

Notes on Boussinesq Equation

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Preface

The purpose of these notes is to give a brief introduction to nonlinear dispersive equations and the issues regarding solutions of these equations.

We have chosen the well know Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxx} + (u^2)_{xx} = 0 \tag{0.1}$$

and one of its generalizations

$$u_{tt} - u_{xx} + u_{xxx} + (\psi(u))_{xx} = 0 \tag{0.2}$$

as models to present the theory.

We will present techniques to deal with problems regarding properties of solutions to the associated initial value problem (IVP). We will discuss the so-called smoothing effect properties of solutions of the linear IVP. Then we will use them to obtain local and global results for solutions of the nonlinear IVP. We next will study the regularity of these solutions. We will also comment on results regarding decay and nonlinear scattering. Finally we will discuss some blow-up results. The material presented here is an amplified version of the contents of the articles [31], [32] and [2].

We have included an appendix containing basic facts from Fourier Analysis, such as, Fourier transform, interpolation, Sobolev spaces and the fractional integral theorem. Most of the content of the appendix was taken from the notes by F. Linares and G. Ponce [33].

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CHAPTER 1

Introduction

We consider the initial value problem (IVP) for a Boussinesq-type equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (\psi(u))_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x). \end{cases} \quad (1.1)$$

This equation arises in the modeling of nonlinear strings which is a generalization of the classical Boussinesq equation. Boussinesq ([7]) reduced into a nonlinear model the equations governing a two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel. Note that the equation in (1.1) is a perturbation of the classical linear wave equation which incorporates the basic idea of nonlinearity and dispersion.

The case $\psi(u) = u^2$ in (1.1) is known as the “good” Boussinesq in comparison with the “bad” Boussinesq equation defined as

$$u_{tt} = u_{xx} + c_1 u_{xxxx} + c_2 (u^2)_{xx}, \quad c_1, c_2 > 0. \quad (1.2)$$

This “bad” version arises in the study of water waves. Specifically, it is used to describe a two-dimensional flow of a body of water over a flat bottom with air above the water, assuming that the water waves have small amplitudes and the water is shallow. It also appeared in a posterior study of Fermi-Pasta-Ulam (FPU) problem, which was performed to show that the finiteness of thermal conductivity of an anharmonic lattice was related to nonlinear forces in the springs but it was not the case. This result motivated N. Zabusky and M. Kruskal [55] to approach the FPU problem from the continuum point of view. They found that the equations governing the dynamics of the lattice in FPU problem are given by equations of type (1.2).

For the “good” Boussinesq and its generalized form in (1.1) it is possible to study local well-posedness for the initial value problem, which is not the case for the “bad” version (1.2). For the model in (1.2) only solutions of soliton type are known. Furthermore, in the side of the Fourier transform we can see that the solution of the linearized equation \hat{u} grows as $\exp(\pm\xi^2 t)$ with time. The same occurs for the nonlinear problem so in order to study well-posedness the component proportional to $\exp(\xi^2 t)$ has to be vanished.

From the inverse scattering approach the following results have been obtained: V. Zakharov in [56] showed that the scattering theory could be applied to solve the equation (1.2). He found a Lax pair for it, namely

$$\frac{dL}{dt} = i[Q, L] = i(QL - LQ)$$

where

$$L = i \frac{d^3}{dx^3} + i \left(u \frac{d}{dx} - \frac{d}{dx} u \right) + i \frac{d}{dx} - av_x$$

$$Q = \left(b \frac{d}{dx^2} + au \right), \quad a, b \text{ constants}$$

which is associated with the equivalent system form of (1.2).

At this point the problem was to construct the theory of the inverse scattering problem for the operator L . This was made successfully by P. Deift, C. Tomei, and E. Trubowitz in [15]. Having developed the theory for L they could construct solutions for the equation (1.2), some global in time, others, blow-up in a finite time. They also showed, using the fact that the equation in (1.1) with $\psi(u) = u^2$ has Lax pair $iL_t = [Q, L]$, where L and Q are essentially as above, that there exists a family of global solutions for that equation. However, these solutions are not real. The difficulties in applying the inverse method to the “good” Boussinesq equation seem to be related to the fact that the “vanishing lemma” fails in this case (see [3]). Hence, from the inverse scattering method point of view, this “bad-good” notation seems to reverse.

In spite of the equation in (1.1) to have Lax pair and be linear stable, V. Kalantarov and O. Ladyzhenskaya proved in [21], that in the periodical case solutions may blow-up in a finite time. They suggested that the same blow-up result holds for the Cauchy problem. But an additional hypothesis was necessary to conclude that claim. This gap was filled by R. Sachs [42]. He showed that under appropriate assumptions on the data the techniques given by Levine in [29] were enough to prove the blow-up in the Cauchy problem case. This latter result deserves a few comments. R. Sachs established that for some finite $T^* > 0$

$$\lim_{t \rightarrow T^*} \int_{-\infty}^{\infty} \frac{|\widehat{u}(\xi, t)|^2}{|\xi|^2} d\xi = \infty. \quad (1.3)$$

where u is a solution of the equation (1.1) with $\psi(u) = u^2$. In the periodical case it is clear how the solution blows-up, since in this case we have (see [21])

$$\lim_{t \rightarrow T^*} \sum \frac{|\widehat{u}(k, t)|^2}{|k|^2} d\xi = \infty$$

implies $\lim_{t \rightarrow T^*} \|u(\cdot, t)\|_2 = \infty$, i.e. a loss of L^2 regularity of u . Consequently, the solution cannot be extended beyond the time T^* in the appropriate class.

But in case (1.3) it is not clear how the blow-up occurs. The integral in (1.3) could diverge for $|\xi| < 1$ or for $|\xi| \geq 1$. In the second case, we have $\lim_{t \rightarrow T^*} \|u(\cdot, t)\|_2 = \infty$, i.e. the solution blows-up and cannot be continued in the appropriate class beyond this time. On the other hand, if the integral for $|\xi| < 1$ diverges, we cannot conclude *a priori* that in a finite time the solution u loses regularity in L^2 . Observe that for appropriate data $\widehat{u}(0, t) = 0$, then it may occur that \widehat{u} does not have finite derivative in zero, i.e. around zero \widehat{u} behaves like $c|\xi|^\theta$ where $0 < \theta < 1/2$. Thus although (1.3) occurs the solution may be extended beyond the time T^* with the same H^s -regularity. In fact, the blow up in this case would affect the decay of the solution instead of its regularity.

We shall notice that the equation in (1.1) admits solitary-wave solutions. As it is known the existence of solitary-wave solutions shows the perfect balance between the dispersion and the nonlinearity of the equation in (1.1). In particular, for α integer, these solutions are given explicitly by

$$U_c(\xi) = A \operatorname{sech}^{2/\alpha-1}(B\xi) \quad (1.4)$$

where

$$A = A(c, \alpha) = \left\{ \frac{(\alpha + 1)(1 - c^2)}{2} \right\}^{1/\alpha-1}, \quad B = B(c, \alpha) = \frac{(1 - c^2)^{1/2}(\alpha - 1)}{2},$$

$\xi = x - ct$, and c is the wave speed satisfying $c^2 < 1$ (see [6]).

The IVP (1.1) can be written in the equivalent system form

$$\begin{cases} u_t = v_x & x \in \mathbb{R}, t > 0, \\ v_t = (u - u_{xx} - \psi(u))_x, \end{cases} \quad (1.5)$$

with

$$\begin{cases} u(x, 0) = u_0(x) \\ v(x, 0) = v_0(x) \end{cases} \quad (1.6)$$

where $\psi \in C^\infty(\mathbb{R})$, $\psi(0) = 0$.

Associated to the system (1.5)-(1.6) we have the Hamiltonian

$$E(u, v) = \frac{1}{2} \int_{-\infty}^{\infty} \{v^2(x, t) + u^2(x, t) + u_x^2(x, t) - 2\Psi(u(x, t))\} dx \quad (1.7)$$

where $\Psi' = \psi$ and $\Psi(0) = 0$. Notice that E is finite at initial time if the velocity is a x -derivative of a L^2 -function. This is the same restriction on the velocity that appears when the IVP (1.1) is written in the system form (1.5)-(1.6).

In these notes, we will present the local existence theory for the IVP (1.1) for data $(f, g) = (f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$ and $(f, g) = (f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, where $\dot{H}^{-1} = (-\Delta)^{-1/2}L^2$. Here h' denotes the derivative of h with respect to the variable x . We will also discuss the global theory established for data $f \in H^1(\mathbb{R})$ and $g = h' \in L^2(\mathbb{R})$ sufficiently small. In addition, we will consider some aspect regarding the asymptotic behavior of solutions to IVP (1.1).

To prove the local results we follow the ideas developed in the study of the semilinear Schrödinger equation. More precisely, consider the initial-value problem for the nonlinear Schrödinger equation

$$\begin{cases} u_t = i\Delta u + \lambda|u|^{\alpha-1}u & x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases} \quad (1.8)$$

$\alpha > 1, \lambda \in \mathbb{R}$. This equation appears in different problems related to physics (see [17], [57]). In [52], Y. Tsutsumi proved that the initial-value problem (1.8) is locally well posed in $L^2(\mathbb{R}^n)$ for any $\lambda \in \mathbb{R}$ and α satisfying $1 < \alpha < 1 + 4/n$, and due to the conservation law (see below) these solutions can be extended globally. The critical case, i.e. $\alpha = 1 + 4/n$ was studied by T. Cazenave and F. Weissler. In [11] they demonstrated the local well-posedness for the IVP (1.8). It is known that for this case $\alpha = 1 + 4/n$, the solution in L^2 cannot be in general extended globally (see [38]).

The theory in H^1 has been studied and developed by several authors, (see for example [23], [17], [12], [20] and for a complete set of references [9]). They showed that the initial-value problem (1.8) in this space is locally well posed for $\lambda \in \mathbb{R}$, and when α satisfies

$$\begin{cases} 1 < \alpha < (n+2)/(n-2) & n > 2 \\ 1 < \alpha < \infty & n = 1, 2. \end{cases}$$

Combining these results with the following ‘‘conservation laws’’

$$\begin{aligned} \|u(\cdot, t)\|_2 &= \|u_0\|_2 \\ \int \left(|\nabla u(x, t)|^2 + \frac{2\lambda}{\alpha+1} |u(x, t)|^{\alpha+1} \right) dx &= \|\nabla u_0\|_2^2 + \frac{2\lambda}{\alpha+1} \|u_0\|_{\alpha+1}^{\alpha+1} \end{aligned}$$

and suitable conditions on λ, α and the size of the data, they were able to extend the local solutions in H^1 to global ones.

The main tool in the proofs of the local results above, is the so called $L^p - L^q$ smoothing effect of Strichartz type [48] present in the Schrödinger equation.

It is known that for the linear Schrödinger equation

$$\begin{cases} u_t = i\Delta u & x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases}$$

the solutions are given by the unitary group $\{e^{it\Delta}\}_{-\infty}^{\infty}$, i.e.

$$e^{it\Delta}u_0(x) = (e^{-4\pi^2it|\xi|^2}\widehat{u}_0(\xi))^\vee$$

where \wedge and \vee denote Fourier and inverse Fourier transforms, respectively. In [48], R. Strichartz showed that

$$\left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |e^{it\Delta}u_0(x)|^{2(n+2)/n} dx dt \right)^{n/2(n+2)} \leq C\|u_0\|_2. \quad (1.9)$$

In particular, this implies that if $u_0 \in L^2(\mathbb{R}^n)$ the solution $u(x, t) = e^{it\Delta}u_0(x)$ belongs to $L^{2+4/n}$ for a.e. t . The proof of (1.9) was based on previous works of P. Thomas [50] and E. M. Stein [44] about restrictions of the Fourier transform.

This result has been generalized and its proof simplified in the works of B. Marshall [37], H. Pecher [40] and J. Ginibre and G. Velo [18]. In fact, for the one-dimensional Schrödinger equation one has that (see [18])

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta}u_0\|_{\infty}^4 dt \right)^{1/4} \leq C\|u_0\|_2.$$

In [26], C. E. Kenig, G. Ponce and L. Vega considered the extension of this result to higher order dispersive equation. In the particular case

$$\begin{cases} \partial_t u = iD^\alpha u & x \in \mathbb{R}, t \in \mathbb{R} \quad \text{and} \quad \alpha > 0, \\ u(x, 0) = u_0(x) \end{cases} \quad (1.10)$$

where $D = (-\partial_x)^{1/2}$, they showed that the solutions $u(t)$ satisfy

$$\left(\int_{-\infty}^{\infty} \|D_x^{(\alpha-2)/4} u(\cdot, t)\|_{\infty}^4 dx dt \right)^{1/4} \leq C \|u_0\|_2. \quad (1.11)$$

Note that for this case the ‘‘curvature’’ of the symbol $P(\xi) = |\xi|^\alpha$ (for $|\xi| > 1$) is grows for large α , which is reflected in the gain of derivatives. We will see that a similar result can be obtained for the linearized equation associated to (1.1).

Returning to the IVP (1.1), we shall consider its integral formulation which for strong solutions are basically equivalent. Our method of proof is based on linear estimates and a contraction mapping argument.

We begin by stating the IVP

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = 0 \\ u_t(x, 0) = h'(x) \end{cases} \quad (1.12)$$

the velocity is a x -derivative function. Then, the formal solution of (1.12) is given by

$$u(x, t) = V(t)h'(x) = (A(\xi)e^{-it|\xi|(1+\xi^2)^{1/2}} + B(\xi)e^{it|\xi|(1+\xi^2)^{1/2}})^\vee(x) \quad (1.13)$$

with

$$A(\xi) = \frac{\operatorname{sgn}(\xi) \widehat{h}(\xi)}{2i(1+\xi^2)^{1/2}} \quad \text{and} \quad B(\xi) = -\frac{\operatorname{sgn}(\xi) \widehat{h}(\xi)}{2i(1+\xi^2)^{1/2}}. \quad (1.14)$$

Using Duhamel’s principle, the solution of the IVP (1.1) with corresponding data $(f, g) = (0, \partial_x h)$ formally satisfies the integral equation

$$u(x, t) = V(t)h'(x) - \int_0^t V(t-\tau)(\psi(u))_{xx} d\tau. \quad (1.15)$$

As was commented above to apply the contraction mapping argument we use the estimates $L^p([0, T]; L^q(\mathbb{R}))$ for the inhomogeneous linear equation. More precisely, it will be proved that (see Theorem 2.5)

$$\left(\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{1/2} \widehat{w}(\xi) d\xi \right\|_{\infty}^4 dt \right)^{1/4} \leq C \|w\|_2 \quad (1.16)$$

where $\phi(\xi) = |\xi|(1+\xi^2)^{1/2}$. This estimate is similar to that for general $\phi(\cdot)$ obtained in [26] (section 2).

The result above allows us to prove that

$$\left(\int_0^T \|V(t)h'\|_{\infty}^4 dt \right)^{1/4} \leq c(1+T^{1/4}) \|h\|_{-1,2}$$

(see Lemma 2.8). In this estimate, the time dependence reflects the hyperbolic character of the linear equation associated to that in (1.1). This appears in the side of the Fourier transform when $|\xi| \leq 1$. Notice that the one-dimensional wave equation does not satisfy

L^p - L^q estimates of the type above described. When \hat{h} is supported away from zero we can obtain the following estimate

$$\left(\int_{-\infty}^{\infty} \|V(t)h'\|_{\infty}^4 dt \right)^{1/4} \leq c \|h\|_{-1,2}$$

where the smoothing effect produces the gain of two derivatives.

In addition to the smoothing effects of Strichartz type (L^p - L^q estimates), it is possible to show that solutions of the linear problem associated to (1.1) satisfy a smoothing effect of Kato type. This smoothing effect was proved by T. Kato in [24] for the Korteweg–de Vries (KdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u \partial_x u = 0 & x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.17)$$

He showed that the solution u of (1.17) satisfies

$$\int_{-T}^T \int_{-R}^R |\partial_x u(x, t)|^2 dx dt \leq C(T, R, \|u_0\|). \quad (1.18)$$

This result was extended by P. Constantin and J. Saut [13], P. Sjolín [43] and L. Vega [53] to linear general dispersive equations in \mathbb{R}^n . In [26] C. Kenig, G. Ponce and L. Vega obtained an improvement of this result in the one-dimensional case (see Theorem 4.1). More precisely, for the IVP

$$\begin{cases} \partial_t u - iP(D)u = 0 & x, t \in \mathbb{R} \\ u(x, 0) = u_0(x) \end{cases} \quad (1.19)$$

a solution u satisfies

$$\sup_x \int_{-\infty}^{\infty} |u(x, t)|^2 dt \leq c \int_{\Omega} \frac{|\hat{u}_0(\xi)|^2}{|\phi'(\xi)|} d\xi \quad (1.20)$$

with $\phi(\xi)$ the real symbol of P considered in a general class.

It is clear that from the group properties and the finite propagation speed a smoothing effect as the one described above cannot be satisfied for solutions of hyperbolic equations.

We will see that solutions of the nonlinear IVP (1.1) also satisfy (locally in time) an estimate similar to that in (1.20).

As was mentioned before, we shall use the contraction principle in the proofs of local well-posedness. One of the advantages of this approach is that it does not require any other theory (for example, Kato's abstract theory of quasilinear evolution equation [22]). In addition, in some cases it provides stronger results, for instance, it can be shown that the dependence of the solutions u on the data (f, g) i.e. the application $(f, g) \mapsto u$ is Lipschitz, instead of just continuous.

The plan of these notes is as follows. In Chapter 2, we will establish all the estimates involving solutions of the linear problem. First, we prove the global smoothing effect present in the Boussinesq equation which allows us to find some estimates for the linear equation

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = h'(x) \end{cases} \quad (1.21)$$

where h is a L^2 -function. As we can see in (1.14) this condition allows us to define $A(\xi)$ and $B(\xi)$ in $L^2(\mathbb{R})$. These estimates are the main ingredients to prove the results in the following sections. Here, we will also show that the solutions of the linear equation above satisfy a smoothing effect of Kato type.

Chapter 3 will be dedicated to study the local well-posedness for the initial-value problem (1.1). In the first section we will treat the IVP (1.1) with data $(f, g) = (f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$. For simplicity in the exposition we shall restrict ourselves to $\psi(u) = |u|^\alpha u$, $\alpha \in \mathbb{R}$. It is clear that for α integer $\psi(u) = u^k$, $k = \alpha + 1$ the same method applies. The argument of proof used a contraction mapping argument combined with the estimates established in section 2. This allows us to conclude the local well-posedness when $0 < \alpha < 4$ similar to the one-dimensional nonlinear Schrödinger equation. In addition, further regularity of the solution of the IVP (1.1) will be established by using the smoothing effect of Kato type commented above.

Then the local well-posedness for the initial-value problem (1.1) with data $(f, g) = (f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ will be established in the next section. Again the estimates in section 2 combined with a contraction mapping argument permit us to achieve this result. In this case we can prove local well-posedness for $\alpha > 0$ this is the same result as for the one-dimensional NLS. Also, we show that the smoothing effect of Kato type is satisfied by the strong solutions constructed here.

We will finish this chapter showing the local well-posedness of the IVP (1.1) in L^2 when $\alpha = 4$. Here we will follow the ideas of T. Cazenave and F. Weissler [11].

In Chapter 4, a global existence result is established for small data $(f, g) = (f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. The main tool used to prove this theorem is the conservation law

$$\|(-\Delta)^{-1/2} u_t\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2 - \frac{2}{\alpha + 1} \|u\|_{\alpha+1}^{\alpha+1} = K_0.$$

We also present some results related to persistence properties and decay of solutions of the IVP (1.1).

The asymptotic behavior of solutions of the IVP (1.1) will be studied in Chapter 5. We first show a result regarding the decay in time of solutions of (1.1). We consider the operator

$$W_\gamma(t)f(x) = \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/2} \hat{f}(\xi) d\xi. \quad (1.22)$$

and prove that

$$\|W_\gamma(t)f\|_p \leq c|t|^{-\gamma/2} \|f\|_{p'} \quad (1.23)$$

where $\gamma \in [0, 1]$ with $\gamma = \frac{1}{p'} - \frac{1}{p}$, $p = \frac{2}{1-\gamma}$ and $p' = \frac{2}{1+\gamma}$. Here $\phi(\xi) = |\xi|\sqrt{1+\xi^2}$. This estimate allows us to obtain similar estimates for solutions of the linear problem (1.21) under some additional hypotheses on the data. Using these estimates we deduce that

solutions of the nonlinear problem (1.1) satisfy

$$\|u(t)\|_p \leq C(1+t)^{-\gamma/2}, \quad t > 0. \quad (1.24)$$

Next we establish a nonlinear scattering result, that is, small solutions of the IVP (1.1) behaves asymptotically like solutions of the associated linear problem.

The result establishes that under some suitable conditions on the data (f, g) and the nonlinearity α , there exist unique solutions u_{\pm} of the linear problem associated to (1.1) such that

$$\|u(t) - u_{\pm}(t)\|_{1,2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (1.25)$$

where u is the solution of the IVP (1.1) with data (f, g) .

