Inverse Problems in Finances
A Short Survey of Calibration Techniques

Jorge P. Zubelli
IMPA, Rio de Janeiro
zubelli@impa.br

Abstract

We survey the problem of calibrating the volatility by using quoted option prices. This is done within the framework of regularization theory provided by inverse problem theory. We start with a simplified model of pure time-dependent volatility to gather insight on the nature of ill-posedness of the problem. We briefly review a number of recent contributions to the field and highlight its activity.

1 Introduction

Pricing and hedging financial instruments, such as derivatives, depends heavily on the knowledge of the volatility. The latter is subject to a tremendous amount of uncertainty due not only to the quality of available data but also on the modeling of the financial instrument and the underlying assets under consideration. In fact, there is a trend in risk management to consider "model risk" at par with other risks such as operational, market, and credit risk. This is corroborated by a report Williams(1999) according to which \$5 billion in derivative losses during 1999 were attributable to model risk. See Cont(2005) and references therein.

In many situations the volatility is computed using historic data on the assets, however this is not always possible or desirable and thus one is faced with an ill-posed problem, in the sense described below, of determining the volatility from quoted data in financial instruments.

In this survey we discuss inverse problems associated to determining the volatility from quoted option prices. We underscore the importance of regularization techniques and how the general methodology of inverse problems can be used to mitigate the difficulties due to often scarce and irregular data. For simplicity we focus on models related to European options, but we emphasize that most of the discussed techniques are also applicable to other instruments such as fixed income ones.

As a paradigmatic example, we consider the problem of determining the local volatility from quoted market option prices at different strikes and maturities. More precisely, let us consider the case of an European call option with maturity T and strike K on an underlying asset $S = S_t$ whose dynamics is described by the stochastic differential equation

$$dS_t = S_t(\rho(t, S_t)dt + \sigma(t, S_t)dW_t) \ , \tag{1}$$

where $\{W_t, \mathcal{F}_t\}_{t\geq 0}$ denotes the standard Brownian motion. Under the usual assumptions of liquidity, absence of arbitrage and transaction cost (perfect market), the call price $C = C(S, t; K, T, r, \sigma)$ satisfies the Black-Scholes equation

$$\left\{ \begin{array}{ll} \partial_t C + \frac{1}{2}\sigma^2(t, S)S^2\partial_S^2 C + r(S\partial_S C - C) = 0 \ , & S > 0 \ , t < T \\ C(S, T) = (S - K)^+ \ , & \end{array} \right. \tag{2}$$

where r is the constant interest rate on a riskless investment.

The inverse problem under consideration here is to determine the local volatility $\sigma(S, t)$ from a set of possibly noisy call prices

$$\left\{ C^\delta(S, t; K, T) \right\}_{(K,T)\in\mathcal{I}} \ , \tag{3}$$

where $\mathcal{I}$ denotes a set of points for which the option is quoted. Although this set in principle could be arbitrary, in practical applications it must be a discrete finite set.

The importance of this problem can be justified in a number of ways. To cite a few: One may want to use the volatility to precify other options (such as for example exotic derivatives). One would also want to evaluate the exposure to risk and hedging proportions of the option issuer.

We refer the reader to the more complete topical review Bouchouev & Isakov(1997) which provides an excellent survey of the subject of inverse problems in financial mathematics up to 1998. Our goal here is to give some background on regularization techniques and to point out more recent developments, in particular concerning regularization techniques such as the approach in Crépey(2003a), Crépey(2003b).

Before closing this Introduction we briefly recall the fundamental concept of *implied volatility*. The classical Black-Scholes model Black & Scholes(1973) assumes that $\sigma(S, t)$ is a constant, say σ_0. This yields the standard Black-Scholes formula for the call price

$$C^{\mathrm{BS}}(S, t; K, T, r, \sigma_0) = SN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{4}$$

where N is the cumulative normal distribution function and

$$d_\pm = \frac{\log(Se^{r(T-t)}/K)}{\sigma_0\sqrt{T-t}} \pm \frac{\sigma_0\sqrt{T-t}}{2} \ . \tag{5}$$

Note that C^{BS} is a strictly increasing function of $\sigma > 0$, with range in the interval bounded by the lower and upper no-arbitrage limits of the call. This implies that there is a 1-1 correspondence between Black-Scholes prices and volatilities, where the prices fall between the no-arbitrage limits and $\sigma > 0$. We thus define the *implied volatility* of a quoted price C at time t and spot S as the unique solution I to the equation

$$C^{\mathrm{BS}}(S, t; K, T, r, I) = C \ .$$

The implied volatility language is widely used by traders and practitioners since it gives a reference upon which option prices can be compared across different strikes, maturities, underlyings, and observation times.

