Affine Skeletons and Monge-Ampère Equations

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Abstract

An important question about affine skeletons is the existence of differential equations that is related to the "affine distance" and "area distance" (hence to the affine skeletons) as the Eikonal equation is related to the "euclidean distance" (and medial axis). We show that some nonlinear second order PDE of Monge-Ampère type are, in fact, related to the affine skeletons. We also discuss some consequences and ideas that the PDE formulation suggests.

Keywords: affine distance, medial axis, skeleton, affine geometry, monge-ampère equation, differential propagation.

1 Introduction

The medial axis the most famous skeleton of shapes and it has been used in a wide range of applications. One of its attractive properties is to be covariant by rigid transformations. In a serie of papers ([6], [7], [5] and [1]), Giblin, Sapiro et al, introduced new skeletons, inspired in the medial axis idea, but having the nice property of being covariant by the larger set of affine transformations. First, they first introduced the affine symmetry sets: the Affine Distance Symmetry Set (ADSS), the Area Distance Symmetry Set (AASS) and the Affine Envelope Symmetry Set (AESS). By an analogy with the (euclidean) symmetry set and the medial axis, they selected a particular subset of the ADSS and AASS to be the affine skeletons, namely, the Affine Distance Skeleton (ADS) and Affine Area Skeleton (AAS). It's not obvious whether the AESS has a subset that can be properly called by "Affine Envelope Skeleton".

The Eikonal equation plays a central role on the research of distance functions and medial axes. Thus, as pointed out by Giblin, Sapiro et al., a natural and important question is the existence of an analogous of the Eikonal to the affine cases.

In this paper we show that some nonlinear second order PDE of Monge-Ampère type that can indeed play a role like the Eikonal. We also discuss some extensions and ideas that come with the PDE formulation.

The paper begins with a brief review of the main properties of medial axis and the definitions of the affine skeletons. In the subsequent section we first show the connection of Monge-Ampère equations with the ADS (and medial axis). The following section is concerned with the AAS, and in the last section we discuss some relation between solutions of Monge-Ampère equations and the AESS.

Some of the ideas of the paper are discussed in PhD thesis of Silva [8]

2 Medial Axis

In this section we list some medial axis topics that will be useful in the comparison with the affine skeletons.

Let Ω be a connected shape (a connected open set of the plane R^2) and Γ the boundary of Ω. The medial axis is (the closure of) the locus of the points X of R^2 where the distance of X to the a point P on the boundary Γ is reached by another point Q of Γ, provided that the distance is a global minimum at X.

$$d(X,P) = d(X,Q) = \min_{T \in \Gamma} d(X,T)$$

The pair $(X,\, d(X,\Gamma))$ - the medial axis points with the distance function to the curve Γ - are usually called Medial Axis Transform (MAT).

In order to compare the medial axis skeleton with the affine ones we point out some alternative definitions:

- (closure of) the centers of maximal disks inside Ω. The distance function is the radius of each disk.
- (closure of) the center of disks inside Ω that are tangents to the curve Γ at least at two points. The distance function is the radius of each disk.
- the singularities of the "distance function of Γ ".
- Evolving each point of Γ along the normals with speed $=1$, the medial axis is the locus of schocks. The distance function is the time of the shock.

The last two items relate the medial axis with the PDE Eikonal (boundary value problem)

$$\begin{cases} |\nabla f| = 1 \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

and its formulation as a initial value problem

$$\begin{cases} C(s,t)_t = N(s,t) \\ C(s,0) = \gamma(s) \end{cases}$$

where $\gamma(s)$ is a parametrization of Γ and $N(s)$ is the normal vector at $\varphi(s)$,

3 Affine Skeletons

In this section we briefly review the definition and main properties of the affine skeletons introduced by Sapiro, Giblin et al ([6], [7] and [1]) . We assume, throughout this section, that Γ is a simple convex curve.

3.1 Affine Distance Skeleton - ADS

Replacing the "distance" by "affine distance" in the definition of medial axis we get the affine distance skeleton - ADS. The ADS was introduced by Giblin and Sapiro in [6], [7]. Let's review the definition of affine distance.

Let $\gamma(s)$ be a affine parametrization of the curve Γ. The affine distance of a point X to the curve point $\gamma(s)$ of Γ is the area of the paralelogram defined by the vectors $\gamma_s(s)$ and $X-\gamma(s)$, that is (figure 1)

$$d(X,\gamma(s)) = \frac{1}{2}[\gamma_s(s), X-\gamma(s)]$$

The notation $[u,\, v]$ means the determinant of the matrix whose the columns are the vectors u and v.

