PURE STRATEGY EQUILIBRIA OF SINGLE AND DOUBLE AUCTIONS WITH INTERDEPENDENT VALUES

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Abstract. We prove the existence of monotonic pure strategy equilibrium for many types of asymmetric auctions between n bidders with unitary demands, interdependent values and independent types. The assumptions require monotonicity only in the own bidder's type and the payments can be function of all bids. Thus, we provide a new equilibrium existence result for asymmetrical double auctions.

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1. Introduction

Since the first works on asymmetrical auctions, by Vickrey (1961) and Griesmer, Levitan and Shubik (1967), many theoretical papers have considered the question on the equilibrium existence for such games, among which we can cite Amman and Leininger (1996), Lebrun (1999), Lizzeri and Persico (2000), Maskin and Riley (2000), Athey (2001), Reny and Zamir (2004) and Jackson and Swinkels (2005).[1] The methods to prove the existence of equilibrium are essentially of two kind. The first papers appeal to a system of differential equations whose solution is shown to be an equilibrium. The later ones discretize the space of types or bids, obtain the equilibrium in this case and then proves that the limit when the grid is made fine is an equilibrium.[2]

In this paper, we take another route to prove the equilibrium existence for monotonic asymmetrical auctions with independent types. We allow assumptions weaker than the usual: we do not assume that the utilities are increasing in all types but only on the own bidder's type and the payment can depend on all bids. Thus, we treat in a single framework many kind of asymmetrical auctions with unitary demand, which includes double auctions.

We work on the set of non-decreasing functions, N, and of smooth increasing functions, I. We prove that the set of best response to functions in I is a unitary subset of N. Since I is dense in N and N is compact in the $\mathcal{L}^1$ topology, we find a perturbation of the best response transformation that is a continuous map and has a fixed point. The limit of these fixed points when the perturbation disappears is shown to be an equilibrium.

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[1] While Vickrey (1961), Athey (2001) and Jackson and Swinkels (2005) consider many kind of games and Amman and Leininger (1996) treat all-pay auctions, the rest of these papers are mainly concerned with first-price auctions, as do Griesmer, Levitan and Shubik (1967).

[2] Jackson and Swinkels (2005) can be considered an exception, because they appeal to more general theorems about existence of Nash equilibrium.

The method has the advantage of being simple, direct and general. Because of the generality of our setting, the possibility of ties with positive probability is unavoidable. We prove the existence with an endogenous tie-breaking rule à la Simon and Zame (1990) and Jackson, Simon, Swinkels and Zame (2002).

In section 2, we describe the model and present the preliminary results. In section 3, we present our main results. Section 4 is a discussion about the contributions of the paper. All proofs are collected in section 5.

2. The Model

There are n players: $1,...,n$. Player $i \in \{1,...,n\}$ receives a private information, t_i, and choose an action that is a real number (i.e., she submits a bid b_i). The auctioneer compares the bids and determines who "wins" and who "looses". The rules for this are standard, but are well specified below (see allocation).

If player i wins, she receives $\overline{u}_i(t,b)$ and if she looses, she receives $\underline{u}_i(t,b)$, where $t=(t_i,t_{-i})$ is the profile of all signals and $b=(b_i,b_{-i})$ is the profile of bids submitted.[3]

2.1. Information.

Types are independent. Because $\overline{u}_i(t,b)$ and $\underline{u}_i(t,b)$ can have any form, we may assume without loss of generality that the private signal of each player, t_i, is a real number uniformly distributed on $[0,1]$.[4] To summarize, we are assuming the following:

(A0) *Types are independent and uniformly distributed on* $[0,1]$.

2.2. Bidding.

After receiving the private information, each player submits a sealed proposal, that is, a bid (or offer) that is a real number. There is a reserve price $b_{\min}$ and a maximum allowed bid ($b_{\max}$), which are commonly know.[5] In addition, the bidders can take a non-participation decision $b_{OUT} < b_{\min}$. Then, the space of bids is $B=\{b_{OUT}\}\cup[b_{\min},b_{\max}]$.

2.3. Allocation.

Given a profile of bids $b_{-i}=(b_1,...,b_{i-1},b_{i+1},...,b_n)$, we assume that there is a bid, denoted by $b_{(-i)}$, which determines the threshold of the winning and losing events for bidder i. For instance, if the auction is an one-object auction where all players are buyers, $b_{(-i)}$ is the maximal bid of the opponents, that is, $b_{(-i)} \equiv \max_{j\neq i} b_j$, provided $b_j \geqslant b_{\min}$ for at least one player $j \neq i$. If there are k objects for selling and a reserve price $b_{\min}$, then $b_{(-i)} \equiv \max\left\{b_{\min}, b^{-i}_{(k)}\right\}$, where $b^{-i}_{(k)}$ is the k-th order statistic of $(b_1,...,b_{i-1},b_{i+1},...,b_n)$, that is, $b^{-i}_{(1)} \geqslant b^{-i}_{(2)} \geqslant ... \geqslant b^{-i}_{(n-1)}$.

[3] We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, as the second price auction, for instance. This also allows us to study "exotic" auctions, i.e., auctions where the payment is an arbitrary function of all bids.

[4] Assume that the original type is h_i, distributed in $\left[\underline{h}_i,\overline{h}_i\right]$ according the strictly increasing and continuous c.d.f. $F_i(\cdot)$ and that the value of the object is given by $v_i(h_i,h_{-i})$. Then, we can define $t_i=F_i(h_i)$ and $u_i(t_i,t_{-i}) \equiv v_i\left(F_i^{-1}(t_i), F_{-i}^{-1}(t_{-i})\right)$. Now, the type t_i is uniformly distributed on $[0,1]$. Thus, our assumption rules out just the cases of atoms or gaps in the distribution of types.

[5] If there is no reserve price (in the usual sense), let $b_{\min}=0$. We are assuming a maximum permitted bid to rule out behaviors (equilibria) in which one bidder bids arbitrarily high and the others bid zero. This could happen in third price auctions, for instance.

In double auctions between k sellers and $l = n - k$ buyers, there are k objects for selling and the k highest bids are "winners" in the sense that they end the auction with one object, being the player a buyer or a seller. Then, for a player i (buyer or seller) $b_{(-i)} \equiv \max\left\{b_{\min}, b_{(k)}^{-i}\right\}$.

If $b_i = b_{OUT} < b_{\min}$ (that is, player i does not participate), the payoff is 0. If $b_i > b_{(-i)}$, player i is "holder of an object" (and she has a ex-post payoff $\overline{u}_i(t,b)$ in this situation). If $b_{\min} \leqslant b_i < b_{(-i)}$, player i receives $\underline{u}_i(t,b)$.[6]

Observe that the model permits to treat buyers and sellers in the similar manner. The difference is just that, if player i is a seller, she begins with a object and if $b_i < b_{(-i)}$, she sells her object. If she is a buyer, the situation $b_i < b_{(-i)}$ corresponds to maintain her previous situation: without the object. Also, the model allows for any specification of the price to be paid by the bidders.

