Examples and structure of CMC surfaces in some Riemannian and Lorentzian homogeneous spaces

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Abstract

It is proved that the holomorphic quadratic differential associated to CMC surfaces in Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ discovered by U. Abresch and H. Rosenberg could be obtained as a linear combination of usual Hopf differentials. Using this fact, we are able to extend it for Lorentzian products. Families of examples of helicoidal CMC surfaces on these spaces are explicitly described. We also present some characterizations of CMC rotationally invariant discs and spheres. Finally, after establish some height and area estimates, we prove the existence of constant mean curvature Killing graphs.

Keywords: constant mean curvature, holomorphic quadratic differentials, Killing graphs
MSC 2000: 53C42, 53A10.

1 Introduction

U. Abresch and H. Rosenberg had recently proved that there exists a quadratic differential for an immersed surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ which is holomorphic when the surface has constant mean curvature. Here, $\mathbb{M}^2(\kappa)$ denotes the two-dimensional simply connected space form with constant curvature κ. This differential Q plays the role of the usual Hopf differential in the theory of constant mean curvature surfaces immersed in space forms. Thus, they were able to prove the following theorem:

Theorem. (Theorem 2, p. 143, [1]) *Any immersed cmc sphere $S^2 \looparrowright \mathbb{M}^2(\kappa)\times\mathbb{R}$ in a product space is actually one of the embedded rotationally invariant cmc spheres $S^2_H \subset \mathbb{M}^2(\kappa)\times\mathbb{R}$.*

The rotationally invariant spheres referred to above were constructed independently by W.-Y. Hsiang and W.-T. Hsiang in [10] and by R. Pedrosa and M. Ritoré in [15] and [16]. The theorem quoted above proves affirmatively a

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conjecture stated by Hsiang and Hsiang in their paper [10]. More importantly, it indicates that some tools often used for surface theory in space forms could be redesigned to more general three dimensional homogeneous spaces, the more natural ones after space forms being $\mathbb{M}^2(k) \times \mathbb{R}$. The price to be paid in abandoning space forms is that the technical difficulties are more involved. The method in [1] is to study very closely the revolution surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in order to guess the suitable differential.

Our idea here is to relate the Q differential on a surface Σ immersed in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with the usual Hopf differential after embedding $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in some Euclidean space $\mathbb{E}^4$. We prove that Q is written as a linear combination of the Hopf differentials Ψ^1 and Ψ^2 associated to two normal directions spanning the normal bundle of Σ in $\mathbb{E}^4$. This fact is also true when the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$ carries a Lorentzian metric. More precisely, if we define r as $r^2 = \epsilon/\kappa$ for $\epsilon = \operatorname{sgn} \kappa$ we state the following result:

Theorem. (Theorem 7, p. 25) *The quadratic differential $Q = 2H\Psi^1 - \varepsilon\frac{\epsilon}{r}\Psi^2$ is holomorphic on $\Sigma \looparrowright \mathbb{M}^2(\kappa) \times \mathbb{R}$ if the mean curvature H of Σ is constant. Inversely, if we suppose that Σ is compact (more generally, if Σ does not admit a function without critical points, or a vector field without singularities), then H is constant if Q is holomorphic.*

Our aim here is to explore geometrical consequences of this alternative presentation of Q. We next give a brief description of this paper. The sections 2 and 3 are concerned with the existence and structure of families of isometric surfaces with same constant mean curvature on both Riemannian and Lorentzian products which are invariant by certain isometry groups of the ambient space. Our construction is inspired by that one presented in [8] and [18]. In Section 4, we present the proof of the Theorem 7 and a variant of the classical Theorem of Joachimstahl which gives a characterization of CMC rotationally invariant discs and spheres in the same spirit of the result by Abresch and Rosenberg mentioned above (see Theorem 8).

We also prove on Section 5 the following result about free boundary CMC surfaces, based on the well-known Nitsche's work on partitioning problem:

Theorem. (Theorem 9, p. 29) *Let Σ be a surface immersed in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ whose boundary is contained in some horizontal plane $\mathbb{P}_a$. Suppose that Σ has constant mean curvature and that its angle with $\mathbb{P}_a$ is constant along its boundary. If $\varepsilon = 1$ and Σ is disc-type, then Σ is a spherical cap. If $\varepsilon = -1$, then Σ is a hyperbolic cap.*

The variational meaning of the conditions on Σ could be seen on Section 5. We end this section with a characterization of stable CMC discs with circular boundary on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ which generalizes a nice result of Alías, López and Palmer (see [3]). Finally, on Section 6, we obtain estimates of some geometrical data of CMC surfaces with boundary lying on vertical planes in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. These estimates are then used to prove the existence on non-negatively curved Riemannian products of CMC Killing graphs with boundary contained in vertical planes:

Theorem. (Theorem 12, p. 35) *Let Π be a vertical plane on the Riemannian product $\mathbb{M}^2(\kappa)\times\mathbb{R}$, $\kappa\geq 0$, determined by an unit vector a in $\mathbb{E}^4$. Let Ω be a domain on Π which does not contain points of the axis $\{\pm a\}\times\mathbb{R}$. If $|H|<\kappa_g/\tilde{\gamma}$, where κ_g is the geodesic curvature of $\partial\Omega$ in Π, then there exists a surface (a Killing graph) with constant mean curvature H and boundary $\partial\Omega$.*

The constant $\tilde{\gamma}$ depends on the maximum and minimum values on Ω of the norm of the Killing vector field generated by rotations fixing a.

In a forthcoming paper (see [11]), one of the authors elaborates versions of the results contained here for constant mean curvature hypersurfaces in some homogeneous spaces and warped products. There, a suitable treatment of Minkowski formulae gives some hints about stability problems and the existence of general Killing graphs.

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2 Screw-motion invariant CMC surfaces

2.1 The mean curvature equation

Let $\mathbb{M}^2(\kappa)$ be a two dimensional simply connected surface endowed with a Riemannian complete metric $\mathrm{d}\sigma^2$ with constant sectional curvature κ. We fix the metric $\varepsilon\mathrm{d}t^2+\mathrm{d}\sigma^2$, $\varepsilon=\pm1$, on the product $\mathbb{M}^2(\kappa)\times\mathbb{R}$. This metric is Lorentzian if $\varepsilon=-1$ and Riemannian if $\varepsilon=1$.

A tangent vector v to $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is projected on horizontal component v^h and vertical component v^t, respectively tangent to the $T\mathbb{M}^2(\kappa)$ and $T\mathbb{R}$ factors. We denote by $\langle\cdot,\cdot\rangle$ and D respectively the metric and covariant derivative in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. The curvature tensor in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is denoted by $\bar{R}$.

Let (ρ,θ) be polar coordinates centered at some point p_0 in $\mathbb{M}^2(\kappa)$ and the corresponding cylindrical coordinates (ρ,θ,t) in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. Fix then a curve $s\mapsto(\rho(s),0,t(s))$ in the plane $\theta=0$. If we rotate this curve at the same time we translate it along the t axis with constant speed b, we obtain a screw-motion invariant surface (for short, an *helicoidal surface*) Σ in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ whose axis is $\{p_0\}\times\mathbb{R}$. This means that this surface has a parametrization X, in terms of the cylindrical cordinates defined above, of the following form:

$$X(s,\theta)=(\rho(s),\theta,t(s)+b\,\theta). \tag{1}$$

For $b=0$ the surface Σ is a *revolution surface*, i.e., it is invariant with respect to the action of $O(2)$ on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ fixing the axis $\{p_0\}\times\mathbb{R}$. Another interesting particular case is obtained when $t(s)=0$ and $s\mapsto\rho(s)$ is just an arbitrary parametrization of the horizontal geodesic $\theta=0$, $t=0$. Here, the resulting surfaces are called *helicoids*. We will see that helicoids are examples with zero mean curvature. Helicoidal surfaces into Riemannian products $\mathbb{M}^2(\kappa)\times\mathbb{R}$ were already extensively studied in [8], [16], [10], [1], [18] and [13], for instance. In Lorentzian products, we will consider only *space-like* helicoidal surfaces, i.e., surfaces for which the metric induced on them is a Riemannian metric.

The tangent plane to Σ at a point $X(s,\theta)$ is spanned by the coordinate vector fields

$$X_s = \dot\rho\,\partial_\rho + \dot t\,\partial_t \quad \text{and} \quad X_\theta = \partial_\theta + b\,\partial_t.$$

Throughout this text, we denote $\mathrm{sn}_\kappa(\rho) = |\langle\partial_\theta,\partial_\theta\rangle|^{1/2}$. For further reference, we still denote $\mathrm{cs}_\kappa(\rho) =: \frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{sn}_\kappa(\rho)$. With this notation, an orientation for $X(\Sigma)$ is given by the unit normal vector field

$$n = \frac{1}{W}\big(\mathrm{sn}_\kappa(\rho)\,\dot t\,\partial_\rho + b\,\mathrm{sn}_\kappa^{-1}(\rho)\,\dot\rho\,\partial_\theta - \varepsilon\mathrm{sn}_\kappa(\rho)\,\dot\rho\,\partial_t\big),$$

where

$$W^2 =: \mathrm{sn}_\kappa^2(\rho)(\dot\rho^2+\varepsilon\dot t^2)+\varepsilon b^2\dot\rho^2.$$

We suppose that $\langle n,n\rangle=\varepsilon$. When $\varepsilon=-1$ this assumption implies that Σ is space-like and that $W^2>0$. It also follows that $U^2 =: \mathrm{sn}_\kappa^2(\rho)+\varepsilon b^2>0$. The induced metric on Σ is given by

$$\begin{aligned}\langle \mathrm{d}X,\mathrm{d}X\rangle &= E\mathrm{d}s^2+2F\mathrm{d}s\mathrm{d}\theta+G\mathrm{d}\theta^2\\ &=: \big(\dot\rho^2+\varepsilon\dot t^2\big)\,\mathrm{d}s^2+2\varepsilon b\dot t\,\mathrm{d}s\,\mathrm{d}\theta+\big(\mathrm{sn}_\kappa^2(\rho)+\varepsilon b^2\big)\,\mathrm{d}\theta^2.\end{aligned}$$

The vector field ∂_t is parallel and $s\mapsto\rho(s)$ parametrizes a geodesic on $\mathbb{M}^2(\kappa)$. So it follows that

$$X_{ss} =: D_{X_s}X_s = \ddot\rho\partial_\rho+\ddot t\partial_t.$$

The remaining two covariant derivatives of the coordinate vector fields are

$$X_{s\theta} =: D_{X_s}X_\theta = \dot\rho\,D_{\partial_\rho}\partial_\theta, \quad X_{\theta\theta} =: D_{X_\theta}X_\theta = D_{\partial_\theta}\partial_\theta.$$

The first coefficient of the second fundamental form $-\langle\mathrm{d}n,\mathrm{d}X\rangle$ of Σ is given by

$$e =: \langle X_{ss},n\rangle = \frac{\mathrm{sn}_\kappa(\rho)}{W}\,(\ddot\rho\dot t-\ddot t\dot\rho)$$

and since $\langle D_{\partial_\rho}\partial_\theta,\partial_\rho\rangle = \langle\partial_\theta,D_{\partial_\rho}\partial_\rho\rangle = 0$ and $\langle D_{\partial_\rho}\partial_\theta,\partial_\theta\rangle = \frac12\frac{\mathrm{d}}{\mathrm{d}\rho}\mathrm{sn}_\kappa^2(\rho) = \mathrm{sn}_\kappa(\rho)\mathrm{cs}_\kappa(\rho)$ it follows that

$$f =: \langle X_{s\theta},n\rangle = \frac{1}{W}\,b\dot\rho^2\mathrm{cs}_\kappa(\rho).$$

Finally, $\langle D_{\partial_\theta}\partial_\theta,\partial_\theta\rangle=0$ implies

$$g =: \langle X_{\theta\theta},n\rangle = -\frac{1}{W}\mathrm{sn}_\kappa(\rho)\dot t\,\langle\partial_\theta,D_{\partial_\rho}\partial_\theta\rangle = -\frac{1}{W}\dot t\,\mathrm{sn}_\kappa^2(\rho)\mathrm{cs}_\kappa(\rho).$$

Thus the formula $H = \frac{1}{2}(eG - 2fF + gE)/(EG - F^2)$ for the mean curvature of X reads

$$\begin{aligned}&2HW^3 = (\ddot\rho\dot t - \ddot t\dot\rho)\mathrm{sn}_\kappa(\rho)(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2) - 2\varepsilon b^2\dot t\dot\rho^2\mathrm{cs}_\kappa(\rho)\\ &-\dot t(\dot\rho^2 + \varepsilon\dot t^2)\mathrm{sn}_\kappa^2(\rho)\mathrm{cs}_\kappa(\rho).\end{aligned} \tag{2}$$

We suppose momentarily that the profile curve $(\rho(s), 0, t(s))$ is given as a graph $t = t(\rho)$. Thus we put $\rho = s$ above and find

$$W^2 = EG - F^2 = \mathrm{sn}_\kappa^2(\rho)(1 + \varepsilon\dot t^2) + \varepsilon b^2. \tag{3}$$

Therefore the mean curvature equation (2) reduces to

$$2HW^3 = -\ddot t\mathrm{sn}_\kappa(\rho)(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2) - 2\varepsilon b^2\dot t\dot{\mathrm{sn}}_\kappa(\rho) - \dot t(1 + \varepsilon\dot t^2)\mathrm{sn}_\kappa^2(\rho)\dot{\mathrm{sn}}_\kappa(\rho), \tag{4}$$

where the derivatives are taken with respect to the parameter ρ. One easily verifies that the expression

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\frac{\dot t\mathrm{sn}_\kappa^2(\rho)}{W}\Big) = -2H\mathrm{sn}_\kappa(\rho)$$

is equivalent to the equation (4) above. This means that

$$\frac{\mathrm{d}t}{\mathrm{d}\rho}\frac{\mathrm{sn}_\kappa^2(\rho)}{W} = I - 2H\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho \tag{5}$$

is a first integral to the mean curvature equation (4) associated to translations on t axis.

2.2 A Bour's type lemma and rotational examples

Next, we will obtain orthogonal parameters for Σ for which one of the families of coordinate curves is given by geodesics on Σ. For this, we write

$$\begin{aligned}\langle \mathrm{d}X, \mathrm{d}X\rangle &= \big(\dot\rho^2 + \varepsilon\dot t^2\big)\,\mathrm{d}s^2 + (\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)\big(\mathrm{d}\theta + U^{-2}\varepsilon b\dot t\mathrm{d}s\big)^2 - U^{-2}b^2\dot t^2\,\mathrm{d}s^2\\ &= \frac{W^2}{U^2}\,\mathrm{d}s^2 + U^2\mathrm{d}\tilde\theta^2 = \mathrm{d}\tilde s^2 + U^2\mathrm{d}\tilde\theta^2,\end{aligned}$$

where $\mathrm{d}\tilde s = \frac{W}{U}\,\mathrm{d}s$ and $\mathrm{d}\tilde\theta = \mathrm{d}\theta + U^{-2}\varepsilon b\dot t\mathrm{d}s$. These differentials could be locally integrated and furnish an actual change of coordinates on Σ. For revolution surfaces (i.e., for $b = 0$) such change of variables is not necessary. More precisely, it consists only in to assume that s is the arc lenght of the profile curve $s \mapsto (\rho(s), 0, t(s))$. For helicoids we have $\dot t = 0$ and then the change of variables is again useless since here we may choose $\rho = s$ along the rules of the helicoid. Since that W and U depend only on s, then $\tilde s$ is a function of s only with $\frac{\mathrm{d}\tilde s}{\mathrm{d}s} = \frac{W}{U}$. Notice that

$$\frac{W^2}{U^2} = \frac{\mathrm{sn}_\kappa^2(\rho)(\dot\rho^2 + \varepsilon\dot t^2) + \varepsilon b^2\dot\rho^2}{\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2} = \dot\rho^2 + \frac{\varepsilon\mathrm{sn}_\kappa^2(\rho)\dot t^2}{\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2}.$$

Thus the functions $\tilde{s}$, $\tilde{\theta}$ satisfy the system

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}\rho^2 + \frac{\varepsilon \mathrm{sn}_\kappa^2(\rho)}{\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2}\, \mathrm{d}t^2, \tag{6}$$

$$U\, \mathrm{d}\tilde{\theta} = (\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)^{1/2}\big(\mathrm{d}\theta + \frac{\varepsilon b}{\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2}\, \mathrm{d}t\big). \tag{7}$$

One easily verifies that the coordinate curves $\tilde{\theta} = \text{cte.}$ are geodesics on Σ. In fact, if we consider the frame $e_1 = \partial_{\tilde{s}}$ and $e_2 = U^{-1}\partial_{\tilde{\theta}}$ and the associated co-frame $\omega^1 = \mathrm{d}\tilde{s}$ and $\omega^2 = U\mathrm{d}\tilde{\theta}$, then $\omega_1^2 = \frac{\dot{U}}{U}\,\omega^2$. So, if ∇ denotes the induced connection on Σ then $\nabla_{e_1} e_1 = \nabla_{\partial_{\tilde{s}}}\partial_{\tilde{s}} = 0$. These geodesics intersect orthogonally the curves $\tilde{s} = \text{cte.}$. This allows us also to prove that the intrinsic Gaussian curvature K_{int} of Σ is simply $-\frac{\ddot{U}}{U}$.