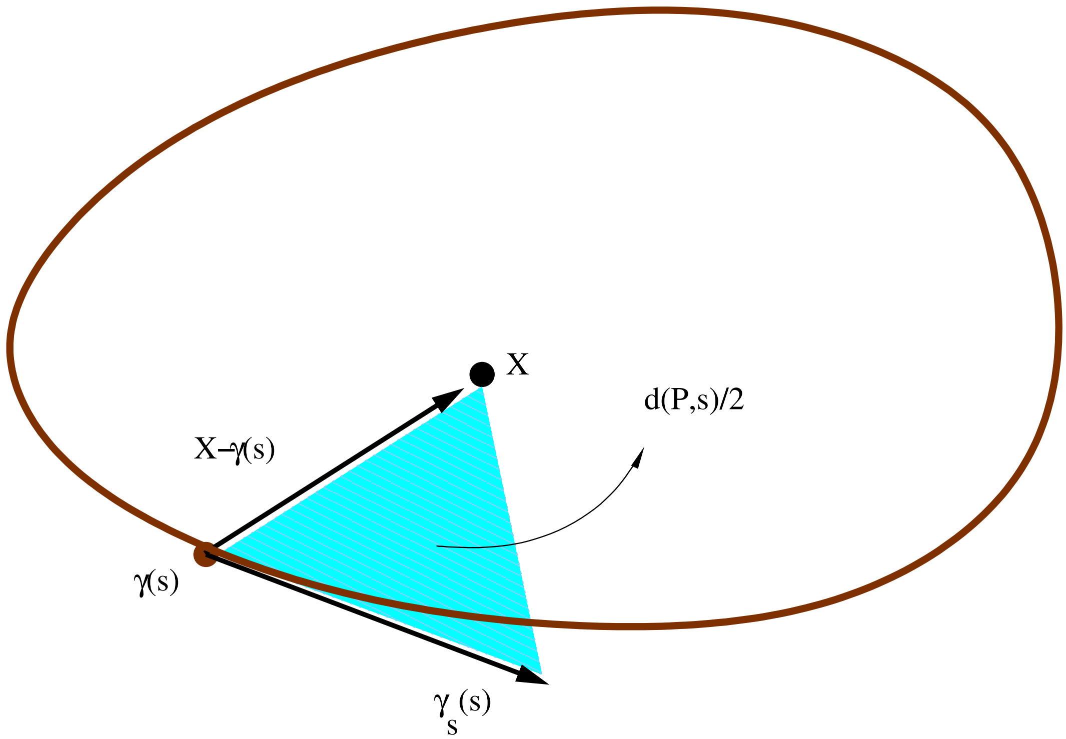

Figure 1. The affine distance

The affine distance of a point X to the convex curve Γ is the minimum distance of X over all points of Γ

$$d(X, \Gamma) = \min_{s} d(X, \gamma(s))$$

The affine distance skeleton (ADS) has the same definition of the medial axis, replacing "euclidean distance" by "affine distance". The ADS of Γ is the locus of points $X \in R^2$ where the affine distance of X to some point of $P \in \Gamma$ is reached by another point Q of Γ, provided that the distance is a global minimum at X.

$$d(X, P) = d(X, Q) = \min_{T \in \Gamma} d(X, T)$$

Like the euclidean distance, the affine distance is zero on Γ and increase linearly along straigth lines, the affine normals. The graph of the affine distance of Γ is a ruled surface. We can formulate the ADS as the shock points of the evolution

$$\begin{cases} C(s,t)_t = N(s,0) \\ C(s,0) = \gamma(s) \end{cases}$$

where $\gamma(s)$ is a parametrization of Γ and $N(s,t)$ is the affine normal vector of the point $C(s,t)$. But this is not an PDE, because the affine normal vector $N(s,0)$ that gives the velocity of the points is the vector at the starting point on Γ, not the affine normal vector at the point $C(s,t)$.

3.2 Affine Area Skeleton - AAS

The affine area skeleton is based on another "distance", namely, the area distance. The area distance was introduced by Moisan [4], and used, with a slight modification, by Giblin and Sapiro et al [5]. To calculate the area distance of a point X to a point $\gamma(s)$ of Γ take the line that pass through $\gamma(s)$ and X. That line - a chord of Γ- meets the curve Γ at another point $\gamma(r)$. The smallest area bounded by Γ and the chord is the area distance (figure 2).

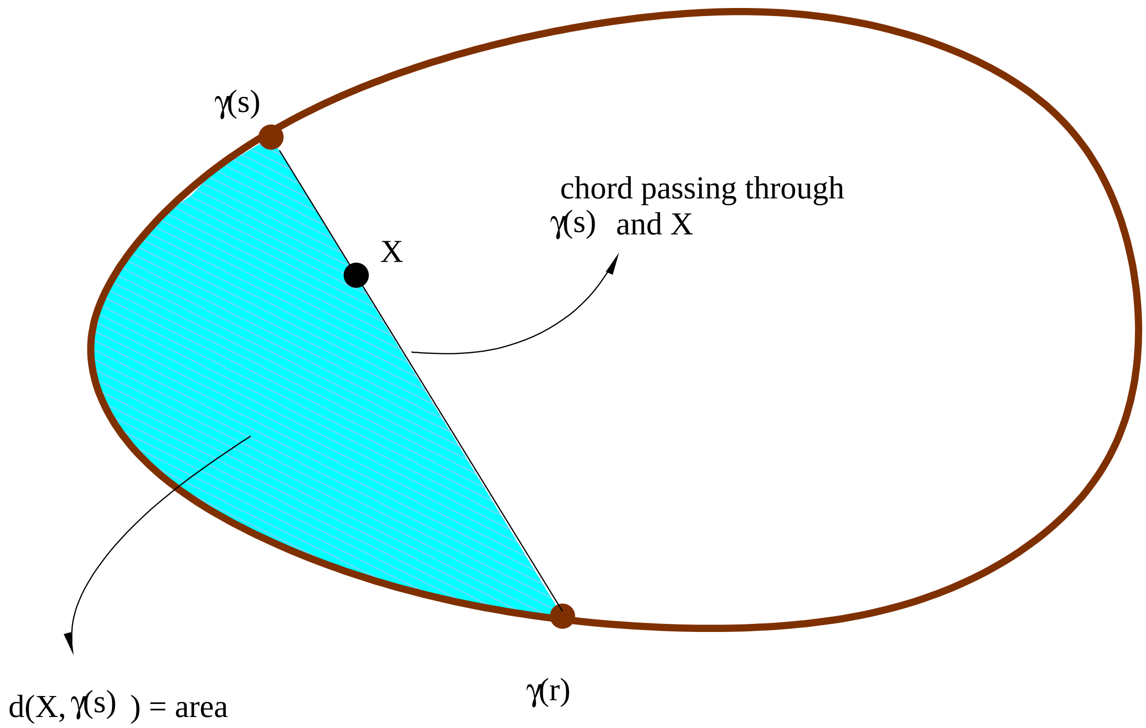

Figure 2. The area distance

The area distance of X to Γ is, as expected, the minimum of the distances of X to all the points of Γ

$$d(X,\Gamma) = \min_s d(X,\gamma(s))$$

The Affine Area Skeleton (AAS) of Γ is the locus of points $X \in R^2$ where the affine distance of X to Γ is reached at two (P and Q) or more points of Γ

$$d(X,P) = d(X,Q) = \min_{T\in\Gamma} d(X,\Gamma)$$

The level curves of the area distance can be considered as the successive evolutions of Γ and the AAS as the shock points of this evolution. The operation of taking the sucessive evolutions of Γ by this way has been called the "affine erosion" of Γ.

