THE TRADE-OFF BETWEEN INCENTIVES AND ENDOGENOUS RISK

By Aloisio Araujo, Humberto Moreira, and Marcos H. Tsuchida[1]

Instituto Nacional de Matemática Pura e Aplicada, Brazil, and

Escola de Pós-Graduação em Economia, Fundação Getulio Vargas, Brazil;

Escola de Pós-Graduação em Economia, Fundação Getulio Vargas, Brazil;

Escola de Economia de São Paulo, Fundação Getulio Vargas, Brazil

Negative relationship between risk and incentives, predicted by standard moral hazard models, has not been confirmed by empirical work. We propose a moral hazard model in which a risk-averse agent can control the mean and variance of the profits. The possibility of risk reduction allows the agent's marginal utility of incentives to be increasing or decreasing in risk aversion. Positive relationship between risk and incentives is found when both marginal utility of incentives and variance are decreasing in risk aversion. Similar results are found if risk aversion is not observable and adverse selection precedes moral hazard.

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1. INTRODUCTION

Moral hazard plays a central role in problems involving delegation of tasks. When the principal cannot perfectly observe the effort exerted by a risk-averse agent, payment must be designed by taking into account the trade-off between incentives and risk sharing. As the optimal level of incentives depends on the variance of output, the relationship between risk and incentives is an important testable implication of incentive models.

Standard models of moral hazard predict a negative relationship between risk and incentives. The central reference is the model presented in Holmstrom and Milgrom (1987). They analyze the conditions in which optimal contracts are linear, that is, the agent's payoff is a fixed part plus a proportion of profits. In their model, the negative relationship between risk and incentives results from the interaction between these two variables in the agent's risk premium. As the agent is risk averse and incentives put risk in the agent's payoff, incentives incur a cost in utility. At the optimal incentive, an increase in risk is balanced by a reduction in incentives.

The empirical work does not verify the negative relationship between risk and incentives, and sometimes finds opposite results. Prendergast (2002) presents a survey of empirical studies in three fields of application, namely, executive compensation, sharecropping and franchising. Positive or insignificant relationships are found in all three fields, while negative relationship is found only in studies on executive compensation. The conclusion is that the evidence is weak. Similarly, in insurance literature, the monotone relationship between risk and coverage is not verified, as reported in Chiappori and Salanié (2000).

The lack of empirical support has stimulated the search for alternative models that are compatible with the observed facts. Prendergast (2002) suggests a theoretical model that assumes monitoring is harder in riskier environments. As incentives are a substitute for monitoring, incentives and risk are positively related. His model departs from the Holmstrom–Milgrom structure and risk aversion plays no role. To analyze contracts in agriculture, Ghatak and Pandey (2000) develop a moral hazard model assuming linear contracts, risk-neutral agents and limited liability. Their model is related to ours, since the agent controls the mean and variance of output. However, as limited liability induces riskier behavior, they obtain the optimization trade-off by assuming that the agent pays a cost to *increase* the risk of the project.

We propose a model with moral hazard and multitask in which the manager can control the mean and the variance of the profits. Note that the resulting variance is endogenous, and we can define two types of risk: the exogenous risk is the natural risk of the firm, and the endogenous risk is the one resulting from the agent's effort in reducing variance. Principal is risk neutral and the agent is risk averse. Multitask models were first developed in Holmstrom and Milgrom (1991), but in these models, effort controls exclusively the mean of the profits. Sung (1995) allows the agent to control the risk and shows that linear contracts are optimal in this class of moral hazard problems. We also examine an extension with adverse selection before moral hazard. In this case, the principal does not know the agent's risk aversion and designs a menu of contracts so that self-selection reveals the type of agent. Sung (2002) shows that linear contracts are optimal for mixed models of adverse selection before moral hazard and controllable variance. However, as he models an observable project choice, variance is assumed to be a contractible variable, while we assume the principal cannot observe the choice of variance. Since the optimality of linear contracts is not established for our model, we assume linearity and restrict the analysis to the space of linear contracts.

When the agent cannot control the risk of the project, the marginal cost of incentive is higher for an agent with more risk aversion. For this reason, the principal assigns lower-powered incentive contracts to more risk-averse agents. However, when agents can exert effort in risk reduction, the marginal cost of incentive may decrease with risk aversion. The principal may assign a high incentive contract to a high risk aversion, because he can reduce risk and the cost associated with risk. Technically speaking, since our model does not have the single-crossing property, the relationship between the incentive given to the agent and his risk aversion is ambiguous. We examined the linear-quadratic specification for the model and mapped the values of parameters that generate positive or negative relationship between risk and incentives. In general terms, the positive relationship is more likely for low risk aversion and intermediate levels of exogenous risk. Then we considered the case in which risk aversion is not observed by the principal. We computed the optimal contracts for representative situations and found that the relationship between endogenous risk and incentives is ambiguous. For a set of agent types with high risk aversion, incentives and endogenous risk are negatively related. Conversely, for a set of agents with low risk aversion, the relationship is positive. With respect to the exogenous risk, the Holmstrom–Milgrom result is preserved: exogenous risk and incentives are negatively related. In Araujo and Moreira (2001b), a model akin to the one presented here is applied to the insurance market and an ambiguous relationship between coverage and risk is found.

The rest of the paper is organized as follows. In Section 2, we present the moral hazard model. In Section 3 we adopt the linear-quadratic specification and analyze the relationship between risk and incentives. Then, as a robustness check, we examine the case in which the agent's risk aversion is not observable. Section 4 presents the model; Section 5 studies the linear-quadratic specifications and computes the optimal contracts for relevant cases. We find positive and negative relationships between risk and incentives. Section 6 states the concluding remarks; in the Appendix, we discuss implementability and optimality in adverse selection models without the single-crossing property, and examine the technical conditions for computing the optimal contract.

2. OBSERVABLE RISK AVERSION

The principal delegates the management of the firm to the agent, whose effort can affect the probability distribution of the profits. The agent may exert effort e increasing the mean, and effort f reducing the variance. The profits, denoted by z, have normal distribution $N(\mu(e), \sigma^2(f))$. Let $c(e)$ and $k(f)$ denote the cost of the efforts for the agent. The agent has exponential utility with risk aversion θ, a public available information. As shown in Sung (1995), linear contract is optimal in the Holmstrom–Milgrom setting where agent has control of risk. So the wage is a linear function of the profits, that is, $w = \alpha z + \beta$, $\alpha \geq 0$. The contract parameter α is the proportion of the profits received by the agent and is called the incentive, or the power, of the contract. The parameter β is the fixed part of the contract which is adjusted in order to induce the agent to participate.

The timing of the problem is as follows: (1) the principal and the agent learn the type

θ, then (2) the principal offers a contract $w = \alpha z + \beta$, (3) the agent may accept or reject the contract. If he accepts, then (4) he exerts effort accordingly, (5) the firm produces profit z, (6) the agent receives $w = \alpha z + \beta$ and the principal earns the net profit, $z - w$. The certainty equivalence of the agent's utility is

$$V_{CE}(\alpha, \beta, \theta, e, f) = \beta + \alpha\mu(e) - c(e) - k(f) - \frac{\alpha^2}{2}\theta\sigma^2(f),$$

that is, the expected wage, minus the cost of the effort and the risk premium. In the traditional moral hazard model, the last term originates the negative relationship between risk and incentives. The risk premium acts as a cost because the principal must compensate the agent to induce him to participate. Since the marginal risk premium with respect to α is increasing in both α and σ^2, the principal compensates an increase in σ^2 by a reduction in α, and equates the marginal cost and the marginal benefit of incentive. In our model, the last term has the same role, but the possibility of variance reduction modifies the relationship between risk and incentives.

The costs are convex and efforts increase mean and reduce variance with diminishing returns to scale. The following assumption summarizes these properties:

ASSUMPTION 1 *$c'(\cdot) > 0$, $c''(\cdot) > 0$, $k'(\cdot) > 0$, $k''(\cdot) > 0$, $\mu'(\cdot) > 0$, $\mu''(\cdot) \leq 0$, $\sigma^{2\prime}(\cdot) < 0$ and $\sigma^{2\prime\prime}(\cdot) > 0$.*

2.1. *Solving the Agent's Problem.* Given the contract (α, β), the agent with risk aversion θ chooses effort levels e^* and f^* that maximize his utility. The first-order conditions for the agent's problem are

$$\alpha\mu'(e^*) = c'(e^*) \qquad \text{and} \qquad k'(f^*) + \frac{\alpha^2}{2}\theta\sigma^{2\prime}(f^*) = 0. \tag{1}$$

Note that e^* and f^* do not depend on β, and that, as a consequence of the separability of costs, e^* does not depend on θ. Let $e^*(\alpha)$ and $f^*(\alpha, \theta)$ denote the agent θ's optimal choice of efforts, given the incentives α. Differentiating the first-order condition, we find that the derivatives of effort with respect to incentives and risk aversion have well defined signs,

$$e^*_\alpha = \frac{\mu'(e^*)}{c''(e^*) - \alpha\mu''(e^*)} > 0,$$

$$f^*_\alpha = -\frac{\alpha\theta\sigma^{2\prime}(f^*)}{k''(f^*) + \frac{1}{2}\alpha^2\theta\sigma^{2\prime\prime}(f^*)} > 0, \tag{2}$$

$$f^*_\theta = -\frac{\frac{1}{2}\alpha^2\sigma^{2\prime}(f^*)}{k''(f^*) + \frac{1}{2}\alpha^2\theta\sigma^{2\prime\prime}(f^*)} > 0. \tag{3}$$

The higher the incentive, the higher the effort in mean increasing and in variance reduction. The higher the risk aversion, the higher the effort in variance reduction. Consequently, the endogenous variance is decreasing in α and in θ. This is the expected result, since higher α provides incentive to the agent increase average profits, but simultaneously increases the risk of his payoff. The risk-averse agent is induced to reduce risk by increasing f^*, and this effect is stronger, the higher is the risk aversion. Using $e^*(\alpha)$ and $f^*(\alpha, \theta)$, the indirect utility is $V(\alpha, \beta, \theta) = \beta + v(\alpha, \theta)$, which is quasi-linear in β. The non-linear term is

$$v(\alpha, \theta) = \alpha\mu(e^*(\alpha)) - c(e^*(\alpha)) - k(f^*(\alpha, \theta)) - \frac{1}{2}\alpha^2\theta\sigma^2(f^*(\alpha, \theta)). \tag{4}$$

By the envelope theorem, we find that $v_\theta(\alpha, \theta) = -\alpha^2\sigma^2(f^*(\alpha, \theta))/2 < 0$. This means that, comparing two agents with marginally distinct risk aversion under the same contract

(α, β), the more risk-averse agent has higher risk premium and lower utility, even when his choice of risk-reduction effort is taken into account.

