MAPS OF CONVEX SETS IN HILBERT SPACES

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Abstract. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{U} \subseteq \mathcal{H}$ an open convex subset, and $f : \mathcal{U} \to \mathcal{H}$ a smooth map. Let Ω be an open convex set in $\mathcal{H}$ with $\overline{\Omega} \subseteq \mathcal{U}$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathcal{H}$. We consider the following questions. First, in case f is Lipschitz, find sufficient conditions such that for $\varepsilon > 0$ sufficiently small, depending only on $\mathrm{Lip}(f)$, the image of Ω by $I \pm \varepsilon f$, $(I \pm \varepsilon f)(\Omega)$, is convex. Second, suppose $df(u) : \mathcal{H} \to \mathcal{H}$ is symmetrizable with $\sigma(df(u)) \subseteq (0, \infty)$, for all $u \in \mathcal{U}$, where $\sigma(df(u))$ denotes the spectrum of $df(u)$. Find sufficient conditions so that the image $f(\Omega)$ is convex. We establish results addressing both questions illustrating our assumptions and results with simple examples. We apply them to finite difference methods for approximating solutions of nonlinear ordinary differential equations in Hilbert spaces and also discuss the invariance of convex-valued maps from measure spaces into Hilbert spaces under certain nonlinear integral operators.

1. Introduction

In this paper we are concerned with the preservation of the convexity of bodies transformed by maps $f : \mathcal{U} \subseteq \mathcal{H} \to \mathcal{H}$ from an open convex set $\mathcal{U}$ of a separable Hilbert space $\mathcal{H}$ into $\mathcal{H}$. The results presented here generalize to the infinite dimensional setting those of [11, 12]. The first type of result we consider is related to Lipshitz maps. So, we assume that f is Lipschitz and, given an open convex Ω, with $\overline{\Omega} \subseteq \mathcal{U}$, we wish to find sufficient conditions on f and $\partial\Omega$ such that $(I \pm \varepsilon f)(\Omega)$ is convex, if $0 < \varepsilon < \varepsilon_0$, with ε_0 depending only on $\mathrm{Lip}(f)$. The link of this problem with the question of the invariance of convex sets under finite difference schemes for systems of conservation laws, not necessarily hyperbolic everywhere, was first discovered in [10].

As in [11], the most important assumption relating f and $\partial\Omega$ is that, for all ω at which $\partial\Omega$ is smooth, $df(\omega)(T_\omega(\partial\Omega)) \subseteq T_\omega(\partial\Omega)$, where $T_\omega(\partial\Omega)$ denotes the tangent space to $\partial\Omega$ at ω. As usual, most of the difficulty for the extension from the finite to the infinite dimensional case is, from the very beginning, to find suitable conditions that allow an adequate adaptation of the finite dimensional techniques to the more general infinite dimensional context. Here, we find necessary to impose the following new assumptions which involve the concept of what we call *standard Fredholm operators*. By this we mean an operator $T : \mathcal{H}_0 \to \mathcal{H}_0$, $\mathcal{H}_0$ a Hilbert space, such that $T = cI + K$, with $c \in \mathbb{R}$ and $K : \mathcal{H}_0 \to \mathcal{H}_0$ a compact operator. When $c \neq 0$ this concept coincides with the simplest example of the usual concept of Fredholm operator (see, e.g., [7]). Roughly speaking, if $\omega \in \partial\Omega$ and locally $\partial\Omega$ is given by the equation $G(v) = 0$, with $G : \mathcal{U} \to \mathbb{R}$, three times continuously Gateaux differentiable, $dG(\omega) \neq 0$, we assume that $df(\omega)|\mathcal{H}_0$ is a standard Fredholm

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operator and the symmetric bilinear forms $d^2G(\omega)|\mathcal{H}_0$ and $dG(\omega)d^2f(\omega)|\mathcal{H}_0$ are also represented by standard Fredholm operators, where $\mathcal{H}_0 = T_\omega(\partial\Omega)$.

The other type of result we consider is concerned with the case when df is symmetrizable everywhere in $\mathcal{U}$ and $\sigma(df(u)) \subseteq (0,\infty)$, for all $u \in \mathcal{U}$, where $\sigma(A)$ denotes the spectrum of the operator $A : \mathcal{H} \to \mathcal{H}$. The question then is to find sufficient conditions on f and $\partial\Omega$ such that $f(\Omega)$ is convex. In the finite dimensional context this question was first addressed by D. Serre [25], who discovered its connection with the question of the invariance of convex sets under continuous relaxation and kinetic approximations for systems of conservation laws. We illustrate our assumptions and results with simple examples and give applications to finite difference approximations of nonlinear ordinary differential equations in Hilbert spaces and to the invariance of convex-valued maps from measure spaces into Hilbert spaces under certain nonlinear integral operators arising in kinetic theory and others fields such as game theory and mathematical economics.

Our main result for the Lipschitz case raises an interesting question, left open, concerning the possibility of improving the upper bound for ε, $\varepsilon_0 = (2\mathrm{Lip}(f))^{-1}$, for which we guarantee the convexity of $(I \pm \varepsilon f)(\Omega)$, to some value in the interval $(\varepsilon_0, 2\varepsilon_0]$, in the interior of which $I \pm \varepsilon f$ is still a bi-Lipschitz diffeomorphism, or, otherwise, to prove the impossibility of such improvement. For our method, the upper bound $(2\mathrm{Lip}(f))^{-1}$ seems optimal. Another interesting question is the possibility of extending the results to the more general context of Banach spaces. In concluding this introduction, we would like to stress the fact that the extension to the infinite dimensional context of the results in [11, 12] provided here not only is far from being a trivial matter but also allows a much neater exposition of the key techniques set forth in the mentioned papers revealing in much clearer way, in particular, their strength.

The remaining of this manuscript is organized as follows. In section 2, we state our main assumptions (A1)-(A6), which will be in force through the whole paper, and establish the main result for the Lipschitz case mentioned above. In section 3, we deal with the symmetrizable case, establishing our corresponding main result. In section 4, we consider the application to finite difference approximations for ordinary differential equations in Hilbert spaces. Finally, in section 5, we consider the application to the invariance of convex-valued maps from measure spaces into Hilbert spaces under certain nonlinear integral operators arising in kinetic theory and others fields such as game theory and mathematical economics.

2. Lipschitz Maps of Convex Bodies

Let L be a real linear space. A subset S of a real linear space L is called convex if, for every pair p, q of its points, it contains the entire segment $[p,q] = \{\theta p + (1-\theta)q : 0 \le \theta \le 1\}$. A subspace V of L has codimension n if there exists a subspace $W \subseteq L$ of dimension n, with $V \cap W = 0$ and $L = V + W$. A hyperplane H in L is the translate of a subspace of codimension 1. If $l : L \to \mathbb{R}$ is a linear functional and $\alpha \in \mathbb{R}$, we denote by $[l = \alpha]$ the set of all points $x \in L$ for which $l(x) = \alpha$. We define analogously the sets $[l \ge \alpha]$ and $[l \le \alpha]$. It is well known that H is a hyperplane of L if and only if there is a linear functional $l : L \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $H = [l = \alpha]$.

H is called a supporting hyperplane of $S \subseteq X$ at the point $p \in S$ if $p \in H$ and S is entirely contained in one of the closed halfspaces bounded by H, that is, either $S \subseteq [l \geq \alpha]$ or $S \subseteq [l \leq \alpha]$, where $H = [l = \alpha]$.

Let $\mathcal{L}$ denote a real topological linear space, that is, a real linear space endowed with a Hausdorff topology with respect to which the operations $(\alpha, u) \mapsto \alpha u$ and $(u, v) \mapsto u + v$ are continuous from $\mathbb{R} \times \mathcal{L}$ to $\mathcal{L}$ and $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$, respectively. The following is a basic fact about convex sets. We refer to [27] for a proof.