Now, given the (natural) parameters $(\tilde{s}, \tilde{\theta})$ on Σ and the function $U(\tilde{s})$ we want to determine a two-parameter family of isometric immersions $X_{m,b} : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ in such a way that the immersed surfaces $X_{m,b}(\Sigma)$ are helicoidal and have induced metric given by $\mathrm{d}\tilde{s}^2 + U^2\mathrm{d}\tilde{\theta}^2$. Moreover, we require that the original immersion X belongs to that family. For this, it suffices that the equations (6) and (7) are satisfied by coordinates ρ, θ, t as functions of $\tilde{s}, \tilde{\theta}$ for some positive constant b. We refer in what follows to the original immersion and its pitch by X_0 and b_0.

From equations (6) and (7) we have $\frac{\partial \rho}{\partial \tilde{\theta}} = \frac{\partial t}{\partial \tilde{\theta}} = 0$ and

$$\frac{\partial \theta}{\partial \tilde{s}} = -\frac{\varepsilon b}{\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2}\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}, \quad \frac{\partial \theta}{\partial \tilde{\theta}} = \frac{U}{(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)^{1/2}} \tag{8}$$

and therefore

$$\frac{\partial^2 \theta}{\partial \tilde{s}\partial \tilde{\theta}} = \frac{\partial^2 \theta}{\partial \tilde{\theta}\partial \tilde{s}} = 0.$$

Hence $\frac{\partial \theta}{\partial \tilde{\theta}} = U(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)^{-1/2}$ does not depend on $\tilde{s}$. Since $U(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)^{-1/2}$ does not depend also on $\tilde{\theta}$ it follows that

$$\frac{U}{(\mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2)^{1/2}} = \frac{1}{m} \tag{9}$$

for some non zero constant m. This defines the first parameter of the family. The other one is the varying pitch b. We have $X_0 = X_{1,b_0}$. Differentiating $m^2U^2 = \mathrm{sn}_\kappa^2(\rho) + \varepsilon b^2$ with respect to $\tilde{s}$ we find

$$m^2 U\dot{U} = \mathrm{sn}_\kappa(\rho)\mathrm{cs}_\kappa(\rho)\dot{\rho}.$$

Thus since $\mathrm{sn}_\kappa^2(\rho) = m^2U^2 - \varepsilon b^2$ and $\mathrm{cs}_\kappa(\rho)^2 + \kappa\mathrm{sn}_\kappa^2(\rho) = 1$ it is clear that

$$\mathrm{cs}_\kappa^2(\rho) = 1 - \kappa(m^2U^2 - \varepsilon b^2).$$

The differential equation for ρ is then

$$\dot{\rho}^2 = \frac{m^4U^2\dot{U}^2}{(m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2))}. \tag{10}$$

From the equation (6) we conclude that t satisfies the equation

$$\dot{t}^2 = \frac{m^2U^2}{(m^2U^2-\varepsilon b^2)^2}\Big(\varepsilon\,\frac{(m^2U^2-\varepsilon b^2)(1-\kappa(m^2U^2-\varepsilon b^2))-m^4U^2\dot{U}^2}{1-\kappa(m^2U^2-\varepsilon b^2)}\Big). \quad (11)$$

Finally we infer from (8) and (9) that

$$\big(\frac{\partial\theta}{\partial\tilde{s}}\big)^2 = \varepsilon b^2\,\frac{(m^2U^2-\varepsilon b^2)(1-\kappa(m^2U^2-\varepsilon b^2))-m^4U^2\dot{U}^2}{m^2U^2(m^2U^2-\varepsilon b^2)^2(1-\kappa(m^2U^2-\varepsilon b^2))} \quad (12)$$

and $\frac{\partial\theta}{\partial\tilde{\theta}} = \frac{1}{m}$. Integrating these equations we obtain

$$\rho(\tilde{s}) = \int\Big(\frac{m^4U^2\dot{U}^2}{(m^2U^2-\varepsilon b^2)(1-\kappa(m^2U^2-\varepsilon b^2))}\Big)^{1/2}\mathrm{d}\tilde{s}, \quad (13)$$

$$t(\tilde{s}) = \int\Big(\varepsilon\,\frac{(m^2U^2-\varepsilon b^2)(1-\kappa(m^2U^2-\varepsilon b^2))-m^4U^2\dot{U}^2}{1-\kappa(m^2U^2-\varepsilon b^2)}\Big)^{1/2}\cdot \quad (14)$$
$$\frac{mU}{m^2U^2-\varepsilon b^2}\,\mathrm{d}\tilde{s},$$

$$\theta(\tilde{s},\tilde{\theta}) = \frac{1}{m}\,\tilde{\theta} + \int\frac{b}{mU(m^2U^2-\varepsilon b^2)}\cdot \quad (15)$$
$$\Big(\varepsilon\,\frac{(m^2U^2-\varepsilon b^2)(1-\kappa(m^2U^2-\varepsilon b^2))-m^4U^2\dot{U}^2}{1-\kappa(m^2U^2-\varepsilon b^2)}\Big)^{1/2}\mathrm{d}\tilde{s},$$

with $\mathrm{sn}^2_\kappa(\rho) = m^2U^2-\varepsilon b^2$.

Theorem 1. *Given a helicoidal surface $X_0:\Sigma\to\mathbb{M}^2(\kappa)\times\mathbb{R}$, with pitch b_0, there exists a two-parameter family of isometric helicoidal surfaces parametrized by $X_{m,b}:\Sigma\to\mathbb{M}^2(\kappa)\times\mathbb{R}$ with pitch b such that $X_0 = X_{1,b_0}$ with coordinates given by (13)-(15).*

We now calculate the components of the second fundamental form and the mean curvature of these surfaces with respect to the parameters $(\tilde{s},\tilde{\theta})$. Under the change of parameters $(s,\theta)\mapsto(\tilde{s},\tilde{\theta})$ the second fundamental form becomes

$$-\langle \mathrm{d}n,\mathrm{d}X\rangle = \Big(e\big(\frac{\partial s}{\partial\tilde{s}}\big)^2 + 2f\frac{\partial s}{\partial\tilde{s}}\frac{\partial\theta}{\partial\tilde{s}} + g\big(\frac{\partial\theta}{\partial\tilde{s}}\big)^2\Big)\mathrm{d}\tilde{s}^2 + 2\,\Big(e\frac{\partial s}{\partial\tilde{s}}\frac{\partial s}{\partial\tilde{\theta}} +$$
$$f\big(\frac{\partial s}{\partial\tilde{s}}\frac{\partial\theta}{\partial\tilde{\theta}} + \frac{\partial s}{\partial\tilde{\theta}}\frac{\partial\theta}{\partial\tilde{s}}\big) + g\frac{\partial\theta}{\partial\tilde{s}}\frac{\partial\theta}{\partial\tilde{\theta}}\Big)\mathrm{d}\tilde{s}\mathrm{d}\tilde{\theta} + \Big(e\big(\frac{\partial s}{\partial\tilde{\theta}}\big)^2 + 2f\frac{\partial s}{\partial\tilde{\theta}}\frac{\partial\theta}{\partial\tilde{\theta}} + g\big(\frac{\partial\theta}{\partial\tilde{\theta}}\big)^2\Big)\mathrm{d}\tilde{\theta}^2$$
$$=: \tilde{e}\,\mathrm{d}\tilde{s}^2 + 2\tilde{f}\,\mathrm{d}\tilde{s}\tilde{\theta} + \tilde{g}\,\mathrm{d}\tilde{\theta}^2. \quad (16)$$

If we choose $s=\rho$, then we have from the expressions (13)-(15) above that

$$\frac{\partial s}{\partial\tilde{s}} = \frac{\mathrm{d}\rho}{\mathrm{d}\tilde{s}},\ \frac{\partial s}{\partial\tilde{\theta}} = 0,\ \frac{\partial\theta}{\partial\tilde{\theta}} = \frac{1}{m}.$$

Turning back to the expression (16) one finds

$$\tilde{f} = f\frac{\mathrm{d}\rho}{\mathrm{d}\tilde{s}}\frac{1}{m} + g\frac{\partial\theta}{\partial\tilde{s}}\frac{1}{m} = \frac{1}{m}\frac{1}{W}b\,\mathrm{cs}_\kappa(\rho)\frac{\mathrm{d}\rho}{\mathrm{d}\tilde{s}} - \frac{1}{m}\frac{1}{W}\mathrm{sn}^2_\kappa(\rho)\frac{\mathrm{d}t}{\mathrm{d}\rho}\mathrm{cs}_\kappa(\rho)\frac{\partial\theta}{\partial\tilde{s}}$$

and

$$\tilde{g} = g\big(\frac{\partial\theta}{\partial\tilde{\theta}}\big)^2 = g\frac{1}{m^2} = -\frac{1}{m^2}\frac{1}{W}\mathrm{sn}_\kappa^2(\rho)\frac{\mathrm{d}t}{\mathrm{d}\rho}\mathrm{cs}_\kappa(\rho).$$

However it holds that

$$\frac{1}{W}\frac{\mathrm{d}t}{\mathrm{d}\rho} = \frac{1}{mU}\,\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}.$$

Thus the expressions

$$\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}} = \frac{mU}{m^2U^2 - \varepsilon b^2}\Big(\varepsilon\,\frac{(m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\Big)^{1/2}$$

and $\mathrm{sn}_\kappa^2(\rho) = m^2U^2 - \varepsilon b^2$ imply that

$$\mathrm{sn}_\kappa^2(\rho)\frac{1}{W}\frac{\mathrm{d}t}{\mathrm{d}\rho} = \Big(\varepsilon\,\frac{(m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\Big)^{1/2}. \tag{17}$$

Notice that this expression is the left-hand side of the first integral (5). Thus we obtain

$$\sqrt{\varepsilon\,\frac{(m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}} = I - 2H\int \mathrm{sn}_\kappa(\rho)\mathrm{d}\rho. \tag{18}$$

Since $\mathrm{cs}_\kappa(\rho) = (1 - \kappa(m^2U^2 - \varepsilon b^2))^{1/2}$ then

$$\tilde{g} = -\frac{1}{m^2}\sqrt{\varepsilon\big((m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2\big)}.$$

Now we calculate $\tilde{f}$ using (8)

$$\begin{aligned}\tilde{f} &= \frac{b}{m^4U^3}\mathrm{cs}_\kappa(\rho)\Big(\mathrm{sn}_\kappa^2(\rho)\big(\frac{\mathrm{d}\rho}{\mathrm{d}\tilde{s}}\big)^2 + \varepsilon b^2\big(\frac{\mathrm{d}\rho}{\mathrm{d}\tilde{s}}\big)^2 + \varepsilon\mathrm{sn}_\kappa^2(\rho)\big(\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}\big)^2\Big)\\ &= \frac{b}{m^4U^3}\mathrm{cs}_\kappa(\rho)m^2U^2 = \frac{b}{m^2U}\sqrt{1 - \kappa(m^2U^2 - \varepsilon b^2)}.\end{aligned}$$

Finally we calculate $\tilde{e}$. For this one uses the Gauss formula $K_{\mathrm{int}} - \bar{K} = K_{\mathrm{ext}}$. Here $\bar{K}$ is the ambient sectional curvature and, by definition, $K_{\mathrm{ext}} = (\tilde{e}\tilde{f} - \tilde{g}^2)/U^2$. So

$$\bar{K} = \frac{\langle \bar{R}(\partial_{\tilde{s}}, \partial_{\tilde{\theta}})\partial_{\tilde{s}}, \partial_{\tilde{\theta}}\rangle}{|\partial_{\tilde{s}}|^2|\partial_{\tilde{\theta}}|^2 - \langle\partial_{\tilde{s}}, \partial_{\tilde{\theta}}\rangle^2} = \frac{\kappa}{U^2}\big(U^2 - \varepsilon U^2\langle\partial_{\tilde{s}}, \partial_t\rangle^2 - \varepsilon\langle\partial_{\tilde{\theta}}, \partial_t\rangle^2\big)$$

However equations (1) and (8) show that

$$\langle\partial_{\tilde{s}}, \partial_t\rangle = \varepsilon\big(\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}} + b\frac{\partial\theta}{\partial\tilde{s}}\big) = \frac{\varepsilon}{m^2U^2}\big(m^2U^2 - \varepsilon b^2\big)\cdot\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}.$$

One also finds

$$\langle\partial_{\tilde{\theta}}, \partial_t\rangle = \varepsilon\,\frac{\partial\theta}{\partial\tilde{\theta}} = \varepsilon b\frac{1}{m}. \tag{19}$$

Then

$$U^2 - \varepsilon U^2 \langle \partial_{\tilde{s}}, \partial_t \rangle^2 - \varepsilon \langle \partial_{\tilde{\theta}}, \partial_t \rangle^2 = U^2 - \frac{\varepsilon}{m^4 U^2} (m^2 U^2 - \varepsilon b^2)^2 \big(\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}\big)^2 - \frac{\varepsilon b^2}{m^2}.$$

Finally $K_{\rm int} = -\frac{\ddot{U}}{U}$ yields

$$\tilde{e}\tilde{g} - \tilde{f}^2 = -U\ddot{U} - \kappa\big(U^2 - \frac{\varepsilon}{m^4 U^2} (m^2 U^2 - \varepsilon b^2)^2 \big(\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}\big)^2 - \frac{\varepsilon b^2}{m^2}\big).$$

Thus

$$\begin{aligned}
&\tilde{e}\tilde{g} = -U\ddot{U} - \kappa U^2 + \kappa \frac{1}{m^2}\Big(m^2U^2 - \varepsilon b^2 - \frac{m^4 U^2 \dot{U}^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\Big) \\
&+\kappa \frac{\varepsilon b^2}{m^2} + \frac{b^2}{m^4 U^2}\big(1 - \kappa(m^2 U^2 - \varepsilon b^2)\big) \\
&= -U\ddot{U} - \frac{\kappa m^2 U^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\,\dot{U}^2 + \frac{b^2}{m^4 U^2}\big(1 - \kappa(m^2 U^2 - \varepsilon b^2)\big).
\end{aligned}$$

So

$$\tilde{e} = \frac{m^2 U\ddot{U} + \frac{\kappa m^4 U^2}{1-\kappa(m^2U^2 - \varepsilon b^2)}\,\dot{U}^2 - \frac{b^2}{m^2U^2}\big(1 - \kappa(m^2U^2 - \varepsilon b^2)\big)}{\sqrt{\varepsilon\big((m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2\big)}}.$$

The mean curvature is expressed in parameters $(\tilde{s}, \tilde{\theta})$ as $2H = \tilde{e} + \frac{\tilde{g}}{U^2}$. Thus we have

$$\begin{aligned}
&2H\,R = m^2U\ddot{U} + \frac{\kappa m^4 U^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\,\dot{U}^2 - \frac{b^2}{m^2U^2}\big(1 - \kappa(m^2U^2 - \varepsilon b^2)\big) \\
&-\frac{1}{m^2U^2}\,R^2 = m^2U\ddot{U} + \big(m^2 + \frac{\kappa m^4 U^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\big)\,\dot{U}^2 - \varepsilon\big(1 - \kappa(m^2U^2 - \varepsilon b^2)\big),
\end{aligned}$$

where

$$R = \sqrt{\varepsilon\big((m^2U^2 - \varepsilon b^2)(1 - \kappa(m^2U^2 - \varepsilon b^2)) - m^4U^2\dot{U}^2\big)}. \tag{20}$$

So, all surfaces $X_{m,b}$ parametrized by the coordinates (13)-(15) have the same constant mean curvature H if and only if U satisfies the following ordinary differential equation

$$2H\,R = m^2U\ddot{U} + \big(m^2 + \frac{\kappa m^4 U^2}{1 - \kappa(m^2U^2 - \varepsilon b^2)}\big)\,\dot{U}^2 - \varepsilon\big(1 - \kappa(m^2U^2 - \varepsilon b^2)\big). \tag{21}$$

It is useful now to consider conformal parameters on Σ by changing variables

$$(\tilde{s}, \tilde{\theta}) \mapsto (u, v) =: (\int \frac{\mathrm{d}\tilde{s}}{U}, \tilde{\theta}).$$