3.3 Affine Envelop Symmetry Set - AESS

The affine envelop symmetry set (AESS) is not based on the distance function. Instead, it is a variation of the definition of the medial axis by its bi-tangential circles. The AESS is the locus of centers of conics that have contact of order 3, at least, at two or more points of Γ. (The contact of order 1 means a mere intersection, the tangents do not coincide. Order 2 means that the curves are tangent at the point of contact. And contact of order 3 means that the curves are tangent and have the same curvature at the point of contact).

Since there is no "distance function" nor level curves envolved, there is no mention to "curve evolution" or "shock points". Besides, the definition gives only a symmetry set. It's not clear how to select a subset of AESS as the natural candidate to be the "affine envelop skeleton".

4 $H=0$ - Medial Axis and ADS

4.1 Ruled Distances

Euclidean distance (medial axis) and affine distance (ADS) of a curve Γ share the properties that its values is zero on Γ and increase linearly along straight lines. Both fit in the scheme

$$f(\gamma(s)+t\cdot w(s))=t$$

where $\gamma(s)$ is a parametrization of Γ and $w(s)$ is a continuous vector field over Γ, the velocity vectors of propagation. In euclidean distance the vector $w(s)$ is the unitary normal vector at $\gamma(s)$. In affine distance the vector $w(s)$ is the affine normal vector at $\gamma(s)$. On both cases we have that $w_s(s)$ is colinear with $\gamma_s(s)$.

A simple calculation shows the following proposition:

Proposition 1. *$D^2 f$ is singular at differentiable points X*

Proof. X is reached by $\gamma(s)+t\cdot w(s)$ for some s and t. Differentiating $f(X=\gamma(s)+t\cdot w(s))=t$ by t and s we obtain

$$Df(X)\cdot(w)=1 \tag{1}$$

$$Df(X)(\gamma_s+t\cdot w_s)=0 \tag{2}$$

Differantiating equation (1) by t and equation (2) by s yelds

$$D^2 f(X)(w)^2=0 \tag{3}$$

$$D^2 f(X)(\gamma_s+t\cdot w_s)^2+Df(X)(\gamma_{ss}+t\cdot w_{ss})=0 \tag{4}$$

And differantiating equation (1) by s (or equation (2) by t) we have

$$D^2 f(X)(w,\gamma_s+t\cdot w_s)+Df(X)\cdot(w_s)=0 \tag{5}$$

Since w_s is colinear with γ_s we can write $w_s=-\lambda\cdot\gamma_s$ and then equation (2) shows that the gradiente Df is ortogonal to γ_s not only at Γ but at all the points on the line $X=\gamma(s)+t\cdot w(s)$. Together with equation (1) we have that Df is the same over the line $X=\gamma(s)+t\cdot w(s)$. We say that the gradient is transported over the characteristics curves (that are lines).

As the gradient is constant along the direction w, this direction is a nullspace of the Hessian. Indeed, using equation (4) and (5), the colinearity $w_s=-\lambda\cdot\gamma_s$ implies that

$$(1+t\cdot\lambda)^2 D^2 f(X)(\gamma_s)^2=-Df(X)(\gamma_{ss}+t\cdot w_{ss}) \tag{6}$$

$$D^2 f(X)(w,\gamma_s)=0 \tag{7}$$

Joining equations (3) and (7) we conclude that $D^2 f(X)\cdot w=0$. □

Equation (6) also gives a nice relation between the curvature of Γ and the curvature of the subsequent level curves. Let's use the u, v coordinates, where u is the unitary vector tangent to the level curve at X and v is the normal to the level curve ($[u, v] = 1$). The gradient at X is the same of the gradiente at $\gamma(s)$. So, if $\gamma(s)$ is a parametrization by arclength, then $\gamma_s = u$ and $\gamma_{ss} = k(P) \cdot v$ (k is the curvature of Γ at $P = \gamma(s)$). On the other had we have that $w_s = -\lambda \cdot \gamma_s$, so $w_{ss} = -\lambda_s \cdot \gamma_s - \lambda \cdot \gamma_{ss} = -\lambda_s \cdot u - \lambda \cdot k \cdot v$. Equation (6) may be rewritten as

$$(1 - t \cdot \lambda)^2 f_{uu} = -f_v \cdot k(P) \cdot (1 - t \cdot \lambda)$$

$$-\frac{f_{uu}}{f_v} = \frac{k(P)}{(1 - t \cdot \lambda)} \tag{8}$$

But $-f_{uu}/f_v = k(X)$, the curvature of the level curve in X. Thus, (8) says that

$$k(X) = \frac{k(P)}{(1 - f(X) \cdot \lambda)} \tag{9}$$

Summarizing, we have that both euclidean and affine distances functions of Γ share the property that the hessian is singular at differentiable points X, i.e., they are solutions of the PDE $\det(D^2 f) = 0$. The functions are, obviously, different at the initial gradients:

- The euclidean distance - we have that $w(s)$ is the unitary normal vector $w(s) = N(s)$ ($N(s)$ is the unitary normal vector at $\gamma(s)$). Equation (1) is $D f(X) \cdot (N) = 1$, so the gradient vector (that is also orthogonal to Γ at $P = \gamma(s)$) must have $\|\nabla f(P)\| = 1$.
- The affine distance - now we have that $w(s)$ is the affine normal vector $w(s) = -\frac{1}{3} k^{-\frac{5}{3}}(P) \cdot T(s) + k^{\frac{1}{3}}(P) \cdot N(s)$ ($T(s)$ and $N(s)$ are the unitary tangent and normal vectors at $P = \gamma(s)$). By equation (1), the gradient vector must have $\|\nabla f(P)\| = k^{-\frac{1}{3}}(P)$.