2.2. *The Principal's Problem.* We assume that the principal is risk-neutral. Her utility is, given the agent's effort choice, the expectation of the net profit, that is, the profit after the wage is paid to the agent,

$$U(\alpha, \beta) = E[z - w] = (1 - \alpha)\mu(e^*(\alpha)) - \beta,$$

where the expectation is taken with respect to the conditional distribution of z, given the agent θ's effort choice under the contract (α, β). Note that the principal's utility does not depend directly on the risk aversion, but the optimal contract may depend.

Denote as $(\alpha(\theta), \beta(\theta))$ the optimal contract for the agent of type θ. For all θ, this is the solution of the principal's maximization problem,

$$(\alpha(\theta), \beta(\theta)) \in \arg\max_{\tilde{\alpha}, \tilde{\beta}} U(\tilde{\alpha}, \tilde{\beta}), \tag{5}$$

subject to

$$V(\tilde{\alpha}, \tilde{\beta}, \theta) \geq 0. \tag{6}$$

The constraint (6) is the participation constraint where the reservation utility is normalized to be zero. As V is increasing and U is decreasing in β, condition (6) holds with equality and β may be eliminated from the objective function. The maximization problem is then, for all θ,

$$\max_{\tilde{\alpha}} S(\tilde{\alpha}, \theta), \tag{7}$$

where $S(\alpha, \theta)$ is the social surplus,

$$S(\alpha, \theta) = (1 - \alpha)\mu(e^*(\alpha)) + v(\alpha, \theta).$$

Whenever first- and second-order conditions are sufficient, the optimal contract is characterized by $S_\alpha(\alpha(\theta), \theta) = 0$ and $S_{\alpha\alpha}(\alpha(\theta), \theta) < 0$.

Note that, as e^* does not depend on θ, $S_{\alpha\theta}(\alpha, \theta) = v_{\alpha\theta}(\alpha, \theta)$, and by the implicit function theorem

$$\frac{d\alpha}{d\theta} = -\frac{v_{\alpha\theta}(\alpha(\theta), \theta)}{S_{\alpha\alpha}(\alpha(\theta), \theta)}, \tag{8}$$

which states that the relationship between incentives and risk aversion has the same sign as $v_{\alpha\theta}(\alpha(\theta), \theta)$, and reveals a close relationship between the cross-derivative of agent's utility and $d\alpha/d\theta$. By straightforward calculation, we find that $S_\alpha(\alpha, \theta) = (1-\alpha)\mu'(e^*(\alpha))e^*_\alpha(\alpha) - \alpha\theta\sigma^2(f^*(\alpha, \theta))$, which implies $S_\alpha(1, 0) = 0$. Therefore, whenever the first-order condition solves (7), the optimal contract $\alpha(\theta)$, for $\theta > 0$, may be found by solving the differential equation (8), for $\theta \geq 0$, with initial condition $\alpha(0) = 1$.

Applying the envelope theorem on α, the marginal utility of incentives is $v_\alpha(\alpha, \theta) = \mu(e^*(\alpha)) - \alpha\theta\sigma^2(f^*(\alpha, \theta))$. The first term does not dependent on risk aversion. The second term represents the marginal cost of risk premium and is related to the risk aversion through two channels. Using Assumption 1 and equation (3), we can study the sign of the cross-derivative of indirect utility,

$$v_{\alpha\theta}(\alpha, \theta) = \underbrace{-\alpha\sigma^2(f^*(\alpha, \theta))}_{<0} \underbrace{- \alpha\theta\sigma^{2\prime}(f^*(\alpha, \theta))f^*_\theta(\alpha, \theta)}_{>0}. \tag{9}$$

It clarifies the relationship between marginal utility of incentives and risk aversion. The first term is the direct effect and the second is the variance-reduction effect. The direct effect occurs as more risk-averse agents are more sensitive to the increase in variance when incentives increase; and the variance-reduction effect reflects the higher risk-reduction effort exerted by more risk-averse agents. As direct and variance-reduction effects have opposite signs, $v_{\alpha\theta}(\alpha, \theta)$ may have any sign. Therefore, in this model, the single-crossing condition does not hold in general.

The interpretation of the relationship between the sign of $d\alpha/d\theta$ and the sign of $v_{\alpha\theta}(\alpha(\theta), \theta)$, provided by equation (8), is that, in the optimal contract, when direct effect dominates, higher risk-aversion agents incur for the principal greater marginal cost of incentives and, for this reason, receives lower incentives; when variance reduction effect dominates, higher risk-aversion agents represent lower marginal cost to the principal as they strongly reduce the variance and the principal provides more incentives.

2.3. *Risk and Incentives.* Our main interest is on the relationship between variance of profits and incentives. The variance is directly related to the risk-reduction effort $f^*(\alpha, \theta)$. Let $\hat{\alpha}(\theta; f)$ be defined by $f^*(\hat{\alpha}(\theta; f), \theta) = f$. For a given f, $\hat{\alpha}(\theta)$ is the constant-risk curve, which is negatively sloped, since

$$\frac{d\hat{\alpha}}{d\theta} = -\frac{f^*_\theta}{f^*_\alpha} = -\frac{\alpha}{2\theta} < 0, \tag{10}$$

by (2) and (3). Constant-risk curves may be plotted in a $\theta \times \alpha$ plane. As $f^*(\alpha, \theta)$ is increasing in α and θ, the constant-risk curves closer to the upper-left corner correspond to lower risk. Inspecting equation (1), it is clear that f^* is determined only by $\alpha^2\theta$. As f^* is constant if and only if $\alpha^2\theta$ is constant, constant-risk curves are defined by $\hat{\alpha}(\theta) = \kappa/\sqrt{\theta}$, where κ is a constant. We reach the same result by solving the diferential equation in (10).

Another related result concerns the locus of $v_{\alpha\theta}(\alpha,\theta)=0$. From equations (3) and (9), $v_{\alpha\theta}(\alpha,\theta)=0$, if and only if

$$\sigma^2(f^*(\alpha,\theta)) - \frac{\frac{1}{2}\alpha^2\theta\left[\sigma^{2\prime}(f^*(\alpha,\theta))\right]^2}{k''(f^*(\alpha,\theta))+\frac{1}{2}\alpha^2\theta\sigma^{2\prime\prime}(f^*(\alpha,\theta))} = 0. \tag{11}$$

As $f^*(\alpha,\theta)$ depends on α and θ only through $\alpha^2\theta$, the whole expression depends on α and θ only through $\alpha^2\theta$. Therefore, if $v_{\alpha\theta}(\alpha_0,\theta_0)=0$, then, for the whole constant-risk curve associated to (α_0,θ_0), $v_{\alpha\theta}(\alpha,\theta)=0$ holds.

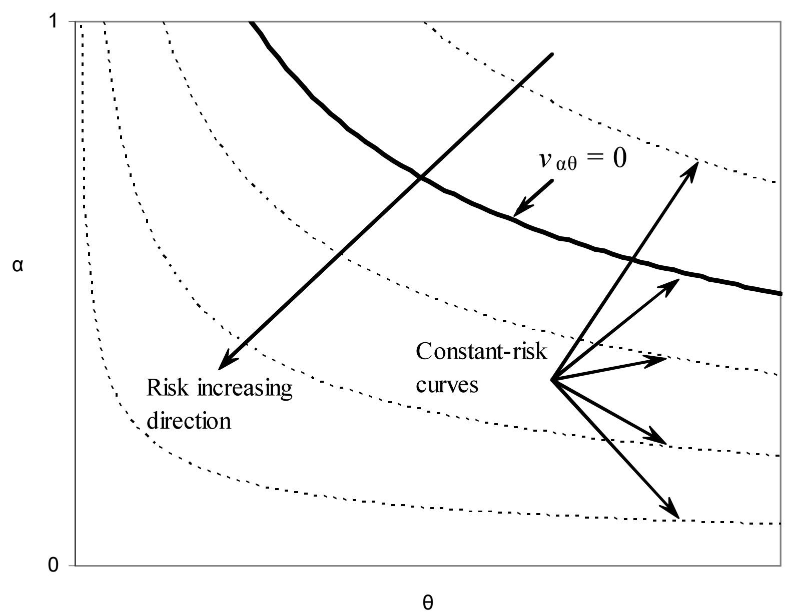

Figure 1: Constant-risk curves.

In Figure 1, the dotted lines are constant-risk curves associated to different risk levels. Risk is greater for low incentives and low risk aversion. Lower risk is associated to high incentives and high risk aversion. The thick line is the set of points where $v_{\alpha\theta}(\alpha,\theta)=0$ and coincides with a constant-risk curve.

The analysis above results from the maximizing effort choices by the agent. The final relationship between incentives and risk must take into account the optimal contract designed by the principal. The following proposition relates the direction of the risk and incentive relationship with the parameters of the model.

PROPOSITION 1 *Let the contract $\alpha(\theta)$ be defined by the first-order condition $S_\alpha(\alpha, \theta) = 0$. If $v_{\alpha\theta}(\alpha(\theta), \theta) \neq 0$ and $S_{\alpha\alpha}(\alpha(\theta), \theta) < 0$,*

$$\text{sign}\left\{\frac{d\sigma^2}{d\alpha}\right\} = \text{sign}\left\{\left[\frac{v_{\alpha\theta}}{S_{\alpha\alpha}} - \frac{\alpha}{2\theta}\right] v_{\alpha\theta}\right\}.$$

PROOF. As the variance of profits is $\sigma^2(f^*(\alpha, \theta))$, the relationship between incentives and risk, in a neighborhood of $\alpha(\theta)$, is given by

$$\frac{d\sigma^2}{d\alpha} = \left[f_\alpha^* + f_\theta^* \left(\frac{d\alpha}{d\theta}\right)^{-1}\right] \sigma^{2\prime}(f^*),$$

provided that $d\alpha/d\theta \neq 0$. As $\sigma^{2\prime} < 0$ and $f_\alpha^* > 0$,

$$\text{sign}\left\{\frac{d\sigma^2}{d\alpha}\right\} = \text{sign}\left\{-\left[\frac{d\alpha}{d\theta} + \frac{f_\theta^*}{f_\alpha^*}\right] \frac{d\alpha}{d\theta}\right\}.$$

By equation (8), $v_{\alpha\theta} \neq 0$ implies $d\alpha/d\theta \neq 0$, and, by equations (2) and (3), $f_\theta^*/f_\alpha^* = \frac{\alpha}{2\theta}$, which establishes the desired result. ■

When incentives are increasing in risk aversion, risk-reduction effort is increasing in incentives. Consequently, risk and incentives are negatively related. On the other hand, when incentives are decreasing in risk aversion, the direction of the relationship between risk and incentives depends on the balance between the risk aversion and incentive effects.

