PURE STRATEGY EQUILIBRIA OF MULTIDIMENSIONAL AND NON-MONOTONIC AUCTIONS

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Abstract. We give necessary and sufficient conditions for the existence of symmetric equilibrium without ties in interdependent value auctions, with multidimensional independent types and no monotonic assumptions. In this case, non-monotonic equilibria might happen. When the necessary and sufficient conditions are not satisfied, there are ties with positive probability. In such case, we are still able to prove the existence of pure strategy equilibrium with an *all-pay auction tie-breaking rule*. As a direct implication of these results, we obtain a generalization of the Revenue Equivalence Theorem. From the robustness of equilibrium existence for all-pay auctions in multidimensional setting, an interpretation of our results gives a new justification to the use of tournaments in practice.

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1. Introduction

The received literature on pure strategy equilibria of single-object auctions is mainly restricted to the setting of unidimensional types and monotonic utilities. Although recent efforts have been made to treat the case of multidimensional types (see McAdams (2003), for instance), the monotonicity assumption is still maintained.

In dealing with multidimensional types, this is obviously restrictive. To see why, consider the auction of a tract of land to construct a building. The private information regards 1) the bidder's capability to construct residential buildings, 2) the bidder's

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capability to construct commercial buildings and 3) how crowded will be the neighborhood. For a bidder with high capability to construct a commercial building, to learn that the place will be crowded is good news, while this is bad news for a firm with good capabilities in constructing just residential buildings. If the information has the features of interdependent (or common) value setting, then non-monotonicities might arise.[1]

In other words, to develop a satisfactory theory of equilibria with multidimensional types, it is necessary to take into account the possibility of non-monotonic utility functions.

However, even in the unidimensional case, non-monotonic auctions are problematic. To see why, consider a symmetric first-price auction between two buyers with independent types where the object is worth $u_i(t) = v(t_i, t_{-i}) = \alpha + t_i + \beta t_{-i}$ to bidder i.

The received theory ensures the existence of a monotonic *pure* strategy equilibrium only if $\beta \geqslant 0$. See Milgrom and Weber (1982), Maskin and Riley (2000) and Athey (2001). If $\beta < 0$, we only know that there exists an *endogenously defined tie-breaking rule* that guarantees the existence of *mixed* strategy equilibrium (see Jackson, Simon, Swinkels and Zame (2002), henceforth JSSZ).

That the case $\beta < 0$ is problematic can be seen through particular examples. Indeed, if $\alpha = 5$, $\beta = -4$ and the distribution is uniform on $[0, 1]$, this is exactly example 1 of JSSZ. If $\alpha = 3$, $\beta = -2$ and types assume values 0 or 1 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$, respectively, it is example 3 of Maskin and Riley (2000). Both cases are counterexamples to the existence of equilibrium, even with special tie-breaking rules. Maskin and Riley (2000) show that there is no equilibrium for their example neither under the standard tie-breaking rule (that assigns the object randomly to tying bidders), nor under the Vickrey auction tie-breaking rule, defined as "if a tie occurs for the high bid, a Vickrey auction is conducted among the high bidders". JSSZ make the claim, corrected in Jackson, Simon, Swinkels and Zame (2004), that there is no type-independent tie-breaking rule ensuring the existence of equilibrium for their example.

Some questions arise from the contrast between the theoretical results for $\beta \geqslant 0$ and $\beta < 0$: For which set of β the standard tie-breaking rule is sufficient to ensure the existence of equilibrium? Is there any negative β such that the equilibrium is unique? Is it possible to define a specific tie-breaking rule for all β? For which set

[1]In the private value case, one can summarize the information of each bidder to the expected value of the object only. Thus, the value can be the type itself and, hence, monotonic.

of β there is no equilibrium in pure strategy? What about the other auction formats: how the negativeness of β impacts the equilibrium existence for them? Is the Revenue Equivalence Theorem still valid for $\beta < 0$?

The framework of this paper includes JSSZ's example as a special case. Our results provide the following answers to the above questions: If $\beta > -1$, there exists equilibrium in pure strategies under the standard tie-breaking rule. If we adopt an *all-pay auction tie-breaking rule*, that consists in conducting an all-pay auction among tying bidders in the case of a tie, then there exists a *pure* strategy equilibrium for all β (provided $\alpha \geqslant \max\{0, -\beta\}$, otherwise the object would have negative values). Moreover, the all-pay auction tie-breaking rule works for all standard type of auctions and the equilibria obtained under it obey the Revenue Equivalence Theorem. We also prove that there is a unique equilibrium if $\beta > -1$, but there are multiple equilibria otherwise.

It is important to note that the all-pay auction tie-breaking rule is *type-independent*, in the sense that it does not require private information. This does not contradict the example of Jackson et. al. (2004), that does not have equilibrium with type-independent tie-breaking rule. The reason is that in this example there is an uncertainty about the number of objects in the auction, while we consider only standard auctions, that is, with a fixed number of objects.

Our results hold for symmetric auctions with independent non-atomic types, for a wide class of auction formats where bidders have unitary demands (first-price, second-price, all-pay, war of attrition).[2] Moreover, we impose no restriction on the dimension of the set of types and make no monotonic assumptions about the value of the object. All the answers provided above for the specific example are given in a general setting (of weakly separable utilities — see assumption H3 in section 5). Of course, the condition for the equilibrium existence is somewhat more complex for the general case, but it is still easy to verify.

From the equilibrium existence for all-pay auctions, an interpretation of our results can give a new justification to the use of tournaments in practice. Indeed, tournaments (for job or research) are well-modeled as all-pay auctions. We prove that the existence of equilibrium for these kind of auctions requires weaker assumptions than other kind of auction mechanisms, because they better reveal information. If we are in a situation

[2] The non-atomicity is a standard but crucial assumption for our method. Thus, example 3 of Maskin and Riley (2000) does not satisfy our assumptions.

where the revelation of information is crucial for a *strategically stable* allocation of the product, then our results say that all-pay auctions perform better.

It is interesting to observe that situations where tournaments are routinely conducted are exactly those where the information is multidimensional and can be non-monotonic for the players. For instance, the better capacity for conducting a research depends on a multidimensional vector of characteristics: technical knowledge and abilities, experience, organizational and financial structures, creativity, motivation and even honesty. Such a complex information environment requires good mechanisms for revelation of information, such as all-pay auctions, implementable via tournaments.

Our results are based on what we call *indirect auctions*, described in subsection 1.1 below. However, one can understand the main findings of this paper in section 2, where we describe them in a simpler set up. The rest of the paper is organized as follows. In section 3, we present the model. Section 4 formally presents the indirect auction. Section 5 develops the theory for general auctions, obtaining necessary and sufficient conditions for the existence of equilibrium. Section 6 particularizes to the case of weakly separable utilities and gives a concise condition for equilibrium existence. Moreover, the all-pay auction tie-breaking rule is introduced and the equilibrium existence proved. As a corollary, we obtain the Revenue Equivalence Theorem. Section 7 concludes with a discussion about the limitations of our results and reviews the contributions of the paper in light of the related literature. All proofs are collected in appendices.

1.1. **The Indirect Auction.** For standard auctions, high bids correspond to higher probabilities of winning. If a bidding function $b(\cdot)$ is fixed and followed by all participants in a symmetric auction, we can associate to each bid (and thus, to each type), the probability of winning. All types that bid in the same manner under $b(\cdot)$ have the same probability of winning. This allows us to introduce the concept of conjugation. If $b(t) = b(s)$ and, hence, t and s have the same probability of winning, we say that t and s are conjugated.

Sometimes in the literature, what we call conjugation is named *reduced form*: "The function relating a bidder's type to his probability of winning is the reduced form of the auction." (Border, 1991, p. 1175).[3] Therefore, what we will call *indirect auction* can be also called *reduced form auction.* The papers about reduced form auctions analyze problems related to the characterization and existence of optimal auctions. Hence, the auction is treated, as in Myerson (1981), only by considering the probability of winning

[3]See also Matthews (1984) and Chen (1986).

and the payments. In turn, our problem is to find the equilibrium for fixed auction rules. Moreover, our indirect auction is not equivalent to the direct one. Thus, it is not merely a reduced form of the auction. (See remarks after Theorem 1 in section 4). Because of these differences and in the attempt not to confuse terms, we decided to use a different terminology.

This terminology comes from the Taxation Principle which allows us to implement direct truthful mechanism through some convenient indirect one.[4] In our case, we are implementing the equilibrium in the auction using an indirect auction obtained from the reparametrization of types through the probability of winning.

As stated, this idea seems to be unpromising at first sight, since the probability of winning will be different for each different bidding function that we begin with. Moreover, if we do not previously fix a bidding function, no conjugation can be defined.

To overcome these problems, we define conjugations (without using bidding functions) as a suitable reparametrization of types. Once conjugations are defined, we can define the Indirect Auction: an auction with the same format of the direct one (for instance, a first-price auction if the original auction is a first-price auction) between just two players with independent signals, uniformly distributed on $[0, 1]$. This makes the analysis of equilibrium existence easier.

This method allows us to deal with non-monotonic bidding in equilibrium. Indeed, we give examples where bidders' types are multidimensional and the values are non-monotonic (see sections 4 and 5). An important part of the method is the necessary condition (i) in Theorem 1, which says that types of a bidder choosing the same bid have the same marginal benefit. A similar property was derived by Araujo and Moreira (2000) for the (monopolistic) principal-agent screening problem where non-monotonic optimal contracts emerge.

2. The main results in a simpler framework

Assume that there are just two bidders in an auction for a single object. Bidder i has a L dimensional signal $t_i \in S = \Pi_{l=1}^{L} \left[\underline{t}^l, \overline{t}^l\right]$ and there is a non-atomic measure μ on S. The signals are drawn independently from the same distribution. The value of the object to bidder 1 is $v(t_1, t_2) = c \cdot t_1 + k \cdot t_2 + d$ and for bidder 2 is $v(t_2, t_1)$, where

[4] See Guesnerie (1998).

$c, k \in \mathbb{R}^L$, $c \neq 0$ and $d \in \mathbb{R}$. The auction can be a first-price, a second-price, an all-pay auction or a war of attrition.[5]

Let us assume that $\inf_{(t_1,t_2)\in S\times S} v(t_1,t_2) \geqslant 0$ (for instance, it is sufficient to take d sufficiently large).

Let us define, for each $t_i \in S$, the sets $S(t_i) = \{s \in S : c \cdot s \leqslant c \cdot t_i\}$ — which will be the set of types that type t_i defeats — and $S^0(t_i) = \{s \in S : c \cdot s = c \cdot t_i\}$ — the set of types that tie with t_i. We say that a strategy $b : S \to \mathbb{R}$ is regular if the *distribution* of bids that it induces is absolutely continuous and strictly increasing (so that b is never constant nor have gaps in its range). Observe that a regular strategy is not necessarily continuous or monotonic.

We will use the all-pay auction tie-breaking rule: in the case of a tie, we conduct an all-pay auction among the tying bidders and the allocation and payment are given by this last auction. If there is another tie in the second auction, we split the object randomly. Nevertheless, this will occur with zero probability.

For the first-price (F), second-price (S), all-pay auctions (A) and war of attrition (W), consider, respectively, the following functions:

$$b(t_i) = \frac{1}{\mu(S(t_i))}\int_{S(t_i)} [(c+k)\cdot z + d]\,\mu(dz); \tag{F}$$

$$b(t_i) = c \cdot t_i + k \cdot E\left[s | s \in S^0(t_i)\right] + d; \tag{S}$$

$$b(t_i) = \int_{S(t_i)} [(c+k)\cdot z + d]\,\mu(dz); \tag{A}$$

$$b(t_i) = \int_{S(t_i)} \frac{[(c+k)\cdot z + d]}{1-\mu(S(z))}\mu(dz). \tag{W}$$

We have the following results:

Theorem A *(Necessary and Sufficient Condition for Equilibrium Existence):* A symmetric pure strategy equilibrium in regular strategies for the first-price (F), second-price (S), all-pay auction (A) or war of attrition exists if and only if the respective functions defined above satisfy

$$b(t_i) \leqslant b(t_i') \Leftrightarrow S(t_i) \subseteq S(t_i') \Leftrightarrow c \cdot t_i \leqslant c \cdot t_i' \tag{M}$$

[5]In a first price auction, the winner pays his bid and the looser pays nothing. In a second-price auction, the winner pays the looser's bid and the looser pays nothing. In an all-pay auction, every bidder pays their bids, no matter if they win or not. In a war of attrition, every bidder pays the looser's bid, no matter if they win or not. (The last definition is valid only in the case of two players.)

In particular, it always exists for all-pay auctions and war of attrition. Moreover, if b satisfies condition (M), it is the unique symmetric pure strategy equilibrium in regular strategies for its corresponding game.

THEOREM B *(Equilibria with ties):* Assume that we use all-pay auction tie-breaking rule. Then there always exists a symmetric equilibrium in pure strategies for the four kind of auctions above. If condition (M) above is not satisfied, there are multiple pure strategy equilibria.

THEOREM C *(Revenue Equivalence Theorem):* If the bidders follow the equilibrium strategies specified in Theorem B, the expected payment for a bidder with type t_i is the same for the four kind of auctions above and given by:

$$p(t_i) = \int_{S(t_i)} [(c+k) \cdot z + d] \, \mu(dz).$$

Observe that Theorem C does not require condition M, that is, it remains valid even if there are ties with positive probability and, hence, multiple equilibria. In this case, all equilibria specified by Theorem B give the same expected payment.

Theorems A, B and C are particular cases of Theorems 3, 4 and 5, respectively.

3. The Model

There are N bidders in an auction of $L < N$ homogenous objects, but each bidder is interested in just one object. Player i $(i = 1, ..., N)$ receives a private information, t_i, possibly multidimensional, and chooses a bid $b_i \in B \equiv \{b_{OUT}\} \cup [b_{\min}, +\infty)$, where $b_{\min} > b_{OUT}$ is the minimal valid bid and if $b_i = b_{OUT}$, bidder i does not participate in the auction and gets a payoff of 0.

Let $t = (t_i, t_{-i})$ be the profile of all signals and $b = (b_i, b_{-i})$, the profile of submitted bids. Let $b^{-i}_{(m)}$ be the m-th order statistic of $(b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$, that is, $b^{-i}_{(1)} \geqslant b^{-i}_{(2)} \geqslant ... \geqslant b^{-i}_{(N-1)}$. Since there are L objects, the value that determines the winning and loosing events for bidder i is $b_{(-i)} \equiv \max\left\{b_{\min}, b^{-i}_{(L)}\right\}$. That is, the bidder i receives an object if $b_i > b_{(-i)}$ and none if $b_i < b_{(-i)}$. If the tie-breaking rule is not explictly mentioned, we assume that ties $(b_i = b_{(-i)})$ are broken by the standard tie-breaking rule, that is, the object(s) are randomly divided among the tying bidders. More specifically, the payoff of bidder i is given by

$$u_i(t,b) = \begin{cases} v(t_i,t_{-i}) - p^W\left(b_i,b_{(-i)}\right), & \text{if } b_i > b_{(-i)} \\ -p^L\left(b_i,b_{(-i)}\right), & \text{if } b_i < b_{(-i)} \\ \frac{v(t_i,t_{-i})-b_i}{m}, & \text{if } b_i = b_{(-i)} \end{cases}$$

where v is the value of the object for all bidders, p^W and p^L are the payments made in the events of winning and losing, respectively, and m is the reason between the number of tying bidders and the number of objects in the tie.

Our setting is given by the following assumptions:

(H0) The types are independent and identically distributed in the non-atomic probabilistic space (S, Σ, μ). The value function $v : S \times S^{N-1} \to \mathbb{R}_+$ is a measurable function (with respect to the product $\sigma-$fields), its range is the compact interval $[\underline{v}, \overline{v}] \subset \mathbb{R}_+$ and it is symmetric in its last $N-1$ arguments, that is, if t'_{-i} is a permutation of t_{-i}, $v\left(t_i, t'_{-i}\right) = v(t_i, t_{-i})$.

The restrictive aspect of (H0) is the symmetry. The others are usually assumed in Auction Theory (although independence is sometimes relaxed to affiliation). Nevertheless, our assumption about the type space is very weak, because we assume just that it is a non-atomic probabilistic space.

The specific auction format is determined by p^W and p^L. We will consider alternatively, two cases. The first one, embodied in (H1)-1 below, cover first-price auctions, for instance. The second case is defined by (H1)-2 and covers second price auctions, among other more exotic formats.

(H1) For $j = W$ or L, $p^j(\cdot,\cdot) \geqslant 0$, $p^j(b_{OUT},\cdot) = 0$, $\partial_1 p^j \geqslant 0$, $p^j(\cdot, b_{OUT}) = p^j(\cdot, b_{\min})$, p^j is differentiable over $(b_{\min},\infty)\times(b_{\min},\infty)$, $p^L(b_{\min}, b) = p^L(b_{\min}, b')$ for all b and b' and one of the two conditions below is satisfied:

(H1)-1: $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$ or

(H1)-2: $\partial_1 p^W = \partial_1 p^L \equiv 0$ and $\partial_2\left(p^W - p^L\right) > 0$.

Observe that assumption (H1) is rather weak. It covers virtually all kind of standard single-object auctions or multi-unit auctions with unitary demands, and allows the use of entry fees. Some examples are:

(F) First-price auctions: $p^W\left(b_i,b_{(-i)}\right) = b_i$ and $p^L\left(b_i,b_{(-i)}\right) = 0$.

