POINCARÉ AND LOGARITHMIC SOBOLEV INEQUALITY FOR GINZBURG-LANDAU PROCESSES IN RANDOM ENVIRONMENT

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Abstract. We consider reversible, conservative Ginzburg–Landau processes in a random environment, whose potential are bounded perturbations of the Gaussian potential, evolving on a d-dimensional cube of length L. We prove in all dimensions that the spectral gap of the generator and the logarithmic Sobolev constant are of order L^{-2} almost surely with respect to the environment.

1. Introduction

Poincaré and logarithmic Sobolev inequalities are powerful tools in the analysis of stochastic processes. A sharp estimate for the spectral gap, for instance, has been shown to be of fundamental importance in the derivation of the hydrodynamic equation of nongradient systems [22], [20]. More recently, in the deduction of the scaling limit of a non-attractive weakly asymmetric process whose hydrodynamic equation is given by a first order hyperbolic equation, a logarithmic Sobolev inequality played a central role [11]. In the same way, the spectral gap and the logarithmic Sobolev inequality are the main tools in the investigation of the decay to equilibrium of conservative systems in infinite volume [1], [2], [14], [12], [17].

We continue in this article the analysis started in [15] on spectral gaps and logarithmic Sobolev inequalities for conservative, reversible Ginzburg-Landau processes evolving on finite cubes and whose potential are bounded perturbations of the Gaussian potential. We present a sharp estimate of the two ergodic constants for systems in disorder. More precisely, we show that for almost all environments, there exists a spectral gap and a logarithmic Sobolev constant of order CL^{-2}, where L is the length of the cube and C is a finite constant depending on the disorder and on the bounded perturbation of the Gaussian potential.

We believe that the dependence of the constant C on the disorder is spurious in the case considered here, where the environment takes values in a bounded interval. One of the crucial steps in the martingale approach is the estimation of a covariance term which appears from a one-site spectral gap or logarithmic Sobolev inequality. The covariance term presents fluctuations due to the environment which can be estimated only almost surely through a Borel-Cantelli type argument.

In fact, Caputo [8] proved a spectral gap for symmetric simple exclusion processes in random environment, based on an alternative method introduced recently by Carlen, Carvalho and Loss [9], in which the constant is uniform over the disorder. Caputo's approach has one advantage and one drawback. On the one hand, it covers

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the convex + bounded perturbations (at least in the non random Ginzburg-Landau case) and provides a uniform result over the disorder for exclusion processes in bounded random environment. On the other hand, it applies only to conservative dynamics with product grand canonical measures and proves only the spectral gap.

Nevertheless, for the purpouses of derivation of convergence to equilibrium, an almost sure result is largely sufficient. Moreover, an advantage of the martingale method is that it provides a diffusive bound on the logarithmic Sobolev constant. In fact, to our knowledge, this article presents the first proof of such an estimate in the disordered context.

Bonds for ergodic constants of conservative dynamics in disordered systems were examined before for Kawasaki dynamics for the diluted Ising model in the Griffith's regime. Cancrini and Martinelli [4] proved a diffusive spectral gap and Cancrini and Roberto [6] obtained a bound for the logarithmic Sobolev constant. Quastel and Yau [21] proved a sharp estimate for the spectral gap of the symmetric exclusion process in random environment using the martingale approach.

As for the Ginzburg–Landau process without environment, the main ingredients needed in the proof are a local central limit theorem, uniform over the parameter and the environment, from which follows the equivalence of ensembles, and sharp large deviations estimates presented in section 7. It turns out that the arguments presented in [15] do not depend on the permutation symmetry of the variables and can be adapted to the present disordered context. However, in the presence of the environment these estimates are technically more demanding and some new arguments are needed. In particular, to bound some terms coming from the fluctuations of the environment, we need to impose the environment to take bounded values and to require a nice behavior of the variance and of the mean as the chemical potential diverges. Moreover, we do not know how to estimate the covariance term in the case where the environment is not random but follows some profile, as in the case of a one-dimensional Ginzburg–Landau processes in contact with infinite reservoirs at different temperature.

The derivation of sharp estimates for the spectral gap and the logarithmic Sobolev inequality for conservative interacting particle systems has started with the pioneering work of Lu and Yau [19] and Yau [23], [24], where the martingale method was introduced. Landim, Sethuraman and Varadhan in [16] extended to unbounded spin systems the sharp estimate of the spectral gap, while Landim, Panizo and Yau [15] proved the Poincaré and the logarithmic Sobolev inequality for Ginzburg-Landau processes where the potential is a bounded perturbation of the Gaussian potential. The estimate of the spectral gap was reobtained by Chafai [10] with a different approach. Cancrini and Martinelli in [3] provided an alternative proof of Lu and Yau result on the spectral gap and Cancrini and Martinelli and Roberto [5] for the logarithmic Sobolev constant.

The article in conceived as follows. In section 2, we present the main results. We prove the spectral gap in section 3 and the logarithmic Sobolev inequality in sections 4. These sections rely on estimates presented in sections 5, 6 and 7. In section 5 we present some consequences of a local central limit theorem, uniform over the parameters. In section 6 we examine the assumptions made on the environment and in section 7 we prove some large deviations estimates.

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2. Notation and Results

For $L \geq 1$, denote by Λ_L the cube $\{1, \dots, L\}^d$. Configurations of the state space $\mathbb{R}^{\Lambda_L}$ are denoted by the Greek letters η, ξ, so that η_x indicates the value of the spin at $x \in \Lambda_L$ for the configuration η. The configuration η undergoes a diffusion on $\mathbb{R}^{\Lambda_L}$ whose infinitesimal generator $\mathcal{L}_{\Lambda_L}$ is given by

$$\mathcal{L}_{\Lambda_L} \;=\; \frac{1}{2} \sum_{\substack{x,y \in \Lambda_L \\ |x-y|=1}} (\partial_{\eta_x} - \partial_{\eta_y})^2 \;-\; \frac{1}{2} \sum_{\substack{x,y \in \Lambda_L \\ |x-y|=1}} (V_y'(\eta_y) - V_x'(\eta_x))(\partial_{\eta_y} - \partial_{\eta_x}) \; .$$

$V_x \colon \mathbb{R} \to \mathbb{R}$ represents the potential $V_x(a) = h_x a + V(a)$, where $V(a) = (1/2)a^2 + F(a)$, $F \colon \mathbb{R} \to \mathbb{R}$ is a smooth bounded function such that $\|F'\|_\infty < \infty$, $\mathbf{h} = \{h_x,\, x \in \mathbb{Z}^d\}$ is a collection of i.i.d. random variables to be specified later and

$$\int e^{-V(x)}\, dx \;=\; 1 \; .$$

Denote by $Z \colon \mathbb{R} \to \mathbb{R}$ the partition function

$$Z(\lambda) \;=\; \int_{-\infty}^{\infty} e^{\lambda a - V(a)}\, da \; , \tag{2.1}$$

by $R \colon \mathbb{R} \to \mathbb{R}$ the density function $\partial_\lambda \log Z(\lambda)$, which is smooth and strictly increasing, and by Φ the inverse of R so that

$$\alpha \;=\; \frac{1}{Z(\Phi(\alpha))} \int_{-\infty}^{\infty} a\, e^{\Phi(\alpha) a - V(a)}\, da$$

for each α in $\mathbb{R}$.

Denote by ν_λ the measure $Z(\lambda)^{-1} \exp\{\lambda a - V(a)\} da$, by $g_\lambda(a) = Z(\lambda)^{-1} \exp\{\lambda a - V(a)\}$ its density with respect to the Lebesgue measure and by $\sigma^2(\lambda)$ the variance of ν_λ: $\sigma^2(\lambda) = \partial_\lambda^2 \log Z(\lambda)$ or

$$\sigma^2(\lambda) \;=\; \int_{-\infty}^{\infty} a^2 g_\lambda(a)\, da \;-\; \Big(\int_{-\infty}^{\infty} a g_\lambda(a)\, da \Big)^2 \; .$$

We assume throughout this article that $\sigma^2(\cdot)$ has limits at $\pm\infty$: There exists $\sigma_\pm^2 < \infty$ such that

$$\lim_{\lambda \to \pm\infty} \sigma^2(\lambda) \;=\; \sigma_\pm^2 \; . \tag{2.2}$$

This is a new assumption with respect to the non-random case needed in order to estimate some terms which appear through the environment (cf. Lemma 6.2). We prove at the end of section 6 that (2.2) holds if, for instance, F has limits at $\pm\infty$:

$$\lim_{a \to \pm\infty} F(a) \;=\; F_\pm \tag{2.3}$$

for some finite values $F_\pm$.

For λ in $\mathbb{R}$, and a finite subset Λ of $\mathbb{Z}^d$, denote by $\nu_{\Lambda,\lambda}^{\mathbf{h}}$ the product measure on $\mathbb{R}^\Lambda$ defined by

$$\nu_{\Lambda,\lambda}^{\mathbf{h}}(d\eta) \;=\; \prod_{x \in \Lambda} \frac{1}{Z(\lambda - h_x)} e^{[\lambda - h_x]\eta_x - V(\eta_x)} d\eta_x$$

Most of the times we omit the superscript $\mathbf{h}$ in $\nu_{\Lambda,\lambda}^{\mathbf{h}}$. Notice that $E_{\nu^{\mathbf{h}}_{\Lambda,\Phi(\alpha)+h_x}}[\eta_x] = \alpha$ for all α in $\mathbb{R}$, x in Λ.

For each M in $\mathbb{R}$, denote by $\mu^{\mathbf{h}}_{\Lambda,M}$ the canonical measure on Λ with total spin equal to M :

$$\mu^{\mathbf{h}}_{\Lambda,M}\Big(\ \cdot\ \Big) \;=\; \nu^{\mathbf{h}}_{\Lambda,\lambda}\Big(\ \cdot\ \Big|\ \sum_{x\in\Lambda}\eta_x = M\Big)\ .$$

Expectation with respect to a measure m is denoted by E_m. Notice that the canonical measure $\mu^{\mathbf{h}}_{\Lambda,M}$ does not depend on λ but depends on $\mathbf{h}$.

An elementary computation shows that the measures $\{\nu^{\mathbf{h}}_{\Lambda_L,\lambda},\ \lambda\in\mathbb{R}\}$, $\{\mu^{\mathbf{h}}_{\Lambda_L,M},\ M\in\mathbb{R}\}$ are reversible for the Markov process with generator $\mathcal{L}_{\Lambda_L}$. The Dirichlet form D_{Λ_L} associated to $\mathcal{L}_{\Lambda_L}$ is given by

$$D_{\Lambda_L}(\mu,f) \;=\; \frac{1}{2}\sum_{\substack{x,y\in\Lambda_L\\|x-y|=1}} <(T^{x,y}f)^2>_\mu\ .$$

In this formula and below, for a probability measure μ, $<\cdot>_\mu$ stands for expectation with respect to μ. Furthermore, for x, y in $\mathbb{Z}^d$, $T^{x,y}$ represents the operator that acts on smooth functions f as

$$T^{x,y}f \;=\; \frac{\partial f}{\partial\eta_x} \;-\; \frac{\partial f}{\partial\eta_y}$$

and μ stands for the invariant measures $\nu^{\mathbf{h}}_{\Lambda_L,\lambda}$, $\mu^{\mathbf{h}}_{\Lambda_L,M}$.

For a positive integer L and M in $\mathbb{R}$, denote by $W(L,M,\mathbf{h})$ the inverse of the spectral gap of the generator $\mathcal{L}_{\Lambda_L}$ with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$:

$$W(L,M,\mathbf{h}) \;=\; \sup_f\frac{<f;f>_{\mu^{\mathbf{h}}_{\Lambda_L,M}}}{D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M},f)}\ .$$

In this formula the supremum is carried over all smooth functions f in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ and $<f;f>_\mu$ stands for the variance of f with respect to μ. We also denote this variance by the symbol $\mathbf{Var}\,(\mu,f)$. Let

$$W(L,\mathbf{h}) \;=\; \sup_{M\in\mathbb{R}} W(L,M,\mathbf{h})\ .$$

We assume that the environment of i.i.d. random variables $\mathbf{h}=\{h_x,\ x\in\mathbb{Z}^d\}$ take bounded values: There exists $A_0<\infty$ such that $|h_x|\le A_0$. This is the only assumption needed on the environment. We denote by $\mathbb{P}$ the probability on $[-A_0,A_0]^{\mathbb{Z}^d}$ corresponding to the environment $\mathbf{h}$ and by $\mathbb{E}$, expectation with respect to $\mathbb{P}$.