Plugging $\partial u/\partial\tilde{s} = \mathrm{d}u/\mathrm{d}\tilde{s} = 1/U$, $\partial v/\partial\tilde{\theta} = \mathrm{d}v/\mathrm{d}\tilde{\theta} = 1$ into (16) implies that its coefficients are now changed as

$$\tilde{e} \mapsto \tilde{e}U^2, \quad \tilde{f} \mapsto \tilde{f}U, \quad \tilde{g} \mapsto \tilde{g}.$$

The metric induced on Σ becomes $U^2(\mathrm{d}u^2 + \mathrm{d}v^2)$. Thus the mean curvature is

$$2HU^2 = \tilde{e}U^2 + \tilde{g}.$$

So the coefficient ψ^1 of the Hopf differential Ψ^1 (see Section 4) in these parameters is written as

$$\psi^1 = \Big(\frac{\tilde{e}U^2 - \tilde{g}}{2}\Big) - i\,\tilde{f}U = \big(HU^2 - \tilde{g}\big) - i\,\tilde{f}U.$$

Since $\tilde{g} = -R/m^2$ and $\tilde{f} = \frac{b}{m^2U}\sqrt{1-\kappa(m^2U^2 - \varepsilon b^2)} = \frac{b}{m^2U}\mathrm{cs}_\kappa(\rho)$ it follows that

$$\psi^1 = \big(HU^2 + \frac{R}{m^2}\big) - i\,\frac{b}{m^2}\mathrm{cs}_\kappa(\rho).$$

However by the very definition of R the expression (18) reads

$$\frac{R}{\mathrm{cs}_\kappa(\rho)} = I - 2H\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho.$$

So, replacing the identity $\frac{\mathrm{d}}{\mathrm{d}\rho}\,\mathrm{cs}_\kappa(\rho) = -\kappa\,\mathrm{sn}_\kappa(\rho)$ gives

$$\kappa R = \mathrm{cs}_\kappa(\rho)\,\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big).$$

We are interested here on $\kappa \neq 0$ (it is a well-known fact that ψ^1 is holomorphic for $\kappa = 0$). In this case it holds that

$$\begin{aligned}\kappa\psi^1 &= \big(\kappa HU^2 + \frac{1}{m^2}\mathrm{cs}_\kappa(\rho)\,\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big)\big) - i\,\frac{\kappa b}{m^2}\,\mathrm{cs}_\kappa(\rho)\\ &= \frac{1}{m^2}\,\big(\kappa H\mathrm{sn}^2_\kappa(\rho) + \kappa H\varepsilon b^2 + \kappa I\mathrm{cs}_\kappa(\rho) + 2H\mathrm{cs}^2_\kappa(\rho)\big) - i\,\frac{\kappa b}{m^2}\,\mathrm{cs}_\kappa(\rho).\end{aligned}$$

Now we want to compute the coefficient ψ^2 of the differential Ψ^2 on the conformal coordinates u, v defined just above. We have

$$\partial_u = X_{\tilde{s}}\frac{\partial\tilde{s}}{\partial u} = X_{\tilde{s}}U.$$

Using equations (1) and (19) one proves that

$$\langle\partial_u, \partial_t\rangle = \varepsilon\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}U\Big(\frac{m^2U^2 - \varepsilon b^2}{m^2U^2}\Big).$$

However

$$\begin{aligned}&\big(m^2U^2 - \varepsilon b^2\big)\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}} = \Big(\frac{\mathrm{sn}^2_\kappa(\rho)}{mU}\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}\Big)\,mU = \Big(\frac{\mathrm{sn}^2_\kappa(\rho)}{W}\frac{\mathrm{d}t}{\mathrm{d}\rho}\Big)\,mU\\ &= mU\big(I - 2H\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho\big).\end{aligned}$$

Therefore

$$\kappa\big(m^2U^2 - \varepsilon b^2\big)\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}} = mU\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big).$$

We conclude that

$$\kappa\langle\partial_u, \partial_t\rangle = \frac{\varepsilon}{m}\,\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big).$$

We also compute

$$\partial_v = X_{\tilde{\theta}} = X_\theta\frac{\partial\theta}{\partial\tilde{\theta}} = \frac{1}{m}\Big(\partial_\theta + b\partial_t\Big)$$

and

$$\langle\partial_v, \partial_t\rangle = \frac{\varepsilon b}{m}.$$

Thus it results that

$$\kappa^2\langle\partial_u, \partial_t\rangle^2 - \kappa^2\langle\partial_v, \partial_t\rangle^2 = \frac{1}{m^2}\,\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big)^2 - \frac{\kappa^2 b^2}{m^2}$$

and

$$\kappa\langle\partial_u, \partial_t\rangle\langle\partial_v, \partial_v\rangle = \frac{b}{m^2}\,\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big).$$

Now since that $\frac{\epsilon}{r^2} = \kappa$ we write

$$\begin{aligned}\frac{\epsilon}{r}\,\psi^2 &= \frac{1}{2}\Big(\kappa\varepsilon\langle\partial_u, \partial_t\rangle^2 - \kappa\varepsilon\langle\partial_v, \partial_t\rangle^2\Big) - i\,\kappa\varepsilon\langle\partial_u, \partial t\rangle\langle\partial_v, \partial_t\rangle\\ &= \frac{\varepsilon}{2\kappa m^2}\Big(\kappa^2 I^2 + 4H^2\mathrm{cs}^2_\kappa(\rho) + 4H\kappa I\mathrm{cs}_\kappa(\rho) - \kappa^2 b^2\Big) - i\,\frac{\varepsilon b}{m^2}\big(\kappa I + 2H\mathrm{cs}_\kappa(\rho)\big).\end{aligned}$$

Therefore

$$2H\psi^1 - \varepsilon\frac{\epsilon}{r}\,\psi^2 = \frac{1}{m^2}\big(2H^2(\frac{1}{\kappa} + \varepsilon b^2) + \frac{1}{2}\kappa(b^2 - I^2)\big) + i\,\frac{b\kappa I}{m^2}.$$

For $\kappa = 0$ we have $\mathrm{cs}_\kappa(\rho) = 1$ and (18) becomes $R = I - H^2\rho^2$. So

$$\psi^1 = \frac{1}{m^2}\big(Hm^2U^2 + R\big) - i\,\frac{b}{m^2} = \frac{1}{m^2}\big(H\varepsilon b^2 + I\big) - i\,\frac{b}{m^2}.$$

Thus, the differential Q has constant coefficient for any surface on the family $X_{m,b} : \Sigma \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ of screw-motion invariant CMC surfaces on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ starting (for $m = 1$) from some given CMC surface. Its final expression is:

$$\psi = -\frac{1}{2m^2\kappa}\big(\kappa^2 I^2 - 4H^2 - \kappa b^2(4H^2\varepsilon + \kappa)\big) + \,i\frac{b\kappa I}{m^2}$$

for $\kappa \neq 0$. From the same calculations, we assure that the Hopf differential has constant coefficient for $\kappa = 0$:

$$\psi^1 = \frac{1}{m^2}\big(H\varepsilon b^2 + I\big) - i\,\frac{b}{m^2}$$

In the case $\kappa \neq 0$, we have for rotational examples ($b = 0$, $m = 1$) that

$$\psi = -\frac{1}{2\kappa}\big(\kappa^2 I^2 - 4H^2\big)$$

Thus $Q = 0$ for CMC rotational examples if and only if $4H^2 - \kappa^2 I^2 = 0$ or

$$I = \pm\frac{2H}{\kappa}.$$

We now determine explicitly the CMC rotational examples with $Q = 0$. In order to do this, we replace $I = \pm 2H/\kappa$ in (5). Since $W^2 = \mathrm{sn}_\kappa^2(\rho)(1 + \varepsilon\frac{\mathrm{d}t}{\mathrm{d}\rho}^2)$ it follows that

$$\frac{\frac{\mathrm{d}t}{\mathrm{d}\rho}}{\sqrt{1+\varepsilon\frac{\mathrm{d}t}{\mathrm{d}\rho}^2}}\,\mathrm{sn}_\kappa(\rho) = -\frac{2H}{\kappa}\,\Big(\pm 1 + \kappa\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho\Big) = -\frac{2H}{\kappa}\big(\pm 1 - \mathrm{cs}_\kappa(\rho)\big).$$

So squaring both sides and taking inverses

$$\frac{1+\varepsilon\frac{\mathrm{d}t}{\mathrm{d}\rho}^2}{\big(\frac{\mathrm{d}t}{\mathrm{d}\rho}\big)^2} = \frac{\kappa}{4H^2}\,\frac{\kappa\mathrm{sn}_\kappa^2(\rho)}{\big(\pm 1 - \mathrm{cs}_\kappa(\rho)\big)^2} = \frac{\kappa}{4H^2}\,\frac{1-\mathrm{cs}_\kappa^2(\rho)}{\big(\pm 1 - \mathrm{cs}_\kappa(\rho)\big)^2}.$$

Thus for $I = -2H/\kappa$ one has

$$\big(\frac{\mathrm{d}\rho}{\mathrm{d}t}\big)^2 + \varepsilon = \frac{\kappa}{4H^2}\,\frac{1+\mathrm{cs}_\kappa(\rho)}{1-\mathrm{cs}_\kappa(\rho)}.$$

However

$$\frac{1+\mathrm{cs}_\kappa(\rho)}{1-\mathrm{cs}_\kappa(\rho)} = \frac{1}{\kappa}\,\mathrm{ct}_\kappa^2(\rho/2) = \frac{1}{\kappa}\,\frac{1}{r^2}.$$

Here $\mathrm{ct}_\kappa(\rho) = \mathrm{sn}_\kappa(\rho)/\mathrm{sn}_\kappa(\rho)$ is the geodesic curvature of the geodesic circle centered at p_0 with radius ρ in $\mathbb{M}^2(\kappa)$ and r is the Euclidean radial distance r measured from p_0 on the Euclidean model for $\mathbb{M}^2(\kappa)$. Thus for $I = -2H/\kappa$ we have

$$\big(\frac{\mathrm{d}\rho}{\mathrm{d}t}\big)^2 + \varepsilon = \frac{1}{4H^2}\,\frac{1}{r^2}.$$

Now

$$\big(\frac{\mathrm{d}\rho}{\mathrm{d}t}\big)^2 = \big(\frac{\mathrm{d}\rho}{\mathrm{d}r}\big)^2\,\big(\frac{\mathrm{d}r}{\mathrm{d}t}\big)^2 = \frac{4}{(1+\kappa r^2)^2}\,\big(\frac{\mathrm{d}r}{\mathrm{d}t}\big)^2.$$

So the resulting equation is

$$\frac{2}{1+\kappa r^2}\,\frac{2Hr\,\mathrm{d}r}{\sqrt{1-4H^2\varepsilon r^2}} = \mathrm{d}t.$$

We change variables defining $1 + \kappa r^2 = u$. We then change variables again by defining ($\kappa < 0$) $v = \varepsilon u - \big(\varepsilon + \kappa/4H^2\big)$ and $v = \big(\varepsilon + \kappa/4H^2\big) - \varepsilon u$ (for $\kappa > 0$). Next, we put $w = \sqrt{v}$. So $\mathrm{d}v = 2w\,\mathrm{d}w$ and the final form of the equation is

$$\frac{2\mathrm{d}w}{w^2 + \big(\varepsilon + \kappa/4H^2\big)} = -\sqrt{-\kappa}\,\mathrm{d}t,\ (\kappa < 0), \quad \frac{2\mathrm{d}w}{w^2 - \big(\varepsilon + \kappa/4H^2\big)} = \sqrt{\kappa}\,\mathrm{d}t,\ (\kappa > 0).$$

We suppose that $4H^2\varepsilon + \kappa > 0$. Then writing $c^2 = \varepsilon + \kappa/4H^2$ one has

$$\frac{2}{c}\arctan(w/c) = -\sqrt{-\kappa}\,t, \quad \kappa < 0$$

and

$$\frac{1}{c}\log\Big|\frac{w+c}{w-c}\Big| = -\sqrt{\kappa}\,t, \quad \kappa > 0$$

In this last case, notice that $|w| < c$ (respectively, $|w| > c$) if and only if $\varepsilon = 1$ (resp., $\varepsilon = -1$). We fix initially $\kappa < 0$. Then necessarily $\varepsilon = 1$ and

$$v = w^2 = -c^2\kappa\,\mathrm{ct}^{-2}_{-\kappa}(-ct/2)$$

so that

$$\varepsilon u = v + c^2 = c^2\left(1 - \kappa\,\mathrm{ct}^{-2}_{-\kappa}(-ct/2)\right) = c^2\mathrm{cs}^{-2}_{-\kappa}(ct/2).$$

Since $u = 1 + \kappa r^2 = 1/\mathrm{cs}^2_\kappa(\rho/2)$ and $\varepsilon c^2 = 1 + \kappa\varepsilon/4H^2 = 1 + \kappa/4H^2\varepsilon$ then

$$\big(4H^2\varepsilon + \kappa\big)\,\mathrm{sn}^2_\kappa(\rho/2) + 4H^2\varepsilon\,\mathrm{sn}^2_{-\kappa}(ct/2) = 1, \quad (\kappa < 0), \tag{22}$$

where $c = \sqrt{\varepsilon + \kappa/4H^2}$ and $\varepsilon = 1$. The same formula holds for $\kappa > 0$, $\varepsilon = 1$. We have for $\kappa > 0$, $\varepsilon = -1$ that

$$4H^2\varepsilon\kappa\,\mathrm{sn}^2_{-\kappa}(ct/2) = -\big(4H^2\varepsilon + \kappa\big)\,\mathrm{cs}^2_\kappa(\rho/2). \tag{23}$$

We now treat the case $\varepsilon + \kappa/4H^2 < 0$. We denote $c^2 = -(\varepsilon + \kappa/4H^2)$. Thus for $\kappa > 0$ and $\varepsilon = -1$ the solution is

$$\big(4H^2\varepsilon + \kappa\big)\,\mathrm{sn}^2_\kappa(\rho/2) - 4H^2\varepsilon\mathrm{sn}^2_\kappa(ct/2) = 1 \tag{24}$$

The same formula holds for $\kappa < 0$, $\varepsilon = -1$ when we have $|w| < c$. For $\varepsilon = 1$, we necessarily have $\kappa < 0$ and $|w| > c$. Thus

$$4H^2\varepsilon\kappa\mathrm{sn}^2_\kappa(ct/2) = \big(4H^2\varepsilon + \kappa\big)\,\mathrm{cs}^2_\kappa(\rho/2). \tag{25}$$

Finally for $\varepsilon + \kappa/4H^2 = 0$ one obtains

$$t^2 = \epsilon\frac{4}{\kappa}\,\mathrm{cs}^2_\kappa(\rho/2). \tag{26}$$

Next, we consider $I = 2H/\kappa$. For this choice we have

$$\big(\frac{\mathrm{d}\rho}{\mathrm{d}t}\big)^2 + \varepsilon = \frac{\kappa}{4H^2}\,\frac{1 - \mathrm{cs}_\kappa(\rho)}{1 + \mathrm{cs}_\kappa(\rho)}.$$

So the resulting equation is

$$\frac{2}{1+\kappa r^2}\,\frac{2H\,\mathrm{d}r}{\sqrt{\kappa^2 r^2-4H^2\varepsilon}}=\mathrm{d}t$$

We change variables considering $u=1/r^2+\kappa=1/\mathrm{sn}_\kappa^2(\rho/2)$. We then change variables again by defining $v=\kappa\big(\varepsilon+\kappa/4H^2\big)-\varepsilon u$. Finally we put $w^2=v$. So

$$\frac{2\mathrm{d}w}{\kappa\big(\varepsilon+\kappa/4H^2\big)-w^2}=\mathrm{d}t.$$

First, we consider $c^2=\varepsilon+\kappa/4H^2>0$. In this case there are no examples with $\kappa<0$. For $\kappa>0$ and $\varepsilon=1$

$$\big(4H^2\varepsilon+\kappa\big)\,\kappa\mathrm{sn}_\kappa^2(\rho/2)=4H^2\varepsilon\,\mathrm{cs}_{-\kappa}^2(\sqrt{\varepsilon+\kappa/4H^2}\,t/2). \tag{27}$$

For $\kappa>0$ and $\varepsilon=-1$

$$\big(4H^2\varepsilon+\kappa\big)\,\mathrm{sn}_\kappa^2(\rho/2)=-4H^2\varepsilon\,\mathrm{sn}_{-\kappa}^2(\sqrt{\varepsilon+\kappa/4H^2}\,t/2). \tag{28}$$

Now, we consider the case $-c^2=\varepsilon+\kappa/4H^2<0$. For $\kappa<0$ and $\varepsilon=1$ we have

$$\big(4H^2\varepsilon+\kappa\big)\kappa\,\mathrm{sn}_\kappa^2(\rho/2)=4H^2\varepsilon\mathrm{cs}_\kappa^2\big(\sqrt{-\big(\varepsilon+\kappa/4H^2\big)}t/2\big) \tag{29}$$

The same expression holds for $\kappa>0$, $\varepsilon=-1$. For $\kappa<0$, $\varepsilon=-1$ we have

$$\big(4H^2\varepsilon+\kappa\big)\,\mathrm{sn}_\kappa^2(\rho/2)=4H^2\varepsilon\mathrm{sn}_\kappa^2\big(\sqrt{-\big(\varepsilon+\kappa/4H^2\big)}t/2\big) \tag{30}$$

Theorem 2. *The revolution surfaces with constant mean curvature H and $Q=0$ on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ correspond to the values $I=-2H/\kappa,\,2H/\kappa$. These surfaces are described by the formulae (22)-(30) just above.*

For $\varepsilon=1$, the formulae above were already obtained in [1] by other integration methods.