If $b_i = b_{(-i)}$, there is a tie and a specific rule (that may include a random device and/or the requirement of a further action) may determine if the player is a winner or a looser.[7] We model this by assuming that there is a function $a : [0,1]^n \to \{0,1\}^n$ such that $\sum_{i=1}^{n} a_i(t) = k \ \forall t \in [0,1]^n$, where k is the number of objects in the auction. Thus, in the case of a tie, bidder i receives $\overline{u}_i(t,b)$ if $a_i(t) = 1$, and $\underline{u}_i(t,b)$ otherwise.[8]

2.4. **Assumptions on the Payoff Functions.** We will assume that the functions $\overline{u}_i$, $\underline{u}_i : [0,1]^n \times B^n \to \mathbb{R}$ satisfy the following conditions for all $i \in I$:

(A1) *$\overline{u}_i$ and $\underline{u}_i$ are absolutely continuous on t and b.*[9]

(A2) *$u_i \equiv \overline{u}_i - \underline{u}_i$ is strictly increasing in t_i.*

(A3) *For all $t = (t_i, t_{-i})$, $t' = (t_i', t_{-i}) \in [0,1]^n$ and $b \in B$, $t_i \leqslant t_i' \Rightarrow \partial_{b_i}\overline{u}_i(b,t) \leqslant \partial_{b_i}\overline{u}_i(b,t')$, when these derivatives exists.*

(A4) *For all $t = (t_i, t_{-i})$, $t' = (t_i', t_{-i}) \in [0,1]^n$ and $b \in B$, $t_i \leqslant t_i' \Rightarrow \partial_{b_i}\underline{u}_i(b,t) \leqslant \partial_{b_i}\underline{u}_i(b,t')$, when these derivatives exists.*

It is worth to discuss the hypotheses. It is standard in auction theory to assume continuity or differentiability of the utility functions. (A1) weaken differentiability, but rules out continuous functions with singular parts. (A2) is a rather weak monotonicity condition. In interdependent value auctions, it is almost always assumed that the functions are increasing in the own bidder's type and nondecreasing in the other types. In contrast, (A2) allows the utility function to be decreasing in the opponents' types.

[6] In most auctions, $\underline{u}_i$ is normalized as 0. However, in double and all-pay auctions or if there is an entry fee, this is not the case.

[7] The required action can be the submission of another bid for a Vickrey auction that will decide who will receive the object (as in Maskin and Riley (2000)) or the announcement of the type (as in Jackson et. al. (2002)). Since the only revealed information in the case of a tie is its occurrence, the action can be required together with the submission of the bid.

[8] The specification of a tie-breaking rule is important for the existence of equilibria, as shown by examples in Simon and Zame (1990) and Jackson *et al.* (2002). (See footnote 10.) With this terminology, the proposal of an "endogenous tie-breaking rule" corresponds to specify endogenously a in order to ensure the equilibrium existence.

[9] Since the domains are compact sets, this implies that the functions are bounded. The absolutely continuity implies the existence of derivatives almost everywhere and that the function is equal to the integral of its derivative.

For instance, it is included in our framework the example 1 of Jackson, Swinkels, Simon and Zame (2002), where $\overline{u}_i(b,t) = 5 + t_i - 4t_{-i} - b_i$ and $\underline{u}_i(b,t) = 0$.[10]

Assumption (A3) and (A4) are weaker versions of super modularity ($\partial^2_{b_i t_i}\overline{u}_i \geqslant 0$ and $\partial^2_{b_i t_i}\underline{u}_i \geqslant 0$). Roughly speaking, this means that a bidder with higher valuation is less sensible to changes in his bid. This assumption is always satisfied in the second price auction. For the first price auction, $\overline{u}_i(t'_i, t_{-i}, b) = U(v(t) - b_i)$, then $\partial^2_{b_i t_i} u_i = U'' \cdot (-1) \cdot v'$. If $v' \geqslant 0$, as usual, then $\partial^2_{b_i t_i} u_i \geqslant 0 \Leftrightarrow U'' \leqslant 0$, i.e., in this setting, super modularity is equivalent to weak risk aversion.

This setting is very general and applies to a broad class of discontinuous games. For example, $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = 0$ correspond to a first price auction with risk neutrality. If $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = -b_i$ we have the all-pay auction. If $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$, this is the war of attrition. As pointed out by Lizzeri and Persico (2000), we can have also combinations of these games. For example, $\overline{u}_i(t,b) = v_i(t) - \alpha b_i - (1-\alpha) b_{(-i)}$ and $\underline{u}_i(t,b) = 0$, with $\alpha \in (0,1)$, gives a combination of the first and second price auctions. Another possibility is the "third price auction" or an auction where the payment is a general function of the others' bids.

The assumptions are easy to be satisfied by double auctions. To see this, assume that bidders 1,..., k are sellers and bidders $k+1$, ..., n are buyers and that the utilities are given as follows:

$$\overline{u}_i(t,b) = \begin{cases} U_i(v_i(t) - e_i), & \text{if } i \in \{1, ..., k\} \\ U_i(v_i(t) - p(b)), & \text{if } i \in \{k+1, ..., n\} \end{cases}$$

and

$$\underline{u}_i(t,b) = \begin{cases} U_i(p(b)), & \text{if } i \in \{1, ..., k\} \\ U_i(-e_i), & \text{if } i \in \{k+1, ..., n\} \end{cases}$$

where $p(b)$ is a payment function that may depend on all bids and e_i is a participation fee (that may be zero). Thus, (A1)-(A4) are satisfied if we have: (i) U_i and v_i are strictly increasing and differentiable for all i; (ii) U_i is concave for all $i \in \{k+1, ..., n\}$; (iii) $p(b)$ is differentiable and non-decreasing in b_i.[11]

2.5. **Notation.** Let $\tilde{N}$ be the set of nondecreasing functions from $[0,1]$ to $B = \{b_{OUT}\} \cup [b_{\min}, b_{\max}]$. Let $\tilde{I}$ be defined as follows the set of functions $\mathbf{g} \in \tilde{N}$ such that $\mathbf{g}$ and is strictly increasing and infinitely differentiable in $(0,1)$ or there is a $\underline{t} \in [0,1]$ such that $\mathbf{g}([0,\underline{t})) = \{b_{OUT}\}$ and $\mathbf{g}$ is strictly increasing and infinitely differentiable in $(\underline{t}, 1)$.

For a function $\mathbf{g} \in \tilde{N}$, let $[\mathbf{g}]$ be the equivalence class of the functions that differ of $\mathbf{g}$ only in a set of zero measure. Now, define N as $\left\{[\mathbf{g}] : \mathbf{g} \in \tilde{N}\right\}$ and I as $\left\{[\mathbf{g}] : \mathbf{g} \in \tilde{I}\right\}$. As usual, we will abuse terminology by calling of functions the equivalence classes. Occasionally, by another abuse of terminology and notation, we may say that functions

[10] For this example, they prove that a standard tie-breaking rule cannot ensure the equilibrium existence for such a game. (See also Jackson et. al. (2004)). As in their case, this example justifies the "endogenous tie-breaking rule" solution concept that we adopt in our proof. See also Araujo, de Castro and Moreira (2004) on another tie-breaking rule.

[11] (A1) and (A2) are immediate. For sellers, (A3) and (A4) hold trivially because $\partial_{b_i}\overline{u}_i$ and $\partial_{b_i}\underline{u}_i$ do not depend on t_i. For buyers, (A4) holds for a similar reason. Now, $t_i \leqslant t'_i$ implies $v_i(t_i, t_{-i}) - p(b) \leqslant v_i(t'_i, t_{-i}) - p(b)$ by (i), which in turn implies $-U'_i(v_i(t_i, t_{-i}) - p(b)) \leqslant -U'_i(v_i(t_i, t_{-i}) - p(b))$ because of (ii). Now, (iii) implies that $\partial_{b_i} p(b) \geqslant 0$ and (A3) follows from the expression $\partial_{b_i}\overline{u}_i(b,t) = -U'_i(v_i(t) - p(b)) \cdot \partial_{b_i} p(b)$.

in I are infinitely differentiable and increasing. We endow N and I with the norm topology of $\mathcal{L}^1([0,1],\mathbb{R})$. It is easy to see that N is compact and I is dense in N.[12,13]