Theorem 2.1. *Assume (2.2). There exists an almost sure event Ω_0 of $[-A_0,A_0]^{\mathbb{Z}^d}$ with the property that for all $\mathbf{h}$ in Ω_0, there exists a finite constant C_0 depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\mathbf{h}$ such that*

$$W(L,\mathbf{h}) \;\le\; C_0(F,\mathbf{h})L^2$$

for all $L\ge 2$.

A lower bound of the same order is easy to derive (cf. [15]). We now turn to the logarithmic Sobolev inequality.

For $L\ge 2$, a probability measure ν on $\mathbb{R}^{\Lambda_L}$ and a function f such that $<f^2>_\nu=1$, denote by $S_{\Lambda_L}(\nu,f)$ the entropy of $f^2d\nu$ with respect to ν:

$$S_{\Lambda_L}(\nu,f) \;=\; \int f^2\log f^2 d\nu\ ;$$

and by $\theta(L,M,\mathbf{h})$ the inverse of the logarithmic Sobolev constant of the Ginzburg-Landau process on the cube Λ_L with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_L,M}$:

$$\theta(L,M,\mathbf{h}) \;=\; \sup_f \frac{S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)}{D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)} \;.$$

In this formula, the supremum is carried over all smooth functions f in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< f^2 >_{\nu^{\mathbf{h}}_{\Lambda_L,M}} = 1$. Let

$$\theta(L,\mathbf{h}) = \sup_{M\in\mathbb{R}} \theta(L,M,\mathbf{h}) \;.$$

To prove the logarithmic Sobolev inequality, we shall require that the function $\Gamma_1(\lambda) = R(\lambda) - \lambda$ has limits at the boundary of $\mathbb{R}$: There exists $\Gamma_\pm$ such that

$$\lim_{\lambda\to\pm\infty} \Gamma_1(\lambda) \;=\; \Gamma_\pm \;. \tag{2.4}$$

We prove at the end of section 6 that (2.4) holds if, for instance, F satisfies (2.3).

Theorem 2.2. *Assume (2.2), (2.4) and that $\|F''\|_\infty < \infty$. There exists an almost sure event Ω_* of $[-A_0, A_0]^{\mathbb{Z}^d}$ with the property that for all $\mathbf{h}$ in Ω_0, there exists a finite constant $C(\mathbf{h})$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$, $\|F''\|_\infty$ and $\mathbf{h}$ such that*

$$\theta(L,\mathbf{h}) \;\le\; C(\mathbf{h}) L^2$$

for all $L \ge 2$.

The proof of these two theorems follows the same lines of [15], where similar results were presented for Ginzburg-Landau processes with constant environment. We assume that the author is acquainted to that work and present here only the modifications needed in the argument.

3. Spectral gap

To fix ideas, we prove Theorem 2.1 in dimension 1. The reader can find in section A.3.3 of [13] the arguments required to extend the proof to higher dimensions and in [15] the reasoning needed to fill the gaps left in Steps 1, 2 and at the beginning of Step 3

All constants in this section denoted by C_0 depend only on $\|F\|_\infty$ and all constants denoted by C_1 depend only on $\|F\|_\infty$, $\|F'\|_\infty$. These constants may change from line to line.

We divide the proof, which goes by induction on the size of the cube, in three steps. We start with $L = 2$.

Step 1: One-site spectral gap. Consider a smooth function $f\colon \mathbb{R}^{\Lambda_2} \to \mathbb{R}$. We want to estimate $< f; f >_{\mu^{\mathbf{h}}_{\Lambda_2,M}}$ in terms of the Dirichlet form of f. Since for the measure $\mu^{\mathbf{h}}_{\Lambda_2,M}$ the total spin is fixed to be equal to M, let $g(a) = f(M-a, a)$ and notice that $< f; f >_{\mu^{\mathbf{h}}_{\Lambda_2,M}}$ is equal to $< g; g >_{\mu^{\mathbf{h}}_{\Lambda_2,M}}$.

The following result is helpful. Fix $L \ge 2$, M in $\mathbb{R}$ and an environment $\mathbf{h}$. Denote by $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ the marginal distribution of η_1 with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$. The Glauber dynamics has a positive spectral gap which is uniform with respect to L, M and $\mathbf{h}$:

Lemma 3.1. *There is a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\mathbf{Var}\,(\mu^{\mathbf{h},1}_{\Lambda_L,M}, f) \;\le\; C_0\, E_{\mu^{\mathbf{h},1}_{\Lambda_L,M}}\Big[\Big(\frac{\partial f}{\partial \eta_1}\Big)^2\Big]$$

for every $L\ge 2$, M in $\mathbb{R}$, environment $\mathbf{h}$ and every smooth function $f\colon \mathbb{R}\to\mathbb{R}$ in $L^2(\mu^{\mathbf{h},1}_{\Lambda_L,M})$.

Applying Lemma 3.1 to the function g defined above, we obtain that $W(2,\mathbf{h})\le C_0$, proving Theorem 2.1 in the case $L=2$.

Step 2: Recursive formula, small values of L. We now obtain a recursive equation for $W(L,\mathbf{h})$. Assume that we have already estimated $W(K,\mathbf{h})$ for $2\le K\le L-1$. By the martingale decomposition and some elementary computation, we obtain that

$$\begin{aligned} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(f - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f]\Big)^2\Big] \;&\le\; \Big(W(L-1,\mathbf{h}) + C_0 L\Big) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)\\ &+ C_0 E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f; \frac{1}{L-1}\sum_{x=1}^{L-1} F'(\eta_x)\,\Big|\,\eta_L\Big]\Big)^2\Big] \end{aligned} \tag{3.1}$$

for some finite constant C_0. Schwarz inequality permits to estimate the second term by $C_1 L^{-1}\mathbf{Var}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)$ so that

$$W(L,\mathbf{h}) \;\le\; \Big(1+\frac{C_1^*}{L}\Big) W(L-1,\mathbf{h}) \;+\; C_1 L\,. \tag{3.2}$$

for some finite constants C_1, C_1^* depending only on $\|F\|_\infty$, $\|F'\|_\infty$. This inequality permits to iterate the bound $W(2,\mathbf{h})\le C_0$ derived in Step 1 to get estimates for $W(L,\mathbf{h})$. We obtain in this way that $W(L,\mathbf{h})\le C_1 L^t$, *uniformly over the environment*, for some power t which depends only on $\|F\|_\infty$, $\|F'\|_\infty$.

Notice that we would obtain the right bound L^2 if we could prove that the constant C_1^*, which appears in the previous inequality, is strictly less than 2. Therefore, to prove the spectral gap, we have to improve our bounds on the covariance term to derive a factor of order εL^{-1} for $\varepsilon<2$.

The bound $W(L,\mathbf{h})\le C_1 L^t$ will be used for small values of L. We now consider large values of L.

Step 3. Recursive formula, large values of L. We have just seen that the crucial point is the estimate of the covariance term

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f; \frac{1}{L-1}\sum_{x=1}^{L-1} F'(\eta_x)\,\Big|\,\eta_L\Big]^2 \;=\; E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f; \frac{1}{L-1}\sum_{x=1}^{L-1} F'(\eta_x)\Big]^2\,. \tag{3.3}$$

Notice that on the right hand side, the variable η_L is not integrated. When we want to stress this point, we denote f by f_{η_L}.

Fix $\varepsilon>0$ small, a random environment $\mathbf{h}$ in some set Ω_0 and a positive integer $K=K(\varepsilon)\ge 2$. Both Ω_0 and K will be specified later. For each $L\ge K^2$, divide the interval $\{1,\dots,L-1\}$ into $\ell=\lfloor (L-1)/K\rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a\rfloor$ represents the integer part of a. Keep in mind that K is large but fixed, while L (and thus ℓ) increases to $+\infty$.

Denote by I_j the j-th interval and by M_j the total spin on I_j: $M_j = \sum_{x\in I_j} \eta_x$. The right hand side of the previous formula is bounded above by

$$2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell} a_j\Big\{\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}\Big]\Big)^2 \quad (3.4)$$

$$+\, 2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2\,,$$

where $a_j = |I_j|/(L-1)$. Taking conditional expectation with respect to M_j, we rewrite the first term as

$$2\Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[f;\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2$$

$$\le\; 2\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},f)\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x))\Big]\,.$$

By the induction assumption, the variance $\mathbf{Var}(\mu^{\mathbf{h}}_{I_j,M_j},f)$ is bounded above by $W(|I_j|,\tau_j\mathbf{h})D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j},f)$, where $\tau_j\mathbf{h}$ stands for the translation of the environment $\mathbf{h}$ for the origin to coincide with the left end of the interval I_j. By the a-priori estimate obtained in Step 2, this expression is bounded by $C_1K^tD_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j},f)$. On the other hand, by Corollary 5.3, the variance of $|I_j|^{-1}\sum_{x\in I_j}F'(\eta_x)$ with respect to $\mu^{\mathbf{h}}_{I_j,M_j}$ is bounded above by $C_0|I_j|^{-1}\|F'\|_\infty^2$ uniformly over M_j, $\mathbf{h}$. The previous expression is thus less than or equal to

$$\frac{C_1K^t}{L}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j},f)\Big]\,.$$

Since the previous sum is bounded by the global Dirichlet form $D_{\Lambda_{L-1}}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f)$, we proved that the first term of (3.4) is bounded above by

$$\frac{C_1K^t}{L}D_{\Lambda_{L-1}}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L},f)\,. \quad (3.5)$$

The next result provides an estimate for the second covariance in (3.4). For λ in $\mathbb{R}$, let $\tilde\sigma^2(\lambda) = \mathbb{E}[\sigma^2(\lambda-h_1)]$.

Lemma 3.2. *Let $\lambda = \lambda(L, M-\eta_L,\mathbf{h})$ so that*

$$E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}\Big[\sum_{x\in\Lambda_{L-1}}\eta_x\Big] \;=\; M-\eta_L\,. \quad (3.6)$$

There exist finite constants C_0, C_1 such that

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f;\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2$$

$$\le\;\Big\{\frac{C_1}{KL} \;+\; \frac{C_0}{L}\frac{1}{\ell}\sum_{j=1}^{\ell}\Big(\frac{1}{|I_j|}\sum_{x\in I_j}\sigma^2(\lambda-h_x)-\tilde\sigma^2(\lambda)\Big)^2\Big\}\mathbf{Var}(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L},f)$$

for all $K\ge 2$, $L\ge K^2$.

It is now easy to conclude the proof of Theorem 2.1. Fix $0 < \varepsilon < 1/2$. Choose K_1 large enough for C_1/K_1 to be smaller that ε, where C_1 is the constant appearing in the previous lemma. Let $K_2 = K_2(\varepsilon C_0^{-1}, \sigma^2)$ be the positive integer given by (6.5) with σ^2 in place of U, where C_0 is the constant appearing in the statement of the previous lemma. Fix $K_0 = \max\{K_1, K_2\}$. For this fixed integer K_0, consider the sequence of disjoint intervals $\{I_1^L, \dots, I_\ell^L\}$ introduced above and an environment $\mathbf{h}$ in the set $\Omega_0(K_0, \sigma^2(\cdot), \varepsilon C_0^{-1})$ defined in (6.5).

