended continuously to the neighborhood U of H_p. Using this, observe that

$$Sl(E_3(h_g(z), g), E_3(z, f)) < Sl(E_3(h_g(z), g), E_3(h_g(z), f)) + \\ + Sl(E_3(h_g(z), f), E_3(z, f))$$

and by lemma 7.5.12 follows that

$$Sl(E_3(h_g(z), f), E_3(z, f)) < dist(h_g(z), z)^\alpha$$

From lemma 7.5.13 follows $dist(h_g(z), z)$ can be taken arbitrarily small if η_0^u is sufficiently small; therefore, to conclude the proof we only need to bound

$$Sl(E_3(h_g(z), f), E_3(z, f))$$

Let N_0 be the minimum positive integer such that $g^{-N_0}(z) \in R(\eta_0^u)$. Observe that if η_0^u is small, by the fact that $z \notin f(B_0) \cap B_0$ follows that N_0 is large.

Let us take $E_3(g^{-N_0}(z), g)$ and $E_3(g^{-N_0}(z), f)$. Observe that if $x \notin R(\eta_0^u)$ follows that $Dg = Df$. Then, by the domination property and previous observation follows that

$$\begin{aligned} Sl(E_3(z,g)), E_3(z,f))) = \\ Sl(Dg^{N_0}(E_3(g^{-N_0}(z),g)), Df^{N_0}(E_3(g^{-N_0}(z),f))) < \\ \lambda^{N_0} Sl(E_3(g^{-N_0}(z),g), E_3(g^{-N_0}(z),f)) \end{aligned}$$

where λ is the constant of domination.

By lemma 7.5.12 follows that $Sl(E_3(g^{-N_0}(z), g), E_3(g^{-N_0}(z), f)) < |g - f|_1$.Then,

$$Sl(E_3(h_g(z), g) E_3(z, f)) < dist(h_g(z), z)^\alpha + |g - f|_1 \lambda^{N_0}$$

So, taking η_0^u such that N_0 is sufficiently large and $dist(h_g(z), z)$ is sufficiently small, it is concluded the proof.

■

In the sequel, we note with $W^u_{[x,x^-]}(x, f)$ the connected arc of $W^u_\epsilon(x)$ that contains x and x^- ($W^u_{[x_g,x_g^-]}(x_g, f)$ is the connected arc of $W^u_\epsilon(x_g)$ that contains x_g and x_g^-.)

Corollary 7.4 *Given η_0 follows that for any $\gamma_1 > 0$ there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ such that if $z \in W^u_{[x_g,x_g^-]}(x_g, g)$ then*

$$Sl(DH[E_3(z, g)], (1, 0, 0)) < \gamma_1$$

The same result follows replacing x_g by y_g.

Proof:

The proof is similar to the previous corollary using that if $z \in W^u_{[x_g,x_g^-]}(x_g, g)$ then if N_0 is the minimum positive integer such that $g^{-N_0}(z) \in R(\eta_0^u)$ follows that N_0 is large.

■

Lemma 7.5.15 *There exists $\alpha > 0$ such that given η_0 then for any $\gamma > 0$ follows that there exists η_0^u and $g = g(\eta_0, \eta_0^u)$ as in lemma 7.5.11 such that for any $z \in H_p \cap R(\eta_0^u)$ and $z' \in \cap_{\{n>0\}} g^n(U) \cap R(\eta_0^u)$ follows that*

$$Slope(E_1(z, f), E_1(z', g)) < dist(z, z')^\alpha + \gamma$$

Proof:

Let z and $z' \in R(\eta_0^u)$. First observe that

$$Sl(E_1(z, f), E_1(z', g)) < |(Df - Dg)(E_1(z, f))| + Sl(E_1(f(z), f), E_1(g(z'), g))$$

It follows that given $\theta > 0$ there exists η_0^u small such that

$$|(Df - Dg)(E_1(z, f))| < \theta$$

In fact, if η_0^u is small then $E_1(z, f)$ is close to the vector $(1, 0, 0)$ and so $|(Df - Dg)(E_1(z, f))|$ is small.

Now, observe that $f(z), g(z') \notin R(\eta_0^u)$ and as in the proof of lemma 7.5.14 we get that

$$\begin{aligned} Sl(E_1(g(z'), g), E_3(f(z), f)) < Sl(E_1(g(z'), g), E_1(g(z'), f)) + \\ + Sl(E_3(g(z'), f), E_3(f(z), f)) \end{aligned}$$

and $Sl(E_3(g(z'), f), E_3(f(z), f)) < dist(f(z), g(z'))^\alpha$ with $dist(f(z), g(z')) < dist(f(z), g(z)) + dist(g(z), g(z')) < \eta_0 \eta_0^u + C_0 dist(z, z')$.

Threfore, follows that

$$Sl(E_1(z, f), E_1(z', g)) < \theta + \eta_0 \eta_0^u + C_0 dist(z, z') + Sl(E_1(g(z'), g), E_1(g(z'), f))$$

To finish, we have to compute $Sl(E_1(g(z'), g), E_1(g(z'), f))$.

Given a positive integer M, there exists η_0^u and $N = N(\eta_0^u) > M$ such that if $g(z') \in f(R(\eta_0^u))$ follows that $g^i(z') \notin R(\eta_0^u)$ and $g^i(z') = f^i(z')$ for $1 \leq i \leq N$. This fact follows from the fact that if η_0^u is small then z' is close to the stable manifold of p_0.

Then using that $Df = Dg$ along the orbit $g^i(z') = f^i(z')$ for $1 \leq i \leq N$ follows that

$$\begin{aligned} Sl(E_1(g(z'), g), E_1(g(z'), f)) = \\ = Sl(Dg^{-N}(E_1(g^{N+1}(z'), g)), Dg^{-N}(E_1(g^{N+1}(z'), f))) < \end{aligned}$$

$$< \lambda^N Sl(E_1(g^{N+1}(z'), g), E_1(g^{N+1}(z'), f))$$

So, if η_0^u is large enough follows that N is large and then the lemma holds.

■

End of proof of proposition 7.11:

Now, we are in condition to finish the proof of proposition 7.11.

Given η_0, we take γ_1 such that $2\gamma_1 + \gamma_1^\alpha < \frac{\eta_0}{2}$.

By corollary 7.4 we can choose η_0^u and $g = g(\eta_0, \eta_0^u)$ such that

$$Sl(DH[E_3(x_g^-, g)], (0,0,1)) < \gamma_1$$

and so, by the construction of g follows that

$$Sl(DH[E_3(g(x_g^-), g)], (0,0,1)) > \eta_0 - \gamma_1 > \frac{\eta_0}{2}$$

Again by corollary 7.4 follows that

$$Sl(DH[E_3(y_g^-, g)], (0,0,1)) < \gamma_1$$

and therefore

$$Sl(DH[Dg(E_3(g(y_g^-), g))], (0,0,1)) < \gamma_1$$

By lemma 7.5.15 follows that for every $z' \in W_\epsilon^{ss}(z, f)$ with $z \in W_{r_0}^u(g(y_g^-), f)$ follows that

$$Slope((1,0,0), DH[E_1(z_g, g)]) < \gamma_1^\alpha + \gamma_1$$

Taking $\theta_1 = \gamma_1$ and $\theta_2 = \gamma_1 + \gamma_1^\alpha$ the proposition follows.

■

8 Proof of theorem 7.1

The proof of the theorem is based on the proof of the Theorem B in the paper [PS1] and also enunciated here in section 6.

We give the steps of the proof, we make the references to the lemmas of the cited paper and we give the proof of the lemma and definition which are different to the one given in [PS1].

To prove that there is a transitive invariant compact subset Λ such that $Df_{|E_3}$ restricted to Λ is uniformly expansive and $\mathcal{T}_\Lambda \neq \emptyset$, we take a compact invariant subset $\Lambda \subset H_p$ which is the minimal set, in the Zorn's lemma sense, such that Λ is not uniform hyperbolic. To prove the existence of this set, it is enough to show that given a sequences of nonhyperbolic compacts invariant sets $\{\Lambda_\alpha\}_{\alpha\in\mathcal{A}}$ ordered by inclusion follows that $\cap_{\alpha\in\mathcal{A}}\Lambda_\alpha$ is a nonhyperbolic compact invariant set.

Related to this set, we prove the following:

Proposition 8.1 *Let H_p be a maximal invariant topologically hyperbolic homoclinic class exhibiting a splitting $E_1 \oplus E_2 \oplus E_3$, such that it is not hyperbolic, $\mathcal{T} \neq \emptyset$ and the interior of $\mathcal{T}$ is empty. Then, the minimal nonhyperbolic set Λ is a compact invariant set, such that verifies:*

1. *it is transitive,*
2. *there is a pair of points $x, y \in \Lambda$ such that $y \in W^{ss}_\epsilon(x)$,*
3. *$Df_{|E_3}$ is expansive restricted to Λ.*

Observe that a similar result holds when we have a dominated splitting $E^s \oplus F$ where E^s is uniformly contractive and F has dimension one (all the time assuming that f is C^2, the periodic points are hyperbolic and the splitting holds in a homoclinic class). In the present case, we only can assume that there is an splitting $E \oplus F$ such that F has dimension one and the center manifolds tangent to the E direction are stable manifolds. We do not know if these hypothesis are enough to guarantee that F is uniformly expansive in H_p. However, in the case that the interior of $\mathcal{T}$ is empty, at least it is possible to find a subset Λ such that $\mathcal{T}_\Lambda$ is not empty and where the direction E_3 is uniformly expansive.