In order to avoid confusion with the bids, we will use bold letters to denote bidding functions, i.e., $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_n) \in N^n$. If we fix the other than i's strategies, $\mathbf{b}_{-i} = (\mathbf{b}_1, ..., \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, ...\mathbf{b}_n)$, let $F_{\mathbf{b}_{(-i)}}(\beta) \equiv \lambda^{n-1}(\{t_{-i} : \mathbf{b}_{-i}(t_{-i}) \leqslant \beta\})$ and $f_{\mathbf{b}_{(-i)}}(\cdot)$ be its Radon-Nykodim derivative with respect to the Lebesgue measure λ^{n-1}, which clearly exists if $\mathbf{b}_{-i} \in I^{n-1}$. Indeed, if $\mathbf{b}_{-i} \in I^{n-1}$ and $\mathbf{b}_{(-i)} = \max_{j\neq i} \mathbf{b}_j(t_j)$, then $F_{\mathbf{b}_{(-i)}}(\beta) = \prod_{j\neq i}\left(\mathbf{b}_j^{-1}(\beta)\right)$ and

$$f_{\mathbf{b}_{(-i)}}(\beta) = \sum_{k\neq i} \frac{1}{\mathbf{b}'_k\left(\mathbf{b}_k^{-1}(\beta)\right)} \prod_{j\neq i, j\neq k} \left(\mathbf{b}_j^{-1}(\beta)\right).$$

For $\mathbf{b}_{-i} \in I^{n-1}$, let us define $b_* \equiv \inf\{\beta \in [b_{\min}, b_{\max}]\colon f_{\mathbf{b}_{(-i)}}(\beta) > 0\}$ and $b^* \equiv \sup\{\beta \in [b_{\min}, b_{\max}]\colon f_{\mathbf{b}_{(-i)}}(\beta) > 0\}$.

If the profile $\mathbf{b}_{-i}$ is fixed, the expected payoff of bidder i of type t_i, when bidding $b_i \in [b_{\min}, b_{\max}]$ is:[14]

$$\begin{aligned}\Pi_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \int &\left[\overline{u}_i(t, b_i, \mathbf{b}_{-i}(t_{-i}))\, 1_{W_i(t_i, b_i, \mathbf{b}_{-i})}\right.\\ &\left.+\underline{u}_i(t, b_i, \mathbf{b}_{-i}(t_{-i}))\, 1_{L_i(t_i, b_i, \mathbf{b}_{-i})}\right] dt_{-i},\end{aligned}$$

where

$$W_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \{t_{-i} \in [0,1]^n : b_i > \mathbf{b}_{(-i)}(t_{-i}) \text{ or } b_i = \mathbf{b}_{(-i)}(t_{-i}) \text{ and } a_i(t) = 1\},$$

and

$$L_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \{t_{-i} \in [0,1]^n : b_i < \mathbf{b}_{(-i)}(t_{-i}) \text{ or } b_i = \mathbf{b}_{(-i)}(t_{-i}) \text{ and } a_i(t) = 0\}.$$

When there is no possibility of confusion, we will write $\Pi_i(t_i, b_i)$ for $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$, $W_i(b_i)$ for $W_i(t, b_i, \mathbf{b}_{-i})$, $L_i(b_i)$ for $L_i(t, b_i, \mathbf{b}_{-i})$ and omit the arguments and the measure (dt_{-i}).

Let $u_i \equiv \overline{u}_i - \underline{u}_i$ be the net payoff.

Finally, we define the interim and the ex-ante best-reply correspondence, respectively, by

$$\Theta_i(t_i, \mathbf{b}_{-i}) \equiv \arg\max_{\beta\in B} \Pi_i(t_i, \beta, \mathbf{b}_{-i}),$$

[12] One way to see the compacity is to remember Helly's Theorem, that says that a sequence of nondecreasing functions has a subsequence that converges pointwise to a nondecreasing function for all the continuity points of the limit function. The pointwise convergence implies the convergence in $\mathbb{L}^1$. Thus, the representative function in each equivalence class $\mathbf{b}_i^m \in N$ has a convergent subsequence that converges to $\mathbf{b}_i \in N$. Another way to see this is to prove that N is totally bounded, constructing, for each $\varepsilon > 0$, a finite covering of N with sets of diameter less than ε. This can be done with step functions for a sufficiently fine grid.

[13] It is well known that $C^\infty$is dense in $\mathcal{L}^p$, for $1 \leqslant p < \infty$. For each function in N, we have a $\underline{t}$ such that in $(\underline{t}, 1)$ the function has values in $[b_{\min}, b_{\max}]$. Thus, one can show that given $\mathbf{g} \in N$ and $\varepsilon > 0$, there is a function $\hat{\mathbf{g}} \in I$ that is strictly increasing and infinitely differentiable in $(\underline{t}, 1)$ such that $\|\mathbf{g} - \hat{\mathbf{g}}\|_1 < \varepsilon$.

[14] We are assuming that $\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = 0$ if $b_i \notin [b_{\min}, b_{\max}]$.

and

$$\Gamma_i\left(\mathbf{b}_{-i}\right) \equiv \arg \max_{\mathbf{b}_i \in \mathcal{L}^1([0,1],B)} V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}\right),$$

where $V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}\right) = \int \Pi_i\left(t_i, \mathbf{b}_i\left(t_i\right), \mathbf{b}_{-i}\right) dt_i$ is the ex-ante payoff.

3. Main Results

Our first result is related to Proposition 1 of Maskin and Riley (2000). Such proposition says that *if* there is a best reply, it is monotonic, but they proved it for first price auctions only. Theorem 1 says that there exists a monotonic best reply to regular functions and it is unique, in the sense made clear in the Remark 1, below.

Theorem 1. Assume (A0)-(A4). Fix a profile $\mathbf{b}_{-i} \in I^{n-1}$. Then, for each t_i, $\Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ is non-empty. Moreover, if $t_i^1 < t_i^2$, $b_i^1 \in \Theta_i\left(t_i^1, \mathbf{b}_{-i}\right)$, $b_i^2 \in \Theta_i\left(t_i^2, \mathbf{b}_{-i}\right)$, b_i^1, $b_i^2 \in [b_*, b^*]$, then $b_i^1 \leqslant b_i^2$.

Remark 1. Theorem 1 has an important consequence. It implies that if $\mathbf{b}_{-i} \in I^{n-1}$, then $\Gamma_i\left(\mathbf{b}_{-i}\right)$ is a unitary set in N. To see why, let $\mathbf{b}_{-i} \in I^{n-1}$. Theorem 1 implies that the sets $\Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ and $\Theta_i\left(t_i', \mathbf{b}_{-i}\right)$ has at most one point in common if $t_i \neq t_i'$. Thus, the set of types t_i where $\Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ has diameter greater than $\varepsilon > 0$ is finite. Then, $\Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ is uni-valued except for a countable set of types t_i. Thus, the correspondence $t_i \longmapsto \Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ has a unique selection in $\mathcal{L}^1$. By the definition of $\Gamma_i\left(\mathbf{b}_{-i}\right)$, we conclude that this selection is the unique function in $\Gamma_i\left(\mathbf{b}_{-i}\right)$, and it is non-decreasing.$\blacksquare$

Remark 1 made clear that the function $t_i \longmapsto \Theta_i\left(t_i, \mathbf{b}_{-i}\right)$ is in N if $\mathbf{b}_{-i} \in I^{n-1}$. Thus, for each $i = 1, ..., n$, the correspondence of best replies to $\mathbf{b}_{-i}$ is, in fact a function $\Gamma_i : I^{n-1} \to N$. Let us prove that Γ_i is continuous.

