ASYMPTOTIC STABILITY OF RIEMANN SOLUTIONS IN BGK APPROXIMATIONS TO CERTAIN MULTIDIMENSIONAL SYSTEMS OF CONSERVATION LAWS

HERMANO FRID AND LEONARDO RENDÓN

ABSTRACT. We prove the asymptotic stability of nonplanar two-states Riemann solutions in BGK approximations of a class of multidimensional systems of conservation laws. The latter consists of systems whose flux-functions in different directions share a common complete system of Riemann invariants, the level surfaces of which are hyperplanes. The asymptotic stability to which the main result refers is in the sense of the convergence as $t \to \infty$ in $L^1_{\rm loc}$ of the space of directions $\boldsymbol{\zeta} = \mathbf{x}/t$. That is, the solution $z(t, \mathbf{x}, \xi)$ of the perturbed Cauchy problem for the corresponding BGK system satisfies $\int_X z(t, t\boldsymbol{\zeta}, \xi) d\mu(\xi) \to R(\boldsymbol{\zeta})$ as $t \to \infty$, in $L^1_{\rm loc}(\mathbb{R}^n)$, where $R(\boldsymbol{\zeta})$ is the self-similar entropy solution of the two-states nonplanar Riemann problem for the system of conservation laws.

1. INTRODUCTION

We consider a system of conservation laws

$$\partial_t u + \sum_{i=1}^{n} \partial_{x_i} f^i(u) = 0 \qquad (t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^n, \tag{1.1}$$

with initial condition

$$u(t, \mathbf{x})|_{t=0} = u_0(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n, \tag{1.2}$$

where $u \in \mathbb{R}^m$, $f^i(u) \in \mathbb{R}^m$ and $u_0 \in L^\infty(\mathbb{R}^n)$.
We are interested in approximations of the problem (1.1),(1.2) of BGK-type which have the form

$$\begin{gathered} \partial_t z + a(\xi).\nabla_{\mathbf{x}} z = M_\xi(u) - z. \\ u(t, \mathbf{x}) = \int_X z(t, \mathbf{x}, \xi) d\mu(\xi) \end{gathered} \tag{1.3}$$

with the initial condition

$$z(0, \mathbf{x}, \xi) = z_0(\mathbf{x}, \xi) \tag{1.4}$$

where $z(t, \mathbf{x}, \xi) \in \mathbb{R}^m$, (X, Ω, μ) is a finite measure space and $a : X \to \mathbb{R}^n$ and $M : X \times \mathbb{R}^m \to \mathbb{R}^m$, with $M(\xi, z) = M_\xi(z)$, are bounded measurable functions satisfying structural conditions described below (see (H5)-(H9)). Here, $a(\xi).\nabla_{\mathbf{x}}$ denotes $\sum_{i=1}^n a^i(\xi) \frac{\partial}{\partial x_i}$.

1991 *Mathematics Subject Classification.* Primary:35B40,35B35;Secondary:35L65,35K55.

Key words and phrases. conservation laws, BGK approximations,Riemann problems, asymptotic stability, multidimensional systems.

More specifically, we are concerned with the asymptotic stability of solutions of problem (1.3),(1.4) when the initial condition satisfies

$$z_0(\mathbf{x},\xi) = M_\xi(u_0(\mathbf{x})), \tag{1.5}$$

$$u_0(\mathbf{x}) = R_0(\mathbf{x}) + P(\mathbf{x}), \tag{1.6}$$

where

$$R_0(\mathbf{x}) = \begin{cases} u_L, & \text{if } x_i < 0, \text{ for some } i \in \{1,\dots,n\} \\ u_R, & \text{if } x_i > 0, \text{ for all } i \in \{1,\dots,n\} \end{cases} \tag{1.7}$$

and $P \in L^\infty(\mathbb{R}^n)$ is such that

$$P(T\mathbf{x}) \to 0 \ \text{ when } T \to \infty \text{ in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{1.8}$$

Namely, we seek to answer the question whether

$$u(t,t\boldsymbol{\zeta}) = \int_X z(t,t\boldsymbol{\zeta},\xi)d\mu(\xi) \to R(\boldsymbol{\zeta}) \text{ when } t \to \infty \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \tag{1.9}$$

where R is a well determined self-similar entropy solution of the non-planar Riemann problem (1.1),(1.7).

We recall that a pair $(\eta(u),\mathbf{q}(u))$ is an entropy-entropy flux pair for (1.1) if

$$\nabla\mathbf{q}(u) = \nabla\eta(u)\nabla\mathbf{f}(u), \tag{1.10}$$

where we denote $\mathbf{f} = (f^1,\dots,f^n)$.

As usual, by an entropy solution of (1.1), we mean a bounded measurable function $u(t,\mathbf{x})$ satisfying the integral inequality

$$\int_{\mathbb{R}^{n+1}_+} \eta(u)\phi_t + \mathbf{q}(u)\cdot\nabla_{\mathbf{x}}\phi\,d\mathbf{x}\,dt + \int_{\mathbb{R}^n}\eta(u_0)\phi(0,\mathbf{x})\,d\mathbf{x} \ge 0, \tag{1.11}$$

for all entropy-entropy flux pair for (1.1), $(\eta(u),\mathbf{q}(u))$, with $\eta(u)$ convex, and for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^{n+1})$. For an account on the basic concepts and recent developments of the theory of conservation laws we refer to [9, 25].

We will consider systems (1.1) where the $f^i(u)$ are in the class of what we call *Temple fields*, after [29], and we further impose some additional conditions on f^i. Namely, we will assume that the fields f^i satisfy the following conditions on a region $\mathcal{O}$ of the phase space where the solutions of (1.1)-(1.2) take values:

(H1) There exists a system of smooth functions $\{w_j(u)\}_{j=1}^m$, Riemann invariants, satisfying

$$\nabla w_j(u)\nabla f^i(u) = \lambda^i_j(u)\nabla w_j(u), \quad u\in\mathcal{O}, \qquad i=1,\dots,n,\ j=1,\dots,m, \tag{1.12}$$

where the λ^i_j are then the eigenvalues of ∇f^i, $i=1,\dots,n$, $j=1,\dots,m$;

(H2) For each $u\in\mathcal{O}$, $\{\nabla w_j(u)\}_{j=1}^m$ is a linearly independent set and the level surfaces $\{w_i = \text{constant}\}$ are hyperplanes;

(H3) For each $i\in\{1,\dots,n\}$, there exist constants $\kappa^i_1 < \kappa^i_2 < \cdots < \kappa^i_{m+1}$, such that

$$\kappa^i_1 < \lambda^i_1(u) < \kappa^i_2 < \cdots < \kappa^i_m < \lambda^i_m(u) < \kappa^i_{m+1}, \qquad \text{for all } u\in\mathcal{O}. \tag{1.13}$$

(H4) There exists a common system of unit right-eigenvectors $\{r_j(u)\}_{j=1}^m$ for $\nabla f^i(u)$, $i=1,\dots,n$, such that

$$\nabla\lambda^i_j(u)\cdot r_j(u) > 0, \quad u\in\mathcal{O}, \qquad i=1,\dots,n,\ j=1,\dots,m. \tag{1.14}$$

The so called Maxwellians functions M_ξ, are of class C^2 in the closure of an open bounded convex set with piecewise smooth boundary, $\mathcal{O}' \subseteq \mathbb{R}^m$. Let us denote by K the closure of $\mathcal{O}'$ and assume $K \subseteq \mathcal{O}$, where $\mathcal{O}$ is the set appearing in the conditions (H1)-(H4). We assume that K is an invariant domain for (1.1), by which we mean:

(H5) if $\nu(\omega)$ is an outer unit normal to ∂K, at a point ω of smoothness of ∂K, then $\nu(\omega)$ is a left eigenvector for all $\nabla f^i(\omega)$, $i = 1, \dots, m$.