It is a well-known empirical remark that calls having different strikes, but otherwise identical, have different implied volatilities. This important observation is called the *smile effect*. It has been subject of intense research. See Berestycki et *al.*(2002), Berestycki et *al.*(2000) for an interesting partial differential equation connecting implied volatilities to the local volatility.

The outline of the paper is the following: We start in Section 2 with some background material on inverse problems and regularization. Then, as a warm up example, we discuss a purely time-dependent volatility calibration problem. In Section 4 we discuss the smile curve, Dupire' s formula, and its implications. In Section 5 we survey different approaches on the calibration of volatility surfaces. In Section 6 we close with some conclusions and directions of research.

2 An Inverse Problem Toolbox

The term *inverse problem* refers to the process of reconstructing or estimating parameters in models from observed data. It is an extremely rich and interdisciplinary subject with applications to several areas ranging from geophysics to medical imaging. See for example Baumeister(1987),Leitão & Baumeister(2005), Engl et *al.*(1996), Natterer & Wübbeling(2001), Zubelli(1999). It uses mathematical tools that include numerical and functional analysis, differential equations, statistics, and high performance computing. Ultimately, most of the problems in finances are inverse ones. Indeed, the process of choosing the appropriate model to describe the financial variables and to identify the parameters for such models is crucial in the success of the theory. In finances the identification of the different parameters is usually called *calibration* of the model.

In what follows we present some of the fundamental tools that are broadly used in inverse problem theory.

2.1 The SVD - Principal Component Analysis

The purpose of this section is to quickly review the *singular value decomposition (SVD)*. This result has a number of important application in numerical analysis, inverse problems, and numerical linear algebra. It has different names in different fields, such as for example, principal component analysis. We follow Simon(1979). For the numerical applications see Golub & Van Loan(1989).

Let $\mathcal{L}(X,Y)$ denote the space of bounded linear operators from X into Y. The so-called "uniform operator topology" is defined by the norm

$$||T||_{X,Y} \stackrel{\text{def}}{=} \sup_{x\neq 0} \frac{||Tx||_Y}{||x||_X} \ .$$

Remark 1 *In these notes we use the convention that our* complex *Hilbert space inner-products are* linear *w. r. t. the first entry and* anti-linear *w. r. t. to the second one.*

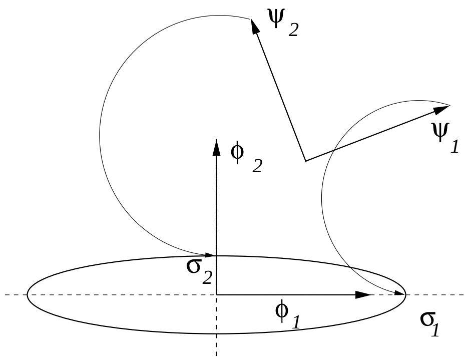

Figure 1: Depiction of the SVD. Notice the unity vectors ψ_n being sent into vectors of length σ_n in the direction ϕ_n

Definition 1 *Let X and Y be Banach spaces and $T : X \to Y$ a linear operator. T is called* compact *(or* completely continuous*) if it maps the unit ball B_X on a pre-compact set, i.e., $T(B_X)$ has compact closure. The set of compact operators from X to Y is denoted by $\kappa(X, Y)$.*

It can be easily shown that the above definition implies that T is a bounded operator. Furthermore, the set of compact operators is closed under limits in the uniform operator topology of $\mathcal{L}(X, Y)$. Typical examples of compact operators are operators of finite dimensional range. Another important class of compact operators is given by integral operators.