Consider a sequence $\left\{\mathbf{b}_{-i}^m\right\}_{m\in\mathbb{N}} \subset I^{n-1}$, $\mathbf{b}_{-i}^m \to \overline{\mathbf{b}}_{-i}$, $\overline{\mathbf{b}}_{-i} \in I^{n-1}$, and let $\overline{\mathbf{b}}_i \equiv \Gamma_i\left(\overline{\mathbf{b}}_{-i}\right)$. Consider also $\{\mathbf{b}_i^m\}_m \subset N$, $\mathbf{b}_i^m = \Gamma_i\left(\mathbf{b}_{-i}^m\right)$. Then, $V_i\left(\mathbf{b}_i^m, \mathbf{b}_{-i}^m\right) \geqslant V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}^m\right)$, $\forall \mathbf{b}_i \in \mathcal{L}^1$. In particular, $V_i\left(\mathbf{b}_i^m, \mathbf{b}_{-i}^m\right) \geqslant V_i\left(\overline{\mathbf{b}}_i, \mathbf{b}_{-i}^m\right)$. Since N is compact, there is a subsequence of $\mathbf{b}_i^m$ converging to a function $\mathbf{b}_i$. Since $\mathbf{b}_{-i}$ is strictly increasing, V_i is continuous at $\left(\mathbf{b}_i, \overline{\mathbf{b}}_{-i}\right)$. Then, we have $V_i\left(\mathbf{b}_i, \overline{\mathbf{b}}_{-i}\right) \geqslant V_i\left(\overline{\mathbf{b}}_i, \overline{\mathbf{b}}_{-i}\right)$ by the continuity and $V_i\left(\overline{\mathbf{b}}_i, \overline{\mathbf{b}}_{-i}\right) \geqslant V_i\left(\mathbf{b}_i, \overline{\mathbf{b}}_{-i}\right)$ because $\overline{\mathbf{b}}_i \equiv \Gamma_i\left(\overline{\mathbf{b}}_{-i}\right)$. But then, $\mathbf{b}_i$ is also a best-reply, what we have seen to be unique. Hence, $\overline{\mathbf{b}}_i \equiv \mathbf{b}_i$ and Γ_i is continuous.

Now, for each $\mathbf{b}_i \in I$, let $U^m\left(\mathbf{b}_i\right)$ be the open set (with respect to N)

$$U^m\left(\mathbf{b}_i\right) = \{\widetilde{\mathbf{b}}_i \in N : \left\|\widetilde{\mathbf{b}}_i - \mathbf{b}_i\right\|_1 < \frac{1}{m}\},$$

where $\|\cdot\|_1$ is the norm of $\mathcal{L}^1$. It is easy to see that $\cup_{\mathbf{b}_i\in I} U^m\left(\mathbf{b}_i\right)$ is a open cover of N. Since N is compact, it has a finite subcover. Let K^m be the finite set of indices λ such that $\cup_{\lambda\in K^m} U^m\left(\mathbf{b}_i^\lambda\right) = N$ and $\mathbf{b}_i^\lambda \in I$, $\forall \lambda \in K^m$. Let $\left\{\psi^\lambda\right\}_{\lambda\in K^m}$ be a partition of the unity subordinate to this finite open cover. That is, $\psi^\lambda : N \to [0,1]$, $\sum_{\lambda\in K^m} \psi^\lambda\left(\mathbf{b}_i\right) = 1$ for all $\mathbf{b}_i \in N$ and $\psi^\lambda\left(\mathbf{b}_i\right) = 0$, unless $\mathbf{b}_i \in U^m\left(\mathbf{b}_i^\lambda\right)$. Define the continuous transformation:

$$\Lambda_i^m\left(\mathbf{b}_i\right) = \sum_{\lambda\in K^m} \psi^\lambda\left(\mathbf{b}_i\right) \mathbf{b}_i^\lambda.$$

Since $\mathbf{b}_i^{\lambda} \in I$, $\forall \lambda \in K^m$, $\Lambda_i^m(\mathbf{b}_i) \in I$.[15]

It is clear that $\Lambda_i^m : N \to I$ is continuous. Now we define $\Lambda_{-i}^m : N^{n-1} \to I^{n-1}$ as $\Lambda_{-i}^m \equiv \times_{j \neq i} \Lambda_j^m$ and $\Lambda^m : N^n \to I^n$ as $\Lambda^m \equiv \left(\Lambda_i^m, \Lambda_{-i}^m\right)$. We can conclude that for all $m \in \mathbb{N}$, the transformation $\Gamma \circ \Lambda^m : N^n \to N^n$, defined by

$$\Gamma \circ \Lambda^m(\mathbf{b}) \equiv \left(\Gamma_i\left(\Lambda_{-i}^m(\mathbf{b}_{-i})\right)\right)_{i=1}^n,$$

is continuous. By the Schauder-Tychonoff Theorem, $\Gamma \circ \Lambda^m$ has a fixed point, which we denote by $\mathbf{b}^m$.[16]

To understand the meaning of $\mathbf{b}_i^m$, suppose that for all $j \neq i$, player j follows $\mathbf{b}_j^m$, but player $i \neq j$ mistakenly considers that every player $j \neq i$ is using strategy $\Lambda_j^m\left(\mathbf{b}_j^m\right)(\cdot)$. Then, the best strategy for bidder i is to follow $\mathbf{b}_i^m$.

Now, since N is compact in the strong topology of $\mathcal{L}^1$, there is a convergent subsequence that converges to a bidding function $\mathbf{b}^*$. Now, we have just to prove that $\mathbf{b}^*$ is equilibrium, that is,

$$V_i\left(\mathbf{b}_i^*, \mathbf{b}_{-i}^*\right) \geqslant V_i\left(\mathbf{b}_i, \mathbf{b}_{-i}^*\right), \forall \mathbf{b}_i \in \mathcal{L}^1([0,1], B), \forall i.$$

Equivalently, we need to show that for all i and almost all $t_i \in [0,1]$,

$$\Pi_i\left(t_i, \mathbf{b}_i^*(t_i), \mathbf{b}_{-i}^*\right) \geqslant \Pi_i\left(t_i, \beta, \mathbf{b}_{-i}^*\right), \forall \beta \in B.$$

If $\mathbf{b}^*$ does not have ties with positive probability, the event $\{t \in T : \mathbf{b}_i^*(t_i) = \mathbf{b}_{(-i)}^*(t_{-i})$ for at least one player $i\}$ has zero measure. Then, the continuity of $\overline{u}_i$ and $\underline{u}_i$ imply that V_i is continuous. The result now follows from the definition of $\mathbf{b}_i^m$, that says that $V_i\left(\mathbf{b}_i^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \geqslant V_i\left(\mathbf{b}_i, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right)$, $\forall \mathbf{b}_i$, $\forall i$. Thus, we need only to deal with the possibility of ties.

To check this, let us define the allocation function $a^m : [0,1]^n \to \{0,1\}^n$ as

$$a^m(t) = \left(a_1^m(t), ..., a_n^m(t)\right),$$

where

$$a_i^m(t_i, t_{-i}) = \begin{cases} 1, & \text{if } \Lambda_i^m(\mathbf{b}_i^m)(t_i) > \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) \\ 0, & \text{if } \Lambda_i^m(\mathbf{b}_i^m)(t_i) < \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) \end{cases}$$

Observe that $a_i^m(t)$ is well defined for almost all t because $\Lambda_i^m(\mathbf{b}_i^m)$ and $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)$ are increasing. Observe also that $\sum_{i=1}^n a_i^m(t) = k$, the number of objects in the auction, as required to an allocation rule.