Similar results have been established for solutions of the nonlinear Schrödinger equation, nonlinear wave equation, Klein-Gordon equation and Korteweg-de Vries equation by Strauss [47], Pecher [40] and Ponce and Vega [41], respectively.

The main ingredients to obtain this result are the decay estimates for solutions of the linear problem and L^p - L^q estimates.

Affirmative results on scattering for small solutions are interpreted as the nonexistence of solitary-wave solutions of arbitrary small amplitude. In this case, we notice that a simple calculation shows that the solitary-wave solutions $U_c(\xi)$ in (1.4) satisfy $\|U_c(\cdot)\|_2 > \epsilon > 0$ for $\alpha > 5$. The result presented here is optimal in this sense.

Finally, we will describe a blow-up result regarding solutions to the IVP (1.1). As we commented above some solutions of the IVP (1.1) might blow-up in finite time [21], [42].

The result we discuss in detail is due to Angulo and Scialom [2] were general conditions are given to show blow-up in finite time for solutions to the IVP (1.1) (see also [35]). The proof uses a result due to Levine [29] to deduce the blow-up. To prepare the setting to apply this theorem one has to study the relationship between the solitary waves solutions and the blow-up phenomena.

We consider the nonlinearity $\varphi(u) = |u|^{\alpha-1}u$ and write the equation in (1.1) in its equivalent system form (1.5). As we commented above, the flow of (1.5) leaves invariant the “energy”,

$$E(u, v) = \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\|v\|^2 - \frac{1}{\alpha+1}\|u\|_{\alpha+1}^{\alpha+1} \quad (1.26)$$

and it is also true for the quantity

$$Q(u, v) = \int_{\mathbb{R}} uv \, dx. \quad (1.27)$$

These two quantities are essential for the analysis below.

The first step to put forward the blow-up theory is to determine the best constant B_{α}^c for the Sobolev inequality

$$\|u\|_{L^{\alpha+1}} \leq B_{\alpha}^c \|u\|_{1,c} \quad (1.28)$$

where $\|\cdot\|_{1,c} = (1-c^2)\|\cdot\|_{H^1}$. This constant is obtained as the minimum of a constrained variational problem and it is given in terms of the corresponding solitary wave solution. To find this minimum one can use the concentration-compactness method (see Lions [34]). We will apply a simplified version of this method due to Lopes [36]. Then we use that the region

$$K_2^c = \{u \in H^1(\mathbb{R}) : L_c(u, -cu) < d(c), R_c(u) < 0\} \quad (1.29)$$

is invariant for the flow (1.5) (see [35] and [39]), where $R_c(u) = \|u\|_{1,c}^2 - \|u\|_{\alpha+1}^{\alpha+1}$, $L_c(u, v) = E(u, v) + cQ(u, v)$ and $d(c) = L_c(\phi_c, -c\phi_c)$. These are the main ingredients to establish the blow-up result given in Theorem 5.6.

CHAPTER 2

Linear Problem

The first part of this chapter is concerned with the smoothing effects called L^p - L^q estimates or Strichartz estimates of solutions of the linear equation associated to that in (1.1). That is,

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x). \end{cases} \quad (2.1)$$

These estimates will be the main ingredient in the proof of local well-posedness of the IVP (1.1).

In the second section, we will prove a smoothing effect (locally in time) of Kato type for solutions the linear problem (2.1). This property for solutions of the linear equation will be used to show the local existence of a stronger solution for the IVP (1.1).

Finally, in the last section we will establish a series of estimates for solutions of the linear problem (2.1) that will allow us to obtain some decay properties and some stronger solutions of the nonlinear problem.

2.1. L^p - L^q estimates

Let $\phi(\xi) = |\xi|(1 + \xi^2)^{\frac{1}{2}}$, then $\phi(\xi)$ is an even C^∞ function in $\mathbb{R} - \{0\}$ with

$$\phi'(\xi) = \operatorname{sgn}(\xi) \frac{1 + 2\xi^2}{(1 + \xi^2)^{1/2}} \quad \text{and} \quad \phi''(\xi) = \operatorname{sgn}(\xi) \frac{\xi(3 + 2\xi^2)}{(1 + \xi^2)^{3/2}}. \quad (2.2)$$

The function $\phi''(\xi)$ has the following properties:

- (i) for $\xi \neq 0$, $\phi''(\xi) \neq 0$.
- (ii) for $|\xi| < \epsilon$, $0 < \epsilon \ll 1$,

$$|\xi| \leq |\phi''(\xi)| \leq 3|\xi|, \quad (2.3)$$

- (iii) for $|\xi| > \frac{1}{\epsilon}$

$$1 \leq |\phi''(\xi)| \leq 2, \quad (2.4)$$

- (iv) $\phi''(\xi)$ has only one change of monotonicity.

For $\phi(\xi)$, define

$$I(x, t) = \int_{-\infty}^{\infty} e^{i(t\phi(\xi) + x\xi)} |\phi''(\xi)|^{1/2 + i\beta} d\xi, \quad x, t \in \mathbb{R}.$$

The following lemma is essential in the proof of the next theorems.

LEMMA 2.1. *There exists $C > 0$ such that for any $x, t \in \mathbb{R}$ we have*

$$|I(x, t)| \leq C(1 + \beta)|t|^{-1/2}, \quad x, t \in \mathbb{R}. \quad (2.5)$$

REMARK 2.2. The result for a general class of functions ϕ was proved by Kenig, Ponce and Vega in [26] (see Lemma 2.7 in [26]).

To prove Lemma 2.1 we will use of the following result.

LEMMA 2.3 (Van der Corput). *Let $\psi \in C_0^\infty(\mathbb{R})$ and $\varphi \in C^2(\mathbb{R})$ satisfy that $\varphi''(\xi) > \lambda > 0$ on the support of ψ . Then*

$$\left| \int e^{i\varphi(\xi)} \psi(\xi) d\xi \right| \leq 10\lambda^{-1/2} \{ \|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \}. \quad (2.6)$$

PROOF. See [44]. □

PROOF OF LEMMA 2.1.

Case 1. We first consider the following situation:

$0 < m \leq |\phi''(\xi)| \leq M$ and Ω bounded and

$$I(x, t) = \int_{\Omega} e^{i(t\phi(\xi) + x\xi)} |\phi''(\xi)|^{1/2+i\beta} d\xi.$$

Lemma 2.3 implies then that

$$\begin{aligned} |I(x, t)| &\leq c(m|t|)^{-1/2} \left\{ M + \int_{\Omega} \left| \frac{1}{2} + i\beta |\phi''(\xi)|^{-1/2} |\phi'''(\xi)| \right| d\xi \right\} \\ &\leq c_{\phi}(1 + |\beta|)|t|^{-1/2}. \end{aligned} \quad (2.7)$$

Case 2: $1 \leq |\phi''(\xi)| \leq 2$, $\Omega = \{\xi \in \mathbb{R} : |\xi| > 1/\epsilon\}$.

$$\begin{aligned} &\left| \int_{|\xi| > 1/\epsilon} e^{it\phi(\xi) + ix\xi} |\phi''(\xi)|^{1/2+i\beta} d\xi \right| \\ &\leq \left| \frac{1}{i(t\phi'(\xi) + x)} e^{it\phi(\xi) + ix\xi} |\phi''(\xi)|^{1/2+i\beta} \right|_{|\xi| > 1/\epsilon} \\ &+ \left| \int_{|\xi| > 1/\epsilon} e^{it\phi(\xi) + ix\xi} \frac{d}{d\xi} \left(\frac{|\phi''(\xi)|^{1/2+i\beta}}{i(t\phi'(\xi) + x)} \right) d\xi \right| \\ &\leq c \int_{|\xi| > 1/\epsilon} |\phi''(\xi)|^{-1/2} \frac{|\phi'''(\xi)|}{|t\phi'(\xi) + x|} + \frac{t|\phi''(\xi)|^{3/2}}{|t\phi'(\xi) + x|^2} d\xi \\ &\leq c \int_{|\xi| > 1/\epsilon} \{ |\xi|^{-6} + |\xi|^{-2} \} d\xi < \infty. \end{aligned} \quad (2.8)$$

Here we have used that $|\phi'(\xi)| = \infty$ as $|\xi| \rightarrow \infty$. This shows that the integral is convergent. Therefore we can write the set Ω as the union of bounded intervals and apply the argument in Case 1.

Case 3: $0 \leq |\phi''(\xi)| \leq 2$, $\Omega = \{\xi \in \mathbb{R} : |\xi| \leq \epsilon\}$.

We consider

$$\tilde{\phi}(\xi) = \begin{cases} \phi(\xi) - \xi, & \xi \geq 0, \\ \phi(\xi) + \xi, & \xi < 0. \end{cases}$$

Thus $\tilde{\phi}(0) = 0$ and $\tilde{\phi}''(0) = \phi''(0) = 0$. Then we have that there exist constants c_1, C_1, c_2, C_2 such that

$$\begin{aligned} c_1 |\xi|^2 &\leq |\tilde{\phi}'(\xi)| \leq C_1 |\xi|^2 \\ c_2 |\xi| &\leq |\tilde{\phi}''(\xi)| \leq C_2 |\xi| \end{aligned} \quad (2.9)$$

for $|\xi| < \epsilon$.

To simplify we use again ϕ instead of $\tilde{\phi}$ and consider

$$\int_{|\xi| \leq \epsilon} e^{it\phi(\xi) + ix\xi} |\phi''(\xi)|^{1/2+i\beta} d\xi \quad (2.10)$$

Define

$$\begin{aligned} \Omega_1 &= \{\xi \in \Omega : |\xi| \leq \min(\epsilon, |t|^{-1/3})\} \\ \Omega_2 &= \{\xi \in \Omega : |\xi| \leq \epsilon \text{ and } |\phi'(\xi) - \frac{x}{t}| \leq \frac{1}{2} \left| \frac{x}{t} \right|\} \\ \Omega_3 &= \{\xi \in \Omega - \Omega_1 \cap \Omega_2 : |\xi| \leq \epsilon\} \end{aligned}$$

In Ω_1 we have that

$$\int_{\Omega_1} |\phi''(\xi)|^{1/2} d\xi \leq c |t|^{-1/2} \quad (2.11)$$

If $\xi \in \Omega_2$, $|\xi|^2 \sim |\phi'(\xi)| \sim |x/t|$. Hence $|\xi| \sim |x/t|^{1/2}$. By Lemma (2.3) we obtain the following chain of inequalities

$$\begin{aligned} &\left| \int_{\Omega_2} e^{i(t\phi(\xi) + x\xi)} |\phi''(\xi)|^{1/2+i\beta} d\xi \right| \\ &\leq c \left(\min_{\xi \in \Omega_2} |\phi''(\xi)| |t| \right)^{-1/2} \\ &\quad \times \left\{ \max_{\xi \in \Omega_2} |\phi''(\xi)|^{1/2} + (1 + |\beta|) \int_{\Omega_2} |\phi''(\xi)|^{-1/2} |\phi'''(\xi)| d\xi \right\} \\ &\leq c (1 + |\beta|) \min_{\xi \in \Omega_2} |\phi''(\xi)|^{-1/2} \max_{\xi \in \Omega_2} |\phi''(\xi)|^{1/2} |t|^{-1/2}. \end{aligned} \quad (2.12)$$

If $\xi \in \Omega_3$ then $|\phi'(\xi) - x/t| \leq c|\phi'(\xi)| \geq c|\xi|^2$ and $|\xi| > |t|^{-1/3}$. Integration by parts yields the results. In fact,

$$\begin{aligned}
& \left| \int_{\Omega_3} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{1/2+i\beta} d\xi \right| \\
& \leq \frac{c}{|t|} \int_{\Omega_3} \left\{ \left(\frac{1}{2} + |\beta| \right) \frac{|\phi''(\xi)|^{-1/2} |\phi'''(\xi)|}{|\phi'(\xi) - x/t|} + \frac{|\phi''(\xi)|^{3/2}}{|\phi'(\xi) - x/t|^2} \right\} d\xi \\
& \leq \frac{c}{|t|} (1 + |\beta|) \int_{|\xi| > |t|^{-1/3}} \frac{|\phi'''(\xi)|}{|\xi|^{3/2}} d\xi + \frac{c}{|t|} \int_{|\xi| > |t|^{-1/3}} |\xi|^{-5/2} d\xi \\
& \leq c(1 + |\beta|) |t|^{-1/2}.
\end{aligned} \tag{2.13}$$

This completes the proof of the lemma. \square

For ϕ define

$$W_\gamma(t)f(x) = \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/2} \hat{f}(\xi) d\xi. \tag{2.14}$$

THEOREM 2.4. *Let ϕ and W_γ be defined as before, we have*

$$\|W_\gamma(t)f\|_p \leq c|t|^{-\gamma/2} \|f\|_{p'} \tag{2.15}$$

where $\gamma \in [0, 1]$ with $\gamma = \frac{1}{p'} - \frac{1}{p}$, $p = \frac{2}{1-\gamma}$ and $p' = \frac{2}{1+\gamma}$.

PROOF. Consider the operator

$$W_{\gamma+i\beta}(t)f(x) = \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/2+i\beta} \hat{f}(\xi) d\xi.$$

Observe that $W_{1+i\beta}(t)f(x) = I(\cdot, t) * f(x)$ and

$$T_{0+i\beta}(t)f(x) = (e^{it\phi(\xi)} |\phi''(\xi)|^{i\beta} \hat{f}(\xi))^\vee.$$

Therefore the Young inequality and Lemma 2.1 imply that

$$\|W_{1+i\beta}(t)f\|_{L^\infty} \leq c|t|^{-1/2} \|f\|_{L^1}.$$

On the other hand, Plancharel's theorem (A.9) gives

$$\|W_{i\beta}(t)f\|_{L^2} = \|f\|_{L^2}.$$

Therefore the Stein interpolation theorem (see Appendix Theorem A.14) yields the result. \square

THEOREM 2.5. *If $W_\gamma(t)$ is defined as in (2.14) and $\gamma \in [0, 1]$ then*

$$\left(\int_{-\infty}^{\infty} \|W_{\gamma/2}(t)f\|_p^q dt \right)^{1/q} \leq c \|f\|_2, \tag{2.16}$$

$$\left\| \int_0^t W_\gamma(t-\tau)g(\cdot, \tau) d\tau \right\|_{L_t^q(\mathbb{R}; L^p)} \leq c \|g\|_{L_t^{q'}(\mathbb{R}; L^{p'})}, \quad (2.17)$$

and

$$\left\| \int_{-\infty}^{\infty} W_{\gamma/2}(-\tau)g(\cdot, \tau) d\tau \right\|_{L_x^2} \leq c \|g\|_{L_t^{q'}(\mathbb{R}; L^{p'})}, \quad (2.18)$$

where $q = 4/\gamma$, $p = 2/(1-\gamma)$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

PROOF. We first show that the inequalities (2.16), (2.17) and (2.18) are equivalent. We will use a duality argument. We recall that

$$\left(\int_{-\infty}^{\infty} \|h(\cdot, t)\|_{L_x^p}^q dt \right)^{1/q} = \sup \left\{ \int_{\mathbb{R}^2} h(x, t)w(x, t) dx dt : \|w\|_{L_t^{q'}(\mathbb{R}; L_x^{p'})} = 1 \right\} \quad (2.19)$$

Using the definition (2.14), Parseval's identity and Fubini's theorem we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\gamma/2}(t)f(x)g(x, t) dx dt = \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} W_{\gamma/2}(t)g(x, t) dt \right) dx \quad (2.20)$$

From (2.19), (2.20) and the Cauchy-Schwarz inequality we obtain

$$\left(\int_{-\infty}^{\infty} \|W_{\gamma/2}(t)f\|_p^q dt \right)^{1/q} \leq c \|f\|_2 \left\| \int_{-\infty}^{\infty} W_{\gamma/2}(t)g(\cdot, t) dt \right\|_2. \quad (2.21)$$

On the other hand, the argument of Stein-Tomas ([50]) and Hölder's inequality give

$$\begin{aligned} & \left\| \int_{-\infty}^{\infty} W_{\gamma/2}(t)g(x, t) dt \right\|_2^2 = \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} W_{\gamma/2}(t)g(x, t) dt \right) \overline{\left(\int_{-\infty}^{\infty} W_{\gamma/2}(t')g(x, t') dt' \right)} dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} W_\gamma(t-t')\overline{g(x, t')} dt' \right) g(x, t) dx dt \\ &\leq \left\| \int_{-\infty}^{\infty} W_\gamma(\cdot - t')g(\cdot, t') dt' \right\|_{L_t^q(\mathbb{R}; L^p)} \|g\|_{L_t^{q'}(\mathbb{R}; L^{p'})}. \end{aligned} \quad (2.22)$$

The equivalence of the inequalities now follows from (2.21) and (2.22). Thus it is enough to prove inequality (2.17) to prove the theorem.

The Minkowskii inequality and Theorem 2.4 yield

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} W_{\gamma}(t-t')g(\cdot, t') dt' \right\|_{L_x^p} &\leq \int_{-\infty}^{\infty} \|W_{\gamma}(t-t')g(\cdot, t')\|_{L_x^p} dt' \\ &\leq c \int_{-\infty}^{\infty} \frac{1}{|t-t'|^{\gamma/2}} \|g(\cdot, t')\|_{L_x^{p'}} dt'. \end{aligned} \quad (2.23)$$

The Hardy-Littlewood Sobolev theorem (see Appendix Theorem A.16) implies that

$$\left\| \int_{-\infty}^{\infty} W_{\gamma}(\cdot-t')g(\cdot, t') dt' \right\|_{L_t^q(\mathbb{R}; L_x^p)} \leq c \|g\|_{L_t^{q'}(\mathbb{R}; L_x^{p'})}. \quad (2.24)$$

Hence the proof of the theorem now is complete. \square

REMARK 2.6. A more general result was established by Kenig, Ponce and Vega in [26]. See Theorem 2.1.

The following estimates are concerned with the linear equation associated with that in (1.1). Here we shall deduce regularity results as a consequence of Theorem 2.5.

LEMMA 2.7. *Consider the following IVP*

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = 0. \end{cases} \quad (2.25)$$

If

$$V_1(t)f(x) \equiv \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi) d\xi \quad (2.26)$$

with $\phi(\xi) = |\xi|(1+\xi^2)^{1/2}$, then

$$\|V_1(t)f\|_2 \leq C\|f\|_2 \quad (2.27)$$

and

$$\left(\int_0^T \|V_1(t)f\|_{\infty}^4 dt \right)^{1/4} \leq c(1+T^{1/4})\|f\|_2. \quad (2.28)$$

PROOF. The proof of (2.27) is immediate.

To show (2.28) we proceed in the following way:

Let $\chi \in C_0^{\infty}(\mathbb{R})$, $\chi \equiv 1$ on $[-1, 1]$ and support of $\chi \subset [-2, 2]$

$$\begin{aligned} V_1(t)f(x) &= \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi)\chi(\xi) d\xi + \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi)(1-\chi(\xi)) d\xi \\ &= V_1^1(t)f(x) + V_1^2(t)f(x). \end{aligned} \quad (2.29)$$

We can write $V_1^2(t)f(x)$ as

$$\int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{1/2} \frac{\hat{f}(\xi)(1-\chi(\xi))}{|\phi''(\xi)|^{1/2}} d\xi.$$

Now using (2.16) in Theorem 2.5 and (2.4) we have

$$\left(\int_{-\infty}^{\infty} \|V_1^2(t)f\|_{\infty}^4 dt \right)^{1/4} \leq c \left\| \frac{\hat{f}(\xi)(1-\chi(\xi))}{|\phi''(\xi)|^{1/2}} \right\|_2 \leq c \|f\|_2. \quad (2.30)$$

For $V_1^1(t)f(x)$ we have that

$$\begin{aligned} \|V_1^1(t)f\|_{\infty} &= \left\| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi)\chi(\xi) d\xi \right\|_{\infty} \\ &\leq c \left\| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi)\chi(\xi) d\xi \right\|_{1,2} \\ &\leq c \|\hat{f}\chi\|_{1,2} \leq c (\|f * \check{\chi}\|_2 + \|\partial_x(f * \check{\chi})\|_2) \\ &\leq c_{\chi} \|f\|_2 \end{aligned}$$

integrating from 0 to T it follows that

$$\left(\int_0^T \|V_1^1(t)f\|_{\infty}^4 dt \right)^{1/4} \leq c_{\chi} T^{1/4} \|f\|_2. \quad (2.31)$$

A combination of (2.30) and (2.31) yields the result. \square

LEMMA 2.8. *Consider the IVP (2.25) with data*

$$u(x, 0) = 0, \quad u_t(x, 0) = h'(x).$$

If

$$V_2(t)h'(x) \equiv \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \frac{\operatorname{sgn}(\xi) \hat{h}(\xi)}{(1+\xi^2)^{1/2}} d\xi \quad (2.32)$$

then

$$\|V_2(t)h'\|_2 \leq c \|h\|_{-1,2} \quad (2.33)$$

and

$$\left(\int_0^T \|V_2(t)h'\|_{\infty}^4 dt \right)^{1/4} \leq c(1+T^{1/4}) \|h\|_{-1,2}. \quad (2.34)$$

PROOF. The proof of (2.33) is immediate.