By (6.5), there exists $\ell_0 = \ell_0(\mathbf{h}, \varepsilon C_0^{-1}, K_0, \sigma^2(\cdot))$ for which the right hand side of the statement of the previous lemma is bounded above by

$$\frac{2\varepsilon}{L} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1}, M-\eta_L}}[f;f]$$

for all $\ell \geq \ell_0$. This bound together with (3.5) gives that (3.4), and therefore (3.3), is less than or equal to

$$\frac{C_1 K_0^t}{L} D_{\Lambda_{L-1}}(\mu_{\Lambda^{\mathbf{h}}_{L-1}, M-\eta_L}, f) \;+\; \frac{2\varepsilon}{L} E_{\Lambda_{L-1}, M-\eta_L}[f_{\eta_L}; f_{\eta_L}]\;.$$

In particular, the second term on the right hand side of (3.1) is bounded above by

$$\frac{C_1 K_0^t}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L, M}, f) \;+\; \frac{C_0 \varepsilon}{L} E_{\mu^{\mathbf{h}}_{\Lambda_L, M}}[f;f]\;.$$

Choose ε small enough for $C_0\varepsilon \leq 1$. Therefore, in view of (3.1),

$$\mathbf{Var}(\mu^{\mathbf{h}}_{\Lambda_L, M}, f) \;\leq\; \Big(1 - \frac{1}{L}\Big)^{-1} \Big(W(L-1, \mathbf{h}) + C_1 L\Big) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L, M}, f)$$

for L large enough. Note that the integer L_0 from which this inequality holds depends on the environment $\mathbf{h}$ because ℓ_0 depends on $\mathbf{h}$.

Taking supremum over smooth functions $f\colon \mathbb{R}^{\Lambda_L} \to \mathbb{R}$ in $L^2(\mu^{\mathbf{h}}_{\Lambda_L, M})$, we obtain that

$$W(L, \mathbf{h}) \;\leq\; \Big(1 - \frac{1}{L}\Big)^{-1} \Big(W(L-1, \mathbf{h}) + C_1 L\Big)\;.$$

It is not difficult to deduce from this recursive relation the existence of a constant $C = C(\|F\|_\infty, \|F'\|_\infty, \mathbf{h})$ such that $W(L, \mathbf{h}) \leq CL^2$ for all $L \geq 2$. This concludes the proof of Theorem 2.1. □

We conclude this section with the

Proof of Lemma 3.2. For each finite subset Λ of $\mathbb{Z}$, denote by $R_\Lambda : \mathbb{R} \to \mathbb{R}$ the smooth strictly increasing function

$$R_\Lambda(\lambda) \;=\; E_{\nu^{\mathbf{h}}_{\Lambda, \lambda}}\Big[\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \eta_x\Big]$$

and denote by Φ_Λ the inverse of R_Λ. For $1 \leq j \leq \ell$, let $A_j : \mathbb{R} \to \mathbb{R}$ be given by

$$A_j(m) \;=\; E_{\nu^{\mathbf{h}}_{I_j, \Phi_{I_j}(m)}}\Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x)\Big]\;, \tag{3.7}$$

let

$$m_j^* \;=\; E_{\nu^{\mathbf{h}}_{\Lambda_{L-1}, \lambda}}\Big[\frac{1}{|I_j|} \sum_{x \in I_j} \eta_x\Big]\;,$$

where λ is given by (3.6), and rewrite the sum

$$\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{I_j,M_j}} \Big[|I_j|^{-1} \sum_{x\in I_j} F'(\eta_x) \Big] \tag{3.8}$$

as

$$\begin{aligned}
&\sum_{j=1}^{\ell} a_j \Big\{ E_{\mu^{\mathbf{h}}_{I_j,M_j}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] - E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] \Big\} \\
&\quad + \sum_{j=1}^{\ell} a_j \Big\{ E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] - A_j(m_j^*) - A_j'(m_j^*)(M_j/|I_j| - m_j^*) \Big\} \\
&\quad + \sum_{j=1}^{\ell} a_j \Big\{ A_j(m_j^*) + A_j'(m_j^*)(M_j/|I_j| - m_j^*) \Big\} \; .
\end{aligned} \tag{3.9}$$

This decomposition is easy to understand. In the first term we compare an expectation with respect to a canonical measure with an expectation with respect to a grand canonical measure. In view of Corollary 5.2 for the difference to be small, the chemical potential should be $\Phi_{I_j}(M_j/|I_j|)$. The second term is a Taylor expansion up to the first order and the last one is what remains.

Notice that $A_j(m_j^*)$ is a function of L, $M-\eta_L$ and $\mathbf{h}$ only. It is therefore a constant with respect to the measure $\mu^{\mathbf{h}}_{\Lambda_{L-1}, M-\eta_L}$. The same statement is true for $\sum_{j=1}^{\ell} a_j(M_j/|I_j| - m_j^*)$. Since we may add constants in covariances, by the previous observation, we may substitute (3.8) by (3.9) where the third term in (3.9) is replaced by

$$\sum_{j=1}^{\ell} a_j (A_j'(m_j^*) - c)(M_j/|I_j| - m_j^*) \; ,$$

where $c = \tilde\sigma^2(\lambda)^{-1} - 1$.

Up to this point, we have replaced the average of $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$ by the sum of three terms. We estimate separately the covariance of f with each term by means of Lemma 3.3 below.

Set first

$$G_j^0 \;=\; E_{\mu^{\mathbf{h}}_{I_j,M_j}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] - E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] \; .$$

Since F' is bounded, by the equivalence of ensembles, Corollary 5.2, $|G_j^0| \le C_1/K$. Hence, by Lemma 3.3, the contribution of the first term in (3.9) to the covariance appearing in the statement of the lemma is bounded by

$$\frac{C_1}{KL} E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}} [f; f] \; . \tag{3.10}$$

Consider now the second term in (3.9):

$$\begin{aligned}
H_j^0 \;&=\; E_{\nu^{\mathbf{h}}_{I_j,\Phi_{I_j}(M_j/|I_j|)}} \Big[\frac{1}{|I_j|} \sum_{x\in I_j} F'(\eta_x) \Big] - A_j(m_j^*) - A_j'(m_j^*)(M_j/|I_j| - m_j^*) \\
&=\; A_j(M_j/|I_j|) \;-\; A_j(m_j^*) \;-\; A_j'(m_j^*)(M_j/|I_j| - m_j^*) \; ,
\end{aligned}$$

which is a first order Taylor expansion. By Lemma 6.1, $\|A_j''\|_\infty \le C_0$. In particular, H_j^0 is absolutely bounded by $C_0(M_j/|I_j| - m_j^*)^2$ and in this case

$$\sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(H_j^0)^2] \;\le\; C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(M_j/|I_j| - m_j^*)^4]\;.$$

By the estimates (5.1), each expectation is bounded by $C_0 K^{-2}$ so that the contribution to the covariance of the second term in (3.9) is bounded again by (3.10).

We turn now to the third term in (3.9) without $A_j(m_j^*)$ and with $A_j'(m_j^*)$ replaced by $A_j'(m_j^*) - c$, as explained above. We need to estimate

$$\sum_{j=1}^{\ell} a_j^2 (A_j'(m_j^*) - c)^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}\Big[\Big(\frac{M_j}{|I_j|} - m_j^*\Big)^2\Big]\;.$$

By the estimates (5.1), this expression is less than or equal to

$$\frac{C_0}{L}\frac{1}{\ell}\sum_{j=1}^{\ell}(A_j'(m_j^*) - c)^2\;.$$

By Lemma 6.1,

$$A_j'(m_j^*) \;=\; \frac{1}{|I_j|^{-1}\sum_{x\in I_j}\sigma^2(\Phi_{I_j}(m_j^*) - h_x)} - 1 \;=\; \frac{1}{|I_j|^{-1}\sum_{x\in I_j}\sigma^2(\lambda - h_x)} - 1 \tag{3.11}$$

because $m_j^* = R_{I_j}(\lambda)$ and $\Phi_{I_j} = R_{I_j}^{-1}$. Since $c = \mathbb{E}[\sigma^2(\lambda - h_1)]^{-1} - 1$, and since $\sigma^2(\cdot)$ is bounded above and below by finite, strictly positive constants, the previous expression is bounded by

$$\frac{C_0}{L}\frac{1}{\ell}\sum_{j=1}^{\ell}\Big(\frac{1}{|I_j|}\sum_{x\in I_j}\sigma^2(\lambda - h_x) - \tilde\sigma^2(\lambda)\Big)^2\;.$$

This concludes the proof of the lemma. □

We now turn to a simple technical lemma needed in the proof of Lemma 3.2 above.

Lemma 3.3. *Fix λ given by (3.6). For $1 \le j \le \ell$, let G_j^0 be a family of functions in $L^2(\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda})$. Assume that each function G_j^0 depends only on the variables $\{\eta_x\,, x \in I_j\}$. There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[f; \sum_{j=1}^{\ell} a_j G_j^0\Big]\Big)^2 \;\le\; C_0 E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f] \sum_{j=1}^{\ell} a_j^2\, E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(G_j^0)^2]$$

for all L, $M - \eta_L$ and $\mathbf{h}$.

Proof. Since we are allowed to add constants in covariances, by Schwarz inequality, the square of the covariance is bounded above by

$$E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[f;f]\, E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}\Big[\Big(\sum_{j=1}^{\ell} a_j G_j\Big)^2\Big]\,, \tag{3.12}$$

where

$$G_j \;=\; G_j^0 \;-\; E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^0]$$

and λ is given by (3.6). The second expectation in (3.12) is equal to

$$\sum_{j=1}^{\ell} a_j^2 E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[G_j^2] \;+\; \sum_{j\neq k} a_j a_k E_{\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}}[G_j G_k]\;. \tag{3.13}$$

We estimate separately the diagonal and the off diagonal terms. Since $2K \le L$, by Corollary 5.4 and since the variance is bounded by the L^2 norm, the sum of the diagonal term is less than or equal to

$$C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2] \;\le\; C_0 \sum_{j=1}^{\ell} a_j^2 E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(G_j^0)^2]$$

for some constant C_0 which depends only on $\|F\|_\infty$.

On the other hand, by the equivalence of ensembles, Corollary 5.2, each off diagonal term is bounded by

$$E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j G_k] \;+\; \frac{C_0 K}{L}\Big\{ E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2 G_k^2]\Big\}^{1/2}$$

for some finite constant C_0 depending only on $\|F\|_\infty$. Since the grand canonical measure $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$ is product and since each G_i has mean zero with respect to $\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}$, the first term vanishes. Since the measure is product, the contribution of the off diagonal terms is bounded by

$$\begin{aligned} &\frac{C_0 K}{L}\sum_{j\neq k} a_j a_k \Big\{ E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_j^2] E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[G_k^2]\Big\}^{1/2} \\ &\le\; \frac{C_0 K}{L}\Big(\sum_{j=1}^{\ell} a_j \Big\{ E_{\nu^{\mathbf{h}}_{\Lambda_{L-1},\lambda}}[(G_j^0)^2]\Big\}^{1/2}\Big)^2 \end{aligned}$$

because the variance is bounded by the L^2 norm. To conclude the proof, it remains to apply Schwarz inequality and to recall that $\ell \le 2L/K$. □

4. Logarithmic Sobolev Inequality

We prove in this section Theorem 2.2. The approach is similar to the one presented in last section for the spectral gap. We will derive a recursive formula for $\theta(L,\mathbf{h})$ in terms of $\theta(L-1,\mathbf{h})$ and L in three steps. As before, for $j = 0, 1, 2$, all constants C_j are allowed to depend only on $\|F^{(i)}\|_\infty$ for $0 \le i \le j$ and may change from line to line. Here $F^{(i)}$ stands for the i-th derivative of F. The arguments of [15] can be adapted to complete the gaps left here in Steps 1, 2 and at the beginning of Step 3.

Step 1. One-site logarithmic Sobolev inequality. We start our proof with the case $L = 2$. Let $f\colon \mathbb{R}^{\Lambda_2} \to \mathbb{R}$ be a smooth function such that $< f^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = 1$. Let $g(\eta_1) = f(\eta_1, M-\eta_1)$. Since the total spin is fixed to be M, we have that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = < f^2 >_{\mu^{\mathbf{h}}_{\Lambda_2,M}} = 1$ and that $S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, g) = S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, f)$. The next lemma permits to estimate the entropy of $S_{\Lambda_2}(\mu^{\mathbf{h}}_{\Lambda_2,M}, g)$ in terms of the Glauber Dirichlet form of g. This result is in fact a logarithmic Sobolev inequality for the Glauber dynamics obtained when restricting the Kawasaki exchange dynamics to one site. Recall that $\mu^{\mathbf{h},1}_{\Lambda_L,M}$ represents the one-site marginal of $\mu^{\mathbf{h}}_{\Lambda_L,M}$.