2.3 Solving the mean curvature equation

We proved on Section 2.2 that a given CMC helicoidal surface could be deformed on isometric helicoidal surfaces with the same mean curvature. In this section, we give explicit parameterizations to these families.

We denote in what follows the variable $\tilde{s}$ simply as s. Squaring both sides of (18) one finds

$$\varepsilon\,\frac{\big(m^2U^2-\varepsilon b^2\big)\big(1-\kappa(m^2U^2-\varepsilon b^2)\big)-m^4U^2\dot{U}^2}{1-\kappa(m^2U^2-\varepsilon b^2)}=(2H\int\mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho-I)^2. \tag{31}$$

In particular, for $\kappa=0$ since $\mathrm{sn}_\kappa(\rho)=\rho$ and $\rho^2=m^2U^2-\varepsilon b^2$ then (31) becomes

$$\varepsilon\left(m^2U^2-\varepsilon b^2-m^4U^2\dot{U}^2\right)=(Hm^2U^2-H\varepsilon b^2-I)^2. \tag{32}$$

For $\varepsilon = 1$ after the substitutions $x =: mU$ and $z =: x^2$ this equation reads

$$\frac{\dot{z}^2}{4} = -H^2 z^2 + (1 + 2Ha)z - (a^2 + b^2), \tag{33}$$

where $a = H\varepsilon b^2 + I$. This equation was solved in [8] and its solutions completely integrated. For $\varepsilon = -1$ the same substitutions show that (32) becomes

$$\frac{\dot{z}^2}{4} = H^2 z^2 + (1 - 2Ha)z + (a^2 + b^2). \tag{34}$$

Completing squares this equation reads

$$\frac{\dot{z}^2}{4H^2} = \Big(z + \frac{1 - 2Ha}{2H^2}\Big)^2 + \frac{4H^2 b^2 + 4Ha - 1}{4H^4}, \tag{35}$$

for $H \neq 0$ and $\dot{z}^2/4 = z + (a^2 + b^2)$ for $H = 0$. This last equation may be rewritten as

$$\frac{\mathrm{d}z}{\sqrt{z + (a^2 + b^2)}} = 2\mathrm{d}s$$

whose solution is of the form $m^2U^2 = z = s^2 - (a^2 + b^2)$, where $a = I$ since $H = 0$. This family contains a Lorentzian catenoid as initial surface. In fact, considering the values $m = 1$ and $b = 0$, we have $U^2 = s^2 - I^2$ and $\rho^2 = U^2$. So $s = \sqrt{\rho^2 + I^2}$ and $\mathrm{d}s = (\rho/\sqrt{\rho^2 + I^2})\,\mathrm{d}\rho$. The expression (14) reads

$$t = \int \frac{I}{\rho}\,\mathrm{d}s = \int \frac{I}{\sqrt{\rho^2 + I^2}}\,\mathrm{d}\rho = I \operatorname{arcsinh}\big(\rho/I\big)$$

Thus the (half of the) catenoid is described as the graph of

$$\rho = \rho(t) = I\,\sinh\big(t/I\big) \tag{36}$$

We remark that this curve is singular at $t = 0$ and asymptotes a light cone there. For the catenoid we have $\tilde{\theta} = \theta$. We now describe the family associated to such a catenoid by the integrals (13)-(15). For the other members of the family that evolves from the Lorentzian catenoid we have $\rho^2 = m^2U^2 + b^2 = s^2 - (I^2 + b^2) + b^2 = s^2 - I^2$ and $s = \sqrt{\rho^2 + I^2}$. So

$$t = \int \sqrt{\frac{\rho^2 + b^2}{\rho^2 + I^2}}\,\frac{I}{\rho}\,\mathrm{d}\rho \tag{37}$$

and the coordinate $\tilde{\theta}(\rho, \theta)$ is given by

$$\tilde{\theta}(\rho, \theta) = m\,\theta - mbI \int \frac{1}{\rho^2\,\sqrt{\rho^2 - I^2}\sqrt{\rho^2 + b^2}}\,\mathrm{d}\rho. \tag{38}$$

Turning back to the Lorentzian equation (34) for $H \neq 0$, if we consider $w = z + \frac{1-2Ha}{2H^2}$ and $c^2 = |\frac{4H^2b^2+4Ha-1}{4H^4}|$ we have

$$\int \frac{\mathrm{d}w}{\sqrt{w^2 \pm c^2}} = 2Hs$$

whose general solutions are, for sign +

$$m^2U^2 = c\,\sinh\big(2H(s-s_0)\big) + \frac{1-2Ha}{2H^2} \tag{39}$$

and for sign −

$$m^2U^2 = c\,\cosh\big(2H(s-s_0)\big) + \frac{1-2Ha}{2H^2}, \tag{40}$$

where

$$c = \Big|\frac{4H^2b^2+4Ha-1}{4H^4}\Big|^{1/2}.$$

We may make explicit the parametrization describing both U and $U\dot{U}$ in terms of these solutions.

Theorem 3. *A family of maximal space-like helicoidal surfaces in $\mathbb{L}^3$ containing a Lorentzian catenoid is described by the formulae (36)-(38). The formulae (39) and (40) describe families of helicoidal CMC surfaces on $\mathbb{L}^3$.*

Now, we consider the case $\kappa \neq 0$. Since $\frac{\mathrm{d}}{\mathrm{d}\rho}\,\mathrm{cs}_\kappa(\rho) = -\kappa\,\mathrm{sn}_\kappa(\rho)$ then

$$(-2H\kappa\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho + \kappa I)^2 = 4H^2\mathrm{cs}^2_\kappa(\rho) + 4H\kappa I\mathrm{cs}_\kappa(\rho) + \kappa^2 I^2. \tag{41}$$

Since $\mathrm{cs}_\kappa(\rho) = (1-\kappa(m^2U^2-\varepsilon b^2))^{1/2}$, defining $z =: (1-\kappa(m^2U^2-\varepsilon b^2))^{1/2}$ for $\kappa \neq 0$ one finds $z^2 - 1 - \varepsilon\kappa b^2 = -\kappa m^2U^2$. Therefore $z\dot{z} = -\kappa m^2 U\dot{U}$ which implies that $\kappa^2 m^4U^2\dot{U}^2 = z^2\dot{z}^2$. Multiplying both sides of the expression (31) by κ^2 and replacing the expression (41) on the right hand side of the resulting equation we obtain a first integral to the equation (21)

$$\kappa^2\varepsilon\,\frac{\big(m^2U^2-\varepsilon b^2\big)\big(1-\kappa(m^2U^2-\varepsilon b^2)\big) - m^4U^2\dot{U}^2}{1-\kappa(m^2U^2-\varepsilon b^2)} = \big(2H\big(1-\kappa(m^2U^2$$
$$-\varepsilon b^2)\big)^{1/2} + \kappa I\big)^2. \tag{42}$$

In terms of z this equation reads

$$\dot{z}^2 = -(4H^2\varepsilon+\kappa)\,z^2 - 4H\kappa I\varepsilon\,z + \kappa(1-\kappa I^2\varepsilon). \tag{43}$$

If we assume that $4H^2\varepsilon+\kappa \neq 0$ then we obtain after completing squares that

$$\frac{\dot{z}^2}{4H^2\varepsilon+\kappa} = -\Big(z + \frac{2H\kappa I\varepsilon}{4H^2\varepsilon+\kappa}\Big)^2 + \frac{\kappa}{(4H^2\varepsilon+\kappa)^2}\,\big(4H^2\varepsilon+\kappa-\kappa^2I^2\varepsilon\big). \tag{44}$$

We first consider the case $4H^2\varepsilon+\kappa < 0$. If $\kappa\big(4H^2\varepsilon+\kappa-\kappa^2I^2\varepsilon\big) < 0$ then putting $w = z + \frac{2H\kappa I\varepsilon}{4H^2\varepsilon+\kappa}$ we get

$$-\frac{\dot{w}^2}{4H^2\varepsilon+\kappa} = w^2 + c^2, \tag{45}$$

where
$$c^2 = \Big|\frac{\kappa}{(4H^2\varepsilon+\kappa)^2}\big(4H^2\varepsilon+\kappa-\kappa^2I^2\varepsilon\big)\Big|.$$
The general solution is in this case
$$z = c\sinh\big((-4H^2\varepsilon-\kappa)^{1/2}(s-s_0)\big) - \frac{2H\kappa I\varepsilon}{4H^2\varepsilon+\kappa}. \tag{46}$$
If $\kappa\big(4H^2\varepsilon+\kappa-\kappa^2I^2\varepsilon\big)>0$ then
$$-\frac{\dot{w}^2}{4H^2\varepsilon+\kappa} = w^2-c^2$$
with solution given by
$$z = c\cosh\big((-4H^2\varepsilon-\kappa)^{1/2}(s-s_0)\big) - \frac{2H\kappa I\varepsilon}{4H^2\varepsilon+\kappa}. \tag{47}$$
Now we consider the case $4H^2\varepsilon+\kappa>0$. Here we necessarily have $\kappa\big(4H^2\varepsilon+\kappa-\kappa^2I^2\varepsilon\big)>0$. The equation becomes
$$\frac{\dot{w}^2}{4H^2\varepsilon+\kappa} = c^2-w^2,$$
whose solution is
$$z = c\sin\big((4H^2\varepsilon+\kappa)^{1/2}(s-s_0)\big) - \frac{2H\kappa I\varepsilon}{4H^2\varepsilon+\kappa}. \tag{48}$$
It remains to see what happens for $4H^2\varepsilon+\kappa=0$. In this case the equation becomes
$$\dot{z}^2 = -4H\kappa I\varepsilon\, z + \kappa(1-\kappa I^2\varepsilon).$$
If $H\kappa I=0$ then we have necessarily $\kappa(1-\kappa I^2\varepsilon)>0$ and
$$z = \big(\kappa(1-\kappa I^2\varepsilon)\big)^{1/2}(s-s_0). \tag{49}$$
When $H\kappa I\neq 0$ then the equation is
$$\frac{\mathrm{d}z}{\sqrt{-4H\kappa I\varepsilon\, z+\kappa(1-\kappa I^2\varepsilon)}} = \mathrm{d}s$$
with solution
$$z = -\frac{1}{4H\kappa I\varepsilon}\big(\frac{1}{4}(s-s_0)^2 - \kappa(1-\kappa I^2\varepsilon)\big). \tag{50}$$

Theorem 4. *The formulae (46)-(50) describe two-parameter families of helicoidal CMC examples on* $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

For $\varepsilon=1$, the formulae above were previously obtained in [18].

3 Rotationally invariant CMC discs on Lorentzian products

3.1 Qualitative description

In this section we consider only space-like revolution surfaces in Lorentzian products $\mathbb{M}^2(\kappa) \times \mathbb{R}$. We assume that the parameter s on (1) is the arc length of the profile curve. So, $\dot{\rho}^2 - \dot{t}^2 = 1$. We denote by φ the hyperbolic angle with the horizontal axis ∂_ρ. So, Σ has constant mean curvature H if and only if $(\rho(s), t(s), \varphi(s))$ is solution to the following ordinary differential equations system

$$\begin{aligned}
\dot{\rho} &= \cosh\varphi, \\
\dot{t} &= \sinh\varphi, \\
\dot{\varphi} &= -2H - \sinh\varphi \,\mathrm{ct}_\kappa(\rho).
\end{aligned} \tag{51}$$

The *flux* I' through an horizontal plane $\mathbb{P}_t = \mathbb{M}^2(\kappa) \times \{t\}$ is, up to a constant, given by the expression for I in terms of s:

$$I' = I + \frac{2H}{\kappa} = \sinh\varphi \,\mathrm{sn}_\kappa(\rho) + 2H \int_0^\rho \mathrm{sn}_\kappa(\tau)\,\mathrm{d}\tau. \tag{52}$$

Integrating the last term on (52) one obtains

$$I' = \sinh\varphi \,\mathrm{sn}_\kappa(\rho) + 4H\mathrm{sn}^2_\kappa(\frac{\rho}{2}). \tag{53}$$

The solutions for (51) for which $Q = 0$ vanishes are those with $I = \pm\frac{2H}{\kappa}$ or $I' = 0, \frac{4H}{\kappa}$. We give later a qualitative description of these solutions.

Since that $\cosh\varphi$ never vanishes on the maximal interval for a solution to (51) it follows that

$$\frac{\mathrm{d}t}{\mathrm{d}\rho} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\rho} = \tanh\varphi.$$

Denoting $u = \sinh\varphi$ we also obtain

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = \frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\rho} = -2H - \sinh\varphi \,\mathrm{ct}_\kappa(\rho).$$

Thus the system (51) above is equivalent to

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}\rho} &= \frac{u}{\sqrt{1+u^2}}, \\
\frac{\mathrm{d}u}{\mathrm{d}\rho} &= -2H - u\,\mathrm{ct}_\kappa(\rho).
\end{aligned} \tag{54}$$

It is clear that solutions to the system (54) are defined on the whole real line and the profile curve may be written as a graph over the ρ-axis. Now, we begin describing the maximal solutions, i.e., solutions for $H = 0$. If we consider a fixed value for I' then the condition $H = 0$ implies that

$$I' = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) \tag{55}$$

So, the horizontal planes are the unique maximal revolution surfaces with $I' = 0$. In fact if we put $I' = 0$ at (55) we have $\sinh\varphi = 0$ for $\rho > 0$. Thus, $\dot{t} = 0$ and we conclude that the solution is an horizontal plane. Hence, we may assume $I \neq 0$. In this case, since that $\mathrm{sn}_\kappa(\rho) \to 0$ if $\rho \to 0$ is follows that $\sinh\varphi \to \infty$ if $\rho \to 0$. So, Σ has a singularity and asymptotes the light cone at p_0 (the light cone corresponds to $\varphi = \infty$). Moreover $\sinh\varphi \to 0$ if $\rho \to \infty$ in the case $\kappa \leq 0$. This means that these maximal surfaces asymptotes an horizontal plane for $\rho \to \infty$, i.e., these surfaces have planar ends. These examples are not complete in the spherical case $\kappa > 0$, since we have $\sinh\varphi \to \infty$ if $\rho \to \frac{\pi}{\sqrt{\kappa}}$.

Consider now the case $H \neq 0$. We observe that the solutions for (54) have no positive minimum for ρ. Otherwise, the solutions must have vertical tangent plane at the minimum points (this is impossible since the solutions are space-like and, in fact, are graphs over the horizontal axis). Hence, the unique possibility for the existence of a isolated singularity is that $\rho \to 0$. In this case the solutions are regular if and only if the $\varphi \to 0$ as $\rho \to 0$ what implies that $\sinh\varphi \to 0$ as $\rho \to 0$. So, necessarily $I' = 0$ as we could see taking the limit $\rho \to 0$ in (53) above. So, examples of solutions for the systems above which touch orthogonally the revolution axis have $I' = 0$. Reciprocally, if we put $I' = 0$ in (53) we get

$$0 = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 4H\mathrm{sn}^2_\kappa(\frac{\rho}{2}).$$

So, dividing the expression above by $2\mathrm{sn}^2_\kappa(\frac{\rho}{2})$ we have

$$\sinh\varphi\,\mathrm{ct}_\kappa(\frac{\rho}{2}) = -2H. \tag{56}$$

One easily verifies that $\sinh\varphi \to 0$ if $\rho \to 0$. So all solutions for (54) with $I' = 0$ reach the revolution axis orthogonally as we noticed earlier. Thus these solutions correspond to initial conditions $t(0) = t_0$, $\rho(0) = 0$ and $\varphi(0) = 0$ for the system (51). Now we have

$$\mathrm{ct}_\kappa(\rho) = \frac{1}{2}\big(-\frac{2H}{\sinh\varphi} + \kappa\,\frac{\sinh\varphi}{2H}\big) = \frac{-4H^2 + \kappa\sinh^2\varphi}{4H\sinh\varphi}.$$

Replacing this on the third equation on (51) we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{1}{4H}\,(-4H^2 - \kappa\sinh^2\varphi). \tag{57}$$

We observe that $\dot{\varphi} = -H$ is the corresponding equation for the case $\kappa = 0$, i.e., for hyperbolic spaces in $\mathbb{L}^3$. This could be obtained as a limiting case if we take $\kappa \to 0$. For $\kappa < 0$, the range for the angle φ is $0 \leq \varphi < \varphi_\infty =$

$\operatorname{arcsinh}(2|H|/\sqrt{-\kappa})$. The surface necessarily asymptotes a spacelike cone with angle φ_∞. Indeed the equation (57) is equivalent to

$$\frac{1}{4H}\int_0^{\varphi_\infty}\frac{\mathrm{d}\varphi}{-4H^2-\kappa\sinh^2\varphi}=\int_0^\infty \mathrm{d}s=\infty.$$

There are no complete solutions for $\kappa>0$ and $H\neq 0$, since that the angle at $\rho=0$ and at $\rho=\frac{\pi}{\sqrt{\kappa}}$ are not the same unless we have $H=0$.