To prove (2.34), we follow a similar argument as in the previous lemma.

Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ on $[-1, 1]$ and support of $\chi \subset [-2, 2]$

$$\begin{aligned} V_2(t)h'(x) &= \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)} \chi(\xi) \frac{\operatorname{sgn}(\xi) \widehat{h}(\xi)}{(1+\xi^2)^{1/2}} d\xi \\ &\quad + \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)} (1-\chi(\xi)) \frac{\operatorname{sgn}(\xi) \widehat{h}(\xi)}{(1+\xi^2)^{1/2}} d\xi \\ &= V_2^1(t)h'(x) + V_2^2(t)h'(x). \end{aligned}$$

Now we can write $V_2^2(t)h'(x)$ as

$$\int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{1/2} \frac{\operatorname{sgn}(\xi) \widehat{h}(\xi) (1-\chi(\xi))}{|\phi''(\xi)|^{1/2} (1+\xi^2)^{1/2}} d\xi$$

making use of (2.16) in Theorem 2.5 and (2.4), it follows

$$\left(\int_{-\infty}^{\infty} \|V_2^2(t)\partial_x h\|_{\infty}^4 dt \right)^{1/4} \leq c \left\| \frac{\widehat{h}(\xi)(1-\chi(\xi))}{|\phi''(\xi)|^{1/2}(1+\xi^2)^{1/2}} \right\|_2 \leq c \|h\|_{-1,2}. \quad (2.35)$$

For $V_2^1(t)h'(x)$ we have that

$$\begin{aligned} \|V_2^1(t)h'\|_{\infty} &= \left\| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \chi(\xi) \frac{\operatorname{sgn}(\xi) \widehat{h}(\xi)}{(1+\xi^2)^{1/2}} d\xi \right\|_{\infty} \\ &\leq c \left\| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \chi(\xi) \frac{\widehat{h}(\xi)}{(1+\xi^2)^{1/2}} d\xi \right\|_{1,2} \\ &\leq c \left(\left\| \left(\frac{\widehat{h}(\xi)}{(1+\xi^2)^{1/2}} \right)^{\vee} * \check{\chi} \right\|_2 + \left\| \left(\frac{\widehat{h}(\xi)}{(1+\xi^2)^{1/2}} \right)^{\vee} * \partial_x \check{\chi} \right\|_2 \right) \\ &\leq c_{\chi} \left\| \left(\frac{\widehat{h}(\xi)}{(1+\xi^2)^{1/2}} \right)^{\vee} \right\|_2 = c_{\chi} \|h\|_{-1,2}. \end{aligned}$$

Now integrating from 0 to T it follows that

$$\left(\int_0^T \|V_2^1(t)h'\|_{\infty}^4 dt \right)^{1/4} \leq C_{\chi} T^{1/4} \|h\|_{-1,2} \quad (2.36)$$

which combined with (2.35) yields the result. \square

LEMMA 2.9. Consider the IVP (2.25) now with data

$$u(x, 0) = 0, \quad u_t(x, 0) = p''(x).$$

If

$$V_2(t)p''(x) \equiv \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} \frac{\operatorname{sgn}(\xi)\xi\widehat{p}(\xi)}{(1+\xi^2)^{1/2}} d\xi$$

then

$$\|V_2(t)p''\|_2 \leq C\|p\|_2 \quad (2.37)$$

and

$$\left(\int_{-\infty}^{\infty} \|V_2(t)p''\|_{\infty}^4 dt \right)^{1/4} \leq c\|p\|_2. \quad (2.38)$$

PROOF. The estimate (2.37) follows directly. To obtain (2.38), we write $V_2(t)p''(x)$ as

$$\int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{1/2} \frac{\operatorname{sgn}(\xi)\xi\widehat{p}(\xi)}{|\phi''(\xi)|^{1/2}(1+\xi^2)^{1/2}} d\xi.$$

The estimate (2.16) in Theorem 2.5 and (2.4) give

$$\left(\int_{-\infty}^{\infty} \|V_2(t)p''\|_{\infty}^4 dt \right)^{1/4} \leq c \left\| \frac{\operatorname{sgn}(\xi)\xi\widehat{p}(\xi)}{|\phi''(\xi)|^{1/2}(1+\xi^2)^{1/2}} \right\|_2 \leq c\|p\|_2.$$

□

2.2. Local Smoothing Effects

The following result will allow us to obtain smoothing effects of Kato type present in solutions of the linear problem (A.12). This result is a particular case of the sharp smoothing effect of Kato type proved in [26] for solutions to one-dimensional linear dispersive equations (see Theorem 4.1 in [26]).

THEOREM 2.10. *Let $\phi(\xi) = |\xi|\sqrt{1+\xi^2}$ and $f \in \mathcal{S}(\mathbb{R}^n)$ define*

$$W(t)f(x) = \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)} \widehat{f}(\xi) d\xi. \quad (2.39)$$

Then

$$\sup_x \int_{\mathbb{R}} |W(t)f(x)|^2 dt \leq C \int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{|\phi'(\xi)|} d\xi. \quad (2.40)$$

PROOF. From (2.2) we have that $\phi'(\xi) \neq 0$, for every $x \in \mathbb{R}$. Then we write

$$\begin{aligned} \int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)} \widehat{f}(\xi) d\xi &= \int_{\xi < 0} e^{i(t\phi(\xi)+x\xi)} \widehat{f}(\xi) d\xi + \int_{\xi \geq 0} e^{i(t\phi(\xi)+x\xi)} \widehat{f}(\xi) d\xi \\ &= I_1 + I_2. \end{aligned} \quad (2.41)$$

Since $\phi'(\xi) \neq 0$, there exists ψ such that $\phi(\psi(\xi)) = \xi$, $\xi \geq 0$. Then making the change of variables $\eta = \psi(\xi)$ we have that

$$I_1 = \int e^{it\eta} e^{ix\psi(\eta)} \hat{f}(\psi(\eta)) \frac{d\eta}{\phi'(\psi(\eta))}. \quad (2.42)$$

where $\tilde{f}(x) = f(x)$, for $x < 0$ and equals 0 otherwise.

Taking the L^2 -norm of I_1 in t , using Plancherel's identity and returning to the previous variables we have that

$$\begin{aligned} \|I_1\|_{L_t^2}^2 &= c \int_{-\infty}^{\infty} \frac{|e^{ix\psi(\eta)} \hat{f}(\psi(\eta))|^2}{|\phi'(\psi(\eta))|^2} d\eta \\ &= c \int_{-\infty}^{\infty} \frac{|\hat{f}(\psi(\eta))|^2}{|\phi'(\psi(\eta))|^2} d\eta \\ &= c \int \frac{|\hat{f}(\xi)|^2}{|\phi'(\xi)|} d\xi. \end{aligned} \quad (2.43)$$

A similar argument can be used to estimate I_2 . Hence the result follows. \square

PROPOSITION 2.11. Let $V_1(t)$ and $V_2(t)$ be defined as in the statements of Lemmas 2.4 and 2.5, then:

For $f \in L^2$

$$\sup_x \left(\int_0^T |D_x^{1/2} V_1(t) f(x)|^2 dt \right)^{1/2} \leq c(1 + T^{1/2}) \|f\|_2. \quad (2.44)$$

If $h' \in \dot{H}^{-1}$

$$\sup_x \left(\int_0^T |D_x^{1/2} V_2(t) h'(x)|^2 dt \right)^{1/2} \leq c(1 + T^{1/2}) \|h\|_{-1,2}. \quad (2.45)$$

And for $p \in L^2$

$$\sup_x \left(\int_0^T |D_x^{1/2} V_2(t) p''(x)|^2 dt \right)^{1/2} \leq c(1 + T^{1/2}) \|p\|_2. \quad (2.46)$$

PROOF. To prove (2.44), (2.45) and (2.46) we shall use the same argument, so we only will sketch the proof of (2.44).

Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ on $[-1, 1]$ and support of $\chi \subset [-2, 2]$ then

$$\begin{aligned} & \sup_x \left(\int_0^T |D_x^{1/2} V_1(t) f(x)|^2 dt \right)^{1/2} \\ & \leq c \sup_x \left(\int_0^T \left| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\xi|^{1/2} \hat{f}(\xi) \chi(\xi) d\xi \right|^2 dt \right)^{1/2} \\ & \quad + c \sup_x \left(\int_0^T \left| \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\xi|^{1/2} \hat{f}(\xi) (1 - \chi(\xi)) d\xi \right|^2 dt \right)^{1/2}. \end{aligned} \quad (2.47)$$

Therefore a direct application of the Theorem 2.8 gives the estimate for the second expression on the right hand side. For the first expression on the right hand side of the inequality above, we use the smoothness of χ and f to make a crude estimate which we have to pay with the dependence on T . \square

2.3. Further Linear Estimates

We begin this section by establishing the needed estimates for the operator $(-\Delta)^{-1/2} \partial_t$. These estimates will be used to prove stronger solutions of the nonlinear problem (1.1) and the global results in Chapter 5.

PROPOSITION 2.12. If

$$(-\Delta)^{-1/2} \partial_t V_1(t) f(x) = \int_{-\infty}^{\infty} i|\xi|^{-1} \phi(\xi) e^{i(t\phi(\xi)+x\xi)} \hat{f}(\xi) d\xi,$$

$$(-\Delta)^{-1/2} \partial_t V_2(t) h'(x) = \int_{-\infty}^{\infty} i|\xi|^{-1} \phi(\xi) e^{i(t\phi(\xi)+x\xi)} \frac{\text{sgn}(\xi) \hat{h}(\xi)}{(1 + \xi^2)^{1/2}} d\xi$$

and

$$(-\Delta)^{-1/2} \partial_t V_2(t) p''(x) = \int_{-\infty}^{\infty} i|\xi|^{-1} \phi(\xi) e^{i(t\phi(\xi)+x\xi)} \frac{\xi \hat{p}(\xi)}{(1 + \xi^2)^{1/2}} d\xi.$$

Then

$$\|(-\Delta)^{-1/2} \partial_t V_1(t) f\|_2 \leq C \|f\|_{1,2}, \quad (2.48)$$

$$\|(-\Delta)^{-1/2} \partial_t V_2(t) h'\|_2 \leq C \|h\|_2 \quad (2.49)$$

and

$$\|(-\Delta)^{-1/2} \partial_t V_2(t) p''\|_2 \leq C \|p\|_{1,2}. \quad (2.50)$$

PROOF. Reminding that $\phi(\xi) = |\xi|(1 + \xi^2)^{1/2}$, Plancherel's theorem gives

$$\begin{aligned} \|(-\Delta)^{-1/2} \partial_t V_1(t) f\|_2 & \leq c \| |\xi|^{-1} \phi(\xi) \hat{f}(\xi) \|_2 \\ & \leq c \|(1 + \xi^2)^{1/2} \hat{f}\|_2 = c \|f\|_{1,2} \end{aligned} \quad (2.51)$$

which is (2.48).

To prove (2.49), we use the same argument as before to obtain

$$\|(-\Delta)^{-1/2}\partial_t V_2(t)h'\|_2 \leq c \left\| |\xi|^{-1}\phi(\xi) \frac{\operatorname{sgn}(\xi)\widehat{h}(\xi)}{(1+\xi^2)^{1/2}} \right\|_2 \leq c \|h\|_2. \quad (2.52)$$

In the same way we can show (2.50)

$$\|(-\Delta)^{-1/2}\partial_t V_2(t)p''\|_2 \leq c \left\| |\xi|^{-1}\phi(\xi) \frac{\operatorname{sgn}(\xi)\xi\widehat{p}(\xi)}{(1+\xi^2)^{1/2}} \right\|_2 = C \|p\|_{1,2}. \quad (2.53)$$

□

Next we will prove linear estimates useful in the proofs of Theorems 5.1 and 5.4 regarding decay and nonlinear scattering for solutions of (1.1).

PROPOSITION 2.13.

(i) Let $g = h' \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$. Then

$$\|V_2(t)g\|_p \leq c(1+t)^{-\gamma/2} (\|h\|_2 + \|h\|_q),$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{2}{1+2\gamma}$ and $\gamma \in (0, 1/2)$.

(ii) Let $f \in H^1(\mathbb{R}) \cap L^q_{\frac{1}{2}}(\mathbb{R})$.

Then

$$\|V_1(t)f\|_p \leq c(1+t)^{-\gamma/2} (\|f\|_{1,2} + \|f\|_{\frac{2}{2},q}),$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{2}{1+2\gamma}$ and $\gamma \in (0, 1/2)$.

PROOF. To show (i) we use the Sobolev embedding theorem for t small to obtain

$$\|V_2(t)g\|_p \leq \|V_2(t)g\|_{1,2} \leq \|h\|_2, \quad (2.54)$$

where p is as above.

For $t > \epsilon > 0$, we write

$$V_2(t)g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/2} \frac{\widehat{h}(\xi)}{|\phi''(\xi)|^{\gamma/2}(1+\xi^2)^{1/2}} d\xi,$$

where $\phi''(\xi) = \frac{|\xi|(2\xi^2+3)}{(1+\xi^2)^{3/2}}$.

Since $\phi(\xi)$ satisfies the conditions in Theorem 2.4 (see [31], [26]), we have that

$$\|V_2(t)g\|_p \leq ct^{-\gamma/2} \|G\|_{\frac{2}{1+\gamma}},$$

here $G(x) = \left(\frac{\widehat{h}(\xi)}{|\xi|^{\gamma/2}}\right)^\vee(x)$.

Applying Hardy-Littlewood-Sobolev theorem (see appendix Theorem A.16) we find that

$$\|G\|_{\frac{2}{1+\gamma}} \leq c \|h\|_q,$$

for $q = \frac{2}{1+2\gamma}$ and $\gamma \in (0, 1/2)$. Thus

$$\|V_2(t)g\|_p \leq ct^{-\gamma/2} \|h\|_q. \quad (2.55)$$

Therefore combining (2.3) and (2.4) we obtain the desired result.

To prove (ii) we can use a similar argument, hence it will be omitted. \square

COROLLARY 2.14. Under the assumptions on f and g in Proposition 2.13. The solution u of the linear problem (2.1) satisfies

$$\|u\|_p \leq c(1+t)^{-\gamma/2} (\|f\|_{1,2} + \|f\|_{\frac{\gamma}{2},q} + \|h\|_2 + \|h\|_q),$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{2}{1+2\gamma}$ and $\gamma \in (0, 1/2)$.

PROOF. A combination of (i) and (ii) gives the result. \square

LEMMA 2.15. Let $f \equiv 0$ and $g(x) = h''(x)$, $h \in L^2(\mathbb{R}) \cap L^{\frac{2}{1+\gamma}}(\mathbb{R})$. Then the solution u of the linear problem (2.1) satisfies

$$\|u\|_p = \|V_2(t)g\|_p \leq ct^{-\gamma/2} \|h\|_{p'} \quad \text{for } t > 0,$$

where $p = \frac{2}{1-\gamma}$, $p' = \frac{2}{1+\gamma}$ and $\gamma \in [0, 1]$.

PROOF. We write $V_2(t)g(x)$ as

$$V_2(t)g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/2} \frac{|\xi| \hat{h}(\xi)}{|\phi''(\xi)|^{\gamma/2} (1+\xi^2)^{1/2}} d\xi$$

then applying Theorem 2.4 the result follows. \square

PROPOSITION 2.16.

(i) Let $g = D^{1+\gamma/4}h \in L^2(\mathbb{R})$. Then

$$\|V_2(t)g\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \leq c\|h\|_{-1+\frac{\gamma}{4},2}.$$

(ii) Let $f = D^{\gamma/4}f_1 \in H^1(\mathbb{R})$. Then

$$\|V_1(t)f\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} \leq c\|f_1\|_{\frac{\gamma}{4},2},$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{4}{\gamma}$ and $\gamma \in [0, 1]$.

PROOF. To prove (i) we write $V_2(t)g(x)$ as

$$V_2(t)g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t\phi(\xi)+x\xi)} |\phi''(\xi)|^{\gamma/4} \frac{|\xi|^{\gamma/4} \hat{h}(\xi)}{|\phi''(\xi)|^{\gamma/4} (1+\xi^2)^{1/2}} d\xi$$

Thus from Theorem 2.5 it follows that

$$\begin{aligned} \|V_2(t)g\|_{L^q(\mathbb{R};L^p(\mathbb{R}))} &\leq c\left\| \left(\frac{|\xi|^{\gamma/4} \hat{h}(\xi)}{|\phi''(\xi)|^{\gamma/4} (1+\xi^2)^{1/2}} \right) \right\|_2 \\ &\leq c\|h\|_{-1+\frac{\gamma}{4},2}. \end{aligned} \tag{2.56}$$

This shows (i). The proof of (ii) is similar, hence it will be omitted. \square

COROLLARY 2.17. Under the assumptions on f and g in Proposition 2.16. The solution u of the linear problem (2.1) satisfies

$$\|u\|_{L^q(\mathbb{R}; L_1^p(\mathbb{R}))} \leq c(\|f_1\|_{1+\frac{\gamma}{4}, 2} + \|h\|_{\frac{\gamma}{4}, 2}),$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{4}{\gamma}$ and $\gamma \in [0, 1]$.

PROOF. The result follows as a consequence of (i) and (ii) in Proposition 2.16. \square

From (2.27), (2.28) and Corollary 2.17 we can conclude that a solution u of the linear problem (2.1) satisfies

$$u \in L^\infty(\mathbb{R}; H^1(\mathbb{R})) \cap L^q(\mathbb{R}; L_1^p(\mathbb{R})),$$

where p, q and the initial data satisfy the conditions in those results.

Nonlinear Problem. Local Theory

In this chapter we establish the local theory for the IVP in the spaces $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$.

3.1. Local Existence Theory in L^2

This section is devoted to prove local well-posedness of the IVP (1.1) for data $(f, g) = (f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$. We remind that $\dot{H}^{-1}(\mathbb{R})$ denotes the space of functions which are x -derivative of L^2 -functions. As we mentioned in the introduction this restriction allows to settle the problem in L^2 . Here, we succeed showing that the IVP (1.1) is locally well posed when $0 < \alpha < 4$. Note that this is the same result obtained for the one-dimensional nonlinear Schrödinger equation (1.8) in the L^2 case. To accomplish this result we consider the integral equation

$$u(t) = V_1(t)f(x) + V_2(t)h'(x) - \int_0^t V_2(t - \tau)(|u|^\alpha u)_{xx}(\tau) d\tau \quad (3.1)$$

where $V_1(t)f(\cdot)$ and $V_2(t)h'(\cdot)$ are defined as in Lemma 2.7 and Lemma 2.8 respectively. Then combining the estimates established in the previous chapter (smoothing effects) with a contraction mapping argument we conclude that the integral equation (3.1) has a unique solution. This solution is a strong solution of the IVP (1.1).

Also we will see below that the same techniques used to show that Φ (see (3.3) below) is a contraction allows us conclude that the solution not only depends on the data continuously, but also that this application is Lipschitz. Finally, we will show that these solutions satisfy the smoothing effect of Kato type previously discussed.

Now consider the following complete metric space:

$$X_T^a = \left\{ u \in C([0, T] : L^2(\mathbb{R})) \cap L^4([0, T] : L^\infty(\mathbb{R})) : \sup_{[0, T]} \|u(t)\|_2 \leq a, \|u\|_{L_T^4 L^\infty} := \left(\int_0^T \|u(t)\|_{L^\infty}^4 dt \right)^{1/4} \leq a \right\}. \quad (3.2)$$

If $0 < \alpha < 4$ define for $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$

$$\Phi_{(f, g)}(u)(t) = \Phi(u)(t) = V_1(t)f(x) + V_2(t)h'(x) - \int_0^t V_2(t - \tau)(|u|^\alpha u)_{xx}(\tau) d\tau \quad (3.3)$$

where $V_1(t)f(\cdot)$ and $V_2(t)h'(\cdot)$ are as above.

PROPOSITION 3.1.

$$\Phi(u(t)) : X_T^a \mapsto X_T^a$$

for some T and a depending on δ and α , where $\delta = \max(\delta_1, \delta_2)$, and $\|f\|_2 \leq \delta_1$, $\|h\|_{-1,2} \leq \delta_2$.

