NEUMANN PROBLEMS FOR QUASILINEAR PARABOLIC SYSTEMS MODELLING POLYDISPERSE SUSPENSIONS

STEFAN BERRES, RAIMUND BÜRGER, AND HERMANO FRID

ABSTRACT. We discuss the well-posedness of a class of Neumann problems for $n \times n$ quasilinear parabolic systems arising from models of sedimentation of polydisperse suspensions in engineering applications. This class of initial-boundary value problems includes the standard (zero-flux) Neumann condition, in the limit as a positive perturbation parameter θ goes to 0. We call, in general, the problem associated with $\theta \geq 0$, the θ-flux Neumann problem. The Neumann boundary conditions, although natural and usually convenient for integration by parts, are nonlinear and couple the different components of the system. An important aspect of our analysis is a time stepping procedure that considers linear boundary conditions for each time step in order to circumvent the difficulties arising from the nonlinear coupling in the original boundary conditions. We prove the well-posedness of the θ-flux Neumann problems for $\theta > 0$ and obtain a solution of the standard (zero-flux) Neumann problem as the limit for $\theta \to 0$ of solutions of the θ-flux Neumann problems. Concerning applications, the analysis developed here supports a new model for the settling of polydisperse suspensions forming compressible sediments.

1. INTRODUCTION

We consider the initial-boundary value problem for certain quasilinear parabolic systems of the (upper triangular) form

$$\partial_t u_i + \partial_x f_i(u) = \sum_{j=i}^{n} \partial_x \big(B_{ij}(u)\partial_x u_j\big), \quad i = 1, \dots, n, \tag{1.1}$$

where $u = (u_1, \dots, u_n)^{\mathrm{T}}$ and $(x, t) \in Q := (-1, 1) \times (0, T)$. We consider the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega := (-1, 1), \tag{1.2}$$

where $u_0 = (u_{01}, \dots, u_{0n})^{\mathrm{T}}$ is a function on Ω for which regularity assumptions are made below. The so-called θ-flux Neumann boundary conditions are given by

$$f_i(u) - \sum_{j=i}^{n} B_{ij}(u)\big(\partial_x u_j \pm \theta(u_j - u_j^{\pm}(t))\big) = 0, \ x = \pm 1, \ i = 1, \dots, n. \tag{1.3}$$

Here, $\theta > 0$ and $u^{\pm} = (u_1^{\pm}, \dots, u_n^{\pm})^{\mathrm{T}}$ are functions on $(0, T)$ for which regularity assumptions are also stated below. In matrix form, (1.1) and (1.3) read

$$\partial_t u + \partial_x f(u) = \partial_x \big(B(u)\partial_x u\big), \qquad (x, t) \in Q, \tag{1.4}$$

Date: December 20, 2005.

1991 *Mathematics Subject Classification.* Primary: 35K50; Secondary: 76S05.

Key words and phrases. Quasilinear parabolic system, boundary value problem, polydisperse suspensions.

$$f(u) - B(u)\big(\partial_x u \pm \theta(u - u^{\pm}(t))\big) = 0, \qquad x = \pm 1, \quad 0 < t < T, \tag{1.5}$$

where we denote $f(u) = (f_1(u), \dots, f_n(u))^{\mathrm{T}}$ and $B(u) = (B_{ij}(u))_{i,j=1}^n$. We impose the parabolicity condition

$$B_{ii}(u) \geq \nu > 0, \quad u \in \Delta, \ i = 1, \dots, n, \tag{1.6}$$

and assume that the matrix $B(u)$ is upper triangular, i.e.

$$B_{ij}(u) \equiv 0 \quad \text{if } i > j. \tag{1.7}$$

When $\theta \to 0$, condition (1.3) reduces to the standard (zero-flux) Neumann boundary condition given by

$$f_i(u) - \sum_{j=i}^{n} B_{ij}(u)\partial_x u_j = 0, \qquad x = \pm 1, \quad t \in (0,T), \quad i = 1, \dots, n. \tag{1.8}$$

We assume that the functions $f_i(u)$ and $B_{ij}(u)$ are smooth on the set

$$\Delta := \big\{ u \in \mathbb{R}^n \, : \, u_1 \geq 0, \dots, u_n \geq 0, \ u_1 + \dots + u_n \leq 1 \big\}, \tag{1.9}$$

and that

$$u_0(x) \in \Delta, \ u^{\pm}(t) \in \Delta \quad \text{for all } x \in \Omega, \, t \in (0,T). \tag{1.10}$$

More precisely, we assume

$$f_i, \ \frac{\partial f_i}{\partial u_j}, \ B_{ij}, \ \frac{\partial B_{ij}}{\partial u_k}, \ \frac{\partial^2 B_{ij}}{\partial u_k \partial u_l} \in H^{\beta}(\Delta), \ \beta \in (0,1), \tag{1.11}$$

where $H^{\beta}(\Delta)$ is the space of Hölder continuous functions (with Hölder exponent β) defined on Δ.

Our interest in problem (1.1), (1.2), (1.8) comes from a mathematical model for the sedimentation of polydisperse suspensions in engineering applications. This model will be further analysed in this paper. Guided by this model, we assume the following conditions, which are relevant for the invariance of Δ:

$$f_i|_{u_i=0} = 0, \quad i = 1, \dots, n; \quad (f_1 + \dots + f_n)\big|_{u_1+\dots+u_n=1} = 0, \tag{1.12}$$

$$\begin{cases} B_{ij}\big|_{u_i=0} = \mu_i \, \delta_{ij}, \quad i,j = 1, \dots, n, \\ B_{11}\big|_{u_1+\dots+u_n=1} = (B_{12} + B_{22})\big|_{u_1+\dots+u_n=1} \\ \qquad = \dots = \sum_{j=1}^n B_{jn}\big|_{u_1+\dots+u_n=1}, \end{cases} \tag{1.13}$$

where $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$, and the μ_i are positive functions defined over the hyperplanes $u_i = 0$.

We denote by $H^{k+\beta}([a,b])$ the space of the functions on the closed interval $[a,b]$ whose derivatives up to k-th order are Hölder continuous with exponent β. By $H^{\alpha,\alpha/2}(\overline{Q})$ we denote the space of Hölder continuous functions on $\overline{Q}$ associated with the norm

$$\begin{aligned} |u|_Q^{(\alpha)} := & \sup_{\overline{Q}} |u(x,t)| + \sup_{x_1,x_2 \in \overline{\Omega}, \ t \in [0,T]} \frac{|u(x_1,t) - u(x_2,t)|}{|x_1 - x_2|^{\alpha}} \\ & + \sup_{x \in \overline{\Omega}, \ t_1,t_2 \in [0,T]} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^{\alpha/2}}. \end{aligned}$$

Let $H^{2+\alpha,1+\alpha/2}(Q)$ be the space of the functions on Q whose first and second x-derivatives and the t-derivative belong to $H^{\alpha,\alpha/2}(K)$ for any compact set $K \subseteq$

Q. For simplicity, we also write e.g. $v \in H^{\alpha,\alpha/2}(K)$ for a vector function $v = (v_1, \ldots, v_n)^{\mathrm{T}}$ if all components of v belong to $H^{\alpha,\alpha/2}$.

The following theorem states our main result concerning the θ-flux Neumann problems (1.1)–(1.3).

Theorem 1.1. *Let $\theta > 0$ be fixed. Assume that*

$$u_0 \in H^{2+\beta}([-1,1]), \quad u^{\pm} \in H^{1+\beta}([0,T]), \tag{1.14}$$

and that the compatibility conditions

$$f_i\big(u_0(\pm 1)\big) - \sum_{j=i}^{n} B_{ij}\big(u_0(\pm 1)\big)\big(\partial_x u_{0j}(\pm 1) \pm \theta(u_{0j}(\pm 1) - u_j^{\pm}(0))\big) = 0, \tag{1.15}$$

$$i = 1, \ldots, n$$

are satisfied. Furthermore, suppose that $f(u), B(u), u_0, u^{\pm}$ satisfy (1.6), (1.7), *and* (1.10)–(1.13)*. Then, the problem* (1.1)–(1.3) *has a unique classical solution u, such that $u \in H^{2+\beta,1+\beta/2}(Q)$ and* (1.1) *is satisfied on Q, $u, u_x \in H^{\alpha,\alpha/2}(\bar{Q})$ for some $0 < \alpha \leq \beta$, and conditions* (1.2), (1.5) *are satisfied in the usual sense of continuous functions. Moreover, $u(x,t) \in \Delta$ for $(x,t) \in \bar{Q}$.*

We also state here our main result for the limit problem (1.1), (1.2), (1.8).