(S) Second-price auctions: $p^W\left(b_i,b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i,b_{(-i)}\right) = 0$.

(A) All-pay auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.
(W) War of attrition: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.

An active reserve price, that is, $b_{\min}$ that excludes some bidders, makes the statement of our equilibrium results more complex. So, we will postpone the analysis of this case to Appendix B and through the paper we will make use of the following assumption:

(H2) $\underline{v}$, p^W, p^L and $b_{\min}$ are such that no bidder plays b_{OUT}, that is, no bidder prefers to stay out of the auction.

We denote the auction described above by (S, μ, v). Observe that we make no restriction about the dimension of S. Also, we are considering just symmetric auctions. Thus, throughout the paper, when we talk about a strategy, we always mean a symmetric one. Under these assumptions we will introduce a new approach to prove existence of equilibria in auctions.

4. The Indirect Auction

In the subsection 4.1, we describe the basic element of our method: the conjugation. In subsection 4.2, the indirect auction is defined and its basic properties derived.

4.1. **Conjugations.**

We will be interested in regular bidding functions as defined below:

Definition 1: A bounded measurable function $b : S \to \mathbb{R}$ is regular if the c.d.f.

$$F_b(c) \equiv \Pr\{s \in S : b(s) < c\}$$

is absolutely continuous and strictly increasing in its support, $[b_*, b^*]$.

From the fact that $F_b(\cdot)$ is absolutely continuous we conclude that $F_b(c) = \Pr\{s \in S : b(s) \leqslant c\}$. Let $\mathcal{S}$ denote the set of regular functions. Observe that $\mathcal{S}$ contains non-monotonic bidding functions. It is formed by functions b that do not induce ties with positive probability (because F_b is absolutely continuous) and that do not have gaps in the support of the bids (because F_b is increasing).

If a bidding function $b \in \mathcal{S}$ is fixed, let $\tilde{P}^b$ denote the c.d.f. of the maximum bid of the opponents, that is, we define the transformation $\tilde{P}^b : \mathbb{R}_+ \to [0,1]$ by:

$$\begin{aligned} \tilde{P}^b(c) &= \left(\Pr\{t_i \in S : b(t_i) < c\}\right)^{N-1} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) < c, j \neq i\right\}. \end{aligned} \tag{1}$$

By the definition of $\mathcal{S}$, $\tilde{P}^b$ is strictly increasing and its image is the whole interval $[0,1]$.

Now, we will denote by $P^b : S \to [0,1]$ the composition $P^b = \tilde{P}^b \circ b$. Thus, for a fixed strategy $b \in \mathcal{S}$, followed by all players, $P^b(t_i)$ is the probability of player i of type t_i wins the auction:

$$\begin{aligned} P^b(t_i) &= \Pr\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) \leqslant b(t_i), \forall j \neq i\right\}. \end{aligned} \tag{2}$$

The following observation is important: from the symmetry required by (H0), the above function does not depend on i and $P^b(t_i) \lesseqqgtr P^b(t_j)$ if and only if $b(t_i) \lesseqqgtr b(t_j)$. Obviously, two players have the same probability of winning if and only if they play the same bids. Thus, we have the following:

$$\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} = \left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\},$$

where $P^b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} P^b(t_j)$. The equality of these events and (2) imply that

$$P^b(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\}.$$

This observation will allow us to define conjugations without mentioning bidding functions. This is important to the statement of our results. We have the following:

Definition 2: A conjugation for the auction (S, μ, v) is a measurable and surjective function $P : S \to [0,1]$ such that for each $i = 1, \ldots N$,

(3)
$$P(t_i) = \Pr\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) \leqslant P(t_i)\} = \left[\Pr\{t_j \in S : P(t_j) < P(t_i), j \neq i\}\right]^{N-1}.$$

Observe that in the above definition, we do not need to mention the strategy $b \in \mathcal{S}$. It is also clear from the previous discussion that definition 2 is not empty, that is, for any regular function $b \in \mathcal{S}$ there exists a conjugation defined by (2) that satisfies the above definition.

Observe also that, since the range of P is $[0,1]$, we have, for all $c \in [0,1]$,

$$\Pr\left\{t_{-i} \in S^{N-1} : P_{(-i)}\left(t_{-i}\right) < c\right\} = c. \tag{4}$$

The above equation will be important in the sequel. It simply means that the distribution of $P_{(-i)}\left(t_{-i}\right)$ is uniform on $[0,1]$.

Given $b \in \mathcal{S}$, equation (2) defines just one conjugation compatible with it. On the other hand, given a conjugation P, any function $b \in \mathcal{S}$ that is an increasing transformation of P is compatible with P. To see this, suppose that there is an increasing function $h : [0,1] \to \mathbb{R}_{+}$, such that $b\left(t_i\right) = h\left(P\left(t_i\right)\right)$ for $\mu-$almost all $t_i \in S$. Then,

$$\begin{aligned} P\left(t_i\right) &= \Pr\left\{t_{-i} : P\left(t_j\right) < P\left(t_i\right), \forall j \neq i\right\} \\ &= \Pr\left\{t_{-i} : h\left(P\left(t_j\right)\right) < h\left(P\left(t_i\right)\right), \forall j \neq i\right\} \\ &= \Pr\left\{t_{-i} : b\left(t_j\right) < b\left(t_i\right), \forall j \neq i\right\}. \end{aligned}$$

That is, given a conjugation P, there are many functions $b \in \mathcal{S}$ compatible with it. In particular, $b = P$ is a bidding function compatible with P.

4.2. **Indirect Auctions.** We proceed to define the indirect auction $\left(\tilde{S}, \tilde{\mu}, \tilde{v}\right)$ related to the direct auction (S, μ, v). The relation between them is given by the conjugation $P : S \to [0,1]$. If the direct type of a player is $t_i \in S$, the indirect type will be $P\left(t_i\right)$. So, $\tilde{S}$ is just $[0,1]$. Each direct strategy $b : S \to \mathbb{R}$ corresponds to an indirect strategy $\tilde{b} : [0,1] \to \mathbb{R}$, such that the direct strategy will be the composition of the indirect strategy and the conjugation, that is, $b = \tilde{b} \circ P$, where $P = P^b$. What is this indirect strategy? Remember that $P^b = \tilde{P}^b \circ b$ and $\tilde{P}^b$ is increasing. So, given b, if we take the indirect strategy as $\tilde{b} \equiv \left(\tilde{P}^b\right)^{-1}$, then $b = \tilde{b} \circ P$, as we want. On the other hand, if it is given an indirect strategy $\tilde{b}$ and a conjugation P, we have the associated direct strategy $b = \tilde{b} \circ P$. So we have just to define the indirect payoffs:

Definition 3: Fix a conjugation P for an auction (S, μ, v). The indirect utility function of bidder i associated to this conjugation is $\tilde{v} : [0,1]^2 \to \mathbb{R}$, given by

$$\tilde{v}(x,y) \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y]. \tag{5}$$

Now, fix a conjugation P and define the following function:

$$\tilde{\Pi}\left(x,c\right) \equiv E\left[\Pi\left(t_i, c\right) | P\left(t_i\right) = x\right], \tag{6}$$

where, $\Pi(t_i, c)$ is the interim payoff of the direct auction. The notation should suggest to the reader that $\tilde{\Pi}(x, c)$ will be the interim payoff of the indirect auction. Indeed, we have the following:

PROPOSITION 1: Assume (H0). Given $b \in \mathcal{S}$, consider the corresponding conjugation $P = P^b$ (defined by (2)) and the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$. Alternatively, given a conjugation P and an indirect bidding function $\tilde{b}$, let $b = \tilde{b} \circ P$ be the corresponding direct bidding function. Thus,

(i)

$$\tilde{\Pi}(x, c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(x, \alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(c)}^1 p^L\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{7}$$

(ii) Assume that P is such that for all s with $P(s) = x$, and for all $x, y \in [0, 1]$,

$$\tilde{v}(x, y) = E[v(t)|P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t)|t_i = s, P_{(-i)}(t_{-i}) = y]. \tag{8}$$

Then, for all t_i such that $P(t_i) = x$ and for all $c \in B$,

$$\tilde{\Pi}(x, c) = \Pi(t_i, c). \tag{9}$$

PROOF: See Appendix A.■

Observe that, because of (7), $\tilde{\Pi}(x, c)$ is formally equivalent to the interim payoff of an auction between two bidders, with signals uniformly distributed on $[0, 1]$, where the opponent is following the strategy $\tilde{b}(\cdot)$ and the (common-value) utility function is given by $\tilde{v}(x, \alpha)$. So, we define the indirect auction as follows:

DEFINITION 4: Given an auction (S, μ, v) and a conjugation P for it, the associated indirect auction is an auction between two players with independent types uniformly distributed on $[0, 1]$ and where the utility function is $\tilde{v}$ defined by (5). The indirect auction is denoted by $\left(\tilde{S}, \tilde{\mu}, \tilde{v}\right)$ where $\tilde{\mu}$ is the Lebesgue measure in $\tilde{S} = [0, 1]$.

The reader should keep in mind that the indirect auction is just an auxiliary and fictitious auction that will help in the analysis of the "direct" one. It is clear through definitions 1-4 how a conjugation relates the direct and the indirect auction. Obviously, a function $\tilde{b} : [0, 1] \to \mathbb{R}_+$ is equilibrium of the indirect auction if for almost all $x \in [0, 1]$, $\tilde{\Pi}\left(x, \tilde{b}(x)\right) \geqslant \tilde{\Pi}(x, c)$, $\forall c \in B = \{b_{OUT}\} \cup [b_{\min}, +\infty)$.

5. Necessary and Sufficient Conditions for Regular Equilibria

The results and definitions of the two previous subsections allow us to show that the existence of a direct equilibrium implies the existence of the indirect one (Theorem 1, below). Conversely, (with an extra relatively weak assumption of consistency of payoffs) the existence of equilibrium in indirect auctions implies the existence in direct ones (Theorem 2).

Theorem 1 *(Necessary Conditions):* Assume (H0)-(H2). If there is a pure strategy equilibrium $b \in \mathcal{S}$ for the direct auction (S, μ, v) and there exists $\partial_b \Pi(\cdot, \cdot)$ at the point $(s, b(s))$ for all s such that $b(s) \in (b_*, b^*)$, then:

(i) the associated conjugation $P = P^b$ (given by (2)) satisfies the following property: if $s \in S$ is such that $P(s) = x$, then[6]

(10)
$$\tilde{v}(x, x) = E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = x];$$

(ii) the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$, where $\tilde{P}^b$ is given by (1), is the increasing equilibrium of the indirect auction.

Moreover, if $\tilde{v}$ is continuous (i.e., if it has a continuous representative), then:

(iii) (H1)-1 implies that $\tilde{b}$ is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}, \tag{11}$$

and (H1)-2 implies that

$$\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0; \tag{12}$$

(iv) the expected payment of a bidder with type t_i is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha, \alpha)\, d\alpha;$$

[6]This condition is related to an analogous one derived by Araujo and Moreira (2000) for the screening problem and Araujo and Moreira (2001) for signaling model. In these papers, the violation of the single crossing property leads to non-monotonicity.

(v) for all x and $y \in [0,1]$,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0. \tag{13}$$

Proof: See Appendix C.■

Theorem 1 says that if a multidimensional auction has a regular equilibrium, then it can be reduced to a unidimensional auction with two players (the indirect one). However, the reader should note that such reduction is non-trivial and that the indirect auction is not equivalent to the direct one. The indirect auction is a "fictitious" game, where each bidder is facing up a "fictitious" player, the "opponent", that does not correspond to a real player. So, the dimension reduction is meant in this particular sense.

The expression in condition (iv) does not depend on the specific format of the payment rules, p^W and p^L, but it does depend on the conjugation. For the class of auctions considered in the next section, we are able to prove that the conjugation is unique and the Revenue Equivalence Theorem holds. On the other hand, condition (iv) plays an important role to prove the existence of equilibrium in the next result.

Theorem 2 is a kind of converse of Theorem 1. The main difference is that we do not require $\tilde{v}$ to be continuous and we need condition (i)', which is slightly stronger than condition (i) in Theorem 1.

Theorem 2 *(Sufficient Conditions):* Assume (H0)-(H2). Consider a direct auction (S,μ,v), a conjugation P and its associated indirect auction $\left(\tilde{S},\tilde{\mu},\tilde{v}\right)$. Assume that

(i)' for all $s \in S$ such that $P(s) = x$, and all $y \in [0,1]$,

$$\tilde{v}(x,y) = E[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t_i,t_{-i})|t_i = s, P_{(-i)}(t_{-i}) = y]; \tag{14}$$

(ii) for all x and $y \in [0,1]$,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0;$$

(iii) there is an increasing function $\tilde{b}$, such that

$$\tilde{p}(y) \equiv \int_0^y p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha,\alpha)\, d\alpha,$$

where $\tilde{p}(P(t_i)) = p(t_i)$ is the expected payment of a bidder of type t_i.

Then, $\tilde{b}$ is the equilibrium of the indirect auction and $b = \tilde{b} \circ P$ is the equilibrium of the direct auction. Moreover, if $\tilde{v}$ is continuous, there exists $\partial_b \Pi(s, b(s))$ for all s (which implies that all conditions of Theorem 1 are satisfied).

Proof: See Appendix C.■

Remark 1: For the four specific formats, namely, the first-price auction (F), second-price auction (S), all-pay auction (A) and war of attrition (W), the function $\tilde{b}$ is given, respectively, by

$$\text{(F)} \quad \tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha \tag{15}$$

$$\text{(S)} \quad \tilde{b}(x) = \tilde{v}(x, x) \tag{16}$$

$$\text{(A)} \quad \tilde{b}(x) = \int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha \tag{17}$$

$$\text{(W)} \quad \tilde{b}(x) = \int_0^x \frac{\tilde{v}(\alpha, \alpha)}{1-\alpha} d\alpha \tag{18}$$

Condition (iii) reduce to the requirement that the function $\tilde{b}$ above is increasing. In particular, the equilibrium may exist for an all-pay auction, for instance, but not for a first-price auction.

Remark 2: Although natural, condition (i)' can be still too restrictive. We need it in order to apply Proposition 1 and reach the conclusion that for all t_i such that $P(t_i) = x$ and for all $c \in \mathbb{R}$, we have: $\tilde{\Pi}(x, c) = \Pi(t_i, c)$. Indeed, condition (i)' is just the assumption (8) and $\tilde{\Pi}(x, c) = \Pi(t_i, c)$ is the conclusion (9) in Proposition 1. (9) is used to conclude that the equilibrium of the indirect auction is equilibrium of the direct auction. Thus, instead of assuming condition (i)' above, it would be sufficient to require the (necessary) condition (i) of Theorem 1 and that (9) is valid. For instance, when $v(t_i, t_{-i})$ is given by $\sum_{k=1}^{K} f_k(t_i)\, g_k(t_{-i})$, condition (i) is sufficient to have (9) and Theorem 2 is valid with the (necessary) condition (i) in the place of the condition (i)'.

Theorem 2 reduces the problem of equilibrium existence to find a conjugation that meets requirements (i)', (ii) and (iii). In the next section we treat a still general case (weakly separable auctions) where such conjugation can be easily defined. Before this, we give two examples where the assumptions of the next section are not satisfied. These examples illustrate a kind of heuristics for the existence problem. In example 1, we have a monotonic equilibrium and also a U-shaped one, which shows that the conjugation is not unique. In example 2, there is no monotonic equilibrium, but there is a bell-shaped equilibrium. Another example where Theorem 2 can be applied is an example provided by Jehiel, Meyer-ter-Vehn, Moldovanu and Zame (2004).

Example 1: Consider a symmetric first-price auction with two bidders, types uniformly distributed on $[0,1]$ and utility function given by

$$v(t_i, t_{-i}) = t_i + \left(3 - 4t_i + 2t_i^2\right) t_{-i}.$$

Observe that $\partial_{t_i} v\left(t_i, t_{-i}\right) = 1 - 4t_{-i} + 4t_i t_{-i}$ can be negative. Thus, the received theory cannot be applied. Nevertheless, there exists a monotonic equilibrium. Indeed, in this case, the conjugation will be given by $P\left(t_i\right) = t_i$ and we obtain

$$\tilde{v}(x, y) = x + \left(3 - 4x + 2x^2\right) y.$$

$\tilde{v}$ clearly satisfies condition (i)' of Theorem 2. Condition (ii) follows from the fact that $x > y$ implies

$$\int_y^x \left[\tilde{v}(x, z) - \tilde{v}(z, z)\right] dz = \frac{(x-y)^2}{6}\left[3 + 3x^2 - 8y + 3y^2 + x\left(-4 + 6y\right)\right] \geqslant 0.$$

Condition (iii) is also satisfied, because the function

$$\tilde{b}\left(x\right) = \frac{1}{x}\int_0^x \tilde{v}\left(z, z\right) dz = \frac{x\left(24 - 16x + 3x^2\right)}{12}$$

is increasing. Clearly, the above function satisfies condition (iv). Thus, there exists a monotonic equilibrium by Theorem 2.