The first two items are easy to prove. From the fact that any proper compact subset of Λ is uniformly hyperbolic, it follows that Λ is transitive. In fact, if it is not the case, follows that for any $x \in \Lambda$ then $\alpha(x)$ and $\omega(x)$ are properly contained in Λ; so it follows that both sets are hyperbolic. This implies that for any $x \in \Lambda$ follows that $|Df^{-n}_{|E_3(x)}| \to 0$ as $n \to +\infty$ and $|Df^n_{|E_1\oplus E_2(x)}| \to 0$ as $n \to +\infty$ and therefore, Λ is hyperbolic, a contradiction.

To prove the second item of proposition 8.1, observe that from theorem 6.1 and theorem 6.2 follows that there is a pair of points $x, y \in \Lambda$ such that $y \in W^{ss}_\epsilon(x)$ follows . In fact, if it is not the case, follows that the set Λ is hyperbolic.

So, it only remains to prove the last item of previous proposition 8.1. To do that, we find a set R such that for any $x \in R \cap \Lambda$ follows that $|Df^{-n}_{|E_3(x)}| \to 0$ as $n \to +\infty$, which would imply that E_3 is uniformly expansive in Λ. In fact, if $x \in \Lambda$ and $\alpha(x)$ (the $\alpha-$limit of x) is a proper subset of Λ, follows that $|Df^{-n}_{|E_3(x)}| \to 0$. If $\alpha(x)$ coincides with Λ, there is $k > 0$ such that $f^{-k}(x) \in R$ and so again $|Df^{-n}_{|E_3(x)}| \to 0$. Then, for any $x \in \Lambda$ follows that $|Df^{-n}_{|E_3(x)}| \to 0$

It remains the question if assuming that the interior of $\mathcal{T}$ is empty is possible to prove that E_2 is hyperbolic.

First we explicit a general strategy extracted from [PS1] about the properties of the mentioned set R. Latter we show that it is possible to follow this general strategy in the hypothesis of the theorem 7.1. The set R is some kind of rectangle in terms of the splitting $E_1 \oplus E_2 \oplus E_3$. So, we start defining a notion of rectangle.

Definition 17 *We say that a set R is a rectangle if*

$$R = int(h([-1,1]^3))$$

where $h : [-1,1]^3 \to M$ is an homeomorphism such that there exists points $x_{-1}, x_1, y_{-1}, y_1, z_{-1}, z_1$ in H_p verifying that

$$h(\{-1\} \times [-1,1]^2) \subset W^{su}_{\epsilon_0}(x_{-1}), \quad h(\{1\} \times [-1,1]^2) \subset W^{su}_{\epsilon_0}(x_1),$$

$$h([-1,1]^2 \times \{-1\}) \subset W^{cs}_{\epsilon}(y_{-1}), \quad h([-1,1]^2 \times \{1\}) \subset W^{cs}_{\epsilon}(x),$$

$$h([-1,1] \times \{-1\} \times [-1,1]) \subset W^{cu}_{\epsilon}(z_{-1}), \quad h([-1,1] \times \{-1\} \times [-1,1]) \subset W^{cu}_{\epsilon}(x)$$

See figure 12.

Given an open set R and point $x \in R$ we denote with $W^u_R(x)$ ($W^{ss}_R(x)$) be the connected component of $W^u_\epsilon(x)$ contained in R (the connected component of $W^{ss}_\epsilon(x)$ contained in R). Moreover, given an unstable segment J we define J_R as the connected component of J contained in R.

Definition 18 *Given a rectangle R, we define*

1. *the stable boundary as $\partial^{cs} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{cs}_R(x))$;*
2. *the strong stable boundary as $\partial^{ss} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{ss}_R(x))$;*
3. *the central boundary as $\partial^{c} R = \cup_{\{x \in R \cap H_p\}} \partial(W^{c}_R(x))$.*

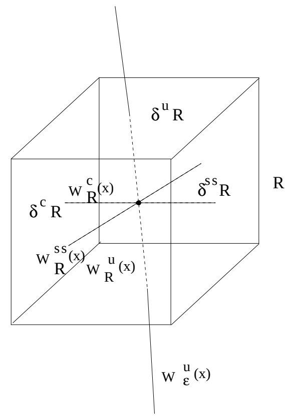


Figure 12

Definition 19 Adapted rectangle *Given a rectangle R we say that it is an adapted rectangle if for any $z \in \Lambda \cap R$ then*

1. *The unstable boundary is given*
2. *$W^u_R(z)$ is a connected component of $W^u_\epsilon(z)$ that intersects the two components of the unstable boundary of R;*
3. *for any positive integer n one of the following holds:*
   (a) *$f^{-n}(W^u_R(x)) \subset R$;*
   (b) *$f^{-n}(W^u_R(x)) \cap R = \emptyset$.*

Related to the notion of rectangle we define the notion of return maps.

Definition 20 Returns.
Let R be an adapted rectangle. A map $\psi : S \to R$ (where $S \subset R$) is called a return of R associated to Λ if:

- *$S \cap \Lambda \neq \emptyset$*
- *there exist $k > 0$ such that $\psi = f^{-k}_{/S}$*
- *$\psi(S) = f^{-k}(S)$ is a connected component of $f^{-k}(R) \cap R$*
- *$f^{-i}(S) \cap R = \emptyset$ for $1 \leq i < k$*

We denote the set of returns of R associated to Λ by $\mathcal{R}(R, \Lambda)$. Moreover, we define with R_ψ the image of Ψ and we say that a return $\psi \in \mathcal{R}(R, \Lambda)$ have $|\psi'| < \xi < 1$ if

$$|D^f - k_{|E_3} y| < \xi \text{ for all } y \in W^u_R(z), z \in dom(\psi) \cap \Lambda,$$

where $\psi = f^{-k}_{/dom(\psi)}$.

To show that if $x \in R$ then $|Df^{-n}_{|E_3(x)}| \to 0$, we prove the following proposition.

Proposition 8.2 *Let R be an adapted rectangle and assume that for every return $\psi \in \mathcal{R}(R,\Lambda)$ we have $|\psi'| < \xi < 1$ for some ξ. Then for all $y \in R \cap \Lambda$ the following holds:*

$$\sum_{n\geq 0} \ell(f^{-n}(J(y))) < \infty$$

$$|D^f - k_{|E_3} y| \to_{n\to\infty} 0.$$

Following this strategy, to conclude the theorem 7.1, it is enough to prove the following proposition.

Proposition 8.3 *Let Λ be the minimal non-hyperbolic set associated to H_p. Then, there exists an adapted rectangle R such that for every return $\psi \in \mathcal{R}(R,\Lambda)$ we have $|\psi'| < \xi$ for some $\xi < 1$.*

So, after showing the proposition 8.2, the goal is to build an adapted rectangle that verifies the hypothesis of 8.2. For that, it is built another kind of special rectangle called well adapted rectangle.

8.1 Proofs of proposition 8.2.

First, we start establishing the relation between sumability of the length of the unstable arcs and the hyperbolicity along the direction E_3. In other words, we show that if the sum of the length of the negative iterates of the unstable leaves is uniformly bounded then the derivative of f along the direction E_3 goes to zero for backward iterates. It is a general argument that follows from smoothness. In our case, since the map is C^2 and 2−domination holds, follows that the unstable discs are C^2. In fact:

Lemma 8.1.1 *There exists $\lambda < 1$ such that $\frac{|Df_{|E_1\oplus E_2}|}{|Df_{|E_3}|^2} < \lambda$. Then follows that the unstable discs $W^u_\epsilon(x)$ are C^2 for any x.*

The proof of the previous lemma is similar to the proof of lemma 6.0.2 ¿From the fact that the unstable arc are C^2 we can get the following lemma:

Lemma 8.1.2 *There exists a constant K_0 such that if $y \in W^u_\epsilon(x)$ follows that*

$$\frac{|Df^{-n}_{E_3(x)}|}{|Df^{-n}_{E_3(y)}|} \leq \exp(K \sum_{i=0}^{n_1} |f^{-i}(x) - f^{-i}(y)|)$$

Moreover

$$|Df^{-n}_{E_3(x)}| \leq \frac{\ell(f^{-n}(J_x))}{\ell(J)} \exp(K\sum_{i=0}^{n_1}\ell(f^{-1}(J)))$$

where $J \subset W^u_\epsilon(x)$.

Proof of proposition 8.2:

The proof is similar to the proof of lemma 3.7.2 given in [PS1] (page 10012) and the key argument is that in each return we have contraction along the E_3-direction, combined with the fact that the sum up to a return of the length of iterates of the unstable arc is uniformly bounded.

More precisely, it is necessary the following lemma which is useful also in the rest of the proof of proposition 8.3. The lemma state the uniform bound of the sum up to a return of the length of the unstable arcs. Moreover, state that the direction E_2 is contractive for sufficently large positive iterates.

Lemma 8.1.3 *Let R be an adapted rectangle. There exists $K_1 = K_1(R)$ such that if $x \in R$, $J = W^u_\epsilon(x) \cap R$ and $f^{-k_0}(J)$ is the first return of J to R then follows that*

$$\sum_{i}^{k_0} \ell(f^{-i}(J)) < K_1$$

Moreover, there exist a positive integer N_0, a positive constant C_0 and $\lambda_0 < 1$ such that if $k_0 > N_0$ then

$$|Df^i_{|E_2(f^{-k_0}(z))}| < C_0\lambda_0^i \quad \forall\, z \in J \; i > N_0$$

The proof of this lemma is similar to the proof of lemma 3.7.1 given in [PS1] (page 1010) and the key argument is the fact that the maximal invariant subset of Λ outside R, i.e.,

$$\Lambda_1 = \bigcap_{n\in Z} f^n(\Lambda - R)$$

is a proper set of Λ and so it verifies that it is a hyperbolic set. More precisely, if the previous set is empty, follows that for any point in R the return time are uniformly bounded, an so the lemma holds immediately. If the set it is no empty, it is possible to get a neighborhood of Λ_1 such that while the iterates remain in this neighborhood follows that the direction E_2 and E_3 are hyperbolic; moreover, the number of iterates that an orbit remains in the complement of the mentioned neighborhood of Λ_1 and R is uniformly bounded. ¿From these facts together follows the conclusion of the lemma.