Theorem 1.2. *Let $f(u)$ and $B(u)$ be as in Theorem 1.1. Assume $u_0 \in L^2(\Omega)$. Then problem* (1.1), (1.2), (1.8) *has a solution $u(x,t)$ in the sense that u satisfies* (1.1) *in the classical sense in Q, u satisfies* (1.2) *in the sense that $u(\cdot,t) \to u_0(\cdot)$ in $L^2(\Omega)$ as $t \to 0$, and the boundary condition* (1.8) *is satisfied in the sense that*

$$\lim_{h\to 0} \frac{1}{h} \int_{\Omega_{h\pm}} \int_0^T \big(B(u)u_x - f(u)\big)\chi(t)\,dt\,dx = 0 \quad \text{for all } \chi \in L^2(0,T), \tag{1.16}$$

where $\Omega_{h+} := (1-h, 1)$ and $\Omega_{h-} := (-1, -1+h)$. Moreover, $u(x,t) \in \Delta$ for $(x,t) \in \bar{Q}$.

1.1. **Brief outline of the proofs of Theorems 1.1 and 1.2.** We briefly outline the proofs of Theorems 1.1 and 1.2. Concerning Theorem 1.1, we first consider the approximate problems for $\varepsilon > 0$

$$\partial_t u_i + \partial_x f_i(u) = \sum_{j=i}^{n} \partial_x\big(B_{ij}(u)\partial_x u_j\big) + \varepsilon\left(\frac{1}{n} - u_i\right), \quad i = 1, \ldots, n, \tag{1.17}$$

$$u(x,0) = u_{0\varepsilon}(x) := (1-\varepsilon)u_0(x) + \frac{\varepsilon}{2n}e, \quad x \in \Omega, \quad e := (1, \ldots, 1)^{\mathrm{T}}, \tag{1.18}$$

$$f_i(u) - \sum_{j=i}^{n} B_{ij}(u)\big(\partial_x u_j \pm \theta(u_j - u_{\varepsilon j}^{\pm}(t))\big) = 0, \quad x = \pm 1, \quad i = 1, \ldots, n, \tag{1.19}$$

where we define

$$u_\varepsilon^{\pm} := (1-\varepsilon)u^{\pm} + \frac{\varepsilon}{2n}e.$$

The importance of introducing the perturbed problem (1.17)–(1.19) is related to the verification of the positive invariance of the region Δ. Indeed, the latter will follow from conditions (1.12) and (1.13) by proceeding as in [6] and [10]. The new point here is the argument on the boundary, which, as in [10], plays with the sign of the space derivative given by the boundary condition (1.19), by using (1.12) and (1.13). Once we prove the invariance of Δ, for the subsequent analysis in the time-stepping procedure we may assume $f(u)$ and $B(u)$ to be defined conveniently outside Δ in order to guarantee the a priori uniform boundedness of the solutions of the step problems, e.g., we may assume $f(u) = 0$ and $B(u) = \nu \, \mathrm{Id}$ outside a bounded open set containing Δ, where Id is the identity matrix.

Because of the nonlinear coupling in the boundary condition (1.19), the proof of Theorem 1.1 requires a *time stepping procedure*. This consists in linearizing the boundary condition in each step, obtaining the solution for each of the resulting problems. The initial-boundary value problem in the k-th step with initial time $t = k\Delta t$, for $k \geq 1$, is given by (1.17) plus the value of the solution of the $(k-1)$-th step as initial condition, i.e.,

$$u(x, k\Delta t) = u^{[k-1]}(x, k\Delta t - 0), \qquad x \in \Omega, \tag{1.20}$$

and the boundary condition

$$\partial_x u \pm \theta\big(u - u_\varepsilon^{\pm}(t)\big) = B\big(u^{[k-1]}\big)^{-1} f\big(u^{[k-1]}\big)\big|_{x=\pm 1,\ t=k\Delta t-0}, \ t > k\Delta t, \tag{1.21}$$

where by $u^{[k]}$ we denote the solution of the k-th step problem. Let us call the just described problem the k-th step problem (1.17), (1.20), (1.21). For $k = 0$, we may define the 0-th step problem (1.17), (1.20), (1.21) as above agreeing to define $u^{[-1]}(x, 0) = u_0(x)$. Since the initial condition (1.20) and the boundary condition (1.21) may not satisfy the appropriate compatibility conditions, it is not possible to guarantee the well-posedness of (1.17), (1.20), (1.21). Nevertheless, a solution can always be obtained as the limit of solutions of well-posed problems obtained through a small regularization of those initial and boundary conditions enforcing the compatibility conditions. The well-posedness of the regularized k-th step problem (1.17), (1.20), (1.21) is proved exactly in the same way as that of the similar problem in [10]. We recall that [10] follows the basic strategy of [13], whose main point is the application of Leray-Schauder's fixed point theorem, combined with bottom to top recursive a priori estimates similar to the procedure in [1]. Actually, the most important point in [10] is related with the regularity at the boundary. In general, in order to prove such regularity (see, e.g., [13]), we need to assume certain smoothness of the coefficients of the second order terms with respect to one of the independent variables, x, t, which is not possible when dealing with system (1.17) since, although we assume that $B(u)$ is upper triangular, we *do not* assume that the B_{ij} depend only on u_k for $k \geq i$; rather, they can depend on all components of u. To surmount this difficulty, the idea introduced in [10] is to use the good sign of the boundary term resulting from integration by parts in the crucial estimates.

We then define $u^\Delta(x, t)$ in Q, assuming $T = N\Delta t$, setting $u^\Delta(x, t) := u^{[k]}(x, t)$ for $(x, t) \in \Omega \times [k\Delta t, (k+1)\Delta t)$, for $k = 0, 1, \dots, N$. Looking carefully at the a priori estimates obtained for the k-th step problem (1.17), (1.20), (1.21), $k = 0, 1, \dots, N$, we show that it is possible to combine the first three of them, which imply that $u^\Delta \in H^{\beta,\beta/2}(\bar{Q})$ for some $\beta \in (0, 1)$, with the Hölder norm bounded by a constant independent of ε and Δt, and that for some $c > 0$, independent of Δt and ε,

$$\int_Q \big(\big|u_{xx}^\Delta\big|^2 + \big|u_x^\Delta\big|^4 + \big|u_t^\Delta\big|^2\big)\, dx\, dt \leq c. \tag{1.22}$$

The uniform boundedness of the Hölder norm of u^Δ in $H^{\beta,\beta/2}(\bar{Q})$ provides the compactness of the sequence $\{u^\Delta\}$ in $H^{\alpha,\alpha/2}(\bar{Q})$ for any $0 < \alpha < \beta$, and so we may make $\Delta t \to 0$ and extract a subsequence $u^{\Delta'}$ converging to a certain u^ε in $H^{\alpha,\alpha/2}(\bar{Q})$ for each $\varepsilon > 0$. The limit u^ε will then satisfy an integral identity which is a weak formulation for problem (1.17)–(1.19). Using this integral identity and the estimate obtained from (1.22) we show that u^ε is in fact the unique smooth solution of (1.17)–(1.19).

Finally, we make $\varepsilon \to 0$ using the relative compactness guaranteed by the regularity of u^ε, which is uniform with respect to ε. It then follows that any vector function which is the limit of a converging subsequence of solutions of problems (1.17)–(1.19) is a classical solution of the problem (1.1)–(1.3). Uniqueness follows in a standard way by taking the difference of the equations for any two solutions, multiplying each equation resulting from the difference of the corresponding components by the difference of these components, integrating over Ω, using integration by parts, and making estimates in a bottom-to-top iterative way.

As for Theorem 1.2, we obtain its proof by compactness taking the limit as $\theta \to 0$ of a subsequence of solutions of (1.1)–(1.3) and proving that the limit function verifies the required properties. As to the uniqueness, for the class of regularity to which the solution obtained in this limit process belongs, in which we miss boundedness of the first space derivative up to the boundary, no method for proving uniqueness is known as yet.