Nevertheless, this is not the unique equilibrium. If we assume that there exists a U-shaped equilibrium, the conjugation can be expressed by $P\left(t_i\right) = \left|c\left(t_i\right) - t_i\right|$, where $c\left(t_i\right)$ is the type that bids as t_i (see Figure 1). Observe that $c \circ c\left(t_i\right) = t_i$. Condition (i) of Theorem 1 requires that

$$s + \left(3 - 4s + 2s^2\right)\frac{s + c\left(s\right)}{2} = c\left(s\right) + \left(3 - 4c\left(s\right) + 2c\left(s\right)^2\right)\frac{s + c\left(s\right)}{2},$$

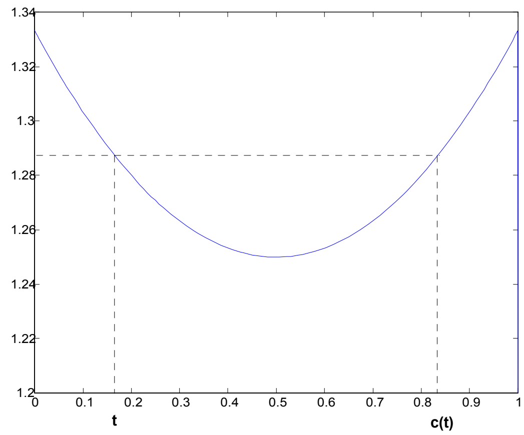

FIGURE 1. Equilibrium bidding function in Example 1.

that is,

$$s - c(s) = [s - c(s)]\,[4 - 2c(s) - 2s]\,\frac{s + c(s)}{2},$$

which simplifies to $[s + c(s)]\,[2 - s - c(s)] = 1 \Rightarrow s + c(s) = 1$. Then, $c(s) = 1 - s$ and $P(s) = |1 - 2s|$. This gives the expression:

$$\tilde{v}(x, y) = \frac{1}{2} + \frac{1}{4}\left[3 - 4\left(\frac{1-x}{2}\right) + 2\left(\frac{1-x}{2}\right)^2 + 3 - 4\left(\frac{1+x}{2}\right) + 2\left(\frac{1+x}{2}\right)^2\right] = \frac{5 + x^2}{4}$$

and condition (i)' and (ii) are easily seen to be satisfied. Also, condition (iii) and (iv) are satisfied, since

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(z, z)\,dz = \frac{5}{4} + \frac{x^2}{12}$$

is increasing. Then, $b(s) = \frac{5}{4} + \frac{(1-2s)^2}{12}$ is a direct equilibrium, plotted in Figure 1.

Observe that no tie rules are needed in this case, because ties occur with zero probability. However, for each equilibrium bid, exactly two types pool and have the same probability of winning.

In Appendix D, we treat a slightly more general case, where the conjugation is more difficult to find than in this example.■

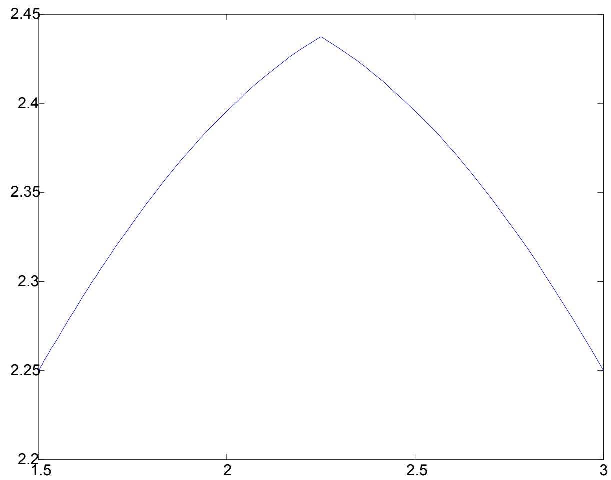

FIGURE 2. Equilibrium bidding function in Example 2.

EXAMPLE 2: Consider again a symmetric first-price auction with two bidders and signals uniformly distributed in $[1.5, 3]$ such that the value of the object is given by $v(t_i, t_{-i}) = t_i \left(t_{-i} - \frac{t_i}{2}\right)$. In Appendix D, we show that this auction does not have monotonic regular equilibria, but there is a bell-shaped equilibrium as shown in Figure 2.■

Example 1 shows that it is possible for a standard auction to have multiple equilibria. Example 2 suggests that the correct conjugation can fail to exist — at least with a fixed shape (that we began assuming). Thus, one would be interested in cases where it is possible to ensure the uniqueness of the equilibrium and where it is possible to find explicitly the conjugation. We do this under the context of assumption H3, to be presented in the next subsection.

6. EQUILIBRIUM EXISTENCE OF WEAKLY SEPARABLE AUCTIONS

Theorem 2 teaches us that the question of equilibrium existence is solved if we are able to find the proper conjugation. In examples 1 and 2 of the previous section we have shown cases where the conjugations could be obtained. However, there we assumed some features of the conjugation that are not necessary and we were able to find the correct conjugation for those settings. Now we will work in a setting where a conjugation always exists: the weakly separable auctions. These are the auctions satisfying the following assumption:

(H3) (Weak Separability). $v(t_i, t_{-i})$ is such that if $v(t_i, t_{-i}) < v(t'_i, t_{-i})$ for some t_{-i} then $v\left(t_i, t'_{-i}\right) < v\left(t'_i, t'_{-i}\right)$ for all t'_{-i}. Moreover, if $C \subset \mathbb{R}$ has zero Lebesgue measure, then $\mu\{s \in S : v^1(s) \in C\} = 0$, where

$$v^1(s) \equiv E\left[v(t_i, t_{-i}) \mid t_i = s\right]$$

is the expected value of the object for bidder with type s.

Assumption (H3) is restrictive, but it is valid in many economic meaningful cases.[7] Of course, private values are included in (H3).

Under (H3), we can explicitly define the conjugation:

$$P(t_i) \equiv \Pr\left\{t_{-i} \in S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\right\}. \tag{19}$$

Moreover, we can give a *necessary and sufficient* condition for the equilibrium existence of the direct auction: that the function $\tilde{b}$ that implies the correct expected payment is increasing. More precisely, we have the following:

Theorem 3 *(Necessary and Sufficient Condition For Equilibrium Existence)*: Assume (H0)-(H3). Let P be defined by (19) and let $\tilde{v}$ be given by (5) for this P. There exists an equilibrium $b \in \mathcal{S}$ *if* there exists an increasing function $\tilde{b}$ that satisfies

$$\int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha)\, d\alpha. \tag{20}$$

If this is the case, the equilibrium of the direct auction is given by $b = \tilde{b} \circ P$ and the expected payment of a bidder of type s is given by

$$p(s) = \int_0^{P(s)} \tilde{v}(\alpha, \alpha)\, d\alpha. \tag{21}$$

Additionally, if $\tilde{v}$ is continuous, then there exists an equilibrium $b \in \mathcal{S}$ and there exists $\partial_b \Pi(s, b(s))$ for all s *if and only if* there exists an increasing function $\tilde{b}$ that satisfies the following:

- For (H1)-1, $\tilde{b}$ is differentiable and

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]};$$

[7] Theorem 3 of Debreu (1960) implies that if S is connected and v satisfies (H3), then $v(t_i, t_{-i})$ can be written as $h\left(u^1(t_i) + u^2(t_{-i})\right)$, where h is an increasing function. In this case, (H3) would also imply that $u^1(t_i)$ does not assume a value with positive probability.

with initial condition $\int_0^1 p^L\left(\tilde{b}(0), \tilde{b}(\alpha)\right) d\alpha = 0$;

- For (H1)-2, $\tilde{b}$ is continuous and satisfies, for all $x \in (0,1)$,

$$\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0.$$

Moreover, if there is a unique $\tilde{b}$ that satisfies such properties, the equilibrium of the direct auction in regular pure strategies is also unique.

PROOF: See Appendix C.■

REMARK 3: As explained in the comments after Theorem 1, if a multidimensional auction has a regular equilibrium, it can always be reduced (in a non-trivial way) to a one dimension auction (the indirect auction). So, for obtaining equilibrium existence, we have to consider auctions that can be "reduced". This is what assumption H3 allows us to explicitly do. It still encompasses cases where such reduction is not trivial, as we show in examples 3 and 4 below. The reduction of the dimension of types is not a novelty in auction theory. While studying the efficiency of auctions, Dasgupta and Maskin (2000) use a condition close to H3 and Jehiel, Moldovanu and Stacchetti (1996) made such reduction for revenue maximization. Nevertheless, to show equilibrium existence in auctions, one cannot use only H3 or the Dasgupta and Maskin's condition, since the received theory would require the monotonicity assumption of $\tilde{v}$ on the reparameterized types. As we show in examples 3 and 4, this is not always possible. So, an important feature of Theorem 3 is that it does not require $\tilde{v}$ to be monotonic.

EXAMPLE 3 (*Spectrum Auction)*:[8] Consider a first-price auction of a spectrum license. The license covers two periods of time:

(1) In the first period, the regulator lets the winner explore its monopoly power. Let t_i^1 be the estimative of bidder i of the monopolist surplus in this first period. Of course, the true surplus will be better approximated by $(t_1^1 + ... + t_N^1)/\ N$. If the bidder i (a firm) wins the auction, it has to invest t_i^2, a privately known amount, to build the network that will support the service. So, in the first period, the license gives to the firm

$$\frac{t_1^1 + ... + t_N^1}{N} - t_i^2.$$

[8]This example is more complex, but formally similar to example 5 of Dasgupta and Maskin (2000).

(2) In the second period, the regulator makes an estimate of the operational costs of the firm. The regulator cannot observe the true operational cost, t_i^3, which is a private information of the firm. Nevertheless, the regulator has a proxy that is a sufficient statistic for the mean operational cost of all participants in the auction, $(t_1^3 + ... + t_N^3)/N$. The regulator will fix a price that will give zero profit for a firm with the mean operational costs.[9] So, in the second period, the license gives to the winner

$$\frac{t_1^3 + ... + t_N^3}{N} - t_i^3.$$

So, the value of the object is given by

$$v(t_i, t_{-i}) = \frac{t_1^1 + ... + t_N^1}{N} - t_i^2 + \frac{t_1^3 + ... + t_N^3}{N} - t_i^3.$$

Let the signals $t_i = (t_i^1, t_i^2, t_i^3)$, $i = 1, ..., N$, be independent. Observe that the problem cannot be reduced to a single dimension. Indeed, if we summarize the private information by, say, $s_i = t_i^1/N - t_i^2 + t_i^3(1/N - 1)$, we lose the information about t_i^1 and t_i^3 that are needed for the value function of bidders $j \neq i$. Also, the model cannot be reparameterized to an increasing one. If we try to put $-t_i^3$ in the place of t_i^3, then the dependence of $v(t_i, t_{-i})$ on the signals t_j^3 will be decreasing. So, the received theory does not ensure the existence of pure strategy equilibrium for this case. Nevertheless, assumption (H3) is trivially satisfied. In Appendix D, we assume the $t_i = (t_i^1, t_i^2, t_i^3)$ are independent and uniformly distributed on $[\underline{s}^1, \overline{s}^1] \times [\underline{s}^2, \overline{s}^2] \times [\underline{s}^3, \overline{s}^3]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3 \geqslant 0$ and we show that a sufficient condition for the existence of equilibrium in pure strategy is

$$\frac{\underline{s}^1}{N} - \overline{s}^2 - \overline{s}^3 \frac{N-1}{N} - 1 \geqslant 0.$$

The derivation in Appendix D indeed provides necessary and sufficient conditions for the existence of equilibrium.■

Example 4 *(Job Market):* We model the job market for a manager as an auction among competing firms, where the object is the job contract. It is natural to assume that the manager has a multidimensional vector of characteristics, $m = (m^1, ..., m^k)$. For the sake of simplicity, we assume that firms learn such characteristics through interviews and curriculum analysis. Each firm has a position to be filled by the manager,

[9] We assume that the regulator is institutionally constrained to follow such a procedure, so the optimality of this regulation is not an issue here.

with specific requirements for each dimension of the characteristics. For instance, if dimension 1 is ability to communicate and the position is to be the manager of a production section, there is level of desirability of this ability. An overly communicative person may not be good. The same goes for the other characteristics. A bank may desire a sufficiently (but not exaggeratedly) high level of risk loving or audacity on the part of the manager, while a family business may desire a much lower level. Even efficiency or qualification can be desirable in different levels. Sometimes, the rejection of a candidate is explained by over-qualification. Therefore, let $t_i = (t_i^1, ..., t_i^K)$ be the value of the characteristics desired by the firm.

Since firms are competitors, then if one hires the employee, the other will remain with a vacant position, at least for a time.[10] In this way, the winning firm also benefits from the fact that its competitors have a vacant position — and, then, are not operating perfectly well. The higher the abilities required for the job, the more the competitor suffers.[11] So, the utility in this auction is as

$$v(t_i, t_{-i}) = \sum_{k=1}^{K} a^k m^k - \sum_{k=1}^{K} b^k \left(t_i^k - m^k\right)^2 + \sum_{j\neq i} \sum_{k=1}^{K} c^k t_j^k,$$

where a^k is the level of importance of characteristic k of the manager, $b^k > 0$ represents how important is the distance from the desired level t_i^k of the characteristic k, and c^k is the weight of the benefit that firm i receives from the fact that the competitors are lacking $\sum_{j\neq i} t_j^k$ of the ability k. As in the previous example, we cannot simplify this model to a unidimensional monotonic model. In Appendix D we analyze the case where there is just one dimension ($K = 1$), 2 players ($N = 2$) and types are uniformly distributed on $[0, 1]$, $b = b^1 \geqslant 0$. We show that when $m^1 = m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \equiv c^1 \geqslant \max\left\{\frac{2b(m-2)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}$$

[10]This model works only for non-competitive job markets. In other words, the buyers (the contracting firms) have no access to a market with many homogenous employees to hire. This is implicit when we model it as an auction. So, this is the reason why a firm that does not contract the manager suffers — it is not possible to find a suitable substitute instantaneously.

[11]If firms act in a oligopolistic market, it is possible to justify such externality through the fact that the vacant position influences the quality of the product delivered by the firms and, hence, the equilibrium in this market. Externalities are an important issue in Auction Theory. See Jehiel, Moldovanu and Stacchetti (1996).

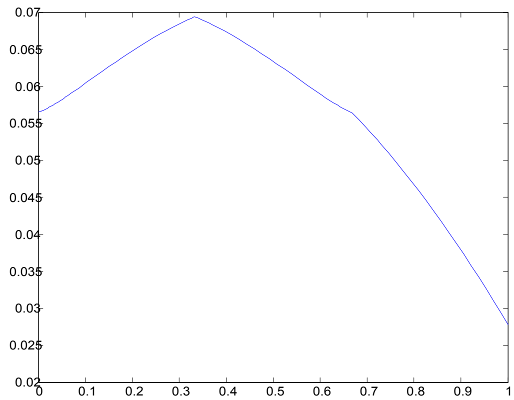

FIGURE 3. Equilibrium bidding function in Example 4.

and when $m < 1/2$, if and only if

$$c \leqslant \min \left\{ \frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3} \right\}.$$

Observe that for both cases the value $c = 0$ ensures the existence of equilibrium. This is expected, since it corresponds to a private value auction. For $a = b = 1/5$, $c = 1/20$ and $m = 1/3$, the equilibrium bidding function is shown in Figure 3.■

Now, we can return to the example given in the introduction. Theorem 3 gives the conditions for the equilibrium existence.

EXAMPLE 5 *(JSSZ, example 1)*: Let us consider a first price auction with two bidders, independent types uniformly distributed on $[0, 1]$. Let $v^1(t_i) = t_i$ and $v^2(t_{-i}) = \alpha + \beta t_{-i}$. It is clear that $P(t_i) = t_i$ in this case and $\tilde{v}(x, y) = \alpha + x + \beta y$. So, $\tilde{v}(x, x) = \alpha + (1 + \beta)x$ and

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(z, z)\,dz = \frac{1}{x}\left[\alpha x + \frac{(1+\beta)x^2}{2}\right] = \alpha + \frac{(1+\beta)x}{2},$$

which is increasing only if $\beta > -1$. Observe that for $\tilde{b}(\cdot) \geqslant 0$, it is necessary $\alpha \geqslant -(1+\beta)x/2$, otherwise negative bids have to be allowed.■

The example above is used by JSSZ to show that equilibrium may fail to exist under the standard tie-breaking rule. They then provide a general existence result based on endogenously defined tie-breaking rules.

Instead of an endogenous rule, consider the following *all-pay auction tie-breaking rule*: if a tie occurs, conduct an all-pay auction among the tying bidders. If another tie occurs, split randomly the object.[12,13]

We show now that the all-pay auction tie-breaking rule ensures the existence of equilibrium for all auctions that we are considering.

THEOREM 4 *(Equilibrium Existence with ties)*: Assume (H0) - (H3) and that the all-pay auction tie-breaking rule is adopted. If there is a absolutey continuous function $\tilde{b}$, not necessarily increasing, such that

$$\int_0^y p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right)d\alpha+\int_y^1 p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right)d\alpha=\int_0^y \tilde{v}(\alpha,\alpha)\,d\alpha,$$

then there exists a pure strategy equilibrium.

PROOF: See Appendix C.■

The function $\tilde{b}$ mentioned in the statement of Theorem 4 is not necessarily the equilibrium. If it is decreasing, as in example 5, it will not be the equilibrium.