Finally, we study the case when $\varphi\to\varphi_0$ as $\rho\to 0$ for some positive value of φ_0. This means that the solution asymptotes a space-like cone at p_0. In this case $\sinh\varphi\to\sinh\varphi_0<\infty$ as $\rho\to 0$. Thus taking the limit $\rho\to 0$ in (53) we obtain $I'=0$. So, as we seen above, necessarily $\varphi_0=0$. This contradiction implies that there are no examples with $\varphi_0>0$.

It remains to give a look at the case $\varphi\to\infty$ as $\rho\to 0$. In this case, the solution asymptotes the light cone at p_0. For any non zero value of I', we obtain after dividing (53) by $\operatorname{sn}^2_\kappa(\rho/2)$ and taking limit for $\rho\to\infty$ that $\sinh\varphi\to 2|H|/\sqrt{-\kappa}$. Moreover, the angle φ is always decreasing in the range $(2|H|/\sqrt{-\kappa},\infty)$ as ρ increases in $(0,\infty)$. For example, consider the values $\kappa<0$ and $I'=\frac{4H}{\kappa}$. Replacing this value for I' in (52) we get

$$0=\sinh\varphi\,\operatorname{sn}_\kappa(\rho)+4H\big(\operatorname{sn}^2_\kappa(\frac{\rho}{2})-\frac{1}{\kappa}\big).$$

So we conclude that

$$\kappa\sinh\varphi=2H\operatorname{ct}_\kappa\big(\frac{\rho}{2}\big). \tag{58}$$

Thus the solution satisfies $\sinh\varphi\to\infty$ if $\rho\to 0$. This means that Σ asymptotes the light cone at the point p_0. Moreover, we have that $\sinh\varphi\to 2|H|/\sqrt{-\kappa}$ if $\rho\to\infty$. Replacing (58) at the third equation in (51) we obtain

$$\operatorname{ct}_\kappa(\rho)=\frac{1}{2}\big(\kappa\frac{\sinh\varphi}{2H}-\frac{2H}{\sinh\varphi}\big)=\frac{-4H^2+\kappa\sinh^2\varphi}{4H\sinh\varphi}$$

and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s}=\frac{1}{4H}\big(-4H^2-\kappa\sinh^2\varphi\big).$$

Since φ satisfies $\sinh\varphi>\sinh\varphi_\infty=\frac{2|H|}{\sqrt{-\kappa}}$ then we conclude that $\dot\varphi<0$ for all s. So, the angle decreases from ∞ at $\rho\to 0$ to its infimum value φ_∞ as $\rho\to\infty$.

We summarize the facts above in the following theorem.

Theorem 5. *Let Σ be a rotationally invariant surface with constant mean curvature H in the Lorentzian product $\mathbb{M}^2(\kappa)\times\mathbb{R}$ with $\kappa\leq 0$. If $H=0$ either Σ is a horizontal plane $\mathbb{P}_t=\mathbb{M}^2(\kappa)\times\{t\}$ or Σ asymptotes a light cone with vertex at some point p_0 of the rotation axis. In this case, Σ has a singularity at p_0 and has horizontal planar ends. We refer to these singular surfaces as* Lorentzian catenoids.

If $H\neq 0$ either Σ is a complete disc-type surface meeting orthogonally the rotation axis or Σ asymptotes a light cone with vertex p_0 at the rotation axis. In *the first case, the angle between the surface and the horizontal planes asymptotes* $2|H|/\sqrt{-\kappa}$ *as the surface goes to the asymptotic boundary* $\partial_\infty \mathbb{M}^2(\kappa) \times \mathbb{R}$*. In the last case, the surface is singular at* p_0 *and asymptotes a space-like cone with vertex at* p_0 *and slope* φ_∞ *where* $\sinh \varphi_\infty = 2|H|/\sqrt{-\kappa}$*.*

3.2 Uniqueness of annular CMC surfaces

We fix $\varepsilon = -1$ and $\kappa \leq 0$ on this section. We then present a version of a theorem proved by R. López (see [12], Theorem 1.2) about uniqueness of annular CMC in Minkowski space $\mathbb{L}^3$.

Let Σ_1 be a connected CMC space-like surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ whose boundary is a geodesic circle Γ in some plane $\mathbb{P}_a$. We suppose that Σ_1 is a graph over $\mathbb{P}_a - \Omega$, where Ω is the domain bounded by Γ on $\mathbb{P}_a$. We further suppose that the angle of Σ_1 with respect to the planes $\mathbb{P}_t$ asymptotes, when Σ_1 approaches $\partial_\infty \mathbb{M}^2(\kappa) \times \mathbb{R}$, a value φ^1_∞ so that $\sinh(\varphi^1_\infty) \geq 2|H|/\sqrt{-\kappa}$. We then consider Σ_2 a revolution surface with same mean curvature, boundary and flux than Σ_1. That this is possible we infer from the description on Theorem 5 above. From the same theorem, we know that the asymptotic angle for Σ_2 is $\varphi^2_\infty = \operatorname{arcsinh}(2|H|/\sqrt{-\kappa})$.

Suppose that $\Sigma_1 \neq \Sigma_2$. Now, we move Σ_1 upwards until there is no contact with Σ_2. This is possible since the asymptotic angle of Σ_1 is greater than or equal to the asymptotic angle of Σ_2. Denote by $\Sigma_1(t)$ the copy of Σ_1 translated t upwards (so that $\Sigma_1(0) = \Sigma_1$). Then we define t_0 as the height where occurs the first contact point. Suppose that $t_0 > 0$. Then, the first contact is not at an interior point. Otherwise, by the interior maximum principle, the surfaces are coincident, what contradicts our hypothesis. If the asymptotic angles are different, there are no point of contact at infinity. If the angles are equal, then for small δ the surfaces $\Sigma_1(t_0 - \delta)$ and Σ_2 intersect transversally. We claim that there exists a connected component Γ' on $S = \Sigma_1(t_0 - \delta) \cap \Sigma_2$ which is not null homologous on both surfaces. Since both graphs have the topology of a punctured plane, this means that Γ' must be homologous to Γ on Σ_2. Suppose by contradiction that all components of S will be null homologous. So, each component Γ' of S bounds a disc on both the graphs with common boundary given by Γ'. These two discs are graphs over a disc on $\mathbb{P}_a$ with the same mean curvature and same boundary. By maximum principle they are equal. By analyticity, this implies that the graphs coincide globally. From this contradiction, we conclude that there exists component Γ' of S not null homologous. The flux of $\Sigma_1(t_0 - \delta)$ and Σ_2 through Γ' are both equal to the flux of Σ_1 and Σ_2 through Γ. However, after crossing Σ_2 along Γ' towards $\partial_\infty \mathbb{M}^2(\kappa) \times \mathbb{R}$, the surface $\Sigma_1(t_0 - \delta)$ remains below Σ_2. Then since ∂_t is a time-like vector, it holds that

$$\langle \eta_2, \partial_t \rangle < \langle \eta_1, \partial_t \rangle.$$

along Γ', where η_1 and η_2 are the *outward* unit co-normal of $\Sigma_a(t_0 - \delta)$ and Σ_2 along Γ'. However, this contradicts the fact that the flux is the same on both surfaces. This contradiction implies that $t_0 = 0$. Now, if the surfaces contact at the boundary, they coincide globally, by the boundary maximum principle. If not, then the angles satisfy again a strict inequality and therefore the flux is not the same for the two surfaces, a contradiction. We conclude from these contradictions that $\Sigma_1 = \Sigma_2$.

Theorem 6. *Let Σ be a space-like CMC surface on $\mathbb{M}^2(\kappa) \times \mathbb{R}$, $\kappa \leq 0$, whose boundary is a geodesic circle on a horizontal plane $\mathbb{P}_a$. We suppose that Σ is a graph over the domain in $\mathbb{P}_a$ outside the disc bounded by $\partial\Sigma$. We further suppose that the angle between Σ and the horizontal planes asymptotes φ_∞ with $\varphi_\infty \geq \operatorname{arcsinh}(2|H|/\sqrt{-\kappa})$. Then, Σ is contained on a revolution surface whose axis passes through the center of $\partial\Sigma$ on $\mathbb{P}_a$.*

A similar reasoning shows, under the same hypothesis on the asymptotic angle, that an entire space-like surface with an isolated singularity and constant mean curvature is a singular revolution surface (v. [12], Theorem 1.3).

4 Hopf differentials in some product spaces

Let Σ be a Riemann surface and $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an isometric immersion. If $\kappa \geq 0$, we may consider Σ as immersed in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. If $\kappa < 0$, we immerse Σ in $\mathbb{L}^3 \times \mathbb{R}$. In fact, we may write $X = (p, t)$, with $t \in \mathbb{R}$ and $p \in \mathbb{M}^2(\kappa) \subset \mathbb{R}^3$, in the first case and $p \in \mathbb{M}^2(\kappa) \subset \mathbb{L}^3$ for $\kappa < 0$. By writing $\mathbb{M}^2(\kappa) \times \mathbb{R} \subset \mathbb{E}^4$ we mean all these possibilities. The metric and covariant derivative in $\mathbb{E}^4$ are also denoted by $\langle \cdot, \cdot \rangle$ and D respectively. We denote by ϵ the sign of κ. Recall that $\varepsilon = 1$ for Riemannian products and $\varepsilon = -1$ for Lorentzian ones.

Let (u, v) be local coordinates in Σ for which $X(u, v)$ is a conformal immersion inducing the metric $e^{2\omega}\,(\mathrm{d}u^2 + \mathrm{d}v^2)$ in Σ. So, denote by ∂_u, ∂_v the coordinate vectors and let $e_1 = e^{-\omega}\partial_u$, $e_2 = e^{-\omega}\partial_v$ be the associated local orthonormal frame tangent to Σ. The unit normal directions to Σ in $\mathbb{E}^4$ are denoted by $n_1, n_2 = p/r$, where $r = (\epsilon\,\langle p, p\rangle)^{1/2}$. We denote by h^k_{ij} the components of h^k, the second fundamental form of Σ with respect to n_k, $k = 1, 2$. Then

$$h^k_{ij} = \langle D_{e_i} e_j, n_k \rangle.$$

It is clear that the h^1_{ij} are the components of the second fundamental form of the immersion $\Sigma \looparrowright \mathbb{M}^2(\kappa) \times \mathbb{R}$. The components of h^2 are

$$\begin{aligned} h^2_{ij} &= \langle D_{e_i} e_j, n_2 \rangle = \langle D_{e^h_i} e^h_j, p/r \rangle = -\frac{1}{r}\langle e^h_i, e^h_j \rangle = \frac{1}{r}\big(\varepsilon \langle e^t_i, e^t_j \rangle - \delta_{ij}\big) \\ &= \frac{1}{r}\big(\varepsilon \langle e_i, \partial_t \rangle \langle e_j, \partial_t \rangle - \delta_{ij}\big) = \frac{\varepsilon}{r}\,\langle e_i, \partial_t \rangle \langle e_j, \partial_t \rangle - \frac{1}{r}\delta_{ij}. \end{aligned}$$

We remark that $\kappa = \epsilon/r^2$. The components of h^1 and h^2 in the frame ∂_u, ∂_v are respectively

$$e = h^1(\partial_u, \partial_u) = e^{2\omega} h^1_{11},\ f = h^1(\partial_u, \partial_v) = e^{2\omega} h^1_{12},\ g = h^1(\partial_v, \partial_v) = e^{2\omega} h^1_{22}$$

and

$$\tilde{e} = h^2(\partial_u, \partial_u) = e^{2\omega} h^2_{11},\ \tilde{f} = h^2(\partial_u, \partial_v) = e^{2\omega} h^2_{12},\ \tilde{g} = h^2(\partial_v, \partial_v) = e^{2\omega} h^2_{22}.$$

The Hopf differential associated to h^k is defined by $\Psi^k = \psi^k \mathrm{d}z^2$, where $z = u+iv$ and the coefficients ψ^1, ψ^2 are

$$\psi^1 = \frac{1}{2}(e-g) - i\, f, \quad \psi^2 = \frac{1}{2}(\tilde{e}-\tilde{g}) - i\, \tilde{f}.$$

The mean curvature of X is by definition $H = (h^1_{11} + h^1_{22})/2$. Differentiating the real part of ψ^1 we obtain

$$\begin{aligned}
&\partial_u\Big(\frac{e-g}{2}\Big) = \partial_u\Big(\frac{e+g}{2} - g\Big) = \partial_u(e^{2\omega}H) - \partial_u g = \partial_u(e^{2\omega}H) - \partial_u\big(h^1(\partial_v, \partial_v)\big)\\
&= \partial_u(e^{2\omega}H) - \big(D_{\partial_u} h^1(\partial_v, \partial_v) + 2h^1(D_{\partial_u}\partial_v, \partial_v)\big)\\
&= \partial_u(e^{2\omega}H) - \big(D_{\partial_v} h^1(\partial_u, \partial_v) + \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle + 2h^1(D_{\partial_u}\partial_v, \partial_v)\big)\\
&= \partial_u(e^{2\omega}H) - \big(\partial_v(h^1(\partial_u, \partial_v)) - h^1(D_{\partial_v}\partial_u, \partial_v) - h^1(\partial_u, D_{\partial_v}\partial_v)\\
&+\langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle + 2h^1(D_{\partial_u}\partial_v, \partial_v)\big)\\
&+\langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle\big)\\
&= \partial_u(e^{2\omega}H) - \big(\partial_v f + \Gamma^1_{12} f + \Gamma^2_{12} g - \Gamma^1_{22} e - \Gamma^2_{22} f + \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle\big)\\
&= \partial_u(e^{2\omega}H) - \big(\partial_v f + f\partial_v\omega + g\partial_u\omega + e\partial_u\omega - f\partial_v\omega + \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle\big)\\
&= \partial_u(e^{2\omega}H) - \big(\partial_v f + (e+g)\partial_u\omega + \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle\big)\\
&= \partial_u(e^{2\omega}H) - 2e^{2\omega}H\partial_u\omega - \partial_v f - \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle\\
&= -\partial_v f + e^{2\omega}\partial_u H - \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle.
\end{aligned}$$

By similar calculations we also obtain

$$\partial_v\Big(\frac{e-g}{2}\Big) = \partial_u f - e^{2\omega}\partial_v H + \langle \bar{R}(\partial_v, \partial_u)n_1, \partial_u\rangle.$$

We used above the Codazzi equation

$$D_{\partial_u} h^1(\partial_v, \partial_v) = D_{\partial_v} h^1(\partial_u, \partial_v) + \langle \bar{R}(\partial_u, \partial_v)n_1, \partial_v\rangle$$

and the following expressions for the Christoffel symbols Γ^k_{ij} for the metric $e^{2\omega}\delta_{ij}$ in Σ

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \partial_u\omega,\ \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \partial_v\omega.$$

An easy calculation yields the components of the curvature tensor

$$\langle \bar{R}(\partial_u, \partial_v)n_1, \partial v\rangle = \kappa\, e^{2\omega}\langle \partial^h_u, n^h_1\rangle, \quad \langle \bar{R}(\partial_v, \partial_u)n_1, \partial u\rangle = \kappa\, e^{2\omega}\langle \partial^h_v, n^h_1\rangle.$$

By this way, we then obtain the following pair of equations

$$\partial_u \Re\psi^1 = \partial_v \Im\psi^1 - \kappa\, e^{2\omega}\langle \partial^h_u, n^h_1\rangle + e^{2\omega}\partial_u H, \tag{59}$$
$$\partial_v \Re\psi^1 = -\partial_u \Im\psi^1 + \kappa\, e^{2\omega}\langle \partial^h_v, n^h_1\rangle - e^{2\omega}\partial_v H. \tag{60}$$