Correlated works besides [9, 10] include the following. In the already mentioned book [13], the theory developed for linear and quasilinear equations is applied to quasilinear systems of the type (1.1) (in the multidimensional case) where the nonlinear diffusion matrix is a scalar multiple of the identity matrix. We also mention the results of H. Amann [1] which require $f_i = f_i(u_i, \dots, u_n)$ and $B_{ij} = B_{ij}(u_i, \dots, u_n)$, $1 \le i \le j \le n$, besides (1.6) and the upper-triangularity of $B(u)$, and assume uniform boundedness of a certain Hölder norm of the local solution, in order to extend it to all times $t > 0$.

1.2. **Brief description of the contents.** This paper is organized as follows. In Section 2, we prove positive invariance of Δ for problem (1.17)–(1.19). In Section 3, we describe our time-stepping procedure and obtain a priori estimates in the space of Hölder continuous functions for the solution of each time-step problem. The Leray-Schauder fixed-point theorem is applied in Section 4 in order to prove existence of solution for the time-step problems. We also discuss the independence of some crucial estimates with respect to the time-step Δt and the parameter enforcing the compatibility conditions at each time-step. In Section 5, we conclude the proof of Theorem 1.1 and also give the proof of Theorem 1.2. Finally, in Section 6, we derive a model for polydisperse suspensions that improves an earlier model considered in [3], which can be cast, by a change of dependent variables, in a form which satisfies, after the addition of artificial viscosity, the assumptions of Theorem 1.1.

2. Invariance of the physical domain Δ

In this section we prove the positive invariance of Δ under the flow of the solution operator for the problem (1.17)–(1.19). More precisely, we have the following result.

Lemma 2.1. *Let the hypotheses of Theorem 1.1 hold. Then, if $u(x,t)$ is a classical solution of the problem* (1.17)–(1.19) *in $\bar{Q}$, we have $u(x,t) \in \Delta$ for all $(x,t) \in \bar{Q}$.*

Proof. We argue by contradiction. So, assume that there is $(x,t) \in \bar{Q}$ such that $u(x,t) \notin \Delta$. Let

$$t_0 := \inf\{\, t \in [0,T] \,:\, u(x,t) \notin \Delta\}.$$

Since $u_0 \in \operatorname{int}\Delta$, we have $t_0 > 0$. Let $x_0 \in [-1,1]$ be such that $u(x_0,t_0) \in \partial\Delta$. Hence, we have either $u_i(x_0,t_0) = 0$, for some $i = 1,\dots,n+1$, where we define $u_{n+1} := 1 - u_1 - \cdots - u_n$. We discuss separately two different cases: (i) $x_0 \in (-1,1)$; (ii) $x_0 = \pm 1$. Let us consider first the case (i). In this case, for some $i \in \{1,\dots,n+1\}$, u_i assumes its minimum value in the region $[-1,1] \times [0,t_0]$ at (x_0,t_0), and so we must have

$$u_{it}(x_0,t_0) \le 0, \qquad u_{ix}(x_0,t_0) = 0, \qquad u_{ixx}(x_0,t_0) \ge 0. \tag{2.1}$$

Now, in light of (1.12) and (1.13) we may easily verify that at (x_0,t_0), the i-th equation from (1.17) reduces to

$$u_{it} + \lambda_i u_{ix} = (\mu_i u_{ix})_x + \varepsilon l_i(u_i), \tag{2.2}$$

where λ_i and μ_i are scalar functions of u, $\mu_i > 0$ by (1.6), $l_i(u_i) := (\frac{1}{n} - u_i)$ if $i \in \{1,\dots,n\}$, $l_{n+1}(u_{n+1}) = u_{n+1}$, and the equation for u_{n+1} is obtained from the n equations of (1.17) in an obvious manner by summation. Hence, by (2.1) we arrive at a contradiction since the left-hand side of (2.2) is nonpositive while the right-hand side is positive. We now consider case (ii) and assume without loss of generality that $x_0 = -1$. Let $i \in \{1,\dots,n+1\}$ be such that $u_i(-1,t_0) = 0$. Applying (1.12) and (1.13) to (1.19), we obtain that

$$u_{ix}(-1,t_0) < 0, \tag{2.3}$$

where, again, the equation for $i = n+1$ is obtained from (1.19) in an obvious manner. Now (2.3) gives a contradiction since $u_i(x,t_0) \ge 0$ for all $x \in [-1,1]$. For $x_0 = 1$ we obtain a contradiction in a similar way. $\square$

In view of the above result, we may assume $f_i(u)$ and $B_{ij}(u)$ to be defined as smooth functions for all $u \in \mathbb{R}^n$ in such way that $f_i(u) \equiv 0$ and $B_{ij}(u) = \nu I$ for $u \notin U$, where $U \subseteq \mathbb{R}^n$ is a bounded open cube with faces parallel to the coordinate axes satisfying $\Delta \subseteq U$. This extension will be needed in what follows, since the invariance of Δ will not hold, in general, for the approximate solutions constructed by our time-stepping procedure.

3. A time-stepping approximation

3.1. **Initial-boundary value problem on short time intervals.** In this section we construct approximate solutions of (1.17)–(1.19), defining them in a recursive way for time intervals $[k\Delta t, (k+1)\Delta t)$, $k = 0,\dots,N-1$, with $T = N\Delta t$. Our one-parameter approximate solution $u^\Delta(x,t)$ will be a solution of the following initial-boundary value problems, which are defined recursively for $(x,t) \in Q_k := [-1,1] \times [k\Delta t, (k+1)\Delta t)$ and consisting of (1.17) with initial and boundary conditions given by

$$u^\Delta(x,k\Delta t) = u^\Delta(x,k\Delta t - 0), \quad x \in \Omega, \tag{3.1}$$

$$\partial_x u \pm \theta\big(u - u_\varepsilon^\pm(t)\big) = \big[B(u^\Delta(k\Delta t - 0))^{-1} f(u^\Delta(k\Delta t - 0))\big]_{x=\pm 1}, \quad t > k\Delta t \tag{3.2}$$

for $k = 1,\dots,N-1$, while, for $k = 0$, we set

$$u^\Delta(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{3.3}$$

$$\partial_x u \pm \theta\big(u - u_\varepsilon^\pm(t)\big) = B(u_{0\varepsilon}(\pm 1))^{-1} f(u_{0\varepsilon}(\pm 1)), \qquad t > 0. \tag{3.4}$$

We remark that the solutions of (1.17), (3.1), (3.2) will in general not be smooth at the corner points $(\pm 1, k\Delta t)$, $k = 1, \ldots, N-1$, since the initial and boundary conditions at each time step, for $t = k\Delta t$, may not be compatible. Nevertheless, we will show that they can be defined in such a way that they are Hölder continuous in $\bar{Q}$, smooth in $\bar{Q} \setminus \{(\pm 1, k\Delta t) : k = 1, \ldots, N-1\}$, and (1.1) is satisfied in the classical sense in the interior of Q, while (3.1) and (3.2) are satisfied pointwise except at those corner points. The corresponding problem for $k = 0$, that is, (1.17), (3.3), (3.4), is well-posed since here the compatibility conditions at the points $(\pm 1, 0)$ hold by assumption.