REMARK 4: The main ingredients in the proof of Theorem 4 are the payment expression and the fact the bidding function of an all-pay auction is always increasing. This is so because in the all-pay auction, the bid

$$\tilde{b}(x)=\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha$$

is exactly the expect payment, which implies (20). Since v is positive by (H0), $\tilde{v}$ is positive which implies that $\tilde{b}$ is increasing. Thus, an auction that has an increasing $\tilde{b}$, as the war of attrition, can be also used as the tie-breaking mechanism.

EXAMPLE 5 (cont.): With the all-pay auction tie-breaking rule, the equilibrium of Example 5 is given, if $\beta\leqslant -1$, by the bid strategy $b^1(x)=\bar{b}$, $\forall x\in[0,1]$, in the first price auction, where $\bar{b}\in\left[\alpha+\frac{1+\beta}{2},\alpha\right]$, and the bid strategy

$$b^2(x)=\alpha x+\frac{1+\beta}{2}x^2$$

for the tie-breaking (all-pay) auction.■

[12] Observe that in the tie-breaking auction, bids and payments may be less than in the first auction.

[13] Maskin and Riley (2000) used a similar tie-breaking rule. They propose to conduct a second price (Vickrey) auction in the case of a tie.

Remark 5: When $\tilde{b}$ is not increasing, there are types that are not ordered correctly (as in Example 5). This can be understood as a failure of $\tilde{b}$ in correctly revealing the information that each bidder possesses. One can say that the tie-breaking rule has exactly the role of revealing information, by sorting the types. Thus, Theorem 4 can be interpreted as saying that all-pay auctions (and war of attrition) are better mechanisms for revealing information than first-price and second-price auctions. This can be another way of justifying the use of research tournaments in practice, since research tournaments are theoretically modeled as all-pay auctions. Indeed, the characteristics needed to compete for a research is very intricate: it is needed information not only on the technical capabilities, but also on discipline, honesty, creativity, etc. Our results show that all-pay auctions perform better in the task of revealing (multidimensional) information and should be preferred in situations where the determination of the best competitor is difficult, as occurs in most tournaments.[14,15]

The reader should note that Theorem 4 does not claim the uniqueness of equilibrium. Indeed, if $\tilde{b}$ is not increasing, there are many equilibria. There are two sources for this multiplicity.

The first source is that under the all-pay auction tie-breaking rule, any level of the bid in the range where $\tilde{b}$ is not increasing can be chosen to be the level of the tie. This is shown in the Figure 4. For instance, any a_0 can be chosen between x_0 and x_1. Once one of the three elements a_k, b_k or c_k is determined, so are the other two. However, these possibilities lead to the same expected payment and payoff for each bidder in the auction.

Another point is that the tie-breaking rule is not unique, in general. It can be shown, for instance, that for cases where $\tilde{b}$ is decreasing (as in example 1 of JSSZ) and for some specifications of v, there is a continuum of tie-breaking rules (like that defined by JSSZ for their example 1), which ensures the existence of equilibrium. Nevertheless, these tie-breaking rules imply different expected revenues.

[14]Standard explanations for the use of tournaments in research also appeals to the role of information. See, for instance, Taylor (1995, p. 872): "Contracting for research is often infeasible because research inputs are unobservable and research outcomes cannot be verified by a court". Our point is somewhat different, but obviously related to this explanation. The comparison among the information revealed by different auctions is the novelty.

[15]Obviously, these observations are valid only under the context of assumptions (H0)-(H3). It is a matter for future research to determine the range of validity of the existence ensured by all-pay auction tie-breaking rule.

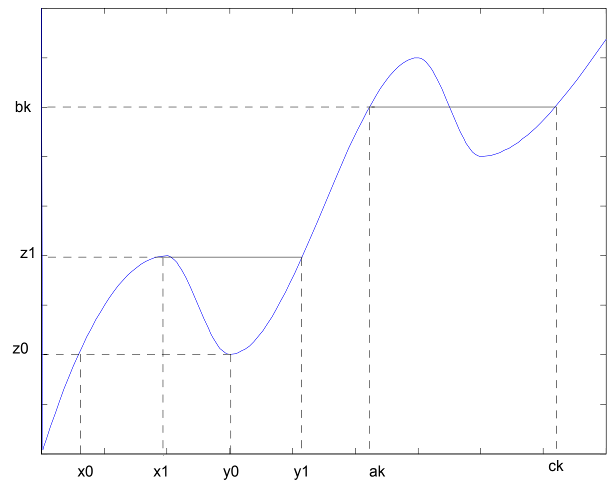

FIGURE 4. Possible specifications for the level of the tie.

The multiplicity of equilibria is in contrast with the "ironing principle" usual in Contract Theory, which gives a unique solution (see, for instance, Guesnerie and Laffont (1984). The uniqueness of the solution through the ironing principle comes from a Lagrangian condition that must be satisfied by the contract, which comes from the principal's problem. Here, we do not have the maximization function of the principal. Thus, there is no additional condition that would fix the level of tieing bids.

The reader may observe that the expression of the payment in Theorem 3 depends only on the conjugation, which is fixed for all kind of auctions. Also, the payment is exactly the same under the all-pay auction tie-breaking rule. Thus, we have the following:

THEOREM 5 (*Revenue Equivalence Theorem*): Assume (H0) - (H3) and that the all-pay auction tie-breaking rule is adopted. If the bidders follow the symmetric equilibrium specified by Theorem 4, then, the expected revenue is the same for any format of the auction.

PROOF: See Appendix C.■

7. CONCLUSION

Now we will briefly highlight what are the most important contributions of this paper and discuss possible extensions.

7.1. **The Contributions.** Our contributions can be summarized as follows:

- Equilibrium Existence in the Multidimensional Setting — We prove the existence of pure strategy equilibrium with no monotonic assumption but with independence of types and symmetry. Working in a monotonic setting, McAdams (2003) generalizes Athey (2001)'s method to multidimensional types and actions and apply this to prove the existence of pure strategy equilibrium of multi-unit asymmetrical uniform price auctions with independent types. He requires two nonprimitive conditions, quasi-supermodularity and single-crossing. (See the details in McAdams (2003).) JSSZ give the existence for multidimensional games, including cases with dependence, while we require independence. Jackson and Swinkels (2005) show the existence of equilibria for a large class of multidimensional private value auctions. They allow asymmetries, dependence of signals and multi-units, but the existence is given in mixed strategies. JSSZ and Jackson and Swinkels (2005) ensure the existence of equilibrium in mixed strategies, while our results are in pure strategies.
- Equilibrium Existence in Non-Monotonic Settings — We are not aware of any general non-monotonic equilibrium existence results in pure strategies.[16] Zheng (2001), Athey and Levin (2001) and Ewerhart and Fieseler (2003) present cases where non-monotonicity arises. Our method develops a theory to deal with the situations where the usual monotonicity is not fulfilled. Araujo and Moreira (2000) use a similar method for the screening problem without the single crossing property and Araujo and Moreira (2001) extend it to signaling model.
- Uniqueness of Equilibrium — We are able to ensure the uniqueness of equilibrium in the symmetric interdependent values auctions that satisfy assumption H3, extending the well known equilibrium uniqueness of unidimensional and monotonic auctions.
- Necessary and Sufficient Conditions for the Existence of Equilibrium without Ties — The results of JSSZ do not allow one to distinguish in which cases a special tie-breaking is needed. Our approach clarifies, under assumption H3, whether ties occur with positive probability (and there is a potential need for special tie-breaking rules).

[16]Papers that provide existence in distributional (mixed) strategies can treat non-monotonic settings as well.

- All-Pay Auction Tie-Breaking Rule — When there is a need for a tie with positive probability, we are able to offer an exogenous tie-breaking rule, which is implemented through an all-pay auction. Moreover, the equilibrium that the rule implements is in pure strategies. For private value auctions, Jackson and Swinkels (2005) show that the equilibrium is invariant for any trade-maximizing tie-breaking rule. Nevertheless, this does not need to hold for the interdependent values auctions that we treat here.
- Information Revelation — In the sense made precise by Remark 5, we show that all-pay auction and war of attrition are better mechanisms to reveal information than first-price and second-price auctions.
- Revenue Equivalence Theorem — We have generalized the Revenue Equivalence Theorem (Theorem 5) to multidimensional types and non-monotonic utilities. Furthermore, Theorem 2 and Appendix B show that there is a deep connection between the revenue equivalence and the existence of equilibrium. Riley and Samuelson (1981) and Myerson (1981) establish that revenue equivalence is a consequence of the equilibrium behavior. Proposition 5 and Corollary 1 in Appendix B show that the revenue equivalence is also sufficient for the existence of equilibrium (if an extra condition is satisfied).

Thus, our results have clarified some aspects of the equilibrium existence problem in auctions. The theory shows that, under assumption H3, there is no additional difficult in working with the more general setting of multidimensional types and non-monotonic utilities besides those difficulties already present in the unidimensional setting.[17] Moreover, this approach allows the equilibrium bidding functions to be expressed in a simple manner. This is so because the equilibrium bidding function of a general auction can be expressed by the equilibrium bidding function of an auction with two bidders and types uniformly distributed on $[0, 1]$.

7.2. **Limitations of the Method.** Our theory makes two important assumptions: independence of the types and symmetry.

The generalization of this approach for dependent types involves some difficulties, because the conjugation would depend in a complicated way on types. It is worth

[17]Theorem 3 shows that the non-existence of the equilibrium comes from the non-monotonicity of the indirect bidding function. This can also occur in an unidimensional setting, although it can be more natural in multidimensional models.

remembering that the problem with dependence is not specific to our approach. Jackson (1999) gives a counter-example for the equilibrium existence of an auction with bidimensional affiliated types. Fang and Morris (2003) also obtain negative results, not only for the existence of equilibrium but also for the revenue equivalence.

On the other hand, asymmetry does not seem to impose severe restriction on the existence of equilibrium. We believe that the approach of the indirect auction can be adapted to this case, although not in a straightforward way. If this can be done, it is unlikely that we will obtain simple expressions as in this paper.

Another limitation of our theory is that it is applied only to auctions with unitary demand. The risk neutrality does not seem to be a fundamental assumption, although complications can arise in extending the approach for risk aversion.

Finally, the relaxation of assumption H3 is an obvious direction to pursue, although H3 seems to encompass many important economical examples.

Appendix A - Proof of the Basic Results

We will need the following result, which was proved, in a more general setting, by de Castro (2004).

LEMMA 1 *(Payoff Characterization)*:— Assume (H0)-(H2). Fix $b(\cdot) \in \mathcal{S}$. The bidder i's payoff can be expressed by

$$\Pi(t_i, b_i, b(\cdot)) = \Pi_i(t_i, b_*) + \int_{(b_*, b_i)} \partial_{b_i} \Pi(t_i, \beta, b(\cdot)) d\beta,$$

where $\partial_{b_i} \Pi(t_i, \beta, b(\cdot))$ exists for almost all β with

$$\begin{aligned}\partial_{b_i} \Pi(t_i, \beta, b(\cdot)) = E\Big[& -\partial_1 p^W \left(\beta, b_{(-i)}(t_{-i})\right) 1_{\left[\beta > b_{(-i)}(t_{-i})\right]} - \partial_1 p^L \left(\beta, b_{(-i)}(t_{-i})\right) 1_{\left[\beta < b_{(-i)}(t_{-i})\right]}\Big] \\ & + E[v(t_i, t_{-i}) - p^W(\beta, \beta) + p^L(\beta, \beta) \,| b_{(-i)}(t_{-i}) = \beta] f_{b_{(-i)}}(\beta).\end{aligned}$$

a.e., where $b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} b(t_j)$.

PROOF: Let us define $b_{(-i)}(t_{-i}) \equiv \max_{j\neq i} b(t_j)$. Since $b_i = b_{(-i)}(t_{-i})$ with probability zero, we have

$$\begin{aligned}\Pi(t_i, b_i, b(\cdot)) &= \int_{S^{N-1}} \left[v(t_i, t_{-i}) - p^W\left(b_i, b_{(-i)}(t_{-i})\right)\right] 1_{\left[b_i > b_{(-i)}(t_{-i})\right]} \prod_{j\neq i} \mu(dt_j) \\ &\quad + \int_{S^{N-1}} \left[-p^L\left(b_i, b_{(-i)}(t_{-i})\right)\right] 1_{\left[b_i < b_{(-i)}(t_{-i})\right]} \prod_{j\neq i} \mu(dt_j).\end{aligned}$$

Take a sequence $a_n \to b_i^-$, i.e., $a^n < b_i$ (the other case is analogous). We want to prove that there exists $\lim_{n\to\infty} D_n(b_i)/(b_i - a_n)$ for almost all b_i, where

$$D_n(b_i) = \Pi(t_i, b_i, b(\cdot)) - \Pi(t_i, a_n, b(\cdot)).$$

In the sequel, we will omit the measure $\prod_{j\neq i} \mu(dt_j)$ and the terms t_{-i}. We have:

$$\begin{aligned}D_n(b_i) &= \int \left[v(t_i, \cdot) - p^W\left(b_i, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i > b_{(-i)}(\cdot)\right]} \\ &\quad + \int \left[-p^L\left(b_i, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i < b_{(-i)}(\cdot)\right]} \\ &\quad - \int \left[v(t_i, \cdot) - p^W\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[a_n > b_{(-i)}(\cdot)\right]} \\ &\quad - \int \left[-p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[a_n < b_{(-i)}(\cdot)\right]} \\ &= \int \left[v(t_i, \cdot) - p^W\left(b_i, b_{(-i)}(\cdot)\right) + p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i > b_{(-i)}(\cdot) > a_n\right]} \\ &\quad + \int \left[-p^W\left(b_i, b_{(-i)}(\cdot)\right) + p^W\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[a_n > b_{(-i)}(\cdot)\right]} \\ &\quad + \int \left[-p^L\left(b_i, b_{(-i)}(\cdot)\right) + p^L\left(a_n, b_{(-i)}(\cdot)\right)\right] 1_{\left[b_i < b_{(-i)}(\cdot)\right]}\end{aligned}$$

Let us call the three last integrals as $D_n^1(b_i)$, $D_n^2(b_i)$ and $D_n^3(b_i)$, respectively. Since p^W and p^L are differentiable, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{D_n^2(b_i)}{b_i - a_n} &= -\lim_{n\to\infty} \int \frac{p^W\left(a_n, b_{(-i)}(\cdot)\right) - p^W\left(b_i, b_{(-i)}(\cdot)\right)}{b_i - a_n} 1_{\left[b_i > b_{(-i)}(\cdot)\right]} \\ &= -\int \partial_1 p^W\left(b_i, b_{(-i)}(\cdot)\right) 1_{\left[b_i > b_{(-i)}(\cdot)\right]}\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{D_n^3(b_i)}{b_i-a_n} &= -\lim_{n\to\infty}\int\frac{p^L\left(a_n,b_{(-i)}(\cdot)\right)-p^L\left(b_i,b_{(-i)}(\cdot)\right)}{b_i-a_n}1_{\left[a_n<b_{(-i)}(\cdot)\right]}\\ &= -\int\partial_1 p^L\left(b_i,b_{(-i)}(\cdot)\right)1_{\left[b_i<b_{(-i)}(\cdot)\right]}.\end{aligned}$$

everywhere. So, if $b_i\geqslant b_*$, the Fundamental Theorem of Calculus gives

$$D_0^2(b_i)=D_0^2(b_*)+\int_{(b_*,b_i)}-\partial_1 p^W\left(\alpha,b_{(-i)}(\cdot)\right)1_{\left[b_i>b_{(-i)}(\cdot)\right]}d\alpha \tag{22}$$

and

$$D_0^3(b_i)=D_0^3(b_*)+\int_{(b_*,b_i)}-\partial_1 p^L\left(\alpha,b_{(-i)}(\cdot)\right)1_{\left[b_i<b_{(-i)}(\cdot)\right]}d\alpha. \tag{23}$$

Now, define the measure ρ over $\mathbb{R}_+$ by

$$\rho(V)\equiv\int_{S^{N-1}}\left[v(t_i,t_{-i})-p^W\left(b_i,b_{(-i)}(t_{-i})\right)+p^L\left(b_i,b_{(-i)}(t_{-i})\right)\right]1_{\left[b_{(-i)}(t_{-i})\in V\right]}.$$

Observe that, since $b\in\mathcal{S}$, ρ is absolutely continuous with respect the Lebesgue measure λ. We have

$$\lim_{n\to\infty}\frac{D_n^1(b_i)}{b_i-a_n}=\lim_{n\to\infty}\frac{\rho([a_n,b_i))}{b_i-a_n}=\lim_{a^n\to b_i}\left\{\frac{\rho([a_n,b_i))}{\lambda([a_n,b_i))}\right\}=\frac{d\rho}{d\lambda}(b_i),$$

where $\frac{d\rho}{d\lambda}(.)$ is the Radon-Nikodym derivative of ρ with respect to λ. Indeed, the existence of such limit is ensured by Theorem 8.6 of Rudin (1966) for almost all b_i, that is,

$$\lambda\left(\left\{b_i:\nexists\lim_{a^n\to b_i}\left\{\frac{\rho([a_n,b_i))}{\lambda([a_n,b_i))}\right\}\right\}\right)=0.$$