Theorem 1 *Let $A \in \kappa(\mathcal{H}, \mathcal{K})$, then we can write*

$$A = \sum_{n=1}^{r} \sigma_n \, (\cdot \mid \psi_n) \, \phi_n, \tag{6}$$

where $r \in \mathbb{N} \cup \{\infty\}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$; the sets $\{\phi_n\}_{n=1}^r$ and $\{\psi_n\}_{n=1}^r$ are orthonormal sets (not necessarily complete) in $\mathcal{K}$ and $\mathcal{H}$, respectively.

As direct consequence of the SVD decomposition we get that the equation $Ax = y$ will have a solution if, and only if,

1. The vector $y \in \ker A^{*\perp}$, and
2. the sum
$$\sum_{n=1}^{r} \frac{1}{\sigma_n^2} |(y \mid \phi_n)|^2 < \infty \ .$$

In this case the solution of x will be given by

$$x = \sum_{n=1}^{r} \frac{1}{\sigma_n} (y \mid \phi_n) \psi_n \ .$$

In the finite dimensional case, the transformation A will be invertible if, and only if, the dimensions of $\mathcal{H}$ and $\mathcal{K}$ equal r. In other words, the eigenvalues of A^*A and AA^* are all nonzero. The presence of singular values close to zero indicates that the solution of the problem $Ax = y$ will be doomed to numerical instability. The condition number of a matrix $A \in \mathbb{C}^{n\times n}$ is the ratio σ_1/σ_r, if $r = n$, and ∞ otherwise. See Golub & Van Loan(1989) for a comprehensive analysis of the numerical aspects related to the SVD and the concept of condition number.

In the infinite dimensional case, if $r = \infty$ then the sequence $\{\sigma_n\}$ must necessarily converge to zero. This shows the inherent instability of solving equations of the form $Ax = y$ when A is a compact operator in an infinite dimensional Hilbert space. For an excellent introduction to these and other methods see Baumeister(1987).

2.2 Well-posed $\times$ Ill-posed problems. Regularization.

A mathematical problem, defined in the form of an equation $F(x) = y$ where F is an operator between two Banach spaces X and Y is said to be well-posed (in the sense of Hadamard) if for every $y \in Y$ the solution $x \in X$ exists, is unique, and depends continuously on y.

Let $A : \mathcal{H} \to \mathcal{K}$ be a compact linear operator. We now analyze the question of solving a linear equation of the form

$$Af = g \ .$$

There are three things that can go wrong:

1. The equation may not be solvable. (i.e. $g \notin \mathbf{Ran}(A)$)
2. The solution may not be unique. (i.e. A is not $1-1$.)
3. The solution may not depend continuously on the data. (i.e. A^{-1} is not continuous.)

The concept of pseudo-inverse (or generalized inverse) $A^\dagger$ is used to handle the cases 1. e 2. above. We define

$$A^\dagger g = \sum_{n=1}^{r} \frac{1}{\sigma_n}(g|\phi_n)\psi_n \ , \ g \in \mathbf{Dom}(A^\dagger) \ ,$$

where

$$\mathbf{Dom}(A^\dagger) \stackrel{\text{def}}{=} \left\{ g \in \mathcal{K} \middle| \sum_{n=1}^{r} \frac{1}{\sigma_n^2} |(g \mid \phi_n)|^2 < \infty \right\} \ .$$

Note that $A^\dagger g$ is the unique solution of $Af = g$ in $(\ker A)^\perp$. To tackle the problem of discontinuous A^{-1}, one needs to introduce the notion of *regularization.*

Let us consider a family of continuous operators $T_\gamma : \mathcal{K} \to \mathcal{H}$ such that

$$\lim_{\gamma \downarrow 0} T_\gamma g = A^\dagger g \tag{7}$$

Note that if $A^\dagger$ is not bounded, then $||T_\gamma|| \to \infty$ when $\gamma \downarrow 0$. If we can solve $Af = g$ approximately in the sense that: Let $g^\epsilon \in \mathcal{K}$ be an approximation to g

and such that $||g - g^\epsilon|| \le \epsilon$. Consider $\gamma(\epsilon)$ such that $\gamma(\epsilon) \downarrow 0$ and $||T_{\gamma(\epsilon)}||\epsilon \to 0$. Thus,

$$\begin{aligned} ||T_{\gamma(\epsilon)} g^\epsilon - A^\dagger g|| &\le ||T_{\gamma(\epsilon)}(g^\epsilon - g)|| + ||T_{\gamma(\epsilon)} g - A^\dagger g|| \\ &\le ||T_{\gamma(\epsilon)}||\epsilon + ||T_{\gamma(\epsilon)} g - A^\dagger g|| \to 0 \end{aligned}$$

So, $T_{\gamma(\epsilon)} g^\epsilon$ is close to $A^\dagger g$ provided g^ϵ is close to g.