Theorem 2.1 (Minkowski [20], Brunn [8], Klee [14]). *If S is a closed subset with nonempty interior in some real Hausdorff topological vectorspace $\mathcal{L}$, S is convex if and only if it possesses a supporting hyperplane at each of its boundary points.*

We say that the subset S of the real topological linear space $\mathcal{L}$ is *strongly locally convex at* $p \in \mathcal{L}$ if there exists a neighborhood U of p in $\mathcal{L}$ such that $S \cap U$ is convex. S is said to be *strongly locally convex* if it is strongly locally convex at each of its points. We recall the following fundamental result. Again, a proof may be found in [27].

Theorem 2.2 (Tietze [26], Klee [14]). *Let S be a closed connected subset of some real topological linear space $\mathcal{L}$. Then S is convex if and only if S is strongly locally convex.*

For many other facts about convex sets we refer to [3], [27], [22], [13] and the references therein.

In what follows we will be working in a real Hilbert space $\mathcal{H}$, that is, a real linear space endowed with an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, which is complete with respect to the metric induced by the norm $\|u\| = \langle u, u \rangle^{1/2}$. We say that $\mathcal{H}$ is separable if it possesses a countable dense subset.

So, we start by assuming:

(A1) $\mathcal{H}$ be a real separable Hilbert space and $\mathcal{U} \subseteq \mathcal{H}$ an open convex subset.

(A2) We consider functions $G_j : \mathcal{U} \to \mathbb{R}$, $j = 1, \ldots, N$, which are in $C^3(\mathcal{U})$, that is, they are 3 times continuously Gateaux differentiable in $\mathcal{U}$. Suppose 0 is a regular value for G_j.

Let

$$S_j = \{u \in \mathcal{U} \,:\, G_j(u) = 0\}, \qquad j = 1, \ldots, N. \tag{2.1}$$

We denote

$$\Omega_j = \{u \in \mathcal{U} \,:\, G_j(u) < 0\}, \qquad j = 1, \ldots, N.$$

We assume

(A3) Ω_j is strongly locally convex at each $\omega \in S_j$, $j = 1, \ldots, N$. If $T_\omega(S_j)$ denotes the tangent space to S_j at $\omega \in S_j$, this assumption is equivalent to the quasiconvexity condition:

$$d^2 G_j(\omega)(\xi, \xi) \geq 0, \qquad \text{for all } \xi \in T_\omega(S_j). \tag{2.2}$$

Let $f : \mathcal{U} \to \mathcal{H}$ be three times continuously Gateaux differentiable, i.e., $f \in C^3(\mathcal{U}, \mathcal{H})$. We now make our most important assumption. Namely:

(A4) For each $\omega \in S_j$, $\quad df(\omega)(T_\omega(S_j)) \subseteq T_\omega(S_j), \qquad j = 1, \ldots, N.$

Finally, set

$$\Omega := \cap_{j=1}^N \Omega_j,$$

and assume

(A5) $\Omega \neq \emptyset$ and $\overline{\Omega} \subseteq \mathcal{U}$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathcal{H}$.

The last assumption that we next state is only needed in the infinite dimensional context and involve the concept of standard Fredholm operator.

Definition 2.1. We will say that a linear operator T on a Hilbert space $\mathcal{H}_0$ is a *standard Fredholm operator* if $T = cI + K$, where $c \in \mathbb{R}$, I is the identity operator of $\mathcal{H}_0$, and K is a linear compact operator on $\mathcal{H}_0$. Also, if $U \subseteq \mathcal{H}_0$ is an open set, we will say that $f : U \to \mathcal{H}_0$ is a *standard Fredholm map* if $f = cI + g$ where $c \in \mathbb{R}$ and $g : U \to \mathcal{H}_0$ is a compact map, that is, g maps bounded sets onto relatively compact sets.

The motivation for the denomination in the above definition is just the fact that when $c \neq 0$ those operators satisfy the Fredholm alternative. Here, we also allow the case $c = 0$ when T is then simply a compact operator. We will use the following basic fact about standard Fredholm operators which follows immediately from the well known spectral theorem for compact symmetric operators (see, e.g., [7]).

Lemma 2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a standard Fredholm operator. Suppose T is symmetric, that is, $\langle T\xi, \eta\rangle = \langle \xi, T\eta\rangle$, for all $\xi, \eta \in \mathcal{H}$. Then there exists an orthonormal basis of $\mathcal{H}$, $\{e_1, e_2, \dots\}$, consisting of eigenvectors of T associated with real eigenvalues, i.e., $Te_j = \lambda_j e_j$, $j = 1, 2, \dots$.*

We also assume:

(A6) For each $j = 1, \dots, N$ and any $\omega \in S_j$, the linear maps $df(\omega)|\mathcal{H}_0, d^2G_j(\omega)|\mathcal{H}_0$, $dG_j(\omega)d^2f(\omega)|\mathcal{H}_0 : \mathcal{H}_0 \to \mathcal{H}_0$ are standard Fredholm operators on $\mathcal{H}_0 = T_\omega(S_j)$.

Here, for $\omega \in S_j$, we denote by $d^2G_j(\omega)|\mathcal{H}_0$ the symmetric linear operator on $\mathcal{H}_0$ such that

$$d^2G_j(\omega)(\xi, \eta) = \langle [d^2G_j(\omega)|\mathcal{H}_0]\, \xi, \eta\rangle, \qquad \text{for all } \xi, \eta \in \mathcal{H}_0, \tag{2.3}$$

and by $dG_j(\omega)d^2f(\omega)|\mathcal{H}_0$ the symmetric linear operator on $\mathcal{H}_0$ representing the symmetric bilinear form on $\mathcal{H}_0$ given by

$$dG_jd^2f(\omega)(\xi, \eta) := dG_j(\omega)\big(d^2f(\omega)(\xi, \eta)\big), \qquad \text{for all } \xi, \eta \in \mathcal{H}_0,$$

that is,

$$dG_jd^2f(\omega)(\xi, \eta) = \langle [dG_jd^2f(\omega)|\mathcal{H}_0]\, \xi, \eta\rangle, \qquad \text{for all } \xi, \eta \in \mathcal{H}_0. \tag{2.4}$$

We say that $\nu(\omega)$ is a vector in the outer normal cone of a convex set Ω at $\omega \in \partial\Omega$ if $\nu(\omega)$ is orthogonal to a supporting hyperplane for Ω at ω and $\omega + \nu(\omega)$ is separated from Ω by the supporting hyperplane.

Theorem 2.3. *Let $\mathcal{H}$, $\mathcal{U}$, $G_j : \mathcal{U} \to \mathbb{R}$, $j = 1, \dots, N$, $f : \mathcal{U} \to \mathcal{H}$ and Ω satisfy the assumptions* (A1)–(A6). *Suppose f is Lipschitz continuous on $\mathcal{U}$ and let $M_0 = \mathrm{Lip}(f)$. Then, $(I \pm \varepsilon f)(\Omega)$ is an open convex subset of $\mathcal{H}$, provided that $0 < \varepsilon < 1/(2M_0)$. Moreover, if $\omega \in \partial\Omega$ and $\nu(\omega)$ is an unit vector in the outer normal cone at ω, we have*

$$|\langle f(u) - f(\omega), \nu(\omega)\rangle| \leq \varepsilon^{-1}\langle \omega - u, \nu(\omega)\rangle, \tag{2.5}$$

for all $u \in \Omega$.