It is easy to see that the Radon-Nikodym derivative $\frac{d\rho}{d\lambda}(b_i)$ is simply

$$E[v(t_i,t_{-i})-p^W(\beta,\beta)+p^L(\beta,\beta)\,|b_{(-i)}(t_{-i})=\beta]f_{b_{(-i)}}(\beta),$$

where $f_{b_{(-i)}}(\beta)$ is the Radon-Nikodym derivative of the distribution of maximum bids, $\int 1_{\left[b_{(-i)}(t_{-i})\in V\right]}$. Moreover, Theorem 8.6 of Rudin says that

$$\rho((b_*,b_i))=\int_{(b_*,b_i)}\frac{d\rho}{d\lambda}(\alpha)\,d\alpha. \tag{24}$$

Thus, (22), (23) and (24) imply that

$$\Pi(t_i,b_i,b(\cdot))=\Pi(t_i,b_*)+\int_{(b_*,b_i)}\partial_{b_i}\Pi(t_i,\beta,b(\cdot))d\beta,$$

where

$$\partial_{b_i}\Pi(t_i,\beta,b(\cdot)) = E\left[-\partial_1 p^W\left(\beta, b_{(-i)}(t_{-i})\right)1_{\left[\beta>b_{(-i)}(t_{-i})\right]} - \partial_1 p^L\left(\beta, b_{(-i)}(t_{-i})\right)1_{\left[\beta<b_{(-i)}(t_{-i})\right]}\right]$$
$$+ E[v(t_i,t_{-i}) - p^W(\beta,\beta) + p^L(\beta,\beta)\,|\max_{j\neq i} b(t_j) = \beta] f_{b_{(-i)}}(\beta).$$

This concludes the proof.■

Proof of Proposition 1: Let us introduce the following notation:

$$\Pi^+(t_i,c) = \int\left[v(t_i,\cdot) - p^W\left(c, b_{(-i)}(\cdot)\right)\right]1_{\left[c>b_{(-i)}(\cdot)\right]}\Pi_{j\neq i}\mu(dt_j)$$
$$\Pi^-(t_i,c) = \int p^L\left(c, b_{(-i)}(\cdot)\right)1_{\left[c<b_{(-i)}(\cdot)\right]}\Pi_{j\neq i}\mu(dt_j),$$
$$\tilde{\Pi}^{+,-}(\phi_i,c) \equiv E\left[\Pi_i^{+,-}(t_i,c)\,|P(t_i)=\phi_i\right].$$

Let us begin with the proof for $\tilde{\Pi}_i^+$ and Π_i^+. Let us denote the conditional expectation by

$$g^{t_i,c}(\alpha) \equiv E\left[v(t_i,t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)|P^b_{(-i)}(t_{-i}) = \alpha\right]. \tag{25}$$

The event $\left[c > b_{(-i)}(t_{-i})\right]$ occurs if and only if $\left[\tilde{P}^b(c) > P^b_{(-i)}(t_{-i})\right]$ occurs. Then, we have

$$\Pi^+(t_i,c) = \int g^{t_i,c}\left(P^b_{(-i)}(t_{-i})\right)1_{\left[\tilde{P}^b(c)>P^b_{(-i)}(t_{-i})\right]}\Pi_{j\neq i}\mu(dt_j).$$

Now we appeal to Lemma 2.2, p. 43, of Lehmann (1959). This lemma says the following: if R is a transformation and if $\mu^*(B) = \mu(R^{-1}(B))$, then

$$\int_{R^{-1}(B)} g[R(t)]\,\mu(dt) = \int_B g(\alpha)\,\mu^*(d\alpha).$$

In our case, $R = P^b_{(-i)}$ and $\mu^*([0,c]) = \mu^*([0,c)) = \tau_{-i}\left(\left(P^b\right)^{-1}_{(-i)}([0,c))\right) = \Pr\{t_{-i} \in S^{N-1} : P^b(t_j) < c\} = c$, by (4). So, μ^* is exactly the Lebesgue measure, and we have

$$\Pi^+(t_i,c) = \int_0^{\tilde{P}^b(c)} g^{t_i,c}(\alpha)\,d\alpha. \tag{26}$$

From this and the definition of $\tilde{\Pi}^{+}$, we have

$$\begin{aligned}\tilde{\Pi}^{+}\left(\phi_i,c\right) &= E\left[\int_0^{\tilde{P}^b(c)} g^{t_i,c}\left(\alpha\right)d\alpha | P^b\left(t_i\right)=\phi_i\right]\\ &= \int_0^{\tilde{P}^b(c)} E\left[g^{t_i,c}\left(\alpha\right)|P^b\left(t_i\right)=\phi_i\right]d\alpha\\ &= \int_0^{\tilde{P}^b(c)}\left[\tilde{v}\left(\phi_i,\alpha\right)-p^W\left(c,\tilde{b}\left(\alpha\right)\right)\right]d\alpha,\end{aligned}$$

where the second line comes from a interchange of integrals (Fubbini's Theorem) and the last line comes from independency and the definition of $\tilde{v}\left(\phi_i,\alpha\right)$ and $g^{t_i,c}\left(\alpha\right)$ (see (5) and (25)). Also from the fact that $\tilde{b}=\left(\tilde{P}^b\right)^{-1}$, we can substitute $\tilde{P}^b$ to obtain

$$\tilde{\Pi}^{+}\left(\phi_i,c\right)=\int_0^{\tilde{b}^{-1}(c)}\left[\tilde{v}\left(\phi_i,\alpha\right)-p^W\left(c,\tilde{b}\left(\alpha\right)\right)\right]d\alpha. \tag{27}$$

Now, we can repeat the above procedures with $\Pi^{-}\left(\phi_i,c\right)$ and obtain:

$$\tilde{\Pi}^{-}\left(\phi_i,c\right)=\int_{\tilde{b}^{-1}(c)}^{1} p^L\left(c,\tilde{b}\left(\alpha\right)\right)d\alpha. \tag{28}$$

Adding up, that is, putting $\tilde{\Pi}\left(\phi_i,c\right)=\tilde{\Pi}^{+}\left(\phi_i,c\right)-\tilde{\Pi}^{-}\left(\phi_i,c\right)$, we obtain the interim payoff of the indirect auction. This concludes the proof of the first part.

For the second part, observe that the equality (8) implies that for all t_i such that $P^b\left(t_i\right)=P^b\left(s\right)=x$,

$$\begin{aligned}E\left[g^{t_i,c}\left(\alpha\right)|P^b\left(t_i\right)=x\right] &= E\left[\left(v\left(t_i,t_{-i}\right)-p^W\left(c,b_{(-i)}\left(t_{-i}\right)\right)\right)|P^b\left(t_i\right)=x,P^b_{(-i)}\left(t_{-i}\right)=\alpha\right]\\ &= E\left[\left(v\left(t_i,t_{-i}\right)-p^W\left(c,b_{(-i)}\left(t_{-i}\right)\right)\right)|t_i=s,P^b_{(-i)}\left(t_{-i}\right)=\alpha\right]\\ &= E\left[\left(v\left(s,t_{-i}\right)-p^W\left(c,b_{(-i)}\left(t_{-i}\right)\right)\right)|P^b_{(-i)}\left(t_{-i}\right)=\alpha\right]\\ &= g^{s,c}\left(\alpha\right).\end{aligned}$$

Then,

$$\begin{aligned}\tilde{\Pi}^{+}(x,c) &= E\left[\int_0^{\tilde{P}(c)} g^{t_i,c}(\alpha)\,d\alpha | P^b(t_i)=x\right] \\ &= \int_0^{\tilde{P}(c)} E\left[g^{t_i,c}(\alpha)\,|P^b(t_i)=x\right]d\alpha \\ &= \int_0^{\tilde{P}(c)} g^{s,c}(\alpha)\,d\alpha \\ &= \Pi^{+}(s,c),\end{aligned}$$

where the last line comes from (26). Obviously, the same can be done for Π^- and $\tilde{\Pi}^-$. So, the proof is complete.■

Appendix B - Indirect Auction Equilibria

In this appendix, we will analyze auctions between two players, with independent types uniformly distributed on $[0,1]$. Since this is the setting of the indirect auction, we will use notation consistent with that, although the results of this appendix are independent from the results of section 4. For $(i,-i)=(1,2)$ or $(2,1)$ let

$$\tilde{u}_i(x,b)=\begin{cases}\tilde{v}(x_i,x_{-i})-p^W(b_i,b_{-i}), & \text{if } b_i>b_{-i}\\ -p^L(b_i,b_{-i}), & \text{if } b_i<b_{-i}\\ \frac{\tilde{v}(x_i,x_{-i})-b_i}{2}, & \text{if } b_i=b_{-i}\end{cases}$$

be the ex-post payoff. Recall that the bids are in $B=\{b_{OUT}\}\cup[b_{\min},+\infty)$, where $b_{OUT}<b_{\min}$. We will assume:

(H0)' The types are independent and uniformly distributed on $[0,1]$. $\tilde{v}$ is positive, measurable and bounded.

Let $\tilde{\mathcal{S}}$ be the set of non-decreasing functions $\tilde{b}:[0,1]\to\{b_{OUT}\}\cup[b_{\min},+\infty)$, such that there exists $x\in[0,1]$ satisfying the following: $\tilde{b}([0,x))=\{b_{OUT}\}$ and $\tilde{b}$ is strictly increasing in $(x,1]$. For a function $\tilde{b}\in\tilde{\mathcal{S}}$, let $b_*=\inf\{\tilde{b}(x)\geqslant b_{\min}:x\in[0,1]\}$ and $b^*=\sup\{\tilde{b}(x)\geqslant b_{\min}:x\in[0,1]\}$.

The interim payoff for a bidder of (indirect) type x who bids $\beta \geqslant b_{\min}$ and faces an opponent following $\tilde{b} \in \tilde{\mathcal{S}}$ is

$$\tilde{\Pi}\left(x, \beta, \tilde{b}\right) = \int_0^{\tilde{b}^{-1}(\beta)} \left[\tilde{v}(x, \alpha) - p^W\left(\beta, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(\beta)}^1 p^L\left(\beta, \tilde{b}(\alpha)\right) d\alpha, \tag{29}$$

where $\tilde{b}^{-1}(\beta) = \inf\left\{x \in [0,1] : \tilde{b}(x) \geqslant \beta\right\} \in [0,1]$. Let us also define

$$\tilde{p}\left(\beta, \tilde{b}\right) = \int_0^{\tilde{b}^{-1}(\beta)} p^W\left(\beta, \tilde{b}(\alpha)\right) d\alpha + \int_{\tilde{b}^{-1}(\beta)}^1 p^L\left(\beta, \tilde{b}(\alpha)\right) d\alpha.$$

Remember that (H1) requires that $p^L(b_{\min}, b) = p^L(b_{\min}, b')$ for all b and b' and $p^W(\cdot, b_{OUT}) = p^W(\cdot, b_{\min})$. Thus, if $\alpha < x \Rightarrow \tilde{b}(\alpha) \leqslant b_{\min}$, the expression of $\tilde{p}\left(b_{\min}, \tilde{b}\right)$ can be simplified to

$$\int_0^{\tilde{b}^{-1}(b_{\min})} p^W(b_{\min}, b_{\min})\, d\alpha + \int_{\tilde{b}^{-1}(b_{\min})}^1 p^L(b_{\min}, b_{\min})\, d\alpha.$$

We will assume the following:

(H2)' There exists $x_0 \in [0,1]$ such that

(i) $x_0 = 0$ and for all $x \in [0,1]$, $\int_0^x \tilde{v}(x, \alpha)\, d\alpha \geqslant \int_0^x p^W(b_{\min}, b_{\min})\, d\alpha + \int_x^1 p^L(b_{\min}, b_{\min})\, d\alpha$;

or

(ii)

$$\int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha = \int_0^{x_0} p^W(b_{\min}, b_{\min})\, d\alpha + \int_{x_0}^1 p^L(b_{\min}, b_{\min})\, d\alpha$$

and $x < x_0 < y$ implies

$$\int_0^x \tilde{v}(x, \alpha)\, d\alpha \leqslant \int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha \leqslant \int_0^y \tilde{v}(y, \alpha)\, d\alpha.$$

Note that (H2)'-(i) corresponds to the original assumption (H2). In (H2)-(ii), the type x_0 represents the type who will bid the minimum bid $b_{\min}$. This type necessarily receives a zero expected payoff, because it must be indifferent between the non-participation and participation. The inequalities in (H2)'-(ii) represent a weak monotonicity condition, that allows to conclude that types below x_0 do not have incentives to bid $b_{\min}$ or above, because the value of the object will be not greater than their expected payments.

We have the following:

PROPOSITION 2*:* Assume (H0)', (H1)-1, that is, $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$ and (H2)'. Let $\tilde{b} \in \tilde{\mathcal{S}}$ be an equilibrium of the indirect auction, increasing on $(x_0, 1)$. Then, $\tilde{b}$ is continuous on $(x_0, 1)$. Moreover, if $\tilde{v}$ is continuous in the second variable, then $\tilde{b}$ is differentiable on $(x_0, 1)$ and satisfies

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}. \tag{30}$$

PROOF: The proof is an adaptation of part of the proof of Theorem 2 of Maskin and Riley (1984).[18] Suppose that player $-i$ follows $\tilde{b}$. The interim payoff of player i with (indirect) type x_i is

$$\begin{aligned}\tilde{\Pi}(x_i, b_i, \tilde{b}(\cdot)) &= \int \left[\tilde{v}(x_i, x_{-i}) - p^W\left(b_i, \tilde{b}(x_{-i})\right)\right] 1_{\left[b_i > \tilde{b}(x_{-i})\right]} dx_{-i} \\ &\quad - \int p^L\left(b_i, \tilde{b}(x_{-i})\right) 1_{\left[b_i < \tilde{b}(x_{-i})\right]} dx_{-i}.\end{aligned}$$

If $\tilde{b}$ is discontinuous, there exists $x^* > x_0$, with

$$\limsup_{x < x^*} \tilde{b}(x) < \lim \inf_{x > x^*} \tilde{b}(x).$$

Consider bidders $x_i^{+\varepsilon}$ that bids $\beta^{+\varepsilon} = \tilde{b}\left(x_i^{+\varepsilon}\right) = \liminf_{x > x^*} \tilde{b}(x) + \varepsilon$ and $x_i^{-\varepsilon}$ that bids $\beta^{-\varepsilon} = \tilde{b}\left(x_i^{-\varepsilon}\right) = \limsup_{x < x^*} \tilde{b}(x) - \varepsilon$. We will prove that, for $\varepsilon > 0$ sufficiently small, the bid $\beta^{-\varepsilon}$ is better than $\beta^{+\varepsilon}$ for a bidder with type $x_i^{+\varepsilon}$. This will be the contradiction. The event $\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i})\right]$ is arbitrarily close to $\left[\beta^{-\varepsilon} > \tilde{b}(x_{-i})\right]$. So, the difference of expected utilities

$$\begin{aligned}&\int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i})\right]} dx_{-i} - \int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{\left[\beta^{-\varepsilon} > \tilde{b}(x_{-i})\right]} dx_{-i} \\ = &\int \tilde{v}(x_i^{+\varepsilon}, x_{-i}) 1_{\left[\beta^{+\varepsilon} > \tilde{b}(x_{-i}) \geq \beta^{-\varepsilon}\right]} dx_{-i}\end{aligned}$$

is arbitrarily small. On the other hand, the difference of expected payments is

[18] Since our assumptions are different from theirs, we will reproduce the proof with details.

$$
\begin{aligned}
& -\int\left[p^W\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{+\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]}+p^L\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{+\varepsilon}<\tilde{b}\left(x_{-i}\right)\right]}\right] d x_{-i} \\
& +\int\left[p^W\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]}+p^L\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(x_{-i}\right)\right]}\right] d x_{-i} \\
= & \int\left[\left(p^W\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right)-p^W\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right)\right)\right] 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +\int\left[p^L\left(\beta^{-\varepsilon}, b\left(x_{-i}\right)\right)-p^L\left(\beta^{+\varepsilon}, b\left(x_{-i}\right)\right)\right] 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(x_{-i}\right)\right]} d x_{-i}+r \\
= & \int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}}-\partial_1 p^W\left(z, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}>\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +\int \int_{\beta^{-\varepsilon}}^{\beta^{+\varepsilon}}-\partial_1 p^L\left(z, b\left(x_{-i}\right)\right) 1_{\left[\beta^{-\varepsilon}<\tilde{b}\left(x_{-i}\right)\right]} d x_{-i} \\
& +r
\end{aligned}
$$

where r denotes the integrals over the event $\left[\beta^{+\varepsilon}>\tilde{b}\left(x_{-i}\right) \geqslant \beta^{-\varepsilon}\right]$. Observe that the sum of the two integrals is negative and bounded away from zero, because $\partial_1 p^W(\cdot)>0$ or $\partial_1 p^L(\cdot)>0$. So, it is not optimal for bidder $x_i^{+\varepsilon}$ to bid $\beta^{+\varepsilon}$ and this contradicts $\tilde{b}$ to be an equilibrium. So, $\tilde{b}$ is continuous.

Now, assume that $\tilde{v}$ is continuous in its second variable. We want to prove that $\tilde{b}$ is differentiable.