We now analyze three techniques that are used in regularization of ill-posed problems.

- Truncated SVD:
$$T_\gamma g = \sum_{\sigma_k \ge \gamma} \frac{1}{\sigma_k}(g|\phi_k)\psi_k \ .$$
- Tikhonov-Phillips regularization:
$$T_\gamma = (A^* A + \gamma I)^{-1} A^* \ ,$$
which is associated to minimizing the quadratic form
$$||Af - g||^2 + \gamma ||f||^2 \ .$$
More generally, to minimizing expressions of the form
$$||Af - g||^2 + \gamma q(f - f_0) \ ,$$
where q is some penalty function designed to keep f close to a prior f_0.
- Early stop of an iterative method: Suppose that
$$f^{k+1} = B_k f^k + C_k g$$
is an iterative method to solve $Af = g$ with B_k and C_k bounded and $\lim_{k\to\infty} f^k = A^\dagger g$. For $\gamma > 0$ let the value of $k(\gamma)$ such that $k(\gamma) \to \infty$ when $\gamma \to 0$. Then, $T_\gamma f \stackrel{\text{def}}{=} f^{k(\gamma)}$ is a regularization of the problem.

2.3 Conditioning and an Example

In the case of infinite dimensional problems for compact operator equations of the form $Af = g$, it is natural to analyze how ill posed the problem is by looking at the rate of convergence of the singular values of A to zero. Problems for which the rate of decay of σ_n is not faster than a polynomial are considered tractable. Problems for which the decay is exponential or faster are considerely severely ill-posed.

We consider an important example, namely the inverse problem associated to integration. Let $A : L^2[0,1] \to L^2[0,1]$ be defined as

$$A\phi(t) = \int_0^t \phi(s)ds \ , \ 0 \le t \le 1$$

Then, it can be shown that the n-th singular value for A is

$$\sigma_n = \frac{2}{(2n-1)\pi} \ , \ n = 1, 2, \cdots \ .$$

Thus the rate of decay of σ_n as $n \to \infty$ is of the form k/n. In fact, it is well-known that numerical differentiation is an ill-posed problem. This rate of decay quantifies the degree of ill-posedness.

We now consider a practical technique that is often used in regularization. Assume that $\mathcal{H} = L^2(G)$, where G is a bounded domain of $\mathbb{R}^n$ and that $||Af|| \geq m||f||_{\mathcal{H}^{-\alpha}(G)}$, where $\mathcal{H}^{-\alpha}$ is the Sobolev space of order $-\alpha$. Then, one can show that

$$||Af|| \leq \epsilon \quad \text{and} \quad ||f||_{\mathcal{H}^{\beta}(G)} \leq \rho$$

implies that

$$||f||_{L^2(G)} \leq C\epsilon^{\frac{\beta}{\alpha+\beta}} \rho^{\frac{\alpha}{\alpha+\beta}} ,$$

for some constant C. If we want to solve $Af = g$ for which we know g with accuracy ϵ and

$$||f||_{\mathcal{H}^{\beta}(G)} \leq \rho$$

Then, it is possible to compute an approximation f with accuracy

$$C\epsilon^{\frac{\beta}{\alpha+\beta}} \rho^{\frac{\alpha}{\alpha+\beta}}$$

In other words, $\frac{\beta}{\alpha+\beta}$ gives us an estimate of the ill-conditioning of the problem. If $\frac{\beta}{\alpha+\beta} \approx 1$ then the problem is not so ill-posed.

3 Time Dependent Volatility - A Case Study

In this section we consider the case of purely time-dependent volatility. In other words, we assume that the volatility $\sigma(t, S)$ is independent of the underlying price S. We follow closely the analysis in Hein & Hofmann(2003). See also Hofmann & Krämer(2005) for further results.