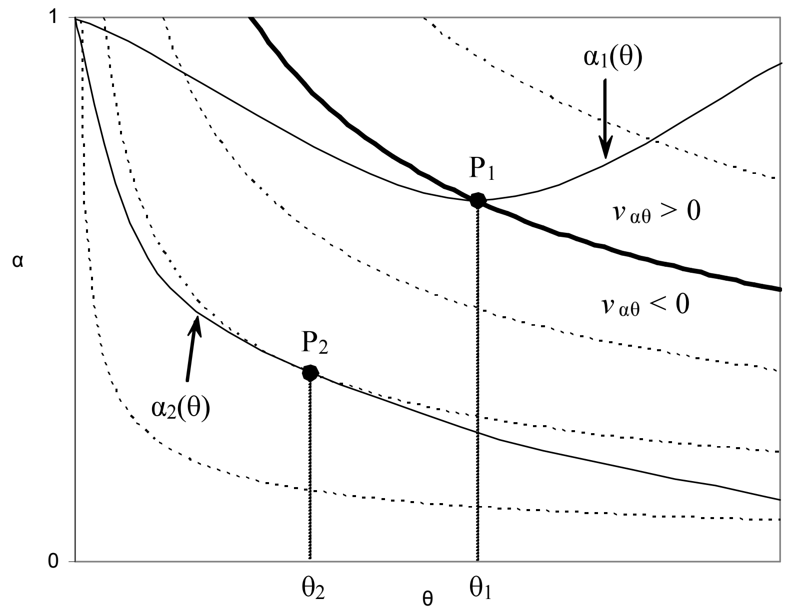

Figure 2: Optimal contracts and constant-risk curves.

Proposition 1 can be graphically interpreted as an interaction between the optimal contracts and constant-risk curves. Figure 2 shows the same constant-risk curves in Figure 1. We assume $v_{\alpha\theta}(\alpha,\theta)$ is positive above and negative below the $v_{\alpha\theta}(\alpha,\theta) = 0$ curve. As an illustration, an optimal contract, $\alpha_1(\theta)$, is plotted. Let P_1 be the point where $v_{\alpha\theta}(\alpha_1,\theta) = 0$. As required by equation (8), $\alpha_1(\theta)$ is increasing when $v_{\alpha\theta}(\alpha_1(\theta),\theta) > 0$ and decreasing when $v_{\alpha\theta}(\alpha_1(\theta),\theta) < 0$. For $\theta > \theta_1$, $v_{\alpha\theta}(\alpha_1(\theta),\theta) > 0$, therefore, for higher risk aversion, incentives are higher and risk-reduction effort increases. As a consequence, incentives and risk are negatively related. For $\theta < \theta_1$, the contract curve is flatter than constant-risk curve. In this case, a moderate reduction in incentives is sufficient to compensate an increase in risk aversion; as risk aversion dominates in the determination of risk-reduction effort, the risk decreases. Therefore, risk and incentives are positively related.

The contract $\alpha_2(\theta)$ illustrate another case. At P_2, the slope of contract curve is equal to the slope of constant-risk curve,

$$\frac{d\alpha}{d\theta} = -\frac{f_\theta^*}{f_\alpha^*}.$$

For $\theta > \theta_2$, the reduction in incentives is strong enough to make the agent reduce effort in risk reduction and risk increases. In this case, risk and incentives are negatively related.

The analysis above is concerned with the endogenous risk, the risk that remains after the risk-reduction effort. Assume now that there is an exogenous parameter of risk. Let σ_0^2 be the exogenous variance; this is the natural risk of the project, which would be observed if the agent exert no effort to reduce risk. We rewrite the endogenous variance as a function of effort and the exogenous variance, $\sigma^2(f, \sigma_0^2)$, and assume $\sigma^2(0, \sigma_0^2) = \sigma_0^2$ and $\partial\sigma^2/\partial\sigma_0^2 > 0$. Analogously, for a given θ, we rewrite the indirect utility and social surplus as $v(\alpha, \sigma_0^2)$ and $S(\alpha, \sigma_0^2)$.

We are interested in the effect of a change in the exogenous risk on incentives. From equation (1), e^* does not depend on σ_0^2 and

$$f_{\sigma_0^2}^* = -\frac{\frac{1}{2}\alpha^2\theta\frac{\partial^2\sigma^2}{\partial f\partial\sigma_0^2}}{k''(f^*) + \frac{1}{2}\alpha^2\theta\frac{\partial^2\sigma^2}{\partial f^2}}.$$

The following proposition relates the exogenous risk and incentives.

Proposition 2 *For a given θ, if the optimal contract α is defined by $S_\alpha(\alpha, \sigma_0^2) = 0$ and $S_{\alpha\alpha}(\alpha, \sigma_0^2) < 0$,*

$$\text{sign}\left\{\frac{d\alpha}{d\sigma_0^2}\right\} = \text{sign}\left\{-\left[\frac{\partial\sigma^2}{\partial f}f_{\sigma_0^2}^* + \frac{\partial\sigma^2}{\partial\sigma_0^2}\right]\right\}.$$

Proof. As the optimal contract is characterized by $S_\alpha(\alpha, \sigma_0^2) = 0$, the relationship between exogenous risk and incentives is given by

$$\frac{d\alpha}{d\sigma_0^2} = -\frac{v_{\alpha\sigma_0^2}(\alpha, \sigma_0^2)}{S_{\alpha\alpha}(\alpha, \sigma_0^2)},$$

that has the same sign as $v_{\alpha\sigma_0^2}$.

By the envelope theorem $v_\alpha(\alpha, \sigma_0^2) = \mu(e^*(\alpha)) - \alpha\theta\sigma^2(f^*(\alpha, \sigma_0^2), \sigma_0^2)$. Then

$$v_{\alpha\sigma_0^2}(\alpha, \sigma_0^2) = -\alpha\theta\left[\frac{\partial\sigma^2(f^*(\alpha, \sigma_0^2), \sigma_0^2)}{\partial f} f^*_{\sigma_0^2}(\alpha, \sigma_0^2) + \frac{\partial\sigma^2(f^*(\alpha, \sigma_0^2), \sigma_0^2)}{\partial\sigma_0^2}\right].$$

■

The following corollary is a straightforward consequence of Proposition 2.

Corollary 1 *If* $\frac{\partial^2\sigma^2}{\partial f\partial\sigma_0^2} > 0$, *then incentives and exogenous risk are negatively related.*

Propositions 1 and 2 and Corollary 1 are helpful for explaining the failure of empirical research in the verification of the Holmstrom–Milgrom prediction. Under certain conditions, exogenous risk and incentives are negatively related for any level of risk aversion. However, empirical work does not measure exogenous risk directly. As agents exert effort in reducing risk, the empirically relevant risk is the endogenous risk that may be positively or negatively related to incentives.

3. THE LINEAR-QUADRATIC SPECIFICATION

To find numerical solutions of optimal contracts, we assume specific functions. The mean of profits is linear, and the variance and the cost functions are quadratic function of efforts.

Assumption 2 *$\mu(e) = me$, $\sigma^2(f) = (\sigma_0 - f)^2$, $c = \frac{1}{2}e^2$ and $k = \frac{1}{2}f^2$.*

Using the first-order conditions of the agent's problem (1), the effort levels are

$$e^* = \alpha m, \qquad \text{and} \qquad f^* = \frac{\alpha^2\theta}{1+\alpha^2\theta}\sigma_0 < \sigma_0. \tag{12}$$

The effort in variance reduction is increasing with respect to the exogenous variance. The endogenous variance is $\sigma^2(f^*(\alpha,\theta)) = \sigma_0^2/(1+\alpha^2\theta)^2$.

The cross-derivative of the indirect utility,

$$v_{\alpha\theta}(\alpha,\theta) = \frac{-\alpha(1-\alpha^2\theta)}{(1+\alpha^2\theta)^3}\sigma_0^2, \tag{13}$$

indicates that $v_{\alpha\theta}$ is positive above and negative below the decreasing curve $\alpha = 1/\sqrt{\theta}$ as illustrated in Figure 2. When $\alpha < 1/\sqrt{\theta}$, the direct effect dominates and the optimal contract $\alpha(\theta)$ is decreasing in risk aversion. Conversely, if $\alpha > 1/\sqrt{\theta}$, the variance-reduction effect dominates and $\alpha(\theta)$ is increasing.

Using the functions specified above, the social surplus becomes

$$S(\alpha,\theta) = \left(1 - \frac{\alpha}{2}\right)\alpha m^2 - \frac{\alpha^2\theta}{2(1+\alpha^2\theta)}\sigma_0^2.$$

The first-order condition for the principal's problem, $S_\alpha(\alpha,\theta) = 0$, is reduced to

$$m^2(1-\alpha)(1+\alpha^2\theta)^2 - \alpha\theta\sigma_0^2 = 0. \tag{14}$$

Note that, for $\theta > 0$, the left-hand side is positive when $\alpha \leq 0$ and negative when $\alpha \geq 1$. Thus, by continuity, there is at least one solution and all solutions are in the interval $(0,1)$.

Assumption 3 *The parameter of risk aversion is restricted to $0 \leq \theta < 4$.*

Under Assumption 3, the solution of equation (14) is unique and gives the optimal level of incentives for each θ.[2]

3.1. *Risk and Incentives.* To analyze the relationship between risk and incentives, we have to examine the properties of

$$\frac{v_{\alpha\theta}(\alpha,\theta)}{S_{\alpha\alpha}(\alpha,\theta)} = \frac{\alpha(1-\alpha^2\theta)\sigma_0^2}{m^2(1+\alpha^2\theta)^3+\theta(1-3\alpha^2\theta)\sigma_0^2}$$
$$= \frac{\alpha(1-\alpha^2\theta)(1-\alpha)}{\theta(1-\alpha^2\theta(3-4\alpha))}.$$

where the parameters m and σ_0^2 were eliminated in the second equality by the use of the first-order condition. Using the expression above and Proposition 1, we find that

$$\text{sign}\left\{\frac{d\sigma^2}{d\alpha}\right\} = \text{sign}\left\{(1-\alpha^2\theta)(2\alpha-1)\right\}.$$

The factor $(1-\alpha^2\theta)$ comes from $v_{\alpha\theta}$, that is, $1-\alpha^2\theta=0$ defines the border between regions where $v_{\alpha\theta}>0$ and $v_{\alpha\theta}<0$. The meaning of factor $(2\alpha-1)$ is that optimal contract curves are tangent to a constant-risk curve at $\alpha=1/2$.