The euclidean distance is then a solution of the Monge-Ampère equation

$$\begin{cases} \det(D^2 f) = 0 \\ \|\nabla f(x)\| = 1, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

and the affine distance is a solution of

$$\begin{cases} \det(D^2 f) = 0 \\ \|\nabla f(x)\| = k^{-1/3}, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

These are particular cases of the "ruled distances" $f(\gamma(s) + t \cdot w(s)) = t$, where $w(s)$ is a smooth vector field over Γ and $w_s(s)$ is colinear with $\gamma(s)$. The calculations made above just show that a "ruled distance" is the a solution of the following equation with double boundary value condition

$$\begin{cases} \det(D^2 f) = 0 \\ \langle \nabla f(x), w(x) \rangle = 1, x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

We must be carefull with the meaning of "solutions" of a such PDE, since we expect the ocurrence of shock points, where the hessian $D^2 f$ is not defined in the classical sense. A good choice is to define the entropy solution: each characteristic curve $\varphi(t) = \gamma(s) + t \cdot w(s)$ starts at $t = 0$ and evolve until the first intersection with another characteristic, at the shock point (both with the same t). At the shock point, the evolution of the characteristics ceases and thus they don't contribute to create any other shock point. So, a function f is a solution in entropy sense if the characteristics curves evolve acording to the differential equation until it has no intersection with another charateristic curve.

At this point, the natural questions that arise are: the homogenous Monge-Ampère equation with double boundary condition

$$\begin{cases} \det(D^2 f) = 0 \\ \|\nabla f(x)\| = h(x), \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases} \tag{10}$$

has a (entropy) solution? It is unique? It is always a “ruled distance” ?

We can exhibit an entropy solution when the function $h(x)$ is smooth and h is never zero. The function $f(\gamma(s) + t \cdot w(s))$ is such a solution, with $w(s) = -\frac{b_s}{k(s)} \cdot T(s) + b \cdot N(s)$. (Here, $b(s) = 1/h(s)$; T and N are the unitary tangent and normal vectors at $\gamma(s)$; $k(s)$ is the curvature of Γ at $\gamma(s)$; and s is the arclength parameter). To see this, one can just check that this solution actually satisfies the boundary condition and that w_s is colinear with γ_s. The previous calculations of this section then shows that $D^2 f = 0$.

If we impose that the solution are suficiently smooth on an open set containing points of the curve Γ, then it can be proved that the solution is unique on that set. *** Is the idea ill detailed? ***

Summarizing, this section shows that the homogeneous Monge-Ampère equation is the “local” description of the ruled distances, a similar role that the Eikonal equation plays to euclidean distance.

4.2 The propagation of Γ by the normal

We can also describe the ruled distances by an evolution of the curve Γ through the normal vectors (euclidean ones). If $C(s,\ t)$ is a parametrization of the level curve t of f, s being the arclength parameter for a fixed t. Then, differentiating $f(C(s,t)) = t$ by s and t, and using that f is a solution of equation (10), we conclude that $C(s,t)$ evolves according

$$\begin{cases} C_t(s,t) = v(s,t) \cdot N(s,t) \\ v_t(s,t) = \dfrac{v_s(s,t)}{k} \end{cases}$$

subject the initial conditions

$$\begin{cases} C(s,0) = \gamma(s) \\ v(s,t) = \dfrac{1}{h(\gamma(s))} \end{cases}$$

Here, $v(s,\ t)$ is the speed of the point $C(s,t)$ in the direction of the normal $N(s,t)$. So, the evolution of Γ that represents the sucessive level curves of ruled distances is a second order differential equation, written above as a coupled system of first order equations.

4.3 The non-convex case

When Γ is a non-convex curve, it has inflexion points, i.e., points where the curvature is zero. Since the boundary condition of the affine distance is $\|\nabla f(x)\| = k^{-1/3}$, we face a problem. Besides the ill definition of the gradiente at the inflexion point, the points very close to the inflexion point run with arbitrary large speed. So, the value of the affine distance along characteristics emanating of a neighborhood of the inflexion point is close to zero, even at points far away from Γ. As consequence of this problem, some “parts” of the skeleton ADS, as defined previously, seems counter intuitive.

To tackle this problem, the idea of an entropy solution is quite usefull. Although the characteristics run very fast when the points are near the inflexion points, they soon quench each other. We ilustrate this idea with the cubic curve (t, t^3), where the inflexion point is at (0,0) (figure 4). We show the piece of the curve with parameter $t \in [-1, 0.1]$. The cubic cuve is the solid thick line at the third quadrant. The characteristics are the thin straight lines. Note that at the second quadrant, the characteristics originated at the first quadrant meet the ones originated at the third quadrant. The locus of the schock points can be visualized easily.

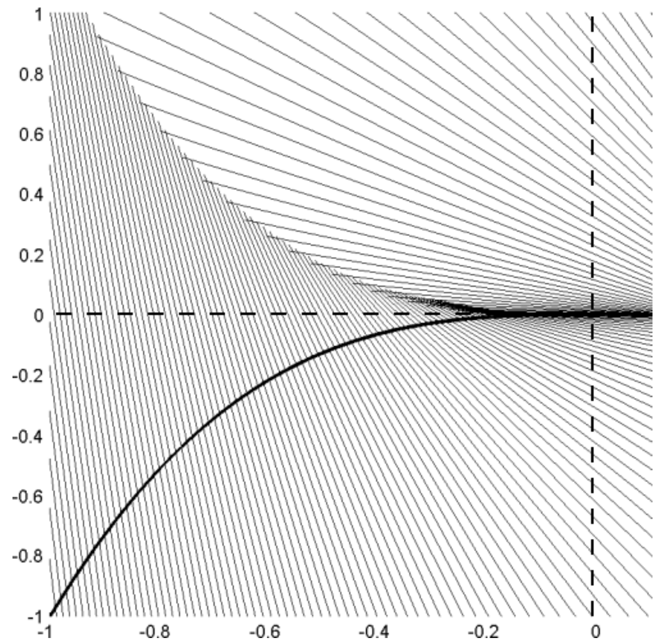

Figure 3. Schocks of the characteristics emanating from the cubic curve

This suggest a sligth modification of the definition of the ADS: instead of “the set of points X where the affine distance of $P \in \Gamma$ to X is the same of the distance of $Q \neq P$ to X, provided that the distance is global in X”, we propose “the set of points where two distinct characteristics intersect, provided that each characteristic has no previous intersection with any other ‘active’ characteristic”.