One also calculates

$$
\begin{aligned}
&\partial_u \Re\psi^2 = \frac{\varepsilon}{2r}\,\partial_u\Big(\langle\partial_u,\partial_t\rangle^2 - \langle\partial_v,\partial_t\rangle^2\Big) = \frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle\\
&-\langle\partial_v,\partial_t\rangle\,\langle D_{\partial_u}\partial_v,\partial_t\rangle\Big) = \frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle - \langle\partial_v,\partial_t\rangle\,\langle D_{\partial_v}\partial_u,\partial_t\rangle\Big)\\
&= \frac{\varepsilon}{r}\Big(\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u,\partial_t\rangle - \partial_v(\langle\partial_v,\partial_t\rangle\,\langle\partial_u,\partial_t\rangle) + \langle D_{\partial_v}\partial_v,\partial_t\rangle\,\langle\partial_u,\partial_t\rangle\Big)\\
&= \frac{\varepsilon}{r}\langle\partial_u,\partial_t\rangle\,\langle D_{\partial_u}\partial_u + D_{\partial_v}\partial_v,\partial_t\rangle - \frac{\varepsilon}{r}\,\partial_v\big(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle\big)\\
&= \frac{1}{r}\,\langle\partial_u,\partial_t\rangle e^{2\omega}\,\Delta t - \frac{\varepsilon}{r}\,\partial_v\big(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle\big)\\
&= 2H\frac{1}{r}e^{2\omega}\langle\partial_u,\partial_t\rangle\,\langle n_1,\partial_t\rangle - \frac{\varepsilon}{r}\,\partial_v\big(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle\big) = -2H\frac{\varepsilon}{r}e^{2\omega}\langle\partial_u^h,n_1^h\rangle\\
&-\frac{\varepsilon}{r}\,\partial_v\big(\langle\partial_u,\partial_t\rangle\,\langle\partial_v,\partial_t\rangle\big) = -2H\frac{\varepsilon}{r}\,e^{2\omega}\langle\partial_u^h,n_1^h\rangle +\,\partial_v\Im\psi^2.
\end{aligned}
$$

We used above the formula $\Delta t = 2H\langle n_1,\partial_t\rangle$, where Δ is the Laplacian on Σ (see Section 6). Similarly, we prove that

$$\partial_v\Re\psi^2 = -\partial_u\Im\psi^2 + 2H\frac{\varepsilon}{r}\,e^{2\omega}\langle\partial_v^h,n_1^h\rangle.$$

Then, using the above mentioned fact that $\kappa = \epsilon/r^2$, we conclude that the function $\psi := 2H\psi^1 - \varepsilon\frac{\epsilon}{r}\,\psi^2$ satisfies

$$
\begin{aligned}
&\partial_u\Re\psi = \partial_v\Im\psi + 2\Re\psi^1 H_u - 2\Im\psi^1 H_v + 2e^{2\omega}HH_u = \partial_v\Im\psi + 2eH_u + 2fH_v,\\
&\partial_v\Re\psi = -\partial_u\Im\psi + 2\Re\psi_1 H_v + 2\Im\psi^1 H_u - 2e^{2\omega}HH_v = -\partial_u\Im\psi - 2gH_v - 2fH_u.
\end{aligned}
$$

Now, using the complex parameter $z = u + iv$ and the complex derivation $\partial_{\bar z} = \frac{1}{2}(\partial_u + i\partial_v)$ we get

$$
\begin{aligned}
\partial_{\bar z}\psi &= (\partial_u\Re\psi - \partial_v\Im\psi) + i(\partial_v\Re\psi + \partial_u\Im\psi)\\
&= 2eH_u + 2fH_v - 2ifH_u - 2igH_v
\end{aligned}
$$

That is, defining the quadratic differential $Q := 2H\,\Psi^1 - \varepsilon\frac{\epsilon}{r}\,\Psi^2$ we prove that Q is holomorphic on Σ if H is constant. Inversely, if Q is holomorphic then

$$e\,H_u + f\,H_v = 0, \quad f\,H_u + g\,H_v = 0$$

We may write this system in the following matrix form

$$\begin{bmatrix} e & f\\ f & g\end{bmatrix}\begin{bmatrix} H_u\\ H_v\end{bmatrix} = \begin{bmatrix} 0\\ 0\end{bmatrix}.$$

This implies that $A\nabla H = 0$, where $A = \langle \mathrm{d}X,\mathrm{d}X\rangle^{-1}\langle \mathrm{d}n_1,\mathrm{d}X\rangle$ is the shape operator for X and ∇H is the gradient of H on Σ. If $\nabla H = 0$, i.e., $H_u = H_v = 0$ on Σ, then H is constant. Thus, we may suppose that $\nabla H \neq 0$ on an (open) set Σ' of Σ. On Σ' we have $K_{\text{ext}} =: \det A = 0$. However, $\det A = 0$ is a closed condition. So, Σ' is clopen and therefore $\Sigma' = \Sigma$. Thus, $e_1 =: \nabla H/|\nabla H|$ is a principal direction with principal curvature $\kappa_1 = 0$. Moreover $H = \kappa_2$, where κ_2 is the principal curvature of Σ calculated on a direction e_2 perpendicular to e_1. So, the only planar (umbilical) points on Σ are the points where H vanishes. Moreover, the integral curves of e_2 are level curves for $H = \kappa_2$ since they are orthogonal to ∇H. Thus, H is constant along such each line. So, we proved

Theorem 7. *The quadratic differential $Q = 2H\Psi^1 - \varepsilon\frac{\epsilon}{r}\,\Psi^2$ is holomorphic on Σ if H is constant. Inversely, if we suppose that Σ is compact (more generally, if Σ does not admit a function without critical points, or a vector field without singularities), then H is constant if Q is holomorphic.*

The considerations above imply that if there exist examples of surfaces with holomorphic Q and non constant mean curvature, these examples must be non compact, have zero extrinsic Gaussian curvature and are foliated by curvature lines along which H is constant. Recently, P. Mira and I. Fernández announced to the authors had constructed such examples.

For $\varepsilon = 1$, the quadratic form Q coincides with that one obtained by U. Abresch and H. Rosenberg in ([1]). It is clear that Q is the complexification of the traceless part of the second fundamental form q corresponding to the normal direction $2Hn_1 - \varepsilon\frac{\epsilon}{r}\,n_2$ on the normal bundle of $\Sigma \hookrightarrow \mathbb{E}^4$.

Using the Theorem 7, we present the following generalization of the theorem of Abresch and Rosenberg quoted in the Introduction:

Theorem 8. *Let $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be a complete CMC immersion of a surface Σ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. If $\varepsilon = 1$ and Σ is homeomorphic to a sphere, then $X(\Sigma)$ is a rotationally invariant spherical surface. If Σ is homeomorphic to a disc and $Q \equiv 0$ on Σ, then $X(\Sigma)$ is a rotationally invariant disc. For $\varepsilon = -1$ and $\kappa \leq 0$, if $X(\Sigma)$ is simply-connected, space-like and $Q \equiv 0$ on Σ, then the same conclusion holds.*

Proof of the Theorem 8. By hypothesis, we have $Q \equiv 0$ (if Σ is homeomorphic with a sphere, this follows from the fact that Q is holomorphic). Thus, $2H\psi^1 \equiv \varepsilon\frac{\epsilon}{r}\,\psi^2$. Given an arbitrary local orthonormal frame field $\{e_1, e_2\}$, we may write this as

$$2Hh^1_{12} = \kappa\,\langle e_1, \partial_t\rangle\langle e_2, \partial_t\rangle, \tag{61}$$

$$2H(h^1_{11} - h^1_{22}) = \kappa\,\langle e_1, \partial_t\rangle^2 - \kappa\,\langle e_2, \partial_t\rangle^2. \tag{62}$$

If $H = 0$, then it follows from these equations that the vector field ∂_t is always normal to Σ. So, the surface is part of a plane $\mathbb{P}_t = \mathbb{M}^2(\kappa) \times \{t\}$, for some $t \in \mathbb{R}$. Since Σ is complete, we conclude that $\Sigma = \mathbb{P}_t$.

We then may consider only CMC surfaces with $H \neq 0$. If (p, t) is an umbilical point of Σ we have for an arbitrary frame that $h^1_{12} = 0$ at this point. So, either $\langle e_1, \partial_t\rangle = 0$ or $\langle e_2, \partial_t\rangle = 0$ at (p, t). Since $h^1_{11} = h^1_{22} = H$ at (p, t) the equation (62) implies that both angles $\langle e_i, \partial_t\rangle$ are null. So, we conclude that if $Q = 0$, then umbilical points are the points where Σ has horizontal tangent plane, and vice-versa.

If (p,t) is not an umbilical point in Σ, we may choose the frame $\{e_1, e_2\}$ as principal frame locally defined (on a neighborhood Σ' of that point). Thus, $h^1_{12} = 0$ and therefore $\langle e_1, \partial_t\rangle = 0$ or $\langle e_2, \partial_t\rangle = 0$ on Σ'. We fix $\langle e_1, \partial_t\rangle = 0$. If we denote by τ the tangential part $\partial_t - \varepsilon\langle \partial_t, n_1\rangle n_1$ of the field ∂_t, then $\tau = \langle e_2, \partial_t\rangle\, e_2$. Thus from (62) it follows that the principal curvatures of Σ are

$$h^1_{11} = H - \frac{\kappa}{4H}\,|\tau|^2, \quad h^1_{22} = H + \frac{\kappa}{4H}\,|\tau|^2.$$

The lines of curvature on Σ' with direction e_1 are locally contained in the planes $\mathbb{P}_t$. Inversely, the connected components of $\Sigma' \cap \mathbb{P}_t$ are lines of curvature with tangent direction given by e_1. Thus, if we parameterize such a line by its arc length s, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle n_1, \partial_t\rangle = \langle D_{e_1} n_1, \partial_t\rangle = h^1_{11}\langle e_1, \partial_t\rangle = 0. \tag{63}$$

We conclude that, for a fixed t, Σ' and $\mathbb{P}_t$ make a constant angle $\theta(t)$ along each connected component of their intersection. So, if a connected component of the intersection between $\mathbb{P}_t$ and Σ has a non umbilical point, then the angle is constant, non zero, along this component, unless that there exists also an umbilical point on this same component. However at this point the angle is necessarily zero. So, by continuity of the angle function, either all points on a connected component $\Sigma \cap \mathbb{P}_t$ are umbilical and the angle is zero, or all points are non umbilical and the angle is non zero. However, supposes that all points on a connected component σ are umbilical points for h^1. Then, as we noticed above, Σ is tangent to $\mathbb{P}_t$ along σ. So, along σ, we have $\langle e_1, \partial_t\rangle = \langle e_2, \partial_t\rangle = 0$ and therefore by equations (61) and (62) we have $h^1_{ii} = 0$ and $H = 0$. From this contradiction, we conclude that the umbilical points may not be on *any* curve on $\Sigma \cap \mathbb{P}_t$. The only possibility is that there exist isolated umbilical points as may occurs on the top and bottom levels $t = a$ and $t = b$ of $X(\Sigma)$.

So, there exists an orthonormal principal frame field $\{e_1, e_2\}$ on a dense subset of Σ. On this dense subset we have $\tau \neq 0$ and then we may choose a positive sign for $\sin\theta(t)$ or $\sinh\theta(t)$, where $\theta(t)$ is the angle between n_1 and ∂_t along a given component of $\Sigma \cap \mathbb{P}_t$. We denote both of these functions by the same symbol $\mathrm{sn}(t)$. Now, we calculate the geodesic curvature of the horizontal curvature lines on $\mathbb{P}_t$. We have

$$e_2 = \frac{\tau}{|\tau|} = \frac{1}{\mathrm{sn}(t)}\,\tau = \frac{1}{\mathrm{sn}(t)}\,(\partial_t - \varepsilon\langle \partial_t, n_1\rangle n_1) = \frac{1}{\mathrm{sn}(t)}\,(\partial_t - \mathrm{s}\dot{\mathrm{n}}(t) n_1)$$

Since $\langle n_1, \partial_t\rangle$ is constant along this curve and therefore $\mathrm{sn}(t)$ is constant we conclude that

$$D_{e_1} e_2 = \frac{1}{\mathrm{sn}(t)}\big(D_{e_1}\partial_t - \mathrm{s}\dot{\mathrm{n}}(t)\, D_{e_1} n_1\big) = \frac{\mathrm{s}\dot{\mathrm{n}}(t)}{\mathrm{sn}(t)}\, h^1_{11} e_1$$

where $\mathrm{s}\dot{\mathrm{n}}(t) = \cos\theta(t)$ for $\varepsilon = 1$ and $\mathrm{s}\dot{\mathrm{n}}(t) = \cosh\theta(t)$ for $\varepsilon = -1$. So the geodesic curvature $\langle D_{e_1} e_1, e_2\rangle$ of the horizontal lines of curvature relatively to Σ is given by $-(\dot{\text{sn}}(t)/\text{sn}(t))\, h^1_{11}$. This means that the horizontal lines of curvature have constant geodesic curvature on Σ. Now, defining $\nu = Je_1 = \varepsilon \text{sn}(t)\, n_1 - \dot{\text{sn}}(t)\, e_2$, we calculate

$$\langle D_{e_1}\nu, e_1\rangle = -\varepsilon \text{sn}(t) h^1_{11} - \dot{\text{sn}}(t)\frac{\dot{\text{sn}}(t)}{\text{sn}(t)}\, h^1_{11} = -\frac{1}{\text{sn}(t)}\, h^1_{11}.$$

Thus, it follows that the geodesic curvature of the horizontal lines of curvature on $\Sigma \cap \mathbb{P}_t$ relatively to the plane $\mathbb{P}_t$ is also constant and equal to $h^1_{11}/\text{sn}(t)$. We conclude that for each t, $\Sigma \cap \mathbb{P}_t$ consists of constant geodesic curvature lines of $\mathbb{P}_t$.

We also obtain $\langle D_{e_2}e_2, e_1\rangle = 0$. So, the curvature lines of Σ with direction e_2 are geodesics on Σ. We then prove that these lines are contained on vertical planes. Fixed a point (p,t) in $\Sigma \cap \mathbb{P}_t$, let $\alpha(s)$ be the line of curvature with $\alpha' = e_2$ passing by (p,t) at $s = 0$. We want to show that α is contained on the vertical geodesic plane Π determined by $e_2(p,t)$ and ∂_t. This is the plane spanned by e_2 and n_1 at (p,t). For each s, consider the vertical geodesic plane Π_s on $\mathbb{M}^2(\kappa)\times\mathbb{R}$ for which $e_2 = \alpha'(s)$ and $D_{e_2}e_2 = D_{\alpha'}\alpha'$ are tangent at $\alpha(s)$. This plane is of the form $\sigma_s \times \mathbb{R}$, where σ_s is some geodesic on $\mathbb{M}^2(\kappa)$ which by its turn is the intersection of $\mathbb{M}^2(\kappa)$ and some plane π_s on $\mathbb{E}^3$ with unit normal $a(s)$. The intersection of the hyperplane $\pi_s \times \mathbb{R}$ of $\mathbb{E}^4$ with $\mathbb{M}^2(\kappa)\times\mathbb{R}$ is then the plane Π_s. Now $p(s)\wedge\alpha'(s)\wedge D_{\alpha'}\alpha'$ is a normal direction to that hyperplane on $\mathbb{E}^4$ where $p(s) = \alpha(s)^h$. However, since α is at the same time line of curvature and geodesic then

$$D_{\alpha'}\alpha' = D_{e_2}e_2 = (D_{e_2}e_2)^T + (D_{e_2}e_2)^N = (D_{e_2}e_2)^N = h^1_{22}\, n_1.$$

Thus we conclude that the unit normal to the hyperplane Π_s is

$$a(s) = p(s)\wedge e_2(s)\wedge n_1(s).$$

Differentiating we obtain $a' = 0$. So $a(s)$ is constant. Thus implies that $\Pi_s = \Pi$ for all s. So, $\alpha(s)$ is a plane curve contained in Π. Notice that Π has normal $e_1(p,t)$ since $e_1(p,t) = a(0)$. We then conclude that the integral curves of e_2 are planar geodesics on Σ.

So, for a fixed t, let $\sigma(s)$ be a component of $\Sigma\cap\mathbb{P}_t$. Then σ is a constant geodesic curvature curve on $\mathbb{P}_t$. Moreover, the vertical plane passing through $\sigma(s)$ with normal $e_1(\sigma(s))$ is a symmetry plane of Σ since contains a geodesic of Σ, namely the curvature line in direction e_2 passing through $\sigma(s)$. Thus, the surface is invariant with respect to the isometries fixing σ. Since the surface is homeomorphic to a disc or a sphere (see Remark 2 below), then we conclude that these isometries are elliptic (their orbits are closed circles). This means that $X(\Sigma)$ is rotationally invariant in the sense of Section 1. So, the proof is concluded.