Figure 3 shows the relevant regions, together with five instances of optimal contracts. Let $s^2=\sigma_0^2/m^2$. In Region A, $v_{\alpha\theta}>0$, incentive is increasing and variance is decreasing

[2]The uniqueness may be established analyzing the properties of the second-order condition, $S_{\alpha\alpha}<0$, which gives

$$m^2(1+\alpha^2\theta)^3+\theta(1-3\alpha^2\theta)\sigma_0^2>0.$$

And using the first-order condition to eliminate m and σ_0^2,

$$1-\alpha^2\theta(3-4\alpha)>0.$$

For $\alpha\in[\frac{3}{4},1]$ this condition is always satisfied, and for $\alpha\in[0,\frac{3}{4})$, the condition is satisfied if $\theta<4$. Therefore, if $\theta<4$, we have $S_{\alpha\alpha}<0$ and $S_\alpha=0$ has a unique solution.

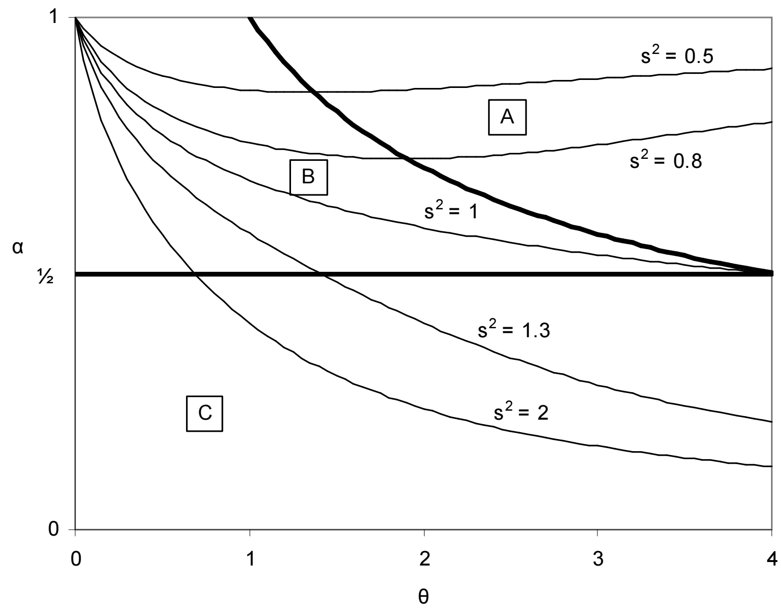

Figure 3: Optimal contracts. $s^2 = \sigma_0^2/m^2$

in θ. In Region B, $v_{\alpha\theta} < 0$, both incentive and variance are decreasing in θ. And, in Region C, $v_{\alpha\theta} < 0$, incentive is decreasing and variance is increasing in θ. Therefore, risk and incentives are positively correlated in Region B and negatively correlated in Regions A and C.

The contract for $s^2 = 1$ is a watershed case, as for $s^2 \geq 1$ contracts are monotonically decreasing with respect to risk aversion. For $s^2 < 1$, incentives may be greater for more risk-averse agents, if risk aversion is sufficiently high.

The relationship between incentives and risk for the contracts in Figure 3 may be directly examined in Figure 4. Risk is normalized as a fraction of the exogenous risk. Curves in Figure 3 are mapped in Figure 4 by $\sigma^2/\sigma_0^2 = 1/(1+\alpha^2\theta)^2$. The $v_{\alpha\theta} = 0$ curve

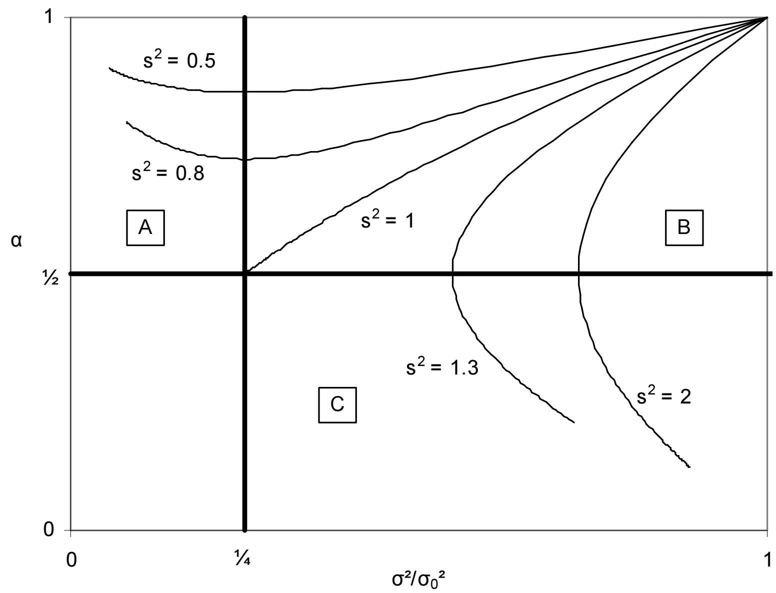

Figure 4: Risk and incentives. $s^2 = \sigma_0^2/m^2$.

corresponds to the vertical line at $\sigma^2/\sigma_0^2 = 1/4$. Variance below $\sigma_0^2/4$, is only possible for $s^2 < 1$. That is, in Region A, low risk is induced by three factors: σ_0 is low, risk aversion is high and incentives are high. For $s^2 > 1$ the variance reduction effort has an upper limit at $\alpha = 1/2$ and variance has a lower limit.

Figure 5 shows the combinations of parameters that results in increasing or decreasing relationship between risk and incentives. The borders between the regions correspond to $\alpha = 1/2$ and $\alpha^2\theta = 1$ mapped by equation (14). Risk and incentives are positively correlated when risk aversion is low and is associated to high incentives. For agents with high risk aversion, positive correlation is possible if σ_0^2/m^2 has intermediate values, but is a less frequent situation, the higher is the risk aversion.

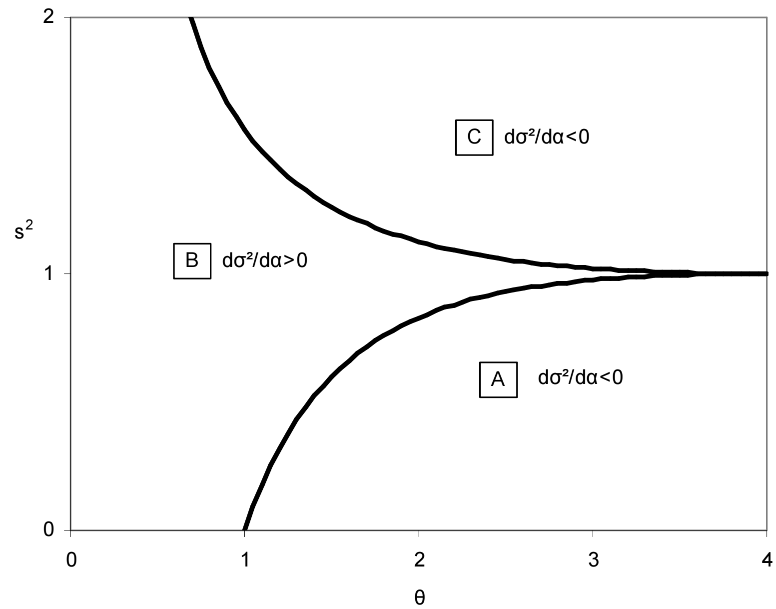

Figure 5: Parameters and the risk-incentive trade-off. $s^2 = \sigma_0^2/m^2$

Figure 3 suggests that the relationship between incentives and exogenous risk is negative. In fact, although the assumption of Corollary 1 does not hold, by Proposition 2, for any θ, the relationship between exogenous risk and incentives is negative as in Holmstrom and Milgrom (1987).

4. ADVERSE SELECTION AND MORAL HAZARD

We now extend the pure moral hazard model presented in Section 2. Assume now that risk aversion is agent's private information. Principal knows that θ is uniformly distributed on $\Theta = [\theta_a, \theta_b]$. So there is adverse selection before moral hazard and the principal must design a menu of contracts taking into account the participation and incentive compatibility constraints. In this section we assume linear contracts.[3]

The timing of the problem is as follows: (1) the agent learns his type, then (2) the principal offers a menu of contracts $\{\alpha(\theta), \beta(\theta)\}_{\theta\in\Theta}$, (3) the agent chooses a contract, and (4) exerts effort accordingly, (5) the firm produces profit z, (6) the agent receives $w = \alpha z + \beta$ and the principal earns the net profit, $z - w$.

We can now divide the problem in two parts. First, for a given contract (α, β), the agent chooses the effort levels $e^*(\alpha)$ and $f^*(\alpha, \theta)$, and we find the indirect utility of the agent. This part is exactly the same as the agent's problem in pure moral hazard problem examined in Section 2. The indirect utility is again $V(\alpha, \beta, \theta) = \beta + v(\alpha, \theta)$, where $v(\alpha, \theta)$ is defined by (4). Second, given the indirect utility of the agent, the principal solves the adverse section problem. The principal designs an incentive compatible menu of contracts that maximizes her expected utility.

The adverse selection problem is to find the functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that

$$(\alpha(\cdot), \beta(\cdot)) \in \arg\max_{\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)} E[U(\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \theta)], \tag{15}$$

subject to

$$V(\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \theta) \geq V(\tilde{\alpha}(\hat{\theta}), \tilde{\beta}(\hat{\theta}), \theta), \text{for all } \theta, \hat{\theta} \in \Theta, \tag{16}$$

$$V(\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \theta) \geq 0, \text{for all } \theta \in \Theta. \tag{17}$$

The expectation in (15) is taken with respect to θ. The constraint (16) is the incentive compatibility condition. A function $\alpha(\cdot)$ is called implementable, if there is a function $\beta(\cdot)$ so that $(\alpha(\cdot), \beta(\cdot))$ is incentive compatible. The constraint (17) is the participation

[3] See Sung (2002) for a discussion on optimality of linear contracts in this setting.

constraint where the reservation utility is normalized to be zero.