After that, as we mentioned, to conclude proposition 8.2 we can repeat the arguments done in lemma 3.7.2 given in [PS1] (page 1012).

■

8.2 Proof of proposition 8.3.

This is done in different steps. First, we need more geometrical properties. More precisely it is introduced some kind of special type of adapted rectangle. This is shown in sub subsection 8.2.1 where also is proved the existences of this kind of rectangle in lemma 8.4. Latter, in subsection 8.2.2 are introduced some techniques called distortion, and which are useful to compare the volume of this rectangle to the length of the local unstable manifold and in subsection 8.2.3 it is study how the distortion changes under iterations. In subsection 8.2.4 it is ended the proof of proposition 8.3.

8.2.1 Well adapted rectangles.

Recall that we want to show the existence of a rectangle that verifies the hypothesis of proposition 8.2. To do that, we need some definitions (see figure 13).

Definition 21 Horizontal rectangle. *Given a rectangle R as the one defined in definition 17, we say that $R^h \subset R$ is an horizontal rectangle if there exist $[a,b] \subset [-1,1]$ such that $R^h = h([-1,1]^2 \times [a,b])$*

Definition 22 Vertical rectangle. *Given a rectangle R as the one defined in definition 17, we say that $R^v \subset R$ is a vertical rectangle if there exist $[a,b] \subset [-1,1]$ and $[c,d] \subset [-1,1]$ such that $R^h = h([a,b] \times [c,d] \times [-1,1])$.*

Remark 8.1 *Given an adapted rectangle R and a return ψ observe that its domain is a vertical rectangle and its image is contained in an horizontal rectangle. Moreover, if the domain is properly contained in R follows that the image is an horizontal rectangle.*

To check the remark, observe that if $x \in S$, where S is the domain of a return associated to a rectangle R, by the definition of adapted box follows that that $W^u_R(x) \subset S$.

Lemma 8.2.1 *Let R be an adapted rectangle. Then for every $\psi \in \mathcal{R}(R,\Lambda)$ follows that $R_\psi = Image(\psi)$ is an adapted rectangle.*

Proof:

Observe that by definition of R_ψ, the bottom and the top of it is given by the center stable manifold of some points in H_p. More precissely, the top and bottom of R_ψ are contained in a connected componet of $f^{-k}(W^{cs}_\epsilon(y_1)) \cap R$ and in a connected componet of $f^{-k}(W^{cs}_\epsilon(y_1)) \cap R$, where $f^{-k} = \psi$ and y_1, y_{-1} are the points such that their center stable manifolds contains the top and bottom of R. To finish, we have to check that if $x \in R_\psi$ follows that $f^{-n}(W^u_{R_\psi}(x)) \subset R_\psi$ or $f^{-n}(W^u_{R_\psi}(x)) \cap R_\psi = \emptyset$. If it is not the case, i.e.: if there is x, and a positive integer n such that $f^{-n}(W^u_{R_\psi}(x)) \cap R_\psi \neq \emptyset$ and $f^{-n}(W^u_{R_\psi}(x))$ is not contained in R_ψ, follows that $f^{-n}(W^u_{R_\psi}(x)) \cap \partial^u R_\psi \neq \emptyset$, and this implies, that $f^{k-n}(W^u_{R_\psi}(x)) \cap \partial^u R \neq \emptyset$ and since $f^k(W^u_{R_\psi}(x)) = W^u_R(f^k(x))$ follows that $f^{-n}(W^u_R(f^k(x))) \cap \partial^u R \neq \emptyset$ which is absurd since R is an adapted rectangle.

■

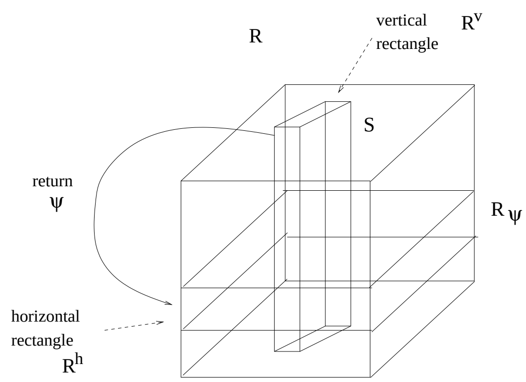


Figure 13

Definition 23 Well adapted rectangle. *Given a rectangle $R = h([-1,1]^3)$, we say that R is a well adapted rectangle if there is a positive integer N_0 such that $f^{-N_0}(W^s_\epsilon(p)) \cap R = h([-1,1]^2 \times \{1\}) \cup h([-1,1]^2 \times \{-1\})$ and there exists a rectangle $\hat{R}$ contained in R such that*

1. $\hat{R} = h([-1,1] \times [a,b] \times [-1,1])$ *for* $-1 < a < b < 1$;
2. $[R \setminus \hat{R}] \cap H_p = \emptyset$.

Moreover, there exist two vertical rectangles R^v_1, R^v_2 such that for each R^v_i follows that $\partial^c(R^v_i) \subset \partial^c(R)$ for $i = 1, 2$ and one of the following options holds:

1. *either* $[W^{ss}_R(R^v_i) \cup R^v_i] \cap H_p = \emptyset$
2. *or there is a horizontal rectangles R^h_i and a returns ψ_i, ψ_2 such that*
   (a) *R^v_i and R^h_i are the domain and image of ψ_1;*
   (b) $[W^{ss}_R(R^v_i) \setminus R^v_i] \cap H_p = \emptyset$ *for* $i = 1, 2$.

Observe that on one hand, if R is a well adapted rectangle then the strong stable boundary of R does not intersect Λ. On the other hand R is a well adapted rectangle if either the set Λ does not intersect the central boundaries of R or if it is not the case, the central boundary is contained in the domain of some return. See figure 14.

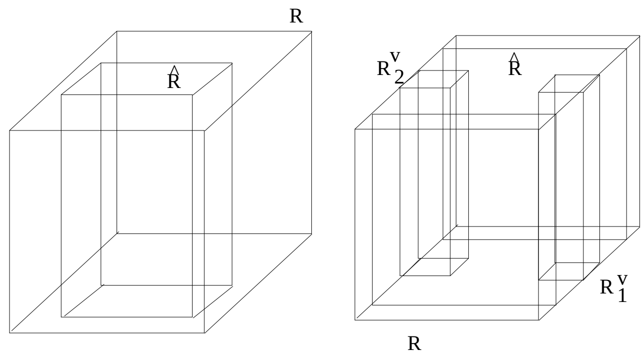


Figure 14

Lemma 8.2.2 *If R is a well adapted rectangle then it is an adapted rectangle.*

Proof: First, we have to check that if $x \in R \cap H_p$ then $W^u_R(x)$ is a connected component of $W^u_\epsilon(z)$ that intersects the top and the bottom of R. If $x \in R^v_1 \cup R^v_2$, follows from the definition. If $x \notin R^v_1 \cup R^v_2$, then $W^u_\epsilon(x) \cap R^v_1 \cup R^v_2 = \emptyset$. In other case, it would imply that $x \in R^v_1 \cup R^v_2$. So, if $W^u_R(x)$ is not a connected component of $W^u_\epsilon(x)$ that intersects the top and the bottom of R follows that $W^u_\epsilon(x)$ intersect $[W^{ss}_R(R^v_i) \setminus R^v_i]$ (for $i = 1$ or $i = 2$) which is an absurd because $W^u_\epsilon(x) \subset H_p$ and $[W^{ss}_R(R^v_i) \setminus R^v_i] \cap H_p = \emptyset$.

To check the second items in the definition of adapted box, observe that for any $x \in R$ and any positive integer k follows that $f^{-k}(W^u_R(x)) \cap \partial^u(R) = \emptyset$. If it is not the case, then follows that $f^k(f^{-N_0}(W^s_\epsilon(p))) \cap interior(R) \neq \emptyset$. Which is an absurd because $f^k(f^{-N_0}(W^s_\epsilon(p))) \subset f^{-N_0}(W^s_\epsilon(p))$ and $f^{-N_0}(W^s_\epsilon(p)) \cap R = h([-1,1]^2 \times \{1\}) \cup h([-1,1]^2 \times \{-1\})$.

■

Lemma 8.2.3 *Given a well adapted rectangle R and a return ψ, follows that R_ψ (the image of ψ) is a well adapted rectangle.*

The proof is immediately and it is similar to the proof of lemma 8.2.1.

Proposition 8.4 *Let H_p be a topologically hyperbolic attracting homoclinic class such that the interior of $\mathcal{T}$ is empty. Then, there exist a well adapted rectangle associated to Λ.*

To prove the proposition 8.4 first we use the following lemma. In this lemma it is used explicitly that the interior of $\mathcal{T}$ is empty

Lemma 8.2.4 *Let H_p be a topologically hyperbolic attracting homoclinic class such that the interior of $\mathcal{T}$ is empty. Then for every ϵ' there is $l_x^+ \subset W_{\epsilon'}^{ss,+}(x)$ such that $l_x^+ \cap H_p = \emptyset$ and there is $l_x^- \subset W_{\epsilon'}^{ss,-}(x)$ such that $l_x^- \cap H_p = \emptyset$, where $W_{\epsilon'}^{ss,+}(x)$ and $W_{\epsilon'}^{ss,-}(x)$ are the both connected components of $W_{\epsilon'}^{ss}(x) \setminus \{x\}$.*

Proof:

Let us assume that the thesis of the lemma is false; i.e: there is $x \in H_p$ such that for instance $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ for some small ϵ_x.