Remark 1. We also prove the Theorem 8 by the following reasoning: denote by Π_s the plane passing through $\sigma(s)$ with normal e_1. This plane contains the curvature line with initial data $\sigma(s)$ for position and $e_2(\sigma(s))$ for velocity. Its plane curvature is given by the derivative of its angle with respect to the (fixed) direction ∂_t, that is, $\theta(t)$. These data, by the fundamental theorem on planar curves, determine completely the curve. Changing the point on σ, the initial data differ by a rigid motion (an isometry on $\mathbb{P}_t$) and the curvature function remains the same at points of equal height. Then, by the uniqueness part on the theorem cited before, the two curves differ only by the same rigid motion. This means that the surface is invariant by the rigid motions fixing σ. Thus, the proof is finished by proving that the only possible isometries are the elliptic ones.

Remark 2. For $\kappa \leq 0$ and $\varepsilon = -1$, since $X(\Sigma)$ is space-like, it is acausal. Thus, the coordinate t is bounded on Σ. Moreover, the projection $(p,t) \in \Sigma \mapsto p \in \mathbb{M}^2(\kappa)$ increases Riemannian distances. So, is a covering map and therefore $X(\Sigma)$ is locally a graph over the horizontal planes. If we suppose Σ simply connected, then $X(\Sigma)$ is globally diffeomorphic with $\mathbb{P}_t$. Is, in fact, a disc-type graph.

Let $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an immersion of a surface with boundary. We suppose that $X|_{\partial\Sigma}$ is a diffeomorphism onto its image $\Gamma = X(\partial\Sigma)$. We further suppose that $X(\partial\Sigma)$ is contained on some plane $\mathbb{P}_t$. So, Γ is a embedded curve on $\mathbb{P}_t$ that bounds a domain Ω. In what follows we always make this hypothesis while treating immersions of surfaces with boundary. Now, we fix $\varepsilon = -1$ and suppose that $X(\Sigma)$ is space-like. We may prove under these assumptions that Σ is simply-connected (disc-type) and $X(\Sigma)$ is a graph over Ω. This conclusion also holds if Γ is supposed to be a graph over some embedded curve on $\mathbb{P}_t$.

Thus, if we suppose either $\varepsilon = 1$ and Σ a disc, or $\varepsilon = -1$ (with the additional hypothesis that $Q = 0$ on both cases) then we are able to prove that if $X(\Sigma)$ is an immersed CMC surface with boundary, then $X(\Sigma)$ is contained on a rotationally invariant CMC disc. In fact, the reasoning on Theorem 8 works well on these cases to show that $X(\Sigma)$ is foliated by geodesic circles and that the angle with a plane $\mathbb{P}_t$ is constant along $\Sigma \cap \mathbb{P}_t$. This suffices to show that $X(\Sigma)$ is rotationally invariant.

5 Free boundary surfaces in product spaces

A classical result of J. Nitsche (see, e.g., [14], [17] and [19]) characterizes discs and spherical caps as equilibria solutions for the free boundary problem in space forms. We will be concerned now about to reformulate this problem in the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

Let Σ be an orientable compact surface with non empty boundary and $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be an isometric immersion. By a *volume-preserving variation of* X we mean a family $X_s : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ of isometric immersions such that $X_0 = X$ and $\int \langle \partial_s X_s, n_s \rangle \mathrm{d}A_s = 0$, where $\mathrm{d}A_s$ and n_s represent respectively the element of area and an unit normal vector field to X_s. In the sequel we set $\xi = \partial_s X_s$ and $f = \langle \xi_s, n_s \rangle$ at $s = 0$. We say that X_s is an *admissible variation* if it is volume-preserving and at each time s the boundary $X_s(\partial\Sigma)$ of $X_s(\Sigma)$ lies on a horizontal plane $\mathbb{P}_a$. We denote by Ω_s the compact domain in $\mathbb{P}_a$ whose boundary is $X_s(\partial\Sigma)$ (in the spherical case $\kappa > 0$, we choose one of the two domains bounded by $X_s(\partial\Sigma)$). A *stationary surface* is by definition a critical point for the following functional

$$E(s) = \int_\Sigma \mathrm{d}A_s + \alpha \int_{\Omega_s} \mathrm{d}\Omega,$$

for some constant α, where $\mathrm{d}\Omega$ is the volume element for Ω_s induced from $\mathbb{P}_a$. The first variation formula for this functional is (see [17] and [5] for the corresponding formulae in space forms)

$$E'(0) = -2\int_\Sigma Hf + \int_{\partial\Sigma} \langle \xi, \eta + \alpha\bar{\eta}\rangle \, \mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the line element for $\partial\Sigma$ and η, $\bar{\eta}$ are the unit co-normal vector fields to $\partial\Sigma$ relatively to Σ and to $\mathbb{P}_a$. If we prescribe $\alpha = -\cos\theta$ in the Riemannian case and $\alpha = -\cosh\theta$ in the Lorentzian case, then we conclude that a stationary surface Σ has constant mean curvature and makes constant angle θ along $\partial\Sigma$ with the horizontal plane.

In what follows, *spherical cap* means that the surface is a part of a CMC revolution sphere bounded by some circle contained in a horizontal plane and centered at the rotation axis. Similarly, the term *hyperbolic cap* means a part of a CMC rotationally invariant disc bounded by a horizontal circle centered at the rotation axis. Granted this, we state the following theorem.

Theorem 9. *Let Σ be a surface with boundary and let $X : \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be a* stationary *immersion for free boundary admissible variations whose boundary lies in some plane $\mathbb{P}_a$. If $\varepsilon = 1$ and Σ is disc-type, then $X(\Sigma)$ is a spherical cap. If $\varepsilon = -1$, then $X(\Sigma)$ is a hyperbolic cap.*

The proof of Theorem 9 follows closely the guidelines of the proof of the Nitsche's Theorem in $\mathbb{R}^3$ as we may found in [14] and [17]. Let Σ denote the disc $|z| < 1$ in $\mathbb{R}^2$, where $z = u + iv$. If we put $\partial_z = \frac{1}{2}(\partial_u - i\partial_v)$, then the $\mathbb{C}$-bilinear complexification of q satisfies

$$q_{\mathbb{C}}(\partial_z, \partial_z) = q(\partial_u, \partial_u) - q(\partial_v, \partial_v) - 2iq(\partial_u, \partial_v) = 2Q(\partial_z, \partial_z).$$

Now, since $X(\partial\Sigma)$ is contained in $\mathbb{P}_a$ then $q(\tau, \eta) = 0$ on $\partial\Sigma$. Here $\tau = e^{-\omega}(-v\partial_u + u\partial_v)$ is the unit tangent vector to $\partial\Sigma$ and $\eta = e^{-\omega}(u\partial_u + v\partial_v)$ is the unit outward co-normal to $\partial\Sigma$. In fact $h^2(\tau, \eta) = 0$ since that τ is a horizontal vector and $h^1(\tau, \eta) = 0$ since that $\partial\Sigma$ is a line of curvature for Σ by Joachimstahl's Theorem.

On the other hand, we have on $\partial\Sigma$ that

$$0 = q(\tau, \eta) = (u^2 - v^2)\, q(\partial_u, \partial_v) - uv\, q(\partial_u, \partial_u) + uv\, q(\partial_v, \partial_v) = \Im\big(z^2 Q(\partial_z, \partial_z)\big)$$

From this we conclude that $\Im(z^2 Q) \equiv 0$ on $\partial\Sigma$. Since $z^2 Q$ is holomorphic on Σ, then $\Im z^2 Q$ is harmonic. So, $\Im z^2 Q = 0$ on Σ and therefore $z^2 Q \equiv 0$ on Σ.

Hence, $Q \equiv 0$ on Σ. This implies that $X(\Sigma)$ is part of a CMC revolution sphere or a CMC rotationally invariant disc. This finishes the proof of the Theorem 9.

We obtain also a result about stable CMC discs in $\mathbb{M}^2(\kappa)\times\mathbb{R}$, following ideas presented in [3]. Here, stability for a CMC surface Σ means that the quadratic form

$$J[f] = \varepsilon \int_\Sigma \Big(\Delta f + \varepsilon\big(|A|^2 + \mathrm{Ric}(n_1, n_1)\big) f\Big)\, f \,\mathrm{d}A,$$

is non-negative with respect to the all variational fields f generating preserving-volume variations (see [6] and [7] for the case $\kappa = 0$). In the formula above, Ric means the Ricci curvature tensor of $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

Theorem 10. *Let Σ be an immersed surface with boundary and constant mean curvature H in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. Suppose that $\partial\Sigma$ is a geodesic circle in some plane $\mathbb{P}_a$ and that the immersion is stable. For $\varepsilon = 1$ we further suppose that Σ is disc-type and for $\varepsilon = -1$ that the immersion is space-like. Then Σ is a spherical or hyperbolic cap, if $H \neq 0$. If $H = 0$ then Σ is a totally geodesic disc.*

We consider the vector field $Y(t, p) = a \wedge \partial_t \wedge p$, where a is the vector in $\mathbb{E}^3$ perpendicular to the plane where $\partial\Sigma$ lies. This is a Killing field in $\mathbb{M}^2(\kappa)\times\mathbb{R}$. Then $f = \langle Y, n_1\rangle$ satisfies trivially $J[f] = 0$. Let η be the exterior unit co-normal direction to Σ along the boundary $\partial\Sigma$.

The normal derivative of f along $\partial\Sigma$ is calculated as

$$\begin{aligned}
&\eta(f) = \eta\langle Y, n_1\rangle = \langle a\wedge\partial_t\wedge D_\eta p, n_1\rangle + \langle a\wedge\partial_t\wedge p, D_\eta n_1\rangle\\
&= \langle a\wedge\partial_t\wedge\eta, n_1\rangle + \langle a\wedge\partial_t\wedge p, D_\eta n_1\rangle = -\langle a\wedge\partial_t\wedge n_1, \eta\rangle + \langle a\wedge\partial_t\wedge p, D_\eta n_1\rangle\\
&= \langle \tau, \eta\rangle + \langle \tau, D_\eta n_1\rangle = \langle \tau, D_\eta n_1\rangle = -h^1(\tau, \eta),
\end{aligned}$$

where $\tau = a\wedge\partial_t\wedge p$ (the restriction of Y to the boundary of Σ) is the tangent positively oriented unit vector to $\partial\Sigma$. Since that $\langle\tau, \partial_t\rangle = 0$ and $\langle\tau, \eta\rangle = 0$ it follows that

$$h^2(\tau, \eta) = -\frac{1}{r}\,\langle\tau^h, \eta^h\rangle = 0.$$

This yields

$$2H\,\eta(f) = -2H\,h^1(\tau, \eta) = -q(\tau, \eta).$$

However, if u, v denote the usual cartesian coordinates on Σ then

$$q(\tau, \eta) = e^{-2\omega}\, q(u\partial_u + v\partial_v, -v\partial_u + u\partial_v) = -\Im(z^2 Q)$$

on $\partial\Sigma$. We conclude that $2H\,\eta(f) = \Im(z^2 Q)$. Proceeding as in ([3]) we verify that $\eta(f)$ vanishes at least three times. Applying Courant's theorem on nodal domains allows us to conclude that f vanishes on the whole disc. So, $X(\Sigma)$ is foliated by the flux lines of Y, i.e. by horizontal geodesic circles centered at the same vertical axis. So, $X(\Sigma)$ is a spherical or hyperbolic cap as we claimed. This proves Theorem 10.

6 Flux formula and Killing graphs

6.1 Flux formula

Let Σ be an immersed surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with constant mean curvature H. We denote by Div and div respectively the divergence operator on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ and on Σ. Consider a Killing vector field Y on $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Thus restricting Y to Σ one finds

$$\mathrm{div} Y =: \sum_i \langle D_{e_i} Y, e_i \rangle = 0.$$

However using the decomposition $Y = Y^T + Y^N = Y^T + \varepsilon \langle Y, n \rangle \, n$ we obtain

$$\begin{aligned}&\mathrm{div} Y = \mathrm{div} Y^T + \mathrm{div} Y^N = \mathrm{div} Y^T + \varepsilon \langle Y, n \rangle \langle D_{e_i} n, e_i \rangle = \langle \nabla_{e_i} Y^T, e_i \rangle \\ &-2H\varepsilon \langle Y, n \rangle = \mathrm{div} Y^T - 2H\varepsilon \langle Y, n \rangle.\end{aligned}$$

Then by Stokes's Theorem on Σ

$$0 = \int_\Sigma \mathrm{div} Y \, \mathrm{d}A = \int_{\partial\Sigma} \langle Y, \nu \rangle \, \mathrm{d}\sigma - 2H\varepsilon \int_\Sigma \langle Y, n \rangle \, \mathrm{d}A,$$

where ν is the *outward* unit co-normal vector field along $\partial\Sigma$. By this way we obtain the *first Minkowski formula*

$$\int_\Sigma \varepsilon H \langle Y, n \rangle \, \mathrm{d}A = \frac{1}{2} \int_{\partial\Sigma} \langle Y, \nu \rangle \, \mathrm{d}\sigma. \tag{64}$$

In the case where Σ and Ω are homologous oriented cycles on $\mathbb{M}^2(\kappa) \times \mathbb{R}$, we conclude from the formula $\mathrm{Div} Y = 0$ and divergence theorem that

$$\int_\Sigma \langle Y, n \rangle \, \mathrm{d}A + \int_\Omega \langle Y, n_\Omega \rangle \, \mathrm{d}\Omega = 0.$$

We then obtain the *flux formula* for Killing vector fields:

$$\int_{\partial\Sigma} \langle Y, \nu \rangle \, \mathrm{d}\sigma + 2H\varepsilon \int_\Omega \langle Y, n_\Omega \rangle \, \mathrm{d}\Omega = 0. \tag{65}$$

6.2 Killing graphs and height estimates

Let n be an unit normal vector field to $\Sigma \looparrowright \mathbb{M}^2(\kappa) \times \mathbb{R}$. We next consider the function $\langle Y, n \rangle$. Let e_1, e_2 be an adapted orthonormal moving frame with $\nabla e_i = 0$ at a point $(p, t) \in \Sigma$. We may suppose that e_i is principal at that point. We have for $v \in T\Sigma$ that

$$\begin{aligned}&v\langle Y, n \rangle = \langle D_v Y, n \rangle + \langle Y, D_v n \rangle = \langle D_v Y, n \rangle + \varepsilon \langle D_v n, n \rangle \langle n, Y \rangle + \langle (D_v n)^T, Y \rangle \\ &= -\langle v, D_n Y \rangle + \langle (D_v n)^T, Y \rangle = -\langle v, D_n Y + A(Y^T) \rangle.\end{aligned}$$

Hence

$$\nabla \langle Y, n \rangle = -A(Y^T) - (D_n Y)^T = \big((D_{Y^T} n) - (D_n Y)\big)^T$$

The restriction of a Killing field to a CMC surface is a Jacobi field for it. Then we have

$$\big(\Delta + \varepsilon|A|^2 + \varepsilon \mathrm{Ric}(n,n)\big)\langle Y, n\rangle = 0.$$

We also compute

$$\mathrm{Ric}(n,n) = \kappa\varepsilon(1 - \langle n, \partial_t\rangle^2).$$

We suppose that the distribution spanned by the vectors orthogonal to Y is integrable (this is a weaker condition than to assume Y is closed). Let $\mathbb{N}$ be the domain in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ free of singularities of Y. So, $\mathbb{N}$ is foliated by surfaces orthogonal to the flow lines of Y. Let s be the flow parameter on the flow lines of Y, so that each leaf is a level surface for s. Taking s as a coordinate on $\mathbb{N}$, it is clear that $\partial_s = Y$. We also have

$$\bar\nabla s = g^{ss}\partial_s = \frac{Y}{|Y|^2} := fY.$$

Then the gradient of s restricted to a surface Σ on $\mathbb{N}$ is $\nabla s = fY^T$ and its Laplacian is calculated as

$$\begin{aligned}\Delta s &= \langle \nabla_{e_i} fY^T, e_i\rangle = \langle D_{e_i} fY^T, e_i\rangle = \langle \nabla f, e_i\rangle\langle Y^T, e_i\rangle + f\langle D_{e_i}Y^T, e_i\rangle\\ &= \langle\nabla f, Y^T\rangle + f\langle D_{e_i}Y, e_i\rangle - \varepsilon f\langle D_{e_i}\langle Y, n\rangle n, e_i\rangle = \langle \nabla f, Y^T\rangle - \varepsilon f\langle Y, n\rangle\langle D_{e_i} n, e_i\rangle.\end{aligned}$$

Thus $\Delta s = 2H\varepsilon f\langle Y, n\rangle + \langle\nabla f, Y\rangle$. However, we easily see that the Killing equation implies that the norm of Y is conserved along the flow lines of Y. Then $\langle \nabla |Y|, Y\rangle = 0$ and therefore $\langle \nabla f, Y\rangle = 0$. So

$$\Delta s = 2H\varepsilon f\langle Y, n\rangle.$$

We also have from Jacobi's equation

$$\Delta\langle Y, n\rangle = -\varepsilon\big(|A|^2 + \mathrm{Ric}(n,n)\big)\langle Y, n\rangle = -\varepsilon\big(|A|^2 + \kappa\varepsilon\,(1 - \langle n, \partial_t\rangle^2)\big)\langle Y, n\rangle.$$