Guesnerie and Laffont (1984) fully characterize the optimal contract under the single-crossing condition, that is, the cross-derivative $v_{\alpha\theta}$ has constant sign. The solution of the model involves the virtual surplus

$$R(\alpha,\theta) = S(\alpha,\theta) + (\theta - \theta_a)v_\theta(\alpha,\theta), \tag{18}$$

which is the social surplus plus the informational rent.[4] The pointwise maximization of $R(\alpha,\theta)$, denoted $\alpha_r(\theta) = \arg\max_\alpha R(\alpha,\theta)$, is the relaxed solution, and this function is fundamental for the characterization of the optimal contract. If $\alpha_r(\theta)$ is an implementable contract, then it is the optimal contract. Otherwise, the optimal contract is the best implementable combination of $\alpha_r(\theta)$ and intervals of bunching. When function $\alpha_r(\theta)$ coincides with the optimal contract, types are separated, that is, distinct levels of incentives are assigned to distinct types of agents.

In our model, the cross-derivative $v_{\alpha\theta}$ may have any sign. That is, the single-crossing property may not hold. The characterization of the optimal contracts in adverse selection problems without the single-crossing property is analyzed in Araujo and Moreira (2001a), and the Appendix contains some results that are relevant for the solution of our model. The distinctive feature of the optimal contract without the single-crossing property is that a discrete set of agent's types may choose the same contract. This situation is called discrete pooling. We can now find a function $\alpha_u(\theta)$ that gives the optimal assignment of incentives when exactly two types are pooled. If the relaxed solution is implementable, then the optimal contract coincides with $\alpha_r(\theta)$. For the more complex situations, as in

[4] See the Appendix for the derivation of the virtual surplus.

the framework examined in Araujo and Moreira (2001a), the optimal contract is the best implementable combination of $\alpha_r(\theta)$, $\alpha_u(\theta)$ and intervals of bunching. Section 5 provides some illustrative examples of optimal contract with the linear-quadratic specification.

4.1. *Single-Task.* Before we study the linear-quadratic specification for the full model, we analyze a simpler specification in which the agent cannot control the variance of the profits. The objective is to certify that the possibility of variance reduction is a necessary element of the model to generate a positive relationship between incentives and risk. In this simple specification, agent's effort controls only the mean of the profits. Let e denote the effort and assume the mean of the profits is linear in e, $\mu(e) = me$, and the cost of effort is quadratic, $c(e) = e^2/2$. In order to fit this model to the previous framework, we may assume $\sigma^2(f) = \sigma_0^2$, $k(f) = f^2/2$ and $k(0) = 0$.

The first-order condition of the agent's problem provides the optimal effort, $e^* = m\alpha$ and $f^* = 0$. As expected, effort increases with the power of incentives. The non-linear term of indirect utility is

$$v(\alpha, \theta) = \frac{\alpha^2}{2}\left(m^2 - \theta\sigma_0^2\right),$$

and the marginal utility of incentives is $v_\alpha = \alpha m^2 - \alpha\theta\sigma_0^2$. An increase in incentives has positive and negative effects on the agent's utility. The positive effect is the increase of the share of profits. The negative effect comes from the increase of risk in the wage. The single-crossing property holds for this case, since $v_{\alpha\theta} = -\alpha\sigma_0^2 < 0$. An agent with low risk aversion has high marginal utility of incentive and may choose a high-powered incentive contract.

The virtual surplus, as defined in (18), is a concave function and the solution of the relaxed problem is given by the first-order condition $R_\alpha(\alpha_r(\theta), \theta) = 0$. Thus,

$$\alpha_r(\theta) = \frac{m^2}{m^2 + (2\theta - \theta_a)\sigma_0^2}.$$

The function α_r is decreasing in θ and $v_{\alpha\theta}$ is negative. In this case, the optimal contract of the problem coincides with the relaxed solution. The variance σ_0^2 has also a negative effect on α, since it increases the marginal cost of incentives present in the risk premium and in the informational rent.

The relationship between α and σ_0^2 is negative, given θ. Therefore, adverse selection before moral hazard is not sufficient to change the traditional risk-incentive trade-off. If agent controls only the mean of the profits, risk does not affect the principal's benefit, because she is risk neutral, but increases the marginal cost, because she has to compensate for the risk premium and has to pay the informational rent. Consequently, the incentives are lower in riskier projects.

5. OPTIMAL CONTRACTS IN THE LINEAR-QUADRATIC MODEL

In this section we return to the linear-quadratic specification in Section 3 and examine the optimal contract and the relationship between incentives and risk, when the risk aversion is a private information of the agent. As the characterization of the optimal contract is complex, we are not able to provide a general analysis as the one developed for the observable risk aversion case. Instead, we provide three illustrative examples that show that positive and negative relationship between incentives and risk may be observed when there is adverse selection before moral hazard.

We now return to the specification in Assumption 2. The variance and the mean are under control of the agent. The agent chooses effort levels in (12) and the non-linear term of indirect utility is (4). The cross-derivative is again $v_{\alpha\theta} = -\alpha(1-\theta\alpha^2)\sigma_0^2/(1+\theta\alpha^2)^3$ and the function $\alpha_0(\theta) = 1/\sqrt{\theta}$ defines a decreasing border between $v_{\alpha\theta} > 0$ and $v_{\alpha\theta} < 0$ regions, with $v_{\alpha\theta} > 0$, for $\alpha > \alpha_0$. For less risk-averse agents, the direct effect dominates and the marginal utility of incentive decreases with risk aversion. For more risk-averse agents, the effort produces a stronger effect, such that the second term dominates and $v_{\alpha\theta} > 0$. This changes the self-selection direction, that is, an agent with a higher degree of risk aversion has a higher marginal utility of incentive, and chooses contracts with more power.

The next step is to define the virtual surplus $R(\alpha, \theta)$ and to find the solution of the relaxed problem $\alpha_r(\theta)$ (see the Appendix). The incentive schedule of the optimal contract is $\alpha_r(\theta)$, whenever it is implementable. As the single-crossing property does not hold, two points have to be observed: first, the incentive compatibility cannot be trivially checked; and, second, if $\alpha_r(\theta)$ is not implementable, the computation of optimal contract must follow the procedure presented in the Appendix. The optimal incentive schedule may have a complex form, resulting from a combination of $\alpha_r(\theta)$, discrete pooling and continuous bunching.

The equations above were numerically implemented for three representative cases that generate increasing, decreasing and mixed relationship between incentives and risk.[5] The

[5] Computer code is provided upon request.

parameter values, $\sigma_0 = 0.91$ and $m = 1$, are the same for the three cases, and the values of θ_a and θ_b change for each case. These values were chosen in order to generate functions that are tractable by the procedure detailed in Araujo and Moreira (2001a). For each case, we compute the optimal contract $\alpha^*(\theta)$ and the endogenous risk $\sigma^2(e^*(\alpha^*(\theta), \theta))$, then we plot the function $\alpha^*(\theta)$, and the risk-incentive curve.

In Figure 6, for $\theta \in [2.5, 3.5]$, the dotted line $\alpha_0(\theta)$ is the border between the $v_{\alpha\theta} < 0$ region to the left, and the $v_{\alpha\theta} > 0$ region to the right. The relaxed solution $\alpha_r(\theta)$ is increasing in Θ, and coincides with the optimal contract. Figure 7 is the corresponding plot for risk and incentives. An agent with higher risk aversion exerts more effort in risk reduction and this behavior reduces the marginal cost from risk premium. This effect more than compensates the increase in marginal cost due to higher risk aversion. The net effect is that more risk-averse agents choose higher-powered incentive contracts and the relationship between risk and incentives is negative as in Holmstrom and Milgrom (1987).

The contract for a set of types with lower risk aversion, $\theta \in [0.5, 1.4]$, is shown in Figure 8. The relaxed solution is implementable as $v_{\alpha\theta}(\alpha_r(\theta_a), \theta_b) < 0$ (see the Appendix). The optimal contract coincides with the relaxed solution, but this time the relationship is reversed. More risk-averse agents have higher marginal cost of incentives, thus they prefer lower-powered incentive contracts. At the same time, more risk-averse agents exert more effort in risk reduction and the variance is lower. As is seen in Figure 9, the risk and incentives are positively related.

For a broader interval of types, that encompasses $v_{\alpha\theta}$ of both signs, the discrete pooling is possible and the optimal contract presents a U-shaped form. In Figure 10, the optimal

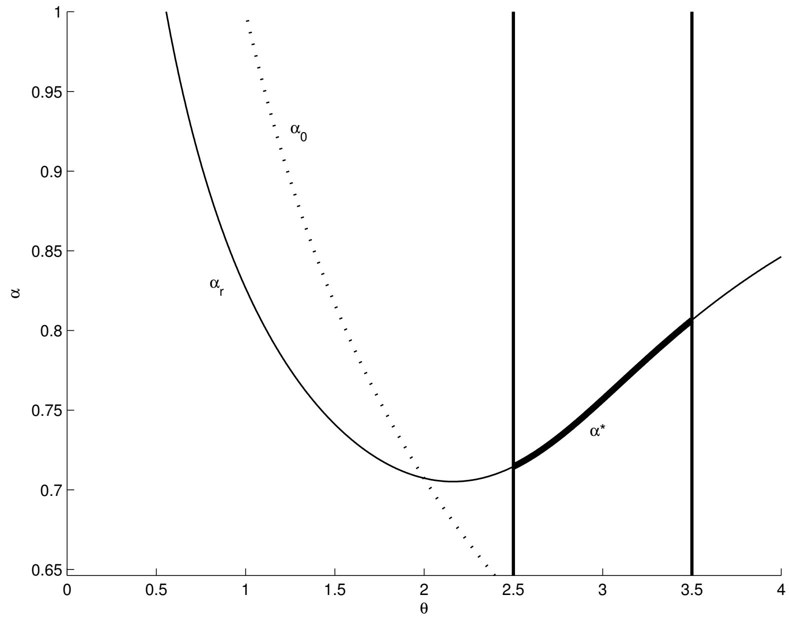

Figure 6: Optimal contract. $\Theta = [2.5, 3.5]$.