3.2. **The k-th step problem.** Here we discuss the solution of the k-th step problem (1.17), (3.1), (3.2), $k \geq 1$. The case $k = 0$ is treated similarly, but does not require regularization of the initial data to enforce compatibility, since this already holds by assumption. As already mentioned, the solution of (1.17), (3.1), (3.2) is obtained as a limit of solutions of regularized problems given by (1.17) with the same boundary condition (3.2) and an initial condition obtained through a regularization of (3.1) in such a way that the new initial condition is compatible with (3.2) at the corners $(\pm 1, k\Delta t)$. So, let $\zeta_\delta \in C_0(\Omega)$ be such that ζ_δ is even, $\zeta_\delta(x) = 1$ for $-1 + \delta \leq x \leq 1 - \delta$, $\zeta_\delta(x) = 0$ for $1 - \frac{\delta}{3} < |x| \leq 1$, and ζ_δ is non-increasing in $[0, 1]$ with $|\zeta_\delta'(x)| \leq 2\delta^{-1}$. We may define regularized initial data by

$$u_\delta^\Delta(x, k\Delta t) = \zeta_\delta(x) u^\Delta(x, k\Delta t - 0) + \big(1 - \zeta_\delta(x)\big)\big(\gamma_{k,\delta}^-\, \chi_{[-1,0)}(x) + \gamma_{k,\delta}^+\, \chi_{[0,1]}(x)\big) \tag{3.5}$$

where, as usual, χ_A denotes the indicator function of the set A and

$$\gamma_{k,\delta}^\pm = u_\varepsilon^\pm(k\Delta t) \pm \frac{1}{\theta}\Big[B\big(u^\Delta(x, k\Delta t - 0)\big)^{-1} f\big(u^\Delta(x, k\Delta t - 0)\big)\Big]_{x=\pm 1}. \tag{3.6}$$

The well-posedness of the regularized k-th problem (1.17), (3.2), (3.5) is proved in [10] in the case $n = 2$. The same result for general n is a straightforward consequence of the case $n = 2$. From the results in [10], we easily deduce that the solution of (1.17), (3.5), (3.2) is bounded in $H^{2+\beta, 1+\beta/2}(K)$, where K is any compact set contained in $\bar{Q}_k \setminus \{(-1, k\Delta t), (1, k\Delta t)\}$ and $Q_k := \Omega \times (k\Delta t, k\Delta t + T)$, with $\beta \in (0, 1)$ and corresponding bound depending, in general, on f, B, T and K but independent of δ, for δ sufficiently small. The latter is clear since the regularization in (3.5) is only visible for points in a small neighborhood of the corner points $(\pm 1, k\Delta t)$ which will not intersect K if δ is small enough.

Using well-known compactness properties of spaces of Hölder continuous functions we easily obtain a solution of (1.17), (3.1), (3.2) as the limit of a subsequence of solutions (1.17), (3.2), (3.5). The solution of (1.17), (3.1), (3.2) obtained in this way is not known to belong to a class of well-posedness with respect to this problem since we do not know whether its space derivative is bounded in the whole domain Q_k. For closer inspection we choose one such solution, which we may do, and we call it $u^{[k]}$.

The following estimate for the solution of the k-th problem (1.17), (3.1), (3.2) is an immediate consequence of the assumptions on f and B stated in Section 2.

Lemma 3.1. *The estimate*

$$\big\| u^{[k]} \big\|_{L^\infty(Q_k)} \leq c \tag{3.7}$$

holds for some constant $c = c(\theta) > 0$, depending on θ, but independent of $\varepsilon, \Delta t$ and k.

Proof. By assumption we have $f(u) = 0$ and $B(u) = \nu I$, if $u \notin U$, where U is a bounded open cube centered at the origin and containing Δ. Let $\tilde{U} \supset U$ be another bounded open cube with faces parallel to the coordinate and such that

$$\inf_{\substack{u \in \tilde{F} \\ t \in (0,T)}} |u - u^{\pm}(t)| > \frac{1}{\theta} \sup_{u \in U} |B(u)^{-1} f(u)|, \tag{3.8}$$

for any face $\tilde{F}$ of $\tilde{U}$. The solution of the k-regularized problem (1.17), (3.2), (3.5) satisfies (3.7) by a reasoning similar to that of the proof of Lemma 2.1, with a constant c determined only by $\tilde{U}$, and so depending on θ, because of (3.8), but independent of ε,Δt, k and δ. Hence (3.7) also holds for $u^{[k]}$ since it is a pointwise limit of solutions to (1.17), (3.2), (3.5) as $\delta \to 0$. □

We collect in the following lemma the a priori integral estimates for the solution of (1.17), (3.1), (3.2) whose bounds may be taken independent of ε, Δt, and k.

Lemma 3.2. *The solution of* (1.17), (3.1), (3.2), $u^{[k]}$, *satisfies*

$$\int_\Omega \bigl|u^{[k]}(t)\bigr|^2 \, dx \le c, \qquad \int_{k\Delta t}^{k\Delta t + \tau} \int_\Omega \bigl|u_x^{[k]}\bigr|^2 \, dx \, dt \le c\tau, \quad 0 < \tau \le T. \tag{3.9}$$

Moreover, there exists $\beta \in (0,1)$, independent of ε, Δt, and k, such that $u^{[k]} \in H^{\beta,\beta/2}(\bar{Q}_k)$, with $Q_k = \Omega \times (k\Delta t, k\Delta t + T)$, and

$$\begin{aligned} &\int_\Omega \bigl|u_x^{[k]}(k\Delta t + \tau)\bigr|^2 \, dx + c \int_{k\Delta t}^{k\Delta t + \tau} \int_\Omega \bigl(\bigl|u_{xx}^{[k]}\bigr|^2 + \bigl|u_x^{[k]}\bigr|^4 + \bigl|u_t^{[k]}\bigr|^2 \bigr) \, dx \, dt \\ &\qquad \le c' \left(\tau + \int_{k\Delta t}^{k\Delta t + \tau} \int_\Omega \bigl|u_x^{[k]}\bigr|^2 \, dx \, dt \right) + \int_\Omega \bigl|u_x^{[k]}(k\Delta t)\bigr|^2 \, dx. \end{aligned} \tag{3.10}$$

The constants c and c' above are independent of $\varepsilon, \Delta t, k$.

Proof. It suffices to consider the case $n = 2$. In this case, the lemma follows from Lemmas 2.2, 2.3, 2.4 and 2.5 of [10]. □

As mentioned above, we define the aproximate solution for the problem (1.17)–(1.19), $u^{\Delta}(x,t)$, by setting

$$u^{\Delta}(x,t) = u^{[k]}(x,t) \quad \text{for } (x,t) \in [-1,1] \times [k\Delta t, (k+1)\Delta t), \quad k = 0, 1, \dots, N-1. \tag{3.11}$$

As a corollary of Lemmas 3.1 and 3.2 we immediately obtain the following lemma.

Lemma 3.3. *The following estimates hold for a constant $c > 0$ independent of ε and Δt:*

$$\|u^{\Delta}\|_{L^\infty(Q)} \le c, \tag{3.12}$$

$$\|u^{\Delta}\|_{H^{\beta,\beta/2}(\bar{Q})} \le c, \tag{3.13}$$

$$\int_Q \bigl|u_x^{\Delta}\bigr|^2 \, dx \, dt \le c, \tag{3.14}$$

$$\int_Q \bigl(\bigl|u_{xx}^{\Delta}\bigr|^2 + \bigl|u_x^{\Delta}\bigr|^4 + \bigl|u_t^{\Delta}\bigr|^2 \bigr) \, dx \, dt \le c. \tag{3.15}$$

Moreover, for all $\varphi \in C_0(\mathbb{R} \times (-\infty, T))$ we have

$$
\begin{aligned}
&\int_Q u^\Delta \varphi_t + f(u^\Delta)\varphi_x - B(u^\Delta) u_x^\Delta \varphi_x \, dx\, dt + \int_\Omega u_{0\varepsilon}(x)\varphi(x,0)\, dx \\
&= \pm \int_{\{\pm 1\}\times(0,T)} \Big((f(u^\Delta([t/\Delta t]\,\Delta t)) - f(u^\Delta)) \\
&\qquad\qquad - \big(B(u^\Delta([t/\Delta t]\,\Delta t)) - B(u^\Delta)\big) u_x^\Delta \Big) \varphi(\pm 1, t)\, dt \\
&\quad \pm \theta \int_{\{\pm 1\}\times(0,T)} B(u^\Delta)(u^\Delta - u_\varepsilon^\pm(t))\varphi(\pm 1, t)\, dt.
\end{aligned} \tag{3.16}
$$

Proof. The first two estimates follow directly from Lemmas 3.1 and 3.2, while the last two also follow from these lemmas by summing up from $k = 0$ up to $k = N - 1$ the inequalities (3.9) and (3.10), respectively, with $\tau = \Delta t$. As to (3.16), we first observe that u^Δ satisfies (1.17) in Q, the boundary condition (3.2), for $t \in [k\Delta t, (k+1)\Delta t)$, $k = 0, 1, \ldots, N-1$, and the initial condition (3.4), for $t = 0$. Hence, (3.16) follows by multiplying (1.17) by φ, integrating over Q, and using integration by parts. □