We further suppose the following consistence conditions:

(H6) $\forall s \in K : \int_X M_\xi(s) d\mu(\xi) = s$;
(H7) $\forall s \in K : \int_X a^i(\xi) M_\xi(s) d\mu(\xi) = f^i(s)$;
(H8) $\forall (s,\xi) \in K \times X$ the spectrum of $(dM_\xi(s))$ is included in $(0,\infty)$;
(H9) $\forall (s,\xi) \in K \times X$ the Jacobian matrices $\nabla M_\xi(s)$ and $\nabla f^i(s)$ commute for each $i = 1, \cdots, n$.

As an example (*cf.* [23]), we observe that if a satisfies

$$\begin{aligned}
&\int_X a^i(\xi) d\mu(\xi) = 0,\ \forall i = 1, \cdots, n,\\
&\int_X a^i(\xi) a^j(\xi) d\mu(\xi) = 0, i \neq j \text{ and}\\
&1 + \sum_{i=1}^n \frac{a^i(\xi)}{\int_X |a^i(\xi)|^2 d\mu(\xi)} \lambda_j^i(s) > 0, \forall s \in K, \forall \xi \in X,\ j = 1 \cdots m
\end{aligned}$$

then

$$M_\xi(s) = s + \textstyle\sum_{i=1}^n \frac{f^i(s) a^i(\xi)}{\int_X |a^i(\xi)|^2 d\mu(\xi)},$$

is an example of Maxwellian function.

So from the conditions (H9),(H5),(H6) and (H7) it is possible to conclude that:

$$\begin{aligned}
&\forall (s,\xi) \in K \times X :\ dM_\xi(s) r_j(s) = \rho_j(\xi,s) r_j(s)\ ,\\
&\forall s \in K :\ \int_X \rho_j(\xi,s) d\mu(\xi) = 1\ ,\\
&\forall s \in K :\ \int_X a^j(\xi) \rho_j(\xi,s) d\mu(\xi) = \lambda_j(s) \text{ and}\\
&\forall (s,\xi) \in K \times X :\ \rho_j(\xi,s) > 0\ ,
\end{aligned}$$

respectively.

We now state our main theorem.

Theorem 1.1. *Suppose the hypotheses* (H1)-(H4) *concerning system* (1.1) *hold. Suppose the Maxwellian* $M_\xi(z)$ *satisfies the conditions* (H5)–(H9) *and* z_0 *satisfies* (1.5)–(1.8)*. Then,* (1.3), (1.4) *admits a unique* L^∞ *solution* $z(t, \mathbf{x}, \xi)$ *and, moreover, it verifies* (1.9).

The analogous result for the parabolic artificial viscosity approximation of (1.1) was obtained in [14]. As in the latter, the method for obtaining (1.9) is motivated by the general approach introduced in [6]. The main point in the strategy is to prove the relation

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |u(t, t\boldsymbol{\zeta}) - R(\boldsymbol{\zeta})|\, dt = 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{1.15}$$

An important aspect of (1.15) is its equivalence to the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ of the scaling sequence $\{u^T\}$, given by $u^T(t,\mathbf{x})=u(Tt,T\mathbf{x})$, to $R(\mathbf{x}/t)$ when $T\to+\infty$. This fact is frequently useful when trying to prove (1.15). Once (1.15) is proved, a standard procedure established in [6] is then used to strengthen (1.15) into (1.9). This strengthening is similar to the ones encountered in [7, 8, 19].

We recall that a well known example of Temple field is given by $f(u)=\frac{1}{D(u)}(k_1u_1,\dots,k_mu_m)^\top$, $0<k_1<\cdots<k_m$, $D(u)=1+u_1+\cdots+u_m$, which is found in a model system in chromatography (see [1]). One-dimensional systems in which the flux functions are Temple fields have been studied by many authors after the first analysis carried out in [29], followed by [22, 26]. The class of systems for which our result applies clearly includes the case where $f^i(u)=a^if(u)$, $a^i\in\mathbb{R}$, $i=1,\dots,n$, and f is any Temple field in $\mathbb{R}^m$. For $m=2$, it was proved in [29] (see also [26]) that the Temple fields whose Riemann invariants are $w_1(u)$, $w_2(u)$, are given by

$$f_1(u)=\frac{h_1(w_1)-h_2(w_2)}{w_1-w_2},\qquad f_2(u)=\frac{w_2h_1(w_1)-w_1h_2(w_2)}{w_2-w_1},\tag{1.16}$$

where h_1 and h_2 are arbitrary smooth functions of one variable. This shows that, at least for $m=2$, it is possible to give examples of systems (1.1) with flux functions satisfying our assumptions, beyond the simplest case when all of them are multiples of one only field.

We also remark that our hypotheses (H1)-(H4) imply, in particular, that the system (1.1) is hyperbolic, in the sense that given any $\boldsymbol{\zeta}=(\zeta_1,\dots\zeta_n)\in\mathbb{R}^n$ the matrix $\zeta_1\nabla f^1(u)+\cdots+\zeta_n\nabla f^n(u)$ is diagonalizable (cf., [9, 26]). Moreover, it is well known that Temple systems admit strictly convex smooth entropies (cf., [26]), and (1.10) implies that the matrices $\nabla^2\eta(u)\nabla f^i(u)$ are symmetric, $i=1,\dots,n$. In particular, (1.1) is symmetrizable (cf., [18, 17, 3]). Hypothesis (H1) implies that the Jacobian matrices $\nabla f^i(u)$ commute. In this connection, we recall that Brenner [5] proved that commutation of the Jacobian matrices is a necessary condition for the stability of L^p norms of the solutions of (1.1) in the case where the f^i's are linear (see also [24]). Conversely, in 2×2 case, using Riemann invariants and entropies, Dafermos [10, 11] proved the stability of the L^p norms of entropy solutions of (1.1), provided they are close to constant state.