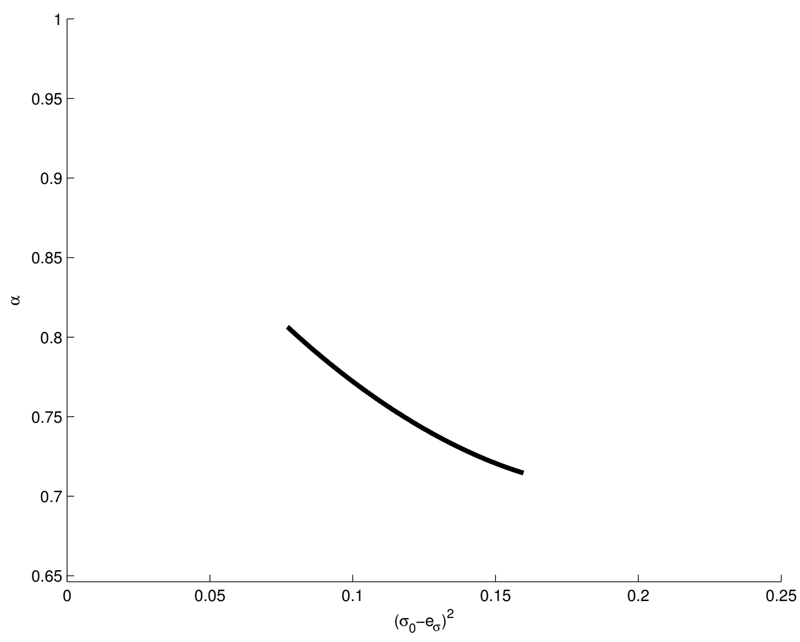

Figure 7: Risk × incentives. $\Theta = [2.5, 3.5]$.

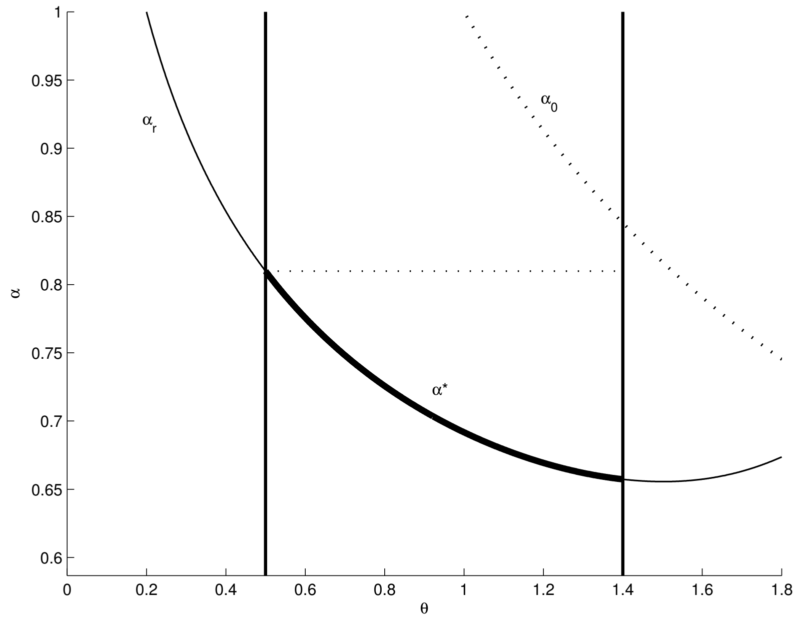

Figure 8: Optimal contract. $\Theta = [0.5, 1.4]$.

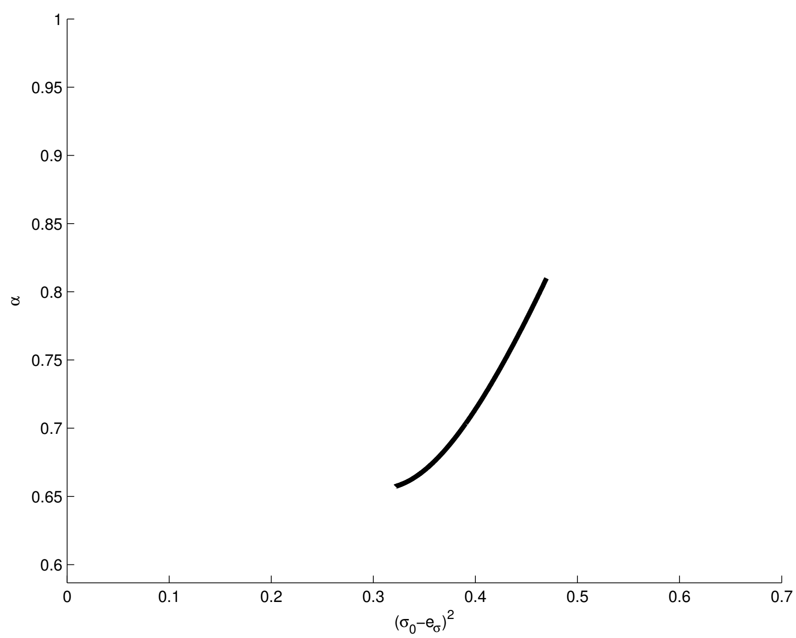

Figure 9: Risk $\times$ incentives. $\Theta = [0.5, 1.4]$.

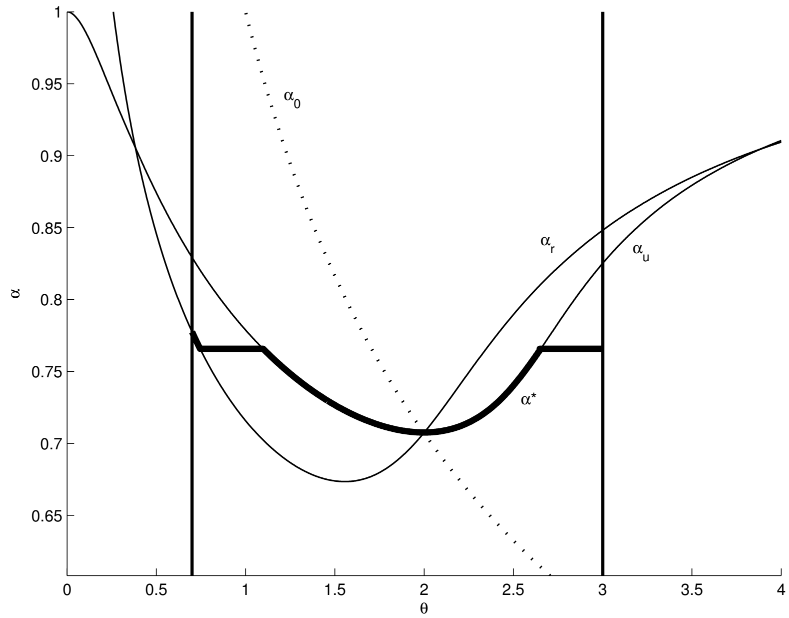

Figure 10: Optimal contract. $\Theta = [0.7, 3.0]$.

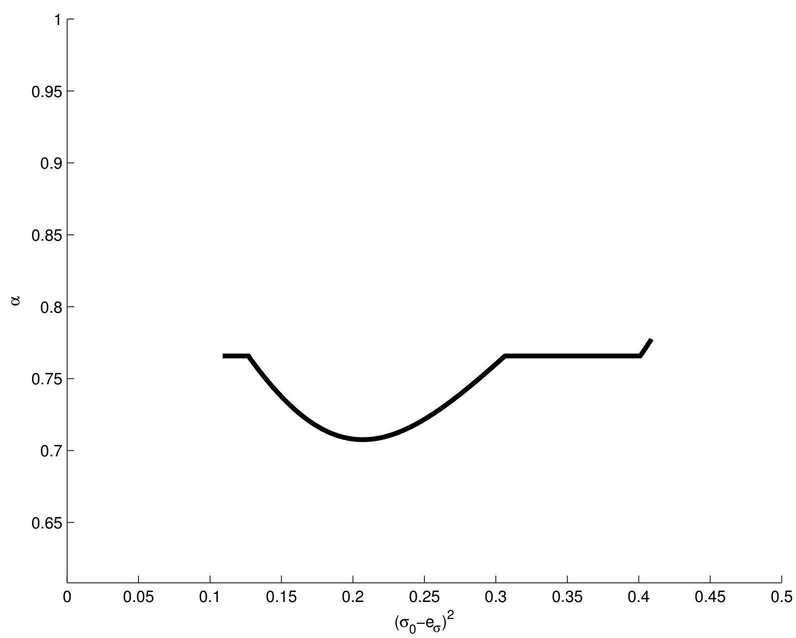

Figure 11: Risk $\times$ incentives. $\Theta = [0.7, 3.0]$.

contract for $\theta \in [0.7, 3.0]$ is plotted.[6] Computational procedures found the optimal contract that combines relaxed solution, discrete pooling and continuous bunching. Incentives and risk aversion are positively related for more risk-averse agents and negatively related for less risk-averse agents. The U-shape of the optimal contract is also present in the risk-incentive graph, as we can see in Figure 11.

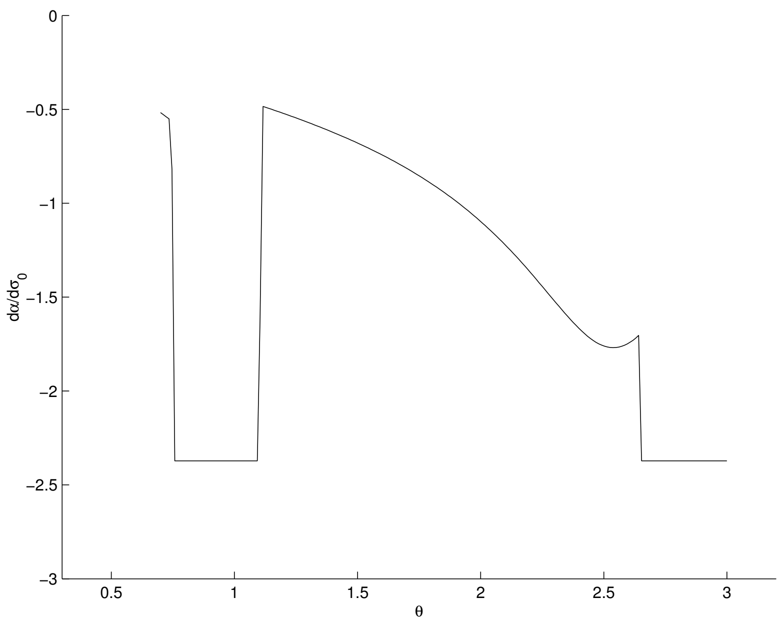

Figure 12: Exogenous risk $\times$ incentives. $\Theta = [0.7, 3.0]$.