Thus, $a^m(t)$ is well defined in $\mathcal{L}^1([0,1]^n, \{0,1\}^n)$ for all $m \in \mathbb{N}$. The set $\{a^m\}_{m \in \mathbb{N}}$ is compact in $\mathcal{L}^1([0,1]^n, \{0,1\}^n)$. To see this, observe that for each i, $a_i^m(t_i, t_{-i})$ is nondecreasing in t_i and non-increasing in t_{-i}. Thus, for each i, $\{a_i^m(\cdot)\}_{m \in \mathbb{N}}$ is compact and the claim follows. So, there is a convergent subsequence (that we will denote by

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[15] If $t_i < t_i'$, $\mathbf{b}_i^{\lambda}(t_i) < \mathbf{b}_i^{\lambda}(t_i')$ and $\psi^{\lambda}(\mathbf{b}_i)\mathbf{b}_i^{\lambda}(t_i) < \psi^{\lambda}(\mathbf{b}_i)\mathbf{b}_i^{\lambda}(t_i')$ for each $\lambda \in K_i^m$ such that $\psi^{\lambda}(\mathbf{b}_i) > 0$. Then, $\Lambda_i^m(\mathbf{b}_i)(t_i) < \Lambda_i^m(\mathbf{b}_i)(t_i')$, since they are finite sums of $\psi^{\lambda}(\mathbf{b}_i)\mathbf{b}_i^{\lambda}(t_i)$ and $\psi^{\lambda}(\mathbf{b}_i)\mathbf{b}_i^{\lambda}(t_i')$, respectively. The differentiability comes from similar argument.

[16] A reference for Schauder-Tychonoff Theorem is Theorem V.10.5, p. 456, of Dunford and Schwartz (1958). Observe that N is convex and compact.

the same superscript), $a^m(t) \to a(t)$, which is also an allocation rule. From now, fix such function $a(t)$.

Tie-Breaking Rule: If there is a tie at β, all bidders are requested to reveal their types and the final allocation is given by $a(t)$, that is, bidder i receives an object if and only if $a_i(t) = 1$. Nevertheless, if a bidder announces an inconsistent type, that is, t_i' such that $\mathbf{b}_i^*(t_i') \neq \beta$, he receives a (sufficiently great) penalty.[17]

With the Tie-Breaking Rule just defined (which is an endogenous tie-breaking rule), we have equilibrium. This will follow from two lemmas. The first shows that there is no profitable deviation from bidding differently of the bid specified by $\mathbf{b}^*$. The second says that it is optimum to state the true type in case of bidding.

Lemma 2. If $\mathbf{b}_i^m(t_i) \to \mathbf{b}_i^*(t_i) = \beta$, there is no $\beta' \in B$ such that

$$\Pi_i\left(t_i, \mathbf{b}_i^*(t_i), \mathbf{b}_{-i}^*\right) < \Pi_i\left(t_i, \beta', \mathbf{b}_{-i}^*\right).$$

The idea of the proof is very simple. If there is such β', then β' would be a profitable deviation along the sequence, which is impossible because $\mathbf{b}_i^m(t_i)$ is the best reply, by definition. For the next lemma, the idea of proof is similar. (See the details in the appendix.)

Lemma 3. In case of a tie, it is optimum for all bidders to reveal their true types.

These two lemmas prove the following:

Theorem 2. Assume (A0)-(A4) and the tie-breaking rule just specified. Then, there exists a pure strategy non-decreasing equilibrium.

One question that can arise is whether we could use Athey's proof to obtain our results. A slight modification of the proof of Theorem 1 can show that the game considered satisfies single crossing of incremental returns (SCP-IR) in (b_i, t_i) (see Athey (2001), Definition 1), for $\mathbf{b}_{-i} \in I^{n-1}$. Nevertheless, we didn't prove that the game in general does not satisfy Athey's Single Crossing Condition, because this requires that SCP-IR holds for non-decreasing $\mathbf{b}_{-i}$.[18] Moreover, even if we could establish the SCC, we are yet not able to apply Athey (2001)'s Theorems 6 or 7, because our setting does not satisfy her assumptions (A2)(iv), which requires that u_i is strictly decreasing with b_i, which rules out double auctions. In sum, our results cannot be derived from Athey (2001).

[17] Thus, the tie-breaking rule depends on the sequence and on the equilibrium bidding functions, $\mathbf{b}^*$. Observe that is never optimal to announce a type t_i' such that $\mathbf{b}_i^*(t_i') \neq \beta$. We could avoid such penalty by the requirement that the allocation rule $a(t)$ also depends on the bids.

[18] Reny and Zamir (2004) modify Athey's SCC for an another condition: individually rational tieless single crossing condition (IRT-SCC). One can modify the arguments in the proof of Theorem 1 to establish this.

4. Conclusion

Now, we review the contributions of this paper in the light of the received literature. Jacskon, Simon, Swinkels and Zame (2002) prove the existence of asymmetrical mixed strategy equilibrium with any distribution of types. They used the "endogenously defined" tie-breaking rule solution concept introduced by Simon and Zame (1990), as we also do. We particularize their assumptions to the independent types' case, but we are able to obtain the existence in monotonic pure strategies.

Athey (2001), for general games, and Reny and Zamir (2004), for first-price auctions, obtained monotonic pure strategy equilibrium without special tie-breaking rules. Nevertheless, they assumed the monotonicity of the utilities with respect to all types and do not consider double auctions, as we do.

Williams (1991) consider symmetric double auctions with independent types. Jackson and Swinkels (2005) consider (multi-unit) asymmetrical double auctions with general distribution of types, but they are restricted to the private values case. In this setting, they are able to prove that the tie-breaking rule does not matter, a result that does not hold in our setting, as we already argued. Reny and Perry (2003) and Fundenberg, Mobius and Szeil (2003) consider symmetrical double-auctions with conditionally independent types but are able to prove the existence of equilibrium just when the number of players is high. Thus, these works do not cover our equilibrium existence result for asymmetrical double auctions with interdependent values.

Appendix

Proof of Theorem 1.

The prove of Theorem 1 requires the following lemma.

Lemma 1. Assume (A0). Fix a profile of bidding functions $\mathbf{b}_{-i} \in I^{n-1}$. The payoff can be expressed by

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \Pi_i(t_i, b_*) + \int_{[b_*, b_i)} \partial_{b_i} \Pi_i(t_i, \beta) d\beta.$$

where $\partial_{b_i} \Pi_i(t_i, \beta)$ exists for almost all $\beta \in (b_*, b^*)$ and is given by

$$\begin{aligned} \partial_\beta \Pi_i(t_i, \beta, \mathbf{b}_{-i}) &= E\left[\partial_{b_i} \overline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]}\right] \\ &+ E\left[\partial_{b_i} \underline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right] \\ &+ E\left[u_i\left(t, \beta, \mathbf{b}_{-i}\left(t_{-i}\right)\right) | \mathbf{b}_{-i}\left(t_{-i}\right) = \beta\right] f_{\mathbf{b}_{-i}}(\beta) . \end{aligned} \tag{1}$$

This lemma can be proved using the Leibiniz's rule. For a proof in a more general setting, see de Castro (2004). Now, we proceed to the proof of Theorem 1.

Fix types $t_i^1 < t_i^2$, $b_i^1 \in \Theta_i\left(t_i^1, \mathbf{b}_{-i}\right)$ and $b_i^2 \in \Theta_i\left(t_i^2, \mathbf{b}_{-i}\right)$, where $\mathbf{b}_{-i}$ is a fixed regular strategy. For a contradiction, suppose that $b_i^2 < b_i^1$ and that the support of the distribution of $\mathbf{b}_{-i}$ has a non-trivial intersection with $\left[b_i^2, b_i^1\right]$ (remember that by assumption, $F_{\mathbf{b}_{(-i)}}\left(b_i^1\right) > 0$). Since $[0,1]^{n-1}$ and B^n are compact and u_i is (absolutely) continuous, there exists $\delta > 0$ such that $u_i\left(t_i^1, t_{-i}, b\right) + 2\delta < u_i\left(t_i^2, t_{-i}, b\right)$ for all $t_{-i} \in$

$[0,1]^{n-1}$ and all $b \in B^n$. For a bid $\beta \in B$, define the functions

$$g^1(t_{-i}) = u_i\left(t_i^1, t_{-i}, \beta, \mathbf{b}_{-i}(t_{-i})\right), \text{ and}$$
$$g^2(t_{-i}) = u_i\left(t_i^2, t_{-i}, \beta, \mathbf{b}_{-i}(t_{-i})\right).$$