Proof. 1. We begin by proving that $(I \pm \varepsilon f)(\Omega)$ is convex if $0 < \varepsilon < 1/(2M_0)$. We prove the assertion for $(I+\varepsilon f)$ being the proof for $(I-\varepsilon f)$ entirely identical. Since $(I+\varepsilon f)$ is clearly a diffeomorphism from $\mathcal{U}$ onto $(I+\varepsilon f)(\mathcal{U})$, in view of (A5) and Theorem 2.2, it suffices to prove that $(I+\varepsilon f)(\Omega_j)$ is strongly locally convex at each $v \in (I+\varepsilon f)(S_j)$, for an arbitrary $j \in \{1,\dots,N\}$. We proceed by contradiction. Suppose, on the contrary, that for some $j \in \{1,\dots,N\}$, there is a point $v_0 \in \partial(I+\varepsilon f)(S_j)$ such that $(I+\varepsilon f)(\Omega_j)$ is not strongly locally convex at v_0. Let $u_0 \in \partial S_j$ be given by $(I+\varepsilon f)(u_0) = v_0$. Set

$$g(u) = u + \varepsilon(f(u) - f(u_0)).$$

Then $g(u_0) = u_0$ and $g(\Omega_j)$ is not strongly locally convex at $u_0 \in g(S_j) \cap S_j$. Now, $g(S_j)$ is a smooth submanifold of codimension 1 in $\mathcal{H}$, and so for $r > 0$ sufficiently small $g(S_j) \cap B(u_0, r)$ is the graph of a non-convex function whose epigraph contains $g(\Omega_j) \cap B(u_0, r)$. So, let us consider such $r > 0$.

2. We observe that, by (A4), g satisfies $dg(\omega)(T_\omega(S_j)) = T_\omega(S_j)$, for all $\omega \in S_j$. Hence, if $\nu(\omega)$ is the unit outer normal to $\partial\Omega_j$ at $\omega \in S_j$, it is also the unit outer normal to $\partial g(\Omega_j)$ at $g(\omega) \in g(S_j)$. Indeed, $\nu(\omega)$ is an eigenvector of dg^*, the adjoint of dg, viewed as a transformation on $\mathcal{H}$ by the usual identification $\mathcal{H}^* \equiv \mathcal{H}$, associated with a positive eigenvalue, and so

$$\langle dg(\omega)\nu(\omega), \nu(\omega)\rangle = \langle \nu(\omega), dg(\omega)^*\nu(\omega)\rangle = \lambda > 0.$$

Hence, since $dg(\omega)\nu(\omega)$ points outwards $g(\Omega_j)$ and $\nu(\omega)$ is normal to $g(S_j)$, $\nu(\omega)$ must point also outwards $g(\Omega_j)$. In particular, for $\omega = u_0$, $\nu(u_0)$ is both the unit outer normal to $\partial\Omega_j$ and $\partial g(\Omega_j)$ at $u_0 \in g(S_j) \cap S_j$.

3. Changing coordinates by means of an orthogonal affine transformation, we may assume $u_0 = 0$, and may take a countable orthonormal basis for $\mathcal{H}$, $\{e_0, e_1, e_2, \dots\}$, with $e_0 = \nu(u_0)$, so that any $u \in \mathcal{H}$ may be written as a square summable sequence $(x_0, x_1, x_2, \cdots)$, and $T_{u_0}(S_j)$ is identified with the Hilbert space $\mathcal{H}_0 \subseteq \mathcal{H}$ consisting of those vectors $\bar{x} = (x_0, x)$, with $x = (x_1, x_2, \dots)$, for which $x_0 = 0$. So, $\{e_1, e_2, \dots\}$ is an orthonormal basis for $\mathcal{H}_0$. Further, $g(S_j) \cap B(u_0, r)$ may be identified with the graph, $x_0 = G(x)$, of a function of class C^3, $G : \mathcal{H}_0 \to \mathbb{R}$, satisfying $G(0) = 0,\ dG(0) = 0$. Moreover, G may be taken so that $d^2G(0)$ is diagonalizable, as we show in the next paragraph. Thus, $\{e_1, e_2, \dots\}$ may be taken as an orthonormal basis of eigenvectors of $d^2G(0)$, where we identify the bilinear form $d^2G(0)$ with the symmetric transformation canonically associated with it. Moreover, for u_0 suitably chosen, as a point at which $g(\Omega_j)$ is not strongly locally convex, we may also assume that e_1 is such that $d^2G(e_1, e_1) > 0$. Let us denote by Π the two-dimensional subspace (plane) of $\mathcal{H}$ having $\{e_0, e_1\}$ as an orthonormal basis.

4. Concerning the fact that G may be chosen so that $d^2G(0)$ is diagonalizable, indeed, we may define G implicitly by $G_j \circ g^{-1}(G(x), x) = 0$, by using the Implicit Function Theorem. The latter also gives

$$d^2G(\cdot,\cdot) = -(dG_j \cdot D_0 g^{-1})^{-1}\Big(d^2G_j\left([D_0 g^{-1} dG + d_{\rm tg} g^{-1}]\,\cdot\,,\ [D_0 g^{-1} dG + d_{\rm tg} g^{-1}]\,\cdot\right)$$
$$+\, dG_j\left(\,[(dG\,\cdot)(dG\,\cdot)] D_0 D_0 g^{-1} + 2[(dG\,\cdot)(d_{\rm tg} D_0 g^{-1}\,\cdot)]_{\rm sym} + d_{\rm tg}^2 g^{-1}(\cdot,\cdot)\,\right)\Big),$$

as may be easily verified, where $d_{\rm tg} g^{-1}$ denotes the restriction of dg^{-1} to $\mathcal{H}_0$, D_0 means the partial derivative in the direction e_0 and $[\,]_{\rm sym}$ means the symmetric part. From this formula, using (A6), it can be seen that d^2G is given by a symmetric standard Fredholm operator and, hence, it is diagonalizable. Indeed, the only terms in the above formula that are not represented by operators of finite rank are $d^2G_j\left(d_{\mathrm{tg}}g^{-1}\cdot, d_{\mathrm{tg}}g^{-1}\cdot\right)$ and $dG_j d_{\mathrm{tg}}^2 g^{-1}(\cdot,\cdot)$. The fact that $df(\omega)|\mathcal{H}_0$ is a standard Fredholm operator, given by (A6), implies that $dg(\omega)|\mathcal{H}_0$ is a standard Fredholm operator, and, since

$$dg_{\mathrm{tg}}^{-1}(g(\omega)) = [dg|\mathcal{H}_0]^{-1} = I + \sum_{k=1}^{\infty}(-\varepsilon[df|\mathcal{H}_0])^k,$$

it follows that $dg_{\mathrm{tg}}^{-1}(g(\omega))$ is also a standard Fredholm operator. On the other hand $dG_j d^2 g|\mathcal{H}_0$ is a standard Fredholm operator by (A6) and

$$dG_j d_{\mathrm{tg}}^2 g^{-1}(\xi,\eta) = -dG_j\left(d^2 g(dg^{-1}\xi, dg^{-1}\eta)\right), \qquad \text{for all } \xi,\eta\in T_\omega(S_j),$$

and so

$$dG_j d_{\mathrm{tg}}^2 g^{-1}|T_\omega(S_j) = -(dg^{-1})^* dG_j d^2 g dg^{-1}|T_\omega(S_j),$$

which shows that $dG_j d_{\mathrm{tg}}^2 g^{-1}$ is also a standard Fredholm operator.