The approximation of (1.1),(1.2) by (1.3),(1.4) was proposed by Natalini in [23] for multidimensional scalar condservation laws and in [2] for systems. A rigorous result of convergence toward the scalar equilibrium was proved in [23]. Sucessivelly, Serre proved in [27] the convergence of that approximation to equilibrium systems satisfying appropriate conditions. The corresponding convergence results in the discrete case were then obtained in [21]. Under more general hypotheses, Bouchut [4] gave a complete classification of BGK models for systems of conservation laws endowed with a family of kinetc entropies, namely, entropies for the semilinear approximating system, which reduce to given entropies of the limit system, as the relaxation parameter tends to zero.

This paper is organized in the following way. In Section 2 we establish some auxiliary results concerning the system (1.3) whose goal is to reduce solution of the question at hand to a problem of showing that certain probability measures reduces to Dirac measures, similar in spirit to the applications of compensated compactness theory to conservation laws first given by Tartar [28] and DiPerna

[12]. This reduction is finally achieved in Section 3, and what remains is the same analysis of the referred probability measures that was developed in [14].

2. Auxiliary Results

We first observe that conditions (H5) and (H8) imply that, for each $\xi \in X$, the map M_ξ is a diffeomorphism of K over its image $K_\xi = M_\xi(K)$, and K_ξ is convex, as a consequence of the general results about maps of convex sets proved by Serre [27] and Frid [15, 16]. As a consequence of these facts we have the following (*cf.* [27]).

Lemma 2.1. *If $w : X \to \mathbb{R}^m$ is measurable and such that $w(\xi) \in K_\xi$, for all $\xi \in X$, then $\int_X w(\xi)\,d\mu(\xi) \in K$.*

Proof. From (H5), (H6) and the fact that K_ξ is convex, for all $\xi \in X$, given $\omega \in \partial K$, a point of smoothness of ∂K, we have

$$\langle \int_X w(\xi)\,d\mu(\xi) - \omega, \nu(\omega)\rangle = \int_X \langle w(\xi) - M_\xi(\omega), \nu(\omega)\rangle\,d\mu(\xi) \le 0,$$

and this proves the assertion. □

As explained in [27], given an entropy-entropy flux pair $(\eta, \mathbf{q})$ for (1.1), we can define the function, so called entropy function for (1.3),

$$e_\xi = q_\xi \circ M_\xi^{-1} \tag{2.1}$$

where $M_\xi^{-1} : K_\xi \to K$ is the inverse map of M_ξ and q_ξ satisfies $\nabla q_\xi(z) = \nabla\eta(z)\nabla M_\xi(z)$, that is, it is the flux of the entropy η for the system

$$\partial_t v + \partial_x M_\xi(v) = 0.$$

It has the fundamental property that

$$\nabla e_\xi(M_\xi(w)) = \nabla\eta(w),$$

which follows from the trivial identity $\nabla e_\xi(z) = \nabla\eta(M_\xi^{-1}(z))$. As in [27], we introduce the entropy functionals

$$E[w] = \int_X e_\xi(w(\xi))d\mu(\xi),\ \ F[w] = \int_X a(\xi)e_\xi(w(\xi))d\mu(\xi), \tag{2.2}$$

defined for $w \in D := \{w : X \to \mathbb{R}^m\ :\ w \text{ is measurable and } w(\xi) \in K_\xi, \text{ for all } \xi \in X\}$. In particular, we have $(E[w], F[w]) = (\eta(u), \mathbf{q}(u))$ if $w = w(\xi) = M_\xi(u)$. We also remark that when η is strictly convex function then E and e_ξ are strictly convex functions. By abuse of nomenclature, we will also refer to (E, F) as an entropy-entropy flux pair associated with (1.3).

We say that $z(t, \mathbf{x}, \xi)$ is a solution of (1.3),(1.4) if $z \in L^\infty(\mathbb{R}^n \times (0,\infty) \times X)$ and, given any entropy η for (1.1), if e_ξ is defined by (2.1), we have

$$\partial_t e_\xi(z) + a(\xi)\cdot\nabla_{\mathbf{x}} e_\xi(z) = de_\xi(z).(M_\xi - z(\xi)), \tag{2.3}$$

in the sense of distributions in $\mathbb{R}^n \times (0,\infty)$, for a.e. $\xi \in X$, and

$$z(\mathbf{x}, t, \xi) \to z_0(\mathbf{x}, \xi) \quad \text{as } t \to 0, \text{ in } L^1_{loc}(\mathbb{R}^n \times X). \tag{2.4}$$

In particular, if $z(\mathbf{x}, t, \xi)$ is a solution of (1.3),(1.4), for any pair (E, F) satisfying (2.2) we have

$$\partial_t E[z] + \nabla_{\mathbf{x}}\cdot F[z] = \int_X de_\xi(z).(M_\xi - z(\xi))d\mu(\xi), \tag{2.5}$$

and, since $de_\xi(M_\xi(u)) = d\eta(u)$ and $d\eta(u)$ does not dependent of ξ, from

$$\int_X (M_\xi(u) - z(\xi))\,d\mu(\xi) = 0,$$

we have

$$\partial_t E[z] + \nabla_{\mathbf{x}} \cdot F[z] = \int_X (de_\xi(z) - de_\xi(M_\xi(u))).(M_\xi - z(\xi))d\mu(\xi). \tag{2.6}$$

Proposition 2.1. *Let hypotheses* (H5) *to* (H9) *hold and let the initial condition* $z_0 : \mathbb{R}^n \times X \to \mathbb{R}^m$ *be bounded, measurable and satisfy* $z(\mathbf{x}, \xi) \in K_\xi$*, for all* $\xi \in X$*. Then the Cauchy problem* (1.3),(1.4) *has a unique global solution* $z : \mathbb{R}^+ \times \mathbb{R}^n \times X \to \mathbb{R}^m$ *satisfying* $z(t, \mathbf{x}, \xi) \in K_\xi$*, for all* $\xi \in X$*.*