Then, $g^1(t_{-i}) + 2\delta < g^2(t_{-i})$. By the positivity of conditional expectations,[19]

$$E\left[g^2 - g^1 - 2\delta | \mathbf{b}_{(-i)} = \beta\right] \geqslant 0.$$

So, by the independence (A1), we conclude that

$$E[u_i\left(t_i^1, \cdot\right) | \mathbf{b}_{(-i)} = \beta] + \delta < E[u_i\left(t_i^2, \cdot\right) | \mathbf{b}_{(-i)} = \beta]. \tag{2}$$

By assumptions A3 and A4,

$$E\left[\partial_{b_i} \overline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]}\right] \leqslant E\left[\partial_{b_i} \overline{u}_i\left(t_i^2, \cdot\right) 1_{\left[\beta > \mathbf{b}_{(-i)}\right]}\right]. \tag{3}$$

and

$$E\left[\partial_{b_i} \underline{u}_i\left(t_i^1, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right] \leqslant E\left[\partial_{b_i} \underline{u}_i\left(t_i^2, \cdot\right) 1_{\left[\beta < \mathbf{b}_{(-i)}\right]}\right]. \tag{4}$$

Then, (2), (3), (4) and the expression of $\partial_{b_i} \Pi_i(t_i, \beta, \mathbf{b}_{-i})$ given by (1) imply that for almost all β,

$$\partial_{b_i} \Pi_i(t_i^2, \beta, \mathbf{b}_{-i}) > \partial_{b_i} \Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) + \delta f_{\mathbf{b}_{(-i)}}(\beta). \tag{5}$$

Since $\mathbf{b}_{-i}$ is regular, the difference $\Pi_i(t_i^2, b_i^1, \mathbf{b}_{-i}) - \Pi_i(t_i^2, b_i^2, \mathbf{b}_{-i})$ can be written as the integral:

$$\begin{aligned}
\int_{\left[b_i^2, b_i^1\right)} \partial_{b_i} \Pi_i(t_i^2, \beta, \mathbf{b}_{-i}) d\beta &> \int_{\left[b_i^2, b_i^1\right)} \partial_{b_i} \Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) d\beta + \delta \int_{\left[b_i^2, b_i^1\right)} f_{\mathbf{b}_{(-i)}}(\beta)\, d\beta \\
&\geqslant \delta\left[F_{\mathbf{b}_{(-i)}}\left(b_i^1\right) - F_{\mathbf{b}_{(-i)}}\left(b_i^2\right)\right] \\
&\geqslant 0,
\end{aligned}$$

where the first inequality comes from (5);[20] the second comes from the fact that $b_i^1 \in \Theta_i\left(t_i^1, \mathbf{b}_{-i}\right)$, that is,

$$\int_{\left[b_i^2, b_i^1\right)} \partial_{b_i} \Pi_i(t_i^1, \beta, \mathbf{b}_{-i}) d\beta \geqslant 0;$$

and the third comes the assumption that $b_i^1 > b_i^2$. Now, this implies that $\Pi_i\left(t_i^2, b_i^1, \mathbf{b}_{-i}\right) > \Pi_i\left(t_i^2, b_i^2, \mathbf{b}_{-i}\right)$, which contradicts the fact that $b_i^2 \in \Theta_i\left(t_i^2, \mathbf{b}_{-i}\right)$.■

Proof of Lemma 2.

For this and the next lemma, we will use the following notation:

$$W_i^m(\beta) = \left\{t_{-i} \in [0,1]^{n-1} : \beta > \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right\};$$
$$L_i^m(\beta) = \left\{t_{-i} \in [0,1]^{n-1} : \beta < \Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right\}.$$

Observe that it is consistent with the allocation rule a^m, as defined in the text.

Fix a type t_i and let us denote $\mathbf{b}_i^*(t_i)$ by β^* and $\Lambda_i^m\left(\mathbf{b}_i^m\right)(t_i)$ by β^m.By contradiction, suppose that there is a bid β' and $\eta > 0$ such that

[19]See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

[20]The fact that is strict depends on $F_{\mathbf{b}_{(-i)}}\left(b_i^1\right) > 0$, otherwise $\Pr\left[b_i^2, b_i^1\right) = 0$ and the integrals would be zero in both sides.

$$\Pi_i(t_i, \beta', \mathbf{b}_{-i}) - \Pi_i\left(t_i, \beta^*, \mathbf{b}^*_{-i}\right) > \eta.$$

To fix ideas, suppose that $\beta' > \beta^*$ (the other case is completely analogous). Then, $W_i\left(\beta^*\right) \subset W_i\left(\beta'\right)$ and $L_i\left(\beta^*\right) \supset L_i\left(\beta'\right)$, where

$$W_i\left(\beta\right) = \{t_{-i} \in [0,1]^n : \beta > \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ or } \beta = \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ and } a_i\left(t\right) = 1\},$$

and

$$L_i\left(\beta\right) = \{t_{-i} \in [0,1]^n : \beta < \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ or } \beta = \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ and } a_i\left(t\right) = 0\}.$$

From the definitions , we have that (passing to a subsequence, if needed) when $m \to \infty$, $\beta^m \to \beta^*$, $1_{W_i^m(\beta^m)} \to 1_{W_i(\beta^*)}$, $1_{L_i^m(\beta^m)} \to 1_{L_i(\beta^*)}$, $\Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right) \to \mathbf{b}^*_{-i}\left(t_{-i}\right)$ for almost all t_{-i}.

Now, observe that

$$\begin{aligned}
&\Pi_i(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)) - \Pi_i\left(t_i, \beta^*, \mathbf{b}^*_{-i}\right) \\
&= \left(\int_{W_i^m(\beta^m)} \overline{u}_i + \int_{L_i^m(\beta^m)} \underline{u}_i\right) - \left(\int_{W_i(\beta^*)} \overline{u}_i + \int_{L_i(\beta^*)} \underline{u}_i\right) \\
&= \int_{W_i(\beta^*)} \left[\overline{u}_i\left(t_i, t_{-i}, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right) - \overline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right] \\
&+ \int_{L_i^m(\beta^m)} \left[\underline{u}_i\left(t_i, t_{-i}, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right) - \underline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right] \\
&+ \int_{W_i^m(\beta^m)\backslash W_i(\beta^*)} \overline{u}_i\left(t_i, t_{-i}, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\left(t_{-i}\right)\right) \\
&- \int_{L_i(\beta^*)\backslash L_i^m(\beta^m)} \underline{u}_i\left(t_i, t_{-i}, \beta^*, \mathbf{b}^*_{-i}\left(t_{-i}\right)\right).
\end{aligned}$$

From the continuity of the $\overline{u}_i$ and $\underline{u}_i$ and the limits, we have that for sufficiently high m,

$$\left|\Pi_i(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)) - \Pi_i\left(t_i, \beta^*, \mathbf{b}^*_{-i}\right)\right| < \frac{\eta}{3}.$$

Similarly, we obtain that

$$\left|\Pi_i(t_i, \beta', \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)) - \Pi_i\left(t_i, \beta', \mathbf{b}^*_{-i}\right)\right| < \frac{\eta}{3}.$$

Thus,

$$\Pi_i(t_i, \beta', \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)) - \Pi_i(t_i, \beta^m, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)) > \frac{\eta}{3} > 0,$$

which is an absurd, since

$$\beta^m \in \Theta_i\left(t_i, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right) = \arg\max_{\beta \in B} \Pi_i\left(t_i, \beta, \Lambda^m_{-i}\left(\mathbf{b}^m_{-i}\right)\right).$$