The results above are concerned with the endogenous risk. We show in the Appendix that the incentive in the relaxed solution is decreasing in σ_0. Therefore, the relationship between incentives and exogenous risk is negative when optimal contract coincides with the relaxed solution, as in the first two cases. For the third case, the sensitivity $d\alpha/d\sigma_0$ was numerically calculated and plotted in Figure 12. Note that the sensitivity is negative,

[6] As prescribed in the Appendix, the validity of assumptions H2 and H3 were checked numerically.

which suggests that incentives decrease with exogenous risk.

6. CONCLUSION

The negative relationship between risk and incentives, found in standard models of moral hazard, may be reversed if we allow the agent to control the variance. In the model with moral hazard in which risk aversion is observable and where the agent may exert costly efforts to increase the mean as well as to reduce variance, we find that effort in variance reduction is an increasing function of both incentives and risk aversion.

The marginal utility of incentives may be decreasing or increasing in risk aversion. As incentives increase the variance of the wage, the marginal utility of incentives tends to be lower for higher risk aversion because the marginal cost from the risk premium is higher. However, as the effort in variance reduction increases with incentives, when risk aversion is sufficiently high, the marginal cost from the risk premium may decrease and the marginal utility of incentives may increase with the risk aversion.

In the moral hazard model, when marginal utility of incentives is decreasing in risk aversion, the principal maximizes the social surplus by giving fewer incentives to a more risk-averse agent. Conversely, for increasing marginal utility, incentives and risk aversion is positively related. In the model with adverse selection before moral hazard, the incentive compatibility constraint leads to the same relationship between the marginal utility and incentives that was found in the pure moral hazard model.

The relationship between risk and incentives is determined by the interaction of the effects described in the previous paragraphs. Three cases may be identified. First, when marginal utility of incentives is increasing in risk aversion, incentives and risk aversion are positively related. And as effort in variance reduction is increasing in both incentives and risk aversion, the relationship between incentives and risk is negative. The other two cases occur when marginal utility of incentives is decreasing in risk aversion. As incentives and risk aversion are negatively related, the effect in the variance-reduction effort is ambiguous. If the incentive effect dominates, the variance-reduction effort increases with incentives and the relationship between incentives and risk is negative. If the risk-aversion effect dominates, the variance-reduction effort decreases with incentives and the relationship between incentives and risk is positive. We found that the positive relationship between incentives and risk is more likely when risk aversion is low and incentives are high.

The analysis above refers to the endogenous risk, which is the empirically relevant case, since endogenous risk is observable. The relationship between the exogenous risk and incentives remains negative for the pure moral hazard model and the numerical calculations suggest the same result for the model with adverse selection before moral hazard.

APPENDIX

A.1. *Adverse Selection without the Single-Crossing Property.* The general model presented in Section 4 reduces to the maximization problem (15) subject to incentive compatibility and participation constraints. It differs from the traditional adverse selection model because the objective function does not have the single-crossing property. We present below the main steps toward the solution, stressing the peculiarities that arise when single-crossing property is absent. Most of the results are developed in Araujo and Moreira (2001a).

When $\alpha(\cdot)$ and $\beta(\cdot)$ are differentiable, the incentive compatibility may be locally checked by the first- and second-order conditions. These conditions are necessary but not sufficient for incentive compatibility. The first-order condition gives

$$v_\alpha(\alpha(\theta),\theta)\alpha'(\theta)+\beta'(\theta)=0, \tag{19}$$

which states that indifference curves of type θ agent must be tangent to an implementable contract on $\alpha \times \beta$ plane, at point $(\alpha(\theta),\beta(\theta))$.

The second-order condition gives

$$v_{\alpha\alpha}(\alpha(\theta),\theta)[\alpha'(\theta)]^2+v_\alpha(\alpha(\theta),\theta)\alpha''(\theta)+\beta''(\theta)\leq 0, \tag{20}$$

and, after differentiating (19) with respect to θ, the expression (20) simplifies to the condition

$$v_{\alpha\theta}(\alpha(\theta),\theta)\alpha'(\theta)\geq 0, \tag{21}$$

which implies the monotonicity of $\alpha(\theta)$, in the single-crossing context.

Given the menu of implementable contracts $\{\alpha(\theta),\beta(\theta)\}_{\theta\in\Theta}$, the level of utility achieved by the agent with risk aversion θ is his informational rent and denoted $r(\theta)$, that is, $r(\theta)=v(\alpha(\theta),\theta)+\beta(\theta)$. Using (19), we get

$$r'(\theta)=v_\theta(\alpha(\theta),\theta), \tag{22}$$

and applying the envelope theorem on equation (4), we have $v_\theta(\alpha,\theta)=-\frac{1}{2}\alpha^2\sigma^2(e^*)<0$. Consequently, the agent with the highest the risk aversion has the lowest informational rent and the participation constraint is active for him, that is, $r(\theta_b)=0$.

Thus, the fixed component of the wage can be isolated by integration of $r'(\theta)$,

$$\beta(\theta) = -\int_{\theta}^{\theta_b} v_\theta(\alpha(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta} - v(\alpha(\theta), \theta), \tag{23}$$

which allows us to eliminate $\beta(\cdot)$ from the problem and focus on the characterization of $\alpha(\cdot)$.

A.2. *Implementability without the Single-Crossing Property.* Since the single-crossing property is not ensured, the first- and the second-order conditions are necessary but they are not sufficient. The following points must be observed:

1. The function $\alpha(\theta)$ may be non-monotone. The same contract may be chosen by a discrete set of agents. We call this situation as discrete pooling. In this case, the pooled types follow the conjugation rule

   $$v_\alpha(\alpha(\theta), \theta) = v_\alpha(\alpha(\theta'), \theta'), \tag{24}$$

   whenever $\alpha(\theta) = \alpha(\theta')$, which states that the indifference curves of θ and θ' are both tangent at the same point to the menu of contracts on $\alpha \times \beta$ plane.

2. The incentive compatibility must be globally checked. When the single-crossing property holds, local incentive compatibility implies global incentive compatibility, that is, if types in the neighborhood of θ is not better with the contract assigned to θ, no other type will be better. This means that the first- and second-order conditions are sufficient for incentive compatibility. On the other hand, when the single-crossing property is violated, types out of the neighborhood of θ may prefer the contract assigned to θ. In this case, the first- and second-order conditions are not sufficient and

further conditions must be imposed to obtain implementability.

3. The function $\alpha(\theta)$ may be discontinuous. The possibility of discrete pooling creates jumps in the optimal assignment of contracts, so we allow the contract to be piecewise continuous. Where jump occurs, the agent must be indifferent between the start and the end point of the jump. If, for example, the agent θ were strictly better with the end point than the start point, then, for a small $\varepsilon > 0$, the agents with type in $[\theta - \varepsilon, \theta]$ would strictly prefer the end point, and no jump could exist in θ.

The following definition will be useful for global analysis of incentive compatibility. For a given contract $\alpha(\theta)$ define the integral $\Phi(\theta, \hat{\theta})$ as

$$\Phi(\theta, \hat{\theta}) = \int_{\theta}^{\hat{\theta}} \left[\int_{\alpha(\tilde{\theta})}^{\alpha(\hat{\theta})} v_{\alpha\theta}(\tilde{\alpha}, \tilde{\theta}) d\tilde{\alpha} \right] d\tilde{\theta}. \tag{25}$$

It can be shown, using (22), that $\Phi(\theta, \hat{\theta}) = V(\alpha(\theta), \beta(\theta), \theta) - V(\alpha(\hat{\theta}), \beta(\hat{\theta}), \theta)$, thus $\Phi(\theta, \hat{\theta})$ is the difference for agent θ between the utility of the contract assigned to himself and the one assigned to $\hat{\theta}$. The incentive compatibility constraint can be stated as

$$\Phi(\theta, \hat{\theta}) \geq 0, \quad \text{for all } \theta, \hat{\theta} \in \Theta,$$

that is, the agent with risk aversion θ is not better by pretending to be an agent with risk aversion $\hat{\theta}$. The function $\Phi(\theta, \hat{\theta})$ is appropriate for a graphical analysis, since the signal of $v_{\alpha\theta}$ is known and the integration is performed in the region between the constant $\alpha(\hat{\theta})$ and the curve $\alpha(\tilde{\theta})$.

A.3. *Virtual Surplus and the Principal's Problem.* The principal's objective function is in expression (15), where $U(\alpha, \beta) = (1 - \alpha)\mu(e^*(\alpha)) - \beta$. We can eliminante $\beta(\theta)$ using

(23). As types are uniformly distributed, applying Fubini's theorem,

$$E\left[\int_{\theta}^{\theta_b} v_\theta(\alpha(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}\right] = E\left[v_\theta(\alpha(\theta), \theta)(\theta - \theta_a)\right],$$

and the principal's objective function can be rewritten as $E[R(\alpha(\theta), \theta)]$, where

$$R(\alpha, \theta) = S(\alpha, \theta) + (\theta - \theta_a) v_\theta(\alpha, \theta). \tag{26}$$

is the virtual surplus. After the optimal incentive $\alpha^*(\theta)$ is found, the fixed part of optimal contract, $\beta^*(\theta)$, can be calculated using (23). The virtual surplus is the social surplus plus the informational rent term. This term is negative and represents a cost that takes into account the rent that is paid to the agents with risk aversion in $[\theta_a, \theta]$, in order to preserve implementability when agent θ receives $\alpha(\theta)$.

The maximization of the virtual surplus without the constraints is called relaxed problem. Its solution, denoted $\alpha_r(\theta)$, satisfies

$$R_\alpha(\alpha_r(\theta), \theta) = 0 \quad \text{and} \quad R_{\alpha\alpha}(\alpha_r(\theta), \theta) < 0.$$

Since $R_\alpha(\alpha_r(\theta), \theta) = S_\alpha(\alpha_r(\theta), \theta) + (\theta - \theta_a) v_{\alpha\theta}(\alpha_r(\theta), \theta)$, the relaxed solution provides less incentive than the first best when $v_{\alpha\theta} < 0$, and more incentive when $v_{\alpha\theta} > 0$. This distortion occurs because the cross derivative is associated with the marginal cost of informational rent. For example, when $v_{\alpha\theta} < 0$, the cost of informational rent is increasing with respect to α, therefore the principal pays less incentive.