Proof. Since the statement follows from well known arguments, we just sketch them. We first assume that $z_0(\cdot, \xi)$ is a Lipschitz continuous function on $\mathbb{R}^n$, for each $\xi \in X$. We consider the operator

$$\mathcal{L}(z) = S_t(z_0(\xi)) + \int_0^t S_{t-\tau}(M_\xi(u(\tau, \mathbf{x})) - z(\tau, \mathbf{x}, \xi))d\tau, \tag{2.7}$$

where S_t is the semigroup associated with the equation $\partial_t z + a(\xi).\nabla_{\mathbf{x}} z = 0$. We view $\mathcal{L}$ as an operator defined in $L^\infty([0, T] \times \mathbb{R}^n \times X)$. For $T > 0$ sufficiently small we easily verify that $\mathcal{L}$ is a contraction. Hence, by Banach's fixed point theorem, $\mathcal{L}$ possesses a unique fixed point in $L^\infty([0, T] \times \mathbb{R}^n \times X)$. Let $z(t, \mathbf{x}, \xi)$ be the fixed point of $\mathcal{L}$ in $L^\infty([0, T] \times \mathbb{R}^n \times X)$. Using the smoothness of z_0, as a function of $\mathbf{x}$, we easily verify that $z(t, \mathbf{x}, \xi)$ is a classical solution of (1.3),(1.4) in $\mathbb{R}^n \times [0, T] \times X$. As a consequence of the convexity of K_ξ, the right hand member of (1.3), that is, the vector $M_\xi(u) - z(t, x, \xi)$, points inwards K_ξ whenever $z(t, x, \xi) \in \partial K_\xi$. Hence, we deduce that $z(t, x, \xi) \in K_\xi$, for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times X$. In particular, since $z(T, \mathbf{x}, \xi) \in K_\xi$ and is Lipschitz continuous as a function of $\mathbf{x} \in \mathbb{R}^n$, we may now consider the operator

$$\mathcal{L}(z(\xi)) = S_t(z(T, ., \xi)) + \int_T^t S_{t-\tau}(M_\xi(u(\tau, \mathbf{x})) - z(\tau, \mathbf{x}, \xi))d\tau,$$

defined in $L^\infty([T, 2T] \times \mathbb{R}^n \times X)$ and repeat the argument above so that the global assertion can be obtained by induction. Now, using Gronwall's lemma and standard arguments we prove the stability property

$$\int_X \int_{|\mathbf{x}|<R} |z_1(t, \mathbf{x}, \xi) - z_2(t, \mathbf{x}, \xi)|\,d\mathbf{x}\,d\mu(\xi)$$
$$\le C(t) \int_X \int_{|\mathbf{x}|<R+M_0 t} |z_{01}(\mathbf{x}, \xi) - z_{02}(\mathbf{x}, \xi)|\,d\mathbf{x}\,d\mu(\xi),$$

valid for any two solutions of the problem (1.3),(1.4), with initial data z_{01} and z_{02}. Hence, for general initial data $z_0(\mathbf{x}, \xi)$, we may approximate $z_0(\mathbf{x}, \xi)$ by functions $z_{0k}(\mathbf{x}, \xi)$, Lipschitz continuous in $\mathbf{x}$, $k \in \mathbb{N}$, and use the above stability property to get a solution of (1.3),(1.4) in the limit as $k \to \infty$. Uniqueness also follows from the above stability property. □

We next introduce several notations defining sets which will play a crucial role in the forthcoming developments. So, let K, M, T_0 be given fixed positive numbers.

We denote:

$$
\begin{aligned}
&\Omega^+(\boldsymbol{\zeta}) = \{(t,\mathbf{x}) \,:\, x_1 > \zeta_1 t, \dots, x_n > \zeta_n t\},\\
&\Omega^-(\boldsymbol{\zeta}) = \{(t,\mathbf{x}) \,:\, x_i < \zeta_i t, \text{ for some } i \in \{1,\dots,n\}\},\\
&|\mathbf{x}|_\infty = \max\{|x_i| \,:\, i = 1,\dots,n\},\\
&\Omega(T) = \{(t,\mathbf{x}) \,:\, |\mathbf{x}|_\infty \le MT_0T + K(T_0T - t),\ 0 \le t \le T_0T\},\\
&\Omega^\pm(\boldsymbol{\zeta},T) = \Omega^\pm(\boldsymbol{\zeta}) \cap \Omega(T),\\
&\nu^i[\boldsymbol{\zeta}] = (\zeta_i, 0, \dots, \underbrace{-1}_{(i+1)\text{-th}}, 0, \dots, 0),\\
&\partial\Omega^+(\boldsymbol{\zeta}) = \cup_{i=1}^n S_i(\boldsymbol{\zeta}), \quad \text{where } \nu^i[\boldsymbol{\zeta}] \perp S_i(\boldsymbol{\zeta})\\
&S_i(\boldsymbol{\zeta},T) = S_i(\boldsymbol{\zeta}) \cap \Omega(T),\\
&\partial^0\Omega^+(\boldsymbol{\zeta},T) = \cup_{i=1}^n S_i(\boldsymbol{\zeta},T),\\
&S_i^\tau(\boldsymbol{\zeta},T) = S_i(\boldsymbol{\zeta},T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\},\\
&\Omega_\tau(T) = \Omega(T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\},\\
&\Omega_\tau^\pm(\boldsymbol{\zeta},T) = \Omega^\pm(\boldsymbol{\zeta},T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\}.
\end{aligned}
$$

Proposition 2.2. *Suppose that there is a strictly convex entropy associated with* (1.1) *and* $z(\xi) \in L^\infty\big(\mathbb{R}_+^{n+1}\big)$ *is solution of* (1.3), (1.4). *Then, for a.e.* $\boldsymbol{\zeta} \in \mathbb{R}^n$, *we have* $\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_X |M_\xi - z(\xi)|^2 (t, t\boldsymbol{\zeta}) d\mu(\xi) dt = 0$. *In particular, for a.e.* $\boldsymbol{\zeta} \in \mathbb{R}^n$, $M, K, T_0 > 0$,

(2.8)
$$
\lim_{T\to\infty} \frac{1}{T^n} \int_0^T dt \int_{S_i^t(\boldsymbol{\zeta},T)} \int_X |M_\xi - z(\xi)|^2 (t,\mathbf{x}) d\mu(\xi) d\mathcal{H}^{n-1} = 0, \qquad i = 1,\dots,n.
$$

Proof. We use the method introduced in [13]. For $\mathbf{x} \in \mathbb{R}^n$ and $i \in \{1,\dots,n\}$ we denote $\overline{\mathbf{x}}^i = (x_1,\dots,x_{i-1})$, if $i \ge 2$, and ${}^i\overline{\mathbf{x}} = (x_{i+1},\dots,x_n)$, if $i \le n-1$. For $i = 1$ (resp., $i = n$) we simply ignore $\overline{\mathbf{x}}^i$ (resp., ${}^i\overline{\mathbf{x}}$) wherever it appears. Let (E,F) be an entropy-entropy flux pair for (1.3), with E strictly convex. Following [13], we divide (2.6) by $(1+t)^{n+\theta}$ with $0 < \theta < 1$ and integrate over

$$
\Omega_T = \{\, (t,\mathbf{x}) \mid |\mathbf{x}|_\infty \le Mt, 0 \le t \le T \,\}.
$$