Under the assumption of purely time-dependent volatility, the call price at an initial time, say $t = 0$, is given by

$$C(T; \sigma^2) = C^{\mathrm{BS}}(S, 0; K, T, r, \overline{\sigma}) \tag{8}$$

where

$$\overline{\sigma} \stackrel{\text{def}}{=} \sqrt{\frac{1}{T}\int_0^T \sigma^2(s)ds} \tag{9}$$

In Hein & Hofmann(2003) the following inverse problem is analyzed: Find the time-dependent volatility $(\sigma(t))_{t\in[0,T_{\max}]}$ from noisy observations $\left(C^{\delta}(T)\right)_{t\in[0,T_{\max}]}$ of the maturity-dependent fair price $C = C(T; \sigma^2)$ given by Equation (8).

The above inverse problem could be rephrased as the problem of calibrating a time-dependent volatility function from a term structure of option prices. Although the above problem is much simpler than the one encounters in practice due to the assumption on the volatility, it is an important benchmark that will give us some insight on the nature of ill-posedness of the inverse problems under consideration.

It is remarked in Hein & Hofmann(2003) that the forward operator

$$F : \sigma^2(\cdot) \mapsto C(T; \sigma^2(\cdot))$$

can be decomposed into

1. An inner linear operator of the form

$$J[h](t) = \int_0^t h(s)ds \ , \ 0 \leq t \leq T_{\max} \ .$$

2. An outer nonlinear functional operator (Nemytskii operator) given by Black-Scholes type function.

The inversion of the outer operator leads to an ill-posedness effect that is localized for small times. Furthermore, the outer operator

$$N[v](T) \stackrel{\text{def}}{=} C^{\text{BS}}(S, 0; K, T, r, \sqrt{v})$$

is a continuous nonlinear operator from the space $C([0,T];\mathbb{R}_{\geq 0})$ into itself endowed with the sup-norm. It is also continuous in the $(L^p; L^q)$ setting on the cone of non-negative functions with $1 \leq p, q < \infty$. Thus, the ill-conditioning of the problem falls onto the linear inner operator J, which has been extensively studied. It is known, for example, that in the Hilbert space setting of $L^2[0, T_{\max}]$ this leads, after discretization to an ill-conditioned problem. The degree of ill-posedness was described in our example of Section 2. See Hein & Hofmann(2003) and references therein for more information. Summing up, at one side we know precisely the ill-conditioning degree of the problem in this setting, at the other one should not expect a better behavior from the local volatility problem when the volatility depends on both S and t.

4 The Smile Curve and Dupire's Equation

As we mentioned in the Introduction, the implied volatility is defined as the value of σ for which the Black-Scholes price given by Equation (4) coincides with the quoted option price for the same expiration and strike. In practice, call with different strikes and otherwise identical have different implied volatilities. For example, in the case of the S-P index, before the 1987 crash, the graph of the implied volatility $I = I(K)$ had a U shape with minimum around $K/S = 1$. Since then, it started displaying a downward slope in the range $0.95 < K/S < 1.05$. Assuming that there exists a local volatility function $\sigma = \sigma(S,t)$ for which (2) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T C - \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 C + rS\partial_K C = 0 \ , & S > 0 \ , \ t < T \\ C(K, T = 0) = (S - K)^+ \ , & \end{cases} \tag{10}$$

This gives us a *theoretical* way of evaluating the local volatility by means of the Dupire's formula

$$\sigma(K,T) = \sqrt{2\left(\frac{\partial_T C + rK\partial_K C}{K^2\partial_K^2 C}\right)} \tag{11}$$

In practice, in order to estimate σ according to this formula, one would have available a limited amount of discrete data and thus interpolate. It is well known that such process is prone to numerical instabilities. In fact, there seems to be no obvious way to interpolate the values so that the square-root argument in (11) remains finite and positive Berestycki et *al.*(2002), Rebonato(1999).

Hence, the use of (10) naturally calls for more sophisticated techniques. In particular in regions where one is close to expiration ($0 < T - t << 1$). An interesting result in Berestycki et *al.*(2002) shows that there is a partial differential equation of degenerate parabolic type linking the local and implied volatilities. Such link has far reaching implications and allows for example asymptotic expressions for the implied volatility near expiration, as well as for, deep in or out-of-the money options. See also Lee(2005) for a survey research activities on the probabilistic interpretation, statics, and dynamics of implied volatility.