5. We may parametrize $\Pi\cap g(S_j)\cap B(u_0,r)$ around u_0 by $\alpha:[-\delta_0,\delta_0]\to g(S_j)$, with $\alpha(s)=(G(x(s)),x(s))$, with $x(s)=(s,0,0,\cdots)$. Set $p=\alpha(-\delta)$, $q=\alpha(\delta)$, for some $0<\delta<\delta_0$. We have

$$\langle\nu(p), q-p\rangle>0, \qquad \langle\nu(q), p-q\rangle>0, \tag{2.6}$$

where $\nu(p)$ and $\nu(q)$ are the unit outer normal vectors to $g(S_j)$ at p and q, respectively (see Figure 1).

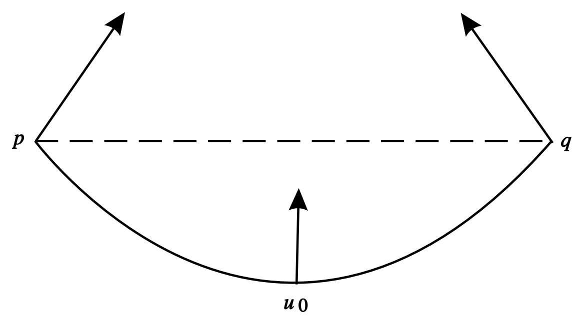

FIGURE 1

On the other hand,

$$\|u-g(u)\|\le \varepsilon M_0\|u-u_0\|\le \frac{\varepsilon M_0}{1-\varepsilon M_0}\|g(u)-u_0\|,$$

from which we deduce

$$\|g^{-1}(v)-v\|\le \frac{\varepsilon M_0}{1-\varepsilon M_0}\|v-u_0\|. \tag{2.7}$$

Now, since $(\varepsilon M_0)/(1-\varepsilon M_0)<1$, (2.7) implies that, if δ is sufficiently small, each of the pairs of points $p, g^{-1}(p)$ and $q, g^{-1}(q)$ lies together in the interior of one of two antipodal and, hence, coaxial convex cones with vertex u_0 and axis parallel to $\alpha'(0)$ (see Figure 2).

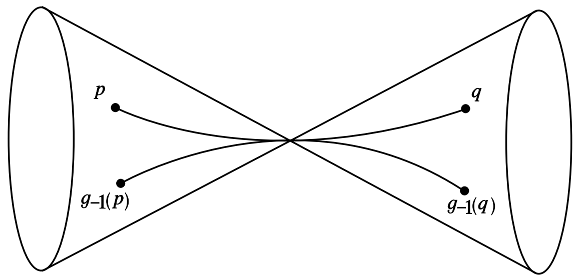

FIGURE 2

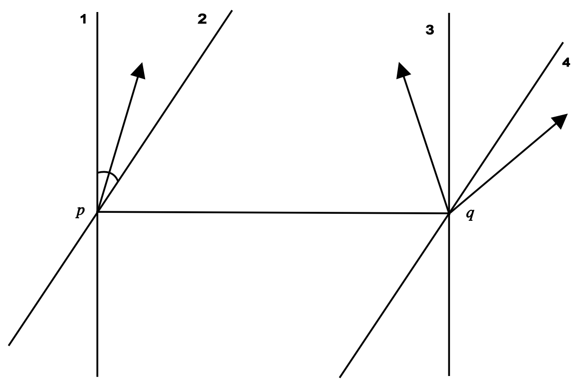

FIGURE 3

6. We first assume that $G(x)$ is quadratic. By the choice of the basis $\{e_1, e_2, \cdots\}$, we then have

$$G(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots ,$$

where $\lambda_1 = d^2G(0)(e_1, e_1) > 0$. In this case, along the curve $\alpha(s)$, the outer unit normal to $g(S_j)$, $\nu(\alpha(s)) \in \mathcal{H}$, is parallel to the plane Π. More specifically,

$$\nu(\alpha(s)) = \frac{1}{\sqrt{1 + 4\lambda_1^2 s^2}}(1, -2\lambda_1 s, 0, 0, \cdots).$$

We then have the diagram described in Figure 3. The lines **1** and **3** are the intersections with Π of the hyperplanes orthogonal to $p-q$, containing p and q, respectively. The lines **2** and **4** are the intersections with Π of the hyperplanes orthogonal to $g^{-1}(p) - g^{-1}(q)$, containing p and q, respectively. Since $g^{-1}(p)$ and $g^{-1}(q)$ are contained in the interior of the antipodal strictly convex cones, the hyperplanes orthogonal to $g^{-1}(p) - g^{-1}(q)$ cannot contain the plane Π, so that the intersection of those hyperplanes with Π must actually be lines as **2** and **4** in Figure 3.

7. Now, the convexity of $\overline{\Omega}$ implies that

$$\langle \nu(p), g^{-1}(q) - g^{-1}(p) \rangle \leq 0, \qquad \langle \nu(q), g^{-1}(p) - g^{-1}(q) \rangle \leq 0, \tag{2.8}$$

where we used the fact that $\nu(p)$ is also an outer unit normal vector to S_j at $g^{-1}(p)$ and similarly for $\nu(q)$ and $g^{-1}(q)$. This means that $\nu(p)$ and $\nu(q)$ should not point toward the interior of the strip bounded by the lines **2** and **4**. But this is impossible because of (2.6). We have then arrived at a contradiction.

8. We now examine the general case dropping the assumption that G is quadratic. In this general case, since G is of class C^3, near $x=0$, we have

$$G(x)=\lambda_1 x_1^2+\sum_{j=2}^{\infty}\lambda_j x_j^2+O(\|x\|^3),$$

again with $\lambda_1>0$. Hence, we get

$$\nu(\alpha(s))=\frac{1}{\sqrt{1+4\lambda_1^2 s^2}}(1,-2\lambda_1 s,0,0,\cdots)+O(|s|^2).$$

Set

$$\nu_*(\alpha(s))=\frac{1}{\sqrt{1+4\lambda_1^2 s^2}}(1,-2\lambda_1 s,0,0,\cdots).$$

So the distance from $\nu(\alpha(s))$ to $\nu_*(\alpha(s))$, which plays the role of $\nu(\alpha(s))$ in the quadratic case, is $\le c|s|^2$. Here and henceforth c will denote a positive constant not depending on $|s|$, whose precise value may change from one occurrence to the subsequent one.

9. On the other hand, for sufficiently small $|s|$, the distance from $\alpha(s)+\nu(\alpha(s))$ to the hyperplane orthogonal to the vector $\alpha(s)-\alpha(-s)$ containing $\alpha(s)$ is $\ge c|s|$, since $\lambda_1>0$. Also, the distance from $\alpha(s)+\nu(\alpha(s))$ to the hyperplane orthogonal to the vector $g^{-1}(\alpha(s))-g^{-1}(\alpha(-s))$ containing $\alpha(s)$ differs from the distance of $\alpha(s)+\nu_*(\alpha(s))$ to the same hyperplane by $O(|s|^2)$. Moreover, because, for s sufficiently small, $g^{-1}(\alpha(s))$ and $g^{-1}(\alpha(-s))$ belong to the interior of the antipodal strictly convex cones with vertice u_0 (see Figure 2), the absolute value of the cosine between the unit vectors in the direction of $\alpha(s)-\alpha(-s)$ and $g^{-1}(\alpha(s))-g^{-1}(\alpha(-s))$, respectively, is bounded below by a positive constant. Now, since $\nu_*(\alpha(s))$ and $\nu_*(\alpha(-s))$ should both point toward the interior of the slab bounded by the hyperplanes orthogonal to the vector $\alpha(s)-\alpha(-s)$ containing $\alpha(s)$ and $\alpha(-s)$, respectively, as in Figure 2, then either $\alpha(s)+\nu_*(\alpha(s))$ will be apart from the hyperplane orthogonal to $g^{-1}(\alpha(s))-g^{-1}(\alpha(-s))$ containing $\alpha(s)$ a distance $\ge c|s|$ (this is the case of $q=\alpha(\delta)$ in Figure 3) or the analogous assertion will hold for $\alpha(-s)+\nu_*(\alpha(-s))$, where we use the observation about the cosine between the unit vectors in the directions of $\alpha(s)-\alpha(-s)$ and $g^{-1}(\alpha(s))-g^{-1}(\alpha(-s))$. Hence, we again arrive at contradiction, similar to the one in the quadratic case, for then either $\nu(\alpha(s))$ or $\nu(\alpha(-s))$ would have to point toward the interior of the slab bounded by the hyperplanes orthogonal to $g^{-1}(\alpha(s))-g^{-1}(\alpha(-s))$ containing $\alpha(s)$ and $\alpha(-s)$, respectively, contradicting (2.8) which must hold by the convexity of Ω.