Lemma 4.1. *There exists a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\int H(\eta_1)^2 \log H(\eta_1)^2 \, \mu^{\mathbf{h},1}_{\Lambda_L,M}(d\eta_1) \;\le\; C_0 E_{\mu^{\mathbf{h},1}_{\Lambda_L,M}}\Big[\Big(\frac{\partial H}{\partial \eta_1}\Big)^2\Big] \tag{4.1}$$

for every $L \ge 2$, M in $\mathbb{R}$, environment $\mathbf{h}$ and smooth function $H\colon \mathbb{R} \to \mathbb{R}$ in $L^2(\mu^{\mathbf{h},1}_{\Lambda_L,M})$ such that $< H^2 >_{\mu^{\mathbf{h},1}_{\Lambda_L,M}} = 1$.

The previous statement applied to $L = 2$ and $H = g$ gives that $\theta(2, \mathbf{h}) \le C_0$ uniformly in $\mathbf{h}$.

Step 2. Recursive formula, small values of L. We now obtain a recursive formula for $\theta(L, \mathbf{h})$ in terms of $\theta(L-1, \mathbf{h})$, L. Assume that $\theta(K, \mathbf{h}) < \infty$ for $2 \le K \le L-1$. The martingale decomposition, Lemma 4.1 and some elementary estimates give that

$$\begin{aligned} S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \;&\le\; \Big\{\theta(L-1, \mathbf{h}) \;+\; C_0 L\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \\ &\quad + \int \frac{1}{2E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[f^2|\eta_L]} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[f^2; \frac{1}{L-1}\sum_{x=1}^{L-1} F'(\eta_x) \,\Big|\, \eta_L\Big]^2 \, d\mu^{\mathbf{h},L}_{\Lambda_L,M} \;, \end{aligned} \tag{4.2}$$

where $\mu^{\mathbf{h},L}_{\Lambda_L,M}$ stands for the one-site marginal of $\mu^{\mathbf{h}}_{\Lambda_L,M}$ at η_L. It follows from the previous bound, Schwarz inequality and simple estimates that

$$\theta(L, \mathbf{h}) \;\le\; C_0 L + (1 + C_2 L^{-1})\theta(L-1, \mathbf{h}) \;.$$

This relation, together with the bound $\theta(2, \mathbf{h}) \le C_0$ proved in the first step, gives that $\theta(L, \mathbf{h}) < L^t$ for some finite t, which depends on C_2. Notice that this estimate is uniform over the environment $\mathbf{h}$.

Step 3. Recursive formula, large values of L. We now give an alternative estimate of the second term of (4.2) that we shall use for large values of L.

Proposition 4.2. *Fix $\delta > 0$. There exists a finite constant C_2 and a set of environments Ω_*, which has $\mathbb{P}$ probability one, such that for any $\mathbf{h}$ in Ω_*, there exists $L_* = L_*(\mathbf{h})$ such that for all $L \ge L_*$,*

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{x=1}^{L} F'(\eta_x)\Big]\Big)^2 \;\le\; \Big\{C_2 L + \frac{\delta\theta(L, \mathbf{h})}{L}\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \tag{4.3}$$

for all M in $\mathbb{R}$ and functions g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$.

We first assume this result to conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. Recall the decomposition (4.2) of the entropy. Fix $\delta < 2$. By Proposition 4.2 applied to the measure $\mu^{\mathbf{h}}_{L-1,M-\eta_L}$ and the function $g^2 = f^2/E_{\Lambda_L,M-\eta_L}[f^2|\eta_L]$, there exists a set Ω_* of $\mathbb{P}$ probability one with the following property. For each $\mathbf{h}$ in Ω_*, there exists $L_* = L_*(\mathbf{h})$ such that for $L \ge L_*$, the expression inside the integral in (4.2) is bounded above by

$$\Big\{C_2 L + \frac{\delta\theta(L-1, \mathbf{h})}{L-1}\Big\} D_{\Lambda_{L-1}}\Big(\mu^{\mathbf{h}}_{\Lambda_{L-1},M-\eta_L}, f/E_{\mu^{\mathbf{h}}_{\Lambda_L,M-\eta_L}}[f^2|\eta_L]^{1/2}\Big)$$

for some finite constant C_2 and all M, η_L. Taking expectations with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$ in the previous formula, we obtain that the second term in (4.2) is less than or equal to

$$\Big\{C_2 L + \frac{\delta\theta(L-1, \mathbf{h})}{L-1}\Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \;.$$

In particular,

$$S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f) \;\le\; \Big\{ C_2 L + (1+\frac{\delta}{L-1})\theta(L-1,\mathbf{h}) \Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, f)$$

or, by definition of $\theta(L,\mathbf{h})$,

$$\theta(L,\mathbf{h}) \;\le\; \Big\{ C_2 L + (1+\frac{\delta}{L-1})\theta(L-1,\mathbf{h}) \Big\}\;.$$

Since $\delta < 2$, it is easy to derive form this inequality the existence of a finite constant $C(\mathbf{h})$ such that $\theta(L,\mathbf{h}) \le C(\mathbf{h})L^2$ for all $L \ge 2$. The constant may depend on $\mathbf{h}$ because ℓ_* depends on the environment. This concludes the proof of Theorem 2.2. $\square$

We now turn to the proof of Proposition 4.2. For clarity reasons, we divide it in several lemmas. We first repeat the procedure presented in Step 3 of the previous section. Fix $K \ge 1$ and divide the interval $\{1,\dots,L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$. Denote by I_j the j-th interval and by M_j the total spin on I_j: $M_j = \sum_{x\in I_j} \eta_x$. As for the proof of the spectral gap, K will be large but fixed, while ℓ will increase to infinity. The left hand side of (4.3) is bounded above by

$$\begin{aligned} &2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{j=1}^{\ell} \Big\{ \sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\sum_{x\in I_j} F'(\eta_x) \Big]\Big\}\Big]\Big)^2 \\ &\quad + 2\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \frac{1}{L}\sum_{j=1}^{\ell} |I_j| E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2\;. \end{aligned} \tag{4.4}$$

Lemma 4.3. *There exists a finite constant C_2 such that*

$$\begin{aligned} &\Big(\frac{1}{L}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \Big\{ \sum_{x\in I_j} F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\sum_{x\in I_j} F'(\eta_x)\Big]\Big\}\Big]\Big)^2 \\ &\quad\le\; \frac{C_2 K^t}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \end{aligned}$$

for all $K \ge 2$, $L \ge K^2$, M in $\mathbb{R}$, environment $\mathbf{h}$ and smooth functions g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$.

Proof. Taking conditional expectation with respect to M_j, $\{\eta_x,\, x\notin I_j\}$, we rewrite the left hand side of the inequality presented in the statement of the lemma as

$$\begin{aligned} &\Big(\frac{1}{L}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\, E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[g_j^2; \sum_{x\in I_j} F'(\eta_x)\Big]\Big]\Big)^2 \\ &\quad\le\; \frac{\ell}{L^2}\sum_{j=1}^{\ell} E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\Big(E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[g_j^2; \sum_{x\in I_j} F'(\eta_x)\Big]\Big)^2\Big]\;, \end{aligned} \tag{4.5}$$

where $g_j^2 = g^2 / E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]$ has mean one with respect to $\mu^{\mathbf{h}}_{I_j,M_j}$. In the expression $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]$, it must be understood that only the variables η_x for x in I_j are integrated so that $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2] = E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[g^2 | M_j, \{\eta_x,\, x\notin I_j\}]$. In the last step we used

Schwarz inequality and the fact $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]] = 1$. Fix $1 \le j \le \ell$. By the entropy inequality, $E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g_j^2; \sum_{x\in I_j} F'(\eta_x)]$ is bounded above by

$$\frac{1}{\beta}\log\int e^{\beta\sum_{x\in I_j} F_j(\eta_x)}\, d\mu^{\mathbf{h}}_{I_j,M_j} \;+\; \frac{1}{\beta} S_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$$

for every $\beta > 0$. Here, $F_j(\eta_x) = F'(\eta_x) - E_{\mu^{\mathbf{h}}_{I_j,M_j}}[F']$. By definition of θ, the second term is bounded above by $\theta(|I_j|, \tau_j\mathbf{h})\beta^{-1} D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$, where $\tau_j\mathbf{h}$ stands for the translation of the environment $\mathbf{h}$ for the origin to be at the left end of the interval I_j. By the a-priori estimate on θ obtained in Step 2, this expression is less than or equal to $C_2|I_j|^t\beta^{-1}D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)$. On the other hand, by Proposition 7.1, the first term in the previous formula is bounded above by $C_0\beta|I_j|$ for some finite constant C_0. Minimizing over β and summing over j, we get that (4.5) is less than or equal to

$$\frac{C_2}{L}\sum_{j=1}^{\ell} |I_j|^t E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[E_{\mu^{\mathbf{h}}_{I_j,M_j}}[g^2]\, D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g_j)\Big] \;\le\; \frac{C_2K^t}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\;.$$

This concludes the proof of the lemma. □

We turn now to the second term of (4.4). Let $a_j = |I_j|/L$ and recall the definition of A_j given in (3.7). Since we may add constants in a covariance, the expectation in the second term of (4.4) is equal to

$$E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j G_j\Big] \;+\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j A_j'(m_j)\Big(\frac{M_j}{|I_j|} - m_j\Big)\Big]\;,$$

where

$$m_j \;=\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}\eta_x\Big]\;,$$

$$\begin{aligned} G_j \;&=\; E_{\mu^{\mathbf{h}}_{I_j,M_j}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[\frac{1}{|I_j|}\sum_{x\in I_j}F'(\eta_x)\Big] \\ &\quad -\; A_j'(m_j)\Big(\frac{M_j}{|I_j|} - m_j\Big)\;. \end{aligned} \tag{4.6}$$

Notice that m_j depends on M, L and the environment $\mathbf{h}$.

To estimate this covariance we need to consider two cases. Let β_0 be the constant given by Proposition 7.3 and fix $0 < \delta < 2$. By Proposition 7.3, there exists $K_0 \ge 2$ for which the left hand side of (7.5) is bounded by $\delta\beta$ for all $\beta \le \beta_0$ and all $K \ge K_0$, $L \ge K^2$, M in $\mathbb{R}$ and environment $\mathbf{h}$.

Lemma 4.4. *Fix $L \ge K^2 \ge K_0^2$, M in $\mathbb{R}$, an environment $\mathbf{h}$ and a smooth function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$. Assume that $\theta(L,\mathbf{h})L^{-1}D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$ $< \delta\beta_0^2$. Then,*

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}\Big[g^2; \sum_{j=1}^{\ell} a_j G_j\Big]\Big)^2 \;\le\; \frac{\delta\theta(L,\mathbf{h})}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)\;.$$

Proof. Fix a density g^2 satisfying the assumptions. By the entropy inequality, the expectation in the statement of the lemma is bounded by

$$\frac{1}{\beta L} \log E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[\exp \Big\{ \beta \sum_{j=1}^{\ell} |I_j| G_j \Big\} \Big] \;+\; \frac{1}{\beta L} S_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \tag{4.7}$$

for every $\beta > 0$. By Proposition 7.3 and our choice of K, L, the first term is bounded above by $\delta\beta$ for all $\beta < \beta_0$. The second one, by definition of θ, is bounded above by $(\theta(L, \mathbf{h})/\beta L) D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$. Therefore, (4.7) is less than or equal to

$$\delta\beta \;+\; \frac{\theta(L,\mathbf{h})}{\beta L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$$

for all $\beta < \beta_0$. The value of β that minimizes this expression is

$$\beta_*^2 \;=\; \frac{\theta(L,\mathbf{h})}{\delta L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \; .$$

By hypothesis, $\beta_* < \beta_0$ and we may therefore minimize in $\beta < \beta_0$ to obtain that the square of (4.7) is bounded above by

$$\frac{\delta\theta(L,\mathbf{h})}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \; ,$$

which concludes the proof of the lemma. □

Recall the explicit formula for $A'_j(m^*_j)$ given in (3.11). Let $c = c(\lambda) = \mathbb{E}[\sigma^2(\lambda - h_x)]^{-1} - 1$. Since $\sigma^2(\cdot)$ is bounded above and below by strictly positive constants,

$$\frac{1}{\ell}\sum_{j=1}^{\ell} |I_j| (A'_j(m^*_j) - c)^2 \;\le\; \frac{C'_0}{\ell} \sum_{j=1}^{\ell} |I_j| \Big(\frac{1}{|I_j|} \sum_{x\in I_j} \sigma^2(\lambda - h_x) - \tilde\sigma^2(\lambda) \Big)^2 \; .$$

Fix $\delta > 0$. Let C^* be the constant given by (6.6) with σ^2 in place of U. Let $K_1 \ge C_0 \exp\{C_0 C'_0 C^*\}\delta^{-1}$, where C_0 is the constant appearing in Lemma 7.4 and C'_0 the one in the previous formula. Fix $K \ge K_1$ and let Ω_1 be the set associated to $(K, \sigma^2(\cdot))$ through (6.6). Fix an environment $\mathbf{h}$ in Ω_1. By (6.6), there exists $\ell_0 = \ell_0(\mathbf{h})$, such that the right hand side of (7.6) is bounded by $\delta\beta$ for all $\ell \ge \ell_0$. The next result follows from these observations and the arguments presented in the proof of the previous lemma.