PROOF. Using the estimates (2.27), (2.33), (2.37) and Holder's inequality we have the following chain of inequalities

$$\begin{aligned} \sup_{[0,T]} \|\Phi(u)(t)\|_2 &\leq C(\|f\|_2 + \|h\|_{-1,2}) + c \int_0^T \|u(\tau)\|_2 \| |u|^\alpha \|_\infty d\tau \\ &\leq c(\|f\|_2 + \|h\|_{-1,2}) + C \sup_{[0,T]} \|u(t)\|_2 \int_0^T \| |u|^\alpha \|_\infty d\tau \\ &\leq c(\|f\|_2 + \|h\|_{-1,2}) + C \sup_{[0,T]} \|u(t)\|_2 T^{(4-\alpha)/4} \|u\|_{L_T^4 L_x^\infty}^\alpha \\ &\leq 2c\delta + ca^{\alpha+1} T^{(4-\alpha)/4} \end{aligned} \quad (3.4)$$

choosing $a = 4c\delta$ the last term above is bounded by the expression

$$2c\delta(1 + 2^{2\alpha+1} c^\alpha \delta^\alpha T^{(4-\alpha)/4}).$$

Now fixing T such that

$$2^{2\alpha+1} c^\alpha \delta^\alpha T^{(4-\alpha)/4} < 1 \quad (3.5)$$

then

$$\sup_{[0,T]} \|\Phi(u)(t)\|_2 \leq 4c\delta.$$

Denote by T_1 the T chosen in (3.5).

On the other hand, the estimates (2.28), (2.34), (2.38) and Holder's inequality give

$$\begin{aligned} \|\Phi(u)(t)\|_{L_T^4 L_x^\infty} &\leq c(1 + T^{1/4})(\|f\|_2 + \|h\|_{-1,2}) + c \int_0^T \|u\|_2 \| |u|^\alpha \|_\infty d\tau \\ &\leq c(1 + T^{1/4})(\|f\|_2 + \|h\|_{-1,2}) + c \sup_{[0,T]} \|u(t)\|_2 T^{(4-\alpha)/4} \|u\|_{L_T^4 L_x^\infty}^\alpha \\ &\leq c(1 + T^{1/4})(\|f\|_2 + \|h\|_{-1,2}) + ca^{\alpha+1} T^{(4-\alpha)/4} \end{aligned} \quad (3.6)$$

choosing a as before, we find that the above term is bounded by

$$2c\delta(1 + T^{1/4} + T^{(4-\alpha)/4} 2^{2\alpha+1} c^{\alpha+1} \delta^\alpha).$$

Now fixing T such that

$$T^{1/4} + 2^{2\alpha+1} c^{\alpha+1} \delta^\alpha T^{(4-\alpha)/4} < 1 \quad (3.7)$$

we have

$$\|\Phi(u)\|_{L_T^4 L_x^\infty} \leq 4c\delta.$$

Denote by T_2 the T in (3.7).

Thus if we take $T = \min(T_1, T_2)$ the proof is completed. \square

THEOREM 3.2. *If $0 < \alpha < 4$ then for all $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$ there exist $T = T(\delta, \alpha) > 0$ and a unique solution u of the integral equation (3.1) in $[0, T]$ with*

$$u \in C([0, T] : L^2(\mathbb{R})) \cap L^4([0, T] : L^\infty(\mathbb{R})).$$

Moreover, for any $T_0 < T$ there exists a neighborhood Y of $(f, g) \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$, where the map $(f, h') \rightarrow u$ is Lipschitz from Y to $C([0, T_0] : L^2(\mathbb{R})) \cap L^4([0, T_0] : L^\infty(\mathbb{R}))$.

PROOF. To prove the first part of this theorem we are going to use a contraction mapping argument. According with Proposition 3.1 we only need to prove that $\Phi(u)(t)$ is a contraction map. Let u and v be in X_T^a , with data f and h' . By the definition of $\Phi(u)(t)$

$$(\Phi(u) - \Phi(v))(t) = - \int_0^t V_2(t - \tau) (|u|^\alpha u - |v|^\alpha v)_{xx}(\tau) d\tau$$

Thus using the the estimate (2.37) and Holder's inequality we get

$$\begin{aligned} \sup_{[0, T]} \left\| (\Phi(u) - \Phi(v))(t) \right\|_2 &\leq c \int_0^T \| |u|^\alpha u - |v|^\alpha v \|_2 d\tau \\ &\leq c \int_0^T \| (|u|^\alpha + |v|^\alpha)(u - v) \|_2 d\tau \\ &\leq c \sup_{[0, T]} \| (u - v)(t) \|_2 \left(\int_0^T \| |u|^\alpha \|_\infty d\tau + \int_0^T \| |v|^\alpha \|_\infty d\tau \right) \\ &\leq c T^{(4-\alpha)/4} \sup_{[0, T]} \| (u - v)(t) \|_2 (\| |u|^\alpha \|_{L_T^4 L_x^\infty} + \| |v|^\alpha \|_{L_T^4 L_x^\infty}) \\ &\leq 2(4c\delta)^\alpha T^{(4-\alpha)/4} \sup_{[0, T]} \| (u - v)(t) \|_2. \end{aligned} \tag{3.8}$$

Now, the use of the estimate (2.38) and Holder's inequality as in (3.5) give

$$\begin{aligned} \| (\Phi(u) - \Phi(v)) \|_{L_T^4 L_x^\infty} &\leq c \int_0^T \| |u|^\alpha u - |v|^\alpha v \|_2 d\tau \\ &\leq 2(4c\delta)^\alpha T^{(4-\alpha)/4} \sup_{[0, T]} \| (u - v)(t) \|_2. \end{aligned} \tag{3.9}$$

From (3.8), (3.9), and the choice of T and a in (3.7) we have that,

$$2(4c\delta)^\alpha T^{(4-\alpha)/4} < 1.$$

Thus Φ is a contraction map. Hence using the contraction mapping principle we establish existence and uniqueness of solutions to (3.1) in X_T^a . However, the uniqueness holds in a

large class

$$X = C([0, T] : L^2(\mathbb{R})) \cap L^4([0, T] : L^\infty(\mathbb{R})).$$

In fact, suppose $\tilde{u} \in X$ satisfying the initial data, then it is easy to see that for $T' < T$ sufficiently small $\tilde{u} \in X_{T'}^\alpha$. Therefore $u = \tilde{u}$ in $\mathbb{R} \times [0, T']$. Reapplying this argument we obtain the desired result.

To prove that for any $T_0 < T$ the map from $Y \mapsto C([0, T_0] : L^2(\mathbb{R})) \cap L^4([0, T_0] : L^\infty(\mathbb{R}))$ is continuous, let us take u and v solutions of (3.1) with data (f_0, h'_0) and (f_1, h'_1) , respectively then

$$\begin{aligned} u(t) - v(t) &= V_1(t)(f_0 - f_1) + V_2(t)(h'_0 - h'_1) \\ &\quad - \int_0^{T_0} V_2(t - \tau) \partial_x^2 (|u|^\alpha u - |v|^\alpha v)(\tau) d\tau. \end{aligned} \quad (3.10)$$

Using the same argument as in (3.4) and (3.8) we have

$$\begin{aligned} \sup_{[0, T]} \|(u - v)(t)\|_2 &\leq c(\|f_0 - f_1\|_2 + \|h_0 - h_1\|_{-1,2}) \\ &\quad + cT_0^{(4-\alpha)/4} (\|u\|_{L_T^4 L_x^\infty}^\alpha + \|v\|_{L_T^4 L_x^\infty}^\alpha) \sup_{[0, T]} \|(u - v)(t)\|_2 \\ &\leq c(\|f_0 - f_1\|_2 + \|h_0 - h_1\|_{-1,2}) \\ &\quad + 2cT_0^{(4-\alpha)/4} (4c\delta)^\alpha \sup_{[0, T]} \|(u - v)(t)\|_2. \end{aligned} \quad (3.11)$$

On the other hand, the arguments used (3.6) and (3.9) imply that

$$\begin{aligned} \|u - v\|_{L_{T_0}^4 L_x^\infty} &\leq c(1 + T_0^{1/4})(\|f_0 - f_1\|_2 + \|h_0 - h_1\|_{-1,2}) \\ &\quad + T_0^{(4-\alpha)/4} (\|u\|_{L_T^4 L_x^\infty}^\alpha + \|v\|_{L_T^4 L_x^\infty}^\alpha) \sup_{[0, T]} \|(u - v)(t)\|_2 \\ &\leq c(1 + T_0^{1/4})(\|f_0 - f_1\|_2 + \|h_0 - h_1\|_{-1,2}) \\ &\quad + 2T_0^{(4-\alpha)/4} (4c\delta)^\alpha \sup_{[0, T_0]} \|(u - v)(t)\|_2. \end{aligned} \quad (3.12)$$

Now using (3.7) we obtain

$$\begin{aligned} \sup\left\{ \sup_{[0, T_0]} \|(u - v)(t)\|_2, \|u - v\|_{L_{T_0}^4 L_x^\infty} \right\} \\ \leq c_\alpha(T_0)(\|f_0 - f_1\|_2 + \|h_0 - h_1\|_{-1,2}) \end{aligned}$$

which yields the result. \square

COROLLARY 3.3. If $0 < \alpha < 4$ then for all $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$ there exist $T = T(\delta, \alpha) > 0$ and a unique solution u of the integral equation (3.1) in $[0, T]$ with

$$u \in C([0, T] : L^2(\mathbb{R})) \cap L^4([0, T] : L^\infty(\mathbb{R}))$$

and

$$D_x^{1/2} u \in L^\infty(\mathbb{R} : L^2([0, T])).$$

Moreover, for any $T_0 < T$ there exists a neighborhood \mathcal{U} of $(f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$, where the map

$$(f, h') \mapsto u$$

is Lipschitz from \mathcal{U} to

$$\mathcal{X}_T = \left\{ u \in C([0, T_0] : L^2(\mathbb{R})) \cap L^4([0, T_0] : L^\infty(\mathbb{R})) / \right. \\ \left. D_x^{1/2} u \in L^\infty(\mathbb{R} : L^2([0, T_0])) \right\}.$$

PROOF. We will only show that $\Phi(u)(t) \in \mathcal{X}_T$, the remaining part of the proof follows from the arguments in the proof of Theorem 3.2 and the estimates in Proposition 2.11.

By Proposition 3.1, we only need to prove that

$$\sup_x \left(\int_0^T |D_x^{1/2} \Phi(u)(t)|^2 dt \right)^{1/2} \leq a, \text{ with } a = 4c\delta.$$

So using Proposition 2.11 and Holder's inequality we obtain

$$\sup_x \left(\int_0^T |D_x^{1/2} \Phi(u)(t)|^2 dt \right)^{1/2} \leq c(1 + T^{1/2}) \{ \|f\|_2 + \|h\|_{-1,2} \} \\ + c(1 + T^{1/2}) \int_0^T \| |u|^\alpha u \|_2 d\tau \\ \leq c(1 + T^{1/2}) \{ \|f\|_2 + \|h\|_{-1,2} \} \\ + c(1 + T^{1/2}) T^{(4-\alpha)/4} \sup_{[0,T]} \|u(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^\alpha.$$

Taking $a = 4c\delta$ and T such that

$$T^{1/2} + (1 + T^{1/2}) T^{(4-\alpha)/4} 2^{2\alpha+1} \delta^\alpha c^\alpha < 1$$

the result follows. \square

REMARK 3.4. It is easy to see that $u_t \in C([0, T] : H^{-2}(\mathbb{R}))$.

REMARK 3.5. Note that when $\alpha = 4$ we cannot control the sizes of the terms involving the $\| \cdot \|$ -norm and $\sup_{[0,T]} \| \cdot \|_2$ -norm, therefore the contraction principle is not applicable with

the same arguments used in Theorem 3.2. However, we will see in section 3.3 that with an additional hypothesis, the local well-posedness can be proved when $\alpha = 4$.

3.2. Local Existence Theory in H^1

Here we study the local well-posedness of the IVP (1.1) with data $(f, g) = (f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. To do this, we follow the ideas used in [11], [23], and [18] to develop the theory in H^1 for the nonlinear Schrödinger equation (1.8). We want to point out that our result in this case is similar to the one-dimensional case for the NLS. More precisely, we show that the IVP (1.1) is locally well posed when $\alpha > 0$. As we noticed in the introduction $H^1 \times L^2$ is the appropriate space to solve the IVP (1.1).

To prove the local existence we will make use of the estimates obtained in Chapter 2 and a contraction mapping argument.

In this section it will be used the following notation.

$$\|u\|_1 = \left(\int_0^T \|u(t)\|_\infty^4 dt \right)^{1/4} + \left(\int_0^T \|u_x(t)\|_\infty^4 dt \right)^{1/4}.$$

Consider the following complete metric space

$$\begin{aligned} Y_T^a = \{ & u \in C([0, T] : H^1(\mathbb{R})) \cap L^4([0, T] : L_1^\infty(\mathbb{R})) / \\ & (-\Delta)^{-1/2} \partial_t u \in C([0, T] : L^2(\mathbb{R})), \\ & \sup_{[0, T]} \|u(t)\|_{1,2} \leq a, \|u(t)\|_1 \leq a, \sup_{[0, T]} \|(-\Delta)^{-1/2} \partial_t u(t)\|_2 \leq a \} \end{aligned}$$

PROPOSITION 3.6. For $0 < \alpha$, $f \in H^1(\mathbb{R})$ and $g = h' \in L^2(\mathbb{R})$ define $\Phi(u)(t)$ as in (3.3). Then

$$\Phi(u)(t) : Y_T^a \rightarrow Y_T^a$$

for some T and a depending on δ and α , where $\delta = \max(\delta_1, \delta_2)$, and $\|f\|_{1,2} \leq \delta_1$, $\|h\|_2 \leq \delta_2$.

PROOF. Using (2.28), (2.34), (2.38) in conjunction with Sobolev's embedding theorem we obtain

$$\begin{aligned} \|\Phi(u)\|_{L_T^4 L_x^\infty} &\leq c(1 + T^{1/4})\|f\|_2 + \|h\|_{-1,2} + c \int_0^T \|u\|_\infty^\alpha \|u\|_2 d\tau \\ &\leq c(1 + T^{1/4})\|f\|_{1,2} + \|h\|_2 + cT \sup_{[0, T]} \|u(t)\|_{1,2}^{\alpha+1}. \end{aligned} \quad (3.13)$$

The same argument as in (3.13) gives us the following

$$\|\partial_x \Phi(u)\|_{L_T^4 L_x^\infty} \leq c(1 + T^{1/4})(\|f\|_{1,2} + \|h\|_2) + cT \sup_{[0, T]} \|u(t)\|_{1,2}^{\alpha+1}. \quad (3.14)$$

Then from (3.13) and (3.14) it follows that

$$\begin{aligned} \|\Phi(u)\|_1 &\leq 2c(1 + T^{1/4})(\|f\|_{1,2} + \|h\|_2) + 2cT \sup_{[0, T]} \|u(t)\|_{1,2}^{\alpha+1} \\ &\leq 2c(1 + T^{1/4})(2\delta) + 2cT a^{\alpha+1} \end{aligned} \quad (3.15)$$

choosing $a = 8c\delta$ and T such that

$$(T^{1/4} + T2^{3\alpha+2} c^{\alpha+1} \delta^\alpha) < 1 \quad (3.16)$$

it follows that

$$\|\Phi(u)\|_1 \leq 8c\delta.$$

Now, if we use (2.27), (2.33), (2.37) and the Sobolev embedding we get

$$\begin{aligned} \sup_{[0,T]} \|\Phi(u)(t)\|_2 &\leq c(\|f\|_2 + \|h\|_{-1,2}) + c \int_0^T \|u\|_\infty^\alpha \|u\|_2 d\tau \\ &\leq c(\|f\|_{1,2} + \|h\|_2) + cT \sup_{[0,T]} \|u(t)\|_{1,2}^{\alpha+1}. \end{aligned} \quad (3.17)$$

Similarly, we obtain

$$\sup_{[0,T]} \|\partial_x \Phi(u)(t)\|_2 \leq c(\|f\|_{1,2} + \|h\|_2) + cT \sup_{[0,T]} \|u(t)\|_{1,2}^{\alpha+1}. \quad (3.18)$$

From (3.17) and (3.18) it follows that

$$\begin{aligned} \sup_{[0,T]} \|\Phi(u)(t)\|_{1,2} &\leq \sup_{[0,T]} \|\Phi(u)(t)\|_2 + \sup_{[0,T]} \|\partial_x \Phi(u)(t)\|_2 \\ &\leq 2c(\|f\|_{1,2} + \|h\|_2) + 2cT \sup_{[0,T]} \|u(t)\|_{1,2}^{\alpha+1} \\ &\leq 2c(2\delta) + 2cT a^{\alpha+1}. \end{aligned}$$

Choosing $a = 8c\delta$ and T such that

$$(2^{3\alpha+2} c^{\alpha+1} \delta^\alpha T) < 1 \quad (3.19)$$

we have that

$$\sup_{[0,T]} \|u(t)\|_{1,2} \leq 8c\delta.$$

Finally, to estimate $\|(-\Delta)^{-1/2} \partial_t \Phi(u)\|_2$ we use the Proposition 2.12, that combined with the Sobolev embedding yield

$$\begin{aligned} \sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t \Phi(u)(t)\|_2 &\leq c(\|f\|_{1,2} + \|h\|_2) + c \int_0^T \| |u|^\alpha u \|_{1,2} d\tau \\ &\leq c(\|f\|_{1,2} + \|h\|_2) + cT \sup_{[0,T]} \|u\|_{1,2}^{\alpha+1}. \end{aligned} \quad (3.20)$$

Therefore,

$$\sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t \Phi(u)\|_2 \leq 2c\delta + cT a^{\alpha+1}.$$

So, taking a and T as in (3.19) it follows that

$$\sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t \Phi(u)\|_2 \leq 4c\delta$$

Hence, we have proved that $\Phi(u) \in Y_T^\alpha$. □ □

THEOREM 3.7. *If $0 < \alpha$ then for all $f \in H^1(\mathbb{R})$ and $g = h' \in L^2(\mathbb{R})$ there exist $T = T(\delta, \alpha) > 0$ and a unique solution u of the integral equation (3.1) in $[0, T]$ with*

$$u \in C([0, T] : H^1) \cap L^4([0, T] : L_1^\infty)$$

and

$$(-\Delta)^{-1/2} \partial_t u \in C([0, T] : L^2).$$

Moreover, for any $T_0 < T$ there exists a neighborhood W of $(f, \partial_x h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, where the map

$$(f, h') \mapsto u$$

is Lipschitz from W to

$$\{u \in C([0, T_0] : H^1) \cap L^4([0, T_0] : L_1^\infty) / (-\Delta)^{-1/2} \partial_t u \in C([0, T_0] : L^2)\}.$$

PROOF. Following the ideas in the proof of the $L^2 \times \dot{H}^{-1}$ case, we will use a contraction mapping argument and the estimates in section 2 to prove the first part of this theorem. The proposition 3.6 assures that $\Phi(u)(t) : Y_T^a \rightarrow Y_T^a$, so we only need to show that Φ is a contraction.