This concludes the proof.■

Proof of Lemma 3.
Now, we have to distinguish the winning events for the announced types. So, let

$$W_i\left(\beta, \tilde{t}_i\right) = \{t_{-i} \in [0,1]^n : \beta > \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ or } \beta = \mathbf{b}^*_{(-i)}\left(t_{-i}\right) \text{ and } a_i\left(\tilde{t}_i, t_{-i}\right) = 1\},$$

and

$$L_i\left(\beta,\tilde{t}_i\right)=\{t_{-i}\in[0,1]^n:\beta<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ or }\beta=\mathbf{b}^*_{(-i)}\left(t_{-i}\right)\text{ and }a_i\left(\tilde{t}_i,t_{-i}\right)=0\},$$

By contradiction assume that there is a type $\hat{t}_i\neq t_i$ such that for $\eta>0$, we have

$$\left(\int_{W_i\left(\beta^*,\hat{t}_i\right)}\overline{u}_i+\int_{L_i\left(\beta^*,\hat{t}_i\right)}\underline{u}_i\right)-\left(\int_{W_i\left(\beta^*,t_i\right)}\overline{u}_i+\int_{L_i\left(\beta^*,t_i\right)}\underline{u}_i\right)>10\eta, \tag{6}$$

where we are denoting $\mathbf{b}^*_i\left(t_i\right)$ by β^*. Let us also denote $\Lambda^m_i\left(\mathbf{b}^m_i\right)\left(t_i\right)$ by β^m and $\Lambda^m_i\left(\mathbf{b}^m_i\right)\left(\hat{t}_i\right)$ by $\hat{\beta}^m$.

To fix ideas, assume that $t'_i>t_i$, so that $W_i\left(\beta^*,t_i\right)\subset W_i\left(\beta^*,t'_i\right)$ and $L_i\left(\beta^*,t_i\right)\supset L_i\left(\beta^*,t'_i\right)$, because $\mathbf{b}^m_{-i}$, $\mathbf{b}^*_{-i}\in N^{n-1}$. Simplifying the expression above, we obtain:

$$\begin{aligned}&\int_{W_i\left(\beta^*,t_i\right)}\left[\overline{u}_i\left(t_i,t_{-i},\beta^*,\mathbf{b}^*_{-i}\left(t_{-i}\right)\right)-\overline{u}_i\left(t_i,t_{-i},\beta^*,\mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right]\\&+\int_{L_i\left(\beta^*,t'_i\right)}\left[\underline{u}_i\left(t_i,t_{-i},\beta^*,\mathbf{b}^*_{-i}\left(t_{-i}\right)\right)-\underline{u}_i\left(t_i,t_{-i},\beta^*,\mathbf{b}^*_{-i}\left(t_{-i}\right)\right)\right]\\&+\int_{W_i\left(\beta^*,t'_i\right)\backslash W_i\left(\beta^*,t_i\right)}\overline{u}_i\left(\beta^*,\cdot\right)-\int_{L_i\left(\beta^*,t_i\right)\backslash L_i\left(\beta^*,t'_i\right)}\underline{u}_i\left(\beta^*,\cdot\right)\end{aligned}$$

The first two integrals are zero. Observe that the set $W_i\left(\beta^*,t'_i\right)\backslash W_i\left(\beta^*,t_i\right)$ is exactly $L_i\left(\beta^*,t_i\right)\backslash L_i\left(\beta^*,t'_i\right)$. Let us call it A. It is easy and useful to see that

$$A\subset\left\{t_{-i}:\mathbf{b}^*_{(-i)}\left(t_{-i}\right)=\beta^*\right\}. \tag{7}$$

If we remember that $u_i\equiv\overline{u}_i-\underline{u}_i$, we can rewrite (6) as

$$\int_A u_i(t_i,t_{-i},\beta^*,\mathbf{b}^*_{-i}\left(t_{-i}\right))dt_{-i}>10\eta. \tag{8}$$

Let M be an upper bound for $\max\left\{\left|u_i\right|,\left|\underline{u}_i\right|,\left|\overline{u}_i\right|\right\}$. Because $\mathbf{b}^*_{-i}\left(\cdot\right)$ is nondecreasing, there exists $\delta_1>0$ such that

$$\Pr\{t_{-i}:\beta^*-2\delta_1<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)<\beta^*\}<\eta/M. \tag{9}$$

and

$$\Pr\{t_{-i}:\beta^*<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)<\beta^*+2\delta_1\}<\eta/M. \tag{10}$$

Indeed, this comes from the continuity of the probability:

$$\begin{aligned}&\lim_{\delta\downarrow 0}\Pr\left(\{t_{-i}:\beta^*-2\delta<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)<\beta^*\}\right)\\&=\Pr\left(\bigcap_{\delta>0}\{t_{-i}:\beta^*-2\delta<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)<\beta^*\}\right)\\&=0,\end{aligned}$$

and analogously for $\Pr\{t_{-i}:\beta^*<\mathbf{b}^*_{(-i)}\left(t_{-i}\right)<\beta^*+2\delta_1\}$.

Since $\overline{u}_i$, $\underline{u}_i$ and u_i are absolutely continuous, there exists $\delta_2>0$, such that for all t_i, t_{-i}, b_i, b_{-i}, b'_{-i}, β'' and β',

$$\left|\beta''-\beta'\right|<4\delta_2\Rightarrow\left|\overline{u}_i(t_i,t_{-i},\beta'',b_{-i})-\overline{u}_i(t_i,t_{-i},\beta',b_{-i})\right|<\eta, \tag{11}$$

$$\left|\beta'' - \beta'\right| < 4\delta_2 \Rightarrow \left|\underline{u}_i(t_i, t_{-i}, \beta'', b_{-i}) - \underline{u}_i(t_i, t_{-i}, \beta', b_{-i})\right| < \eta. \tag{12}$$

There exists δ_3 such that

$$\max_{k \neq i} \left|b_k - b'_k\right| < 4\delta_3 \Rightarrow \left|u_i(t_i, t_{-i}, b_i, b_{-i}) - u_i(t_i, t_{-i}, b_i, b'_{-i})\right| < \eta. \tag{13}$$

Fix $0 < \delta < \min\{\delta_1, \delta_2, \delta_3\}$.

The functions $\Lambda_j^m\left(\mathbf{b}_j^m\right)$ are nondecreasing and converge to $\mathbf{b}_j^*$. Moreover, there exists a set $U \subset [0,1]^{n-1}$ such that $\Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right) \to \mathbf{b}_{-i}^*$ uniformly on U and such that $\Pr\left([0,1]^{n-1} \backslash U\right) < \eta/M$. So, there exists m_1 such that $m \geqslant m_1$ implies that

$$\sup_{t_{-i} \in U} \max_{j \neq i} \left|\Lambda_j^m\left(\mathbf{b}_j^m\right)(t_j) - \mathbf{b}_j^*(t_j)\right| < \delta. \tag{14}$$

Also, there is m_2 such that $m \geqslant m_2$ implies $|\beta^m - \beta^*| < \delta$. We will define

$$\begin{aligned} A^m &\equiv W_i^m\left(\hat{\beta}^m\right) \backslash W_i^m\left(\beta^m\right) \\ &= \left\{t_{-i} : a^m\left(t'_i, t_{-i}\right) = 1 \text{ and } a^m\left(t_i, t_{-i}\right) = 0\right\}. \end{aligned}$$

Remember that, since $\mathbf{b}_i^*\left(t'_i\right) = \mathbf{b}_i^*\left(t_i\right) = \beta^*$,

$$\begin{aligned} A &\equiv W_i\left(\beta^*, t'_i\right) \backslash W_i\left(\beta^*, t_i\right) \\ &= \left\{t_{-i} : a\left(t'_i, t_{-i}\right) = 1 \text{ and } a\left(t_i, t_{-i}\right) = 0\right\}. \end{aligned}$$