Lemma 4.5. *There exists a constant K_1 such that for all $K \ge K_1$, there exists a set $\Omega_1 = \Omega_1(K)$ with the following property. For all $\mathbf{h}$ in Ω_1, there exists $\ell_1 = \ell_1(K, \mathbf{h})$ such that for $\ell \ge \ell_1$, M in $\mathbb{R}$ and a smooth function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$,*

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[g^2 ; \sum_{j=1}^{\ell} a_j A'_j(m_j) \Big(\frac{M_j}{|I_j|} - m_j \Big) \Big] \Big)^2 \;\le\; \frac{\delta\theta(L,\mathbf{h})}{L} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$$

if $\theta(L,\mathbf{h}) L^{-1} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) < \delta\beta_0^2$.

It remains to consider the case where the Dirichlet form of g is large.

Lemma 4.6. *Fix $\delta > 0$. There exist $K_2 = K_2(\delta)$ and a finite constant C_2 with the following property. For each $K \ge K_2$, there exists a set of environments $\Omega_2 =$*

$\Omega_2(K)$ with $\mathbb{P}$ probability one such that for each $\mathbf{h}$ in Ω_2, there exists $\ell_2 = \ell_2(\mathbf{h})$, such that for all $\ell \ge \ell_2$,

$$\Big(E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[g^2 ; \sum_{j=1}^{\ell} a_j G_j \Big] \Big)^2 \;\le\; \Big\{ \frac{\delta \theta(L,\mathbf{h})}{L} \;+\; C_2 K^t L \Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \tag{4.8}$$

for all M in $\mathbb{R}$ and smooth function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L,M})$ satisfying $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L,M}} = 1$,

$$\theta(L,\mathbf{h}) L^{-1} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \;\ge\; \delta \beta_0^2 \; .$$

Proof. The covariance $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[g^2; \sum_{1\le j \le \ell} a_j G_j]$ is equal to the covariance of g^2 and $\sum_{1\le j\le \ell} a_j H_j$, where $H_j = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j},M_j}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$. Since g^2 is a density with respect to $\mu^{\mathbf{h}}_{\Lambda_L,M}$, by Schwarz inequality, the left hand side of (4.8) is bounded above by

$$2 \Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[g^2 \Big(H_j - \tilde H_j \Big) \Big] \Big)^2 \;+\; 2 \Big(\sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[H_j - \tilde H_j \Big] \Big)^2 \; , \tag{4.9}$$

where $\tilde H_j = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j}, m|I_j|}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$, $m = M/L$. Let $\bar M_j = M_j/|I_j|$. Since g^2 is a density, and since by Lemma 7.5 $R(\theta) = E_{\mu^{\mathbf{h}}_{\Lambda_{I_j},\theta|I_j|}}[|I_j|^{-1}\sum_{x\in I_j} F'(\eta_x)]$ is Lipschitz continuous, uniformly in L and $\mathbf{h}$, by Schwarz inequality the first term is bounded above by

$$C_0 \sum_{j=1}^{\ell} a_j E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[g^2 [\bar M_j - m]^2 \Big] \;\le\; C_0 \sum_{1\le i\neq j\le \ell} a_j \, a_k \, E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[g^2 [\bar M_j - \bar M_i]^2 \Big]$$

for some finite constant C_0 because m is just the average of the densities m_i.

By Lemma 4.7 below, each expectation is bounded by

$$\begin{aligned} & 6 \Big(\Gamma_{I_i}(\lambda,\mathbf{h}) - \tilde\Gamma_1(\lambda) \Big)^2 \;+\; 6 \Big(\Gamma_{I_j}(\lambda,\mathbf{h}) - \tilde\Gamma_1(\lambda) \Big)^2 \;+\; \frac{C_0}{K} \\ & \quad + 6 \Big(W_{I_i}(\mathbf{h}) - \mathbb{E}[h_1] \Big)^2 \;+\; 6 \Big(W_{I_j}(\mathbf{h}) - \mathbb{E}[h_1] \Big)^2 \\ & \qquad + C_2 K^t \Big\{ D_{I_i}(\mu^{\mathbf{h}}_{I_i,M_i}, g) + D_{I_j}(\mu^{\mathbf{h}}_{I_j,M_j}, g) + E_{\mu^{\mathbf{h}}_{\Lambda_L,M}} \Big[\Big\{ \frac{\partial g}{\partial \eta_{y_i}} - \frac{\partial g}{\partial \eta_{x_j}} \Big\}^2 \Big] \Big\} \; . \end{aligned}$$

In this formula, Γ_Λ, $\tilde\Gamma_1$ and W_Λ are defined in (4.10). Here we are assuming that the cubes I_j are ordered, that $i < j$ and that y_i is the rightmost site in I_i and x_j is the leftmost site in I_j. An elementary computation shows that the expectation in the previous formula is bounded above by $L D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g)$. Therefore, the first term in (4.9) is less than or equal to

$$\begin{aligned} & \frac{C_0}{K} \;+\; \frac{C_0}{\ell} \sum_{j=1}^{\ell} \Big(\Gamma_{I_j}(\lambda,\mathbf{h}) - \tilde\Gamma_1(\lambda) \Big)^2 \;+\; \frac{C_0}{\ell} \sum_{j=1}^{\ell} \Big(W_{I_j}(\mathbf{h}) - \mathbb{E}[h_1] \Big)^2 \\ & \quad + C_2 K^t L D_{\Lambda_L}(\mu_{\Lambda^{\mathbf{h}}_L,M}, g) \; . \end{aligned}$$

There exists K_2 large enough for C_0/K to be smaller than $\delta^2\beta_0^2/3$ and for the constant given by (6.5), with $\Gamma_1(\cdot)$ and $Id(\cdot)$ in place of $U(\cdot)$ to be smaller than $\delta^2\beta_0^2/6C_0$ (cf. the remark stated in the penultimate paragraph of Section 6). Here Id stands for the identity and C_0 for the constant which appears in the previous formula. Fix $K \geq K_2$ and let Ω_2 be the set of environments given by (6.5) associated to $(K, \Gamma_1(\cdot), \delta^2\beta_0^2/6C_0)$ and to $(K, Id, \delta^2\beta_0^2/6C_0)$. For any $\mathbf{h}$ in Ω_2, there exists $\ell_2 = \ell_2(\mathbf{h})$ such that for $\ell \geq \ell_2$, the previous expression is bounded by

$$\delta^2\beta_0^2 \ + \ C_2 K^t L D_{\Lambda_L}(\mu_{\Lambda_L^{\mathbf{h}},M}, g) \ .$$

Since, by assumption,

$$\theta(L,\mathbf{h}) L^{-1} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \ \geq \ \delta\beta_0^2 \ ,$$

the previous expression is less than or equal to

$$\Big\{ \frac{\delta\theta(L,\mathbf{h})}{L} + C_2 K_1^t L \Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \ .$$

The second term in (4.9) is easier to estimate since one need just to take $g = 1$ in the previous argument. Replacing δ by $\delta/2$, we conclude the proof of the lemma. □

Proof of Proposition 4.2. Let $K_* = \max\{K_0, K_1, K_2\}$, where K_i are the integers given by Lemmas 4.4, 4.5, 4.6. Let $\Omega_* = \Omega_1 \cap \Omega_2$, where Ω_i are the sets of environments given by Lemmas 4.5, 4.6 associated to K_*. Fix an environment $\mathbf{h}$ in Ω_* and let $\ell_*(\mathbf{h}) = \max\{\ell_1(\mathbf{h}), \ell_2(\mathbf{h})\}$, where $\ell_1(\mathbf{h})$ are the positive constants given by Lemmas 4.5, 4.6. For $\ell \geq \ell_*$, In view of Lemmas 4.3, 4.4, 4.5, 4.6 and the the decomposition (4.4), the left hand side of (4.3) is bounded above by

$$\Big\{ \frac{C_2 K_*^{t-1}}{\ell} + C_2 K_*^{t-1} L + \frac{8\delta\, \theta(L,\mathbf{h})}{L} \Big\} D_{\Lambda_L}(\mu^{\mathbf{h}}_{\Lambda_L,M}, g) \ .$$

Since K_* is fixed, this concludes the proof. □

We conclude this section with a technical result needed above. For a finite subset Λ of $\mathbb{Z}$, let

$$\Gamma_\Lambda(\lambda, \mathbf{h}) \ = \ \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \Gamma_1(\lambda - h_x) \ , \quad W_\Lambda(\mathbf{h}) \ = \ \frac{1}{|\Lambda|} \sum_{x\in\Lambda} h_x \tag{4.10}$$

and let $\tilde\Gamma_1(\lambda) = \mathbb{E}[\Gamma_1(\lambda - h_1)]$. Consider a cube Λ_{2K} as the union of two intervals of size K. For $i = 1$, 2, denote by $\bar M_i$, the average spin over the i-th cube: $\bar M_1 = K^{-1}\sum_{1\leq x\leq K} \eta_x$, $\bar M_2 = K^{-1}\sum_{K+1\leq x\leq 2K} \eta_x$. Let

$$U_\Lambda(\lambda, \mathbf{h}) \ = \ \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \gamma_1(\lambda - h_x) \ .$$

Lemma 4.7. *There exist finite constants C_0, C_2, such that*

$$\begin{aligned} E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[g^2(\bar M_1 - \bar M_2)^2\Big] \ &\leq \ 6\Big(W_{\Lambda_K}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2 \ + \ 6\Big(W_{\Lambda_{K+1,2K}}(\mathbf{h}) - \mathbb{E}[h_1]\Big)^2 \\ &+ 6\Big(\Gamma_{\Lambda_K}(\lambda,\mathbf{h}) - \tilde\Gamma_1(\lambda)\Big)^2 \ + \ 6\Big(\Gamma_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h}) - \tilde\Gamma_1(\lambda)\Big)^2 \\ &+ \frac{C_0}{K} \ + \ C_2 K^t D_{\Lambda_{2K}}(\mu^{\mathbf{h}}_{\Lambda_{2K},M}, g) \end{aligned} \tag{4.11}$$

for every $K \geq 1$, M in $\mathbb{R}$, environment $\mathbf{h}$ and function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_{2K},M})$ such that $E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}[g^2] = 1$.