One obtains

$$
\begin{aligned}
&\int_{|\mathbf{x}|_\infty \le MT} \frac{E[z](T,\mathbf{x})}{(1+T)^{n+\theta}}\, d\mathbf{x} + \iint_{\Omega_T} \frac{E[z]}{(1+t)^{n+\theta+1}}\, d\mathbf{x}dt\\
&\quad + \sum_{i=1}^n \int_0^T dt \int_{|(\overline{\mathbf{x}}^i,\, {}^i\overline{\mathbf{x}})|_\infty \le Mt} \left[\frac{-\zeta_i E[z] + F^i[z]}{(1+t)^{n+\theta}}\right]_{\zeta_i=-M}^{\zeta_i=M} d\overline{\mathbf{x}}^i d\,{}^i\overline{\mathbf{x}}\\
&\quad = -\iint_{\Omega_T} \int_X \frac{d^2 e_\xi(\gamma_\xi).(M_\xi - z(\xi))^2}{(1+t)^{n+\theta}} d\mu(\xi) d\mathbf{x}dt,
\end{aligned}
$$

for some $\gamma_\xi \in K_\xi$. It follows that the left-hand side of the above equation is bounded independently of $T > 0$. We then deduce that the right-hand side is also bounded independently of $T > 0$, and, so, from the strict convexity of e_ξ we get

$$
\int_0^\infty dt \int_{|\mathbf{x}|_\infty \le Mt} \int_X \frac{|M_\xi - z(\xi)|^2}{(1+t)^{n+\theta}}\, d\mu(\xi) d\mathbf{x} < \infty.
$$

Therefore, changing coordinates, one has

$$\int_0^\infty dt \int_{|\boldsymbol{\zeta}|_\infty \le M} \int_X \frac{|M_\xi - z(\xi)|^2(t, \boldsymbol{\zeta} t)}{(1+t)^\theta} \, d\mu(\xi) d\boldsymbol{\zeta} < \infty.$$

Hence, for a.e. $\boldsymbol{\zeta} \in \mathbb{R}^n$, with $|\boldsymbol{\zeta}| \le M$, we have

$$\int_0^\infty \int_X \frac{|M_\xi - z(\xi)|^2(t, \boldsymbol{\zeta} t)}{(1+t)^\theta} \, d\mu(\xi) dt < \infty. \tag{2.9}$$

But, given $\boldsymbol{\zeta}$ satisfying (2.9) one obtains for $T > 1$,

$$\frac{1}{T} \int_0^T \int_X |M_\xi - z(\xi)|^2(t, \boldsymbol{\zeta} t) \, d\mu(\xi) dt \le \frac{2^\theta}{T^{1-\theta}} \int_0^\infty \int_X \frac{|M_\xi - z(\xi)|^2(t, \boldsymbol{\zeta} t)}{(1+t)^\theta} \, d\mu(\xi) dt,$$

and then using (2.9) we get

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X |M_\xi - z(\xi)|^2(t, \boldsymbol{\zeta} t) \, d\mu(\xi) dt = 0. \tag{2.10}$$

We observe that (2.10) and Fubini's theorem imply that

$$\lim_{T \to \infty} \frac{1}{T^n} \int_{\epsilon_0}^T dt \int_{S_i^t(\boldsymbol{\zeta}, T)} \int_X |M_\xi - z(\xi)|^2 (t, \mathbf{x}) d\mu(\xi) d\mathcal{H}^{n-1} = 0, \qquad i = 1, \dots, n. \tag{2.11}$$

For any $\epsilon_0 > 0$. Then we have (2.8). $\square$

Proposition 2.3. *Suppose the hypotheses of Proposition 2.2 hold then for fixed $K, T_0 > 0$, and $M_0 > 0$ sufficiently large, given any subsequence $T_k \to \infty$, there exists a further subsequence $T_l = T_{k_l} \to \infty$, such that, for a.e. $\boldsymbol{\zeta} \in \mathbb{R}^n$ and $M > M_0 > 0$, we have*

$$\lim_{T_l \to \infty} \frac{1}{T_l^n} \int_{\partial\Omega_t(\boldsymbol{\zeta}, T_l)} dt \int_X |M_\xi - z(\xi)|^2 \, d\mu(\xi) d\mathcal{H}^{n-1} = 0. \tag{2.12}$$

Proof. We define $z^T(t, \mathbf{x}, \xi) = z(Tt, T\mathbf{x}, \xi)$, $u^T = \int_X z^T(\xi) \, d\mu(\xi)$, and $M_\xi^T = M_\xi(u^T)$. We have that $\int_X |M_\xi^T - z^T(\xi)|^2 d\mu(\xi)$ converges to 0 in $L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$ as $T \to \infty$ if and only if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X |M_\xi - z(\xi)|^2 \, d\mu(\xi) dt = 0, \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^n),$$

as observed in [13]. Hence, by Proposition 2.2, $\int_X |M_\xi^T - z^T(\xi)|^2 d\mu(\xi)$ converges to 0 in $L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Therefore, given any subsequence $T_k \to \infty$, we can then extract a further subsequence $T_l = T_{k_l} \to \infty$, such that $\int_X |M_\xi^{T_l} - z^{T_l}(\xi)|^2 d\mu(\xi) \to 0$ a.e. in $\mathbb{R}^{n+1}_+$. By Fubini's theorem we then have that $\int_X |M_\xi^{T_l} - z^{T_l}(\xi)|^2 d\mu(\xi) \to 0$ a.e. in $\partial\Omega^+(\boldsymbol{\zeta})$ for a.e. $\boldsymbol{\zeta} \in \mathbb{R}^n$, and also $\int_X |M_\xi^{T_l} - z^{T_l}(\xi)|^2 d\mu(\xi) \to 0$ a.e. in $\partial\Omega(1)$ for a.e. $M > M_0$, which imply that $\int_X |M_\xi^{T_l} - z^{T_l}(\xi)|^2 d\mu(\xi) \to 0$ a.e. in $\partial\Omega(\boldsymbol{\zeta}, 1)$ for a.e. $\boldsymbol{\zeta} \in \mathbb{R}^n$ and $M > M_0$. Since (2.12) may be rewrite as

$$\lim_{T_l \to \infty} \frac{1}{T_l} \int_0^{T_0} dt \int_{\partial\Omega_t(\boldsymbol{\zeta}, 1)} \int_X |M_\xi^{T_l} - z^{T_l}(\xi)|^2 d\mu(\xi) d\mathcal{H}^{n-1} = 0,$$

the result follows by dominated convergence $\square$

Proposition 2.4. *Assume again that the hypotheses of Proposition 2.2 are valid, and that $z_0(\xi)$ satisfies* (1.5), (1.7), (1.8). *Then, for $M_0>0$ large enough, we have $u^T=\int_X z^T(\xi)d\mu(\xi)\to u_L$ in $L^1_{loc}\big(\Omega^-(-\mathbf{M}_0)\big)$, and $u^T\to u_R$ in $L^1_{loc}\big(\Omega^+(\mathbf{M}_0)\big)$, as $T\to\infty$, where $\mathbf{M}_0=(M_0,\dots,M_0)$.*