We know that $a^m \to a$ in $\mathcal{L}^1$. Finally, there is m_3 such that $m \geqslant m_3$ implies

$$\Pr\left(A^m \Delta A\right) < \frac{\eta}{M}. \tag{15}$$

Fix some $m > \max\{m_1, m_2, m_3\}$. We have:

$$\begin{aligned} &\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\ &= \int_{W_i^m\left(\hat{\beta}^m\right)} \overline{u}_i(t_i, t_{-i}, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})) \\ &+ \int_{L_i^m\left(\hat{\beta}^m\right)} \underline{u}_i(t_i, t_{-i}, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})) \\ &- \int_{W_i^m\left(\beta^m\right)} \overline{u}_i(t_i, t_{-i}, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})) \\ &- \int_{L_i^m\left(\beta^m\right)} \underline{u}_i(t_i, t_{-i}, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})) \end{aligned}$$

From now on, we will substitute the arguments $(t_i, t_{-i}, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))$ and $(t_i, t_{-i}, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))$ by $(\hat{\beta}^m, \cdot)$ and $(\beta^m, \cdot)$, respectively. Since $u_i > -M$, $\underline{u}_i >$

$-M$ and $\overline{u}_i > -M$, we have:

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&> \int_{U\cap W_i^m(\hat{\beta}^m)} \overline{u}_i(\hat{\beta}^m, \cdot) + \int_{U\cap L_i^m(\hat{\beta}^m)} \underline{u}_i(\hat{\beta}^m, \cdot) \\
&- \int_{U\cap W_i^m(\beta^m)} \overline{u}_i(\beta^m, \cdot) - \int_{U\cap L_i^m(\beta^m)} \underline{u}_i(\beta^m, \cdot) \\
&+ \int_{[0,1]^{n-1}\setminus U} (-M)
\end{aligned}$$

Since $\Pr\left([0,1]^{n-1}\setminus U\right) < \eta/M$, the last integral is greater than $-\eta$. Rearranging the terms,

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&> -\eta + \int_{U\cap W_i^m(\hat{\beta}^m)\setminus W_i^m(\beta^m)} \left[\overline{u}_i(\hat{\beta}^m, \cdot) - \underline{u}_i(\beta^m, \cdot)\right] \\
&+ \int_{U\cap W_i^m(\beta^m)} \left[\overline{u}_i(\hat{\beta}^m, \cdot) - \overline{u}_i(\beta^m, \cdot)\right] \\
&+ \int_{U\cap L_i^m(\hat{\beta}^m)} \left[\underline{u}_i(\hat{\beta}^m, \cdot) - \underline{u}_i(\beta^m, \cdot)\right]
\end{aligned}$$

From (14) and (11),

$$\int_{U\cap W_i^m(\beta^m)} \left[\overline{u}_i(\hat{\beta}^m, \cdot) - \overline{u}_i(\beta^m, \cdot)\right] > \int_{U\cap W_i^m(\beta^m)} (-\eta) \geqslant -\eta.$$

Analogously, from (14) and (12),

$$\int_{U\cap L_i^m(\hat{\beta}^m)} \left[\underline{u}_i(\hat{\beta}^m, \cdot) - \underline{u}_i(\beta^m, \cdot)\right] > \int_{U\cap L_i^m(\hat{\beta}^m)} (-\eta) \geqslant -\eta$$

Remember that $A^m \equiv W_i^m\left(\hat{\beta}^m\right) \setminus W_i^m\left(\beta^m\right)$. Thus,

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&\geqslant \int_{U\cap A^m} \left[\overline{u}_i(\hat{\beta}^m, \cdot) - \underline{u}_i(\beta^m, \cdot)\right] - 3\eta
\end{aligned}$$

From (11) and (12), for $t_{-i} \in U \cap A^m$,

$$\begin{aligned}
&\overline{u}_i\left(t_i, t_{-i}, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) - \underline{u}_i\left(t_i, t_{-i}, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) \\
&\geqslant u_i\left(t_i, t_{-i}, \beta^*, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i})\right) - 2\eta.
\end{aligned}$$

We obtain:

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \\
&\geqslant \left(\int_{U\cap A^m} u_i(\beta^*, \cdot)\right) - 5\eta.
\end{aligned}$$

For $t_{-i} \in U \cap A^m$, we have $\max_{k \neq i} \left|\Lambda_k^m\left(\mathbf{b}_k^m\right)(t_k) - \mathbf{b}_k^*(t_k)\right| < \delta$, from (14). Also, $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) \in [\beta^m, \beta^* + \delta)$, because in the event A^m, $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) > \beta^m$. Thus, $\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) \in (\beta^* - \delta, \beta^* + \delta)$, that is, $\left|\Lambda_{(-i)}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}) - \beta^*\right| < \delta$.

So, for $t_{-i} \in U \cap A^m$, $\left|\mathbf{b}_{(-i)}^*(t_{-i}) - \beta^*\right| < 2\delta$. The event $U \cap A^m$ is contained in the union of the following events:

$$\begin{aligned}
U_- &= U \cap A^m \cap \left[\beta^* - 2\delta_1 < \mathbf{b}_{(-i)}^*(t_{-i}) < \beta^*\right];\\
U_0 &= U \cap A^m \cap \left[\mathbf{b}_{(-i)}^*(t_{-i}) = \beta^*\right];\\
U_+ &= U \cap A^m \cap \left[\beta^* < \mathbf{b}_{(-i)}^*(t_{-i}) < \beta^* + 2\delta_1\right].
\end{aligned}$$

By (9) and (10), $\Pr U_- < \eta/M$ and $\Pr U_+ < \eta/M$. Thus, we have

$$\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \geqslant \int_{U_0} u_i(\beta^*, \cdot) - 7\eta.$$

The argument in the function above is $(t_i, t_{-i}, \beta^*, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)(t_{-i}))$. Observe that in U_0, $\max_{k \neq i} \left|\Lambda_k^m\left(\mathbf{b}_k^m\right)(t_k) - \mathbf{b}_k^*(t_k)\right| < \delta$. So, (13) implies that

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right)\\
&\geqslant \int_{U_0} u_i(t_i, t_{-i}, \beta^*, \mathbf{b}_{-i}^*(t_{-i})) - 8\eta.
\end{aligned}$$

From (7), we know that $A \subset \left[\mathbf{b}_{(-i)}^*(t_{-i}) = \beta^*\right]$. So, we have

$$\begin{aligned}
\int_{U_0} u_i &= \int_{U \cap A^m \cap \left[\mathbf{b}_{(-i)}^*(t_{-i}) = \beta^*\right]} u_i\\
&= \int_{U \cap A^m \cap A} u_i + \int_{U \cap A^m \cap \left[\mathbf{b}_{(-i)}^*(t_{-i}) = \beta^*\right] \setminus A} u_i\\
&= \int_A u_i - \int_{A \setminus (U \cap A^m)} u_i + \int_{(A^m \setminus A) \cap U \cap \left[\mathbf{b}_{(-i)}^*(t_{-i}) = \beta^*\right]} u_i\\
&\geqslant \int_A u_i - \int_{A \setminus A^m} M - \int_{A^m \setminus A} M\\
&= \int_A u_i - M \Pr\left(A \Delta A^m\right)\\
&> \int_A u_i - \eta,
\end{aligned}$$

where the last line comes from (15). Now we can use (8) to conclude that

$$\begin{aligned}
&\Pi_i\left(t_i, \hat{\beta}^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right)\\
&> \int_A u_i(t_i, t_{-i}, \beta^*, \mathbf{b}_{-i}^*(t_{-i})) dt_{-i} - 9\eta\\
&> 10\eta - 9\eta\\
&= \eta > 0.
\end{aligned}$$

But the fact that $\beta^m \in \Theta_i\left(t_i, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right)$, implies

$$\Pi_i\left(t_i, \beta', \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) - \Pi_i\left(t_i, \beta^m, \Lambda_{-i}^m\left(\mathbf{b}_{-i}^m\right)\right) \leqslant 0$$

for all β'. This contradiction concludes the proof.■

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