Proof. Fix an environment $\mathbf{h}$ and a charge M. By the entropy inequality and by definition of θ, the left hand side of (4.11) is bounded above by

$$\frac{1}{\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{\beta(\bar{M}_1-\bar{M}_2)^2\Big\}\Big] \;+\; \frac{\theta(2K,\mathbf{h})}{\beta} D_{\Lambda_{2K}}(\mu_{\Lambda^{\mathbf{h}}_{2K},M}, g)\;.$$

By the a-priori estimate obtained in Step 2, the second term is less than or equal to $CK^t\beta^{-1}D_{\Lambda_{2K}}(\mu_{\Lambda^{\mathbf{h}}_{2K},M}, g)$. To estimate the first term, choose λ according to

$$M \;=\; E_{\nu^{\mathbf{h}}_{\Lambda_{2K},\lambda}}\Big[\sum_{x\in\Lambda_{2K}}\eta_x\Big]\;.$$

Since $(\bar{M}_1-\bar{M}_2)^2$ is bounded above by

$$\begin{aligned}
&6\Big(\Gamma_{\Lambda_K}(\lambda,\mathbf{h})-\tilde{\Gamma}_1(\lambda)\Big)^2 \;+\; 6\Big(W_{\Lambda_K}(\mathbf{h})-\mathbb{E}[h_1]\Big)^2\\
&\quad + 6\Big(W_{\Lambda_{K+1,2K}}(\mathbf{h})-\mathbb{E}[h_1]\Big)^2 \;+\; 6\Big(\Gamma_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h})-\tilde{\Gamma}_1(\lambda)\Big)^2\\
&\qquad + 6\Big(\bar{M}_1-U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2 \;+\; 6\Big(\bar{M}_2-U_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h})\Big)^2\;,
\end{aligned}$$

by Schwarz inequality, the first term is bounded by the sum of the previous four expressions which are constants with respect to $\mu^{\mathbf{h}}_{\Lambda_{2K},M}$ with

$$\begin{aligned}
&\frac{1}{2\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{12\beta\Big(\bar{M}_1-U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]\\
&\quad+\frac{1}{2\beta}\log E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\exp\Big\{12\beta\Big(\bar{M}_2-U_{\Lambda_{K+1,2K}}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]\;.
\end{aligned}$$

Recall that $e^x \le 1+xe^x$ for $x>0$ and that $\log(1+x)\le x$ to estimate the first term by

$$6E_{\mu^{\mathbf{h}}_{\Lambda_{2K},M}}\Big[\Big(\bar{M}_1-U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\exp\Big\{12\beta\Big(\bar{M}_1-U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]\Big\}\;.$$

By Corollary 5.4, we may replace the expectation with respect to the canonical measure by an expectation with respect to a grand canonical measure, paying the price of a finite constant. Since the grand canonical measures are product, by Schwarz inequality, by the definition of $U_{\Lambda_K}(\lambda,\mathbf{h})$ and by the uniform estimates (5.1), the previous expression is bounded above by

$$\frac{C_0}{K}E_{\nu^{\mathbf{h}}_{\Lambda_{2K},\lambda}}\Big[\exp\Big\{24\beta\Big(\bar{M}_1-U_{\Lambda_K}(\lambda,\mathbf{h})\Big)^2\Big\}\Big]^{1/2}\;.$$

Since $\exp\{ax^2\}$ is a convex function for $a>0$, this expression is less than or equal to

$$\frac{C_0}{K}\sup_{\lambda}E_{\nu_\lambda}\Big[\exp\Big\{24\beta\{\eta_1-\gamma_1(\lambda)\}^2\Big\}\Big]^{1/2}\;.$$

For β small enough, the previous expectation is bounded, uniformly in λ. This proves the lemma. □

5. Local Central Limit Theorem

We prove in this section some estimates which follow from the local central limit theorem and which play a central role in the proof of the spectral gap and the logarithmic Sobolev inequality. All constants in this section depend only on $\|F\|_\infty$.

Fix a sequence $\mathbf{b}=\{b_k,,\, k\ge 1\}$ of real numbers. For λ in $\mathbb{R}$, denote by $P_{\mathbf{b},\lambda}$ the probability measure on the product space $\mathbb{R}^{\mathbb{N}}$ that makes the coordinates $\{X_k,\, k\ge$

$1\}$ independent random variables with X_k having density $Z(\lambda - b_k)^{-1}\exp\{[\lambda - b_k]x - V(x)\}$. Denote by $E_{\mathbf{b},\lambda}$ expectation with respect to $P_{\mathbf{b},\lambda}$.

Denote by $\gamma_1(\lambda)$, $\sigma^2(\lambda)$, $\{\gamma_k(\lambda),\, k \ge 3\}$ the expectation, the variance and the k-th truncated moment of a random variable with density $Z(\lambda)^{-1}\exp\{\lambda x - V(x)\}$:

$$\gamma_1(\lambda) \;=\; \frac{1}{Z(\lambda)}\int_{\mathbb{R}} x\, e^{\lambda x - V(x)}\, dx\ , \quad \sigma^2(\lambda) \;=\; \frac{1}{Z(\lambda)}\int_{\mathbb{R}} [x-\gamma_1(\lambda)]^2\, e^{\lambda x - V(x)}\, dx\ ,$$
$$\gamma_k(\lambda) \;=\; \frac{1}{Z(\lambda)}\int_{\mathbb{R}} [x-\gamma_1(\lambda)]^k\, e^{\lambda x - V(x)}\, dx\ .$$

We proved in [15] that there exist finite strictly positive constants $\{C_j,\, j \ge 1\}$, depending only on j and $\|F\|_\infty$, such that

$$\begin{gathered} C_1^{-1} \;\le\; \frac{Z(\lambda)}{e^{\lambda^2/2}} \;\le\; C_1\ , \quad |\gamma_1(\lambda) - \lambda| \;\le\; C_1\ , \\ C_1^{-1} \;\le\; \sigma^2(\lambda) \;\le\; C_1\ , \quad C_j^{-1} \;\le\; \gamma_{2j}(\lambda) \;\le\; C_j \end{gathered} \tag{5.1}$$

for all λ in $\mathbb{R}$. In particular,

$$\sup_{\lambda\in\mathbb{R}} \Big| \frac{\gamma_j(\lambda)}{\sigma^2(\lambda)^{j/2}} \Big| \;\le\; \tilde{C}_j \tag{5.2}$$

for all $j \ge 3$, which is the estimate needed in order to prove the uniform local central limit theorem.

For a finite subset Λ of $\mathbb{N}$, denote by $f_{\mathbf{b},\lambda,\Lambda}$ the density of the random variable

$$\frac{1}{\sigma^2(\mathbf{b},\lambda,\Lambda)^{1/2}} \sum_{j\in\Lambda} [X_j - \gamma_1(\lambda - b_j)]\ ,$$

where, for $k \ge 3$,

$$\sigma^2(\mathbf{b},\lambda,\Lambda) \;=\; \sum_{j\in\Lambda} \sigma^2(\lambda - b_j)\ , \quad \gamma_k(\mathbf{b},\lambda,\Lambda) \;=\; \sum_{j\in\Lambda} \gamma_k(\lambda - b_j)\ . \tag{5.3}$$

Theorem 5.1. *Assume that $\|F\|_\infty < \infty$. There exists $N_0 \ge 1$ and a finite constant C_0 depending only on $\|F\|_\infty$ such that*

$$\Big| f_{\mathbf{b},\lambda,\Lambda_N}(x) \;-\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big\{ 1 - \frac{\gamma_3(\mathbf{b},\lambda,\Lambda_N)x}{6\sigma^2(\mathbf{b},\lambda,\Lambda_N)^{3/2}} \Big\} \Big| \;\le\; \frac{C_0}{N}$$

for all $N \ge N_0$, x in $\mathbb{R}$, environment $\mathbf{b}$ and λ in $\mathbb{R}$.

In virtue of (5.1), both $\sigma^2(\mathbf{b},\lambda,\Lambda_N)$ and $\gamma_3(\mathbf{b},\lambda,\Lambda_N)$ are of order N. The second term in the expansion is therefore of order $N^{-1/2}$. For a fixed parameter λ this is just the usual statement of the local central limit theorem for independent random variables with finite fourth moments. The important point here is the uniformity over the parameter λ. This uniformity can be obtained in virtue of (5.2) and the estimates (5.1).

The local central limit theorem gives asymptotic expansions of the expectation of a function with respect to a canonical measure. This is the content of the next result. Recall that $E_\nu[G;G]$ stands for the variance of G with respect to ν.

Corollary 5.2. *Fix $\ell \ge 1$ and fix a function $G\colon \mathbb{R}^\ell \to \mathbb{R}$. There exist $N_0 \ge 1$ and a finite constant C_0, depending only on $\|F\|_\infty$, such that for all $N \ge N_0$, environment*

$\mathbf{b}$, and M in $\mathbb{R}$

$$\Big| E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[G] \;-\; E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[G] \Big| \;\le\; \frac{C_0 \ell}{|\Lambda_N|} \, \|G\|_\infty \quad \text{if } G \text{ is bounded and}$$
$$\Big| E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[G] \;-\; E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[G] \Big| \;\le\; \frac{C_0 \ell}{|\Lambda_N|} \sqrt{E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[G;G]} \; .$$

In these formulas, the chemical potential λ, which depends on N, M and $\mathbf{b}$, is chosen so that

$$M \;=\; E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}} \Big[\sum_{x\in\Lambda_N} \eta_x \Big] \; . \tag{5.4}$$

The proof of this result is elementary (cf. Corollary A2.1.4 in [13]). Of course, by changing the value of the constant C, the first inequality remains valid for all values of $N \ge \ell$.

Corollary 5.3. *Let $G\colon \mathbb{R} \to \mathbb{R}$ be a smooth bounded function and let $G_{\mathbf{b},N,M} = G - E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[G]$. There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$E_{\mu^{\mathbf{b}}_{\Lambda_N,M}} \Big[\Big(\frac{1}{N} \sum_{x=1}^N G_{\mathbf{b},N,M}(\eta_x) \Big)^2 \Big] \;\le\; C_0 \frac{\|G\|_\infty^2}{N}$$

for all $N \ge 1$, environment $\mathbf{b}$ and M in $\mathbb{R}$.

The proof of this result is identical to the one of Corollary 5.4 in [15].

Recall the definition of the variance $\sigma^2(\mathbf{b},\lambda,\Lambda)$ and of the density $f_{\mathbf{b},\lambda,\Lambda}$ given in (5.3). For $1 \le K < N$, denote by $\mu^{\mathbf{b}}_{\Lambda_N,K,M}$ the marginal on $\mathbb{R}^{\Lambda_K}$ of the canonical measure $\mu^{\mathbf{b}}_{\Lambda_N,M}$. An elementary computation shows that $\mu^{\mathbf{b}}_{\Lambda_N,K,M}$ is absolutely continuous with respect to the Lebesgue measure and that its Radon-Nikodym derivative $R^{\mathbf{b}}_{N,K,M}(\mathbf{x}_K)$ is given by

$$\frac{R^{\mathbf{b}}_{N,K,M}(\mathbf{x}_K)}{g^{\mathbf{b}}_{\lambda,\Lambda_K}(\mathbf{x}_K)} \;=\; \frac{\sigma^2(\mathbf{b},\lambda,\Lambda_N)}{\sigma^2(\mathbf{b},\lambda,\Lambda_{K,N})} \frac{f_{\mathbf{b},\lambda,\Lambda_{K,N}}\Big(\frac{\sum_{j=1}^K [\gamma_1(\lambda-b_j) - x_j]}{\sigma^2(\mathbf{b},\lambda,\Lambda_{K,N})^{1/2}} \Big)}{f_{\mathbf{b},\lambda,\Lambda_N}(0)} \;,$$

where $\mathbf{x}_K = (x_1,\dots,x_K)$, $\Lambda_{K,N} = \{K+1,\dots,N\}$, for a finite set Λ,

$$g^{\mathbf{b}}_{\lambda,\Lambda}(\mathbf{x}_K) \;=\; \prod_{j\in\Lambda} \frac{1}{Z(\lambda-b_j)} e^{[\lambda-b_j]x_j - V(x_j)}$$

and λ is chosen according to (5.4). The next result shows that the ratio $R^{\mathbf{b}}_{N,K,M}/g^{\mathbf{b}}_{\lambda,\Lambda_K}$ is bounded above, uniformly over λ, provided K/N is bounded away from 1.

Corollary 5.4. *There exists a finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$\frac{R^{\mathbf{b}}_{N,K,M}(\mathbf{x}_K)}{g^{\mathbf{b}}_{\lambda,\Lambda_K}(\mathbf{x}_K)} \;\le\; C_0 \; .$$

for all $N/2 \ge K \ge 1$, M in $\mathbb{R}$, environment $\mathbf{b}$ and $\mathbf{x}_K$ in $\mathbb{R}^{\Lambda_K}$. In this formula, λ is chosen for (5.4) to hold. In particular, if $K \le N/2$, for any local function $H\colon \mathbb{R}^{\Lambda_K} \to \mathbb{R}$,

$$E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[H(\eta_1,\dots,\eta_K)] \;\le\; C_0 E_{\nu^{\mathbf{b}}_{\Lambda_K,\lambda}}[\,|H(\eta_1,\dots,\eta_K)|\,] \tag{5.5}$$

for λ satisfying (5.4).