Proof. We prove the assertion for $\Omega^-(-\mathbf{M}_0)$; the proof for $\Omega^+(\mathbf{M}_0)$ is completely analogous. We consider the quadratic entropy associated with the strictly convex entropy η, defined by taking its quadratic part around u_L. That is, we take

$$\eta_*(u)=\eta(u)-\eta(u_L)-\nabla\eta(u_L)(u-u_L),$$

for which the corresponding entropy-flux is

$$\mathbf{q}_*(u)=q(u)-q(u_L)-\nabla\eta(u_L)(\mathbf{f}(u)-\mathbf{f}(u_L)).$$

For simplicity of notation we denote $(\eta_*,\mathbf{q}_*)$ also by $(\eta,\mathbf{q})$. The strict convexity guarantees the existence of M_0 such that

$$|\mathbf{q}(u)|_\infty\le\frac{M_0}{n}\eta(u). \tag{2.13}$$

We choose M_0 such that

$$M_0>nP \tag{2.14}$$

where P is such that $|a(\xi)|<P,\ \forall\xi\in X$.
Since u^T is uniformly bounded it suffices to prove that $u^T(t,\cdot)$, for any fixed $t>0$, converges to u_L as $T\to\infty$ in $L^1_{\rm loc}\big(\Omega_t^-(-\mathbf{M}_0)\big)$, where

$$\Omega_\tau^-(-\mathbf{M}_0)=\Omega^-(-\mathbf{M}_0)\cap\{(t,\mathbf{x})\,:\,t=\tau\}.$$

We fix an arbitrary $\tilde t>0$. Given any point $(\tilde t,\tilde{\mathbf{x}})\in\Omega_{\tilde t}^-(\mathbf{M}_0)$, we take $r>0$ such that $\{\,(\tilde t,\mathbf{x})\mid|\mathbf{x}-\tilde{\mathbf{x}}|<r\,\}\subseteq\Omega_{\tilde t}^-(-\mathbf{M}_0)$, and considerer the sets

$$\mathcal{S}_r=\{\,(t,\mathbf{x})\mid|\mathbf{x}-\tilde{\mathbf{x}}|=r+M_0(\tilde t-t),\ 0<t<\tilde t\,\},$$

$$\mathcal{T}_r=\{\,(t,\mathbf{x})\mid|\mathbf{x}-\tilde{\mathbf{x}}|<r+M_0(\tilde t-t),\ 0<t<\tilde t\,\}.$$

Now, for $T>0$ fixed, u^T satisfies

$$\begin{aligned}\partial_t\eta(u^T)+\nabla_{\mathbf{x}}\cdot\mathbf{q}(u^T)&=\partial_t\left[\eta(u^T)-E[u^T]\right]+\nabla_{\mathbf{x}}\cdot\left[\mathbf{q}(u^T)-F[u^T]\right]\\&\quad+\int_X de_\xi(z^T(\xi)).(M_\xi(u^T)-z^T(\xi))d\mu(\xi)\end{aligned} \tag{2.15}$$

where $z^T(\xi)=z(Tt,Tx,\xi)$.
Integrating (2.15) over $\mathcal{T}_r$ and applying Green's formula (where we use the fact that

$z^T(t,\cdot,\xi) \to z_0^T(\cdot,\xi)$ as $t \to 0$) we obtain

$$\int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r} \eta(u^T(\tilde t,\mathbf{x}))\,d\mathbf{x} - \int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r+M_0\tilde t} \eta(u_0^T(\mathbf{x}))\,d\mathbf{x} \tag{2.16}$$
$$+\int_{\mathcal{S}_r} \{\nu_t\eta(u^T)+\nu_{\mathbf{x}}\cdot\mathbf{q}(u^T)\}\,d\mathcal{H}^n$$
$$= \int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r} (\eta(u^T)-E[z^T])(\tilde t,\mathbf{x})\,d\mathbf{x} \tag{2.17}$$
$$+\int_{\mathcal{S}_r} \{\nu_t(\eta(u^T)-E[z^T])+\nu_{\mathbf{x}}\cdot(\mathbf{q}(u^T)-F[z^T])\}\,d\mathcal{H}^n$$
$$+\int_{\mathcal{S}_r}\int_X de_\xi(z^T(\xi)).(M_\xi(u^T)-z^T(\xi))d\mu(\xi)$$

where $(\nu_t,\nu_{\mathbf{x}})$ represents an outward unity vector normal to $\mathcal{S}_r$. Now, to get rid of the third term on the left hand side of (2.16), we use the fact that $|\nu_{x_i}|\le M_0^{-1}\nu_t$, $i=1,\dots,n$, together with (2.13). The first term on the right hand side of (2.16) is negative. Indeed, if we use

$$\eta(u^T)-E[z^T]=E[M(u^T)]-E[z^T], \tag{2.18}$$

we get

$$E[M(u^T)]-E[z^T] \tag{2.19}$$
$$=\int_x \left\{e_\xi.(M_\xi(u^T))-e_\xi(z^T)-de_\xi(M(u^T)).(M_\xi(u^T)-z^T(\xi))\right\}d\mu(\xi).$$

Therefore,

$$E[M(u^T)]-E[z^T]=-\frac12\int_X d^2e_\xi(\gamma_\xi).(M_\xi(u^T)-z^T(\xi))^2d\mu(\xi), \tag{2.20}$$

for some $\gamma_\xi\in K_\xi$, and the follows from the strict convexity of the function e_ξ. Using a similar argument it is possible to prove that

$$\nu_t(\eta(u^T)-E[z^T])+\nu_{\mathbf{x}}\cdot(\mathbf{q}(u^T)-F[z^T]) \tag{2.21}$$
$$=-\frac12\int_X\{(\nu_t+\nu_{\mathbf{x}}\cdot a(\xi))d^2e_\xi(\gamma_\xi)\cdot(M_\xi-z^T(\xi))^2\}d\mu(\xi).$$

By the choice of M_0 this expression is negative. Puting together the above facts and using (1.8) we then conclude that

$$\lim_{T_l\to\infty}\int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r}|u^{T_l}(\tilde t,\mathbf{x})-u_L|\,d\mathbf{x}=0. \tag{2.22}$$

From (2.22) one readily gets that u^T converges to u_L in $L^1_{\mathrm{loc}}\big(\Omega^-_{\tilde t}(-\mathbf{M}_0)\big)$ since $\tilde t>0$ is arbitrary this implies the convergence of u^T in $L^1_{\mathrm{loc}}\big(\Omega^-(-\mathbf{M}_0)\big)$. □

The following lemma is an important tool that allows to reduce the study of the asymptotic behavior of the solution to a question of proving that the supports of certain probability measures are concentrated in the state corresponding to the exact value of the Riemann solution.