4. Convergence of the approximate solutions

In this section we discuss the convergence of the approximate solutions $u^\Delta(x,t)$. From (3.13) in Lemma 3.1 we see that the family $\{u^\Delta\}$ is compact in $H^{\alpha,\alpha/2}(\bar{Q})$ for any $0 < \alpha < \beta$. Hence, we may extract a subsequence $u^{\Delta'}$ converging to some u^ε in $H^{\alpha,\alpha/2}(\bar{Q})$ as $\Delta t \to 0$. Also from Lemma 3.3 we deduce that u^ε satisfies

$$\|u^\varepsilon\|_{L^\infty(Q)} \le c, \tag{4.1}$$

$$\|u^\varepsilon\|_{H^{\beta,\beta/2}(\bar{Q})} \le c, \tag{4.2}$$

$$\int_Q |u_x^\varepsilon|^2 \, dx\, dt \le c, \tag{4.3}$$

$$\int_Q \big(|u_{xx}^\varepsilon|^2 + |u_x^\varepsilon|^4 + |u_t^\varepsilon|^2\big)\, dx\, dt \le c, \tag{4.4}$$

and, for any $\varphi \in C_0(\mathbb{R} \times (-\infty, T))$,

$$
\begin{aligned}
&\int_Q \big(u^\varepsilon \varphi_t + f(u^\varepsilon)\varphi_x - B(u^\varepsilon)u_x^\varepsilon \varphi_x\big)\, dx\, dt + \int_\Omega u_{0\varepsilon}(x)\varphi(x,0)\, dx \\
&= \pm\theta \int_{\{\pm 1\}\times(0,T)} B(u^\varepsilon)(u^\varepsilon - u_\varepsilon^\pm(t))\varphi(\pm 1, t)\, dt.
\end{aligned} \tag{4.5}
$$

5. Conclusion of the proofs of Theorem 1.1 and 1.2

Conclusion of the proof of Theorem 1.1. Estimates (4.1)–(4.5) allow the application of Lemmas 2.7, 2.8 and 2.9 of [10] (these lemmas are stated for $n = 2$, but they are obviously also valid for general n) to conclude that

$$u_x^\varepsilon \in H^{\alpha,\alpha/2}(\bar{Q}) \quad \text{and} \quad |u_x^\varepsilon|^{(\alpha)} \le c, \tag{5.1}$$

for some $\alpha \in (0,1)$ and $c > 0$ independent of ε. Hence, u^ε is a classical solution of (1.17)–(1.19), and by Lemma 2.1,

$$u^\varepsilon(x,t) \in \Delta \quad \text{for all } (x,t) \in \bar{Q}. \tag{5.2}$$

Making $\varepsilon \to 0$, using well-known compactness arguments in spaces of Hölder continuous functions, we may extract a subsequence $u^{\varepsilon'}$ converging in $H^{\gamma,\gamma/2}(\bar{Q})$, $0<\gamma<\alpha$ to some u satisfying $u, u_x \in H^{\alpha,\alpha/2}(\bar{Q})$ and $u(x,t) \in \bar{Q}$ for all $(x,t) \in \bar{Q}$, which is a classical solution of (1.1), (1.2), (1.3). Clearly, we have

$$u(x,t) \in \Delta \quad \text{for all } (x,t) \in \bar{Q}. \tag{5.3}$$

Uniqueness of u in the class of such solutions is proved by the standard arguments already mentioned in Section 1. □

Conclusion of the proof of Theorem 1.2. Let u^θ denote the solution of (1.1), (1.2), (1.3) obtained above. Given any compact subset $K \subseteq Q$ we have

$$u^\theta \in H^{2+\alpha,1+\alpha/2}(K) \quad \text{and} \quad \|u^\theta\|_{H^{2+\alpha,1+\alpha/2}(K)} \leq c \tag{5.4}$$

for some $\alpha \in (0,1)$ and $c = c(K) > 0$ independent of θ. Moreover, we also have that u^θ satisfies (5.3) and

$$\|u_x^\theta\|_{L^2(Q)} \leq c \tag{5.5}$$

for some $c > 0$ also independent of θ. Hence, we may make $\theta \to 0$ and extract a subsequence $u^{\theta'}$ converging in $L^1(Q)$ to a certain function $u \in H^{2+\alpha,1+\alpha/2}(Q)$, which satisfies (1.1) in Q. By (5.5) we may choose $u^{\theta'}$ so that $u_x^{\theta'} \rightharpoonup u_x$ in the weak topology of $L^2(Q)$. It satisfies

$$\int_Q \big(u\varphi_t + f(u)\varphi_x - B(u)u_x\varphi_x\big)\,dx\,dt + \int_\Omega u_0(x)\varphi(x,0)\,dx = 0 \tag{5.6}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R}\times(-\infty,T))$, which follows from (4.5) by making first $\varepsilon \to 0$ and then $\theta \to 0$, and using the already mentioned convergences. Now, (1.16) follows easily from (5.6) by choosing $\varphi = \chi(t)\zeta_h(x)$, with $\chi \in C_0^1((0,T))$, $0<h<1/2$, and $\zeta_h(x) = 1$ for $x \in [-1+h, 1-h]$, $\zeta_h(x) = h^{-1}(1+x)$ for $x \in [-1,-1+h)$, and $\zeta_h(x) = h^{-1}(1-x)$ for $x \in (1-h,1]$, and making $h \to 0$. Finally, it also satisfies (1.2) in the sense that

$$\lim_{t\to 0}\int_\Omega \big|u(x,t)-u_0(x)\big|^2\,dx = 0. \tag{5.7}$$

Indeed, from (5.6) we easily deduce

$$\lim_{t\to 0}\int_\Omega \big(u(x,t)-u_0(x)\big)\zeta(x)\,dx = 0 \tag{5.8}$$

for all $\zeta \in C_0^1(\mathbb{R})$. On the other hand, also from (5.6) in a standard way (see, e.g., [13, Chapter III, §2]), we deduce the energy estimate

$$\int_\Omega |u(x,t)|^2\,dx + \frac{\nu}{2}\int_0^t\int_\Omega |u_x(x,s)|^2\,dx\,ds \leq \int_\Omega |u_0(x)|^2 + c\int_0^t\int_\Omega |u(x,s)|^2\,dx\,ds,$$

from which it follows that

$$\limsup_{t\to 0}\int_\Omega |u(x,t)|^2\,dx \leq \int_\Omega |u_0(x)|^2\,dx. \tag{5.9}$$

On the other hand, from (5.8) and convexity (see, e.g., [21]), we deduce

$$\liminf_{t\to 0}\int_\Omega |u(x,t)|^2\,dx \geq \int_\Omega |u_0(x)|^2\,dx. \tag{5.10}$$

Now, (5.8), (5.9) and (5.10) imply as usual (5.7). □

6. Sedimentation of polydisperse suspensions

6.1. **Balance equations and model assumptions.** Our interest in the initial-boundary value problem (1.1), (1.2), (1.8) is motivated by a model of gravity sedimentation of polydisperse suspensions that form compressible sediments. After a change of variables, which ensures an upper triangular form, this model provides the coefficient functions $f_1(u),\dots,f_n(u)$ and $(B_{ij}(u))_{i,j=1,\dots,n}$. It is a variant of the model introduced in [3], and will be briefly derived here.