5 Volatility Surfaces

In this section we focus on the inverse problem for the case of volatility that depends on both the spot price and time. Before that, however, we mention the work Bouchouev & Isakov(1997) which deals with the case pure spot-price dependence, i.e, $\sigma(S,t)$ is independent of t. In Bouchouev & Isakov(1999), under suitable conditions, uniqueness results were obtained for the identification of $\sigma(S)$ for S in an interval provided option price data was available on a smaller interval. They also obtain upper bounds on the Hölder norm of order λ of $(\sigma_1 - \sigma_2)$ by the Hölder norm of order $\lambda + 2$ of the data. Furthermore, they present an iterative way of reconstructing the volatility. Further extensions and numerical results are presented in Bouchouev & Isakov(1999). See also Jiang & Tao(2001).

The use of optimization techniques to deal with the local volatility identification problem has been proposed by a number of authors. We mention, in particular, the contribution of Rubinstein(1994) for a discretized version of the inverse problem and of Rubinstein & Jackwerth(1996) that included a penalty function. We also mention the work of Lagnado & Osher(1997) who suggested minimizing the gradient of the local volatility subject to the data constraints, i.e.,

$$\min_{\sigma} \left\{ ||\mathrm{grad}\sigma||_2^2 + \lambda \sum_{(K_i,T_j)\in\mathcal{I}} (C_{i,j} - C(S,t;K_i,T_j,r,\sigma))^2 \right\}$$

In Avellaneda et *al.*(1997) an interesting dynamic programming approach was applied so as to minimize an entropy functional. More precisely, their algorithm yields an arbitrage-free diffusion process that minimizes the Kullback-Leibler relative entropy distance to a prior diffusion. It thus solves a constrained (minimax) optimal control problem using a finite-difference scheme for a Bellman parabolic equation combined with a gradient-based optimization routine. The method is applied to to market prices of Dollar-Deutschmark over-the-counter options.

Crépey(2003a) establishes Sobolev space $W_p^{1,2}$ estimates for the Black-Scholes Equation (2) and the Dupire Equation (10) with measurable ingredients. It is then shown that the the inverse problem under consideration fits into the Tikhonv regularization scheme. Furthermore, this holds even when the set of measurements $\mathcal{I}$ is discrete. See also Crépey(2003b) for the discretization, numerical implementation, and extension to American option prices.

6 Conclusion

We close this short survey by mentioning that we hope to have demonstrated an evergrowing impact of inverse problem techniques in model calibration problems in quantitative finances. Such techniques, when conjoined with optimization and high performance computing ones, lead to potentially porwerful tools in risk management, hedging, and exotic option pricing.

Finally, we mention that this survey did not touch upon the extremely important and active topic of stochastic volatility and multiple scale analysis. See for example Fouque et *al.*(2000), Cotton et *al.*(2004), Fouque et *al.*(2003), Fouque et *al.*(2001a), Fouque et *al.*(2001b) and references therein. It should be mentioned, however, that this apprach leads to powerful insights into the effective solution of the calibration problem and the understanding of the smile curve under suitable regimes.

References

Avellaneda, M., C. Friedman, R. Holmes, & D. Samperi (1997) Calibrating volatility surfaces via relative-entropy minimization.. *Appl. Math. Finance* **4**, 37–64.

Baumeister, J. (1987) *Stable solution of inverse problems*Advanced Lectures in Mathematics. Braunschweig: Friedr. Vieweg & Sohn.

Berestycki, H., J. Busca, & I. Florent (2000) An inverse parabolic problem arising in finance. *C. R. Acad. Sci. Paris Sér. I Math.* **331**, 965–969.

Berestycki, H., J. Busca, & I. Florent (2002) Asymptotics and calibration of local volatility models. *Quant. Finance* **2**, 61–69. Special issue on volatility modelling.

Black, F. & M. Scholes (1973) The pricing of options and corporate liabilities. *Journal of Political Economy* **81**, 637–659.

Bouchouev, I. & V. Isakov (1997) The inverse problem of option pricing. *Inverse Problems* **13**, L11–L17.