To fix ideas, choose $y>x$. We have

$$
\begin{aligned}
& \tilde{\Pi}(x, \tilde{b}(x), \tilde{b}(\cdot))-\tilde{\Pi}(x, \tilde{b}(y), \tilde{b}(\cdot)) \\
= & \int_0^x\left[\tilde{v}(x, \alpha)-p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha-\int_x^1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) d \alpha \\
& -\int_0^y\left[\tilde{v}(x, \alpha)-p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d \alpha+\int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d \alpha \\
= & \int_x^y\left[-\tilde{v}(x, \alpha)+p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)-p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d \alpha \\
& +\int_0^x\left[p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)-p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha \\
& +\int_x^1\left[p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right)-p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right)\right] d \alpha .
\end{aligned}
$$

Since the first integrand is continuous, by the mean value theorem, there exists x^* between x and y such that

$$\begin{aligned}&\int_x^y \left[-\tilde{v}(x,\alpha)+p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right)-p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right)\right]d\alpha\\ =\ &(y-x)\left[-\tilde{v}(x,x^*)+p^W\left(\tilde{b}(y),\tilde{b}(x^*)\right)-p^L\left(\tilde{b}(y),\tilde{b}(x^*)\right)\right]\end{aligned}$$

Since p^W and p^L are differentiable and $\tilde{b}$ is continuously increasing, there exist x^W and x^L between x and y such that, for $j=W,L$,

$$\begin{aligned}&\int_0^x \left[p^j\left(\tilde{b}(y),\tilde{b}(\alpha)\right)-p^j\left(\tilde{b}(x),\tilde{b}(\alpha)\right)\right]d\alpha\\ =\ &\left[\tilde{b}(y)-\tilde{b}(x)\right]\int_0^x \partial_1 p^j\left(\tilde{b}\left(x^j\right),\tilde{b}(\alpha)\right)d\alpha.\end{aligned}$$

Thus, since $\tilde{\Pi}(x,\tilde{b}(x),\tilde{b}(\cdot))-\tilde{\Pi}(x,\tilde{b}(y),\tilde{b}(\cdot))\geqslant 0$, we have, since $y>x$,

$$\frac{\tilde{b}(y)-\tilde{b}(x)}{y-x}\geqslant\frac{\left[\tilde{v}(x,x^*)-p^W\left(\tilde{b}(y),\tilde{b}(x^*)\right)+p^L\left(\tilde{b}(y),\tilde{b}(x^*)\right)\right]}{\int_0^x\partial_1 p^W\left(\tilde{b}(x^W),\tilde{b}(\alpha)\right)d\alpha+\int_x^1\partial_1 p^L\left(\tilde{b}(x^L),\tilde{b}(\alpha)\right)d\alpha}.\tag{31}$$

Analogously, interchanging x and y, and using $\tilde{\Pi}(y,\tilde{b}(y),\tilde{b}(\cdot))-\tilde{\Pi}(y,\tilde{b}(x),\tilde{b}(\cdot))\geqslant 0$, with $y>x$, we obtain, the existence of $\hat{x}$, $\hat{x}^W$ and $\hat{x}^L$ in $[x,y]$ such that

$$\frac{\tilde{b}(y)-\tilde{b}(x)}{y-x}\leqslant\frac{\left[\tilde{v}(y,\hat{x})-p^W\left(\tilde{b}(x),\tilde{b}(\hat{x})\right)+p^L\left(\tilde{b}(x),\tilde{b}(\hat{x})\right)\right]}{\int_0^y\partial_1 p^W\left(\hat{x}^W,\tilde{b}(\alpha)\right)d\alpha+\int_y^1\partial_1 p^L\left(\hat{x}^L,\tilde{b}(\alpha)\right)d\alpha}.\tag{32}$$

(If $x>y$, we obtain reverse inequalities.) When we make $y\to x$, the right hand side in (31) and (32) both converge to

$$\frac{\tilde{v}(x,x)-p^W\left(\tilde{b}(x),\tilde{b}(x)\right)+p^L\left(\tilde{b}(x),\tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x),\tilde{b}(\alpha)\right)1_{\left[\tilde{b}(x)>\tilde{b}(\alpha)\right]}+\partial_1 p^L\left(\tilde{b}(x),\tilde{b}(\alpha)\right)1_{\left[\tilde{b}(x)<\tilde{b}(\alpha)\right]}\right]}.$$

So, $\tilde{b}$ is differentiable at $x\in(x_0,1)$ and $\tilde{b}'(x)$ is equal to the expression above.■

Proposition 3*:* Assume (H0)', (H1)-2, (H2)' and that $\tilde{v}$ is continuous. Let $\tilde{b}\in\tilde{\mathcal{S}}$ be an equilibrium of the indirect auction, increasing on $(x_0,1)$. Then, $\tilde{b}$ is continuous on $(x_0,1)$ and

$$\tilde{v}(x,x)-p^W\left(\tilde{b}(x),\tilde{b}(x)\right)+p^L\left(\tilde{b}(x),\tilde{b}(x)\right)=0,\forall x\in(x_0,1).\tag{33}$$

PROOF: The proof is based on the proof of Theorem 3 of Maskin and Riley (1984). Given b, b', let us define the function h as

$$h\left(z\right)\equiv p^{W}\left(b,z\right)-p^{L}\left(b',z\right).$$

Since $\partial_1 p^W=\partial_1 p^L\equiv 0$ and $\partial_2\left(p^W-p^L\right)>0$, h does not depend on b or b' and is differentiable and increasing.

By contradiction, assume that (33) is false, that is, there exists $x^*\in(x_0,1)$ such that

$$\tilde{v}\left(x^*,x^*\right)>h\left(\tilde{b}\left(x^*\right)\right). \tag{34}$$

(The other case will be considered later.) Because $\tilde{v}$ is continuous and $\tilde{b}$ is increasing, for sufficiently small $\delta>0$, we have

$$\tilde{v}\left(x^*-\delta,x^*-\delta\right)>h\left(\tilde{b}\left(x^*\right)\right)>h\left(\tilde{b}\left(x^*-\delta\right)\right).$$

Since the set of discontinuity points of $\tilde{b}$ is enumerable, we may assume that (34) holds for a point x^* where $\tilde{b}$ is continuous. Thus, for sufficiently small ε and $\delta>0$, $\forall\alpha\in[x^*,x^*+\varepsilon]$,

$$\tilde{v}\left(x^*,\alpha\right)>h\left(\tilde{b}\left(x^*\right)+\delta\right)>h\left(\tilde{b}\left(\alpha\right)\right).$$

Consider the following difference:

$$\begin{aligned}
&\tilde{\Pi}(x^*,\tilde{b}\left(x^*+\varepsilon\right),\tilde{b}\left(\cdot\right))-\tilde{\Pi}(x^*,\tilde{b}\left(x^*\right),\tilde{b}\left(\cdot\right))\\
=&\int_0^{x^*+\varepsilon}\left[\tilde{v}\left(x^*,\alpha\right)-p^W\left(\tilde{b}\left(x^*+\varepsilon\right),\tilde{b}\left(\alpha\right)\right)\right]d\alpha-\int_{x^*+\varepsilon}^1 p^L\left(\tilde{b}\left(x^*+\varepsilon\right),\tilde{b}\left(\alpha\right)\right)d\alpha\\
&-\int_0^{x^*}\left[\tilde{v}\left(x^*,\alpha\right)-p^W\left(\tilde{b}\left(x^*\right),\tilde{b}\left(\alpha\right)\right)\right]d\alpha+\int_{x^*}^1 p^L\left(\tilde{b}\left(x^*\right),\tilde{b}\left(\alpha\right)\right)d\alpha\\
=&\int_{x^*}^{x^*+\varepsilon}\left[\tilde{v}\left(x^*,\alpha\right)-h\left(\tilde{b}\left(\alpha\right)\right)\right]d\alpha>0
\end{aligned}$$

where we used the property $\partial_1 p^W=\partial_1 p^L\equiv 0$ in order to obtain the last equality. This contradicts the optimality of $\tilde{b}\left(x^*\right)$ for x^*.

Analogously, if we assume that there is a $x^*\in(x_0,1)$ such that $\tilde{v}\left(x^*,x^*\right)<h\left(\tilde{b}\left(x^*\right)\right)$ and that x^* is a point of continuity of $\tilde{b}$, we have for $\varepsilon>0$ sufficiently small,

$$\begin{aligned}
&\tilde{\Pi}(x^*,\tilde{b}\left(x^*\right),\tilde{b}\left(\cdot\right))-\tilde{\Pi}(x^*,\tilde{b}\left(x^*-\varepsilon\right),\tilde{b}\left(\cdot\right))\\
=&\int_{x^*-\varepsilon}^{x^*}\left[\tilde{v}\left(x^*,\alpha\right)-h\left(\tilde{b}\left(\alpha\right)\right)\right]d\alpha<0.
\end{aligned}$$

This completes the proof of (33). So, we have

$$\tilde{b}(x) = h^{-1}(\tilde{v}(x,x)), \tag{35}$$

which shows that $\tilde{b}$ is continuous. Moreover, $\tilde{b}$ is increasing if and only if $x \longmapsto \tilde{v}(x,x)$ is also increasing.■

We have the following:

LEMMA 2: Assume (H0)', (H1), (H2)' and that $\tilde{v}$ is continuous. Let $\tilde{b} \in \tilde{\mathcal{S}}$ be an equilibrium of the indirect auction, increasing on $(x_0, 1)$. Then $\tilde{\Pi}\left(x, \beta, \tilde{b}\right)$ is differentiable in β on (b_*, b^*).

PROOF: Under (H1)-1, Proposition 2 implies that $\tilde{b}$ is differentiable and, since $\tilde{b}$ is increasing on $(x_0, 1)$, $\tilde{b}^{-1}$ is differentiable on (b_*, b^*). We also have $\tilde{v}$ is continuous and p^W and p^L differentiable. Thus, one can easily see from (29) that $\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right)$ and

$$\begin{aligned}\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right) &= \tilde{v}\left(x, \tilde{b}^{-1}(\beta)\right) - p^W(\beta,\beta) - p^L(\beta,\beta) \\ &\quad - \int_0^{\tilde{b}^{-1}(\beta)} \partial_1 p^W\left(\beta, \tilde{b}(\alpha)\right) \cdot \left[\tilde{b}^{-1}(\beta)\right]' d\alpha - \int_{\tilde{b}^{-1}(\beta)}^1 \partial_1 p^L\left(\beta, \tilde{b}(\alpha)\right) \cdot \left[\tilde{b}^{-1}(\beta)\right]' d\alpha.\end{aligned}$$

Under (H2)-2, we have $\partial_1 p^W(\cdot,\cdot) = \partial_1 p^L(\cdot,\cdot) = 0$ and again there exists $\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right)$, with

$$\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right) = \tilde{v}\left(x, \tilde{b}^{-1}(\beta)\right) - p^W(\beta,\beta) - p^L(\beta,\beta).$$

The lemma is proved.■

Now, we want to consider cases where $\tilde{b}$ is not monotonic, because this is exactly the setting of Theorem 4. To treat non-increasing $\tilde{b}$, we define the following:

MODIFIED AUCTION: Fix a function $\tilde{b}$. The bidder submits a type $y \in [0,1]$. In any event, the payment is determined as if the bidder has submitted the bid $\tilde{b}(y)$. The bidder wins against opponents who announce types below y and loses against opponents who announce types above y. If there is a tie, the object is given with probability $1/2$ for each bidder.

Observe that if $\tilde{b}$ is increasing, the modified auction is a standard auction if all bidders follow $\tilde{b}$. If $\tilde{b}$ is not increasing, the difference is that the winning events are not determined by $\tilde{b}$ but by the announced type y. The rule of the modified auction implies the following interim payoff:

$$\hat{\Pi}(x,y) = \begin{cases} \int_0^y \left[\tilde{v}(x,\alpha) - p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right)\right] d\alpha - \int_y^1 p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0 \end{cases}$$

We can simplify the above expression to

$$\hat{\Pi}(x,y) = \begin{cases} \int_0^y \tilde{v}(x,\alpha)\, d\alpha - \hat{p}(y), & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0 \end{cases} \tag{36}$$

where

$$\hat{p}(y) \equiv \begin{cases} \int_0^y p^W\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y),\tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0, & \text{if } y < x_0 \end{cases}$$

We have the following:

Proposition 4 *(Payment Rule):* Assume (H0)', (H1) and (H2)' and that $\tilde{v}$ is continuous. If truth-telling is equilibrium for the modified auction, then:

$$\hat{p}(y) = \begin{cases} \hat{p}(x_0) + \int_{x_0}^y \tilde{v}(\alpha,\alpha)\, d\alpha, & \text{if } y > x_0 \\ \int_0^{x_0} \tilde{v}(x_0,\alpha)\, d\alpha, & \text{if } y = x_0 \\ 0, & \text{if } y < x_0 \end{cases} \tag{37}$$

Proof: In case (H1)-1, $\tilde{b}$, p^W and p^L are differentiable on $(x_0, 1)$. Thus, $\hat{p}$ and $\hat{\Pi}$ are also differentiable. So, for every $y \in (x_0, 1)$, we have

$$\hat{p}'(y) = \partial_y \left\{ \int_0^y \tilde{v}(x,\alpha)\, d\alpha - \hat{\Pi}(x,y) \right\} = \tilde{v}(x,y) - \partial_y \hat{\Pi}(x,y).$$

Truth-telling is always optimal if

$$\hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0. \tag{38}$$

Under (H1)-1, this is equivalent to

$$\int_y^x \partial_y \hat{\Pi}(x,\alpha)\, d\alpha \geqslant 0$$

if $x, y \geqslant x_0$. Also, if $x, y \geqslant x_0$, $\partial_y \hat{\Pi}(x,y)\mid_{y=x}$ must be zero, so that

$$\partial_y \hat{\Pi}(x,y)\mid_{y=x} = 0 \Rightarrow \hat{p}'(x) = \tilde{v}(x,x). \tag{39}$$

Indeed, these are simply the second- and the first-order conditions, respectively. So, for $y \geqslant x_0$,

$$\hat{p}(y) = \int_{x_0}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha + \hat{p}(x_0).$$

Now, let us turn to (H1)-2. Since $\tilde{b}$ is only continuous, $\hat{p}$ is not necessarily differentiable. Nevertheless, if $y \geqslant x_0$,

$$\begin{aligned}
\hat{p}(y) &= \int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha \\
&= \int_{x_0}^{y} \left[p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) - p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^{y} h\left(\tilde{b}(\alpha)\right) d\alpha + \hat{p}(x_0) \\
&= \int_{x_0}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha + \hat{p}(x_0).
\end{aligned}$$

Observe that the expression above is exactly the same of case 1. For $y < x_0$, the payment is zero. For $y = x_0$, $\hat{p}(y)$ is obtained from (**??**):

$$\int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha - \int_0^{x_0} p^W\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha + \int_{x_0}^{1} p^L\left(\tilde{b}(x_0), \tilde{b}(\alpha)\right) d\alpha = 0.$$

So, the proposition is proved.■

Now, we turn to the equilibrium existence. We will not assume that $\tilde{v}$ is continuous. Instead, we assume only the validity of the payment expression (37).

Proposition 5 *(Equilibrium).* Assume (H0)', (H1), (H2)' and (37). Then, truth-telling is equilibrium of the modified auction if and only if, for all $(x, y) \in [0,1] \times [0,1] \backslash [0, x_0) \times [0, x_0)$,

$$(40) \quad \begin{cases}
\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)]\, d\alpha \geqslant 0, & \text{if } x, y \geqslant x_0 \\
\int_{x_0}^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)]\, d\alpha + \int_0^{x_0} [\tilde{v}(x, \alpha) - \tilde{v}(x_0, \alpha)]\, d\alpha \geqslant 0, & \text{if } x \geqslant x_0 > y \\
0 \geqslant \int_{x_0}^y [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)]\, d\alpha + \int_0^{x_0} [\tilde{v}(x, \alpha) - \tilde{v}(x_0, \alpha)]\, d\alpha & \text{if } y \geqslant x_0 > x
\end{cases}$$

Proof. Given (37), the optimality condition for truth-telling, namely $\hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0$, is equivalent to

$$\begin{aligned} & \int_0^x \tilde{v}(x,\alpha)\,d\alpha - \int_{x_0}^x \tilde{v}(\alpha,\alpha)\,d\alpha - \hat{p}(x_0) \\ & - \int_0^y \tilde{v}(x,\alpha)\,d\alpha + \int_{x_0}^y \tilde{v}(\alpha,\alpha)\,d\alpha + \hat{p}(x_0) \\ = & \int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\,d\alpha \geqslant 0 \end{aligned} \tag{41}$$

if $x, y \geqslant x_0$. The other cases are similar.■

As we have said before, if $\tilde{b}$ is increasing, the modified auction is just the original (unmodified) auction. Thus, we have:

Corollary 1. Assume (H0)', (H1), (H2)' and that $\tilde{b}$ is increasing on $(x_0, 1)$ and implies (37). If (40) holds, then $\tilde{b}$ is equilibrium of the indirect auction.

Observe that Proposition 5 and Corollary 1 does not require $\tilde{v}$ to be continuous.

Appendix C - Proofs of the Theorems

Proof of Theorem 1.