We consider two cases:

1. For some x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (some small ϵ_x), there is $y \in W_{\epsilon}^{ss,+}(x)$ such that $\Pi^{ss}(W_\epsilon^u(y))$ does not coincide with $W_\epsilon^u(x)$ or

2. For any x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (some small ϵ_x) holds that for any $y \in W_\epsilon^{ss,+}(x)$ follows that $\Pi^{ss}(W_\epsilon^u(y)) = W_\epsilon^u(x)$; i.e.: $W_\epsilon^u(y) \subset W_\epsilon^{su}(x) = \cup_{\{z \in W_\epsilon^{ss,+}(x)\}} W_\epsilon^u(z)$

In other words, we are considering if for any x such that $W_{\epsilon_x}^{ss,+}(x) \subset H_p$ (some small ϵ_x) given the point x then the strong foliation associated to x are either jointly integrable or it is not the case.

Let us take

$$W_\epsilon^{us}(x) = \cup_{\{z \in W_\epsilon^{ss,+}(x)\}} W_\epsilon^u(z)$$

From the fact that we are assuming that $W_\epsilon^{ss,+}(x)$ is contained in H_p and from the fact that H_p is an attractor, follows that $W_\epsilon^{us}(x) \subset H_p$.

Let us take a point z_0 in $W_\epsilon^{us}(x)$. Let us consider the set

$$W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$$

and observe that there is z_0 such that $W_\epsilon^{ss}(z_0)$ intersect transversally $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ in the sense, that $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ intersects both components of $W_\epsilon^s(z_0) \setminus W_\epsilon^{ss}(z_0)$. To check this assertion, it is enough to take z_0 such that that $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ intersects only one components of $W_\epsilon^s(z_0) \setminus W_\epsilon^{ss}(z_0)$ and $W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ it is not contained in $W_\epsilon^{ss}(z_0)$. This point z_0 exists because otherwise it follows that the strong foliations are jointly integrable. Then, it holds immediately that we can choose another point $z_0' \in W_\epsilon^s(z_0) \cap W_\epsilon^{us}(x)$ such that $W_\epsilon^{ss}(z_0') \cap W_\epsilon^{us}(x)$ intersect both components of $W_\epsilon^s(z_0') \setminus W_\epsilon^{ss}(z_0')$.

Now, let z be any point close to z_0 contained in $H_p \cap W_\epsilon^s(z_0)$, so it follows that $W_\epsilon^{ss}(z)$ intersect transversally $W_\epsilon^s(z) \cap W_\epsilon^{us}(x)$ (this follows from the fact that $W_\epsilon^{ss}(z)$ is C^1-close to $W_\epsilon^{ss}(z_0)$). Now we have two options: If for some z

close to z_0 holds that $z \notin W^{us}_\epsilon(x)$ or for any z close to z_0 holds that $z \in W^{us}_\epsilon(x)$. In the former, taking an small neighborhood of z follows that for any z' in this small neighborhood of z holds that $[W^{ss}_\epsilon(z') \setminus \{z'\}] \cap H_p \neq \emptyset$, and this implies that the interior of $\mathcal{T}$ is not empty, which is absurd. In the latter, from the fact that any point z close to z_0, follows that we can take a periodic point q such that $q \in W^{us}_\epsilon(x)$. We take $W^s_\epsilon(q)$ and $W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ and we can assume that $W^{ss}_\epsilon(q)$ intersect transversally the set $W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ (otherwise, using that the periodic points are dense it can be argued as before). Then we take $f^{n_q k}(W^s_\epsilon(q) \cap W^{su}_\epsilon(x))$ where n_q is the period of q and k is large positive integer. Observe that $f^{n_q k}(W^s_\epsilon(q) \cap W^{us}_\epsilon(x)) \subset W^s_\epsilon(q)$. Since $W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ intersect transversally $W^{ss}_\epsilon(q)$ follows that $W^s_\epsilon(q) \cap W^{us}_\epsilon(x)$ is not invariant by $f^{n_q k}$, then there is $z \in f^{n_q k}(W^s_\epsilon(q) \cap W^{us}_\epsilon(x)) \setminus [W^s_\epsilon(q) \cap W^{su}_\epsilon(x)]$ close to q. Taking an small neighborhood of z follows that for any z' in this small neighborhood holds that $[W^{ss}_\epsilon(z') \setminus \{z'\}] \cap H_p \neq \emptyset$, and this implies that the interior of $\mathcal{T}$ is not empty, which is absurd. See figure 20.

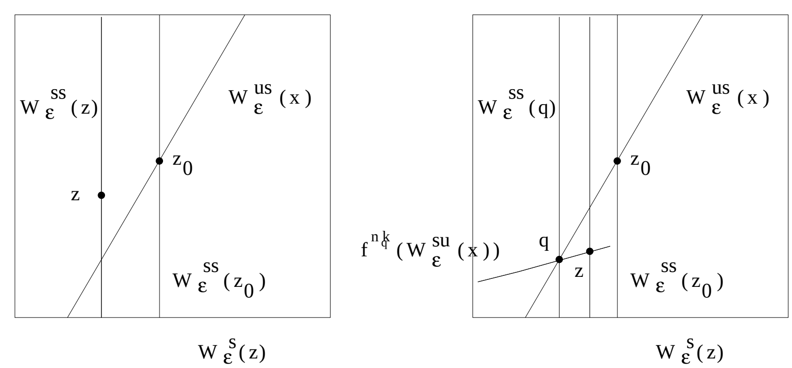


Figure 15

In the second situation we have that $W^{us}_\epsilon(x) \cap H_p = W^{su}_\epsilon(x)$. Let us consider

$$W^{us}(x) = Closure(\cup_{\{n>0\}} \cup_{\{y \in W^{ss}_\epsilon(x)\}} f^n(W^u_\epsilon(f^{-n}(y))))$$

Observe that for any $y \in W^{us}(x)$ follows that there is ϵ_y such that $W^{us}_{\epsilon_y}(y) = W^{su}_{\epsilon_y}(y) \subset W^{us}(x)$. Let us take also

$$\Lambda_0 = Closure(\cup_{\{y \in W^{us}(x)\}} \alpha(y))$$

Observe that Λ_0 is a topologically hyperbolic compact invariant set such that for any $z \in \Lambda_0$ follows that $W^u(z) \subset \Lambda_0$. To prove that, observe that there is $\epsilon' > 0$ such that for any $y \in W^{ss,+}_\epsilon(x)$ if n is large enough, then $W^{us}_{\epsilon'}(f^{-n}(y)) \subset f^{-n}(W^{us}(x))$.

From the fact that $W^u(z) \subset \Lambda_0$ for any $z \in \Lambda_0$, follows that Λ_0 has local product structure. So $\Lambda_0 = \cap_n f^n(V)$ for some V. Since it holds that whole unstable manifold of each points is contained in Λ_0 and the unstable manifolds are dense, follows that $H_p = \Lambda_0$.

On the other hand, observe that if $z \in \Lambda_0$, then $W^{us}_\epsilon(z) \subset \Lambda_0$ and $W^{us}_\epsilon(z) = W^{su}_\epsilon(z)$. It follows from the fact that $W^{us}_{\epsilon'}(f^{-n}(y)) \subset f^{-n}(W^{us}(x))$ and from the fact that for any $z \in W^{us}(x)$ there is ϵ_z such that $W^{us}_{\epsilon_z}(x) \subset W^{us}(x)$ and $W^{us}_{\epsilon_z}(x) = W^{su}_{\epsilon_z}(x)$. Therefore, $\mathcal{T} = H_p$ and so the interior is not empty, which is an absurd.

■

Now, we proceed to give the proof of proposition 8.4.

Proof of proposition 8.4:

Let us start taking a point $x \in H_p$. By the previous lemma, for any ϵ' there exist l^+_x and l^-_x contained in opposite connected components of $W^{ss}_{\epsilon'}(x) \setminus \{x\}$ and such that $l^+_x \cap H_p = \emptyset$ and $l^-_x \cap H_p = \emptyset$. So, there exists $\gamma_x > 0$ such that $W^c_{\gamma_x}(l^+_x) \cap H_p = \emptyset$ and $W^c_{\gamma_x}(l^-_x) \cap H_p = \emptyset$, where $W^c_{\gamma_x}(l^{+-}_x) = \cup_{\{z \in l^{+-}_x\}} W^c_{\gamma_x}(z)$. Let z^-_x and z^+_x (they depends on the point x) in opposite connected components of $W^c_\epsilon(x)$ such that $W^{ss}_\epsilon(z^-_x) \cap W^c_{\gamma_x}(l^+_x) \neq \emptyset$, $W^{ss}_\epsilon(z^+_x) \cap W^c_{\gamma_x}(l^+_x) \neq \emptyset$ and $W^{ss}_\epsilon(z^-_x) \cap W^c_{\gamma_x}(l^-_x) \neq \emptyset$, $W^{ss}_\epsilon(z^+_x) \cap W^c_{\gamma_x}(l^-_x) \neq \emptyset$. Let us consider the region B_x in $W^s_\epsilon(x)$ bounded by $W^{ss}_\epsilon(z^-_x)$, $W^c_{\gamma_x}(l^+_x)$, $W^{ss}_\epsilon(z^+_x)$ and $W^c_{\gamma_x}(l^-_x)$.

We take B^+_x and B^-_x the two connected components of $B_x \setminus W^{ss}_\epsilon(x)$.