Bouchouev, I. & V. Isakov (1999) Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets. *Inverse Problems* **15**, R95–R116.

Cont, R. (2005) Model uncertainty and its impact on the pricing of derivative instruments. *Mathematical Finance* **To appear**.

Cotton, P., J.-P. Fouque, G. Papanicolaou, & R. Sircar (2004) Stochastic volatility corrections for interest rate derivatives. *Math. Finance* **14**, 173–200.

Crépey, S. (2003) Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization. *SIAM J. Math. Anal.* **34**, 1183–1206 (electronic).

Crépey, S. (2003) Calibration of the local volatility in a trinomial tree using Tikhonov regularization. *Inverse Problems* **19**, 91–127.

Dupire, B. (1994) Pricing with a smile. *Risk Magazine* **7**, 18–20.

Engl, H. W., M. Hanke, & A. Neubauer (1996) *Regularization of inverse problems*Volume 375 of *Mathematics and its Applications.* Dordrecht: Kluwer Academic Publishers Group.

Fouque, J.-P., G. Papanicolaou, & K. R. Sircar (2000) *Derivatives in Financial Markets with Stochastic Volatility* Cambridge University Press.

Fouque, J.-P., G. Papanicolaou, & K. R. Sircar (2001) From the implied volatility skew to a robust correction to Black-Scholes American option prices. *Int. J. Theor. Appl. Finance* **4**, 651–675.

Fouque, J.-P., G. Papanicolaou, & R. Sircar (2001) Stochastic volatility and epsilon-martingale decomposition. In *Mathematical finance (Konstanz, 2000)*, Trends Math., pp. 152–161. Basel: Birkhäuser.

Fouque, J.-P., G. Papanicolaou, R. Sircar, & K. Solna (2003) Multiscale stochastic volatility asymptotics. *Multiscale Model. Simul.* **2**, 22–42 (electronic).

Golub, G. H. & C. F. Van Loan (1989) *Matrix computations* (Second ed.).Johns Hopkins University Press, Baltimore, MD.

Hein, T. & B. Hofmann (2003) On the nature of ill-posedness of an inverse problem arising in option pricing. *Inverse Problems* **19**, 1319–1338.

Hofmann, B. & R. Krämer (2005) On maximum entropy regularization for a specific inverse problem of option pricing. *J. Inverse Ill-Posed Probl.* **13**, 41–63.

Jiang, L. & Y. Tao (2001) Identifying the volatility of underlying assets from option prices. *Inverse Problems* **17**, 137–155.

Lagnado, R. & S. Osher (1997) A technique for calibrating derivative security pricing models: numerical solution of the inverse problem. *J. Comput. Finance* **1**, 13–25.

Lee, R. W. (2005) Implied volatility: statics, dynamics, and probabilistic interpretation.. In *Baeza-Yates, Ricardo (ed.) et al., Recent advances in applied probability. Papers presented at the second international workshop for Asian public key infrastructures, IWAP 2002, Taipei, Taiwan, October 30 - November 1, 2002. New York, NY: Springer. 241-268* .

Leitão, A. & J. Baumeister (2005) *Topics in Inverse Problems*25º Colóquio Brasileiro de Matemática. [25th Brazilian Mathematics Colloquium]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro.

Natterer, F. & F. Wübbeling (2001) *Mathematical methods in image reconstruction* SIAM Monographs on Mathematical Modeling and Computation. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Rebonato, R. (1999) *Volatility and Correlation in the Pricing of Equity, FX and Interest-Rate Options*Chichester: Wiley.

Rubinstein, M. (1994) Implied binomial trees. *J. Finance* **49**, 771–818.

Rubinstein, M. & J. Jackwerth (1996) Recovering probability distributions from option prices. *J. Finance* **51**, 1611–1631.

Simon, B. (1979) *Trace ideals and their applications*Cambridge University Press, Cambridge-New York.

Williams, D. (1999). Models vs the market: Survival of the fittest, report FIN514, Meridien Research. .

Zubelli, J. P. (1999) *An introduction to inverse problems. Examples, methods and questions* 22^{o} Colóquio Brasileiro de Matemática. [22nd Brazilian Mathematics Colloquium]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro. Appendix A by the author and Luis Orlando Castellano Pérez.