We consider small solid particles belonging to n different species having sizes $d_1 \geq d_2 \geq \cdots \geq d_n$ and densities $\varrho_1,\dots,\varrho_n$, where $d_i \neq d_j$ or $\varrho_i \neq \varrho_j$ for $i \neq j$, $1 \leq i,j \leq n$, which are dispersed in a viscous fluid of density ϱ_{f} and viscosity μ_{f}. We consider the solids and the fluid as $n+1$ superimposed continuous phases, and start from the multi-dimensional mass and linear momentum balances

$$\partial_t\phi_i + \nabla\cdot(\phi_i\mathbf{v}_i) = 0, \quad i=1,\dots,n; \quad \partial_t\phi - \nabla\cdot\bigl((1-\phi)\mathbf{v}_{\mathrm{f}}\bigr) = 0, \tag{6.1}$$

where $\Phi := (\phi_1,\dots,\phi_n)$ is the vector of the local volume fractions of the solid phases, $\phi := \phi_1+\cdots+\phi_n$ is the total solids volume fraction, and $\mathbf{v}_1,\dots,\mathbf{v}_n$ and $\mathbf{v}_{\mathrm{f}}$ are the solids and fluid phase velocities, respectively. Summation over all mass balances implies $\nabla\cdot\mathbf{q} = 0$, where $\mathbf{q} := \phi_1\mathbf{v}_1+\cdots+\phi_n\mathbf{v}_n+(1-\phi)\mathbf{v}_{\mathrm{f}}$ is the volume average mixture velocity. Introducing the relative velocities (or slip velocities) $\mathbf{u}_i := \mathbf{v}_i - \mathbf{v}_{\mathrm{f}}$ for $i=1,\dots,n$, we obtain $\mathbf{q} = \phi_1\mathbf{u}_1+\cdots+\phi_n\mathbf{u}_n+\mathbf{v}_{\mathrm{f}}$ and

$$\mathbf{v}_i = \mathbf{u}_i + \mathbf{v}_{\mathrm{f}} = \mathbf{u}_i + \mathbf{q} - (\phi_1\mathbf{u}_1+\cdots+\phi_n\mathbf{u}_n), \quad i=1,\dots,n.$$

The momentum balance equations for the n solid species and the fluid are

$$\varrho_i\phi_i(\partial_t\mathbf{v}_i + (\mathbf{v}_i\cdot\nabla)\mathbf{v}_i) = \nabla\cdot\mathbf{T}_i - \varrho_i\phi_i g\mathbf{k} + \mathbf{m}_i, \quad i=1,\dots,n, \tag{6.2}$$

$$\varrho_{\mathrm{f}}(1-\phi)(\partial_t\mathbf{v}_{\mathrm{f}} + (\mathbf{v}_{\mathrm{f}}\cdot\nabla)\mathbf{v}_{\mathrm{f}}) = \nabla\cdot\mathbf{T}_{\mathrm{f}} - \varrho_{\mathrm{f}}(1-\phi)g\mathbf{k} + \mathbf{m}_{\mathrm{f}}, \tag{6.3}$$

where $\mathbf{m}_i$ is the interaction force per unit volume between solid species i and the fluid and $\mathbf{m}_{\mathrm{f}} = -(\mathbf{m}_1+\cdots+\mathbf{m}_n)$. Interaction forces due to solid-solid contacts are assumed to be negligible compared with the solid-fluid transfer of momentum [5]. Moreover, $\mathbf{T}_i$ denotes the stress tensor of solids phase i, $i=1,\dots,n$, $\mathbf{T}_{\mathrm{f}}$ that of the fluid, g is the acceleration of gravity, and $\mathbf{k}$ is the upwards directed unit vector.

We assume that the stress tensors of the solid and fluid phases take the respective forms $\mathbf{T}_i = -p_i\mathbf{I} + \mathbf{T}_i^{\mathrm{E}}$ for $i=1,\dots,n$ and $\mathbf{T}_{\mathrm{f}} = -p_{\mathrm{f}}\mathbf{I} + \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$, where p_i denotes the phase pressure of particle species i, p_{f} that of the fluid, $\mathbf{I}$ denotes the identity tensor, and $\mathbf{T}_i^{\mathrm{E}}$ and $\mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$ are the extra (or viscous) stress tensors of particle species i and the fluid, respectively. Since viscous effects due to the motion of the mixture are not considered to be dominant for our analysis, we neglect the viscous stress tensors.

The theoretical phase pressures $p_1,\dots,p_n$ and p_{f} are now expressed in terms of the pore pressure p and the effective solids stress σ_{e}, which can be measured. As in [3], we assume that σ_{e} is a known material specific function of ϕ, which satisfies

$$\sigma_{\mathrm{e}}(\phi)\begin{cases} = 0 & \text{for } \phi \leq \phi_{\mathrm{c}}, \\ > 0 & \text{for } \phi > \phi_{\mathrm{c}}, \end{cases} \qquad \sigma_{\mathrm{e}}'(\phi) := \frac{d\sigma_{\mathrm{e}}(\phi)}{d\phi}\begin{cases} = 0 & \text{for } \phi \leq \phi_{\mathrm{c}}, \\ > 0 & \text{for } \phi > \phi_{\mathrm{c}}, \end{cases} \tag{6.4}$$

where ϕ_{c} is the critical concentration at which the solid particles touch each other.

In this paper, the relationship between the theoretical pressures $p_1,\dots,p_n$ and p_{f} and the experimental variables p and $\sigma_{\mathrm{e}}(\phi)$ slightly differs from the one given in [3].

In that paper, it is argued that the portions of the total pressure $p_1+\cdots+p_n+p_{\mathrm{f}} = \phi p+\sigma_{\mathrm{e}}(\phi)$ can be assigned to species i and to the fluid by

$$p_i = \phi_i\bigl(p+\sigma_{\mathrm{e}}(\phi)/\phi\bigr), \quad i=1,\dots,n; \quad p_{\mathrm{f}} = (1-\phi)p. \tag{6.5}$$

In this work, we do not fix $p_1,\dots,p_n$ and p_{f} as volume quantities, as expressed by (6.5); rather, the gradient of the pressure of each phase is specified as a surface quantity. Thus, the gradients of the solid and fluid phase pressures introduced by the stress tensors in (6.2) and (6.3) are replaced by the respective expressions

$$\nabla p_i = \phi_i\bigl(\nabla p+\nabla\sigma_{\mathrm{e}}(\phi)/\phi\bigr) \quad\text{and}\quad \nabla p_{\mathrm{f}} = (1-\phi)\nabla p. \tag{6.6}$$

Our preference of (6.6) (instead of (6.5)) is in part motivated by the discussion of sediment diffusivity in Section 7.5 of [3], where (6.5) is used. In fact, the numerical examples of [3] (see also [2]) show a strong differential relative movement of solid species within the sediment, which is driven by a diffusive term involving the gradient $\nabla(\phi_i/\phi)$. In particular, as is shown in [3], this term leads to a probably unrealistic equi-distribution of the solid species within the sediment at steady state. The model analyzed herein is nearly the same as that of [3], but it does not include that particular term, and is expected to predict more realistic results.

We emphasize herein that a polydisperse sedimentation model based on (6.6) is supported by mathematical analysis, while a detailed quantitative (numerical) comparison with the predictions generated by using (6.5) is not within the scope of this paper. From an experimental point of view, both alternatives, (6.5) or (6.6), are discussed in the literature. Both variants have been scrutinized [25], considered in parallel [17], and (6.6) is postulated a priori for example in [19]. This ambiguity is still an unresolved issue, as is emphasized in [22].

Our analysis is valid only for the case of (6.6). In fact, based on (6.5), the final solid momentum balances would include the additional term $\nabla((\phi_i/\phi)\sigma_{\mathrm{e}}(\phi))$, which in turn later appears in the "Darcy-type law" expression for the slip velocity (6.10) as $-(\sigma_{\mathrm{e}}(\phi)/\phi_i)\nabla(\phi_i/\phi)$. Even in the diffusive flux term, the gradient of ϕ_i remains and precludes recasting the diffusion matrix to an upper triangular matrix by a change of variables. This particular form is, however, essential for our treatment. The introduction of $\phi=\phi_1+\cdots+\phi_n$ as a new variable requires an extra equation for ϕ, which can be generated by summing the equations for $\phi_1,\dots,\phi_n$. If (6.5) is used, then the resulting equation will involve all variables $\phi_1,\dots,\phi_n,\phi$, while starting from (6.6), the resulting equation depends on ϕ only.