We consider two cases:

1. there exist x such that, there exist y^-, y^+ in opposite connected components of $W^c_\epsilon(x) \setminus \{x\}$ such that $W^{ss}_\epsilon(y^-) \cap B_x \cap H_p = \emptyset$ and $W^{ss}_\epsilon(y^+) \cap B \cap H_p = \emptyset$;
2. for every x holds that either for every $y \in B^+_x$ follows that $W^{ss}_\epsilon(y) \cap H_p \cap B^+_x \neq \emptyset$ or for every $y \in B^-_x$ follows that $W^{ss}_\epsilon(y) \cap H_p \cap B^-_x \neq \emptyset$

First case.

In the first case, we claim that the we can built a rectangle as in the option one of lemma 8.4. To avoid notation in this part we do not write the dependence of the points on x.

In fact, let us take arcs l_{y^-} and l_{y^+} in opposite connected components of $W^c_\epsilon(x) \setminus \{x\}$ such that $W^{ss}_\epsilon(l_{y^-}) \cap B^s \cap H_p = \emptyset$ and $W^{ss}_\epsilon(l_{y^+}) \cap B^s \cap H_p = \emptyset$.

Then, there are arcs l_{y^-} and l_{y^+} in opposite connected components of $W^c_\epsilon(x) \setminus \{x\}$ such that $W^{ss}_\epsilon(l_{y^-}) \cap B^s \cap H_p = \emptyset$ and

$W^{ss}_\epsilon(l_{y^+}) \cap B^s \cap H_p = \emptyset$.

Now we take x^+_1, x^+_2, the boundary points of l^+_x; x^-_1, x^-_2, the boundary points of l^-_x; y^+_1, y^+_2, the boundary points of l_{y^+} and y^-_1, y^-_2, the boundary points of l_{y^-}. We order then by distances to the point x. We take the rectangle R^s bounded by $W^{ss}_\epsilon(y^-_1)$, $W^{ss}_\epsilon(y^+_1)$, $W^c_\epsilon(x^-_1)$ and $W^c_\epsilon(c^+_1)$. We take the rectangle $\hat{R}^s$ bounded by

$W^{ss}_\epsilon(y_2^-)$, $W^{ss}_\epsilon(y_2^+)$, $W^c_\epsilon(x_2^-)$ and $W^c_\epsilon(c_2^+)$. Observe that $R^s \subset \hat{R}^s$. Now, we take z^+ and z^- in opposite components of $W^u_\epsilon(x) \setminus \{x\}$. For each $z \in \hat{R}^s$ we define $W^u_{z^+,z^-}(z)$ as the connected component of $W^u_\epsilon(z) \setminus \{W^s_\epsilon(z^+) \cup W^s_\epsilon(z^-)\}$ bounded by $W^s_e(z^+)$ and $W^s_\epsilon(z^-)$. Now we define

$$\hat{R} = \cup_{\{z\in\hat{R}^s\}} W^u_{z^+,z^-}(z) \quad R = \cup_{\{z\in R^s\}} W^u_{z^+,z^-}(z)$$

Observe that these rectangles verified the items 1 and 2 of the proposition. To get that the bottom and the top are contained in the stable manifold of p and that R is adapted, we use the following claim:

Claim 11 *Let p be a periodic point. For every $\delta > 0$, there exist $N_0 = N_0(\delta)$ such that for every z follows that $f^{-N_0}(W^s_\epsilon(p))$ intersects both connected components of $W^u_\delta(x) \setminus \{x\}$.*

Then, using the previous claim, we can cut the rectangle by the stable manifold of a periodic point.

Second case.

We start with the following lema:

Lemma 8.2.5 *Assuming that we are in the second case follows that given two pair of points x_1 and x_2 in H_p such that $x_1 \in W^{ss}_\epsilon(x_2)$ follows that $W^u_\epsilon(x_1)$ and $W^u_\epsilon(x_2)$ cannot s-intersect transversally.*

Proof:

Let us assume that the lemma is false. Let $x_1' \in W^u_\epsilon(x_1)$ and $x_2' \in W^u_\epsilon(x_2)$ such that $x_2' \in W^{ss}_\epsilon(x_1')$ and for any r small follows that $W^u_r(x_1')$ manifold s-intersect transversally $W^u_r(x_2')$.

Let us take $B_{x_2'}$ small enough such that $x_1' \notin W^u_\epsilon(B_{x_2'}) = \cup_{\{z\in B_{x_2'}\}} W^u_\epsilon(z)$. Let us assume that for any $y \in B^+_{x_2'}$ follows that $W^{ss}_\epsilon(y) \cap B^+_{x_2'} \cap H_p \neq \emptyset$.

Let us take $W^{cu,+}_r(x_2') = \cup_{\{z\in W^{c,+}_\epsilon(x_2')\}} W^u_r(z)$ where $W^{c,+}_\epsilon(x_2')$ is the connected component of $W^c_\epsilon(x_2') \setminus \{x_2'\}$ that intersects $B^+_{x_2'}$. Le t us take

$$\Pi^{ss} : B_{x_2'} \to W^{cu}_\epsilon(x)$$

If r is small, observe that

$$W^{cu,+}_r(x_2') \subset \Pi^{ss}(W^u_\epsilon(B_{x_2'}) \cap H_p)$$

Moreover, form the fact that we are assuming that the local unstable manifold of x_2' and x_1' s-intersect transversally each other, follows that for some r' holds that

$$\Pi^{ss}(W^{u,+}_{r'}(x_1')) \subset interior(W^{cu,+}_r(x_2'))$$

where $W^{u,+}_{r'}(x'_1)$ is one of the connected components of $W^u_{r'}(x'_1) \setminus \{x'_1\}$.

Then, taking any point z_0 close to a point $z \in int(W^{u,+}_r(x'_1))$ such that $\Pi^{ss}(z)$ is in the interior of $W^{cu,+}_r(x'_2)$ follows that $[W^{ss}_\epsilon(z_0) \setminus \{z_0\}] \cap H_p \neq \emptyset$ which implies that the interior of $\mathcal{T}$ is not empty, which is an absurd.

■

Coming back to the proof of the proposition, we take a point x such that for every $y \in B_x$ follows that $W^{ss}_\epsilon(y) \cap H_p \cap B^+_x \neq \emptyset$. Let us define

$$R_x = \cup_{\{z \in B_x\}} W^u_\epsilon(z)$$

We can find two periodic points q_1 and q_2 with large period in each side of $R_x \setminus W^{su}_\epsilon(x)$ such that for each q_i holds that $dist(q_i, x) \leq dist(f^j(q_i), x)$ for any j. Take the connected component of $R_x \setminus W^{su}_\epsilon(q_1) \cup W^{su}_\epsilon(q_2)$ that contains x. Now, we can use the claim 11 and we cut this connected component by the stable manifold of the p. We claim that the remaining rectangle is a well adapted rectangle. To check that, first observe that it is an adapted rectangle and the proof is similar to the previous case. To check that it is well adapted, it is necessary to show that associated to each q_i it can be constructed a vertical rectangle with its associated return map. Firts, for each q_i, it is taken the connected component of $R'_i = f^{n_1}(W^{su}_\epsilon(q_i) \cap R) \cap R$ that contains q_i and where n_i is the period of q_i. Later, we take the connected component of $f^{-n_i}(W^c_\epsilon(R'_i) \cap R) \cap R$ that contains q_i. This connected component is the horizontal rectangle R^h_i; the vertical rectangle is $f^{n_i}(R^v_i)$ and the return $\psi_i = f^{-n_i}_{|R^v_i}$. This finish the proof of the proposition.

■

8.2.2 Rectangle: volume and length. Distortions.

Now we adapt to dimension three, a series of definition given in [PS1].

Definition 24 *We say that a rectangle R has distortion (or s-distortion) C if for any two intervals J_1, J_2 in R transversal to the $E_1 \oplus E_2$-direction and whose endpoints are in $\partial^u R$ the following holds:*

$$\frac{1}{C} \leq \frac{\ell(J_1)}{\ell(J_2)} \leq C.$$

Remark 8.2 *If a box has distortion C, then, for any $y, z \in R$*

$$\frac{1}{C} \leq \frac{\ell(W^u_R(z))}{\ell(W^u_R(y))} \leq C$$

and

$$\ell(W^u_R(z)) Area(W^{cs}_R(z)) < C Volume(R)$$

Notice that, in order to guarantee distortion C on a rectangle R, it is sufficient to find a C^1 foliation $\mathcal{F}^s$ by two dimensional embeddings with tangent planes close to the $E_1 \oplus E_2$-direction ($T_x\mathcal{F}^s$ lies in a b-cone for b small along the center-stable), such that, for any two intervals $J_1,\ J_2$ (taken as in the definition 24),

$$\frac{1}{C} \leq \|\Pi'\| \leq C$$

holds, where $\Pi = \Pi(J_1, J_2)$ is the projection along the foliation between these intervals.

Given a point $z \in R \cap \Lambda_0$, for any positive integer n we can take the rectangle R_n around $f^{-n}(x)$ defined as the connected component of

$$R_n = f^{-n}(R) \cap B_\epsilon(f^{-n}(x))$$

that contains $f^{-n}(x)$. Observe that inside R_n we can define the foliation $\mathcal{F}^s_n$ taking the negative iterations of the foliation $\mathcal{F}^s$; i.e.: given $z \in R_n$ we take the the connected component of

$$\mathcal{F}^s_n(z) = f^{-n}(\mathcal{F}^s(f^n(z))) \cap R_n$$

that contains z.

The following lemma will be useful in the sequel. The proof of this lemma is similar to the proof of lemma 3.4.1 in [PS1].

Lemma 8.2.6 *Let R, $\mathcal{F}^s$ and C be as above. If for any $z \in R_n$ follows that*

$$|Df^k_{T_x f^{-n}(\mathcal{F}^s_{f^n(x)})}| < C_0 \lambda_0^k$$

for any $N_0 \leq k \leq k_0$. Then R_n has distortion $C_1 = C_1(C, C_0, \lambda_0, N_0)$.