5 $H = -4$ and the AAS

The definition of Affine Area Skeleton of a curve Γ is based on the “area distance function” of Γ. The aim of this section is to show the that the area distance function is a solution of the following Monge-Ampère equation with double boundary condition

$$\begin{cases} \det(D^2 f) = -4 \\ \|\nabla f(x)\| = 0, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases} \tag{11}$$

We assume, throughout this section, that Γ is a simple and strictly convex curve. The treatment of non-convex curves are subtle and deserves a special care.

5.1 The gradient and hessian at regular points

As presented in [4] and [5], it’s not difficult to show that the the k-level curve of the area distance is the envelope of the chords that bound a region of area k with Γ and it is also the set of the midpoints of that chords. So, the gradients of the area distance at a point X are orthogonal to the chord that has X as its midpoint (figure 3).

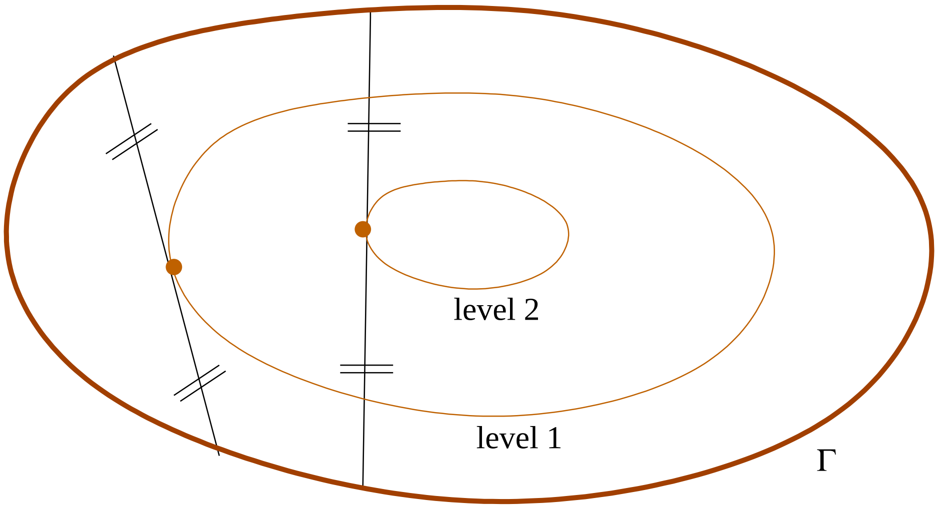

Figure 4. The midpoint property

The *generator points* of X are the two points on Γ that form a chord having X at its midpoint. Let us see how the generator points vary when we move the point X (figure 5).

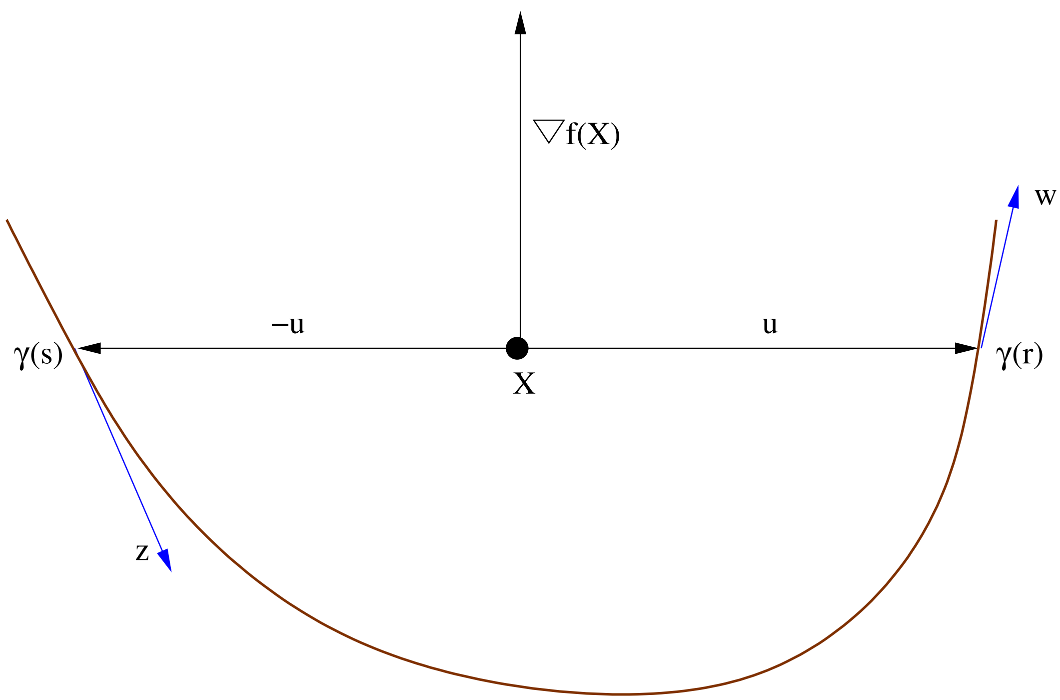

Figure 5. The gradient of the area distance at a regular point

The pieces of Γ that contains the generator points are parametrized by $\gamma(s)$ and $\gamma(r)$. Let the tangent vector be $z = \gamma'(s)$ and $w = \gamma'(r)$. Assume that z and w are linearly independent. So, we can write the vector $u = \gamma(r) - X$ as a linear combination of z and w.

$$u = \frac{\langle R\,u, w\rangle}{\langle R\,z, w\rangle}\,z - \frac{\langle R\,u, z\rangle}{\langle R\,z, w\rangle}$$

where R is the matrix of the anti-clockwise rotation by $\pi/2$.