10. This completes the proof that $(I\pm\varepsilon f)(\Omega_j)$ is strongly locally convex at each point of $(I\pm\varepsilon f)(S_j)$, for each $j=1,\dots,N$. Since, by (A5), $(I\pm\varepsilon f)(\Omega)=\bigcap_{j=1}^N(I\pm\varepsilon f)(\Omega_j)$ and $\partial(I\pm\varepsilon f)(\Omega)\subseteq\bigcup_{j=1}^N(I\pm\varepsilon f)(S_j)$, applying Theorem 2.2, we easily deduce the convexity of $(I\pm\varepsilon f)(\Omega)$, as desired, concluding the proof of the first part of the theorem.

11. Concerning the inequality (2.5), by the convexity of $(I\pm\varepsilon f)(\Omega)$, we have for all $u\in\overline{\Omega}$ and $\omega\in\partial\Omega$, using (A4),

$$\langle (I\pm\varepsilon f)(u)-(I\pm\varepsilon f)(\omega),\nu(\omega)\rangle\le 0,$$

which gives

$$-\varepsilon^{-1}\langle \omega - u, \nu(\omega)\rangle \leq \langle f(u) - f(\omega), \nu(\omega)\rangle \leq \varepsilon^{-1}\langle \omega - u, \nu(\omega)\rangle.$$

Now, the convexity of $\overline{\Omega}$ implies that $\langle \omega - u, \nu(\omega)\rangle \geq 0$ and so we get

$$|\langle f(u) - f(\omega), \nu(\omega)\rangle| \leq \varepsilon^{-1}\langle \omega - u, \nu(\omega)\rangle,$$

which is the desired inequality. The proof now is complete. □

Remark 2.1. Perhaps it should be natural to expect that the result would hold already for $\varepsilon < (\mathrm{Lip} f)^{-1}$, instead of $\varepsilon < (2\mathrm{Lip} f)^{-1}$, which is true in some cases where we assume df to be symmetrizable, as we will see in the next section. On the other hand, the upper bound $(2\mathrm{Lip} f)^{-1}$ for ε seems optimal for our method. It is an interesting open problem to know whether it is possible or not to improve the upper bound for ε up to its seemingly natural value.

2.1. **A simple example.** We consider here the following very simple example. Let $\mathcal{H}$ be any real separable Hilbert space and $f \in C^3(\mathcal{H}, \mathcal{H})$ such that

$$f(u) = \begin{cases} \rho(\|u\|^2)u, & \text{if } u \in \bigcup_1^N S_j, \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

where $\rho \in C^3([0, \infty))$, and

$$S_j = \{u \in \mathcal{H} : \langle u, \xi_j\rangle = 0\}, \ j = 1, \dots, N-1,$$
$$S_N = \{u \in \mathcal{H} : \|u\|^2 = R^2\},$$

for some fixed linearly independent set of vectors $\{\xi_1, \dots, \xi_{N-1}\} \subseteq \mathcal{H}$. Setting $G_j(u) = \langle u, \xi_j\rangle$, $j = 1, \dots, N-1$, $G_N(u) = \|u\|^2 - R^2$, and

$$\Omega = \{u \in \mathcal{H} : \|u\| < R, \ \langle u, \xi_j\rangle < 0, \ j = 1, \dots, N-1\},$$

it is easy to verify that all assumptions (A1)-(A6) are trivially satisfied and f is Lipschitz on any open bounded convex $\mathcal{U} \subseteq \mathcal{H}$, say, $\mathcal{U} = B(0, \bar{R})$, with $\bar{R} > R$.

3. Maps with Symmetrizable Differential

In this section we analize the convexity of $f(\Omega)$ for f and Ω satisfying (A1)-(A6) but now, instead of assuming f to be Lipschitz, as in Theorem 2.3, we assume that $df(u)$ is symmetrizable, for all $u \in \mathcal{U}$.

Before stating our theorem concerning this context, we establish an elementary lemma about standard Fredholm maps.

Lemma 3.1. *Let $f \in C^1(\mathcal{U}, \mathcal{H})$ be a standard Fredholm map. Then, for each $u \in \mathcal{U}$, $df(u) : \mathcal{H} \to \mathcal{H}$ is a standard Fredholm operator.*

Proof. We have that $f = cI + g$, where $g \in C^1(\mathcal{U})$ is a compact map, and so the lemma reduces to the fact that the differential $dg(u) : \mathcal{H} \to \mathcal{H}$ of a differenciable compact map $g \in C^1(\mathcal{U})$ is a compact operator, which follows directly from the definition of differential. Indeed, given $u \in \mathcal{U}$ and $\delta > 0$, the image by $g_{u,\delta} = (g(u + \cdot) - g(u))/\delta$ of the sphere $S_\delta = \{v \in \mathcal{H} : \|v\| = \delta\}$, $g_{u,\delta}(S_\delta)$, is a relatively compact set, whose distance to $dg(u)(S_1)$ is less than $\varepsilon > 0$, for sufficiently small $\delta > 0$, where $S_1 = \{v \in \mathcal{H} : \|v\| = 1\}$. Since $\varepsilon > 0$ is arbitrary, we get that $dg(u)(S_1)$ is relatively compact. The latter clearly implies the compactness of the operator $dg(u)$ as desired. □

We now state the main result of this section. In order to do that, if $h:\mathcal{O}\to\mathcal{H}$ is a non-compact standard Fredholm map ($c\neq 0$), let us say for short that the pair $h,\mathcal{O}$, formed by such a map h and an open convex set $\mathcal{O}\subseteq\mathcal{H}$, has the properties (P1), (P2) or (P3) if it satisfies:

(P1) $h:\mathcal{O}\to\mathcal{H}$ is proper, that is, the pre-image of a compact set is compact.
(P2) For any vector $\xi\in\mathcal{H}$, $\sup_{u\in\mathcal{O}}\xi\cdot u<+\infty$ implies $\sup_{u\in\mathcal{O}}\xi\cdot h(u)<+\infty$.
(P3) $h(\mathcal{O})$ is simply connected.

It is an easy exercise to check that, for non-compact standard Fredholm maps, property (P1') below implies property (P1).

(P1') If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}$ with $\|u_n\|\to\infty$ then $\|h(u_n)\|\to\infty$.

Also, properties (P1') and (P2) are trivially satisfied if $\mathcal{O}$ is bounded.