Applying previous lemma to lemma 8.1.3 we can conclude the following corollary:

Corollary 8.1 *Let R be an adapted box with distortion C. Then, for any $\psi \in \mathcal{R}(R, \Lambda)$ follows that R_ψ has distortion $C_1 = C_1(C, C_0, \lambda_0, N_0)$.*

8.2.3 Control of distortion.

The next lemma is similar to the lemma 3.7.3 of [PS1] (page 1014). However, in the present context the proof is simpler than the one done in [PS1] and use explicitly the properties of the adapted rectangle.

Lemma 8.2.7 *Let R be a well adapted rectangle. There exists $K = K(R)$ with the following property: for every $\psi \in \mathcal{R}(R, \Lambda)$ and $z \in B_\psi = Image(\psi)$ (denoting $J_\psi(z) = J(z) \cap B_\psi$) follows that*

$$\sum_{j=0}^{n} \ell(f^{-j}(J_\psi(z))) \leq K$$

whenever $f^{-j}(z) \notin R_\psi, 1 \leq j \leq n$.

Proof: Let R be a well adapted rectangle and let $\psi \in \mathcal{R}(R, \Lambda)$. Let $z \in R_\psi = Image(\psi)$ and a positive integer n such that $f^{-j}(z) \notin R_\psi, 1 \leq j \leq n$. Let also C_1 be as in corollary 8.1.3 and consider C_2 from corollary 8.1 (corresponding to $C_2 = C_1$). This means, that R_ψ has distortion C_2.

Let $0 < n_1 < n_2 < ... < n_s \leq n$ be the set $\{0 < j \leq n : f^{-j}(z) \in R\}$. For every n_i we have associated a return $\psi_i \in \mathcal{R}(R, \Lambda)$ such that $f^{-n_i}(z) \in R_{\psi_i}$, i.e., $f^{-n_i}(z) = \psi_i(f^{-n_{i-1}}(z))$ where $\psi = f^{-k_i}$ for some k_i.

We consider (if exists) the sequence $0 = m_0 < m_1 < m_2 < ... < m_l \leq n$ such that

$$|Df^j_{/E_1 \oplus E_2(f^{-m_i}(z))}| < \lambda_2^j,\ 0 \leq j \leq m_i,\ \forall i = 1, ..., l$$

We claim the following:

Claim 12 *There exists $C_4 = C_4(R)$ such that*

$$\sum_{i=0}^{l} \ell(f^{-m_i}(J_\psi(z))) \leq C_4$$

where $J_\psi(z) = W^u_\epsilon(z) \cap R_\psi$.

Proof of the claim: To show that, we construct a rectangle associated to each m_j. First we select a series of constants:

Recall that the rectangle R contains a sub rectangle $\hat{R}$ such that $[R \backslash \hat{R}] \cap H_p = \emptyset$. Let us take $\hat{\hat{R}} = \hat{R} \cup W^{ss}_\epsilon(R^v_1) \cup W^{ss}_\epsilon(R^v_2)$. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the following positive constants

$$\gamma_1 < \frac{1}{2} \min_{z \in \hat{\hat{R}}} dist(\partial^s(W^{cs}_\epsilon(z) \cap R), \partial^s(W^{cs}_e(z) \cap \hat{\hat{R}}))$$

$$\gamma_2 < \ell(f^{-n}(W^c_{\gamma_1}(z)) \cap W^c_\epsilon(f^{-n}(z)))\ \ \forall\ z \in H_p$$

$$\gamma_3 < \min_{z \in R^h_1 \cup R^h_2} \{\ell(W^c_{R^h_1}(z)), \ell(W^c_{R^h_2}(z))\}$$

$$\gamma_4 < \ell(f^{-n}(W^c_{\gamma_3}(z)) \cap W^c_\epsilon(f^{-n}(z))) \ \forall\ z \in H_p$$

Now, let

$$\gamma_0 = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$$

For each m_j, we take n_{i_j} of the sequences $\{n_1, ..., n_s\}$, such that $f^{-k}(z) \notin R$ for $n_{i_j} < k \leq m_j$. Given $z \in R_\psi$, we consider

$$i_0 = \min\{i_j > 0 : f^{-n_{i_j}}(z) \notin R^v_1 \cup R^v_2\}$$

and

$$j_0 = \min\{j : m_j \geq m_{i_0}\}$$

We assert first that in this case

$$\Sigma^l_{j=j_0} l(f^{-m_j}(J_\psi(z))) < K_1$$

To show that, we consider the rectangle $R(n_{i_j}) = R_{\psi_{i_j}}$ and we take the rectangle $R(j)$ as the connected component of

$$f^{-(m_j - n_{i_j})}(R(n_{i_j})) \cap B_{\gamma_0}(f^{-m_j}(J_\psi(z)))$$

that contains $f^{-m_j}(z)$.

On one hand, we show that for $j_1 \neq j_2$ and larger than j_0 follows that

$$R(j_1) \cap R(j_2) = \emptyset$$

On the other hand, from corollary 8.1 follows that R_i has distortion C_2 and so the area is compare to the length in the following way:

$$\ell(f^{-m_j}(J_\psi(z)))Area(W^{cs}_{R(j)}(f^{-m_j}(z)) < C_2 Vol(R(j))$$

and

$$Area(W^{cs}_{R(j)}(f^{-m_j}(z)) > \gamma_0$$

So,

$$\ell(f^{-m_j}(J_\psi(z))) < C_2 \frac{1}{\gamma_0} Vol(R(j))$$

Therefore

$$\sum_{j=j_0}^{l} \ell(f^{-m_j}(J_\psi(z))) \leq C_2 \frac{1}{\gamma_0} \sum_{j=0}^{j_0-1} Vol(R(j)) \leq \frac{1}{\gamma_0} C_2 K$$

where K is such that

$$\sum_{j=0}^{j_0-1} Vol(R(j)) < K$$

The constant K exists because the rectangle $R(j)$ are disjoint. So, the claim is proved.

So, to conclude that $\Sigma_{j=j_0}^{l} l(f^{-m_j}(J_\psi(z))) < K_1$, we have to show that the rectangles $R(j)$ are disjoint. To show that, first observe that if $f^{-n_i}(z) \in R$, then $f^{n_i}(W_R^s(f^{-n_i}(z)) \subset R_\psi$. Let us suppose that $R(j_1) \cap R(j_2) \neq \emptyset$. It follows that $W_{\gamma_0}^s(f^{-m_{j_1}}(z)) \cap f^{-m_{j_2}}(J_\psi(z)) \neq \emptyset$ so (assuming that $j_1 < j_2$) it follows that $W_{\gamma_0}^s(f^{-n_{i_0}}(z)) \cap f^{k_0-m_{j_2}}(J_\psi(z)) \neq \emptyset$ where $k_0 = m_{j_1} - n_{i_0}$. By the election of γ_0 and from the fact that $f^{-n_{i_0}}(z) \in R \setminus R_1^v \cup R_2^v$ follows that $f^{k_0-m_{j_2}}(J_\psi(z)) \subset R$. Then, $f^{k_0-m_{j_2}}(J_\psi(z)) \cap W_R^s(f^{-n_{i_0}}(z)) \neq \emptyset$ which implies that $f^{n_{i_0}}(f^{k_0-m_{j_2}}(J_\psi(z))) \subset R_\psi$, i.e: $f^{-(m_{j_1}-m_{j_2})}(J_\psi(z)) \subset R_\psi$ which is an absurd because the first return is $f^{-n}(z)$ and $m_{j_2} - m_{j_1} < n$.

Now, to finish the proof of the claim, we have to control the sum

$$\Sigma_{j=0}^{j_0} l(f^{-m_j}(J_\psi(z)))$$

In this case we have that $f^{-n_i}(z) \in R_1^v \cup R_2^v$ for any $i < i_0$. Observe that in particular $f^{-n_{i+1}}(z) \in R_1^v \cup R_2^v$ for any $i < i_0$

Define $B(n_i)$ as the connected component of $f^{-n_i}(R_\psi) \cap R_l^v$ which contains $f^{-n_i}(z)$, and l is equal to 1 or 2 depending if $f^{n_i}(z) \in R_1^v$ or $f^{n_i}(z) \in R_2^v$. Observe that, for $B(n_i)$ follows that

$$(*) \;\; f^{-k}(B(n_i)) \cap R = \emptyset \;\; \forall \; 0 < k < n_{i+1} - n_i$$

In this case, for each m_j such that $n_{i_j} < n_{i_0}$ we consider the rectangle $R(j)$ as the connected component of

$$f^{-(m_j - n_{i_j})}(B(n_{i_j})) \cap B_{\gamma_0}(f^{-m_j}(J_\psi(z)))$$

Again, we have that for this rectangle we can uniformly compare the length with the volume. So, we have to show that the rectangles $R(j)$ in this case are also disjoint. To show that, observe that if $R(j_1) \cap R(j_2) \neq \emptyset$ then $f^{-(m_{j_1}-n_{i_{j_1}})}(B(n_{i_{j_1}})) \cap f^{-(m_{j_2}-n_{i_{j_2}})}(B(n_{i_{j_2}})) \neq \emptyset$. Assuming that $m_{j_1} - n_{i_{j_1}} \leq m_{j_2} - n_{i_{j_2}}$ follows that $B(n_{i_{j_1}}) \cap f^{-k}(B(n_{i_{j_2}})) \neq \emptyset$ with $0 \leq k < m_{j_2} - n_{i_{j_2}} \leq n_{i_{j_2}+1} - n_{i_{j_2}}$. Which is a contradiction with $(*)$. Then, we have concluded that

$$\Sigma_{j=0}^{l} l(f^{-m_j}(J_\psi(z)) < K_1$$

■

To finish the proof of the lemma, we must control the sum between consecutive $m_i's$. Let $N = N(\lambda_1, \lambda_2)$ from Pliss's lemma and consider $K_2 = \sup\{\|Df^j\| : 1 \leq j \leq N\}$. There are two possibilities: $m_{i+1} - m_i < N$ or $m_{i+1} - m_i \geq N$. If $m_{i+1} - m_i < N$, then

$$\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \leq NK_2\ell(f^{m_i}(J_\psi(z))).$$