Let u and v in Y_T^a , with data f and $g = \partial_x h$, by the definition of Φ it follows that

$$(\Phi(u) - \Phi(v))(t) = - \int_0^t V_2(t - \tau) (|u|^\alpha u - |v|^\alpha v)_{xx}(\tau) d\tau. \quad (3.21)$$

Using a similar argument as in (3.8), and (3.13) we obtain

$$\|\Phi(u) - \Phi(v)\|_{L_T^4 L_x^\infty} \leq CT \sup_{[0, T]} \|(u - v)(t)\|_{1,2} \left(\sup_{[0, T]} \|u\|_{1,2}^\alpha + \sup_{[0, T]} \|v\|_{1,2}^\alpha \right) \quad (3.22)$$

To estimate $\partial_x \Phi(u) - \partial_x \Phi(v)$ in $L_T^4 L_x^\infty$ -norm we will use the following inequality

$$||u|^\alpha u_x - |v|^\alpha v_x| \leq c \{ (|u|^{\alpha-1} + |v|^{\alpha-1}) |u - v| |u_x| + |v|^\alpha |u_x - v_x| \}. \quad (3.23)$$

Then combining the estimate (2.38), the inequality (3.23), Sobolev embedding and Young's inequality we obtain

$$\begin{aligned} \|\partial_x \Phi(u) - \partial_x \Phi(v)\|_{L_T^4 L_x^\infty} &\leq c \int_0^T \| |u|^\alpha u_x - |v|^\alpha v_x \|_2 d\tau \\ &\leq c \int_0^T \| (|u|^{\alpha-1} + |v|^{\alpha-1}) \|_\infty \|u - v\|_\infty \|u_x\|_2 d\tau + c \int_0^T \| |v|^\alpha \|_\infty \|u_x - v_x\|_2 d\tau \\ &\leq cT \left(\sup_{[0, T]} \|u(t)\|_{1,2}^\alpha + \sup_{[0, T]} \|v(t)\|_{1,2}^\alpha \right) \sup_{[0, T]} \|(u - v)(t)\|_{1,2}. \end{aligned} \quad (3.24)$$

Combining (3.22) and (3.24) we have

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_1 &\leq cT \sup_{[0, T]} \|(u - v)(t)\|_{1,2} \left\{ 2 \sup_{[0, T]} \|u(t)\|_{1,2}^\alpha + 2 \sup_{[0, T]} \|v(t)\|_{1,2}^\alpha \right\} \\ &\leq cT (4a^\alpha) \sup_{[0, T]} \|(u - v)(t)\|_{1,2}. \end{aligned} \quad (3.25)$$

Following the arguments in (3.13) and (3.18) it is obtained

$$\sup_{[0, T]} \|(u - v)(t)\|_2 \leq cT \sup_{[0, T]} \|(u - v)(t)\|_{1,2} \left\{ \sup_{[0, T]} \|u(t)\|_{1,2}^\alpha + \sup_{[0, T]} \|v(t)\|_{1,2}^\alpha \right\}. \quad (3.26)$$

The estimate (2.37), inequality (3.23), Sobolev embedding and Young's inequality give

$$\begin{aligned} \sup_{[0,T]} \|(\partial_x \Phi(u) - \partial_x \Phi(v))(t)\|_2 \\ \leq cT \sup_{[0,T]} \|(u-v)(t)\|_{1,2} \left\{ \sup_{[0,T]} \|u(t)\|_{1,2}^\alpha + \sup_{[0,T]} \|v(t)\|_{1,2}^\alpha \right\}. \end{aligned} \quad (3.27)$$

Combining (3.26) and (3.27)

$$\begin{aligned} \sup_{[0,T]} \|(\Phi(u) - \Phi(v))(t)\|_{1,2} \\ \leq 2cT \sup_{[0,T]} \|(u-v)(t)\|_{1,2} \left\{ \sup_{[0,T]} \|u(t)\|_{1,2}^\alpha + \sup_{[0,T]} \|v(t)\|_{1,2}^\alpha \right\} \\ \leq cT(4a^\alpha) \sup_{[0,T]} \|(u-v)(t)\|_{1,2}. \end{aligned} \quad (3.28)$$

Finally, using the same arguments as in (3.20) and (3.24) we obtain

$$\begin{aligned} \sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t \Phi(u) - (-\Delta)^{-1/2} \partial_t \Phi(v)\|_2 \\ \leq c \int_0^T \|(|u|^\alpha + |v|^\alpha) |u-v|\|_{1,2} d\tau \\ \leq c \int_0^T \|(|u|^\alpha + |v|^\alpha)\|_\infty \|u-v\|_2 d\tau + c \int_0^T \|(|u|^\alpha u_x - |v|^\alpha v_x)\|_2 d\tau \\ \leq cT \sup_{[0,T]} \|(u-v)(t)\|_{1,2} \left\{ 2 \sup_{[0,T]} \|u(t)\|_{1,2}^\alpha + 2 \sup_{[0,T]} \|v(t)\|_{1,2}^\alpha \right\} \\ \leq cT(4a^\alpha) \sup_{[0,T]} \|(u-v)(t)\|_{1,2}. \end{aligned} \quad (3.29)$$

Combining (3.25), (3.28), (3.29) and the choice of T and a in (3.19) it follows that $4cTa^\alpha < 1$, therefore Φ is a contraction mapping. The proof of the first part of the theorem can be completed by using the argument given in the previous section.

To show that for any $T_0 < T$, the map $(f, h') \mapsto u$ is Lipschitz from W to

$$\{u \in C([0, T_0] : H^1) \cap L^4([0, T_0] : L_1^\infty) / (-\Delta)^{-1/2} \partial_t u \in C([0, T_0] : L^2)\}$$

let us take u and v solutions of (3.1) with data (f_0, h'_0) and (f_1, h'_1) in W respectively, then

$$\begin{aligned} u(t) - v(t) = V_1(t)(f_0 - f_1) + V_2(t)(h'_0 - h'_1) \\ - \int_0^{T_0} V_2(t-\tau) (|u|^\alpha u - |v|^\alpha v)_{xx}(\tau) d\tau. \end{aligned}$$

From (3.17) and (3.25)

$$\begin{aligned}
\sup_{[0,T]} \|(u-v)(t)\|_{1,2} &\leq c(\|f_0 - f_1\|_{1,2} + \|h_0 - h_1\|_2) \\
&\quad + 2cT_0 \sup_{[0,T]} \|(u-v)(t)\|_{1,2} \left\{ \sup_{[0,T]} \|u(t)\|_{1,2}^\alpha + \sup_{[0,T]} \|v(t)\|_{1,2}^\alpha \right\} \\
&\leq c(\|f_0 - f_1\|_{1,2} + \|h_0 - h_1\|_2) + 4cT_0 a^\alpha \sup_{[0,T]} \|(u-v)(t)\|_{1,2}.
\end{aligned} \tag{3.30}$$

Similarly, from (3.13) and (3.24) it follows that

$$\begin{aligned}
\|u-v\|_1 &\leq c(1+T_0^{1/4})(\|f_0 - f_1\|_{1,2} + \|h_0 - h_1\|_2) \\
&\quad + 4cT_0 a^\alpha \sup_{[0,T]} \|(u-v)(t)\|_{1,2}.
\end{aligned} \tag{3.31}$$

Finally, making use of (3.20) and (3.28) we get

$$\begin{aligned}
\sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t u(t) - (-\Delta)^{-1/2} \partial_t v(t)\|_2 \\
\leq c(\|f_0 - f_1\|_{1,2} + \|h_0 - h_1\|_2) + cT_0(4a^\alpha) \sup_{[0,T]} \|(u-v)(t)\|_{1,2}.
\end{aligned} \tag{3.32}$$

Now, by (3.19) it has that $4cT_0 a^\alpha < 1$, therefore

$$\begin{aligned}
\sup_{[0,T]} \left\{ \sup_{[0,T]} \|(u-v)(t)\|_{1,2}, \|u-v\|_1, \sup_{[0,T]} \|(-\Delta)^{-1/2} \partial_t u - (-\Delta)^{-1/2} \partial_t v\|_2 \right\} \\
\leq c_T(\|f_0 - f_1\|_{1,2} + \|h_0 - h_1\|_2)
\end{aligned}$$

Thus, if $\|f_0 - f_1\|_{1,2}$ and $\|h_0 - h_1\|_2$ are sufficiently small, the result follows. \square

The following result shows that the above solution satisfies the Kato smoothing effect commented in the introduction.

COROLLARY 3.8. If $0 < \alpha$ then for all $f \in H^1(\mathbb{R})$ and $g = h' \in L^2(\mathbb{R})$ there exist $T = T(\delta, \alpha) > 0$ and a unique solution u of the integral equation (3.1) in $[0, T]$ with

$$u \in C([0, T] : H^1) \cap L^4([0, T] : L_1^\infty)$$

and

$$D_x^{3/2} u \in L^\infty(\mathbb{R} : L^2([0, T]))$$

Moreover, for any $T_0 < T$ there exists a neighborhood \mathcal{W} of $(f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, where the map

$$(f, h') \mapsto u$$

is Lipschitz from \mathcal{W} to

$$\{u \in C([0, T_0] : H^1(\mathbb{R})) \cap L^4([0, T_0] : L_1^\infty(\mathbb{R})) / D_x^{3/2} u \in L^\infty(\mathbb{R} : L^2([0, T_0]))\}.$$

PROOF. Using similar arguments as in Proposition 3.6, Theorem 3.7, and Proposition 2.11 the result follows. \square

3.3. Critical case in L^2

In this section we prove a theorem that assures the local existence, uniqueness, and continuous dependence for IVP (1.1) in $L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$ when $\alpha = 4$. Here, we follow the ideas in [11]. Note that the proof of well-posedness in this case is basically the same as in the $0 < \alpha < 4$ case. More precisely, a contraction mapping argument combined with estimates of type $L^p([0, T]; L^q)$ will be used in the proof.

To solve the problem we consider the following complete metric space

$$Z_T^a = \{u \in C([0, T] : L^2) \cap L^4([0, T] : L^\infty) / \\ \|u\|_{L_T^4 L_x^\infty} \leq a, \sup_{[0, T]} \|(u - u_0)(t)\|_2 \leq a\},$$

where $u_0(t) = V_1(t)f(x) + V_2(t)h'(x)$, and the data satisfy

$$\|V_1(t)f\|_{L_T^4 L_x^\infty}, \|V_2(t)h'\|_{L_T^4 L_x^\infty} \leq \delta$$

for some a and δ to be chosen below and $(f, g) = (f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$.

Observe that for fixed data $(f, h') \in L^2 \times \dot{H}^{-1}$ given $\delta > 0$ there exists $t_0 > 0$ such that

$$\|V_1(t_0)f\|_{L_T^4 L_x^\infty} \leq \delta \\ \|V_2(t_0)h'\|_{L_T^4 L_x^\infty} \leq \delta$$

and that the same estimates hold in a neighborhood of $(f, h') \in L^2 \times \dot{H}^{-1}$.

Define for $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$

$$\Phi_{(f, g)}(u)(t) = \Phi(u)(t) = V_1(t)f(x) + V_2(t)h'(x) - \int_0^t V_2(t - \tau) \partial_x^2(u^5)(\tau) d\tau. \quad (3.33)$$

Our main result in this section is the following

THEOREM 3.9. *For $\alpha = 4$, $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$ as above, there exist $T > 0$ and a unique solution u in Z_T^a of the integral equation (3.1).*

Moreover, for $T_0 < T$ there exists a neighborhood Y of $(f, g) = (f, h') \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})$ where the map $(f, h') \mapsto u$ is Lipschitz from Y to

$$\{u \in C([0, T] : L^2(\mathbb{R})) \cap L^4([0, T] : L^\infty(\mathbb{R}))\}.$$

Before proving the theorem, we shall show the following.

PROPOSITION 3.10. *For a $f \in L^2(\mathbb{R})$ and $g = h' \in \dot{H}^{-1}(\mathbb{R})$ as above*

$$\Phi(u(t)) : Z_T^a \mapsto Z_T^a$$

where a depends on δ .

PROOF. From the definition of Φ and estimate (2.38)

$$\begin{aligned}
\|\Phi(u)\|_{L_T^4 L_x^\infty} &\leq \|V_1(t)f\|_{L_T^4 L_x^\infty} + \|V_2(t)h'\|_{L_T^4 L_x^\infty} \\
&\quad + c(\sup_{[0,T]} \|(u - u_0)(t)\|_2 + \sup_{[0,T]} \|u_0(t)\|_2) \|u\|_{L_T^4 L_x^\infty}^4 \\
&\leq 2\delta + c \sup_{[0,T]} \|(u - u_0)(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^4 + c \sup_{[0,T]} \|u_0(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^4 \\
&\leq 2\delta + ca^4(a + M)
\end{aligned} \tag{3.34}$$

where $M = \sup_{[0,T]} \|u_0(t)\|_2$.

Choosing $a = 4\delta$ and then δ such that

$$2^7 \delta^3 c(4\delta + M) < 1 \tag{3.35}$$

we have that

$$\|\Phi(u)\|_{L_T^4 L_x^\infty} \leq 4\delta.$$

On the other hand, the estimate (2.37) gives

$$\begin{aligned}
\sup_{[0,T]} \|(\Phi(u) - u_0)(t)\|_2 &\leq c \sup_{[0,T]} \|u(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^4 \\
&\leq c \sup_{[0,T]} \|(u - u_0)(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^4 + c \sup_{[0,T]} \|u_0(t)\|_2 \|u\|_{L_T^4 L_x^\infty}^4 \\
&\leq a^4 c(a + M).
\end{aligned} \tag{3.36}$$

Taking $a = 4\delta$ with δ such that

$$2^6 \delta^3 c(4\delta + M) < 1 \tag{3.37}$$

it follows that

$$\sup_{[0,T]} \|(\Phi(u) - u_0)(t)\|_2 \leq 4\delta$$

This shows that $\Phi(u(t)) : Z_T^a \mapsto Z_T^a$. \square

Now we will prove the theorem

PROOF OF THEOREM 3.9. Applying estimate (2.38), mean value theorem, we obtain the following chain of inequalities.

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L_T^4 L_x^\infty} &\leq \int_0^T \|u^5 - v^5\|_2 d\tau \\
&\leq c \int_0^T \|u^4 + v^4\|_\infty \|u - v\|_2 d\tau \\
&\leq c \sup_{[0,T]} \|(u - v)(t)\|_2 (\|u\|_{L_T^4 L_x^\infty}^4 + \|v\|_{L_T^4 L_x^\infty}^4) \\
&\leq 2a^4 c \sup_{[0,T]} \|(u - v)(t)\|_2.
\end{aligned} \tag{3.38}$$

On the other hand, the estimate (2.37) applied to $\Phi(u) - \Phi(v)$ and the same argument above exposed give

$$\begin{aligned} \sup_{[0,T]} \|\Phi(u) - \Phi(v)\|_2 &\leq c \sup_{[0,T]} \|u - v\|_2 (\|u\|_{L_T^4 L_x^\infty}^4 + \|v\|_{L_T^4 L_x^\infty}^4) \\ &\leq 2a^4 c \sup_{[0,T]} \|u - v\|_2. \end{aligned} \tag{3.39}$$

Noticing that by (3.35), $2a^4 c < 1$ we can conclude that Φ is a contraction.

The remaining part of the proof uses a previous argument. Therefore it will be omitted. \square

Global Theory. Persistence

4.1. Global Theory in H^1

In this section our aim is to prove that for small data the solution of the IVP (1.1) in $H^1 \times L^2$ can be extended globally ($t > 0$). To show this, we combine the conservation law (4.2) (see below) and one of the results obtained by J. Bona and R. Sachs in [6]. As we pointed out in the introduction, Kato's theory [22] allowed them to show the local well-posedness of the system (1.5)-(1.6) for smooth data, as we will see below.

We begin by stating the following result of Bona and Sachs in [6].

Denote by

$$Y_s(T) = C([0, T] : H^{s+2}(\mathbb{R})) \cap C^1([0, T] : H^s(\mathbb{R})) \cap C^2([0, T] : H^{s-2}(\mathbb{R})).$$

THEOREM 4.1. *Let $u_0 \in H^{s+2}(\mathbb{R})$ and $v_0 \in H^{s+1}(\mathbb{R})$ for some $s > 1/2$. Then there exists a $T > 0$, depending only upon of $(u_0, v_0) \in H^{s+2}(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, and a unique function $u \in Y_s(T)$ which is solution of the equation in (1.1) in the distributional sense on $\mathbb{R} \times [0, T]$, and for which $u(\cdot, 0) = u_0$ and $u_t(\cdot, 0) = v_0'$. The solution depends continuously upon the data (u_0, v_0) in the sense that the mapping that associates to (u_0, v_0) the solution u is continuous from $H^{s+2}(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ into $Y_s(T)$. If $s > 5/2$ the solution is classical.*

PROOF. See Corollary 2 in [6]. □

Now, consider the equation

$$u_{tt} - u_{xx} + u_{xxxx} + (|u|^{\alpha-1}u)_{xx} = 0. \quad (4.1)$$

Suppose that u is a solution of the initial-value problem (1.1) given by Theorem 4.1 for s sufficiently large, thus we can proceed as follows.

Apply the operator $(-\Delta)^{-1}$ to the equation (4.1) and multiply by u_t , then integrating respect to x , we obtain the following

$$\frac{1}{2} \frac{d}{dt} \left\{ \|(-\Delta)^{-1/2} u_t\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2 - \frac{2}{\alpha+1} \|u\|_{\alpha+1}^{\alpha+1} \right\} = 0$$

or

$$\|(-\Delta)^{-1/2} u_t\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2 - \frac{2}{\alpha+1} \|u\|_{\alpha+1}^{\alpha+1} = K_0 \quad (4.2)$$

where $K_0 = K_0(f_0, g_0)$.

The relation (4.2) is the main tool to show that $\|u(t)\|_{1,2}$ remains bounded on the interval $[0, T')$ for small data, and so $\|(-\Delta)^{-1/2} u_t\|_2$ does. Hence we can apply the Theorem 4.2 again to continue the solution. So we have the following

THEOREM 4.2. *Suppose that $\|f\|_{1,2}$ and $\|g\|_2 = \|\partial_x h\|_2$ are sufficiently small, then for $0 < \alpha$, the solution of the integral equation (3.1) given in Theorem 4.2 extends to any time interval in the same class.*

PROOF. From (4.2) we have

$$\|u\|_{1,2}^2 - c\|u\|_{\alpha+1}^{\alpha+1} \leq K_0.$$

Sobolev embedding gives

$$\|u\|_{1,2}^2 - c\|u\|_{1,2}^{\alpha+1} \leq K_0.$$

Set $X(t) = \|u(t)\|_{1,2}^2$, then let f_0, g_0 be initial data with $\|f_0\|_{1,2}, \|g_0\|_2 \ll 1$ such that K_0 satisfies

$$0 < X(0) - cX(0)^{(\alpha+1)/2} \leq K_0. \quad (4.3)$$

The inequality

$$X(t) \leq K_0 + cX(t)^{1+\epsilon}, \quad \epsilon > 0$$

is satisfied if $X(t) \in [0, \beta_1] \cup [\beta_2, \infty)$ with $0 < \beta_1 < \beta_2 < \infty$, since K_0 is small.

Now, condition (4.3) ensures that $X(0) (= \|u(\cdot, 0)\|_{1,2}^2) \in [0, \beta_1]$ then the continuity of $X(t)$ allows to conclude that $X(t)$ remains in that interval for $t < T'$ which means that $\sup_{[0, T']} \|u(t)\|_{1,2}$ will be bounded.

To complete the proof we need to see that $\|(-\Delta)^{-1/2}u_t\|_2$ is bounded, but this follows from the identity (4.2) and the fact that $\sup_{[0, T']} \|u(t)\|_{1,2}$ is bounded.

Applying Theorem 3.7 we can continue the solution, thus the result follows. \square

In the proof of the theorem we made use of the identity (4.2), so we shall justify this to complete the validity of Theorem 4.2. To do that we shall use Kato's techniques in [24].

Let u be the solution of (3.1) given in Theorem 3.7 with data $\phi = (f, g) = (f, h') \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, then we approximate ϕ by a sequence $\phi_j = (f_j, g_j) = (f_j, h'_j) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, such that

$$\|\phi_j - \phi\| = \|f_j - f\|_{1,2} + \|h'_j - h'\|_2 \rightarrow 0.$$

Now, let u_j be the solution to (1.1) with data

$$u_j(\cdot, 0) = f_j(\cdot), \quad \partial_t u_j(\cdot, 0) = h'_j(\cdot).$$

By Theorem 4.1 u_j exists on $[0, T]$ for sufficiently large j and $u_j \rightarrow u$ in $C([0, T] : H^1(\mathbb{R}))$. The identity (4.2) is formally justified for $u_j(t) \in H^s(\mathbb{R})$ if s is sufficiently large, letting $j \rightarrow \infty$ and noting that $u_j(t) \rightarrow u(t)$ in H^1 and $(-\Delta)^{-1/2}\partial_t u_j \rightarrow (-\Delta)^{-1/2}\partial_t u$ in L^2 , we obtain (4.2) for u .

4.2. Persistence

In this section we present some results concerning the persistence properties in H^s and the decay for solutions of the IVP (1.1). Since the case of interest for the scattering theory is when $\psi(u) = u^2$ in (1.1) we will restrict to consider the problem

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = f(x) \\ u_t(x, 0) = h'(x). \end{cases} \quad (4.4)$$

We begin by stating the following theorem that deals with the persistence properties in H^s for solutions of the IVP (4.4). In this section we use g to denote $g = h'$.

THEOREM 4.3. *Let $(f, g) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$, with $\|f\|_{1,2}$ and $\|g\|_2$ small, then there exists a unique solution $u \in C(\mathbb{R}^+ : H^1(\mathbb{R}))$ of the IVP (4.4) such that*

$$u \in C_b(\mathbb{R}^+ : H^1(\mathbb{R})) \cap C(\mathbb{R}^+ : H^{s+1}(\mathbb{R})).$$

PROOF. Applying ∂_x^α to the equation in (4.4), multiplying by $\partial_x^\alpha u_t$, and integrating with respect to x , and then integrating by parts we have

$$\frac{1}{2} \frac{d}{dt} \int \left\{ |\partial_x^\alpha u_t|^2 + |\partial_x^\alpha u_x|^2 + |\partial_x^\alpha u_{xx}|^2 \right\} dx = \int \partial_x^\alpha (u^2)_{xx} \partial_x^\alpha u_t dx.$$

Now, using the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities we have

$$\frac{d}{dt} \left\{ \|\partial_x^\alpha u_t\|_2^2 + \|\partial_x^\alpha u_x\|_2^2 + \|\partial_x^\alpha u_{xx}\|_2^2 \right\} \leq c \|u\|_\infty \|\partial_x^\alpha u_{xx}\|_2 \|\partial_x^\alpha u_t\|_2.$$

Gronwall's inequality gives

$$\begin{aligned} & \sup_{[0, T]} \left(\|\partial_x^\alpha u_t\|_2 + \|\partial_x^\alpha u_x\|_2 + \|\partial_x^\alpha u_{xx}\|_2 \right) \\ & \leq \left\{ \|\partial_x^\alpha g\|_2 + \|\partial_x^\alpha f_x\|_2 + \|\partial_x^\alpha f_{xx}\|_2 \right\} \exp \left(C \int_0^T \|u(t)\|_\infty dt \right). \end{aligned}$$

As usual the above formal computation can be justified by using the continuous dependence on the data.