Lemma 2.2. *Suppose η is a nonnegative convex entropy for* (1.1), *with associated entropy flux* $\mathbf{q}$, *and* $z(\xi) \in L^\infty(\mathbb{R}^{n+1}_+)$ *is solution of* (1.3), (1.4). *Let T_l be a subsequence as in Proposition 2.3 and $\mathbf{M}_0$ as in Proposition 2.4 such that $u^{T_l} \to u_L$ a.e. in $\Omega^-(-\mathbf{M}_0)$, and $u^{T_l} \to u_R$ a.e. in $\Omega^+(\mathbf{M}_0)$. Then, for a.e. $\boldsymbol{\zeta} \in \mathbb{R}^n$, and $K > 0$ such that*

$$|\mathbf{q}(u)|_\infty \le K\eta(u) \tag{2.23}$$

and any $M > 0$ we have: **(a)** *if $\eta(u_L) = 0$,*

$$\limsup_{T_l\to\infty} \frac{1}{T_l^n} \sum_{i=1}^n \int_0^{T_l} \int_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{(-\zeta_i\eta + q_i)\,(u(t,\mathbf{x}))}{\sqrt{1+\delta_i^2}}\, d\mathcal{H}^{n-1}dt \le 0; \tag{2.24}$$

(b) *if $\eta(u_R) = 0$,*

$$\limsup_{T_l\to\infty} \frac{1}{T_l^n} \sum_{i=1}^n \int_0^{T_l} \int_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{(\zeta_i\eta - q_i)\,(u(t,\mathbf{x}))}{\sqrt{1+\delta_i^2}}\, d\mathcal{H}^{n-1}dt \le 0. \tag{2.25}$$

Proof. We consider first the domain $\Omega^-(\boldsymbol{\zeta}, T)$ where K is as in the statement of the lemma, M and $\boldsymbol{\zeta}$ are chosen so that (2.8) and (2.12) hold. The choice of such $\boldsymbol{\zeta}$ and M is possible due to Propositions 2.3-2.4. So, integrating (2.6) over $\Omega^-(\boldsymbol{\zeta}, T_l)$ one obtains

$$\begin{aligned}
&\int\limits_{\Omega^-_{T_l}(\boldsymbol{\zeta},T_l)} \eta(u)\,d\mathbf{x} - \int\limits_{\Omega^-_0(\boldsymbol{\zeta},T_l)} \eta(u)\,d\mathbf{x} + \sum_{i=1}^n \int_0^{T_l} \int\limits_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{(-\zeta_i\eta + q_i)\circ u(t,\mathbf{x})}{\sqrt{1+\zeta_i^2}}\, d\mathcal{H}^{n-1} \\
&+ \int_0^{T_l} \int\limits_{\partial\Omega_t^-(\boldsymbol{\zeta},T_l)\cap\partial\Omega_t(T_l)} (\nu_t\eta + \nu_{\mathbf{x}}\cdot\mathbf{q})\circ u(t,\mathbf{x})\, d\mathcal{H}^{n-1} \\
&= \int\limits_{\Omega^-_{T_l}(\boldsymbol{\zeta},T_l)} (\eta(u) - E[z])\,d\mathbf{x} + \sum_{i=1}^n \int_0^{T_l} \int\limits_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{-\zeta_i(\eta(u) - E[z]) + q_i(u) - F_i[z]}{\sqrt{1+\zeta_i^2}}\, d\mathcal{H}^{n-1} \\
&+ \int_0^{T_l} \int\limits_{\partial\Omega_t^-(\boldsymbol{\zeta},T_l)\cap\partial\Omega_t(T_l)} (\nu_t(\eta(u) - E[z]) + \nu_{\mathbf{x}}\cdot(\mathbf{q}(u) - F[z]))d\mathcal{H}^{n-1} \\
&+ \int_{\Omega^-(\boldsymbol{\zeta},T_l)} \int_X de_\xi(z(\xi)).(M_\xi - z(\xi))d\mu(\xi).
\end{aligned}$$

From the above equation we obtain

$$\begin{aligned}
&\sum_{i=1}^n \int_0^{T_l} \int\limits_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{(-\zeta_i\eta + q_i)\circ u(t,\mathbf{x})}{\sqrt{1+\zeta_i^2}}\, d\mathcal{H}^{n-1} \\
&\le \int\limits_{\Omega^-_0(\boldsymbol{\zeta},T_l)} (E[z] - \eta(u))\,d\mathbf{x} + \int\limits_{\Omega^-_0(\boldsymbol{\zeta},T_l)} \eta(u)\,d\mathbf{x} \\
&+ \sum_{i=1}^n \int_0^{T_l} \int\limits_{S_i^t(\boldsymbol{\zeta},T_l)} \frac{(-\zeta_i(\eta(u) - E[z]) + (q_i(u) - F_i[z])(t,\mathbf{x})}{\sqrt{1+\zeta_i^2}}\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Then, we divide the above inequality by T_l^n and take lim sup when $T_l \to \infty$. So, from our choice of $\boldsymbol{\zeta}, M$ we get (2.24). The relation (2.25) is proved by an entirely analogous procedure, this time integrating (2.6) over $\Omega^+(\boldsymbol{\zeta}, T_l)$. This concludes the proof of the lemma. $\square$

3. Conclusion of the Proof of Theorem 1.1

A crucial step in the proof of Theorem 1.1 is as follows. Let $\{T_k\}$ be an increasing sequence such that $T_k \to \infty$ be given. We consider the probability measures defined for all $g \in C(\mathcal{O})$ by

(3.1)
$$\langle \mu_{i,T}^{\zeta}, g(u)\rangle \equiv c_i^{-1}(\zeta, K, M, T_0)\frac{1}{T^n}\int_0^T \int_{S_i^t(\boldsymbol{\zeta},T)} g\left(u(t,\mathbf{x})\right)\, d\mathcal{H}^{n-1}dt, \quad i = 1, \dots, n,$$

where $c_i(\boldsymbol{\zeta}, K, M, T_0)$ is a normalizing positive constant such that $\langle \mu_{i,T}^{\boldsymbol{\zeta}}, 1\rangle = 1$. We prove the existence of a piecewise continuous function $\bar{R} : \mathbb{R} \to \mathbb{R}^m$ and a function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{h}(\zeta) = (h_1(\zeta), \dots, h_n(\zeta))$, such that each $h_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bi-Lipschitz homeomorphism of $\mathbb{R}$, independent of K, M, T_0 and of the sequence $T_k \to \infty$, such that

$$\mu_{i,T_k}^{\mathbf{h}(\zeta)} \rightharpoonup \delta_{\bar{R}(\zeta)}, \qquad \text{as } T_k \to \infty, \text{ for a.e. } \zeta \in \mathbb{R},\ i = 1, \dots, n. \tag{3.2}$$

Here and in what follows δ_v denotes the Dirac measure with unit mass concentrated at v. We then define

$$R(\boldsymbol{\zeta}) = \bar{R}(\zeta), \qquad \text{if } (t, t\boldsymbol{\zeta}) \in \partial\Omega^+(\mathbf{h}(\zeta)).$$