We know that X is the midpoint of the chord $\gamma(s)$ and $\gamma(r)$, so $X(s,r) = \frac{\gamma(s)+\gamma(s)}{2}$. Hence, the jacobian matrix is $D\,X = \frac{1}{2}\cdot(z,\,w)$, putting the vectors z and w as the columns. As the vector z and w are independents, the jacobian is invertible. Thus, by the inverse function theorem, we have functions $s(Y)$ and $r(Y)$ that give the parameters of the points $\gamma(r)$ and $\gamma(s)$ for a point Y in a neighborhood of X. Differentiating $\frac{\gamma(s(X))+\gamma(r(X))}{2} = X$ by X, we get the equations for ∇s and ∇r

$$D\,X\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix} = I$$

$$\frac{1}{2}(z,w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix} = I \tag{12}$$

where I is the identity matrix and $\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}$ is the matrix whose lines are the gradient vectors of $s(X)$ and $r(X)$. Solving this equation for ∇s and ∇r, we get

$$\nabla s = \frac{-2\,R\,w}{\langle R\,z,w\rangle} \text{ and } \nabla r = \frac{2\,R\,z}{\langle R\,z,w\rangle} \tag{13}$$

Therefore, we can evaluate the gradient $\nabla f(X)$ at X. The value of f is the area calculated by the integral

$$f(X) = \frac{1}{2}\int_{s(X)}^{r(X)} \langle R\,(\gamma(m)-X,\gamma'(m)\rangle\,\mathrm{dm}$$

Differentiating we get

$$2\,\nabla f(X) = \langle R\,(\gamma(r)-X,w\rangle\cdot\nabla r - \langle R\,(\gamma(s)-X,z\rangle\cdot\nabla s + \int_s^r R\cdot\gamma'(m)\,\mathrm{dm}$$

Simplifying this expression and using (13) we get $2\,\nabla f(X) = 4\cdot R\cdot u$. That is, the gradient is

$$\nabla f(X) = R\cdot(c(r)-c(s)) \tag{14}$$

Thus, the modulus of the gradient is exactly the length of the chord. It is easy to see that the length of the chord goes to zero as the point X gets close to the boundary Γ. So, assuming the continuity of the gradient on the boundary curve Γ, the gradiente must be zero on Γ.

Differentiating (14) again, we get

$$D^2 f(X) = R\cdot(z,-w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}$$

but equation (12) is $(z,w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix} = 2\cdot I$. Hence,

$$\det(D^2 f) = -4$$

Thus, we have the following theorem

Theorem 2. *Let $f\colon\Omega\to R$ be the area distance function to a simple and strictly convex curve $\Gamma(=\partial\Omega)$. Then, the function f is a solution of the Monge-Ampéère equation*

$$\begin{cases}\det(D^2 f) = -4, \text{ if } x\in\Omega\\ D\,f(x) = 0, \text{if } x\in\Gamma\\ f(x) = 0, \text{if } x\in\Gamma\end{cases}$$

As with the affine distance, we are expecting singular points (the skeleton) and then we must be carefull with the meaning of "a solution" of this equation.

5.2 Example - Circle and Ellipse

By a simple calculation we can find the expression of the solution when the curve Γ is a circle (the elipses follow by an affine transformation). When the radius of the circle is one, the equation is

$$\begin{cases} \det(D^2 f) = -4, & \text{if } x^2 + y^2 < 1 \\ D f(x, y) = 0, & \text{if } x^2 + y^2 = 1 \\ f(x) = 0, & \text{if } x^2 + y^2 = 1 \end{cases} \tag{15}$$

Using polar coordinates $(r,\ \theta)$, the solution is, by rotational symmetry, a function of the radius only, that is, $f(x,\ y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. The gradient vector and hessian matrix of f are

$$\nabla f(x, y) = (g' \cdot r_x, g' \cdot r_y)$$

$$D^2 f(x, y) = \begin{pmatrix} g'' \cdot r_x^2 + g' \cdot r_{\mathrm{xx}} & g'' \cdot r_x r_y + g' \cdot r_{\mathrm{xy}} \\ g'' \cdot r_x r_y + g' \cdot r_{\mathrm{xy}} & g'' \cdot r_y^2 + g' \cdot r_{\mathrm{yy}} \end{pmatrix}$$

choosing the point $(r = 1,\ \theta = 0)$, we have $r_x = -1$, $r_{\mathrm{xx}} = 0$, $r_y = 0$, $r_{\mathrm{xy}} = 0$ and $r_{\mathrm{yy}} = 1/r$. Thus,

$$\nabla f(x, y) = (-g_x', 0)$$

$$D^2 f(x, y) = \begin{pmatrix} g'' & 0 \\ 0 & \frac{g'}{r} \end{pmatrix}$$

The partial differential equation (15) is translate to the ordinary differential equation

$$\begin{cases} g''(r)\, g'(r) = -4r \\ g'(1) = 0 \\ g(1) = 0 \end{cases}$$

Observing that $g''(r)\, g'(r) = \frac{1}{2}((g')^2)'$, it's easy to solve the ODE. The solution is

$$g(r) = \pi/2 - r\sqrt{1 - r^2} - \arcsin(r)$$

It should be noted that g assume real values only inside the circle Γ.

5.3 Area distance for non-convex curves

The selection of sucessive level sets of the area distance function is a operator called "affine erosion", in analogy with the erosion operator of morphology. A candidate of a extension of the area distance function for non-convex curves and the extension for the region outside the shape Ω must behave as an "affine dilation" operator.

Taking the example of the circle above, we can try a solution of the equation

$$\begin{cases} \det(D^2 f) = 4 \\ D f(x) = 0, \text{if } x \in \Omega \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

changing the constant -4 to +4. This equation is well defined for the region outside the convex shape Ω. Similar calculations made for $\det(D^2 f) = -4$ inside the circle yeld the following solution f for $\det(D^2 f) = 4$: $f(x, y) = h(r)$ where

$$h(r) = -r\sqrt{r^2 - 1} + \log(r + \sqrt{r^2 - 1})$$

that has real values outside the circle only. Gluing the functions g and h we get the following function (figure 6).