Now, using Theorem 4.2 the result follows. \square

THEOREM 4.4. *Consider the IVP (4.4) with $f \in H^3(\mathbb{R})$, $g \in H^2(\mathbb{R})$. If xf_{xx} , xf_x and $xg \in L^2(\mathbb{R})$, then there exists $T > 0$ such that*

$$xu \in C([0, T] : H^2(\mathbb{R})).$$

Moreover, if $\|f\|_{1,2}$, $\|g\|_2 \ll 1$, the result is global, i.e.

$$xu \in C(\mathbb{R}^+ : H^2(\mathbb{R})).$$

PROOF. From (4.4) we have

$$(xu)_{tt} = (xu)_{xx} - (xu)_{xxx} - 2(xu)_x u_x - 2(xu)_{xx} u - 2u_x + 4u_{xxx} + 6uu_x$$

multiplying by $(xu)_t$ and integrating with respect to x , we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \int [(xu)_t]^2 dx + \int [(xu)_x]^2 dx + \int [(xu)_{xx}]^2 dx \right\} \\ & = -2 \int u_x (xu)_x (xu)_t dx - 2 \int u (xu)_{xx} (xu)_t dx \\ & \quad + 4 \int u_{xxx} (xu)_t dx - 2 \int u_x (xu)_t dx + 6 \int uu_x (xu)_t \end{aligned}$$

The Cauchy-Schwarz and Young inequalities yield

$$\begin{aligned} & \frac{d}{dt} \{ \|(xu)_t\|_2^2 + \|(xu)_x\|_2^2 + \|(xu)_{xx}\|_2^2 \} \\ & \leq c (\|u_{xxx}\|_2^2 + \|u_x\|_2^2 + \|u\|_{L^\infty}^2 \|u_x\|_2^2) \\ & \quad + c (1 + \|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \{ \|(xu)_{xx}\|_2^2 + \|(xu)_x\|_2^2 + \|(xu)_t\|_2^2 \} \\ & \leq c (\|u\|_{3,2}^2 + \|u\|_{1,2}^4) + c (1 + \|u\|_{2,2}) \{ \|(xu)_{xx}\|_2^2 + \|(xu)_x\|_2^2 + \|(xu)_t\|_2^2 \} \end{aligned}$$

Gronwall's inequality and Theorem 4.3 give the desired result. \square

COROLLARY 4.5. Consider the IVP (4.4), with $f \in H^4(\mathbb{R})$, $g \in H^3(\mathbb{R})$. If xf_{xxx} , xf_{xx} , and $xg_x \in L^2(\mathbb{R})$ then

$$(xu_x)_{xx} \in C([0, T] : L^2(\mathbb{R})).$$

Moreover, if $\|f\|_{1,2}$, $\|g\|_2 \ll 1$, the result is global.

PROOF. It is an immediate application of Theorem 4.4. \square

COROLLARY 4.6. Same hypotheses of Corollary 4.5 and x^2f_{xx} , x^2f_x , $x^2g \in L^2(\mathbb{R})$, then

$$x^2u \in C([0, T] : H^2(\mathbb{R})).$$

Moreover, if $\|f\|_{1,2}$, $\|g\|_2 \ll 1$ the result is global i.e.

$$x^2u \in C(\mathbb{R}^+ : H^2(\mathbb{R})).$$

PROOF. The proof is similar to the proof of Theorem 4.4. \square

As a consequence of the Theorem 4.3 and Theorem 4.4, we have the following result for functions in the Schwartz class \mathcal{S} . i.e.

$$\mathcal{S}(\mathbb{R}) = \left\{ h \in C^\infty(\mathbb{R}) / \sup_{x \in \mathbb{R}} |x^\gamma D^\beta h(x)| < \infty, \gamma, \beta \in \mathbb{Z}^+ \right\}$$

with the topology defined by the family of seminorms $\rho_{\gamma\beta}$

$$\rho_{\gamma\beta}(h) = \sup_{x \in \mathbb{R}} |x^\gamma D^\beta h(x)|.$$

COROLLARY 4.7. Let $(f, g) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ with $\|f\|_{1,2}$, $\|g\|_2 \ll 1$, then there exists a unique solution $u \in C(\mathbb{R} : H^1(\mathbb{R}))$ of the IVP (4.4) such that

$$u \in C(\mathbb{R}^+ : H^1(\mathbb{R})) \cap C(\mathbb{R} : \mathcal{S}(\mathbb{R})).$$

Asymptotic Behavior of Solutions

In this chapter we study some aspects related to the asymptotic behavior of solutions to the IVP (1.1). In particular, we study the decay of solutions with time. We will show that the decay of solutions of the nonlinear problem is the same inherited from the linear problem. The decay result will allow us to show the existence of solutions to the linear problem that approximate to solutions of the nonlinear problem. This is called nonlinear scattering. In the last section of this chapter we present a blow-up result for solutions of the IVP (1.1)

5.1. Decay

One interesting question concerning solutions of evolution equations is the behavior of these regarding the time. In this direction we have the following result.

THEOREM 5.1. *Let $f \in H^1(\mathbb{R}) \cap L^q_{\frac{1}{2}}(\mathbb{R})$, $g = h'$, $h \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$, and $\alpha > \frac{4-3\gamma-\gamma^2}{\gamma}$. If $\|f\| + \|g\| \equiv \|f\|_{1,2} + \|f\|_{\frac{1}{2},q} + \|h\|_2 + \|h\|_q < \delta$ small. Then there exists $c > 0$ such that the solution u of the IVP (1.1) satisfies*

$$\|u\|_p \leq c(1+t)^{-\gamma/2}, \quad t > 0,$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{2}{1+2\gamma}$ and $\gamma \in (0, 1/2)$.

PROOF. The solution of the IVP (1.1) is written as

$$u(t) = V_1(t)f(x) + V_2(t)g(x) - \int_0^t V_2(t-\tau)(|u|^{\alpha-1}u)_{xx}(\tau) d\tau, \quad (5.1)$$

$V_1(t) \cdot$ and $V_2(t) \cdot$ are defined as in (2.26) and (2.32), respectively.

From (5.1) it follows that

$$\|u(t)\|_p \leq \|V_1(t)f\|_p + \|V_2(t)g\|_p + \int_0^t \|V_2(t-\tau)(|u|^{\alpha-1}u)_{xx}(\tau)\|_p d\tau,$$

then use of Proposition 2.13 and Lemma 2.15 leads to

$$\|u(t)\|_p \leq c(1+t)^{-\gamma/2}(\|f\| + \|g\|) + c \int_0^t (t-\tau)^{-\gamma/2} \| |u|^{\alpha-1}u(\tau) \|_{\frac{2}{1+\gamma}} d\tau. \quad (5.2)$$

On the other hand, Gagliardo-Nirenberg interpolation (see appendix inequality (A.19)) yields the inequality

$$\|u\|_{\frac{2\alpha}{1+\gamma}}^\alpha \leq c \|u_x\|_2^{\frac{\alpha(1-\gamma)-(1+\gamma)}{2-\gamma}} \|u\|_p^{\frac{\alpha+1+\gamma}{2-\gamma}}.$$

Hence the integral in (5.2) can be bounded as follows

$$\begin{aligned} & \int_0^t (t-\tau)^{-\gamma/2} \| |u|^{\alpha-1} u(\tau) \|_{\frac{2}{1+\gamma}} d\tau \leq c \left(\sup_{[0,T]} \|u(t)\|_{1,2} \right)^a \\ & \times \int_0^t (t-\tau)^{-\gamma/2} \|u(\tau)\|_p^{\frac{\alpha+1+\gamma}{2-\gamma}} d\tau \end{aligned} \quad (5.3)$$

where $a = \frac{\alpha(1-\gamma)-(1+\gamma)}{2-\gamma}$.

Next we define

$$M(T) = \sup_{[0,T]} (1+t)^{\gamma/2} \|u(t)\|_p.$$

Therefore combining (5.2) and (5.3), and the definition of $M(T)$ we obtain

$$\begin{aligned} M(T) & \leq c(\|f\| + \|g\|) \\ & + c \left(\sup_{[0,T]} \|u(t)\|_{1,2} \right)^a (1+t)^{\gamma/2} M(T)^{\frac{\alpha+1+\gamma}{2-\gamma}} \int_0^t (t-\tau)^{-\gamma/2} (1+\tau)^{-\frac{\gamma}{2} \left(\frac{\alpha+1+\gamma}{2-\gamma} \right)} d\tau \end{aligned}$$

Now using the hypothesis $\alpha > \frac{4-3\gamma-\gamma^2}{\gamma}$ we have that

$$M(T) \leq c(\|f\| + \|g\|) + c' \left(\sup_{[0,T]} \|u(t)\|_{1,2} \right)^a M(T)^{\frac{\alpha+(1+\gamma)}{2-\gamma}}$$

or

$$M(T) \leq c\delta + c'\delta^a M(T)^{\frac{\alpha+(1+\gamma)}{2-\gamma}}.$$

Therefore for δ sufficiently small we will have

$$M(T) \leq c$$

for any $T > 0$, where c is the smallest positive zero of the function $f(x) = c\delta + c'\delta x^{\frac{\alpha+(1+\gamma)}{2-\gamma}} - x$. Thus we obtain the desired result. \square

Some remarks are listed.

REMARK 5.2. If we assume in Theorem 5.1 $f = D^{\gamma/2} f_1 \in H^1(\mathbb{R})$, $f_1 \in H^{1+\gamma/2}(\mathbb{R}) \cap L^q_{\frac{1}{2}}(\mathbb{R})$, and $g = D^{1+\gamma/2} h \in L^2(\mathbb{R})$, $h \in H^{\gamma/2}(\mathbb{R}) \cap L^q(\mathbb{R})$, we obtain

$$\|u(t)\|_p \leq c(1+t)^{-\gamma/2}, \quad t > 0,$$

where $p = \frac{2}{1-\gamma}$, $q = \frac{2}{1+\gamma}$, and $\alpha > \max \left\{ \frac{1+\gamma}{1-\gamma}, \frac{4-3\gamma-\gamma^2}{\gamma} \right\}$, for $\gamma \in (0, 1)$. See [51] for related results.

REMARK 5.3. Moreover, if we assume $f = D^{1/2}f_1 \in H^1(\mathbb{R})$ and $g = D^{3/2}h \in L^2(\mathbb{R})$, where $f_1 \in H^{3/2}(\mathbb{R}) \cap L^1_{\frac{1}{2}}(\mathbb{R})$ and $h \in H^{1/2}(\mathbb{R}) \cap L^1(\mathbb{R})$, it can be proven that the solutions of the IVP (1.1) satisfy for any $x \in \mathbb{R}$, $t > 0$ and $\alpha > 4$

$$|u(x, t)| \leq c(1+t)^{-1/2}.$$

The proof of this follows an argument similar to the proof of Theorem 5.1 but easier.

5.2. Nonlinear Scattering

In this section we establish a result concerning nonlinear scattering for small data for solutions of the initial value problem (1.1). More precisely, under some suitable conditions on the initial data and the nonlinearity we obtain a result which ensures that small solutions of the IVP (1.1) behave asymptotically like solutions of the associated linear problem.

THEOREM 5.4. *Let $(f, g) = (D^{\gamma/4}f_1, D^{1+\gamma/4}h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|f\|_{1,2} + \|g\|_2 < \delta$ small, $\alpha = \frac{4}{\gamma}$, $\gamma \in (0, 4/5)$ and u be the solution of the IVP (1.1). Then there exist unique solutions u_{\pm} of the linear problem associated to (1.1) such that*

$$\|u(t) - u_{\pm}(t)\|_{1,2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (5.4)$$

In the proof of this theorem we follow ideas used by Strauss [47], Pecher [40] and Ponce and Vega [41] to establish similar results for the nonlinear Schrödinger equation, nonlinear wave equation, Klein-Gordon equation and Korteweg-de Vries equation, respectively.

Affirmative results on scattering for small solutions are interpreted as the nonexistence of solitary-wave solutions of arbitrary small amplitude. In this case, we notice that a simple calculation shows that the solitary-wave solutions $U_c(\xi)$ in (1.4) satisfy $\|U_c(\cdot)\|_2 > \epsilon > 0$ for $\alpha > 5$. So according to the statement above we should expect to have scattering as in (5.4) when the nonlinearity power α satisfies the previous constrain. The results in Theorem 5.4 show that scattering for small solutions of (1.1) occurs when the restriction on α above mentioned is satisfied. In this sense we could say that the scattering results presented here are optimal.

To prove Theorem 5.4 we first need to establish an existence result for the IVP (1.1) under some additional conditions on the initial data. In the proof of this result we will use the integral equation form of the IVP (1.1) (see (5.5) below).

THEOREM 5.5. *Let $(f, g) = (D^{\gamma/4}f_1, D^{1+\gamma/4}h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|f\|_{1,2} + \|g\|_2 < \delta$ sufficiently small and $\alpha = \frac{4}{\gamma}$, $\gamma \in (0, 4/5)$. Let ψ denote the solution in $H^1(\mathbb{R})$ of the linear problem (2.1). Then the integral equation*

$$u(t) = \psi(t) - \int_0^t V_2(t-\tau)(|u|^{\alpha-1}u)_{xx}(\tau) d\tau \quad (5.5)$$

has a unique solution in $X = L^\alpha(\mathbb{R} : L^p_1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R}))$, where $p = \frac{2}{1-\gamma}$.

PROOF. To establish this result we will use the contraction mapping principle.

Define

$$\Phi_{(f,g)}(u)(t) = \Phi(u)(t) = \psi(t) - \int_0^t V_2(t-\tau)(|u|^{\alpha-1}u)_{xx}(\tau) d\tau \quad (5.6)$$

and

$$\mathcal{Y}_a = \{v \in L^\alpha(\mathbb{R} : L_1^p(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R})) : \Lambda(v) \leq a\}$$

where $\Lambda(v) = \max\{\sup_t \|v(t)\|_{1,2}, \|v\|_{L^q(\mathbb{R} : L_1^p(\mathbb{R}))}\}$.

We first shall show that $\Phi : \mathcal{Y}_a \rightarrow \mathcal{Y}_a$ and next that Φ is a contraction in \mathcal{Y}_a .

$$\Phi : \mathcal{Y}_a \mapsto \mathcal{Y}_a.$$

Notice that the hypothesis $\alpha = \frac{4}{\gamma}$ with $\gamma \in (0, 4/5)$ implies $\alpha > \frac{1}{1-\gamma}$. With this observation we can begin the proof of the previous statement.

Use of the definition (5.6), (2.27), (2.33), Hölder's inequality in the x -variable and the fact that $L_1^p(\mathbb{R}) \not\subset L^{\frac{2(\alpha-1)}{\gamma}}(\mathbb{R})$ for $\alpha > \frac{1}{1-\gamma}$, $\gamma \in (0, 4/5)$, lead to

$$\begin{aligned} \|\Phi(u)(t)\|_{1,2} &\leq \|\psi(t)\|_{1,2} + c \int_0^t (\| |u|^{\alpha-1}u(\tau) \|_2 + \| |u|^{\alpha-2}u_x(\tau) \|_2) d\tau \\ &\leq \|\psi(t)\|_{1,2} + c \int_0^t (\| u(\tau) \|_{\frac{2(\alpha-1)}{\gamma}}^{\alpha-1} \| u(\tau) \|_p + \| u(\tau) \|_{\frac{2(\alpha-1)}{\gamma}}^{\alpha-1} \| u_x(\tau) \|_p) d\tau \\ &\leq \|\psi(t)\|_{1,2} + c \int_{-\infty}^{\infty} \| u(\tau) \|_{1,p}^\alpha dt. \end{aligned} \quad (5.7)$$

On the other hand, definition (5.6), Lemma 2.15, Hölder's inequality in the x -variable and the embedding $L_1^p(\mathbb{R}) \subset L^{\frac{2(\alpha-1)}{\gamma}}(\mathbb{R})$ for $\alpha > \frac{1}{1-\gamma}$, $\gamma \in (0, 4/5)$, yield

$$\begin{aligned} \|\Phi(u)(t)\|_p &\leq c\|\psi(t)\|_p + c \int_0^t (t-\tau)^{-\gamma/2} \| |u|^{\alpha-1}u(\tau) \|_q d\tau \\ &\leq c\|\psi(t)\|_p + c \int_0^t (t-\tau)^{-\gamma/2} \| u(\tau) \|_{\frac{2(\alpha-1)}{\gamma}}^{\alpha-1} \| u(\tau) \|_2 d\tau \\ &\leq c\|\psi(t)\|_p + c \int_0^t (t-\tau)^{-\gamma/2} \| u(\tau) \|_{1,p}^{\alpha-1} \| u(\tau) \|_2 d\tau \end{aligned}$$

setting $\frac{\gamma}{2} = \frac{2}{\alpha}$ and applying Hardy-Littlewood-Sobolev theorem (see Theorem A.16 in the Appendix) we obtain

$$\|\Phi(u)\|_{L^\alpha(\mathbb{R} : L^p(\mathbb{R}))} \leq c\|\psi(t)\|_{L^\alpha(\mathbb{R} : L^p(\mathbb{R}))} + c \sup_t \|u(t)\|_{1,2} \|u\|_{L^\alpha(\mathbb{R} : L_1^p(\mathbb{R}))}^{\alpha-1}. \quad (5.8)$$

A similar argument leads to

$$\|(\Phi(u))_x\|_{L^\alpha(\mathbb{R};L^p(\mathbb{R}))} \leq c\|\psi_x(t)\|_{L^\alpha(\mathbb{R};L^p(\mathbb{R}))} + c\sup_t\|u(t)\|_{1,2}\|u\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}^{\alpha-1}. \quad (5.9)$$

Therefore a combination of (5.7), (5.8) and (5.9) gives

$$\Lambda(\Phi(u)) \leq c\Lambda(\psi) + 2c(\Lambda(u))^\alpha.$$

So if $\|f\|_{1,2} + \|g\|_2$ is sufficiently small such that $c\Lambda(\psi) \leq \frac{a}{2}$ with $2ca^{\alpha-1} < \frac{1}{4}$, we obtain that

$$\Lambda(\Phi(u)) \leq a,$$

this shows that $\Phi : \mathcal{Y}_a \rightarrow \mathcal{Y}_a$.

Next step is to prove that Φ is in fact a contraction. We consider u and v in \mathcal{Y}_a , thus

$$(\Phi(u) - \Phi(v))(t) = - \int_0^t V_2(t - \tau)(|u|^{\alpha-1}u - |v|^{\alpha-1}v)_{xx}(\tau) d\tau.$$

To estimate $\sup_t\|(\Phi(u) - \Phi(v))(t)\|_{1,2}$ we use (2.27), (2.33) to have

$$\begin{aligned} \|(\Phi(u) - \Phi(v))(t)\|_{1,2} &\leq c \int_0^t \|(|u|^{\alpha-1} + |v|^{\alpha-1})(u - v)(\tau)\|_2 d\tau \\ &\quad + \int_0^t \|(|u|^{\alpha-1}u - |v|^{\alpha-1}v)_x(\tau)\|_2 d\tau \\ &= I_1 + I_2. \end{aligned}$$

Using a similar argument as in (5.7) it follows that

$$\begin{aligned} I_1 &\leq c \int_0^t (\|u(\tau)\|_{1,p}^{\alpha-1} + \|v(\tau)\|_{1,p}^{\alpha-1})\|(u - v)(\tau)\|_p d\tau \\ &\leq c(\|u\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}^{\alpha-1} + \|v\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}^{\alpha-1})\|u - v\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}. \end{aligned} \quad (5.10)$$

The last inequality follows from Hölder's inequality in t .