(i) If $b \in \mathcal{S}$, it defines a conjugation P^b by (2). The bid $b(t_i) = \beta$ is optimal for bidder t_i against the strategy $b(\cdot)$ of the opponents. This, the fact that $\partial_b \Pi(s, b(s))$ $= 0$ and Lemma 1 in Appendix A imply that

$$\begin{aligned} & E\left[v(t_i,\cdot)\,|t_i = s, b_{(-i)}(t_{-i}) = \beta\right] \\ & \qquad = p^W(\beta,\beta) - p^L(\beta,\beta) - \frac{E_{t_{-i}}\left[\partial_{b_i} p^W 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \partial_{b_i} p^L 1_{\left[b_i < \mathbf{b}_{(-i)}\right]}\right]}{f_{b_{(-i)}}(\beta)}. \end{aligned}$$

Observe that the right-hand side does not depend on s (it depends on it only by the fact that $\beta = b(s)$ is the optimum bid for such bidder). Thus, the left-hand side has to be the same for all s that are bidding the same bid in equilibrium, which implies that (10) holds.

(ii) If $b(t_i)$ maximizes $\Pi(t_i, c)$ for t_i, and $P(t_i') = P(t_i)$, then $b(t_i') = b(t_i)$. Then, $b(t_i)$ maximizes $\tilde{\Pi}(P(t_i'), c)$ for all t_i' such that $P(t_i') = P(t_i)$, from the definition of $\tilde{\Pi}(P(t_i), c)$ given by (6). In other words, $\tilde{b}(x) = \left(\tilde{P}^b\right)^{-1}(x) = b\left(P^{-1}(x)\right)$ is the equilibrium of the indirect auction.

If $\tilde{v}$ is continuous, we appeal to the results of Appendix B. Propositions 2 and 3 prove (iii), Proposition 4 proves (iv) and Proposition 5 gives (v).■

Proof of Theorem 2. Corollary 1 of Appendix B proves that conditions (ii) and (iii) are sufficient for $\tilde{b}$ to be the equilibrium of the indirect auction. Now, Proposition 1 proves that condition (i)' implies that for all s such that $P(s) = x$, $\tilde{\Pi}(x, c) = \Pi(s, c)$ (see (9)). Now, if we put $b(s) = \tilde{b}(P(s))$, then

$$\Pi(s, b(s)) = \tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \text{ and}$$
$$\Pi(s, c) = \tilde{\Pi}(P(s), c).$$

But this is sufficient to show the equilibrium existence in the direct auction, since $\tilde{b}$ is the equilibrium in the indirect auction, which implies that

$$\tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \geqslant \tilde{\Pi}(P(s), c),$$

for all $c \in \mathbb{R}$. If $\tilde{v}$ is continuous, Lemma 2 in Appendix B shows that $\tilde{\Pi}(x, c)$ is differentiable at all $c \in \mathbb{R}$. This concludes the proof.■

Through the proof of Theorem 3, we will make successive use of the following fact:

Lemma 3. Assume (H1), (H2) and (H3). For any $\sigma-$field Ξ on S^{N-1}, we have

$$\begin{aligned}
&\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\
&\quad \Leftrightarrow \forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\
&\quad \Leftrightarrow E[v(t_i, t_{-i})\,|t_i = s', \Xi] > E[v(t_i, t_{-i})\,|t_i = s, \Xi], \text{ a.s.}
\end{aligned}$$

Proof. (H3) gives the first equivalence. By (H2), v is continuous over a compact. So, if $\forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$, there is $\delta > 0$ so that $d(t_{-i}) \equiv v(s', t_{-i}) - v(s, t_{-i}) - \delta \geqslant 0$ for all t_{-i}. Then, for any Ξ, $E[d(t_{-i})\,|\Xi] \geqslant 0$ almost surely.[19] This implies that $E[v(t_i, t_{-i}) \mid t_i = s', \Xi] > E[v(t_i, t_{-i}) \mid t_i = s, \Xi]$, a.s. On the other hand, $E[v(t_i, t_{-i}) \mid t_i = s', \Xi] > E[v(t_i, t_{-i}) \mid t_i = s, \Xi]$ a.s. implies that $\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$.■

[19]See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

Proof of Theorem 3. *Equilibrium Existence.* If we define P by (19), it is a conjugation. Let us prove that it satisfies condition (i)' of Theorem 2. If for some x, y and s, such that $P(s) = x$, we have

$$\tilde{v}(x,y) = E[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = y] < E[v(t_i,t_{-i})|t_i = s, P_{(-1)}(t_{-i}) = y],$$

then, for at least one t_{-i} and s', $P(s') = x$, $v(s,t_{-i}) > v(s',t_{-i})$. But then, by (H3), $v(s,t_{-i}) > v(s',t_{-i})$ for all t_{-i} which implies $v^1(s) > v^1(s')$ and $P(s) > P(s')$, a contradiction with the assumption that $P(s) = P(s') = x$. So, condition (i)$'$ is satisfied.

Let us prove condition (ii) of Theorem 2. If $x > y$, for all t_i and t_i' such that $P(t_i') = x$ and $P(t_i) = y$, we have $v(t_i',t_{-i}) > v(t_i,t_{-i})$ for all t_{-i}, by (H3). Then, for all $z \in [0,1]$,

$$\begin{aligned}\tilde{v}(x,z) &\equiv E\left[v(t_i,t_{-i})|P(t_i) = x, P_{(-i)}(t_{-i}) = z\right] \\ &> E\left[v(t_i,t_{-i})|P(t_i) = y, P_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y,z).\end{aligned}$$

Then, if $y < \alpha < x$, $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) > 0$ and we have:

$$\int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0.$$

Now if $x < \alpha < y$, we have $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) < 0$ so that condition (ii) is satisfied. Since our assumption is the condition (iii) of Theorem 2, this implies the existence of equilibrium, with the equilibrium bidding function given by $b = \tilde{b} \circ P$.

Sufficiency. Conditions (i)' and (ii) of the Theorem 2 was shown in the first part, above. Proposition 4 in appendix B proves condition (iii) of Theorem 2. Then, there exists a equilibrium $b = \tilde{b} \circ P$. Since $\tilde{v}$ is continuous, Theorem 2 shows the existence of $\partial_b \Pi(s, b(s))$ for all s.

Necessity. According to Theorem 1, given a $b \in \mathcal{S}$, the associated conjugation P^b (given by (2)) is such that for all $s \in \left(P^b\right)^{-1}(x)$,

$$E[v(t_i,t_{-i})|P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = x] = E[v(t_i,t_{-i})|t_i = s, P^b_{(-i)}(t_{-i}) = x].$$

If $P^b(s) = P^b(s')$ and there is some t_{-i} such that $v(s,t_{-i}) < v(s',t_{-i})$, Lemma 3 implies that

$$E[v(t_i,t_{-i})|t_i = s, P^b_{(-i)}(t_{-i}) = x] < E[v(t_i,t_{-i})|t_i = s', P^b_{(-i)}(t_{-i}) = x],$$

which contradicts the previous equality between the conditional expectations. We conclude that

$$P^b(s) = P^b(s') \Rightarrow v(s, t_{-i}) = v(s', t_{-i}) \text{ for all } t_{-i}. \tag{42}$$

Let us define $\tilde{v}^1(x)$ as $E\left[v(t_i, t_{-i})\,|P^b(t_i) = x\right]$ and prove that it is non-decreasing. Suppose by absurd that there exist x and y, $x > y$, such that $\tilde{v}^1(x) < \tilde{v}^1(y)$.

First, we claim that for all t_i and t_i' such that $P^b(t_i) = x$ and $P^b(t_i') = y$, we have $v(t_i, t_{-i}) < v(t_i', t_{-i})$ for all t_{-i}. Otherwise, $v(t_i, t_{-i}) \geqslant v(t_i', t_{-i})$ for some t_{-i} and, by (H3), $v\left(t_i, t_{-i}'\right) \geqslant v\left(t_i', t_{-i}'\right)$ for all t_{-i}'. Then, Lemma 3 and (42) would imply that $\tilde{v}^1(x) = E\left[v(t_i, t_{-i})\,|P^b(t_i) = x\right] \geqslant E\left[v(t_i, t_{-i})\,|P^b(t_i) = y\right] = \tilde{v}^1(y)$, a contradiction with our (absurd) assumption. Thus, the claim is proved.

This claim and Lemma 3 imply that

$$\begin{aligned}\tilde{v}(x, z) &\equiv E\left[v(t_i, t_{-i})\,|P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = z\right] \\ &< E\left[v(t_i, t_{-i})\,|P^b(t_i) = y, P^b_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y, z),\end{aligned}$$

for all $z \in [0, 1]$, a.s. Thus,

$$\int_y^x \left[\tilde{v}(x, \alpha) - \tilde{v}(y, \alpha)\right] d\alpha < 0.$$

By condition (v) of Theorem 1, we also have that

$$\int_y^x \left[\tilde{v}(y, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \leqslant 0.$$

Summing up these two integrals, we obtain

$$\int_y^x \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha < 0,$$

which contradicts condition (v) of Theorem 1. This contradiction establishes that $x > y \Rightarrow \tilde{v}^1(x) \geqslant \tilde{v}^1(y)$.

Suppose now that there exists $x > y$ such that $\tilde{v}^1(x) = \tilde{v}^1(y)$. Then, the monotonicity of $\tilde{v}^1$ (just proved) gives

$$\forall \phi \in [y, x], \tilde{v}^1(\phi) = \tilde{v}^1(x) = \tilde{v}^1(y). \tag{43}$$

Let $S' = \left\{s \in S : \tilde{b}(y) \leqslant b(s) < \tilde{b}(x)\right\}$. From (2), for all $s \in S'$, $P^b(s) \in [y, x]$. Then, (42) and (43) imply that $s \in S' \Rightarrow v^1(s) = \tilde{v}^1(x)$. Assumption (H3) requires that $\mu(S') = 0$. Observe that $S' = A \backslash B$, where $A \equiv \left\{s \in S : b(s) < \tilde{b}(x)\right\}$ and $B =$

$\left\{s \in S : b(s) < \tilde{b}(y)\right\}$. But then, $\mu(A) = \mu(B)$. However, from the definition of $\tilde{b}$ as the inverse of $\tilde{P}^b$, we have the following:

$$0 < x - y = \tilde{P}^b\left(\tilde{b}(x)\right) - \tilde{P}^b\left(\tilde{b}(y)\right) = (\mu(A))^{N-1} - (\mu(B))^{N-1},$$

which is a contradiction. So, we have proved that $x = P^b(s') > P^b(s) = y$ implies $v^1(s') = \tilde{v}^1(x) > \tilde{v}^1(y) = v^1(s)$ and $P^b(s') = P^b(s)$ implies $v^1(s') = v^1(s)$. In other words, $P^b(s') \lesseqqgtr P^b(s)$ if and only if $v^1(s') \lesseqqgtr v^1(s)$ which allows us to conclude that

$$P^b(t_i) = \Pr\left\{t_{-i} \in T_{-i} = S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\right\},$$

as we have defined in (19). In other words, the conjugation is unique.

Now, $\tilde{v}$ and $\tilde{b}$ in Theorem 1 are exactly those defined in the statement of Theorem 3. So, Theorem 1 implies the claims about $\tilde{b}$.

Uniqueness. Since $\tilde{v}$ is continuous, Propositions 2 and 3 in Appendix B says that any equilibrium $\tilde{b}$ satisfy the conditions given. If there is just one $\tilde{b}$ that satisfy such conditions, then the equilibrium of the indirect auction is unique. Since the previous step (necessity) shows that the conjugation is unique, the equilibrium of the direct auction is unique.■

Proof of Theorem 4. If $\tilde{b}$ is strictly increasing, then $b = \tilde{b} \circ P$ is an equilibrium of direct auction, by Theorem 3.

So, we have to show that an equilibrium exists if $\tilde{b}$ is not increasing. For future use, remember that in the first part of the proof of Theorem 3, we have established conditions (i)' and (ii) of Theorem 2 and that

$$x > y \Rightarrow \tilde{v}(x,z) > \tilde{v}(y,z), \forall z \in [0,1]. \tag{44}$$

Let us define $\overline{b}(x) = \sup_{\alpha \in [0,x]} \tilde{b}(\alpha)$. As we discussed after the statement of Theorem 4, this is just one of the possible specification for the equilibrium bidding function. The only exception is when the tie is to occur including the highest bidder. In such a case, it is mandatory to have the bid of tying bidders following the above definition. The reason will become clear in the sequel.

Remember that $\tilde{b}$ is absolutely continuous. Then, there is an enumerable set of intervals $[a_k, c_k]$ where $\overline{b}(x)$ is constant. Let $b_k \equiv \overline{b}(x)$ for $x \in [a_k, c_k]$. (See Figure 5.)

Therefore, there is a tie among the indirect types in $[a_k, c_k]$ for the bidding function $\overline{b}$. Let b_k be the specified bid for indirect types in $[a_k, c_k]$, that is, $\overline{b}([a_k, c_k]) = \{b_k\}$. The tie is solved by an all-pay auction among tying bidders.

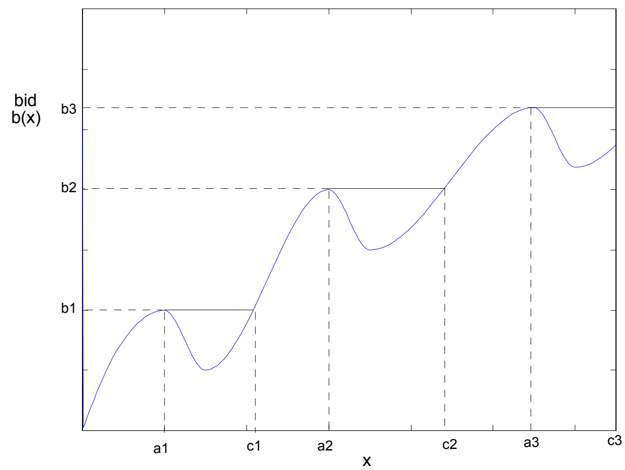

FIGURE 5. Indirect Equilibrium Bidding Function

The unique information that bidders have for the second auction is that there is a tie in b_k, that is, $P_{(-i)}(t_{-i}) \in [a_k, c_k]$.

By the definition of P in (19), $P_{(-i)}$ satisfies the following:

$$\Pr\left(\left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < x\right\} | P_{(-i)}(t_{-i}) \in [a_k, c_k]\right) = \frac{x - a_k}{c_k - a_k}.$$

So, in the tie-breaking auction, the (direct) type t_i of bidder i is competing against players t_j in the set $\{s \in S : P(s) \in [a_k, c_k]\}$ and the equilibrium is to bid the increasing function[20]

$$\tilde{b}^2(x) = \frac{1}{c_k - a_k} \int_{a_k}^{x} \tilde{v}(\alpha, \alpha)\, d\alpha.$$

Indeed, from condition (ii) of Theorem 2, we have that

$$\begin{aligned} & \frac{1}{c_k - a_k}\left[\int_{a_k}^{x} \tilde{v}(x, \alpha)\, d\alpha - \int_{a_k}^{x} \tilde{v}(\alpha, \alpha)\, d\alpha\right] \\ \geqslant \ & \frac{1}{c_k - a_k}\left[\int_{a_k}^{y} \tilde{v}(x, \alpha)\, d\alpha - \int_{a_k}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha\right] \end{aligned}$$

for any $x, y \in [a_k, c_k]$.

[20]It is increasing because $\tilde{v}$ is positive.

Thus, in the whole auction, the bidder of indirect type $x \in [a_k, c_k]$ who follows the strategy $\overline{b}(x)$ and, in case of a tie, the above strategy, will receive the expected payoff

$$\begin{aligned} & \int_0^{a_k} [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha + (c_k - a_k)\left\{ \frac{1}{c_k - a_k} \int_{a_k}^{x} [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \right\} \\ = & \int_0^{x} [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \end{aligned}$$

Deviation in the second auction is suboptimal. By deviating from $\overline{b}$, but bidding in the range of $\overline{b}$, that is, bidding $\overline{b}(y) \neq \overline{b}(x)$, he will get

$$\tilde{\Pi}_i\left(x, \overline{b}(y)\right) = \int_0^{y} [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha,$$

if $\overline{b}(y)$ is not a bid with positive probability. This cannot be profitable by condition (ii). If it is a bid with positive probability, the second stage will be again an all-pay auction, where the bidder cannot improve its payoff, again by condition (ii).

Now, if x bids $\beta < \inf_{x\in[0,1]} \overline{b}(x)$, then his payoff will be

$$\int_0^1 p^L\left(\beta, \overline{b}(\alpha)\right) d\alpha \leqslant 0,$$

because $p^L \leqslant 0$. Therefore, this deviation cannot be profitable.

If x bids $\beta > \sup_{x\in[0,1]} \overline{b}(x) = \tilde{b}(\overline{x}) \geqslant \tilde{b}(1)$, for some $\overline{x}$. Since $\partial_1 p^W(\cdot) \geqslant 0$, $p^W\left(\beta, \overline{b}(z)\right) \geqslant p^W\left(\tilde{b}(1), \overline{b}(z)\right)$. Then,

$$\int_0^1 p^W\left(\beta, \overline{b}(\alpha)\right) \geqslant \int_0^1 p^W\left(\tilde{b}(1), \overline{b}(\alpha)\right) d\alpha = \int_0^1 \tilde{v}(\alpha,\alpha)\, d\alpha.$$

Then, the payoff of the bidder with indirect type x that bids β will be

$$\begin{aligned} & \int_0^1 \left[\tilde{v}(x,\alpha) - p^W\left(\beta, \overline{b}(\alpha)\right)\right] d\alpha \\ \leqslant & \int_0^1 [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \\ = & \int_0^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha + \int_x^1 [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \\ < & \int_0^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha, \end{aligned}$$

where the last inequality comes from (44). Thus, the deviation to β is unprofitable.