Under the present assumptions, and in particular neglecting the viscous stress tensors, the momentum balances for the solid and fluid phases become

$$\begin{aligned}\phi_i\varrho_i\bigl(\partial_t\mathbf{v}_i+(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i\bigr) &= \mathbf{m}_i-\varrho_i\phi_i g\mathbf{k}-\phi_i\nabla p-\frac{\phi_i}{\phi}\nabla\sigma_{\mathrm{e}}(\phi),\\ (1-\phi)\varrho_{\mathrm{f}}\bigl(\partial_t\mathbf{v}_{\mathrm{f}}+(\mathbf{v}_{\mathrm{f}}\cdot\nabla)\mathbf{v}_{\mathrm{f}}\bigr) &= \mathbf{m}_{\mathrm{f}}-\varrho_{\mathrm{f}}(1-\phi)g\mathbf{k}-(1-\phi)\nabla p.\end{aligned} \tag{6.7}$$

As a result of a dimensional analysis [3, 5] and due to the reduction to one space dimension, the left-hand sides in (6.7) can be neglected. After applying this simplification, summation of all equations in (6.7) yields the momentum balance of the suspension

$$0 = -\varrho(\Phi)g\mathbf{k}-\nabla p-\nabla\sigma_{\mathrm{e}}(\phi), \tag{6.8}$$

where $\varrho(\Phi) := \phi_1\varrho_1+\cdots+\phi_n\varrho_n+(1-\phi)\varrho_{\mathrm{f}}$ is the local density of the mixture. Next, we insert ∇p of the suspension momentum balance (6.8) into a solids momentum balance and assume the proportionality $\mathbf{m}_i = (\lambda_i(\Phi))^{-1}\mathbf{u}_i$ for $i = 1,\dots,n$, where the mobilities $\lambda_1(\Phi),\dots,\lambda_n(\Phi)$ are given by [3]

$$\lambda_i(\Phi) = \frac{d_i^2 \tilde{V}(\phi)}{18\mu_{\mathrm{f}}\phi_i}, \quad i = 1,\dots,n, \tag{6.9}$$

where μ_{f} is the dynamic viscosity of the pure fluid and $\tilde{V}(\phi)$ is a hindered settling factor. This yields the Darcy-type law

$$\mathbf{u}_i = \lambda_i(\Phi)\phi_i \left(\big(\varrho_i - \varrho(\Phi)\big) g\mathbf{k} - \frac{1-\phi}{\phi}\sigma_{\mathrm{e}}'(\phi)\nabla\phi \right) \tag{6.10}$$

for $i = 1,\dots,n$. Wherever $\phi \leq \phi_{\mathrm{c}}$, the solid effective stress term $((1-\phi)/\phi)\sigma_{\mathrm{e}}'(\phi)\nabla\phi$ vanishes, and (6.10) may be viewed as a generalization of Stokes' law, while for $\phi > \phi_{\mathrm{c}}$, it represents a version of Darcy's law. The hindered settling factor $\tilde{V}(\phi)$ is supposed to satisfy $\tilde{V}(0) = 1$ and $\tilde{V}(1) = 0$ and can e.g. be chosen as [16] $\tilde{V}(\phi) = (1-\phi)^{N-2}$ for $0 < \phi \leq 1$ with $N > 2$, and $\tilde{V}(\phi) = 0$ otherwise.

In one space dimension, the volume average velocity q vanishes for batch settling in a closed vessel. Then, the only equations that actually need to be solved are the continuity equations $\partial_t \phi_i + (\phi_i v_i)_x = 0$, $i = 1,\dots,n$, which now assume the form

$$\partial_t \phi_i + \partial_x \tilde{f}_i(\Phi) = \sum_{j=1}^{n} \partial_x \big(\tilde{B}_{ij}(\Phi)\partial_x \phi_j\big), \quad i = 1,\dots,n, \tag{6.11}$$

with the functions

$$\tilde{f}_i(\Phi) = \phi_i \big(d_i^2\big(\varrho_i - \varrho(\Phi)\big) - \tilde{S}(\Phi)\big)V(\phi), \quad i = 1,\dots,n, \tag{6.12}$$

where, for convenience, we define

$$V(\phi) := \frac{\tilde{V}(\phi)}{18\mu_{\mathrm{f}}}, \quad \tilde{S}(\Phi) := \sum_{k=1}^{n} \phi_k d_k^2 \big(\varrho_k - \varrho(\Phi)\big),$$

and use the regularized diffusion matrix

$$\tilde{B}_{ij} := \delta_{ij}\nu + \hat{B}_i(\Phi), \quad i,j = 1,\dots,n, \tag{6.13}$$

where $\nu > 0$ is a viscous regularization parameter, and we define

$$\hat{B}_i(\Phi) := \phi_i \left(d_i^2 - \sum_{k=1}^{N} \phi_k d_k^2 \right) \left(\frac{1-\phi}{\phi}\sigma_{\mathrm{e}}'(\phi) \right) V(\phi), \quad i = 1,\dots,n. \tag{6.14}$$

Finally, we mention that a very similar model for polydisperse sedimentation, which equally gives rise to a parabolic system of the type (6.11), was proposed and solved numerically in a recent paper by Watson et al. [24]. Summarizing the approaches of [3], [8] and [24], one may view the initial-boundary value problem studied herein as a generic model for polydisperse sedimentation.

Remark 6.1. The diffusion constant $\nu > 0$ represents the hydrodynamic diffusion of the solids phases. This phenomenon is based on the observation that in reality, solid particles at a given concentration vector Φ do not settle at precisely the same velocity. A detailed account of this variability is given in Section 7.4 of [3], but its main effect (as compared to non-diffusive models, e.g. those studied in [3, 4, 5]) is a blurring of otherwise sharp concentration discontinuities. This effect can be most easily modeled by the simple diffusion constant introduced here. In fact, Esipov [8] proposes a similar diffusive model that is approximated by constant diffusivities.

The identification of nonlinear but positive diffusion functions accounting for hydrodynamic diffusion is, however, a topic of current research, see e.g. [5, 7, 11, 12, 23] and the references cited in these papers.

6.2. **Initial and boundary conditions.** We consider batch settling of a suspension with a given initial composition $\Phi^0(x)$ in a closed cylindrical vessel, where the height of the suspension column has been normalized to 2 and $x \in [-1,1]$ is the corresponding downwards-increasing depth variable. Thus, we prescribe the initial condition

$$\Phi(x,0) = \Phi^0(x) = \big(\phi_1^0(x),\dots,\phi_n^0(x)\big)^{\mathrm{T}} \in \tilde{\Delta}, \quad x \in [-1,1], \tag{6.15}$$

where

$$\tilde{\Delta} := \big\{\Phi \in \mathbb{R}^n \,:\, \phi_1 \geq 0,\dots,\phi_n \geq 0,\ \phi_1 + \dots + \phi_n \leq 1\big\}, \tag{6.16}$$

is the phase space of the original variables. At $x=-1$, the suspension surface and at $x=1$, the vessel bottom are located. Both levels give rise to zero flux boundary conditions, which means that all solids fluxes through $x=-1$ and $x=1$ vanish:

$$(\phi_i v_i)|_{x=-1} = (\phi_i v_i)|_{x=1} = 0, \quad i = 1,\dots,n. \tag{6.17}$$

Equations (6.11) represent the solids continuity equations, therefore

$$\phi_i v_i = \tilde{f}_i(\Phi) - \sum_{m=1}^{n} \tilde{B}_{im}(\Phi)\partial_x \phi_m, \quad i = 1,\dots,n, \tag{6.18}$$

and the boundary conditions (6.17) can be written as the zero-flux or Neumann boundary conditions

$$\left(\tilde{f}_i(\Phi) - \sum_{m=1}^{n} \tilde{B}_{im}(\Phi)\partial_x \phi_m\right)\bigg|_{x=\pm 1} = 0. \tag{6.19}$$

Thus, (6.11), (6.15), (6.19) is the relevant initial-boundary value problem in our application.