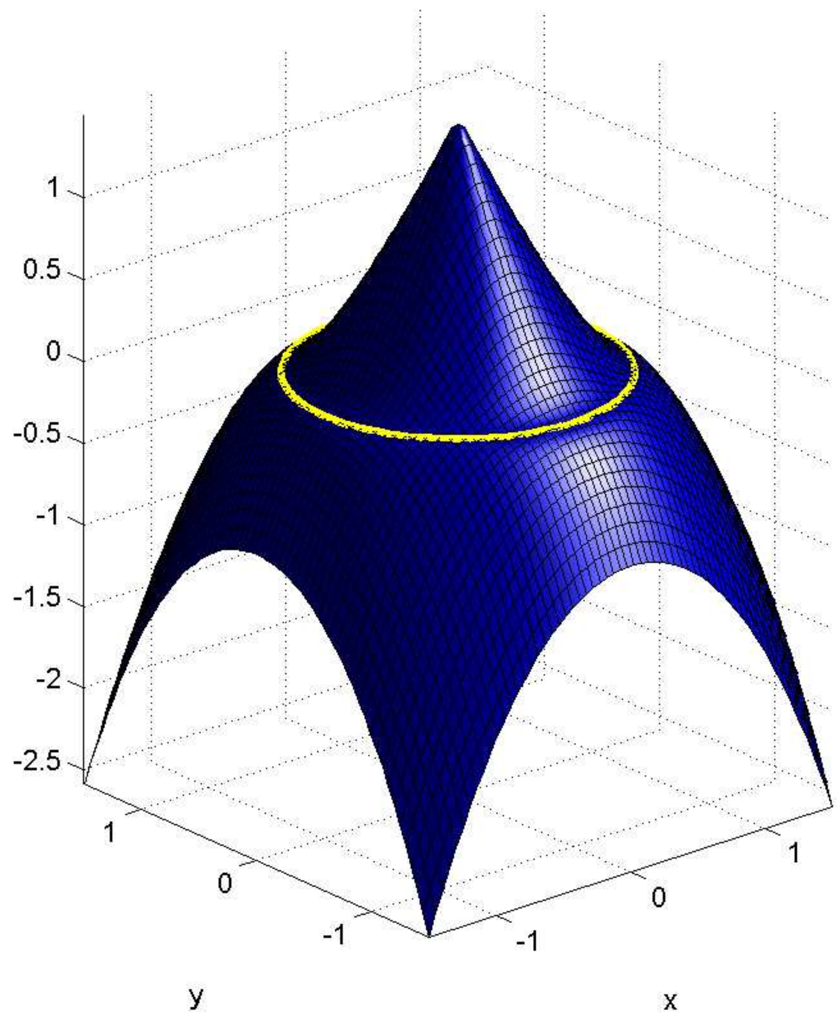

Figure 6. $H = -4$ and $H = 4$ together. The circle Γ is also ploted.

It's intersting to note that $h(r) = -i \cdot g(r)$. So, the solution on both regions is $\varphi(x, y) = \mathrm{Re}\{g(r) + h(r)\} = \mathrm{Re}\{(1 - i)\, g(r)\}$. But $f(x, y) = g(r)$ satisfies $\det(D^2 f) = -4$, thus $\varphi(x, y) = (1 - i)\, f(x, y)$ satisfies $\det(D^2\varphi) = -4 \cdot (1 - i)^2 = 8\, i$. Hence, we can describe the function generated by gluing h and g, as the real part of the solution of the equation

$$\begin{cases} \det(D^2\varphi) = 8i \\ D\,\varphi(x) = 0, \text{if } x \in \Omega \\ \varphi(x) = 0, \text{if } x \in \Gamma \end{cases}$$

Whether this equation makes sense in the case of a generic curve Γ is a question to be investigated.

*** O “chute” é pretensioso demais? ***

*** incluir interpretação geométrica de H = 4 ?***

6 H=1 - AES

The Affine Envelop Symmetry Set (AESS) has also a connection with Monge-Ampère equations. Let Γ be, as usual, a simple and convex curve, with regularity C^∞. A remarkable result of Caffarelli, Nirenberg and Spruck [2] states that if $H\colon \bar{\Omega} \to R$ and $\rho\colon \Gamma \to R$ are C^∞ and $H > 0$, the Monge-Ampère problem

$$\begin{cases} \det(D^2 f) = H(x) & \text{if } x \in \Omega \\ f(x) = \rho(x), & \text{if } x \in \Gamma \end{cases}$$

has a unique convex solution $f\colon \bar{\Omega} \to R$.

Let f be the solution of the Monge-Ampère problem above, with $H \equiv 1$ and $\rho \equiv 0$. On a point $X_0 = (x_0, y_0)$ take the unitary orthogonal pair of eigenvectors p and q, related to the minimum and the maximum eigenvalues of the hessian on X_0 respectively. Taking these vectors as the coordinate system at X_0, let's look to the functionsf_{qq}, f_{pp} and f_{pq}. Let $w(X_0)$ be the unitary vector tangent to the level curve of f_{qq} at X_0. That is

$$f_{\mathrm{qqw}}(X_0) = 0 \tag{16}$$

Since for every point in Ω we have $f_{\mathrm{pp}} \cdot f_{\mathrm{qq}} - f_{\mathrm{pq}}^2 = 1$ then, differentiating by w,

$$f_{\mathrm{ppw}} \cdot f_{\mathrm{qq}} + f_{\mathrm{pp}} \cdot f_{\mathrm{qqw}} - f_{\mathrm{pqw}} \cdot f_{\mathrm{pq}} = 0$$

and since $f_{\mathrm{qq}} > 0$ and $f_{\mathrm{pq}} = 0$ at X_0 this equality simplifies to

$$f_{\mathrm{ppw}}(X_0) = 0 \tag{17}$$

Hence, the vector w is also tangent to the level curve of f_{pp} at X_0

We have also $f_{\mathrm{pqw}}(X_0) = 0$. To check this, look at the values of the functions at $X_0 + h \cdot w$

$$\begin{aligned} f_{\mathrm{pp}}(X_0 + h \cdot w) &= f_{\mathrm{pp}}(X_0) + f_{\mathrm{ppw}}(X_0) \cdot h + o(h^2) = f_{\mathrm{pp}} + o(h^2) \\ f_{\mathrm{qq}}(X_0 + h \cdot w) &= f_{\mathrm{qq}}(X_0) + f_{\mathrm{qqw}}(X_0) \cdot h + o(h^2) = f_{\mathrm{qq}} + o(h^2) \\ f_{\mathrm{pq}}(X_0 + h \cdot w) &= f_{\mathrm{pq}}(X_0) + f_{\mathrm{pqw}}(X_0) \cdot h + o(h^2) = f_{\mathrm{pqw}} \cdot h + o(h^2) \end{aligned}$$