On the other hand, if $m_{i+1} - m_i \geq N$, then

$$|Df^j_{/E_1\oplus E_2(f^{m_i-j}(z))}| \geq \lambda_2^j \text{ for } N \leq j \leq m_{i+1} - m_i.$$

Thus, by the dominated splitting,

$$|Df^{-j}_{/E_3(f^{m_i}(z))}| \leq \lambda_1^j \text{ for } N \leq j \leq m_{i+1} - m_i.$$

Then, by Pliss's lemma, there exists $\tilde{n}_i, \tilde{n}_i - m_i < N$ such that

$$|Df^{-j}_{/E_3(f^{\tilde{n}_i}(z))}| \leq \lambda_2^j \text{ for } 0 \leq j \leq m_{i+1} - \tilde{n}_i$$

and so, for any $y \in f^{\tilde{n}_i}(J(z))$ we have, setting $\tilde{E}_2(y) = T_y f^{\tilde{n}_i}(J(z))$, that

$$|Df^{-j}_{/\tilde{E}_3(y)}| \leq \lambda_3^j \text{ for } 0 \leq j \leq m_{i+1} - \tilde{n}_i.$$

Hence

$$\begin{aligned}
\sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) &\leq \sum_{j=m_i}^{\tilde{n}_i-1} \ell(f^{-j}(J_\psi(z))) + \sum_{j=\tilde{n}_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \\
&\leq NK_2\ell(f^{-m_i}(J_\psi(z))) \\
&\quad + \sum_{j=0}^{m_{i+1}-\tilde{n}_i-1} K_2\ell(f^{-m_i}(J_\psi(z)))\lambda_3^j \\
&\leq \left(NK_2 + K_2\frac{1}{1-\lambda_3}\right)\ell(f^{-m_i}(J_\psi(z))).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j\geq 0} \ell(f^{-j}(J_\psi(z))) &= \sum_i \sum_{j=m_i}^{m_{i+1}-1} \ell(f^{-j}(J_\psi(z))) \\
&\leq \left(NK_2 + K_2\frac{1}{1-\lambda_3}\right) \sum_i \ell(f^{-m_i}(J_\psi(z))) \\
&\leq \left(NK_2 + K_2\frac{1}{1-\lambda_3}\right) K_1 = K_3.
\end{aligned}$$

Finally, if the sequence $m_i's$ does not exist, the same argument shows that

$$\sum_{j\geq 0} \ell(f^{-j}(J_\psi(z))) \leq \left(NK_2 + K_2\frac{1}{1-\lambda_3}\right)\ell(J_\psi(z))$$
$$\leq \left(NK_2 + K_2\frac{1}{1-\lambda_3}\right)L = K_4$$

where $L = \sup\{\ell(J_\psi(z)) : z \in R \cap \Lambda\}$. Taking $K = \max\{K_3, K_4\}$ we conclude the proof.

■

8.2.4 Finishing the proof of proposition 8.3.

We shall finish the proof of the proposition 8.3 in two cases: one, when Λ is not a topological minimal set, and the other when it is. Remember that a compact invariant set is topological minimal if it has no properly compact invariant subset, or equivalently, if any orbit is dense.

Case: Λ is not a minimal set

Lemma 8.2.8 *Let R be an adapted rectangle such that $\#\mathcal{R}(R, \Lambda) = \infty$. Then there exists a return $\psi_0 \in \mathcal{R}(R, \Lambda)$ such that the adapted rectangle $R_{\psi_0} = Image(\psi_0)$ satisfies the conditions of lemma 8.2, i.e., for every $\psi \in \mathcal{R}(R_{\psi_0}, \Lambda)$, $|\psi'| < \frac{1}{2}$ holds.*

The proof of the this lemma is similar to the proof of lemma 3.7.4 given in [PS1] (page 1016). The central idea is that if there are infinitely many returns, we can get one, namely ψ_0 such that $\frac{J_{\psi_0}(z)}{J_R(z)}$ is small, so $|\psi_0'|$ is small and then it is showed that for any ψ such that $\psi \in \mathcal{R}(R_{\psi_0}, \Lambda)$, follows that $|\psi'| < \frac{1}{2}$

Case: Λ is a minimal set.

The proof of the this lemma is similar to the proof for the minimal case proved in [PS1] (page 10018). However, we give some overview details. We begin remarking that we cannot expect to do the same argument here as in the preceding case, due to the fact that if Λ is a minimal set, then the set of returns to R is always finite. Nevertheless we shall exploit the fact that in the case Λ is a minimal set, then there are unstable "boundary points". First, we introduce some notations. Given an unstable arc J, we order J in some way and we denote $J^+ = \{y \in J : y > x\}$, $J^- = \{y \in J : y < x\}$. Also, giving $x \in R$ we shall denote by R^+ (say the upper part of the box) the connected component of $R - W^s_{\epsilon(x)}$ which contains J^+, and by R^- (the bottom one) the one containing J^-.

Lemma 8.2.9 *Assume Λ is minimal set. Then, reducing R in the unstable direction such that $R^+ \cap \Lambda = \emptyset$ or $R^- \cap \Lambda = \emptyset$.*

The idea to show that is that if the lemma does not follows, we would get that there is a periodic point in Λ which is a contradiction since Λ is minimal. See the proof of lemma 3.7.5 in [PS1] (page 1018).

■

Related to this rectangle we will get the following lemma that will imply the Main Lemma when Λ is minimal:

Lemma 8.2.10 *Let R be an adapted rectangle such that $R^+ \cap \Lambda = \emptyset$. Then there exist K such that for every $y \in R \cap \Lambda$,*

$$\sum_{j \geq 0} \ell(f^{-j}(J^+(y))) < K.$$

In particular there exist $J_1(y), J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty.$$

The proof of this lemma, use the following one:

Lemma 8.2.11 *Assume that Λ is a minimal set and let R be an adapted rectangle such that $R^+ \cap \Lambda = \emptyset$. Then R^+ verifies that for all $y \in R \cap \Lambda$,*

$$f^{-n}(J^+(y)) \cap R^+ = \emptyset \text{ or } f^{-n}(J^+(y)) \subset R^+$$

where $J^+(y) = J(y) \cap R^+$. Moreover, there exist K_1 such that if $y \in R \cap \Lambda$ and $f^{-j}(J^+(y)) \cap R^+ = \emptyset, 1 \leq j < n$ then

$$\sum_{j=0}^{n} \ell(f^{-j}(J^+(y))) < K_1.$$

Again, the proof are similar to the equivalent lemmas proved in [PS1] See the lemma 3.7.7 for the first and lemma 3.7.6 for the second one in [PS1] (page 1019).

Now we can prove the proposition 8.3 when Λ is a minimal set. Take

$$R_0 = \bigcup_{y \in B \cap \Lambda} J_1(y).$$

Notice that R_0 is an open set of Λ, and for every $y \in R_0 \cap \Lambda$ (i.e. $y \in J_1(y)$), we have

$$\sum_{n=0}^{\infty} \ell(f^{-n}(J_1(y))) < \infty$$

and so

$$|Df^{-n}_{/E_3(y)}| \to_{n\to\infty} 0.$$

Let z be any point in Λ. Since Λ is a minimal set there exist $m_0 = m_0(z)$ such that $f^{-m_0}(z) \in R_0$ and so

$$|Df^{-n}_{/E_3(f^{-m_0}(z))}| \to_{n\to\infty} 0$$

implying that

$$|Df^{-n}_{/E_3(z)}| \to_{n\to\infty} 0.$$

This completes the proof of the proposition 8.3.

9 Further results.

The next two theorems give a better description of the kind o homoclinic bifurcation that could follows in the case that the homoclinic class is not hyperbolic. This description it is related to the different kind of splitting that the attractor could exhibit. We state different theorems for the case that the point p has stable index either one or two.

Theorem B: *Let $f \in Diff^2(M^3)$ be a Kupka-Smale system.*
Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point of stable index one. Then, the following options holds:

1. *If H_p does not exhibit any dominated splitting, then there exists a g C^1- close to f such that g has a homoclinic tangency and a heterodimensional cycle in U.*

2. *If H_p does not exhibit any dominated splitting $E \oplus F$ with $dim(F) = 2$ then there exists a g C^1- close to f having a heterodimensional cycle and a homoclinic tangency in U.*

3. *If H_p has a dominated splitting $E \oplus F$ with $dim(F) = 2$ and F cannot be decomposed in two directions then follows that either*

    - *H_p is hyperbolic or*
    - *there exists a g C^1- close to f exhibiting a homoclinic tangency associated to a point of stable index one and exhibiting a heterodimensional cycle in U.*

4. *If H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that either*

    - *H_p is hyperbolic or*
    - *there exists a g C^1- close to f exhibiting a heterodimensional cycle in U.*

Remark 9.1 *Observe that in the previous theorem, any time that it can be created a tangency by a C^1-perturbation it also can be created a heterodimensional cycle.*

Let us assume now that p has stable index two. In this case it is not possible to get a strong version as in theorem B.