In Theorem 3, we also proved condition (i)'. Then, the equilibrium in the indirect auction gives the equilibrium for the direct one.■

Proof of Theorem 5. If y is an indirect type that is not involved in ties, the payment is given by

$$\int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha = \int_0^y \tilde{v}(\alpha, \alpha)\, d\alpha.$$

If $x \in [a_k, c_k]$, in the notation of the previous proof, the expected payment of x will be

$$\int_0^{a_k} \tilde{v}(\alpha, \alpha)\, d\alpha + (c_k - a_k)\left\{\frac{1}{c_k - a_k}\int_{a_k}^{x} \tilde{v}(\alpha, \alpha)\, d\alpha\right\} = \int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha.$$

So, if the equilibrium specified in the proof of Theorem 4 is followed, the expected payment does not depend on the auction format.■

Appendix D - Proofs for the Examples

Generalization of Example 1.

As in Example 1, consider a symmetric auction with two bidders whose utility functions are given by:

$$v(t_i, t_{-i}) = t_i + \alpha(t_i)t_{-i}$$

where $\alpha : [0,1] \to \mathbb{R}$ and t_i is uniformly distributed on $[0,1]$. If there are only two pooling types, that is, types which bid the same for each equilibrium bidding, then, for each i and t, the respective pooling type of t, $\varphi = \varphi(t)$, in a symmetric equilibrium (b^*, b^*) satisfies the condition (i) of Theorem 1:

$$t + \alpha(t)E[t_2|b^*(t) = b^*(t_2)] = \varphi + \alpha(\varphi)E[t_2|b^*(t) = b^*(t_2)].$$

Since $E[t_2|b^*(t) = b^*(t_2)] = (t+\varphi)/2$, because of the symmetry and the uniform distribution, then φ is the implicit solution of

$$(t+\varphi)(\alpha(\varphi) - \alpha(t)) = 2(t - \varphi).$$

Claim. Assume the following conditions:
(i) α is differentiable, decreasing and convex such that $\alpha(0) - \alpha(\bar{v}) = 2$;
(ii) α' is strictly convex and $\alpha'(x) \geq -1/x$ for all $x \in (0,1]$;
Then there exists an U-shaped symmetric equilibrium.
Proof. Define the conjugation

$$P(t) = \frac{\varphi(t) - t}{2}.$$

It is easy to see that P is decreasing.

Define

$$\begin{aligned} \widetilde{v}(x,y) &\equiv E\left[v(t_1,t_2)|P\left(t_1\right)=x,P\left(t_2\right)=y\right] \\ &= \frac{x+\varphi\left(x\right)}{2}+\frac{\alpha(x)+\alpha\left(\varphi\left(x\right)\right)}{2}\frac{y+\varphi\left(y\right)}{2} \end{aligned}$$

Observe that $v(t_1,t_2)$ is of the form $\sum_{k=1}^{2} f_k\left(t_i\right) g_k\left(t_{-i}\right)$. Therefore, by Remark 2 and Theorem 2, the bidding function is an equilibrium if

$$x \geqslant y \Rightarrow \widetilde{v}(x,y) \geqslant \widetilde{v}(y,y).$$

Dividing by $(y+\varphi(y))/2$, we rewrite the above condition as

$$\frac{x+\varphi(x)}{y+\varphi(y)}+\frac{\alpha(x)+\alpha(\varphi(x))}{2} \geqslant 1+\frac{\alpha(y)+\alpha(\varphi(y))}{2} \tag{45}$$

For each $w \in (0,1]$, define $g_w(z)=\frac{z}{w}+\alpha(z)$, for $z \in [w,1]$. It is easy to see that g is non-decreasing (because $g_w'(z)=\frac{1}{w}+\alpha'(z) \geqslant \frac{1}{w}-\frac{1}{z} \geqslant 0$). Let y and $w=\frac{y+\varphi(y)}{2}$ be fixed and take $x \geqslant y$. Since α is convex,

$$\begin{aligned} \frac{x+\varphi(x)}{y+\varphi(y)}+\frac{\alpha(x)+\alpha(\varphi(x))}{2} &\geqslant \frac{x+\varphi(x)}{y+\varphi(y)}+\alpha\left(\frac{x+\varphi\left(x\right)}{2}\right) \\ &= g_w\left(\frac{x+\varphi\left(x\right)}{2}\right), \end{aligned}$$

if $x+\varphi\left(x\right) \geqslant y+\varphi\left(y\right)$. So, to show (45), it is enough to show that $t+\varphi(t)$ is non-increasing in t or, equivalently, $\varphi'(t) \leq -1$.

The implicit derivative of φ with respect to t is:

$$\varphi'\left(t\right)=\frac{\alpha(t)-\alpha(\varphi\left(t\right))+(t+\varphi\left(t\right))\alpha'(t)+2}{\alpha(\varphi\left(t\right))-\alpha(t)+(t+\varphi\left(t\right))\alpha'(\varphi)+2}.$$

If $\varphi'\left(t\right) \leqslant -1$, then the numerator and denominator of the fraction above should have opposite sign. Without loss of generality (because $\varphi \circ \varphi(t)=t$), we can assume that the denominator is negative and $\varphi > t$. Thus,

$$\varphi'(t) \leqslant -1 \Leftrightarrow \frac{\alpha'(t)+\alpha'(\varphi)}{2} \geqslant \frac{\alpha(\varphi)-\alpha(t)}{\varphi-t}.$$

Since α' is a convex function, for the inequality above be attended it is enough to show

$$\alpha'\left(\frac{t+\varphi}{2}\right) \geq \frac{\alpha(\varphi)-\alpha(t)}{\varphi-t}=-\frac{2}{t+\varphi}$$

where the equality comes from the implicit definition of φ. However, this last equality is true because $\alpha'(x) \geq -1/x$ for all $x \in (0,1]$. ■

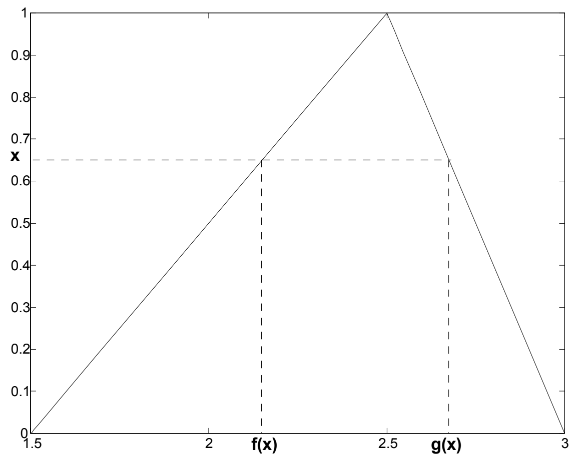

Figure 6. Equilibrium bidding function in Example 1.

Proof of the claims in Example 2.

First, let us show that there is no monotonic equilibria for this auction. By contradiction, assume that there is an increasing equilibrium bidding function. Then, $P(t_i) = \frac{t_i - 1.5}{1.5}$ and condition (i)' is trivial. We have

$$\begin{aligned}\tilde{v}(x,y) &= (1.5x + 1.5)\left[1.5y + 1.5 - \frac{1.5x + 1.5}{2}\right] \\ &= \frac{9(x+1)(2y - x + 1)}{8}.\end{aligned}$$

Thus, the necessary condition (ii) is not satisfied, because $x > y$ implies

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha = -\frac{3(x-y)^3}{8} < 0.$$

Thus, there is no monotonic equilibrium.

Now, we will show that there are multiple equilibria non-monotonic for this auction. Assume that there exists a bell-shaped equilibrium and that, for each x, there are two types, $f(x)$ and $g(x)$, such that $P(t_i) = x = \frac{3 - g(x) + f(x) - 1.5}{1.5}$, which implies that $g(x) = f(x) + 1.5(1-x)$. (See Figure 6).

Condition (i)' requires

$$f(x)\left(\frac{f(y)+g(y)}{2}-\frac{f(x)}{2}\right)=\frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right)-\frac{f^2(x)+g^2(x)}{4}$$
$$\Leftrightarrow \frac{f(y)+g(y)}{2}\left[f(x)-\frac{f(x)+g(x)}{2}\right]=\frac{f(x)^2-g(x)^2}{4}$$
$$\Leftrightarrow \frac{f(y)+g(y)}{2}=\frac{f(x)+g(x)}{2}$$

Then, $f(y)+g(y)$ is a constant, and we have $f(x)=k+3/4x$. Since $f(0)=1.5$, $k=1.5$. We obtain:

$$\begin{aligned}\tilde{v}(x,y)&=\frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right)-\frac{f^2(x)+g^2(x)}{4}\\&=\left(\frac{9}{4}\right)^2-\frac{(3/2+3/4x)^2+(3-3/4x)^2}{4}\\&=\left(\frac{9}{4}\right)\left[1+\frac{x}{4}-\frac{x^2}{8}\right],\end{aligned}$$

which satisfies condition (ii) because it is increasing in x on $[0,1]$. Condition (iii) and (iv) are also satisfied, since

$$\tilde{b}(x)=\frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha=\frac{3\left(24+3x-x^2\right)}{32}$$

is increasing on $[0,1]$.■

Proof for Example 3 - Spectrum Auction

Let us assume that $t_i=(t_i^1,t_i^2,t_i^3)$ are independent and uniformly distributed on $[\underline{s}^1,\overline{s}^1]\times[\underline{s}^2,\overline{s}^2]\times[\underline{s}^3,\overline{s}^3]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3\geqslant 0$. We have

$$v^1\left(t_i^1,t_i^2,t_i^3\right)=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]$$

Let us denote by $\overline{v}^1$ the expression in the first line above, that is,

$$\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3.$$

The conjugation P and the c.d.f. $\tilde{P}$ are given by:

$$P\left(t_i^1,t_i^2,t_i^3\right)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)<\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)\right\}\right]^{N-1}.$$

and

$$\tilde{P}(k)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]<k\right\}\right]^{N-1}.$$

We can reparameterize the problem so that

$$\tilde{P}(k)=\left[\Pr\left\{(x,y,z)\in[0,1]^3:ax+by+cz<l(k)\right\}\right]^{N-1},$$

where $a=\left(\overline{s}^1-\underline{s}^1\right)/N>0$, $b=-\left(\overline{s}^2-\underline{s}^2\right)<0$, $c=-(N-1)\left(\overline{s}^3-\underline{s}^3\right)/N<0$ and

$$l(k)=k-\frac{\underline{s}^1}{N}+\overline{s}^2+\frac{N-1}{N}\underline{s}^3-\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right].$$

It is elementary to obtain that, for a uniform distribution on $[0,1]^3$ and $a>0$, $b<0$, $c<0$ and $k>b+c$,

$$\Pr\left\{(x,y,z)\in[0,1]^3:ax+by+cz<l\right\}=\frac{(l-b-c)^3}{6abc}.$$

So,

$$\tilde{P}(k)=\frac{[l(k)-b-c]^{3(N-1)}}{(6abc)^{N-1}}$$

and

$$\tilde{v}(x,y)=\left\{\tilde{P}^{-1}(x)-\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]\right\}y$$
$$+E\left[\frac{\sum_{j\neq i}\left(t_j^1+t_j^3\right)}{N}|\max_{j\neq i}P\left(t_j\right)=y\right].$$

The candidate for the equilibrium on the first-price indirect auction is

$$\tilde{b}(x)=\frac{1}{x}\int_0^x\tilde{v}(\alpha,\alpha)\,d\alpha,$$

which is differentiable, with $\tilde{b}'(x)=[\tilde{v}(x,x)-x]/x$. Then, Theorem 3 tells us that there exists an equilibrium in regular pure strategies for this auction if and only if

$$\tilde{v}(x,x)-x=\left\{\tilde{P}^{-1}(x)-\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]-1\right\}x$$
$$+E\left[\frac{\sum_{j\neq i}\left(t_j^1+t_j^3\right)}{N}|\max_{j\neq i}v^1\left(t_j\right)=\tilde{P}^{-1}(x)\right]\geqslant 0.$$

Depending on the values of $\underline{s}^n$, $\overline{s}^n$, for $n = 1, 2, 3$, the above expression can be positive or negative. If it is always positive, $\tilde{b}$ is increasing and it is the equilibrium of the indirect auction. In the other case, there is no equilibrium without ties. For instance, a sufficient condition for the existence of equilibrium in pure strategy is $\frac{\underline{s}^1}{N} - \overline{s}^2 - \overline{s}^3 \frac{N-1}{N} - 1 \geqslant 0$.■

Proof for Example 4 - Job Market

We assume that there are two players with unidimensional signals uniformly distributed on $[0,1]$ and that $m \in [0,1]$, $b \geqslant 0$. Following the method given by Theorem 3, we first obtain

$$v^1(t_i) = am + \frac{c}{2} - b(t_i - m)^2.$$

We will consider two cases.

First case: $m \leqslant 1/2$. We have

$$P(t_i) = \begin{cases} 1 - 2m + 2t_i, & \text{if } 0 \leqslant t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leqslant t_i < 2m \\ 1 - t_i, & \text{if } 2m \leqslant t_i \leqslant 1. \end{cases}$$

So,

$$\tilde{v}(x,y) = \begin{cases} am + c(1-y) - b(1-x-m)^2, & \text{if } 0 \leqslant x, y < 1 - 2m \\ am + c(1-y) - \frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 1 - 2m \leqslant x \leqslant 1 \\ (a+c)m - b(1-x-m)^2, & \text{if } 0 \leqslant x < 1 - 2m \leqslant y \leqslant 1 \\ (a+c)m - \frac{b}{4}(1-x)^2, & \text{if } 1 - 2m \leqslant x, y \leqslant 1. \end{cases}$$

Now, it is easy to obtain, for $x < 1 - 2m$,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x}\int_0^x \left[am + c(1-y) - b(1-y-m)^2\right] dy, \\ &= am + c - b(1-m)^2 - x\left[\frac{c}{2} + b(m-1)\right] - \frac{b}{3}x^2, \end{aligned}$$

which is increasing if $c \leqslant \frac{2b(m+1)}{3}$. For $x > 1 - 2m$,

$$\begin{aligned}
\tilde{b}(x) &= \frac{1}{x}\left\{\frac{1}{6}(1-2m)\left[6am+3c(1+2m)-2b\left(1-m+m^2\right)\right]\right. \\
&\quad \left. + \int_{1-2m}^{x}\left[(a+c)m-\frac{b}{4}(1-y)^2\right]dy\right\} \\
&\Rightarrow \tilde{b}(x) = \frac{(1-2m)\left[2c-b(1-2m)\right]}{4x} + m(a+c) - \frac{b}{4} + \frac{b\left(3x-x^2\right)}{12}
\end{aligned}$$

whose derivative can be simplified to

$$\tilde{b}'(x) = -\frac{(1-2m)\left[2c-b(1-2m)\right]}{4x^2} + \frac{b(3-2x)}{12}.$$

Since the term $x^2(3-2x)$ is increasing, the bidding function will be increasing if and only if $\tilde{b}'(1-2m) \geqslant 0$, that is,

$$c \leqslant \frac{2b(1-2m)(1+m)}{3}.$$

We conclude that in the case of $m < 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leqslant \min\left\{\frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}.$$

Second case: $m > 1/2$. We have

$$P(t_i) = \begin{cases} t_i, & \text{if } 0 \leqslant t_i < 2m-1, \\ 1-2m+2t_i, & \text{if } 2m-1 \leqslant t_i < m \\ 1-2t_i+2m, & \text{if } m \leqslant t_i \leqslant 1 \end{cases}$$

and

$$\tilde{v}(x,y) = \begin{cases} am+cy-b(x-m)^2, & \text{if } 0 \leqslant x, y < 2m-1 \\ am+cy-\frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 2m-1 \leqslant x \leqslant 1 \\ (a+c)m-b(x-m)^2, & \text{if } 0 \leqslant x < 2m-1 \leqslant y \leqslant 1 \\ (a+c)m-\frac{b}{4}(1-x)^2, & \text{if } 2m-1 \leqslant x, y \leqslant 1. \end{cases}$$

For $x < 2m-1$,

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \left[am + cy - b(y-m)^2\right] dy,$$
$$= am - bm^2 + x\left(\frac{c}{2} + bm\right) - \frac{b}{3}x^2,$$

which is increasing in the considered interval if and only if $c \geqslant \frac{2}{3}b(m-2)$.

For $x > 2m-1$,

$$\tilde{b}(x) = \frac{-2c(2m-1) - b(2m-1)^2}{4x} + \frac{12(a+c)m - b(3-3x+x^2)}{12}$$

which gives

$$\tilde{b}'(x) = \frac{2c(2m-1) + b(2m-1)^2}{4x^2} + \frac{b(3-2x)}{12}.$$

Following the same procedure of the first case, $\tilde{b}'(x) \geqslant 0, \forall x \in [2m-1,1]$ if and only if

$$c \geqslant -\frac{2b(2m-1)(1+m)}{3}.$$

We conclude that, if $m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \geqslant \max\left\{\frac{2}{3}b(m-2), \frac{2b(1-2m)(1+m)}{3}\right\}.$$

■

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