Calculating $\det(D^2 f(X_0 + h \cdot w))$ we get

$$\det \begin{pmatrix} f_{\mathrm{pp}} + o(h^2) & f_{\mathrm{pqw}} \cdot h + o(h^2) \\ f_{\mathrm{pqw}} \cdot h + o(h^2) & f_{\mathrm{qq}} + o(h^2) \end{pmatrix} = 1$$

$$f_{\mathrm{pp}} \cdot f_{\mathrm{qq}} - f_{\mathrm{pqw}}^2 \cdot h + o(h^2) = 1$$

$$f_{\mathrm{pqw}}(X_0) = 0 \tag{18}$$

Therefore, the eigenvalues and eigenvectors don't change in the direction pointed by w.

Let $\psi(t)$ be a parametrization by arclength of the level curve of f_{qq} that starts at some point of Γ, say $\psi(0) = \gamma(s_0)$. Assume that that $\psi'(0)$ is pointing inwards Ω. The curve $\psi(t)$ traverses the region $\bar{\Omega}$ and ends on another point of Γ, say $\psi(t_1) = \gamma(s_1)$. We have that the hessian matrix $D^2 f$ is constant along the curve $\psi(t)$, a consequence of the equations (16), (17) and (18).

Now, look at the unique quadratic polynomial $C(x, y)$ that coincides with f at the point $\gamma(s_0)$ up to the second derivatives, i.e.,

$$\begin{aligned} C(\gamma(s_0)) &= f(\gamma(s_0)) \\ DC(\gamma(s_0)) &= Df(\gamma(s_0)) \\ D^2C(\gamma(s_0)) &= D^2 f(\gamma(s_0)) \end{aligned}$$

The zero level curve of this polynomial is a conic Q that has a contact with Γ of order three at $\gamma(s_0)$. As the hessians $D^2 f$ and $D^2 C$ are constants along $\psi(t)$, f and C coincide up to the second derivatives along the whole curve $\psi(t)$, until it reaches the point $\gamma(s_1)$. Thus, the conic Q has also a contact or order three with Γ at $\gamma(s_1)$. The points $\gamma(s_0)$ and $\gamma(s_1)$ are then the "generators" of a point of the AESS of Γ.

Therefore, given a point $\gamma(s_0)$ of Γ, we have a scheme to seek for his mate $\gamma(s_1)$: construct the function $\lambda\colon \bar{\Omega} \to R$ taking the maximum eigenvalue of the hessian of f at each point of $\bar{\Omega}$. That is, $\lambda(x, y) = \max(\lambda_1, \lambda_2)$, where the λ_i are the eigenvalues of $D^2 f$.

Then, track the level curve of λ that begins on $\gamma(s_0)$ until it reaches another point $\gamma(s_1)$ of Γ. This point is the mate of $\gamma(s_0)$.

The original definition of AESS, as the center of the conics, can exhibits points laying outside the region Ω. Besides, it's not clear how to select a subset of the AESS to be the "Affine Envelop Skeleton". To avoid such difficulties, we propose the Affine Envelop Skeleton (AES) to be the points of $\bar{\Omega}$ where the level curves of f (the solution of $D^2 f = 1$) are tangent to the level curves of λ (the maximum of the eigenvalues of f). *** por que esta definição é boa?***

It should be remarked a fundamental difference between the equation of this section and the equations of the previous sections. Here, the equation

$$\begin{cases} \det(D^2 f) = 1 & \text{if } x \in \Omega \\ f(x) = 0, & \text{if } x \in \Gamma \end{cases}$$

presents only one boundary condition, the value of the solution on Γ. But it requires a smooth solution and thus the solution cannot be obtained by a simply propagation of the values on Γ inward Ω. A "more global" reasoning must be used to obtain the solutions.

*** faltam exemplos para justificar a definição e para exibir a função f e λ em funcionamento. Claro que usar uma cônica para ser o bord Γ (e aí f será um polinômio quadrático) é muito trivial ***

7 Conclusion

We have shown the relations between the affine skeletons and the Monge-Ampère equations. Let's write these equations, as a review, using the differential operators $H = D^2 f$ and $J = -k \cdot |\nabla f|$ (k is the curvature of the level curve). These differential operators are usefull in affine image processing and are the unique differential operators of second order that are affine invariant (see [3] for instance).

- Affine Distance Skeleton - shock points of the solution of the homogeneous Monge-Ampère equation, with double boundary condition

$$\begin{cases} H = 0 \\ J = 1, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

- Affine Area Skeleton - shock points of the solution the Monge-Ampère equations, with double boundary condition

$$\begin{cases} H = -4 \\ J = 0, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

The Monge-Ampère equations open the toolbox of PDE for the affine skeletons issues. As with the Eikonal equation to euclidean distance and medial axis, the Monge-Ampère equations make possible the development of fast algorithms (similar to the Fast Marching Method) to compute the distances and the skeletons.

The Monge-Ampère equations suggest also some extensions:

- The outer Affine Area Skeleton (or the inner skeleton of a non-convex curve Γ) - shock points of the solution of

$$\begin{cases} H = 4 \\ J = 0, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

- The Affine Envelope Skeleton - ridges of the smooth solution of

$$\begin{cases} H = 1 \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

The merits of such proposed extensions, if any, are currently being investigated.

In summary, the PDE formulation of the skeletons issues can enligthen the theory and suggest some extensions. There is a lot of questions to be investigate, such as an appropriate treatment of shocks, the treatment of non-convex curves and the development of good numerical methods.

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