On the other hand, we have that

$$\begin{aligned} I_2 &\leq c \int_0^t \|(|u|^{\alpha-2} + |v|^{\alpha-2})(u - v)u_x(\tau)\|_2 d\tau \\ &\quad + c \int_0^t \|v|v|^{\alpha-2}(u_x - v_x)(\tau)\|_2 d\tau \\ &= I_2^1 + I_2^2. \end{aligned}$$

The argument in (5.7) leads to

$$I_2^2 \leq c\|v\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}^{\alpha-1} \|u - v\|_{L^\alpha(\mathbb{R};L^p_1(\mathbb{R}))}. \quad (5.11)$$

To bound I_2^1 , we first apply the generalized Hölder's inequality to obtain

$$\|(|u|^{\alpha-2} + |v|^{\alpha-2})(u-v)u_x(\tau)\|_2 \leq (\|u\|_{p'}^{\alpha-2} + \|v\|_{p'}^{\alpha-2})\|u-v\|_{p'}\|u_x\|_p \quad (5.12)$$

where $p' = \frac{2p(\alpha-1)}{p-2}$, $p = \frac{2}{1-\gamma}$. Then using the fact that $L_1^p \subset L^{p'}$ for $\alpha > \frac{1}{1-\gamma}$, $\gamma \in (0, 4/5)$, and Hölder's inequality in t it follows that

$$\begin{aligned} I_2^1 &\leq c \int_0^t (\|u(\tau)\|_{1,p}^{\alpha-1} + \|v(\tau)\|_{1,p}^{\alpha-2}\|u(\tau)\|_{1,p})\|(u-v)(\tau)\|_{1,p} d\tau \\ &\leq c(\|u\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-1} + \|v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-2}\|u\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))})\|u-v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}. \end{aligned} \quad (5.13)$$

Gathering (5.10), (5.11) and (5.13) up, we find that

$$\begin{aligned} \sup_t \|(\Phi(u) - \Phi(v))(t)\|_{1,2} &\leq c(2\|u\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-1} + 2\|v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-1} \\ &\quad + \|v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-2}\|u\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))})\|u-v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}. \end{aligned} \quad (5.14)$$

Next we shall estimate $\|\Phi(u) - \Phi(v)\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}$, so we begin by estimating $\|\Phi(u) - \Phi(v)\|_{L^\alpha(\mathbb{R};L^p(\mathbb{R}))}$, it can be done using a similar argument as in (5.8), thus we obtain

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{L^\alpha(\mathbb{R};L^p(\mathbb{R}))} &\leq c(\|u\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-1} + \|v\|_{L^\alpha(\mathbb{R};L_1^p(\mathbb{R}))}^{\alpha-1}) \\ &\quad \times \sup_t \|(u-v)(t)\|_{1,2}. \end{aligned} \quad (5.15)$$

It remains to estimate $\|(\Phi(u) - \Phi(v))_x\|_{L^\alpha(\mathbb{R};L^p(\mathbb{R}))}$. Lemma 2.15 leads to

$$\begin{aligned} \|(\Phi(u) - \Phi(v))_x(t)\|_p &\leq c \int_0^t (t-\tau)^{-\gamma/2} \|(|u|^{\alpha-2} + |v|^{\alpha-2})(u-v)u_x(\tau)\|_q d\tau \\ &\quad + c \int_0^t (t-\tau)^{-\gamma/2} \| |v|v|^{\alpha-2}(u_x - v_x)(\tau)\|_q d\tau \\ &= I^1 + I^2 \quad \text{here } q = \frac{2}{1+\gamma}. \end{aligned}$$

To bound I^2 we use a similar argument as in (5.8) to obtain

$$I^2 \leq c \int_0^t (t-\tau)^{-\gamma/2} \|v(\tau)\|_{1,p}^{\alpha-1} \|(u-v)(\tau)\|_{1,2} d\tau. \quad (5.16)$$

Now to estimate I^1 we follow the argument in (5.12) to have

$$I^1 \leq c \int_0^t (t-\tau)^{-\gamma/2} (\|u(\tau)\|_{1,p}^{\alpha-2} + \|v(\tau)\|_{1,p}^{\alpha-2})\|(u-v)(\tau)\|_{1,p}\|u_x(\tau)\|_{1,2} d\tau.$$

Setting $\frac{\gamma}{2} = \frac{2}{\alpha}$ and using Hardy-Littlewood-Sobolev theorem it follows that

$$\begin{aligned} \|(\Phi(u) - \Phi(v))_x\|_{L^\alpha(\mathbb{R}; L^p(\mathbb{R}))} &\leq c \|v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-1} \sup_t \|(u-v)(t)\|_{1,2} \\ &+ (\|u\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-2} + \|v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-2}) \sup_t \|u(t)\|_{1,2} \|u-v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}. \end{aligned} \quad (5.17)$$

Therefore a combination of (5.15), (5.16) and (5.17) leads to

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))} \\ &\leq c (\|u\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-1} + 2\|v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-1}) \sup_t \|(u-v)(t)\|_{1,2} \\ &+ (\|u\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-2} + \|v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}^{\alpha-2}) \sup_t \|u(t)\|_{1,2} \|u-v\|_{L^\alpha(\mathbb{R}; L_1^p(\mathbb{R}))}. \end{aligned} \quad (5.18)$$

Thus from (5.14) and (5.18) it follows that

$$\begin{aligned} \Lambda(\Phi(u) - \Phi(v)) &\leq c \{2(\Lambda(u))^{\alpha-1} + 2(\Lambda(v))^{\alpha-1} + (\Lambda(v))^{\alpha-2} \Lambda(u)\} \Lambda(u-v) \\ &\leq \Lambda(u-v), \quad \text{by the choice of } a. \end{aligned}$$

This shows that Φ is a contraction. Therefore the contraction mapping principle gives the existence and uniqueness in Y_a . It is not difficult to prove the uniqueness in X , this completes the proof. \square

Now we are in position to prove Theorem 5.4

PROOF OF THEOREM 5.4. We define

$$u_\pm(t) = u(t) + \int_{\pm\infty}^t V_2(t-\tau) (|u|^{\alpha-1}u)_{xx}(\tau) d\tau,$$

where $u(t)$ is given by Theorem (5.5).

From (2.27), (2.33) and Hölder's inequality it follows that

$$\begin{aligned} \|u(t) - u_\pm(t)\|_{1,2} &\leq c \int_{\pm\infty}^t \| |u|^{\alpha-1}u(\tau) \|_{1,2} d\tau \\ &\leq c \int_{\pm\infty}^t \|u(\tau)\|_{2\alpha}^\alpha d\tau + c \int_{\pm\infty}^t \|u(\tau)\|_{\frac{2(\alpha-1)}{\gamma}}^{\alpha-1} \|u_x(\tau)\|_p d\tau. \end{aligned}$$

But $L_1^p(\mathbb{R}) \subset L^{2\alpha}(\mathbb{R})$ and $L_1^p(\mathbb{R}) \subset L^{\frac{2(\alpha-1)}{\gamma}}(\mathbb{R})$ for $\alpha > \frac{1}{1-\gamma}$ and $\gamma \in (0, 4/5)$. Therefore

$$\|u(t) - u_\pm(t)\|_{1,2} \leq 2c \int_{\pm\infty}^t \|u(\tau)\|_{1,p}^\alpha d\tau.$$

By Theorem 5.5, the integral on the right hand side approaches to zero as $t \rightarrow \pm\infty$. This gives the desired result. \square

5.3. Blow-up

As we commented in the introduction the Boussinesq equation

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0,$$

has a large set of initial data for which there are no global (in time) smooth solutions.

In this section we present a result obtained by Angulo and Scialom [2] showing the finite time blow-up of solutions to (1.1). This is an extension of a previous result obtained by Liu [35]. The main tool used in [35] was the application of general methods introduced by Payne and Sattinger in [39].

We will consider $\psi(u) = u|u|^{\alpha-1}$, $\alpha > 1$, and we write (1.1) as the first-order system given in (1.5), (1.6), that is,

$$\begin{cases} u_t = v_x, & x \in \mathbb{R}, \quad t \geq 0 \\ v_t = (u - u_{xx} - \psi(u))_x, \\ (u(x, 0), v(x, 0)) = (u_0, v_0). \end{cases} \quad (5.19)$$

One of the main idea is to find regions invariant for the flow and such that for initial data in these sets, it is possible either to find global bounded solutions or solutions blowing-up in finite time. Here, we mean by blow-up of solutions the existence of $t^* < \infty$ such that

$$\lim_{t \rightarrow t^*-} \|u(t)\|_1 = +\infty.$$

We recall the time invariant quantities for the flow of (5.19),

$$\begin{cases} E(u, v) = \int_{\mathbb{R}} \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}v^2 - \frac{1}{\alpha+1}|u|^{\alpha+1} dx \\ \equiv \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{\alpha+1}\|u\|_{\alpha+1}^{\alpha+1} \\ Q(u, v) = \int_{\mathbb{R}} uv dx. \end{cases}$$

Define the set K_2^c as:

$$K_2^c = \{u \in H^1(\mathbb{R}) : L_c(u, -cu) < d(c), R_c(u) < 0\},$$

where $R_c(u) = \|u\|_{1,c}^2 - \|u\|_{\alpha+1}^{\alpha+1}$, $L_c(u, v) = E(u, v) + cQ(u, v)$ and $d(c) = L_c(\phi_c, -c\phi_c)$. The $\|\cdot\|_{1,c}^2$ -norm is defined by $\|u\|_{1,c}^2 \equiv (1-c^2)\|u\|_2^2 + \|u_x\|_2^2$, $|c| < 1$ and ϕ_c is the solution (up translation) of the nonlinear equation,

$$-\phi_c'' + (1-c^2)\phi_c - \phi_c|\phi_c|^{\alpha-1} = 0, \quad |c| < 1. \quad (5.20)$$

As it was noted in Bona and Sachs [6] the function ϕ_c can be found explicitly as

$$\phi_c(\xi) = \left[\frac{(\alpha+1)(1-c^2)}{2} \right]^{\frac{1}{\alpha-1}} \operatorname{sech}^{\frac{2}{\alpha-1}} \left(\frac{(\alpha-1)\sqrt{1-c^2}}{2} \xi \right).$$

The main result regarding blow-up in finite time of solutions of (5.19) is next.

THEOREM 5.6. *Let $\alpha > 1$ and $|c| < 1$. Suppose,*

- (a) $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and $\xi^{-1}\widehat{u_0}(\xi) \in L^2(\mathbb{R})$.
- (b) $u_0 \in K_2^c$, $E(u_0, v_0) < d(c)$ and $L_c(u_0, v_0) < d(c)$.

Let $\vec{u} = (u, v)$ be the solution of (5.19) with $\vec{u}_0 = (u_0, v_0)$ such that $\vec{u} \in C([0, T_{\max}); H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, where T_{\max} is the maximum time of existence of the solution. Then, $T_{\max} < +\infty$ and

$$\lim_{t \rightarrow T_{\max}^-} \|u(t)\|_1 = \lim_{t \rightarrow T_{\max}^-} \|u(t)\|_{\alpha+1} = +\infty.$$

REMARK 5.7. The above theorem considered in the special case of $c = 0$ recovers Liu's result (see Theorem 4.2 in [35]).

The first step in the proof of Theorem 5.6 is to characterize the best constant for the inequality

$$\|f\|_{\alpha+1} \leq B_\alpha^c \|f\|_{1,c}, \quad (5.21)$$

with $|c| < 1$, $1 < \alpha < \infty$. The value of the best constant B_α^c is obtained as the minimum of a constrained variational problem naturally associated with (5.21). This is made by using the method of concentration-compactness introduced by Lions in [34] (see also [1], [10] and Lopes in [36]). Once this characterization is obtained we use some of the ideas in ([39], [35]) to establish the theorem.

5.3.1. Variational Problem and the Best Constant. In this section we show that the best constant for the inequality (5.21) is $B_\alpha^c = \|\phi_c\|_{1,c}^{-\frac{\alpha-1}{\alpha+1}}$, where ϕ_c is the solution of (5.20) (see Theorem 5.10 below). This value can be obtained for instance using the concentration-compactness method [34]. Here we will use the next result due to Lopes [36].

Let

$$V(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^n} F(u(x)) dx \quad (5.22)$$

and

$$I(u) = \int_{\mathbb{R}^n} G(u(x)) dx = \lambda \neq 0. \quad (5.23)$$

For $n \geq 2$ let $\ell(n) = \frac{2n}{n-2}$. Denote by $\mathcal{M} = \{u \in H^1(\mathbb{R}^n) : I(u) = \lambda\}$ the admissible set. Denote by $f(u)$ and $g(u)$ the derivatives of $F(u)$ and $G(u)$, respectively. Set $F(u) = mu^2 + F_1(u)$ and $G(u) = m_0u^2 + G_1(u)$.

The following assumptions are made:

(H₁) $F_1(u)$ and $G_1(u)$ are C^2 functions with $F_1(0) = G_1(0) = 0 = F_1'(0) = G_1'(0)$ and for some constant k and $2 < q \leq p < \ell(N)$ we have

$$|F_1''(u)|, |G_1''(u)| \leq k(|u|^{q-2} + |u|^{p-2}).$$

(H₂) V is bounded below on \mathcal{M} and any minimizing sequence is bounded in $H^1(\mathbb{R}^n)$.

(H₃) If $u \in H^1(\mathbb{R}^n)$ and $u \neq 0$, then $g(u(\cdot)) \neq 0$

THEOREM 5.8. Assume (H₁), (H₂) and (H₃). If u_j is a minimizing sequence and u_j converges weakly in $H^1(\mathbb{R}^n)$ to $u \neq 0$, then u_j converges to u strongly in $L^r(\mathbb{R}^n)$, $2 < r < \ell(n)$ (for $n = 1$ this interval becomes $2 < r \leq \infty$). Moreover, if $m_0 = 0$ and $m > 0$. Then modulo translation in the x variable any minimizing sequence is precompact in $H^1(\mathbb{R}^n)$.

First we will see that the inequality (5.21) is satisfied for $|c| < 1$, indeed, considering the classical Sobolev embedding theorem, we obtain

$$\|f\|_{\alpha+1} \leq C_\alpha \|f\|_{1,2} \leq B_\alpha^c \|f\|_{1,c}, \quad (5.24)$$

where $B_\alpha^c = \frac{C_\alpha}{\sqrt{1-c^2}}$.

Next we state the variational problem which leads to the best constant.

THEOREM 5.9. *If $\alpha > 1$ and $|c| < 1$, then for $J(f) = \frac{\|f\|_{1,c}^2}{\|f\|_{\alpha+1}^2}$ we obtain*

$$\min\{J(f) : f \in H^1(\mathbb{R}), f \neq 0\} = J(\phi_c),$$

where ϕ_c is the solution of (5.20).

PROOF. Observe that since for $f \in H^1(\mathbb{R}), f \neq 0$, $J(\frac{f}{\|f\|_{\alpha+1}}) = J(f)$, then

$$\min\{J(f) : f \in H^1(\mathbb{R}), f \neq 0\} = \min\{J(f) : f \in H^1(\mathbb{R}), \|f\|_{\alpha+1} = 1\}.$$

Therefore to find the minimum in Theorem 5.9, it is sufficient to minimize the functional

$$V(f) \equiv \frac{1}{2} \int_{\mathbb{R}} [f'(x)]^2 dx + \frac{1}{2} \int_{\mathbb{R}} (1-c^2)f^2(x) dx \quad (5.25)$$

subject to the constraint,

$$I(f) \equiv \int_{\mathbb{R}} |f(x)|^{\alpha+1} dx = 1, \quad (5.26)$$

in the space $H^1(\mathbb{R})$. The proof of existence of this minimum is a straightforward application of Theorem 5.8.

Therefore, let $\varphi \in H^1(\mathbb{R})$ be such that $\|\varphi\|_{\alpha+1} = 1$ and $V(\varphi) = \inf_{f \in M} V(f)$. Then φ satisfies the Euler-Lagrange equation

$$V'(\varphi) = \lambda I'(\varphi),$$

for some $\lambda \in \mathbb{R}$ ($\lambda > 0$). This implies that φ satisfies the equation

$$-\varphi'' + (1-c^2)\varphi = \lambda(\alpha+1)\varphi|\varphi|^{\alpha-1},$$

in the distributions sense on \mathbb{R} . Taking $\tilde{\phi}_c(\xi) = [\lambda(\alpha+1)]^{\frac{1}{\alpha-1}} \varphi(\xi)$, we obtain a distribution solution of (5.20), which, after a bootstrapping argument, is in fact a C^∞ -function and satisfies (5.20) pointwise. Then by the uniqueness (up translations) of solution for (5.20) (see [4], Theorem 5) we must have that $\tilde{\phi}_c(\cdot) = \phi_c(\cdot + r)$. Thus

$$\begin{aligned} J(\phi_c) &= J([\lambda(\alpha+1)]^{\frac{1}{\alpha-1}} \varphi) = J(\varphi) = 2V(\varphi) \\ &= 2 \min\{V(f) : f \in H^1(\mathbb{R}), \|f\|_{\alpha+1} = 1\} \\ &= \min\{J(f) : f \in H^1(\mathbb{R}), f \neq 0\}. \end{aligned}$$

This completes the proof of Theorem 5.24. \square

As a consequence of this result we obtain the value of the best constant stated above.

THEOREM 5.10. *Let $\alpha > 1$. The smallest constant for which inequality (5.24) holds is given by $B_\alpha^c = \|\phi_c\|_{1,c}^{-\frac{\alpha-1}{\alpha+1}}$, where ϕ_c is the solution of (5.20).*

PROOF. Since ϕ_c satisfies (5.20) we obtain that $\|\phi_c\|_{1,c}^2 = \|\phi_c\|_{\alpha+1}^{\alpha+1}$. Moreover, by Theorem 5.9 $J(\phi_c) = \|\phi_c\|_{1,c}^{\frac{2(\alpha-1)}{\alpha+1}}$, thus $B_\alpha^c = \|\phi_c\|_{1,c}^{-\frac{\alpha-1}{\alpha+1}}$. \square

5.3.2. Finite Blow-up Time. In this section we use the best constant result to obtain the prove Theorem 5.9.

The next lemma is concerned with the invariance of the region K_2^c for the flow governed by (5.19).

LEMMA 5.11 (Invariant sets). *Suppose $f(s) = s|s|^{\alpha-1}$ with $\alpha > 1$. Let $|c| < 1$, $(u_0, v_0) \in K_2^c \times L^2(\mathbb{R})$ and $L_c(u_0, v_0) < d(c)$. Let $\vec{u} = (u, v)$ be the solution of (5.19) with $\vec{u}(0) = (u_0, v_0)$ such that $\vec{u} \in C([0, T]; H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ for $T > 0$. Then*

$$u(t) \in K_2^c$$

and

$$R_c(u(t)) < 2L_c(u_0, v_0) - 2d(c) \quad \text{for } t \in [0, T].$$

PROOF. See Lemma 5.3 in [35]. \square

The next differential inequality will play a crucial role in the proof of finite blow-up in time for (5.19).

LEMMA 5.12. *Suppose that a twice-differentiable function $\Psi(t)$ is positive and satisfies for $t \in [0, T)$ the inequality*

$$\Psi(t)\Psi''(t) - (1 + \beta)(\Psi'(t))^2 \geq 0$$

where $\beta > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then

$$\lim_{t \uparrow t_1^*} \Psi(t) = +\infty,$$

where $t_1^* \leq \frac{\Psi(0)}{\beta\Psi'(0)}$.

PROOF. See Sachs ([31]) or Levine ([29]). \square

We are ready to establish Theorem 5.6.

PROOF OF THEOREM 5.6. Suppose that $T_{max} = +\infty$. A contradiction will be obtained from Lemma 5.12, choosing $\Psi(t) = \|\xi^{-1}\hat{u}(t)\|^2$. In fact, as $\xi^{-1}\hat{u}_t \in L^2(\mathbb{R})$, we obtain

$$\Psi'(t) = 2\text{Re} \langle \xi^{-1}\hat{u}, \xi^{-1}\hat{u}_t \rangle.$$

From (1.1) and (5.19) it follows that $\Psi''(t) = 2\|v(t)\|^2 - 2\|u(t)\|_1^2 + 2\|u(t)\|_{\alpha+1}^{\alpha+1}$. We now see that, $\Psi''(t) > (\alpha + 3)\|v(t)\|^2$. In fact,

$$\begin{aligned} \Psi''(t) &= (\alpha + 3)\|v(t)\|^2 + (\alpha - 1)\|u(t)\|_{1,2}^2 - 2(\alpha + 1)E(u_0, v_0) \\ &= (\alpha + 3)\|v\|^2 + (\alpha - 1)\|u\|_{1,c}^2 + (\alpha - 1)c^2\|u\|^2 - 2(\alpha + 1)E(u_0, v_0) \\ &> (\alpha + 3)\|v\|^2 + (\alpha - 1)\|u\|_{1,c}^2 - 2(\alpha + 1)d(c) + (\alpha - 1)c^2\|u\|^2. \end{aligned}$$

Now, from assumption (b) and the result concerning the best constant (Theorem 5.10), we obtain

$$\|u\|_{1,c}^2 < \|u(t)\|_{\alpha+1}^{\alpha+1} \leq \|\phi_c\|_{1,c}^{-(\alpha-1)} \|u\|_{1,c}^{\alpha+1}.$$

Thus,

$$\begin{aligned}
\Psi''(t) &> (\alpha + 3)\|v\|^2 + (\alpha - 1)\|\phi_c\|_{1,c}^2 - 2(\alpha + 1)d(c) + (\alpha - 1)c^2\|u\|^2 \\
&= (\alpha + 3)\|v\|^2 + (\alpha - 1)\|\phi_c\|_{1,c}^2 - 2(\alpha + 1)\left[\frac{\alpha - 1}{2(\alpha + 1)}\|\phi_c\|_{1,c}^2\right] \\
&\quad + (\alpha - 1)c^2\|u\|^2 \\
&= (\alpha + 3)\|v\|^2 + (\alpha - 1)c^2\|u\|^2
\end{aligned} \tag{5.27}$$

where we used that $d(c) = L_c(\phi_c, -c\phi_c) = \frac{\alpha-1}{2(\alpha+1)}\|\phi_c\|_{1,c}^2$.