We observe that

$$\frac{1}{T^n}\int_{\partial^0\Omega^+(\mathbf{h}(\zeta),T)} g\left(u(t,\mathbf{x})\right)\, d\mathcal{H}^{n-1} = \sum_{i=1}^n c_i(\zeta)\langle \mu_{i,T}^{\mathbf{h}(\zeta)}, g(u)\rangle, \tag{3.3}$$

where we omit the dependence of c_i on K, M, T_0, as will be frequently done henceforth. Hence, (3.2) implies, in particular, that

$$\lim_{T_k\to\infty} \frac{1}{T_k^n}\int_{\partial^0\Omega^+(\mathbf{h}(\zeta),T_k)} |u(t,\mathbf{x}) - R(\boldsymbol{\zeta})|\, d\mathcal{H}^n = 0, \quad \text{for a.e. } \zeta \in \mathbb{R}. \tag{3.4}$$

The point is that (3.4) implies that $u^{T_k}(t,\mathbf{x})$ converges to $R(\boldsymbol{\zeta})$ in $L^1_{loc}(\mathbb{R}^{n+1}_+)$, and since the increasing sequence T_k is arbitrary, we obtain that $u^T(t,\mathbf{x})$ converges to $R(\boldsymbol{\zeta})$ in $L^1_{loc}(\mathbb{R}^{n+1}_+)$. The remaining of the proof follows exactly as in [14].

Acknowledgements

The research of H. Frid was partially supported by CNPq, grant 352871/96-2, and FAPERJ, grant E-26/152.192/2002. The present paper was prepared while L. Redón was visiting IMPA, partially supported by a post-doctoral fellowship from FAPERJ through the grant E-26/150.587/2003.

References

[1] Aris, R., Amundson, N., “Mathematical Methods in Chemical Engineering,” Vol.2, Prentice-Hall, Englewood Cliffs, N. J., 1966.

[2] D. Aregba-Driollet and R. Natalini. *Discrete kinetic schemes for multidimensional conservation laws.* SIAM J. Numer. Analysis **37** (2000), 1973-2004.

[3] Boillat, G., *Sur l éxistence et la recherche d'équations de conservation supplémentaires pour les systèmes hyperboliques*, C.R. Acad. Sc. Paris **278** A (1974), 909-912.

[4] Bouchut, F., *Construccion of BGK models with family of kinetic entropies for a given system of conservation laws*, J. Statist. Phys. **95** (1999), 113-170.

[5] Brenner, P., *The Cauchy problem for symmetric hyperbolic system in L^p*,Math. Scand. 19 (1966),27-37.

[6] Chen, G.-Q., Frid, H., *Large-time behavior of entropy solutions of conservation laws*, Journal of Differential Equations **152** (1999), 308-357.

[7] Chen, G.-Q., Frid, H., *Asymptotic decay of solutions to conservation laws*, C. R. Acad. Sci. Paris, Serie I (1996), 257–262.

[8] Chen, G.-Q., Frid, H., *Decay of entropy solutions of nonlinear conservation laws*, Arch. Rational Mech. Anal. **146** (1999), 95-127.

[9] Dafermos, C. M., "Hyperbolic Conservation Laws in Continuum Physics", Springer-Verlag: Berlin, 1999.

[10] Dafermos, C. M., *Entropy and the stability of classical solutions of hyperbolic conservation laws*, Lecture Notes in Mathematics **1640**, Springer Verlag, 1996, 48-69.

[11] Dafermos, C. M., *Stability for systems of conservation laws in several space dimensions*, SIAM Jr. Math. Anal. **26**, No. 6 (1995), 1403-1414.

[12] DiPerna, R., *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal. **82** (1983), 27-70.

[13] Frid, H., *On a new concept of asymptotic stability for Riemann solutions*, Matemática Contempoânea **10** (1996), 99-121.

[14] Frid, H., *Asymptotic stability of Riemann solutions for a class of multi-D viscous systems of conservation laws*. to appear in Arch.Rational Mech. Anal. Preprint available at http://www.preprint.impa.br/Shadows/SERIE_A/2002/153.html.

[15] Frid, H., *Maps of Convex Sets and Invariant Regions for Finite Difference Systems of conservation Laws*. Arch.Rational Mech. Anal.**160**(2001),245-269.

[16] Frid, H., *Correction to: "Maps of convex sets and invariant regions for finite-difference systems of conservation laws" [Arch. Ration. Mech. Anal. 160 (2001), no. 3, 245–269]*. Arch. Ration. Mech. Anal. **171** (2004), no. 2, 297–299.

[17] Friedrichs, K.O., Lax, P.D., *Systems of conservation laws with convex extension*. Proc. Nat. Acad. Sci. USA **68** (1971), 1686-1688.

[18] Godunov, S.K., *An interesting class of quasilinear systems*, Sov. Math. **2** (1961), 947-948.

[19] Hsiao, L., Serre, D., *Asymptotic behavior of large weak entropy solutions of the damped p-system*, J. Partial Diff. Eqs., **10** (1997), 355–368.

[20] Lattanzio C. and Natalini R.,*Convergence of Diffusive BGK Approximations for Nonlinear Strongly Parabolic Systems*, Proc. Roy. Soc. Edinburgh Sect. A, **132** (2002), 341-358.

[21] Lattanzio C. and Serre D., *Convergence of a Relaxation Scheme for Hyperbolic Systems of Conservation Laws*. Numer. Math., **88** (2001), 121-134.

[22] Leveque, R. J. and Temple, B., *Stability of Godunov's method for a class of 2×2 systems of conservation laws*, Trans. Amer. Math. Soc. **288** (1985), 115–123.

[23] Natalini, R., *A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws*, Journal of Differential Equations. **148**(1998), 292-317.

[24] Rauch, J., *BV estimates fail for most quasilinear hyperbolic system in dimensions greater than one*, Commun. Math. Phys. 106(1986), 481-484

[25] Serre, D., *Systems of Conservation Laws, Vols. 1,2*, Cambridge University Press: Cambridge, 1999, 2000.

[26] Serre, D., *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservations*, J. Diff. Eqs. **68** (1987), 137–169.

[27] Serre, D.,*Relaxations semi-linéaire et cinetique des systémes de lois de conservations*,Ann. Inst. Henri Poincairé. Analyse non linéaire, **17**, (2000), 169-192

[28] Tartar, L., *Compensated compactness and applications to partial differential equations*, In: Research Notes in Mathematics, Nonlinear Analysis and Mechanics ed. R. J. Knops, **4**(1979), Pitman Press, New York, 136–211.

[29] Temple, B.,*Systems of conservation laws with invariant submanifolds*,Trans. Amer. Math. Soc. **280** (1983), 781–795.

Instituto de Matemática Pura e Aplicada -IMPA Estrada Dona Castorina, 110, Rio de Janeiro, RJ, 22460-320, Brazil
E-mail address: hermano@impa.br

Universidad Nacional de Colombia, Bogotá. Ciudad Universitaria, Departamento de Matemáticas
E-mail address: lrendona@unal.edu.co