Theorem C: *Let $f \in Diff^2(M^3)$ be a Kupka-Smale system.*
Let $H_p = \cap_{n>0} f^n(U)$ be an attracting homoclinic class associated to a periodic point of stable index two. Then, the following options holds:

1. *If H_p does not exhibit any dominated splitting, then there exists a g C^1- close to f such that g exhibits a homoclinic tangency and a heterodimensional cycle in U.*

2. *If H_p has a dominated splitting $E \oplus F$ with dim$(E) = 2$ and E cannot be decomposed in two directions then follows that either*

   - *H_p is hyperbolic or*
   - *there exists a g C^1- close to f exhibiting a homoclinic tangency associated to a point of stable index one and exhibiting a heterodimensional cycle in U.*

3. *If H_p has a dominated splitting $E \oplus F$ with dim$(E) = 1$ such that F cannot be decomposed in two directions. Then follows that either:*

   (a) *there is a g C^1- close to f exhibiting a homoclinic tangency and a heterodimensional cycle in U;*

   (b) *all the periodic points in H_p has stable index two, E is uniformly contractive and one of the following options holds:*

      - *there exists a g C^1- close to f exhibiting a sectional dissipative homoclinic tangency in U and the set H_p is contained in a normally hyperbolic submanifold;*
      - *the set $\mathcal{T}$ is not empty, (i.e.: there exists x such that $[W^{ss}_\epsilon(x) \setminus \{x\}] \cap H_p \neq \emptyset$) and there exists a g C^1- close to f exhibiting a homoclinic tangency in U;*

4. *If H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that either*

   - *H_p is hyperbolic or*
   - *there exists a g C^1- close to f exhibiting a heterodimensional cycle in U.*

Remark 9.2 *To get a better description it remains the question that if in the case* 3.*b when $\mathcal{T}$ is not empty it also follows that a heterodimensional cycle can be created.*

Observe that from the maim theorem follows the last case of theorem B and C. In fact, if H_p has a dominated splitting $E_1 \oplus E_2 \oplus E_3$ then follows that it can not be created a tangency in U and so follows that either H_p is hyperbolic or there exists a g C^1- close to f exhibiting a heterodimensional cycle.

9.1 Proof of Theorem B:

9.1.1 There is not a dominated splitting.

Following the techniques in [BDP] it can be proved that if H_p has not a dominated splitting then there is a diffeomorphisms g close to f and a periodic point q with orbit arbitrarily close to H_p such that $Dg_q^{n_q}$ has three eigenspaces with arbitrarily small angle, having only real eigenvalues and at most one eigenvalue with modulus smaller than one.

From the fact that the angle between all the eigenspaces is small, follows that by a C^1-perturbation, a tangency can be created between the strong directions. From there, follows that we get a periodic point having a strong homoclinic connection. Moreover, and again from the fact the the angle between the eigenspaces is small, by another C^1-perturbation follows that the center eigenvalue has a weak expansion. Then, by perturbation follows the existences of a heterodimensional cycle.

9.1.2 There is not a dominated splitting $T_{H_p}M = E\oplus F$ with $dim(F) = 2$.

First we state a lemma that shows that if H_p is an attractor and p has stable index one, then it can be created a heterodimensional cycle.

Lemma 9.1.1 *Let us assume that H_p is an attractor. If p has stable index one, then there exist g arbitrarily C^1-close to f and a periodic point q of f such that the analytic continuation q_g is homoclinically connected with p_g and it exhibits a strong homoclinic intersection.*

Proof:

Let us assume first that p has real eigenvalues. In this case, we can show that the point p as the one that verifies the thesis of the lemma. In fact, from the fact that H_p is an attractor follows that the strong unstable manifold of p is contained in the homoclinic class. On the other hand, there are orbits in the homoclinic class accumulating in the stable manifold of p, which is one dimensional, so its coincide with the strong stable manifold. Then, using the C^1-connecting lemma we can perturb the systems in a way to connect the strong stable and unstable manifold of p.

If p has complex expanding eigenvalue we use the following lemma which is a consequences of lemma 2.1.7:

Lemma 9.1.2 *Let p be a periodic point with complex eigenvalue and stable index one. Let us assume that there is a transversal intersection of the stable and unstable manifold of p. Then, there exists a periodic point q of f and a diffeomorphisms g C^1-close to f such that q has real eigenvalues and the strong unstable manifold of q intersect the stable manifold of p.*

To conclude, if p has an expanding complex eigenvalue, first it is created a transversal homoclinic intersection and later it is applied the previous lemma. Now, observe that by theorem 2.1, if there is not a splitting $T_{H_p}M = E \oplus F$ with $dim(F) = 2$ then it implies that there exist g C^1-close to f and a periodic point q' of f such that the analytic continuation q'_g has a tangency between the stable and unstable manifold and q'_g. Using that we are dealing with homoclinic classes, it can be proved that q'_g is homoclinically connected with p_g. In particular, it is connected with the point q_g obtained in the previous lemma.

Unfolding the tangency, we can get another tangency and another periodic point $\hat{q}_g$ homoclinically connected with q'_g and q_g, and exhibiting a weak expansion along the center direction. Then, we have a periodic point with real eigenvalues having a weak expansion along the center direction and homoclinically connected with a periodic point having a strong homoclinic connection. Then, we can create a heterodimensional cycle applying the proposition 7.2 and keeping at the same time the tangency.

■

9.1.3 There is a dominated splitting $T_{H_p}M = E \oplus F$ with $dim(F) = 2$.

¿From theorem 2.1 follows that if F cannot be decomposed in two direction then either F is expansive or it can be created a tangency. Moreover, by lemma 2.1.2 applied to f^{-1}, follows that there is a periodic point of stable index one with real eigenvalues and having a weak expansion along the center direction. On the other hand, by lemma 9.1.2, we get a periodic point with real eigenvalues and exhibiting a strong homoclinic connection. Using proposition 7.2 it is conclude the existence of a heterodimensional cycle for a perturbation of the initial map.

To deal with the situation that the splitting is decomposed in three direction we proceed as in the proof of the main theorem.

9.2 Proof of theorem C.

In the case that there are not a dominated splitting we proceed as in the previous theorem.

9.2.1 There is a dominated $E \oplus F$ with $dim(E) = 2$, and E cannot be decomposed in two directions.

In this case, by theorem 2.1 follows that either E is contractive or there exists a g C^1- close to f exhibiting a homoclinic tangency. To finish the proof, remains to shaw that it can also be created a heterodimensional cycle. Since we are assuming that E cannot be decomposed in two invariant directions and E is not contractive, by lemma 2.1.2 follows that for any $\gamma > 0$ and $\delta > 0$ there exists a periodic point q for a diffeomorphisms g C^1-close to f such that

1. q has two real contractive eigenvalues;
2. $(1-\delta_2)^{n_q} < |Df^{n_q}_{|E^s_2(q)}| < 1$;
3. $\alpha(E^s_1(q), E^s_2(q)) < \gamma$
4. q has a transversal homoclinic point

Then, using that the angle between $E^s_1(q)$ and $E^s_2(q)$ it can be shown that after a second perturbation it is possible to get a strong homoclinic connection. Since one of the stable eigenvalues is close to one, then it can be performed a perturbation to get a heterodimensional cycle.

9.2.2 $T_{H_p}M = E \oplus F$ with $dim(E) = 1$.

Let us assume that there is a periodic point q in H_p with stable index one. using that E is onedimensional and therefore the local manifold tangent to E is dynamically defined, we can argue as in lemma 4.1.2 and it is proved that by a C^1-perturbation it is created a heterodimensional cycle. ¿From the fact that F cannot be decomposed in two direction, follows also the existences of a tangency.

Now we deal with the case that all the periodic points in H_p has stable index one. Firts it is proved that in this case, the direction E is uniformly contractive.

Lemma 9.2.1 *Let us assume that all the periodic points in H_p has stable index two. Then follows that for any $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that for any $x \in H_p$ and $n > n_0$ follows that*

$$\Pi_{i=0}^{n_1}|Df_{F(f^i(x))}| < (1+\delta)^n$$

Proof:

The proof is similar to the proof of the lemma 5.0.1. In fact, if the thesis does not hold, using the Pliss's lemma there it is possible to find a point x such that there is a sequences of integers $n_k \to +\infty$ such that

$$\Pi_{i=0}^{n_1}|Df_{F(f^{-i}(f^{-n_k}(x)))}| < (1+\delta)^{-n}$$

for some δ, any $n > n_0$ for some n_0 and any n_k. Observe that this implies that there exists $\epsilon_0 = \epsilon_0(\delta, f, n_0) > 0$ for any $f^{-n_k}(x)$ follows that

$$W^F_{\epsilon_0}(f^{-n_k}(x)) \subset W^u_{\epsilon_0}(f^{-n_k}(x))$$

where$W^F_{\epsilon_0}(y)$ is the manifold tangent to the direction F. Taking two integers n_{k_1} and n_{k_2} such that $f^{-n_{k_1}}(x)$ and $f^{-n_{k_2}}(x)$ are close, using that the manifold tangent to the direction E is dynamically defined (because the direction E is one dimensional), that we can find a periodic point q of stable index one close to $f^{-n_{k_1}}(x)$.

■

As a consequences of the previous lemma and using the domination property, follows that E is contractive.

Then, we have two options: either E is involved in the dynamics or it is not the case. In the second case, we can apply the theorem 6.1 and follows that there exists a C^1 two dimensional normally hyperbolic submanifold S such that the homoclinic class is contained in S. Since H_p is an attractor, follows that $Df_{|S}$ cannot be volume expanding. Moreover, follows that there is a periodic point q which is dissipative restricted to S. Since the direction E is contractive, follows that q is dissipative. From the fact that F cannot be decomposed in two direction having a dominated splitting, follows that there is g C^1-close to f exhibiting a tangency. Using that we are dealing with a homoclinic class, it is possible to show that the tangency is associated to a periodic point q' that remains homoclinically connected with q. Therefore, we can get a tangency associated to q, which is a sectional dissipative periodic point.

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