We then fix $\varepsilon = 1$. Suppose that Σ has boundary on the leaf Π given by $s = 0$ and that $\langle Y, n\rangle \geq 0$ on Σ. So, $H \leq 0$ when we consider n pointing outwards Π. Next, for a given constant c, the function $\phi =: Hc\,s + \langle Y, n\rangle$ satisfies $\phi|_{\partial\Sigma} \geq 0$ and

$$\Delta\phi = \big(2H^2\frac{c}{|Y|^2} - |A|^2 - \kappa(1-\nu^2)\big)\langle Y, n\rangle,$$

where $\nu = \langle n, \partial_t\rangle$. We want to choose c so that ϕ is super-harmonic. It suffices that

$$2H^2\frac{c}{|Y|^2} - |A|^2 - \kappa(1-\nu^2) \leq 0. \tag{66}$$

However

$$|A|^2 = k_1^2 + k_2^2 = (k_1+k_2)^2 + (k_1-k_2)^2 - k_1^2 - k_2^2 = 4H^2 + 4|\psi^1|^2 - |A|^2.$$

So, $|A|^2 = 2H^2 + 2|\psi^1|^2$. For further reference we point that (for any sign of ε)

$$|\psi^2|^2 = \frac{\kappa^2}{4}(1-\nu^2)^2.$$

Thus, (66) is rewritten as

$$2H^2\big(\frac{c}{|Y|^2} - 1\big) \leq 2|\psi^1|^2 + \kappa(1-\nu^2)$$

If $\kappa \geq 0$ it suffices to take $0 < c \leq \inf_\Sigma |Y|^2$. For $\kappa < 0$, if we rather suppose that $2H^2 + \kappa > 0$, then we obtain the super-harmonicity of ϕ when

$$0 < c \leq \inf_\Sigma |Y|^2 \frac{H^2 + \frac{\kappa}{2}}{H^2}.$$

Thus, for these choices for c we have $\Delta\phi \leq 0$ and

$$s \leq \frac{1}{|H|}\frac{\sup_\Sigma |Y|}{\inf_\Sigma |Y|^2}, \ (\kappa > 0), \quad s \leq \frac{|H|}{H^2 + \frac{\kappa}{2}}\frac{\sup_\Sigma |Y|}{\inf_\Sigma |Y|^2}, \ (\kappa < 0). \tag{67}$$

Theorem 11. *Let Y be a Killing field on $\mathbb{M}^2(\kappa) \times \mathbb{R}$, $\varepsilon = 1$, which determines an integrable orthogonal distribution $\mathcal{D}$. Let Σ be an immersed CMC surface on $\mathbb{N}$ whose boundary lies on a integral leaf of $\mathcal{D}$. If s is the parameter of the flow lines of Y, then it holds the estimates on (67). If Σ is a compact closed embedded CMC surface on $\mathbb{N}$, then Σ is symmetric with respect to some integral leaf of $\mathcal{D}$.*

The proof of the second statement on the theorem above is similar to that one presented in Proposition 1 of [9]. It is based on Aleksandrov reflection method with respect to the integral leaves of $\mathcal{D}$. That this makes sense we could see noticing that the flux of Y is, at fixed s, an ambient isometry.

We remark that the integrability condition on $\mathcal{D}$ imposes that the form $\omega = \langle Y, \cdot\rangle$ satisfies $\mathrm{d}\omega = 0$ on $\mathcal{D} \times \mathcal{D}$. This implies that $\langle D_v Y, w\rangle = 0$ for all vector fields v, w on $\mathcal{D}$. So, the integral leaves for $\mathcal{D}$ are totally geodesic on $\mathbb{N}$.

Next, we use the height estimates on Theorem 11 to show the existence of CMC Killing graphs for $\kappa \geq 0$. We observe that if Σ is a Killing graph, then each flow line through Σ meets Ω. Since the norm of Y is constant along the flow lines, we have $\inf_\Sigma |Y| = \inf_\Omega |Y|$ and so on. Thus, the height estimates on Theorem 11 for the particular case of graphs depend only on data of the domain and the mean curvature.

Killing graphs corresponding to the vertical field $Y = \partial_t$ were previously studied in [9] (see also [2]). We then restrict ourselves to consider the horizontal field $Y = a \wedge \partial_t \wedge p$. Let Ω be a domain on $\Pi \cap \mathbb{N}$. We may write $\Pi = \sigma \times \mathbb{R}$, where σ is a horizontal geodesic parametrized by ρ. Let $s = u(\rho, t)$ be a function defined on Ω, which specifies a point on the flow line of Y starting at the point with coordinates (ρ, t) on Π. Let Σ be the (Killing) graph of u. We fix boundary data $u = 0$ on $\partial\Omega$. The tangent vectors to Σ are

$$X_\rho = \partial_\rho + \frac{\partial u}{\partial \rho}\,\partial_s, \quad X_t = \partial_t + \frac{\partial u}{\partial t}\,\partial_s$$

Recall that $\partial_s = Y$. The unit normal vector field to Σ so that $\langle Y, n\rangle \geq 0$ is

$$n = \frac{1}{W}\big(-\frac{\partial u}{\partial \rho}\,\partial_\rho - \frac{\partial u}{\partial t}\,\partial_t + f\,\partial_s\big)$$

where $W^2 = f + (\partial_\rho u)^2 + (\partial_t u)^2 = f + |\nabla u|^2$. Thus

$$\langle Y, n\rangle = \frac{1}{W}.$$

So, a lower estimate for $\langle Y, n\rangle$ gives an upper estimate for W and therefore for $|\nabla u|$. However, $\langle Y, n\rangle$ is super-harmonic for $\kappa \geq 0$. Then

$$\min_\Omega \langle Y, n\rangle = \min_{\partial\Omega} \langle Y, n\rangle.$$

Then $|u|_{1,\Omega} \leq C\,|u|_{1,\partial\Omega}$ for some constant C independent of u. We must then estimate $|\nabla u|$ on the boundary. As we said before, this means to get a lower estimate for $\langle Y, n\rangle$ on the boundary. Since $s = 0$ on $\partial\Omega$ and ϕ is also super-harmonic, then

$$\min_\Omega \langle Y, n\rangle = \min_{\partial\Omega} \langle Y, n\rangle = \min_{\partial\Omega} \phi = \min_\Omega \phi = \phi(p_0)$$

for a given p_0 on $\partial\Omega$. Thus, if η is the unit interior co-normal then at p_0

$$\begin{aligned}
&0 \leq \langle \nabla\phi, \eta\rangle = Hc\langle \nabla s, \eta\rangle + \eta\langle Y, n\rangle = Hc\,f\langle Y, \eta\rangle + \langle D_\eta Y, n\rangle + \langle Y, D_\eta n\rangle \\
&= Hcf\,\langle Y, \eta\rangle + \langle a \wedge \partial_t \wedge \eta, n\rangle + \langle D_\eta n, \eta\rangle\langle Y, \eta\rangle \\
&= Hcf\,\langle Y, \eta\rangle + \langle \tau, n\rangle - \langle A(\eta), \eta\rangle\langle Y, \eta\rangle = \big(Hcf - \langle A\eta, \eta\rangle\big)\,\langle Y, \eta\rangle,
\end{aligned}$$

where τ is the unit positively oriented vector tangent to $\partial\Omega$. However by choice of c we have $cf \leq 1$ and $cf \geq \gamma =: \inf_\Omega |Y|^2/\sup_\Omega |Y|^2$. Then $0 \leq \big(H\gamma - \langle A\eta, \eta\rangle\big)\,\langle Y, \eta\rangle$. Since $\langle Y, \eta\rangle \geq 0$ and $2H = \langle A\eta, \eta\rangle + \langle A\tau, \tau\rangle$ then $\langle A\tau, \tau\rangle \geq H(2-\gamma)$. However, $n = \langle n, \bar\eta\rangle\bar\eta + \langle n, fY\rangle Y$ where $\bar\eta$ is the unit interior normal to $\partial\Omega$ on Π. Then

$$\langle A\tau, \tau\rangle = \langle D_\tau n, \tau\rangle = \langle n, \bar\eta\rangle\langle D_\tau \bar\eta, \tau\rangle + \langle n, fY\rangle\langle D_\tau Y, \tau\rangle = \langle n, \bar\eta\rangle\,\kappa_g,$$

where κ_g is the geodesic curvature on $\partial\Omega$, which we suppose to be strictly positive. Since $H \leq 0$, $\langle n, \bar\eta\rangle \leq 0$ and $\kappa_g > 0$ then at p_0 we have

$$H^2\tilde\gamma^2 \geq \kappa_g^2\langle n, \bar\eta\rangle^2 = \kappa_g^2\big(1 - f\,\langle n, Y\rangle^2\big)$$

where $\tilde\gamma = 2 - \gamma$. Denoting $\inf_\Omega |Y| = \widetilde{c}$ we then obtain

$$|\langle Y, n\rangle| \geq \widetilde{c}\sqrt{\frac{\kappa_g^2 - H^2\tilde\gamma^2}{\kappa_g^2}}$$

We then verified that there exists a non-trivial gradient estimate for u if we suppose $|H| < \kappa_g/\tilde{\gamma}$. By Schauder theory on quasi-linear elliptic equations, we conclude that there exists a CMC Killing graph on $\mathbb{N}$ with boundary on Π.

Theorem 12. *Let Π be a vertical plane on the Riemannian product $\mathbb{M}^2(\kappa)\times\mathbb{R}$, $\kappa \geq 0$, determined by an unit vector a in $\mathbb{E}^4$. Let Ω be a domain on Π which does not contain points of the axis $\{\pm a\}\times\mathbb{R}$. If $|H| < \kappa_g/\tilde{\gamma}$, where κ_g is the geodesic curvature of $\partial\Omega$ in Π and $\tilde{\gamma} = 2 - \sup_\Omega f/\inf_\Omega f$, then there exists a surface (a Killing graph) with constant mean curvature H and boundary $\partial\Omega$.*

Notice that $Y = \partial_\theta$, where θ is the polar coordinate centered at $r\,a$ as defined on Section 1. Thus $|Y| = \mathrm{sn}_\kappa(\rho)$. Thus, a simple application to the flux formula gives us the following area estimate:

$$|H| \leq \frac{\max_{\partial\Omega}\mathrm{sn}_\kappa(\rho)}{\min_\Omega \mathrm{sn}_\kappa(\rho)}\,\frac{|\partial\Omega|}{2|\Omega|}.$$

This estimate also holds for surfaces in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ satisfying the condition that its boundary bounds a domain on a vertical plane which does not contain singularities of Y.

Next, we fix $\varepsilon = -1$ and $\kappa \leq 0$. Let Σ be a CMC surface whose boundary is a geodesic circle on some horizontal plane $\mathbb{P}_t$. Thus considering the Killing field ∂_t, the function ϕ we defined above becomes $\phi = Ht - \nu$, where $\nu = \langle n, \partial_t\rangle$. Then we have as before

$$\Delta\phi = \big(2H^2 - |A|^2 + \kappa(1-\nu^2)\big)\,\nu$$

We recall that

$$|A|^2 = 2H^2 + 2|\psi^1|^2, \quad 4|\psi^2|^2 = \kappa^2(1-\nu^2)^2$$

and since $\kappa \leq 0$ and $1-\nu^2 \leq 0$ we have $2|\psi^2| = \kappa(1-\nu^2)$. Replacing this above and assuming that $|\psi^1|^2 - |\psi^2| \geq 0$ we have

$$\Delta\phi = -2\big(|\psi^1|^2 - |\psi^2|\big)\,\nu \geq 0,$$

since that we choose n pointing upwards (which implies that $H \leq 0$). By Stokes's theorem

$$-2\int_\Sigma \big(|\psi^1|^2 - |\psi^2|\big)\,\nu\,\mathrm{d}A = \int_{\partial\Sigma}\langle\nabla\phi, \eta\rangle\,\mathrm{d}\sigma$$

where η is the outward unit co-normal to Σ along $\partial\Sigma$. However

$$\langle\nabla\phi,\eta\rangle = H\langle\nabla t,\eta\rangle - \langle\nabla\nu,\eta\rangle = -H\langle\partial_t,\eta\rangle + \langle\partial_t, A\eta\rangle$$

Therefore, $\langle\nabla\phi,\eta\rangle = \big(\langle A\eta,\eta\rangle - H\big)\,\langle\eta,\partial_t\rangle$. However, $\langle A\eta,\eta\rangle = 2H - \langle A\tau,\tau\rangle$, where τ is the unit tangent vector to $\partial\Sigma$. Let $\bar{\eta}$ be the outwards unit normal to

$\partial\Sigma$ with respect to $\mathbb{P}_t$. Since $n = \langle n, \bar{\eta}\rangle\bar{\eta} - \langle n, \partial_t\rangle\, \partial_t$ and since τ is orthogonal to both ∂_t and $\bar{\eta}$ it follows that

$$-\langle A\tau, \tau\rangle = \langle D_\tau n, \tau\rangle = \langle n, \bar{\eta}\rangle\langle D_\tau\bar{\eta}, \tau\rangle = -\kappa_g\langle n, \bar{\eta}\rangle = \kappa_g\langle\partial_t, \eta\rangle$$

Thus we conclude that $\langle\nabla\phi, \eta\rangle = \big(H + \kappa_g\langle\eta, \partial_t\rangle\big)\,\langle\eta, \partial_t\rangle$. So by flux formula

$$\int_{\partial\Sigma}\langle\nabla\phi, \eta\rangle\,\mathrm{d}\sigma = H\int_{\partial\Sigma}\langle\eta, \partial_t\rangle\mathrm{d}\sigma + \int_{\partial\Sigma}\kappa_g\langle\eta, \partial_t\rangle^2\mathrm{d}\sigma = 2H^2|\Omega| + \int_{\partial\Sigma}\kappa_g\langle\eta, \partial_t\rangle^2\mathrm{d}\sigma$$

Gathering the expressions, we have

$$-2\int_\Sigma\big(|\psi^1|^2 - |\psi^2|\big)\,\nu\,\mathrm{d}A = 2H^2|\Omega| + \int_{\partial\Sigma}\kappa_g\langle\eta, \partial_t\rangle^2\mathrm{d}\sigma.$$

Now again by flux formula

$$\Big(\int_{\partial\Sigma}\langle\eta, \partial_t\rangle\,\mathrm{d}\sigma\Big)^2 = 4H^2|\Omega|^2$$

But by Cauchy-Schwarz on L^2 functions we have

$$\Big(\int_{\partial\Sigma}\langle\eta, \partial_t\rangle\,\mathrm{d}\sigma\Big)^2 \le |\partial\Sigma|\int_{\partial\Sigma}\langle\eta, \partial_t\rangle^2\,\mathrm{d}\sigma$$

So

$$\frac{4H^2|\Omega|^2}{|\partial\Sigma|} \le \int_{\partial\Sigma}\langle\eta, \partial_t\rangle^2\,\mathrm{d}\sigma$$

Thus

$$-2\int_\Sigma\big(|\psi^1|^2 - |\psi^2|\big)\,\nu\,\mathrm{d}A \le 2H^2\frac{|\Omega|}{\partial\Omega}\big(|\partial\Omega| + 2|\Omega|\,\kappa_g\big) \tag{68}$$

with equality if and only of $\langle\eta, \partial_t\rangle$ is constant along $\partial\Sigma$. Now, the geodesic curvature of $\partial\Sigma$ calculated with respect to $\bar{\eta}$ is $\kappa_g = -\mathrm{ct}_\kappa(\rho)$. Thus

$$|\partial\Omega| + 2|\Omega|\,\kappa_g = \frac{2\pi}{\kappa}\,\mathrm{sn}_\kappa(\rho)\,\big(\mathrm{cs}_\kappa(\rho) - 1\big)^2 \le 0$$

since $\kappa \le 0$. So, occurs equality on (68). Then, the angle between Σ and the horizontal plane is constant along $\partial\Sigma$. So, Σ is a stationary surface for the energy defined on Section 5. Thus, by Theorem 9, $Q = 0$ and the surface is a hyperbolic cap.

Theorem 13. *Fix $\varepsilon = -1$ and $\kappa \le 0$. Let Σ be a immersed CMC surface whose boundary is a geodesic circle on some horizontal plane $\mathbb{P}_t$. If we suppose that $|\psi^1|^2 - |\psi^2| \ge 0$, then $Q = 0$ and the surface is part of a hyperbolic cap or a planar disc.*

This theorem is a partial answer to a Lorentzian formulation of the well-known *spherical cap conjecture* which was positively proved on [4] for the case $\kappa = 0$.

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