6.3. **Transformation to upper triangular form.** The analysis of Sections 1 to 5 applies to our model if we change variables such that the resulting diffusion matrix becomes upper triangular. To this end, we rewrite the model in terms of the new variables $u = (u_1,\dots,u_n)^{\mathrm{T}}$ defined by

$$u_1 := \phi_1,\dots,u_{n-1} := \phi_{n-1}, \quad u_n := 1-\phi = 1 - (\phi_1 + \dots + \phi_n). \tag{6.20}$$

The phase space Δ is then as defined in (1.9). With a slight abuse of notation, let us define

$$\varrho(u) = \varrho(\Phi) := u_1(\varrho_1 - \varrho_n) + \dots + u_{n-1}(\varrho_{n-1} - \varrho_n) + (1-u_n)\varrho_n + u_n \varrho_{\mathrm{f}},$$

and $S(u) = \tilde{S}(\Phi)$. Noting that the quantity $1-\phi$ satisfies the equation

$$\begin{aligned} \partial_t(1-\phi) - \partial_x\left(\sum_{i=1}^{n} \tilde{f}_i(\Phi)\right) &= -\partial_x\left\{\nu \sum_{i=1}^{n} \partial_x \phi_i + \sum_{i=1}^{n} \hat{B}_i(\Phi)\left(\sum_{m=1}^{n} \partial_x \phi_m\right)\right\} \\ &= \nu \partial_x^2(1-\phi) + \partial_x\left(\left[\sum_{i=1}^{n} \hat{B}_i(\Phi)\right]\partial_x(1-\phi)\right), \end{aligned} \tag{6.21}$$

we then obtain, after the change of variables (6.20),

$$\begin{aligned} f_i(u) &= u_i\big(d_i^2(\varrho_i - \varrho(u)) - S(u)\big)V(1-u_n), \quad i = 1,\dots,n-1,\\ f_n(u) &= -u_n S(u) V(1-u_n), \end{aligned} \tag{6.22}$$

where

$$S(u) := \sum_{i=1}^{n-1} u_i d_i^2\big(\varrho_i - \varrho(u)\big) + \big(1-(u_1+\dots+u_n)\big)d_n^2\big(\varrho_n - \varrho(u)\big).$$

We easily see that (1.12) is satisfied. Defining

$$d^2(u) := \sum_{i=1}^{n} \phi_i d_i^2 = \big(1-(u_1+\dots+u_n)\big)d_n^2 + \sum_{i=1}^{n-1} u_i d_i^2,$$

in view of (6.21) and

$$\partial_t \phi_i + \partial_x \tilde{f}_i(\Phi) = \partial_x\Big(\nu\partial_x\phi_i - \hat{B}_{in}(\Phi)\partial_x(1-\phi)\Big), \quad i = 1,\dots,n-1 \tag{6.23}$$

we see that the new diffusion matrix $(B_{ij})_{1\le i,j\le n}$ is given by

$$\begin{aligned} B_{ii} &= \nu, \quad i = 1,\dots,n-1,\\ B_{in} &= -u_i(d_i^2 - d^2(u))\frac{u_n}{1-u_n}\sigma_{\mathrm{e}}'(1-u_n)V(1-u_n), \quad i = 1,\dots,n-1,\\ B_{nn} &= \nu + d^2(u)\frac{u_n^2}{1-u_n}\sigma_{\mathrm{e}}'(1-u_n)V(1-u_n), \end{aligned} \tag{6.24}$$

and $B_{ij} = 0$ otherwise. Note that the matrix has nonzero entries on its diagonal and in its last column only (as a special case of a triangular matrix). Satisfaction of conditions (1.13) can easily be verified.

The original problem is recast in the new variables, which means that we refer to the system (1.1) with the flux vector f and the matrix B given by (6.22) and (6.24), respectively, and use the initial datum

$$u(x,0) = u^0(x) := \big(\phi_1^0(x),\dots,\phi_{n-1}^0(x), 1-(\phi_1^0(x)+\dots+\phi_n^0(x))\big)^{\mathrm{T}} \in \Delta, \quad x \in \overline{\Omega},$$

where $\phi_1^0(x),\dots,\phi_n^0(x)$ are the initial solids concentrations. Finally, it is easy to check from (6.19) and (6.21) that for the variables $u_1,\dots,u_n$, the boundary conditions are again the Neumann (zero-flux) conditions

$$\left.\left(f_i(u) - \sum_{m=1}^{n} B_{im}(u)\partial_x u_m\right)\right|_{x=\pm 1} = 0. \tag{6.25}$$

Acknowledgements

S. Berres and R. Bürger acknowledge support by the Sonderforschungsbereich 404 at the University of Stuttgart and by German Acadamic Exchange Service (DAAD) and CONICYT (Chile) through project Alechile/DAAD/CONICYT 2003154. R. Bürger is supported by Fondecyt project 1050728 and Fondap in Applied Mathematics. The research of H. Frid was partially supported by CNPq through the grants 352871/96-2, 46.5714/00-5, 479416/2001-0, and FAPERJ through the grant E-26/152.192/2002.

References

[1] H. Amann, *Dynamic theory of quasi-linear parabolic systems III. Global existence*, Math. Z., 202 (1989), pp. 219–250.

[2] S. Berres and R. Bürger, *On gravity and centrifugal settling of polydisperse suspensions forming compressible sediments*, Int. J. Solids Structures, 40 (2003), pp. 4965–4987.

[3] S. Berres, R. Bürger, K.H. Karlsen and E.M. Tory, *Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression*, SIAM J. Appl. Math., 64 (2003), pp. 41–80.

[4] R. Bürger, K.-K. Fjelde, K. Höfler and K.H. Karlsen, *Central difference solutions of the kinematic model of settling of polydisperse suspensions and three-dimensional particle-scale simulations*, J. Eng. Math., 41 (2001), pp. 167–187.

[5] R. Bürger, K.H. Karlsen, E.M. Tory, and W.L. Wendland, *Model equations and instability regions for the sedimentation of polydisperse suspensions of spheres*, Z. Angew. Math. Mech., 82 (2002), pp. 699–722.

[6] K.N. Chueh, C.C. Conley and J.A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J., 26 (1977), pp. 373–392.

[7] R.H. Davis, *Hydrodynamic diffusion of suspended particles: a symposium*, J. Fluid Mech., 310 (1996), pp. 325–335.

[8] S.E. Esipov, *Coupled Burgers equations: A model of polydispersive sedimentation*, Phys. Rev. E, 52 (1995), pp. 3711–3718.

[9] H. Frid and V. Shelukhin, *A quasilinear parabolic system for three-phase capillary flow in porous media*, SIAM J. Math. Anal., 35 (2003), pp. 1029–1041.

[10] H. Frid and V. Shelukhin, *Initial boundary value problems for a quasi-linear parabolic system in three-phase capillary flow in porous media*, SIAM J. Math. Anal., 36 (2005), pp. 1407–1425.

[11] K. Höfler, *Simulation and Modeling of Mono- and Bidisperse Suspensions*, Doctoral Thesis, Institute of Computer Applications, University of Stuttgart, 2000.

[12] A.J.C. Ladd, *Dynamical simulations of sedimenting spheres*, Phys. Fluids A, 5 (1993), pp. 299–310.

[13] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968.

[14] O.A. Ladyzhenskaya and N.N. Ural'ceva, *Linear and Quasilinear Equations of Elliptic Type*, Academic Press, New York, 1968.

[15] J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[16] J.F. Richardson and W.N. Zaki, *Sedimentation and fluidization: Part I*, Trans. Instn. Chem. Engrs. (London), 32 (1954), pp. 35–53.

[17] K. Rietema, *Science and technology of dispersed two-phase systems I and II*, Chem. Eng. Sci,. 37 (1982), pp. 1125–1150.

[18] D. Serre, *Systèmes de Lois de Conservation, vol. II.* Diderot Editeur, Arts et Sciences, Paris, New York, Amsterdam, 1996.

[19] Y.T. Shih, D. Gidaspow and D.T. Wasan, *Hydrodynamics of sedimentation of multisized particles*, Powder Technol., 50 (1987), pp. 201–215.

[20] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York 1983.

[21] L. Tartar, *Compensated compactness and applications to partial differential equations.* Research Notes in Mathematics, Nonlinear Analysis and Mechanics, ed. R. J. Knops, vol. **4**, Pitman Press, New York, 1979, pp. 136–211.

[22] C. Tien, S.K. Teoh and R.B.H. Tan, *Cake filtration analysis — the effect of the relationship between the pore liquid pressure and the cake compressive stress*, Chem. Eng. Sci., 56 (2001), pp. 5361–5369.

[23] E.M. Tory, *Stochastic sedimentation and hydrodynamic diffusion*, Chem. Eng. J., 80 (2000), pp. 81–89.

[24] A.D. Watson, G.C. Barker and M.M. Robins, *Sedimentation in bidisperse and polydisperse colloids*, J. Colloid Interface Sci., 286 (2005), pp. 176–186.

[25] M.S. Willis, M. Shen, and K.J. Gray, *Investigations of the fundamental assumptions relating compression-permeability data with filtration*, Canad. J. Chem. Eng., 52 (1974), pp. 331–337.

Institut für Angewandte Analysis und Numerische Simulation, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: berres@mathematik.uni-stuttgart.de

Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: rburger@ing-mat.udec.cl

Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Jardim Botânico, CEP 22460-320, Rio de Janeiro, RJ, Brazil
E-mail address: hermano@impa.br