A.4. *Optimality without the Single-Crossing Property.* In the standard adverse selection model, the single-crossing property ensures that $\alpha_r(\theta)$ is the optimal contract if (21) is satisfied, that is, $\alpha_r(\theta)$ is non-increasing when $v_{\alpha\theta} < 0$, or non-decreasing when

$v_{\alpha\theta} > 0$. When $\alpha_r(\theta)$ is non-monotone, the optimal contract is the best combination of $\alpha_r(\theta)$ and intervals of bunching so that (21) is satisfied. Such procedure is not suitable in the absence of the single-crossing property. As before, $\alpha_r(\theta)$ is the optimal contract if it is implementable. However, monotonicity condition (21) is not sufficient for implementability and global incentive condition must be checked.

When $v_{\alpha\theta}$ changes its sign, the discrete pooling is possible and $\alpha_r(\theta)$ is not the optimal contract for the pooled types. The assignment of contracts to the discretely pooled types must take into account the conjugation of types according to the constraint (24). Let $\alpha_u(\theta)$ denote the optimum assignment of contracts with discrete pooling. Then the joint maximization of pooled types results in the condition

$$\frac{R_\alpha(\alpha_u(\theta),\theta)}{v_{\alpha\theta}(\alpha_u(\theta),\theta)} = \frac{R_\alpha(\alpha_u(\theta'),\theta')}{v_{\alpha\theta}(\alpha_u(\theta'),\theta')}. \tag{27}$$

where θ' is given by $v_\alpha(\alpha_u(\theta),\theta) = v_\alpha(\alpha_u(\theta'),\theta')$ and $\alpha_u(\theta) = \alpha_u(\theta')$. The optimal contract will be a combination of $\alpha_r(\theta)$, bunching and $\alpha_u(\theta)$.

We follow Araujo and Moreira (2001a) and restrict the solution $\alpha^*(\theta)$ to the closure of the continuous functions. It means that when there is a jump in $\alpha(\theta)$ all the intermediate contracts in the jump is offered to the agent. The optimal contract with discrete pooling can be characterized under the following assumptions:

H1. $v_{\alpha\theta}(\alpha,\theta) = 0$ defines a decreasing function $\alpha_0(\theta)$, $v_{\alpha\theta}$ is positive above and negative below $\alpha_0(\theta)$, for all $\theta \in \Theta$.

H2. α_r is U-shaped, crosses α_0 in an increasing way, $\alpha_r(\theta_a) \leq \alpha_r(\theta_b)$, $R_\alpha(\alpha,\theta)$ is negative above and positive below $\alpha_r(\theta)$, for all $\theta \in \Theta$.

H3. For each $\hat{\theta}$, the equations $v_\alpha(\alpha_r(\cdot), \cdot) = v_\alpha(\alpha_r(\cdot), \hat{\theta})$ have at most one solution in the decreasing part of α_r, on $v_{\alpha\theta} < 0$ region.

Under these assumptions, the optimal contract, $\alpha^*(\theta)$, will have one of the following forms:

$$\alpha^*(\theta) = \begin{cases} \alpha_u(\theta), & \text{if } \theta < \theta_1, \\ \alpha_r(\theta), & \text{if } \theta \geq \theta_1, \end{cases} \tag{28}$$

where θ_1 is defined by $\alpha_u(\theta_1) = \alpha_u(\theta_a)$, [7] or

$$\alpha^*(\theta) = \begin{cases} \alpha_r(\theta), & \text{if } \theta < \theta_2, \\ \min\{\bar{\alpha}, \alpha_u(\theta)\}, & \text{if } \theta \geq \theta_2, \end{cases} \tag{29}$$

where $\bar{\alpha}$ is the incentive of the continuous bunching and θ_2 is defined by $\alpha_r(\theta_2) = \bar{\alpha}$. The set of bunched types, $J = \{\theta \in \Theta : \alpha(\theta) = \bar{\alpha}\}$, satisfies

$$\int_J R_\alpha(\bar{\alpha}, \theta) p(\theta) d\theta = 0.$$

A.5. *Optimal Contract in the Linear-Quadratic Specification.* The following expression is the virtual surplus of the problem,

$$R(\alpha, \theta) = \frac{\alpha(2-\alpha)}{2} m^2 - \frac{\alpha^2(\alpha^2\theta^2 + 2\theta - \theta_a)}{2(1+\alpha^2\theta)^2} \sigma_0^2.$$

[7] To be rigorous, we should consider the case in which the jump transition from α_u-segment to α_r-segment takes place in $\theta_j < \theta_1$. In this case, the contracts for $[\theta_a, \hat{\theta}_j]$, where $\hat{\theta}_j$ is the conjugate of θ_j, are the conjugates of the contracts in the vertical line, at the jump. For the examples developed in this paper, the characterization above suffices. For further details see Araujo and Moreira (2001a)

The derivative with respect to α is

$$R_\alpha(\alpha,\theta) = (1-\alpha)m^2 - \frac{\alpha[\theta(1+\alpha^2\theta_a)+(\theta-\theta_a)]}{(1+\alpha^2\theta)^3}\sigma_0^2, \tag{30}$$

and the relaxed solution $\alpha_r(\theta)$ is given by $R_\alpha(\alpha_r(\theta),\theta)=0$ and $R_{\alpha\alpha}(\alpha_r(\theta),\theta)<0$. Note that $R_\alpha(0,\theta)>0$ and $R_\alpha(1,\theta)<0$, so the relaxed problem has an interior solution and $R_\alpha(\cdot,\theta)$ has at least one root in the interval $[0,1]$. If $R(\cdot,\theta)$ is not concave in α, the incentive that maximizes the virtual surplus must be correctly chosen among the solutions of the first-order condition.

Writing R_α as a function of σ_0, we can see that $\partial R_\alpha/\partial\sigma_0<0$, and, as $R_{\alpha\alpha}(\alpha_r(\theta),\theta)<0$, the application of the implicit function theorem to $R_\alpha(\alpha_r(\theta),\theta)=0$ gives $d\alpha_r/d\sigma_0<0$. That is, for a given θ, an increase in exogenous risk reduces incentives in the relaxed solution.

When $v_{\alpha\theta}(\alpha_r(\theta),\theta)$ has ambiguous sign, the optimal contract must consider the possibility of discrete pooling. When θ and $\hat{\theta}$ are discretely pooled at incentive α, the conjugation rule (24) relates the pooled types by $\hat{\theta}(\alpha,\theta)=1/\theta\alpha^4$. Then, working on condition (27), we obtain the discrete pooling segment $\alpha_u(\theta)$ as the solution of the equation

$$(1-\alpha)(1+\theta\alpha^2)^2(1+\theta^2\alpha^4) = 2\theta^2\alpha^3\frac{\sigma_0^2}{m^2}.$$

The numerical examples presented in Section 5 correspond to three cases for which we can characterize the optimal contract.

(a) $\alpha_r(\theta)$ is increasing and $v_{\alpha\theta}(\alpha_r(\theta),\theta)>0$.

Since $\alpha_0(\theta)$ is decreasing, the integral in $\Phi(\theta,\hat{\theta})$ takes values in $v_{\alpha\theta}>0$ region. Therefore, $\Phi(\theta,\hat{\theta})>0$ and $\alpha_r(\theta)$ is the optimal contract.

(b) $\alpha_r(\theta)$ is decreasing and $v_{\alpha\theta}(\alpha_r(\theta),\theta) < 0$.

A sufficient condition for implementability is $v_{\alpha\theta}(\alpha_r(\theta_a),\theta_b) < 0$. As $\alpha_0(\theta)$ is a decreasing function, the integral in $\Phi(\theta,\hat{\theta})$ takes values in $v_{\alpha\theta} < 0$ region. Then $\Phi(\theta,\hat{\theta}) > 0$ and $\alpha_r(\theta)$ is the optimal contract.

(c) $v_{\alpha\theta}(\alpha_r(\theta),\theta)$ changes sign only once.

In this case, the optimal contract can be computed, if assumptions H1, H2 and H3 hold. Assumption H1 holds since, from equation (13), the function $\alpha_0(\theta) = 1/\sqrt{\theta}$ defines a decreasing border between $v_{\alpha\theta} > 0$ and $v_{\alpha\theta} < 0$ regions, with $v_{\alpha\theta} > 0$, for $\alpha > \alpha_0$. The following lemma shows that the first part of assumption H2 holds.

LEMMA 1 *Let θ_x be defined by $\alpha_r(\theta_x) = \alpha_0(\theta_x)$. If θ_x exists, $\alpha_r'(\theta_x) > 0$.*

PROOF. By definition, $\alpha_r(\theta)$ satisfies $R_\alpha(\alpha_r(\theta),\theta) = 0$. Using the implicit function theorem,

$$\alpha_r'(\theta) = -\frac{R_{\alpha\theta}(\alpha_r(\theta),\theta)}{R_{\alpha\alpha}(\alpha_r(\theta),\theta)},$$

and, as second-order condition states that $R_{\alpha\alpha}(\alpha_r(\theta),\theta) < 0$, $\alpha_r'(\theta)$ has the same sign as $R_{\alpha\theta}(\alpha_r(\theta),\theta)$. Differentiating R_α with respect to θ,

$$R_{\alpha\theta}(\alpha,\theta) = \frac{-2\alpha[1 - 2\alpha^2(\theta-\theta_a) - \alpha^4\theta\theta_a]}{(1+\alpha^2\theta)^4}$$

and manipulating this expression, we conclude that $\alpha_r'(\theta)$ has the same sign as

$$h(\alpha,\theta) = \theta - \frac{1+2\alpha^2\theta_a}{\alpha^2(2+\alpha^2\theta_a)}.$$

On $\alpha_0(\theta)$, $\alpha = 1/\sqrt{\theta}$. Then $h(\alpha_0(\theta_x), \theta_x) = \theta_x(1-\theta_a/\theta_x)(2+\theta_a/\theta_x)$, which is positive for $\theta_x > \theta_a$. Therefore, $\alpha_r'(\theta_x) > 0$. ■

However, the second part of H2 and H3 are not valid for every value of parameters and must be checked numerically.

The following lemma proves that the relaxed solution is a decreasing function of exogenous variance.

LEMMA 2 *For a given θ, $d\alpha_r/d\sigma_0^2 < 0$.*

PROOF. Redefine R_α in (30) as a function of α and σ_0^2. Thus

$$\frac{d\alpha_r}{d\sigma_0^2} = -\frac{R_{\alpha\sigma_0^2}(\alpha, \sigma_0^2)}{R_{\alpha\alpha}(\alpha, \sigma_0^2)},$$

and the theorem is proved as $R_{\alpha\sigma_0^2}(\alpha, \sigma_0^2) = -\alpha[\theta(1+\alpha^2\theta_a)+(\theta-\theta_a)]/(1+\alpha^2\theta)^3 < 0$.

■

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