Thus, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\Psi(t)\Psi''(t) - \frac{\alpha + 3}{4}(\Psi'(t))^2 &> (\alpha + 3)\|\xi^{-1}\widehat{u}\|^2\|v\|^2 \\
&\quad - \frac{\alpha + 3}{4}(2 \operatorname{Re} \langle \xi^{-1}\widehat{u}, \xi^{-1}\widehat{u}_t \rangle)^2 \\
&\geq (\alpha + 3)\|\xi^{-1}\widehat{u}\|^2\|v\|^2 - (\alpha + 3)\|\xi^{-1}\widehat{u}\|^2\|v\|^2 = 0.
\end{aligned}$$

Now, we suppose that $\Psi'(t_1) > 0$ for some $t_1 > 0$ (this will imply that $\Psi(t_1) > 0$). Then from Lemma 5.12 we obtain that

$$\lim_{t \rightarrow t_2^-} \|\xi^{-1}\widehat{u}(t)\|^2 = \lim_{t \rightarrow t_2^-} \Psi(t) = +\infty$$

for $t_1 < t_2 \leq t^* = \frac{\Psi(t_1)}{\beta\Psi'(t_1)} + t_1$, where $\beta = \frac{\alpha-1}{4}$. Hence there exists a sequence $\{t_n\}_{n \geq 1}$, $t_n \rightarrow t_2^-$ such that $\lim_{t_n \rightarrow t_2^-} \|v(t_n)\| = +\infty$. This contradicts the fact that $v \in C([0, +\infty); L^2(\mathbb{R}))$.

Thus $T_{max} < +\infty$ and from the local existence theory (see [31]), we have that

$$\lim_{t \rightarrow T_{max}^-} \|u(t)\|_1 + \|v(t)\| = +\infty.$$

Finally, since $E(u, v) = E(u_0, v_0)$ we obtain

$$\lim_{t \rightarrow T_{max}^-} \|u(t)\|_1 = \lim_{t \rightarrow T_{max}^-} \|u(t)\|_{\alpha+1} = +\infty.$$

Next we prove that $\Psi'(t_1) > 0$ for some $t_1 > 0$. Suppose not, then, $\Psi'(t) \leq 0$ for all $t \geq 0$. Since $\Psi''(t) > 0$ and Ψ' is continuous, the limit $\lim_{t \rightarrow +\infty} \Psi'(t)$ exists. Then,

$$\lim_{t \rightarrow +\infty} \Psi'(t) = \Psi'(0) + \int_0^{+\infty} \Psi''(s) ds < +\infty.$$

Hence, we obtain a sequence $\{t_n\}_{n \geq 1}$ such that $\lim_{t_n \rightarrow +\infty} \Psi''(t_n) = 0$. Then it follows from (5.27) that $\lim_{t_n \rightarrow +\infty} \|v(t_n)\| = \lim_{t_n \rightarrow +\infty} \|u(t_n)\| = 0$. Moreover,

$$\begin{aligned}
0 &= \lim_{t_n \rightarrow +\infty} \Psi''(t_n) \\
&= \lim_{t_n \rightarrow +\infty} [(\alpha + 3)\|v(t_n)\|^2 + (\alpha - 1)\|u(t_n)\|_{1,c}^2 + (\alpha - 1)c^2\|u(t_n)\|^2] \\
&\quad - 2(\alpha + 1)E(u_0, v_0) \\
&= (\alpha - 1) \lim_{t_n \rightarrow +\infty} \|u(t_n)\|_{1,c}^2 - 2(\alpha + 1)E(u_0, v_0).
\end{aligned}$$

Thus,

$$\lim_{t_n \rightarrow +\infty} \|u(t_n)\|_{1,c}^2 = \frac{2(\alpha+1)}{\alpha-1} E(u_0, v_0).$$

Now, since $L_c(u_0, v_0) < d(c)$, Lemma 5.11 implies that

$$\begin{aligned} R_c(u) &= \|u\|_{1,c}^2 - |u|_{\alpha+1}^{\alpha+1} < 2L_c(u_0, v_0) - 2d(c) \\ &\leq 2E(u_0, v_0) + 2|c|\|u\|\|v\| - 2d(c), \end{aligned} \quad (5.28)$$

and the result concerning the best constant yields

$$\begin{aligned} \limsup |u(t_n)|_{\alpha+1}^{\alpha+1} &\leq \|\phi_c\|_{1,c}^{-(\alpha-1)} \limsup \|u(t_n)\|_{1,c}^{\alpha+1} \\ &= \|\phi_c\|_{1,c}^{-(\alpha-1)} (\limsup \|u(t_n)\|_{1,c}^2)^{\frac{\alpha+1}{2}} \\ &\leq \|\phi_c\|_{1,c}^{-(\alpha-1)} \left[\frac{2(\alpha+1)}{\alpha-1} \right]^{\frac{\alpha+1}{2}} [E(u_0, v_0)]^{\frac{\alpha+1}{2}}. \end{aligned} \quad (5.29)$$

Therefore, from (5.28) and the relation $\limsup(a_n - b_n) \geq \limsup a_n - \limsup b_n$ we deduce that

$$\begin{aligned} \frac{2(\alpha+1)}{\alpha-1} E(u_0, v_0) - \|\phi_c\|_{1,c}^{-(\alpha-1)} \left[\frac{2(\alpha+1)}{\alpha-1} \right]^{\frac{\alpha+1}{2}} [E(u_0, v_0)]^{\frac{\alpha+1}{2}} \\ \leq \limsup \|u(t_n)\|_{1,c}^2 - \limsup |u(t_n)|_{\alpha+1}^{\alpha+1} \\ \leq 2E(u_0, v_0) - 2d(c). \end{aligned} \quad (5.30)$$

Since $d(c) = \frac{\alpha-1}{2(\alpha+1)} \|\phi_c\|_{1,c}^2$ and $E(u_0, v_0) < d(c)$ the left hand side of (3.4) is positive, therefore $E(u_0, v_0) > d(c)$. This contradiction completes the proof of Theorem 5.6. \square

APPENDIX A

A.1. Fourier Transform

DEFINITION A.1. The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$, denoted by \widehat{f} , is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx, \quad \text{for } \xi \in \mathbb{R}^n, \quad (\text{A.1})$$

where $x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n$.

We list some basic properties of the Fourier transform in $L^1(\mathbb{R}^n)$.

THEOREM A.2. Let $f \in L^1(\mathbb{R}^n)$. Then

- (1) $f \mapsto \widehat{f}$ defines a linear transformation from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ with

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \quad (\text{A.2})$$

- (2) \widehat{f} is continuous.

- (3) $\widehat{f}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ (Riemann–Lebesgue).

- (4) If $\tau_h f(x) = f(x - h)$ denotes the translation by $h \in \mathbb{R}^n$, then

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi i (h \cdot \xi)} \widehat{f}(\xi), \quad (\text{A.3})$$

and

$$\widehat{(e^{-2\pi i (x \cdot h)} f)}(\xi) = (\tau_{-h} \widehat{f})(\xi). \quad (\text{A.4})$$

- (5) If $\delta_a f(x) = f(ax)$ denotes a dilation by $a > 0$, then

$$\widehat{(\delta_a f)}(\xi) = a^{-n} \widehat{f}(a^{-1} \xi). \quad (\text{A.5})$$

- (6) Let $g \in L^1(\mathbb{R}^n)$ and $f * g$ be the convolution of f and g . Then

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi). \quad (\text{A.6})$$

- (7) If $x_k f \in L^1(\mathbb{R}^n)$, then

$$\frac{\partial \widehat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k \widehat{f}(x))(\xi). \quad (\text{A.7})$$

- (8) If $\frac{\partial f}{\partial x_k} \in L^1(\mathbb{R}^n)$, then

$$\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \widehat{f}(\xi). \quad (\text{A.8})$$

THEOREM A.3 (Plancherel). *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f} \in L^2$ and*

$$\|\widehat{f}\|_2 = \|f\|_2. \quad (\text{A.9})$$

Theorem A.3 shows that the Fourier transform defines a linear bounded operator from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Indeed, this operator is an isometry. Thus, there is a unique bounded extension \mathcal{F} defined in all $L^2(\mathbb{R}^n)$. \mathcal{F} is called the Fourier transform in $L^2(\mathbb{R}^n)$. We shall use the notation $\widehat{f} = \mathcal{F}(f)$ for $f \in L^2(\mathbb{R}^n)$. In general, this definition \widehat{f} is realized as a limit in L^2 of the sequence $\{\widehat{h}_j\}$, where $\{h_j\}$ denotes any sequence in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which converges to f in the L^2 -norm. It is convenient to take h_j equals to f for $|x| \leq j$ and vanishing for $|x| > j$. Then,

$$\widehat{h}_j(\xi) = \int_{|x| < j} f(x) e^{2\pi i x \cdot \xi} dx = \int_{\mathbb{R}^n} h_j(x) e^{2\pi i x \cdot \xi} dx$$

and so

$$\widehat{h}_j(\xi) \rightarrow \widehat{f}(\xi) \quad \text{in } L^2 \text{ as } j \rightarrow \infty.$$

An isometry which is also onto defines a unitary operator. Theorem A.3 affirms that \mathcal{F} is an isometry and if fact \mathcal{F} is also onto.

THEOREM A.4. *The Fourier transform defines an unitary operator in $L^2(\mathbb{R}^n)$.*

THEOREM A.5. *The inverse of the Fourier transform \mathcal{F}^{-1} can be expressed as*

$$(\mathcal{F}\mathcal{F}f)(x) = f(-x), \quad \text{for any } f \in L^2(\mathbb{R}^n). \quad (\text{A.10})$$

From the definitions of the Fourier transform on $L^1(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$ there is a natural extension to $L^1 + L^2$. It is not hard to see that $L^1 + L^2$ contains the spaces $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$. On the other hand, as we shall prove, any function in $L^p(\mathbb{R}^n)$ for $p > 2$ has a Fourier transform in the distribution sense. However, they may not be functions, they are **tempered distributions**.

For each $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ we denote the semi-norm $\|\cdot\|_{(\nu, \beta)}$ defined as

$$\|f\|_{(\nu, \beta)} = \|x^\nu \partial_x^\beta f\|_\infty.$$

Now we define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of the C^∞ -functions decaying at infinity, i.e.,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{(\nu, \beta)} < \infty \text{ for any } \nu, \beta \in (\mathbb{Z}^+)^n\}.$$

Observe that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. The topology in $\mathcal{S}(\mathbb{R}^n)$ is given by the family of semi-norms $\|\cdot\|_{(\nu, \beta)}$, $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$.

DEFINITION A.6. Let $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$. Then $\varphi_j \rightarrow 0$ as $j \rightarrow \infty$ if for any $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ one has that

$$\|\varphi_j\|_{(\nu, \beta)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

The relationship between the Fourier transform and the function space $\mathcal{S}(\mathbb{R}^n)$ is described in the formulae (A.7), (A.8). More precisely, we have the following result:

THEOREM A.7. *The map $\varphi \mapsto \widehat{\varphi}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ into itself.*

The space $\mathcal{S}(\mathbb{R}^n)$ appears naturally associated to the Fourier transform. By duality we can define the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

DEFINITION A.8. We say that $\Psi : \mathcal{S}(\mathbb{R}^n) \mapsto \mathbb{C}$ defines a *tempered distribution* if

- (1) Ψ is linear,
- (2) Ψ is continuous, i.e. if $\varphi_j \rightarrow 0$ as $j \rightarrow \infty$ then the numerical sequence $\Psi(\varphi_j) \rightarrow 0$ as $j \rightarrow \infty$.

Now given a $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ its Fourier transform can be defined in the following natural form.

DEFINITION A.9. Given $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ its *Fourier transform* $\widehat{\Psi} \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$\widehat{\Psi}(\varphi) = \Psi(\widehat{\varphi}), \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (\text{A.11})$$

DEFINITION A.10. Let $\{\Psi_j\} \subset \mathcal{S}'(\mathbb{R}^n)$. Then $\Psi_j \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{S}'(\mathbb{R}^n)$, if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that $\Psi_j(\varphi) \rightarrow 0$ as $j \rightarrow \infty$.

As a consequence of the definitions A.8, A.10 we get the following extension of Theorem A.7.

THEOREM A.11. *The map $\mathcal{F} : \Psi \mapsto \widehat{\Psi}$ is an isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ into itself.*

A.2. Interpolation of Operators

Let (X, \mathcal{A}, μ) be a measurable space (i.e. X is a set, \mathcal{A} denotes a σ -algebra of measurable sub-sets of X and μ is a measure defined on \mathcal{A}). $L^p = L^p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$ denotes the space of complex valued functions f that are σ -measurable such that

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu \right)^{1/p} < \infty.$$

Functions in $L^p(X, \mathcal{A}, \mu)$ are defined almost everywhere with respect to μ . Similarly, we have $L^\infty(X, \mathcal{A}, \mu)$ the space of functions f that are μ -measurable complex valued and essentially μ -bounded, with $\|f\|_\infty$ the essential supremum of f .

Let T be a linear operator from $L^p(X)$ to $L^q(Y)$. If T is continuous or bounded, i.e.,

$$M = \sup_{f \neq 0} \frac{\|Tf\|_q}{\|f\|_p} < \infty.$$

we call the number M the norm of the operator T .

THEOREM A.12 (Riesz-Thorin). *Let $p_0 \neq p_1$, $q_0 \neq q_1$. Let T be a bounded linear operator from $L^{p_0}(X, \mathcal{A}, \mu)$ to $L^{q_0}(Y, \mathcal{B}, \nu)$ with norm M_0 and from $L^{p_1}(X, \mathcal{A}, \mu)$ to $L^{q_1}(Y, \mathcal{B}, \nu)$ with norm M_1 . Then T is bounded from $L^{p_\theta}(X, \mathcal{A}, \mu)$ in $L^{q_\theta}(Y, \mathcal{B}, \nu)$ with norm M_θ such that*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta,$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0, 1). \quad (\text{A.12})$$

An extension of this result to a family of linear operators is given next. Let S be the strip defined by

$$S = \{z = x + iy : 0 \leq x \leq 1\}$$

and $z = x + iy \in S$. Suppose that for each $z \in S$ corresponds a linear operator T_z defined on the space of simple functions in $L^1(X, \mathcal{A}, \mu)$ into measurable functions on Y in such a way $(T_z f)g$ is integrable on Y provided f is a simple function in $L^1(X, \mathcal{A}, \mu)$ and g is a simple function in $L^1(Y, \mathcal{B}, \nu)$.

DEFINITION A.13. The family of operators $\{T_z\}$ is called *admissible* if the mapping

$$z \mapsto \int_Y (T_z f)g \, d\nu$$

is analytic in the interior of S , continuous on S and there exists a constant $a < \pi$ such that

$$e^{-a|y|} \log \left| \int_Y (T_z f)g \, d\nu \right|$$

is uniformly bounded above in the strip S .

THEOREM A.14 (Stein). *Suppose $\{T_z\}$, $z \in S$, is an admissible family of linear operators satisfying*

$$\|T_{iy}f\|_{q_0} \leq M_0(y) \|f\|_{p_0} \quad \text{and} \quad \|T_{1+iy}f\|_{q_1} \leq M_1(y) \|f\|_{p_1}$$

for all simple functions f in $L^1(X, \mathcal{A}, \mu)$, where $1 \leq p_j, q_j \leq \infty$, $M_j(y)$, $j = 0, 1$, are independent of f and satisfy

$$\sup_{-\infty < y < \infty} e^{b|y|} \log M_j(y) < \infty$$

for some $b < \pi$. Then, if $0 \leq t \leq 1$, there exists a constant M_t such that

$$\|T_t f\|_{q_t} \leq M_t \|f\|_{p_t}$$

for all simple functions f provided

$$\frac{1}{p_t} = \frac{(1-t)}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{(1-t)}{q_0} + \frac{t}{q_1}.$$

PROOF. For the proof of this theorem we refer the reader to [46]. □

A.3. Fractional Integral Theorem

Next we describe some continuity properties of the Riesz potentials. We recall that the fundamental solution of the laplacian Δ is given by the following formula describing the newtonian potential

$$Uf(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} \, dy, \quad \text{for } n \geq 3.$$

The Riesz potentials generalize this expression.

DEFINITION A.15. Let $0 < \alpha < n$. The Riesz potential of order α , denoted by I_α is defined as

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = c_\alpha k_\alpha * f(x) \quad (\text{A.13})$$

where $c_\alpha = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma(n/2 - \alpha/2)$.

Since the Riesz potentials are defined as integral operators it is natural to study their continuity properties in $L^p(\mathbb{R}^n)$.

THEOREM A.16 (Hardy-Littlewood-Sobolev). Let $0 < \alpha < n$, $1 \leq p < q < \infty$, with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

(1) If $f \in L^p(\mathbb{R}^n)$, then the integral (A.13) is absolutely convergent for almost every $x \in \mathbb{R}^n$.

(2) If $p > 1$, then I_α is of type (p, q) , i.e.,

$$\|I_\alpha(f)\|_q \leq c_{p,\alpha,n} \|f\|_p. \quad (\text{A.14})$$

A proof of this result can be seen in [45].

A.4. Sobolev Spaces

DEFINITION A.17. Let $s \in \mathbb{R}$, we define Sobolev space of order s , denoted by $H^s(\mathbb{R}^n)$, as

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s f(\xi) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^n) \right\}, \quad (\text{A.15})$$

with norm $\|\cdot\|_{s,2}$ defined as

$$\|f\|_{s,2} = \|\Lambda^s f\|_2. \quad (\text{A.16})$$

From the definition of Sobolev spaces we deduce the following properties.

PROPOSITION A.18.

- (1) If $s < s'$, then $H^{s'}(\mathbb{R}^n) \not\subseteq H^s(\mathbb{R}^n)$.
- (2) $H^s(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_s$ defined as follows:

$$\text{If } f, g \in H^s(\mathbb{R}^n), \text{ then } \langle f, g \rangle_s = \int_{\mathbb{R}^n} \Lambda^s f(\xi) \overline{\Lambda^s g(\xi)} d\xi.$$

We can see, via the Fourier transform, that $H^s(\mathbb{R}^n)$ is equal to $L^2(\mathbb{R}^n; (1 + |\xi|^2)^s d\xi)$.

- (3) For any $s \in \mathbb{R}$, the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, is dense in $H^s(\mathbb{R}^n)$.
- (4) If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1 - \theta) s_2$, $0 \leq \theta \leq 1$, then

$$\|f\|_{s,2} \leq \|f\|_{s_1,2}^\theta \|f\|_{s_2,2}^{1-\theta}.$$

The relationship between the spaces $H^s(\mathbb{R}^n)$ and the differentiability of functions in $L^2(\mathbb{R}^n)$, is given in the next result.

THEOREM A.19. *If k is a positive integer, then $H^k(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives (in the distribution sense) $\partial_x^\alpha f$ belong to $L^2(\mathbb{R}^n)$ for every $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$.*

In this case the norms $\|f\|_{k,2}$ and $\sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_2$ are equivalent.

THEOREM A.20 (Embedding). *If $s > n/2 + k$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C_\infty^k(\mathbb{R}^n)$ the space of functions with k continuous derivatives vanishing at infinity. In other words, if $f \in H^s(\mathbb{R}^n)$, $s > n/2 + k$, then (after a possible modification of f in a set of measure zero) $f \in C_\infty^k(\mathbb{R}^n)$ and*

$$\|f\|_{C^k} \leq c_s \|f\|_{s,2}. \quad (\text{A.17})$$

From the point of view of non linear analysis the next property is essential.

THEOREM A.21. *If $s > n/2$, then $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$ with*

$$\|fg\|_{s,2} \leq c_s \|f\|_{s,2} \|g\|_{s,2}. \quad (\text{A.18})$$

Next we list some useful inequalities. We begin with the so called the Gagliardo-Nirenberg inequality, that is,

$$\|\partial_x^\alpha f\|_p \leq c \sum_{|\beta| \leq m} \|\partial_x^\beta f\|_q^\theta \|f\|_r^{1-\theta}, \quad (\text{A.19})$$

with $|\alpha| = j$, $c = c(j, m, p, q, r)$, $\frac{1}{p} - \frac{j}{n} = \theta(\frac{1}{q} - \frac{m}{n}) + (1 - \theta)\frac{1}{r}$, $\theta \in [\frac{j}{m}, 1]$. For the proof of this inequality we address the reader to the reference [16].

For the general case $s > 0$ where the usual pointwise Leibniz rule is not available, the inequality

$$\|fg\|_{s,2} \leq c_s (\|f\|_{s,2} \|g\|_\infty + \|f\|_\infty \|g\|_{s,2}). \quad (\text{A.20})$$

holds (see [25]).

In many applications the following commutator estimates are often used.

$$\begin{aligned} \sum_{|\alpha|=s} \|[\partial_x^\alpha; g] f\|_2 &= \sum_{|\alpha|=s} \|\partial_x^\alpha(gf) - g\partial_x^\alpha f\|_2 \\ &\leq c_{n,s} (\|\nabla g\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta f\|_2 + \|f\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta g\|_2), \end{aligned} \quad (\text{A.21})$$

(see [25], [28]). Similarly, for $s \geq 1$ one has

$$\|[\Lambda^s; g] f\|_2 \leq c (\|\nabla g\|_\infty \|\Lambda^{s-1} f\|_2 + \|f\|_\infty \|\Lambda^s g\|_2), \quad (\text{A.22})$$

(see [25]).

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