Theorem 3.1. *Let $\mathcal{H}$, $\mathcal{U}$, $G_j:\mathcal{U}\to\mathbb{R}$, $j=1,\dots,N$, and $f:\mathcal{U}\to\mathcal{H}$ and Ω satisfy the assumptions* (A1)–(A6). *Suppose, for each $u\in\mathcal{U}$, $df(u):\mathcal{H}\to\mathcal{H}$ is continuously symmetrizable, that is, there exists a symmetric positive definite bounded operator $P(u):\mathcal{H}\to\mathcal{H}$, depending continuously on $u\in\mathcal{U}$, such that $P(u)df(u)$ is symmetric. Further, assume that, for each $u\in\mathcal{U}$, the spectrum of $df(u)$, $\sigma(df(u))$, satisfies $\sigma(df(u))\subseteq(0,\infty)$. Then, f is a diffeomorphism from Ω onto $f(\Omega)$ and the latter set is convex, provided that, in addition, one of the following is satisfied:*

(i) *$\mathcal{U}=\mathcal{H}$, $\sigma(df(u))\subseteq(\varepsilon_0,\infty)$ and $\mu I\le P(u)\le MI$, for all $u\in\mathcal{H}$, for certain $\varepsilon_0,\mu,M>0$.*
(ii) *$\mathcal{U}=\mathcal{H}$ and the pair $f,\mathcal{H}$ has the property* (P1).
(iii) *The pair f,Ω has the properties* (P1) *and* (P3).
(iv) *f is a non-compact standard Fredholm map, $f=cI+g$, with g compact and $c>0$, and the pair f,Ω has the properties* (P1') *and* (P2).

Moreover, if $\omega\in\partial\Omega$ and $\nu(\omega)$ is an unit vector in the outer normal cone at ω, we have

$$\langle f(u)-f(\omega),\nu(\omega)\rangle\le 0, \tag{3.1}$$

for all $u\in\Omega$.

Proof. 1. We first prove that f is a diffeomorphism from Ω onto $f(\Omega)$ in each of the cases (i)-(iv). We observe that, since $\sigma(df(u))\subseteq(0,\infty)$, we immediately have that f is a local diffeomorphism on $\mathcal{U}$.

2. In case (i), we easily verify that there exists $\alpha>0$ such that $\|df(u)\xi\|\ge\alpha\|\xi\|$, for all $u,\xi\in\mathcal{H}$. The fact that $f:\mathcal{H}\to\mathcal{H}$ is a diffeomorphism then follows from a straightforward infinite dimensional version of a well known lemma of Hadamard (see, e.g., [2], p. 222).

3. In case (ii), we have that $f:\mathcal{H}\to\mathcal{H}$ is a local diffeomorphism which is closed and proper, in view of property (P1). Hence, $f(\mathcal{H})=\mathcal{H}$ and f is a covering map from $\mathcal{H}$ onto $\mathcal{H}$. Since, $\mathcal{H}$ is simply connected, it follows that f is a diffeomorphism of $\mathcal{H}$ onto itself (see, e.g., [16, 19]).

4. Similarly, in case (iii), f is a local diffeomorphism which is proper, by property (P1), and, so, it is a covering map (see, e.g., [16, 19]), whose image is simply connected, by property (P3). Hence, again, f is a diffeomorphism from Ω onto its image and the assertions follow as above.

5. As for case (iv), first we prove that $f(\partial\Omega)=\partial f(\Omega)$. Since f is a local diffeomorphism, clearly $f(\partial\Omega)\supset\partial f(\Omega)$. Therefore, it is enough to prove that

there can be no point of $f(\partial\Omega)$ in the interior of $f(\overline{\Omega})$. Indeed, suppose v_0 is such a point, and let $\omega_0 \in \partial\Omega$ be such that $f(\omega_0) = v_0$, and let $\nu(\omega_0)$ be the outer unit normal to $\partial\Omega$ at ω_0, which we may assume to be well defined by properly choosing v_0. Then $\nu(\omega_0)$ is also local outer normal to $f(\partial\Omega)$ at v_0 by (A4). Since v_0 is in the interior of $f(\overline{\Omega})$, $\nu(\omega_0) \cdot f(u)$ cannot assume a maximum at $u = \omega_0$. Hence, because of the property (P2), there exists $\omega_1 \in \partial\Omega$ for which

$$\nu(\omega_0) \cdot f(\omega_1) = \sup_{u \in \Omega} \nu(\omega_0) \cdot f(u). \tag{3.2}$$

It then follows that $\nu(\omega_0) \cdot u = \nu(\omega_0) \cdot \omega_1$ is a supporting hyperplane to $\overline{\Omega}$ and $\nu(\omega_0) \cdot u = \nu(\omega_0) \cdot f(\omega_1)$ is a supporting hyperplane to $f(\overline{\Omega})$. It follows by convexity that the supporting hyperplanes $\nu(\omega_0) \cdot u = \nu(\omega_0) \cdot \omega_0$ and $\nu(\omega_0) \cdot u = \nu(\omega_0) \cdot \omega_1$ must coincide and so both ω_0 and ω_1 must lie in this hyperplane. Again by convexity, the line segment connecting ω_0 to ω_1 is entirely contained in $\partial\Omega$. But then the image by f of this line segment must be contained in a hyperplane normal to $\nu(\omega_0)$ and containing both $f(\omega_0)$ and $f(\omega_1)$, which is an absurd, and so we actually have $f(\partial\Omega) = \partial f(\Omega)$.

6. Now, for $\theta \in [0,1]$ let $f_\theta = (1-\theta)\, I + \theta\, f$; clearly each f_θ also satisfies properties (P1) and (P2). We obtain analogously $f_\theta(\partial\Omega) = \partial f_\theta(\Omega)$. Let $v_0 \in f(\Omega)$ and $u_0 \in \Omega$ be such that $f(u_0) = v_0$. Define $g_\theta(u) = f_\theta(u) - f_\theta(u_0)$. We notice that $0 \notin g_\theta(\partial\Omega)$, for $\theta \in [0,1]$. We also observe that the Leray-Schauder topological degree $\deg(g_\theta, \Omega, 0)$ is well defined since, by property (P1), $g_\theta^{-1}(0)$ is finite, and it coincides with the number of elements of $g_\theta^{-1}(0)$ because of the positiveness of the spectrum of $dg_\theta(u)$, everywhere in $\mathcal{U}$. Since $\theta \mapsto g_\theta$ is a homotopy with $g_0 = I - u_0$ and $g_1 = f - v_0$, we conclude that $\deg(f - v_0, \Omega, 0) = 1$, and since this holds for all $v_0 \in f(\Omega)$, it follows that f is a diffeomorphism of Ω over its image, and the proof is finished.

7. We now pass to the proof that $f(\Omega)$ is convex. We proceed as in the proof of Theorem 2.3 and assume that $v_0 \in f(S_j)$ is a point at which $f(S_j)$ is not strongly locally convex, suitably chosen, and $u_0 \in S_j$ is given by $f(u_0) = v_0$. Let $r > 0$ be small enough so that $0 < \varepsilon_0 \equiv \inf\{\lambda \in \sigma(df(u)) \,:\, u \in B(v_0, r)\}$. Define

$$h(u) = u_0 + \frac{1}{\varepsilon_0}(f(u) - f(u_0)).$$

Let $\alpha : [-\delta_0, \delta_0] \to h(S_j)$, with $\alpha(0) = u_0$, $p = \alpha(-\delta)$, $q = \alpha(\delta)$, for some $0 < \delta < \delta_0$, as in the proof of Theorem 2.3. Given $\xi, \eta \in \mathcal{H}$, define

$$\langle \xi, \eta \rangle_u = \langle P(u)\xi, \eta \rangle, \qquad \|\xi\|_u = \langle P(u)\xi, \xi \rangle^{1/2}.$$

We have

$$\langle dh^{-1}(u_0)\alpha'(0), \alpha'(0) \rangle_{u_0} > \varepsilon_0 M_0^{-1} \langle \alpha'(0), \alpha'(0) \rangle_{u_0}, \tag{3.3}$$

where M_0 is the least upper bound of the eigenvalues of $df(u_0)$. Obviously, a similar inequality holds for $-\alpha'(0)$. Also, clearly

$$\|dh^{-1}(u_0)\alpha'(0)\|_{u_0} \leq \|\alpha'(0)\|_{u_0},$$

which, from (3.3), gives

$$\langle dh^{-1}(u_0)\alpha'(0), \alpha'(0) \rangle_{u_0} > \varepsilon_0 M_0^{-1} \|dh^{-1}(u_0)\alpha'(0)\|_{u_0} \|\alpha'(0)\|_{u_0}. \tag{3.4}$$