For N large, the proof is an elementary consequence of the explicit formula for the ratio $R^{\mathbf{b}}_{N,K,M}/g^{\mathbf{b}}_{\lambda,\Lambda_K}$, the estimates for the variance presented in (5.1) and the local central limited theorem stated in Theorem 5.1. For N small, the local central limited theorem is replaced by direct inspection. More details can be found in [15].

This Corollary provides an estimate on the variance of functions with respect to canonical measures in terms of the variance of the same function with respect to grand canonical measures. Indeed, fix a function $H : \mathbb{R}^{\Lambda_K} \to \mathbb{R}$, assume that $N \geq 2K$ and choose λ according to (5.4). By Corollary 5.4,

$$E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}[H;H] \;\leq\; E_{\mu^{\mathbf{b}}_{\Lambda_N,M}}\Big[\Big(H - E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[H]\Big)^2\Big] \;\leq\; C_0 E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[H;H]\,. \tag{5.6}$$

Theorem 5.1 and its corollaries permit to estimate expectations with respect to a canonical measure $\mu^{\mathbf{b}}_{\Lambda_L,M}$, provided L is large. The next result provides an estimate for small values of L. The important point in this result is once again the uniformity over the parameter λ and the environment $\mathbf{b}$. For a finite set Λ of $\mathbb{N}$, denote by $\tilde{f}_{\mathbf{b},\lambda,\Lambda}$ the density of the random variable $|\Lambda|^{-1/2}\sum_{j\in\Lambda}\{X_j - (\lambda - b_j)\}$ under the measure $P_{\mathbf{b},\lambda}$. Note that we are not renormalizing by $\sigma^2(\mathbf{b},\lambda,\Lambda)$ and that we are subtracting $\lambda - b_j$ instead of $\gamma_1(\lambda - b_j)$.

Lemma 5.5. *There exists a positive and finite constant C_0, depending only on $\|F\|_\infty$, such that*

$$C_0^{-N}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \;\leq\; \tilde{f}_{\mathbf{b},\lambda,\Lambda_N}(x) \;\leq\; C_0^{N}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

for every λ in $\mathbb{R}$, $N \geq 1$ and environment $\mathbf{b}$.

The proof is similar to the one of Lemma 5.6 in [15] and therefore omitted. The same argument shows that $g_\lambda(x) = Z(\lambda)^{-1}\exp\{\lambda x - V(x)\}$ is bounded above and below by a Gaussian density. More precisely, there exists a finite, strictly positive constant C_0 depending only on $\|F\|_\infty$, such that

$$C_0\frac{1}{\sqrt{2\pi}}e^{-(x-\lambda)^2/2} \;\leq\; g_\lambda(x) \;\leq\; C_0^{-1}\frac{1}{\sqrt{2\pi}}e^{-(x-\lambda)^2/2} \tag{5.7}$$

for every λ in $\mathbb{R}$.

We conclude this section with an important estimate, essential in the proof of Proposition 7.3. The proof of this lemma follows closely the one of Lemma 5.7 in [15].

Lemma 5.6. *There exists $\beta_1 > 0$ and a finite constant C_0 such that*

$$E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}\Big[\exp\Big\{\beta_1|\Lambda_N|\{m_{\Lambda_N} - R_{\mathbf{b},\Lambda_N}(\lambda)\}^2\Big\}\Big] \;\leq\; C_0$$

for every λ in $\mathbb{R}$, environment $\mathbf{b}$ and $N \geq 1$. In this formula, m_{Λ_N} stands for the charge average in Λ_N: $m_{\Lambda_N} = |\Lambda_N|^{-1}\sum_{x\in\Lambda_N}\eta_x$ and $R_{\mathbf{b},\Lambda_N}(\lambda) = E_{\nu^{\mathbf{b}}_{\Lambda_N,\lambda}}[m_{\Lambda_N}]$.

6. The environment

We present in this section all results needed on the environment in the proof of the spectral gap and the logarithmic Sobolev inequality. We start with simple bounds on the derivative of a function. For a finite subset Λ of $\mathbb{Z}$, and an environment $\mathbf{b}$, denote by $R_\Lambda : \mathbb{R} \to \mathbb{R}$ the smooth strictly increasing function

$$R_\Lambda(\lambda) \;=\; E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x\Big]$$

and denote by Φ_Λ the inverse of R_Λ. Let $A_\Lambda : \mathbb{R} \to \mathbb{R}$ be given by

$$A_\Lambda(m) \;=\; E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(m)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big]\;.$$

Lemma 6.1. *Fix a finite subset Λ and an environment $\mathbf{b}$. We claim that*

$$A'_\Lambda(m) \;=\; \frac{1}{|\Lambda|^{-1}\sum_{x\in\Lambda}\sigma^2(\Phi_\Lambda(m)-b_x)} - 1\;\cdot$$

Moreover, there exists a constant C_0, depending only on $\|F\|_\infty$, such that $\|A''_\Lambda\|_\infty \le C_0$ for all environments $\mathbf{b}$.

Proof. We claim that

$$E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big] \;=\; \Phi_\Lambda(\theta) \;-\; \theta \;-\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}b_x\;. \tag{6.1}$$

Indeed, by definition of the product measure $\nu^{\mathbf{b}}_{\Lambda,\lambda}$,

$$\begin{aligned}
E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big] \;&=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\frac{1}{Z(\lambda-b_x)}\int F'(a)e^{(\lambda-b_x)a-V(a)}\,da\\
&=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\{(\lambda-b_x)-R_{\{x\}}(\lambda)\}\\
&-\;\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\frac{1}{Z(\lambda-b_x)}\int\partial_a\{(\lambda-b_x)a-V(a)\}e^{(\lambda-b_x)a-V(a)}\,da\;.
\end{aligned}$$

Since $R_\Lambda(\lambda) = |\Lambda|^{-1}\sum_{x\in\Lambda}R_{\{x\}}(\lambda)$ and since the each term in the last sum vanishes, the previous expression is equal to

$$\lambda \;-\; R_\Lambda(\lambda) \;-\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}b_x\;,$$

which proves (6.1).

Since $\Phi_\Lambda = R_\Lambda^{-1}$, and since

$$\begin{aligned}
\frac{d}{d\lambda}R_\Lambda(\lambda) \;&=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}E_{\nu^{\mathbf{b}}_{\Lambda,\lambda}}[\eta_x;\eta_x] \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma^2(\lambda-b_x)\;,\\
\frac{d^2}{d\lambda^2}R_\Lambda(\lambda) \;&=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\gamma_3(\lambda-b_x)\;,
\end{aligned}$$

by (6.1),

$$\begin{aligned}
\frac{d}{d\theta}E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big] \;&=\; \frac{1}{|\Lambda|^{-1}\sum_{x\in\Lambda}\sigma^2(\Phi_\Lambda(\theta)-b_x)} - 1\;,\\
\frac{d^2}{d\theta^2}E_{\nu^{\mathbf{b}}_{\Lambda,\Phi_\Lambda(\theta)}}\Big[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}F'(\eta_x)\Big] \;&=\; -\,\frac{|\Lambda|^{-1}\sum_{x\in\Lambda}\gamma_3(\Phi_\Lambda(\theta)-b_x)}{\Big(|\Lambda|^{-1}\sum_{x\in\Lambda}\sigma^2(\Phi_\Lambda(\theta)-b_x)\Big)^3}\;.
\end{aligned}$$

The statement of the lemma follows from these identities and from estimates (5.1). □

We proceed with a lemma which explains the assumption made on $\sigma^2(\cdot)$, $\Gamma_1(\cdot)$. Let $U : \mathbb{R} \to \mathbb{R}$ be a continuous functions with limits at the boundary:

$$\lim_{\lambda\to\infty} U(\lambda) = U_+ \ , \quad \lim_{\lambda\to-\infty} U(\lambda) = U_- \tag{6.2}$$

for some finite values U_-, U_+. Let $\tilde U(\lambda) = \mathbb{E}[U(\lambda - h_1)]$.

Recall that for $K \ge 2$, $L \ge K^2$, we decompose the cube Λ_L in ℓ disjoint cubes $\{I_1, \dots, I_\ell\}$ of length K or $K+1$. One should think that K is large but fixed. The key point in the next result is the uniformity in λ.

Lemma 6.2. *Fix a continuous function U satisfying (6.2), $\varepsilon > 0$, $K \ge 2$ and an increasing function t. There exists a measurable set Ω_0 of sequences $\mathbf{h}$ such that*

- $\mathbb{P}[\Omega_0] = 1$,
- *For any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that*

$$\frac{1}{\ell}\sum_{j=1}^{\ell} t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2 \ \le\ c + \varepsilon \tag{6.3}$$

for all $\ell \ge \ell_0$ and all λ in $\mathbb{R}$. In this formula,

$$c \ =\ \max_{n=K,K+1}\Big\{t(n)\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{x\in\Lambda_n} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2\Big]\Big\}$$

Proof. Fix $\delta_0 > 0$. Since $U(\cdot)$ is a continuous bounded function with limits at the boundaries $\lambda = \pm\infty$, for each $\delta_0 > 0$, there exists a finite set of chemical potentials $\Gamma_{\delta_0} = \{\lambda_1, \dots, \lambda_r\}$ with the property that for every λ in $\mathbb{R}$, there exists λ_i in Γ_{δ_0} such that

$$\sup_{b, |b|\le A_0}\Big|U(\lambda - b) - U(\lambda_i - b)\Big| \ \le\ \frac{\delta_0}{t(K+1)^{1/2}} \ .$$

In other words,

$$\sup_{\lambda\in\mathbb{R}}\ \min_{\lambda_i\in\Gamma_{\delta_0}}\ \sup_{b,|b|\le A_0}\Big|U(\lambda - b) - U(\lambda_i - b)\Big| \ \le\ \frac{\delta_0}{t(K+1)^{1/2}} \ . \tag{6.4}$$

It is here and only here that we need the environment $\mathbf{h}$ to assume bounded values and the assumption (6.2) on the asymptotic behavior of $U(\cdot)$.

In view of (6.4), the left hand side of (6.3) is less than or equal to

$$6\delta_0^2 \ +\ 3\max_{\lambda_i\in\Gamma_{\delta_0}} \frac{1}{\ell}\sum_{j=1}^{\ell} t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde U(\lambda_i)\Big)^2 \ .$$

Fix $\delta_1 > 0$ and λ_i in Γ_{δ_0}. Since the intervals $\{I_j\,, 1 \le j \le \ell\}$ are disjoints, it is not difficult to show that the sequence

$$\mathbb{P}\Big[\frac{1}{\ell}\sum_{j=1}^{\ell}\Big\{t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde U(\lambda_i)\Big)^2 - c_j\Big\} \ge \delta_1\Big]$$

is summable in ℓ if

$$c_j \ =\ t(|I_j|)\mathbb{E}\Big[\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - \tilde U(\lambda_i)\Big)^2\Big] \ .$$

In particular, since $c_j \le c$, by the Borel-Cantelli lemma, there exists a set $\Omega_0 = \Omega_0(K, \delta_1, U)$ with the following properties: $\mathbb{P}[\Omega_0] = 1$ and for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell}\sum_{j=1}^{\ell} t(|I_j|)\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda_i - h_x) - U(\lambda_i)\Big)^2 \;\le\; c + \delta_1 \ .$$

for $\ell \ge \ell_0$. Since Γ_{δ_0} is a finite set, we can make the previous inequality to be uniform in λ_i.

In conclusion, for any sequence $\mathbf{h}$ in Ω_0, λ in $\mathbb{R}$ and $\ell \ge \ell_0$, the expression on the left hand side of (6.3) is bounded by $c + 6\delta_0^2 + 3\delta_1$. This concludes the proof of the lemma. □

This lemma is used in two different situations: with $t(n) = 1$ and with $t(n) = n$. In the first case, since U is bounded,

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{x\in\Lambda_n} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2\Big]$$

vanishes as $n \uparrow \infty$, uniformly in λ. In particular, taking K large enough, we may turn the constant c appearing in the right hand side of (6.3) as small as we wish. Hence, for every $\varepsilon > 0$, there exists $K_0 \ge 2$ and a $\mathbb{P}$-measure one set Ω_0 such that for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell}\sum_{j=1}^{\ell} \Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2 \;\le\; \varepsilon \tag{6.5}$$

for all $\ell \ge \ell_0$ and all λ in $\mathbb{R}$.