8. Inequality (3.4) means that $dh^{-1}(u_0)\alpha'(0)$ lies in the interior of a strictly convex cone symmetric around the axis passing through u_0 in the direction of

$\alpha'(0)$, in the geometry induced in $\mathcal{H}$ by the inner product $\langle\cdot,\cdot\rangle_{u_0}$. Replacing $\alpha'(0)$ for $-\alpha'(0)$, we get that $-dh^{-1}(u_0)\alpha'(0)$ lies in the interior of the strictly convex cone antipodal to the one just described, in the referred geometry. It follows that for $\delta > 0$ sufficiently small, $h^{-1}(p)$ and p lie together in the interior of one of these strictly convex cones and $h^{-1}(q)$ and q lie together in the antipodal one, as depicted in Figure 2, with g replaced for h. From this point on the proof of the convexity of $f(\Omega)$ follows exactly as the proof of the convexity of $(I \pm \varepsilon f)(\Omega)$ in Theorem 2.3. The inequality (3.1) follows directly from the convexity of $f(\Omega)$ as was the case for inequality (2.5). The proof is complete. $\square$

3.1. **A simple example.** Let $\mathcal{H}$ be any real separable Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ be a linear compact symmetric operator, with $\sigma(T) \subseteq [0,\infty)$, $f = cI + g$, with $c > 0$ to be chosen later, and $g \in C^3(\mathcal{H},\mathcal{H})$ defined by

$$g(u) = \rho(\|T^{1/2}u\|^2)Tu,$$

where $\rho \in C^3 \cap L^\infty \cap \mathrm{Lip}([0,\infty))$. Let $\{\xi_1,\dots,\xi_N\}$ be a linearly independent set of eigenvectors of T,

$$S_j = \{u \in \mathcal{H} \,:\, \langle u,\xi_j\rangle = 0\},$$

set $G_j(u) = \langle u,\xi_j\rangle$, $j = 1,\dots,N$, and

$$\Omega = \{u \in \mathcal{H} \,:\, \langle u,\xi_j\rangle > 0,\ j = 1,\dots,N\}.$$

It is easy to verify that all assumptions (A1)-(A6) are trivially satisfied. Moreover, f is a standard Fredholm map such that $df(u)$ is a symmetric standard Fredholm operator, for all $u \in \mathcal{H}$, and $\sigma(df(u)) \subseteq (0,\infty)$ if $c > 0$ is sufficiently large. Finally, since

$$\|u\|\|f(u)\| \geq \langle u, f(u)\rangle = c\|u\|^2 + \rho(\|T^{1/2}u\|^2)\langle Tu,u\rangle \geq (c - \|\rho\|_\infty\|T\|)\|u\|^2,$$

we deduce that, if $c > \|\rho\|_\infty\|T\|$, (P1') and, hence, item (ii) of Theorem 3.1 is satisfied.

4. Application to Finite Difference Approximations

In order to apply our results to finite difference approximations for ordinary differential equations in $\mathcal{H}$, we establish the following corollary of Theorem 2.3.

Corollary 4.1. *Let the hypotheses of Theorem 2.3 be satisfied. Let* $M_0 = \mathrm{Lip}(f)$ *and* $g(u) = u + \varepsilon f(u)$*, for some* $\varepsilon \leq 1/(2M_0)$*. Suppose further that*

$$\langle f(\omega), \nu(\omega)\rangle \leq 0, \tag{4.1}$$

for all $\omega \in \partial\Omega$ *and* $\nu(\omega)$ *in the outer normal cone of* Ω *at* ω*. Then* $g(\overline{\Omega}) \subseteq \overline{\Omega}$*. Moreover, when equality holds in* (4.1) *we get* $g(\overline{\Omega}) = \overline{\Omega}$*, for* $\varepsilon \leq 1/(2M_0)$*.*

Proof. The proof follows from the fact that if $u \in \overline{\Omega}$, $\omega \in \partial\Omega$ and $\nu(\omega)$ is in the outer normal cone of Ω at ω then, by Theorem 2.3, one has

$$\langle g(u) - \omega, \nu(\omega)\rangle = \langle g(u) - g(\omega), \nu(\omega)\rangle + \varepsilon\langle f(\omega), \nu(\omega)\rangle \leq 0,$$

which in turn implies that $g(u) \in \overline{\Omega}$ for any $u \in \overline{\Omega}$. Finally, in case the equality holds in (4.1), using the first part for both f and $-f$ we conclude that, for any $\omega \in \partial\Omega$, both $\omega + \varepsilon f(\omega)$ and $\omega - \varepsilon f(\omega)$ belong to $\overline{\Omega}$. But, since $\omega \in \partial\Omega$ is in the line segment joining these two points, convexity of $\overline{\Omega}$ implies that they both should also belong to $\partial\Omega$. Hence, for $\varepsilon \leq 1/(2M_0)$, we have that g is obviously bijective, $g(\overline{\Omega}) \subseteq \overline{\Omega}$ and $g(\partial\Omega) \subseteq \partial\Omega$. Since $g|\partial\Omega : \partial\Omega \to \partial\Omega$ provides a homeomorphism between $\partial\Omega$ and $g(\partial\Omega)$, we have that $g(\partial\Omega)$ is open and closed in $\partial\Omega$. Since, by convexity, $\partial\Omega$ is connected, we easily conclude that $g(\partial\Omega) = \partial\Omega$, which immediately implies $g(\overline{\Omega}) = \overline{\Omega}$. □

We apply the above corollary to prove the invariance of $\overline{\Omega}$ under Euler and Runge-Kutta type schemes applied to the system of ordinary differential equations $\dot{u} = f(u)$, for $\overline{\Omega}$ and f satisfying its hypotheses. Indeed, we recall that the Euler scheme is given by

$$u_{n+1} = u_n + hf(u_n),$$

where $h = \Delta t$, while the fourth-order Runge-Kutta type scheme we consider is given by

$$u_{n+1} = u_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where,

$$\begin{aligned} k_1 &= hf(u_n) \\ k_2 &= hf(u_n + \frac{k_1}{2}) \\ k_3 &= hf(u_n + \frac{k_2}{2}) \\ k_4 &= hf(u_n + k_3). \end{aligned}$$

So, the invariance of $\overline{\Omega}$ under the Euler scheme follows immediately from the corollary if we choose $h \le (2M_0)^{-1}$.

Concerning the Runge-Kutta scheme, instead of (4.1), we make the stronger assumption that

$$\langle f(\omega), \nu(\omega)\rangle = 0, \tag{4.2}$$

for all $\omega \in \partial\Omega$ and $\nu(\omega)$ in the outer normal cone of Ω at ω. The invariance of $\overline{\Omega}$ now follows by first observing that we may write

$$u_{n+1} = \frac{1}{6}(u_n + k_1) + \frac{1}{3}(u_n + k_2) + \frac{1}{3}(u_n + k_3) + \frac{1}{6}(u_n + k_4). \tag{4.3}$$

We claim that the expressions inside the parentheses belong to $\overline{\Omega}$, for $h \le (2M_0)^{-1}$. Indeed, that $u_n + k_1 \in \overline{\Omega}$ follows directly from Corollary 4.1. Moreover,

$$\begin{aligned} k_2 &= hJ_2(u_n) := hf \circ (I + \frac{h}{2} f)(u_n), \\ k_3 &= hJ_3(u_n) := hf \circ (I + \frac{h}{2} J_2)(u_n), \\ k_4 &= hJ_4(u_n) := hf \circ (I + hJ_3)(u_n), \end{aligned}$$

and J_2, J_3, J_4 so defined also satisfy (4.2) and the other hypotheses of Corollary 4.1, as can be recursively verified by applying iteratively the corollary itself. Therefore, the claim follows. Hence, u_{n+1} is a convex combination of points in $\overline{\Omega}$ and, hence, it is a point in $\overline{\Omega}$.

5. Application to Convex-Valued Maps

In this section we briefly analyze a class of convex-valued maps whose motivation, in our case, comes from the theory of kinetic approximation shemes, which also arise in game theory and mathematical economics (see,e.g. [1]). The result we establish here in the setting of Hilbert spaces, was first obtained in the finite dimensional case by D. Serre [25], analyzing the invariance of convex sets under kinetic approximation schemes. In this connection, we also refer to [21, 4, 5, 15] and to [6, 18, 17], the latter relating to the so called kinetic formulation of conservation laws.

So, let $(S, \mu, \mathcal{A})$ be a finite measure space, that is, S is a set, μ is a finite measure on S and $\mathcal{A}$ is a σ-algebra of μ-measurable subsets of S. For each $\xi \in S$, let $M_\xi : \mathcal{U} \to \mathcal{H}$ and $\Omega \subseteq \mathcal{U}$ satisfy the assumptions of Theorem 3.1, with M_ξ playing the role of f. From Theorem 3.1, we obtain that

$$\Omega_\xi := M_\xi(\Omega) \quad \text{is convex, for each } \xi \in S,$$

provided that one of the assumptions (i)–(iv) thereof is satisfied.

(A8) Assume that, for each fixed $w \in \mathcal{U}$, $\xi \mapsto M_\xi(w)$ is a bounded weakly measurable map from S to $\mathcal{H}$ satisfying

$$w = \int_S M_\xi(w)\, d\mu(\xi). \tag{5.1}$$

Because of (5.1) $M_\xi(u)$ is usually called *Maxwellian distribution*.

Theorem 5.1. *Let $X : S \to \mathcal{H}$ satisfy $X(\xi) \in \overline{\Omega}_\xi$, $\sup_{\xi \in S} \|X(\xi)\| < \infty$. Suppose X is weakly μ-measurable, that is, for all $v \in \mathcal{H}$, $\langle X(\xi), v\rangle$ is a μ-measurable function from S to $\mathbb{R}$. Let*

$$u = \int_S X(\xi)\, d\mu(\xi). \tag{5.2}$$

Then, $u \in \overline{\Omega}$.

Proof. First of all we observe that u is well defined by (5.2) since

$$v \mapsto \int_S \langle v, X(\xi)\rangle\, d\mu(\xi)$$

clearly defines a bounded linear functional on $\mathcal{H}$. To prove (5.2) it suffices to show that

$$\langle u - \omega, \nu(\omega)\rangle \le 0, \tag{5.3}$$

for all $\omega \in \partial\Omega$ and all $\nu(\omega)$ in the outer normal cone of Ω at ω. Now, given such ω and $\nu(\omega)$, from (A6) we deduce that $\nu(\omega)$ is also in the outer normal cone of Ω_ξ at $M_\xi(\omega) \in \partial\Omega_\xi$, for all $\xi \in S$. Using also (A8) we get

$$\langle u - \omega, \nu(\omega)\rangle = \int_S \langle X(\xi) - M_\xi(\omega), \nu(\omega)\rangle\, d\mu(\xi),$$

which implies (5.3) because of the convexity of Ω_ξ, for all $\xi \in S$. $\square$

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References

[1] Aubin, J.-P., Ekeland, I., "Applied Nonlinear Analysis". John Wiley & Sons, New York, 1984.

[2] Berger, M.S., "Nonlinearity and Functional Analysis". Lectures on Nonlinear Problems in Mathematical Analysis, Academic Press, New York, 1977.

[3] Bonnesen, T., Fenchel, W., "Theorie der konvexen Körper", Springer-Verlag, Berlin, 1934.

[4] Bouchut, F., *Construction of BGK models with a family of kinetic entropies for a given system of conservation laws.* J. Statist. Phys. **95** (1999), 113–170.

[5] Bouchut, F., Guarguaglini, F., and Natalini, R., *Discrete kinetic approximation to multidimensional parabolic equations.* Indiana Univev. Math. J. **49** (2000), 723-749.

[6] Brenier, Y., *Résolution d'équations d'évolutions quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension $N+1$.* J. Diff. Eqns. **50** (1983), 375-390.

[7] Brezis, H., "Analyse Fonctionnelle". Masson, Paris, 1983.

[8] Brunn, H., *Fundamentalsatz von den Stützen eines Eigebietes*, Math. Ann. **100**, 634-637.

[9] Dafermos, C.M., "Hyperbolic Conservation Laws in Continuum Physics", Springer-Verlag, 1999.

[10] Frid, H., *Invariant regions under Lax-Friedrichs scheme for multidimensional systems of conservation laws*, Discrete and Continuous Dynamical Systems **1**, 4 (1995), 585-593.

[11] Frid, H., *Maps of convex sets and finite difference systems of conservation laws.* Arch. Rational Mech. Anal. **160** (2001) 245-269.

[12] Frid, H., *Correction to "Maps of convex sets and finite difference systems of conservation laws"*. Arch. Rat. Mech. Anal. **171** (2004), 297-299.

[13] Gruber, P.M., Wills, J.M., "Handbook of Convex Geometry", North-Holland, Amsterdam, 1993.

[14] Klee, V., *Convex sets in linear spaces*, Duke Math. J. **18**, 443-466.

[15] Lattanzio, C., Natalini, R., *Convergence of diffusive BGK approximations for nonlinear strongly parabolic systems.* Proc. Royal Soc. Edinburgh 132A (2002), 341-358.

[16] Lima, E., "Grupo Fundamental e Espaços de Recobrimento" (Fundamental Groups and Covering Spaces; in Portuguese). Instituto de Matemática Pura e Aplicada, Projeto Euclides, 1998.

[17] Lions, P.-L., Perthame, B., Souganidis, P.E., *Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Apll. Math., **49** (1996), 599-638.

[18] Lions, P.-L., Perthame, B., Tadmor, E., *Kinetic formulation of the isentropic gas dynamics and p-systems*, Comm. Math. Phys., **163** (1994), 415-431.

[19] Massey, W., "Algebraic Topology: An Introduction". Harcourt, Brace and World, New York, 1967; Springer, New York, 1986.

[20] Minkowski, H., "Gesammelte Abhandlungen", Teubner, Leipzig, 1911.

[21] Natalini, R., *A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws.* J. Diff. Eqs. **148** (1998), 292–317.

[22] Rockafeller, T., "Convex Analysis", Princeton University Press, Princeton, 1970.

[23] Rudin, W, "Functional Analysis", second ed. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, 1991.

[24] Serre, D., "Systèmes de Lois de Conservation II", Diderot Editeur Arts et Sciences, Paris, 1996.

[25] Serre, D., *Relaxation semi-linéaire et cinétique des systèmes de lois de conservation*, Ann. Inst. Henri Poincaré, Analyse non linéaire **17**, 2 (2000), 169-192.

[26] Tietze, H., *Über Konvexheit im kleinen und im grossen und über gewisseden Punkten einer Menge zugeordnete Dimensionszahlen*, Proc. Amer. Math. Soc. **11**, 112-116.

[27] Valentine, F.A., "Convex Sets", McGraw-Hill Book Company, New York, 1964.

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