On the other hand, if $t(n) = n$, since U is bounded,

$$n\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{x\in\Lambda_n} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2\Big] \;\le\; 4\|U\|_\infty^2 \ .$$

Thus, we may take the constant c in the right hand side of (6.3) to be equal to $4\|U\|_\infty^2$ and ε to be 1. In this case, the statement of the lemma becomes: There exists $C = C(\|U\|_\infty)$ such that for all $K \ge 2$, there exists a $\mathbb{P}$-measure one set $\Omega_0 = \Omega_0(K, U)$ such that for any $\mathbf{h}$ in Ω_0, there exists $\ell_0 = \ell_0(\mathbf{h})$ such that

$$\frac{1}{\ell}\sum_{j=1}^{\ell} |I_j|\Big(\frac{1}{|I_j|}\sum_{x\in I_j} U(\lambda - h_x) - \tilde U(\lambda)\Big)^2 \;\le\; C \tag{6.6}$$

for all $\ell \ge \ell_0$ and all λ in $\mathbb{R}$.

Notice that the previous results (6.5), (6.6) hold for $U = Id$, where Id stands for the identity. Of course, the identity does not satisfy (6.2) but we need this assumption only to guarantee uniformity over λ, a parameter which disappears in the case of the identity.

We conclude this section proving that (2.2), (2.4) hold if F converges at $\pm\infty$. Indeed, assume that there exists $F_\pm$ such that

$$\lim_{a\to\pm\infty} F(a) \;=\; F_\pm \ .$$

In this case, by the dominated convergence theorem, $Z(\lambda)\exp\{-\lambda^2/2\}$ converges to $\sqrt{2\pi}\exp\{-F_\pm\}$ at $\pm\infty$. The same argument shows that Γ_1 converge to 0 at the boundary and that σ^2 converges to 1.

7. Large deviations estimates

In this section, for $j = 0, 1, 2$, all constants C_j are allowed to depend only on $\|F^{(i)}\|_\infty$ for $0 \le i \le j$ and may change from line to line. Here $F^{(i)}$ stands for the i-th derivative of F.

Fix a differentiable function $H\colon \mathbb{R} \to \mathbb{R}$ with bounded derivative: $\|H'\|_\infty < \infty$. For each $L \ge 1$, x in Λ_L, λ, M in $\mathbb{R}$ and environment $\mathbf{h} = \{h_x, x \in \mathbb{Z}\}$, let

$$H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) = H(\eta_x) - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] \;,$$
$$H_{\mathbf{h},\Lambda_L,M}(\eta_x) = H(\eta_x) - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)] \;.$$

Although the notation is slightly ambiguous, the context will always clarify if we are subtracting the average with respect to the canonical measure or with respect to the grand canonical one. Notice that $H_{\mathbf{h},\Lambda_L,M}(\eta_x) - H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x) = E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)]$. In particular, if λ is chosen according to (5.4), by Corollary 5.2,

$$\Big|H_{\mathbf{h},\Lambda_L,M}(\eta_x) - H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)\Big| \;=\; \Big|E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x)] - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[H(\eta_x)]\Big| \;\le\; \frac{C_0\|H'\|_\infty}{|\Lambda_L|} \tag{7.1}$$

for some finite constant C_0 depending only on $\|F\|_\infty$ because $E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[H(\eta_x); H(\eta_x)]$ $\le E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\{H(\eta_x) - H(\theta_x)\}^2] \le \|H'\|^2_\infty \sigma^2(\lambda - h_x)$. Here, θ_x stands for $E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x] = \gamma_1(\lambda - h_x)$.

We claim that there exists a finite constant C_0 depending only on $\|F\|_\infty$ for which

$$|H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \;\le\; C_0\|H'\|_\infty\Big(1 + \Big|\eta_x - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x]\Big|\Big) \;, \tag{7.2}$$
$$|H_{\mathbf{h},\Lambda_L,M}(\eta_x)| \;\le\; C_0\|H'\|_\infty\Big(1 + \Big|\eta_x - E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[\eta_x]\Big|\Big)$$

for all $L \ge 1$, M, λ in $\mathbb{R}$ and environment $\mathbf{h}$. Consider first the grand canonical case. By definition of $H_{\mathbf{h},\Lambda_L,\lambda}$ and by Schwarz inequality,

$$\begin{aligned} |H_{\mathbf{h},\Lambda_L,\lambda}(\eta_x)| \;&\le\; E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\big|H(\eta_x) - H(\xi_x)\big|\Big] \;\le\; \|H'\|_\infty E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}\Big[\big|\eta_x - \xi_x\big|\Big] \\ &\le\; \|H'\|_\infty\Big\{\big|\eta_x - E_{\nu^{\mathbf{h}}_{\Lambda_L,\lambda}}[\eta_x]\big| \;+\; \sigma^2(\lambda - h_x)^{1/2}\Big\} \;. \end{aligned}$$

Of course, in the previous formulas, only the variable ξ_x is integrated. By (5.1), the second term inside braces in the last expression is bounded above by some finite constant C_0 that depends on $\|F\|_\infty$ only. This proves the claim in the grand canonical case. The same arguments apply to the canonical case provide we show that $E_{\mu^{\mathbf{h}}_{\Lambda_L,M}}[\eta_x;\eta_x]$ is uniformly bounded. But this is follows from estimate (5.6) of Corollary 5.4.

Proposition 7.1. *Fix a differentiable function $H\colon \mathbb{R} \to \mathbb{R}$ with bounded derivative. There exists a constant C_0, depending only on $\|F\|_\infty$, such that*

$$\frac{1}{\beta|\Lambda_L|}\log\int \exp\Big\{\beta\sum_{x\in\Lambda_L} H_{\mathbf{h},\Lambda_L,M}(\eta_x)\Big\}\, d\mu^{\mathbf{h}}_{\Lambda_L,M} \;\le\; C_0\|H'\|^2_\infty\beta \tag{7.3}$$

for all $\beta > 0$, $L \ge 2$, M in $\mathbb{R}$ and environment $\mathbf{h}$.

The proof of this statement is similar to the one of Lemma 6.1 in [15] and thus omitted.

Fix $K \ge 1$, $L \ge K^2$ and divide the interval $\{1, \dots, L\}$ into $\ell = \lfloor L/K \rfloor$ adjacent intervals of length K or $K+1$, where $\lfloor a \rfloor$ represents the integer part of a. For $1 \le j \le \ell$, denote by I_j the j-th interval, by M_j the total spin on I_j: $M_j = \sum_{x \in I_j} \eta_x$ and let

$$m_j \;=\; E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} \eta_x \Big] \;, \quad m^*_j \;=\; E_{\nu^{\mathbf{h}}_{\Lambda_L, \lambda}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} \eta_x \Big] \;,$$
$$A_j(m) \;=\; E_{\nu^{\mathbf{h}}_{I_j, \Phi_{I_j}(m)}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \;,$$

where λ is given by (5.4). Notice that

$$|m_j - m^*_j| \;\le\; C_0 L^{-1} \tag{7.4}$$

in virtue of Lemma 5.2 and estimates (5.1).

Lemma 7.2. *For $1 \le j \le \ell$, let $G_j = G_j(M_j; M, L, \mathbf{h})$ be given by*

$$G_j \;=\; E_{\mu^{\mathbf{h}}_{I_j, M_j}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] - E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] - A'_j(m_j) \Big(\frac{M_j}{|I_j|} - m_j \Big) \;.$$

There exists a finite constant C^*_0, depending only on $\|F\|_\infty$, and a finite constant C_1, depending only on $\|F\|_\infty$, $\|F'\|_\infty$, such that

$$|G_j| \;\le\; C^*_0 (M_j/|I_j| - m^*_j)^2 + \frac{C_1}{|I_j|} \;.$$

Proof. Fix j and rewrite G_j as

$$\begin{aligned}
& E_{\mu^{\mathbf{h}}_{I_j, M_j}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \;-\; E_{\nu^{\mathbf{h}}_{I_j, \Phi_{I_j}(M_j/|I_j|)}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \\
&\quad + E_{\nu^{\mathbf{h}}_{I_j, \Phi_{I_j}(m_j)}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \;-\; E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \\
&\quad + A_j(M_j/|I_j|) \;-\; A_j(m_j) \;-\; A'_j(m_j)(M_j/|I_j| - m_j) \;.
\end{aligned}$$

The first two lines will be estimated by the equivalence of ensembles, while the third is a Taylor expansion up to the second order.

By Corollary 5.2, the first line is bounded above by $C_1 |I_j|^{-1}$ for some finite constant C_1. By Lemma 6.1, $\|A''_j\|_\infty$ is finite. In particular, in view of (7.4), the third term is bounded above by $C_0 (M_j/|I_j| - m^*_j)^2 + C_0 L^{-2}$. Taking conditional expectation with respect to $\sum_{x \in I_j} \eta_x$, $\{\eta_z, \, z \notin I_j\}$ in the second expectation, the second line can be rewritten as

$$E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[E_{\nu^{\mathbf{h}}_{I_j, \Phi_{I_j}(m_j)}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \;-\; E_{\mu^{\mathbf{h}}_{I_j, M_j}} \Big[\frac{1}{|I_j|} \sum_{x \in I_j} F'(\eta_x) \Big] \Big] \;.$$

Using the equivalence of ensembles, Corollary 5.2, we may replace the second term inside the expectation by an expectation with respect to the grand canonical measure with chemical potential given by $\Phi_{I_j}(M_j/|I_j|)$. We may also add inside the expectation $A'_j(m_j)(M_j/|I_j| - m_j)$, because this expression has mean zero. We recover in this way the same Taylor expansion up to the second order, which was shown above to be bounded by $C_0 (M_j/|I_j| - m^*_j)^2 + C_0 L^{-2}$. Applying Corollary 5.4 to replace the canonical measure $\mu^{\mathbf{h}}_{\Lambda_L, M}$ by the grand canonical, we obtain that the second term in the decomposition of G_j is bounded above by $C_0 |I_j|^{-1}$. $\square$

Proposition 7.3. *There exist $\beta_0 > 0$ and a finite constant C_1 depending only on $\|F\|_\infty$, $\|F'\|_\infty$ such that*

$$\frac{1}{\beta L} \log E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[\exp \Big\{ \beta \sum_{j=1}^{\ell} |I_j| G_j \Big\} \Big] \;\le\; \frac{C_1 \beta}{K} \tag{7.5}$$

for all $\beta \le \beta_0$, all $L \ge K^2$, all M in $\mathbb{R}$ and all environment $\mathbf{h}$.

The proof of this statement is similar to the one of Lemma 6.5 in [15] and thus omitted.

Lemma 7.4. *Fix $K \ge 2$, $L \ge K^2$, an environment $\mathbf{h}$, c, M in $\mathbb{R}$ and a smooth function g in $L^2(\mu^{\mathbf{h}}_{\Lambda_L, M})$ such that $< g^2 >_{\mu^{\mathbf{h}}_{\Lambda_L, M}} = 1$. There exists a finite constant C_0 depending on $\|F\|_\infty$ such that*

$$\begin{aligned} &\frac{1}{\beta L} \log E_{\mu^{\mathbf{h}}_{\Lambda_L, M}} \Big[\exp \Big\{ \sum_{j=1}^{\ell} |I_j| [A'_j(m_j) - c] \Big(\frac{M_j}{|I_j|} - m_j \Big) \Big\} \Big] \\ &\quad \le\; \frac{C_0 \beta}{K} \exp \Big\{ \frac{C_0}{\ell} \sum_{j=1}^{\ell} |I_j| (A'_j(m_j) - c)^2 \Big\} \,. \end{aligned} \tag{7.6}$$

for every $\beta > 0$.

We conclude this section with a technical result needed in the proof of the logarithmic Sobolev inequality.

Lemma 7.5. *Fix a bounded function $H : \mathbb{R} \to \mathbb{R}$, an environment $\mathbf{h}$ and $L \ge 2$. The function $\tilde{H}_{L,\mathbf{h}} \colon \mathbb{R} \to \mathbb{R}$ defined by $\tilde{H}_{L,\mathbf{h}}(m) = E_{\mu^{\mathbf{h}}_{\Lambda_L, m|\Lambda_L|}}[H(\eta_1)]$ is Lipschitz continuous on $\mathbb{R}$ and the Lipschitz constant does not depend on L.*

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